Technische Universität München

TUM School of Computation, Information and Technology

A dynamic p-Laplacian: theory, computational aspects, and numerical experiments

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Vollständiger Abdruck der von der TUM School of Computation, Information and Technology der Technischen Universität München zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften (Dr. rer. nat.)

genehmigten Dissertation.

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Die Dissertation wurde am 25.09.2023 bei der Technischen Universität München eingereicht und durch die TUM School of Computation, Information and Technology am 26.01.2024 angenommen.

Abstract

We introduce a dynamic p-Laplacian, a generalization of the dynamic Laplacian introduced by Froyland, in a similar way that the well-known p-Laplacian is derived from the standard 2-Laplacian. This p-Laplacian has connections to a geometric problem called the Cheeger problem. These get more pronounced as p approaches 1. We transfer known results about these connections to the dynamic setting, study an associated numerical approximation, and perform numerical experiments.

Zusammenfassung

Wir führen einen dynamischen p-Laplace als Verallgemeinerung des von Froyland eingeführten dynamischen Laplaces ein, ähnlich zur bekannten Konstruktion des p-Laplace-Operators aus dem klassischen 2-Laplace-Operator. Dieser p-Laplace lässt sich mit einem geometrischen Problem, dem Cheeger-Problem, in Verbindung bringen. Für p nahe 1 wird diese Verbindung ausgeprägter. Wir übertragen bekannte Resultate darüber auf den dynamischen Fall, untersuchen damit verbundene numerische Approximation und führen numerische Experimente durch.

Acknowledgements

First and foremost, I would like to thank my supervisor, Oliver Junge, for his continued support and vital feedback in countless discussions and, at the same time, for being open and considerate when I had requests.

Following right behind, I thank Gary Froyland and Péter Koltai for frequent and thorough feedback and critique about topics of this thesis in connection with the accompanying publication [DFJK23]. Having someone so dedicated do discuss with has been a great joy. There have also been other people I had the opportunity to talk to about my ideas and problems; in particular, I would like to thank David Hien, Nathanael Schilling and Christian Ludwig.

Representative of all of my friends, who have given me essential support when I needed it, I thank Ronja Dudda, Marcel Maier, Leopold Zoller, Hannah Zoller, Leander Stecker and Kilian Lieret, not only for help with proofreading but also for all the other things they did to aid me in this project.

Finally, the biggest thanks go out to my parents and sister, whom I owe so much I cannot begin to list it here. Thank you for making this possible!

Contents

Chapter 1

Introduction

Dynamical systems often exhibit complex behavior. A prominent example of this are systems that one encounters in fluid dynamics, which commonly show complicated phenomena where mixing and turbulence is combined with the appearance of long-lived, coarse structures. In this thesis, we seek to sharpen a tool that was developed to help understand such systems: The *dynamic Laplacian* Δ^D introduced by Froyland in [Fro15]. Our approach is to combine this generalization of the standard Laplacian with another one, namely the *p-Laplacian* Δ_p.

The goal is to introduce and understand a *dynamic p-Laplacian* Δ_p^D. This is motivated by known theory about connections between Δ_p and a certain geometric problem, the so-called Cheeger problem. These connections get more pronounced as $p \to 1$, and we hope to encounter similar behavior in the dynamic p-Laplacian Δ_p^D. Ultimately, we try to improve the dynamic Laplacian as a tool for understanding nonautonomous dynamical systems.

In this chapter, we will elaborate on the preceding paragraphs with increasing detail, introducing the objects at play and touching some known theory about them. Chapter 2 is dedicated to introducing the dynamic p-Laplacian Δ_p^D and establishing some properties analogous to the known theory for Δ_p. Chapter 3 treats numerical approximation of the relevant associated nonlinear eigenvalue problem and Chapter 4 contains numerical experiments about the connections between this eigenvalue problem and the Cheeger problem.

The results of this thesis have partially been published in [DFJK23] in collaboration with Oliver Junge, Gary Froyland and Péter Koltai.

1.1 Outline

We give a more detailed but still informal description of the setup that motivates this thesis. Formal introductions will be given in Sections 1.2 to 1.5.

Say we are given some (Riemannian) manifold M and a one-parameter family of volume-preserving diffeomorphisms $T_t : M \to M$, describing, for example, an incompressible fluid in motion. In order to gain a coarse description of transport of material within that system, a wide range of schemes has been developed [Had+17], which try to find structures that exhibit some kind of *coherence* over long periods of time. The question of what a good exact notion of coherence constitutes, has been the subject of debate, and there are different approaches in characterizing it. We will touch this briefly in Section 1.2; here we only present the approach that motivates this thesis:

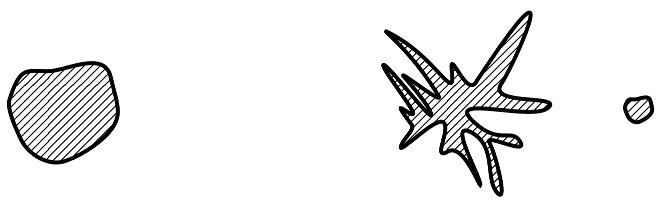
In [Fro15], Froyland proposed to measure coherence of a set by averaging the ratio of its perimeter and its volume over the course of the dynamics. Denoting by ℓ_k the k-dimensional Hausdorff measure and a subset of M by D, this ratio is

$$\frac{\ell_{d-1}(\partial D)}{\ell_d(D)}.$$

It is a well-known quantity that is sometimes called the *Cheeger ratio* of D [Leo15]. The notion of coherence proposed by Froyland[1] says that a coherent set should on average have a small Cheeger ratio, i.e.

$$\frac{1}{L}\int_0^L \frac{\ell_{d-1}(\partial(T_t(D)))}{\ell_d(T_t(D))}dt = \frac{\frac{1}{L}\int_0^L \ell_{d-1}(\partial(T_t(D))dt}{\ell_d(D)}$$

should be small if we are to call D a coherent set. This penalizes filamentation but also rules out very small sets, as the volume of the boundary scales with a smaller order than the volume of the interior:



Small Cheeger ratio *Big Cheeger ratios*

It is often enough to only average the Cheeger ratio over the start time $t = 0$ and end time $t = L$. (In most cases, the Cheeger ratio is unlikely to decrease substantially once it becomes big). The simplified quantity to minimize then becomes

$$\frac{1}{2}\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{\ell_d(D)},$$

where $T := T_L$ is the flow T_t at time $t = L$. Figure 1.1 illustrates sets having small and big averaged Cheeger ratios. We see a set as coherent if both its Cheeger ratios at time $t = 0$ and at time $t = L$ are small. This produces the geometric problem that we deal with in this thesis (we will state this more formally in Definition 2.2.1):

Problem 1.1.1 (informal)**.** *Given a manifold M and a volume-preserving diffeomorphism $T : M \to M$, find a subset $D \subset M$ that minimizes the* dynamic Cheeger ratio

$$\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)}$$

under all subsets of M.

[1] using the "Dirichlet" version from [FJ18, Section 2.2]; see Section 1.2.3 for details.

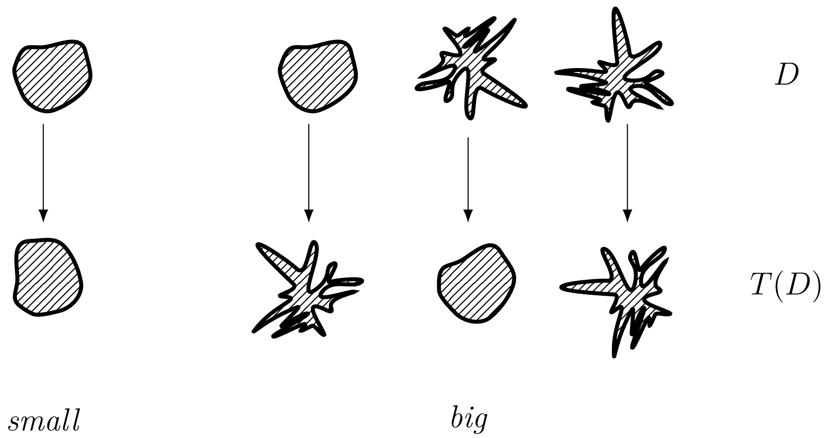


Figure 1.1: Illustration of sets having small (left) and big (right) average Cheeger ratio. We are looking for sets where it is small.

If $T = id$ is the identity, Problem 1.1.1 reduces to the well-known isoperimetric problem called the *Cheeger problem*, which asks for the minimum of the Cheeger ratio

$$\frac{\ell_{d-1}(\partial D)}{\ell_d(D)}.$$

Sets that minimize the Cheeger ratio are called *Cheeger sets* of M, and the value of their Cheeger ratio is called the *Cheeger constant* $h(M)$. Problem 1.1.1 is a true generalization of this, which we will call the *dynamic Cheeger problem*, following the naming convention of [Fro15]. Solutions of the dynamic Cheeger problem will be called *dynamic Cheeger sets* and their dynamic Cheeger ratio determines the *dynamic Cheeger constant* $h^D(M,T)$. Sometimes, we will denote the situation $T = id$ as the *classical* or *static* case.

It is well known that the (classical) Cheeger problem is equivalent to a variational problem. By the *Federer-Fleming theorem* [FF60; Cha01; Leo15] the Cheeger constant $h(M)$ coincides with the so-called *Sobolev constant*

$$s(M) = \inf_{u \neq 0} \frac{\|\nabla u\|_1}{\|u\|_1} \tag{1.1}$$

(we will later deal with the exact function space over which the infimum can be taken). Further, we can recover a solution for the Cheeger problem from a minimizer u of $\|\nabla u\|_1/\|u\|_1$ (where, in the limit, one has to define the gradient distributionally in the space of functions bounded variation, see Section 1.3): it can be shown that superlevel sets of u must be Cheeger sets [Par11]. Under the right circumstances, u is in fact a suitably scaled characteristic function on a Cheeger set.

The L^1 variational problem (1.1) is generally hard to solve numerically without certain regularizations [FP03]. It becomes easier to solve if one replaces $\|\cdot\|_1$ with $\|\cdot\|_2^2$. The new quantity

$$\lambda_2 := \inf_{u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} \tag{1.2}$$

coincides with the smallest eigenvalue of $-\Delta$ with Dirichlet boundary conditions, and a minimizer of (1.2) is given by its first eigenfunction. At the same time, λ_2 is connected to $h(M)$ via the well-known *Cheeger inequality*

$$h(M) \leq 2\sqrt{\lambda_2}.$$

Froyland generalized these results to a dynamic setting, introducing a *dynamic Sobolev constant*

$$s^D(M,T) := \inf_{u\neq 0} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1} \tag{1.3}$$

(where $T_* u := u \circ T^{-1}$ is the transfer operator) and proving a dynamic Federer-Fleming theorem stating that $h(M,T) = s(M,T)$. He also generalized the Cheeger inequality to a *dynamic Cheeger inequality*, showing[2] that

$$h^D(M,T) \leq 2\sqrt{\lambda_2^D},$$

where λ_2^D arises as the first Dirichlet eigenvalue of the so-called *dynamic Laplacian*

$$\Delta^D := \frac{1}{2}(\Delta + T^* \Delta T_*).$$

In the above expression, T^* is the dual of the transfer operator T_* and maps a function f to $T^* f = f \circ T$. This first eigenvalue λ_2 also arises in a variational problem, namely

$$\lambda_2^D = \inf_{u\neq 0} \frac{\|\nabla u\|_2^2 + \|\nabla(T_* u)\|_2^2}{2\|u\|_2^2}. \tag{1.4}$$

As in the static case, the eigenvalue problem for Δ^D is easier to solve numerically than the L^1 problem, e.g. by finite element methods [FJ18; SFJ20], or radial basis functions [FJ15].

The effect of replacing $\|\cdot\|_1$ with $\|\cdot\|_2^2$ We will illustrate[3] heuristically how the minimizer of the modified variational problem

$$\inf_{u\neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}$$

relates to a minimizer of (1.1) in the example of the unit square $M = [0,1]^2 \subset \mathbb{R}^2$. It can be shown that the Cheeger constant of $[0,1]^2$ is

$$h([0,1]^2) = 2 + \sqrt{\pi} \approx 3.772\ldots$$

and the Cheeger set of M is unique and coincides with a square with rounded corners of radius

$$R = \frac{1}{h([0,1]^2)} \approx 0.265\ldots$$

(see Example 1.3.6). In this setting, it can be shown [e.g. Par11] that a minimizer u in (1.1) is a multiple of the characteristic function χ_D on a Cheeger set D (see Figure 1.2). In particular, all of its superlevel sets

$$A_t := \{x \in M \mid u(x) > t\}, \quad (t > 0)$$

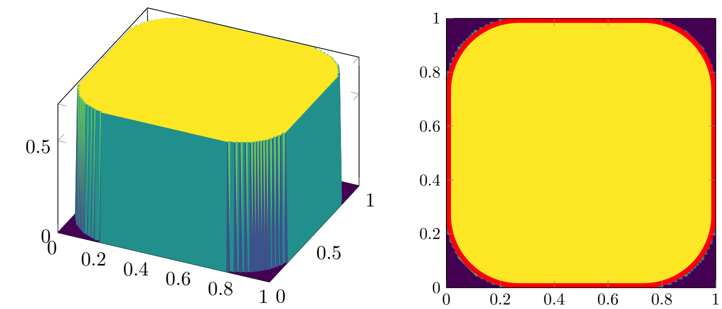


Figure 1.2: A minimizer u of the variational problem in (1.1) on the unit square $[0,1]^2$. the Cheeger set of $[0,1]^2$ is indicated in yellow, its boundary in red (figure also appears in [DFJK23]).

are Cheeger sets.

In contrast, if we replace $\|\cdot\|_1$ by $\|\cdot\|_2^2$, a minimizer coincides with the first eigenfunction of $-\Delta$ on M with Dirichlet boundary conditions, which is (see also Figure 1.3)

$$u_2(x) = \sin(\pi x_1)\sin(\pi x_2).$$

The level sets of u_2 are more distributed in M, although the superlevel set with the best Cheeger ratio is close to the actual Cheeger set. Informally speaking, one can hope that by having made $\|\nabla u\|_2^2/\|u\|_2^2$ small, one forces the Cheeger ratios of the superlevel sets to be not too big on average. This heuristically explains u_2 as a somewhat smoothed version of χ_D.

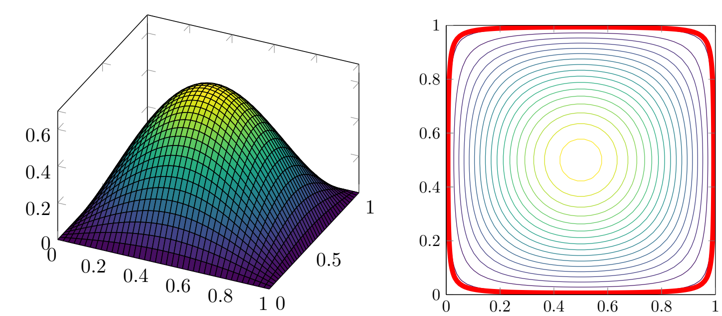


Figure 1.3: The first eigenfunction u_2 of Δ on $[0,1]^2$ with Dirichlet boundary conditions (left) and the boundaries of its superlevel sets (right). The boundary of the superlevel set with the lowest Cheeger ratio is indicated in red (figure also appears in [DFJK23]).

However, the variational problems (1.1) and (1.4) are still different problems, and there is no tight bound on the Cheeger ratios of individual level sets.

The goal of this thesis In this thesis, we seek to investigate what happens if one tries to bridge the gap between $\|\cdot\|_1$ and $\|\cdot\|_2^2$ by considering $\|\cdot\|_p^p$ with $1 < p < 2$ in the variational problem (1.4).

[2]for the Dirichlet case this was done in [FJ18]

[3]in a similar way to the introduction of [DFJK23]

In the classical case, there are known results describing what happens. The variational problem is solved by the first eigenfunction of a nonlinear operator, the *p-Laplacian* Δ_p, i.e., the a solution of the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda |u|^{p-2} u$$

with minimal λ (see Section 1.4). The corresponding minimal eigenvalue λ_p of Δ_p converges to $h(M)$ for $p \to 1$ (see Section 1.5). Under the right conditions, first eigenfunctions of Δ_p converge to characteristic functions on the Cheeger set [KF03].

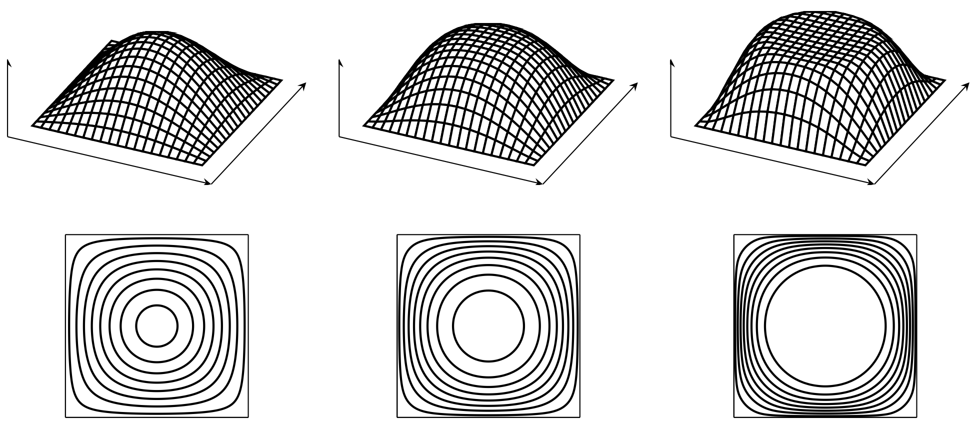


Figure 1.4: Schematic behavior of an eigenfunction of Δ_p (top row) and its level sets (bottom row) for p approaching 1 (from left to right): empirically, the level sets get closer to each other, and the eigenfunction starts to look more like a characteristic function.

We will present the known theory of interest to us in Sections 1.3 to 1.5. The hope is that similar behavior occurs if one replaces $\|\cdot\|_1$ by $\|\cdot\|_p^p$ in the dynamic case and that we can get better approximate solutions to the dynamic Cheeger problem with eigenfunctions of a *dynamic* p-Laplacian.

We will indeed be able to generalize theory from the classical case to the dynamic case in Chapter 2, and in Chapter 4 we will empirically see the behavior that eigenfunctions do get more "plateau-like" and have, on average, superlevel sets of lower dynamic Cheeger ratio, although we find that the superlevel set with the best dynamic Cheeger ratio does not improve substantially.

Conventions Throughout this thesis, if not mentioned otherwise, we will use the following assumptions on the domain M and the diffeomorphism T.

Setup 1.1.2. *Let $M \subset \mathbb{R}^d$ be a compact, d-dimensional submanifold with Lipschitz boundary. We assume the map $T : M \to M$ to be a volume-preserving diffeomorphism on the interior $\mathring{M}$ and T as well as T^{-1} to be Lipschitz continuous on M. In particular, T and T^{-1} have bounded derivatives in $\mathring{M}$ and map the boundary ∂M onto itself. A subset $D \subset M$ is generally assumed to be at least Borel-measurable. The exponent denoted by $p \in \mathbb{R}$ will assumed to be strictly between 1 and 2. We denote its conjugate exponent by q, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.*

1.2 The dynamic Laplacian Δ^D

1.2.1 Motivation

Fluid flows are ubiquitous in nature. They also can be challenging to model and understand. Highly complicated behavior like turbulence or mixing emerges easily and poses challenges not only to numerical simulation but also to the interpretation of simulated or measured data.

As far as the applications of this thesis go, we are exclusively interested in the problem of gaining insight from a flow that is already fully determined – either through measurement or numerical simulation[4]. The situation is often better than it first seems. For example, in ocean flows, one can often find structure in the form of "eddies, jet streams and other coherent structures" [Van+18, p. 4.2].

Specifically, the question that motivates the study of so-called *Lagrangian coherent structures* (LCS), where the dynamic Laplacian emerged from, is the question of how to understand the structure of material transport within a fluid in a coarse-grained way. Methods analyzing this structure promise, for example, to help understand the distribution of pollutants and other tracer materials in ocean water or the atmosphere [PH13a; HSM13]. They can also serve as general tools for understanding of a flow and have, for example, been used to illustrate the wakes behind flying and swimming animals [DGCC05; PD08].

While the appearance of structure is often visually apparent – distinguished features appear in all sorts of diagnostic scalar fields, fields [HS11; Had+17, section II.D] – there is no consensus about one single formal definition. In the mathematical community, Lagrangian coherent structures have been the object of study starting with [HY00] and have produced a variety of approaches. The general intuition is that one looks for subsets of the material whose boundaries should pose persistent barriers to mixing, so that one obtains a high-level "skeleton" of the transport and mixing within the fluid [HS11; PH13b]. There are many approaches for making this more precise, which start with Haller's original work in [HY00], where a local characterization of the boundaries (as opposed to the subsets themselves) is proposed. In [Hal02], for example, the finite-time Lyapunov exponent field is used to indicate the boundaries of coherent structures. Later [HB13], closed curves in the material are identified in as the boundaries of coherent sets if they are local optima of the averaged stretching during the dynamics. In [HHFH16], Haller et al. use what they call *Lagrangian Averaged Vorticity Deviation*(LAVD) to find *rotationally coherent Lagrangian vortices*. The boundaries of the latter are identified as level sets of the LAVD.

Lately, also purely data-based approaches have been of interest, like in [HKTH16], where a clustering of the trajectories based on the average distance of two material points is used.

Transfer operator based approaches like the ones collected in [FP14] analyze the nonlinear transport within the material by means of the linear *Perron-Frobenius* (transfer) operator (see also Section 1.2.4). The transfer operator describes the evolution of densities under the dynamics, which is why approaches using it are also called *probabilistic*. In the autonomous setting, sets that lose little material under the dynamics are called *almost*

[4] there are methods to work with the dynamic Laplacian on incomplete data like, e.g. [FJ18], but we will not handle them here.

invariant sets and can be characterized and approximated using spectral theory of the transfer operator [DJ99]. In the nonautonomous setting, one can use the singular vectors of a stochastically perturbed transfer operator to characterize finite-time coherent sets, which leak little material over a fixed finite time interval [FLS10; Fro13].

For a more extensive overview over existing LCS methods, we refer the reader to the survey [Had+17].

1.2.2 Dynamic isoperimetry

In [Fro15], Froyland proposed a geometric characterization of a coherent set: the boundary should be persistently small compared to its interior. Intuitively, such a set will lose little material through its boundary if one adds a small amount of diffusion to the advective transport. This global characterization of a coherent set is close in spirit to the elliptic LCS of [HB13], which also uses the stretching of the material. The intuition about the loss of material through small-scale diffusion can be made more precise [Fro15, Theorem 5.1], connecting this method to transfer operator approaches like [Fro13].

Froyland considered a smooth codimension-one submanifold $\Gamma \subset M$ that splits the domain into two parts M_1, M_2 and associated the quantity

$$\frac{\ell_{d-1}(\Gamma) + \ell_{d-1}(T(\Gamma))}{2\min\{\ell_d(M_1), \ell_d(M_2)\}}$$

to it (where ℓ_d denotes the d-dimensional Hausdorff measure). In reference to the classical field of isoperimetry, he coined the term *dynamic isoperimetry* for the study of this quantity. He defined the *dynamic Cheeger constant* $\mathbf{h}^D$ by

$$\mathbf{h}^D \coloneqq \inf_{\Gamma} \frac{\ell_{d-1}(\Gamma) + \ell_{d-1}(T(\Gamma))}{2\min\{\ell_d(M_1), \ell_d(M_2)\}} \tag{1.5}$$

and showed its equality with a *dynamic Sobolev constant*:

$$\mathbf{h}^D = \mathbf{s}^D \coloneqq \inf_{f \in C^\infty(M)} \frac{\|\nabla f\|_1 + \|\nabla(T_* f)\|_1}{\inf_{\alpha\in\mathbb{R}} \|f - \alpha\|_1}, \tag{1.6}$$

where T_* is the *transfer operator*, also called the *Perron-Frobenius operator* mapping f to $T_* f \coloneqq f \circ T^{-1}$ (see Section 1.2.4). This constitutes a generalization of the known *Federer-Fleming* theorem [FF60], which equates the classical *Cheeger constant* $\mathbf{h}$ and the *Sobolev constant* $\mathbf{s}$,

$$\mathbf{h} = \inf_{\Gamma} \frac{\ell_{d-1}(\Gamma)}{\min\{\ell_d(M_1), \ell_d(M_2)\}} \tag{1.7}$$

$$\mathbf{s} = \inf_{f \in C^\infty(M)} \frac{\|\nabla f\|_1}{\inf_{\alpha\in\mathbb{R}} \|f - \alpha\|_1}. \tag{1.8}$$

Froyland also generalized a classical result known as the *Cheeger inequality*, which says that

$$h \le 2\sqrt{\lambda_2},$$

where λ_2 is defined as the first[5] nontrivial Neumann eigenvalue of $-\Delta$, i.e., it is the smallest λ such that there is some nonconstant u with

$$-\Delta u = \lambda u \quad \text{on } \mathring{M} \tag{1.9}$$
$$u \cdot n \equiv 0 \quad \text{on } \partial M, \tag{1.10}$$

where n is the outward pointing unit normal on ∂M (note that the constant function is a solution of (1.9) with $\lambda = 0$). The *dynamic Cheeger inequality* proven in [Fro15] states that

$$\mathbf{h}^D \leq 2\sqrt{\lambda_2^D},$$

where λ_2^D now is the first nontrivial Neumann eigenvalue of a new operator $-\Delta^D$, defined as

$$\Delta^D u \coloneqq \frac{1}{2}(\Delta u + T^* \Delta T_*),$$

with the dual $T^* : f \mapsto f \circ T$ of the transfer operator operator, also called the *Koopman operator* (see Section 1.2.4). The dynamic Laplacian also arises in the limit $\varepsilon \to 0$ when adding ε-scale diffusion to the purely advective transport: in [Fro15, Theorem 5.1] it is shown that

$$\lim_{\varepsilon \to 0} \frac{(\mathcal{L}_\varepsilon^* \mathcal{L}_\varepsilon - I) f(x)}{\varepsilon^2} = c \Delta^D f(x),$$

where $\mathcal{L}_\varepsilon \coloneqq \mathcal{D}_{T(M),\varepsilon} T_* \mathcal{D}_{M,\varepsilon}$ is defined by adding a small amount of smoothing $\mathcal{D}$ before and after the transfer operator.

The eigenvalue problem of Δ^D is much easier approximated numerically than solutions of the variational problem (1.6): for example, in [FJ18; SFJ20] finite element methods have been used, an approach using radial basis functions appears in [FJ15].

Eigenfunction of Δ^D have since been used to detect coherent structures in a number of ways: the first approach to extracting coherent sets from eigenfunctions was to look for sets with small dynamic Cheeger ratio in the familiy of level sets of the eigenfunctions [Fro15; FK17]. A more sophisticated approach can be found, e.g. in [FJ18]. It allows for the detection of more than one set and uses spectral clustering techniques coming from graph clustering (see e.g. [Von07] for an introduction). This is done by using higher eigenfunctions $u^{(1)}, \dots u^{(k)}$ of Δ^D to get an embedding $x \mapsto u^{(1)}(x), \dots, u^{(k)}(x)) \in \mathbb{R}^k$. On the embedded points, some clustering scheme like k-means can be applied. This approach is refined in [FRS19], where the clustering post-processing is replaced by a procedure that approximates the space spanned by the eigenfunctions of Δ^D with vectors of higher sparsity than the eigenfunctions themselves, making the basis vectors automatically indicate coherent sets.

1.2.3 Dirichlet vs. Neumann boundary conditions

In this thesis, we study a slightly different geometric problem than (1.5): we do not allow Γ to touch the boundary of M and exploit that this implies that only one of the sets M_1 and M_2 can intersect the boundary. Then, instead of the normalization $\min\{\ell_d(M_1), \ell_d(M_2)\}$

[5] the subscript 2 indicates $p = 2$, following the notation λ_p in later sections

in the denominator, we take the volume of the set that does not intersect the boundary. The modified problem consists, in the static case, of finding the so-called *Cheeger constant*,

$$h(M) \coloneqq \inf_{D \subset \mathring{M}} \frac{\ell_{d-1}(\partial D)}{\ell_d(D)}$$

and, in the dynamical case, of finding the *dynamical Cheeger constant*

$$h^D(M,T) \coloneqq \inf_{D \subset \mathring{M}} \frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)},$$

where D varies over subsets of M with smooth boundary that does not touch ∂M.[6] This version of the problem was originally introduced in [Che70] for the case that M has nonempty boundary, using the version in (1.5) only for empty boundary. In the dynamic case, it was first introduced in [FJ18]. The reason we focus on this version of the problem is that this is the version on which most of the literature on connections between the p-Laplacian and the Cheeger problem to the p-Laplacian focuses.

In Section 1.3 and Section 2.3, we will state the Cheeger problem and the dynamic Cheeger problem precisely. Here, we only summarize how they relate to (1.5): A classical [Leo15] and dynamic [FJ18] Federer-Fleming still holds, yielding the variational characterizations

$$h(M) = \inf_{\substack{u \in C_0^\infty(M) \\ u \neq 0}} \frac{\|\nabla u\|_1}{\|u\|_1}$$

and

$$h^D(M,T) = \inf_{\substack{u \in C_0^\infty(M) \\ u \neq 0}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1}$$

In the eigenvalue problems, the boundary conditions change from Dirichlet to Neumann boundary conditions. This means that there is no eigenvalue 0, and the value of the modified variational problem

$$\lambda_2^D(M,T) = \inf_{\substack{u \in C_0^\infty(M) \\ u \neq 0}} \frac{\|\nabla u\|_2^2 + \|\nabla(T_* u)\|_2^2}{2\|u\|_2^2} \tag{1.11}$$

coincides with the smallest eigenvalue of Δ^D with Dirichlet boundary conditions (a general version of this result, which is classical for $T = id$, is proved in Corollary 2.1.4). Classical [LW97, appendix] and dynamical [FJ18, Theorem 2] Cheeger inequalities also still hold, stating

$$h(M) \leq 2\sqrt{\lambda_2} \;\text{ and }\; h^D(M) \leq 2\sqrt{\lambda_2^D}.$$

In Section 2.3 we will show Theorem 2.3.1, which includes these as special cases. The different versions of the quantities introduced in this section are recapitulated in Table 1.1. For the rest of the thesis, we will restrict ourselves to the Dirichlet case.

[6] note that there might not be a minimizer fulfilling these restrictions. See Section 1.3 and section 2.2 for details.

	Neumann	Dirichlet
Classical case		
Cheeger constant	$\inf_\Gamma \frac{\ell_{d-1}(\Gamma)}{\min\{\ell_d(M_1),\ell_d(M_2)\}}$	$\inf_D \frac{\ell_{d-1}(\partial D)}{\ell_d(D)}$
Sobolev constant	$\inf_{u\in C^\infty} \frac{\|\nabla u\|_1}{\inf_\alpha \|u-\alpha\|_1}$	$\inf_{u\in C_0^\infty} \frac{\|\nabla u\|_1}{\|u\|_1}$
eigenvalue problem	$-\Delta u = \lambda u; \quad u\cdot n \equiv 0 \text{ on } \partial M.$	$-\Delta u = \lambda u; \quad u \equiv 0 \text{ on } \partial M.$
Dynamical case		
Cheeger constant	$\inf_\Gamma \frac{\ell_{d-1}(\Gamma)+\ell_{d-1}(T(\Gamma))}{2\min\{\ell_d(M_1),\ell_d(M_2)\}}$	$\inf_D \frac{\ell_{d-1}(\partial D)+\ell_{d-1}(\partial(T(D)))}{2\ell_d(D)}$
Sobolev constant	$\inf_{u\in C^\infty} \frac{\|\nabla u\|_1+\|\nabla(T_* u)\|_1}{2\inf_\alpha \|u-\alpha\|_1}$	$\inf_{u\in C_0^\infty} \frac{\|\nabla u\|_1+\|\nabla(T_* u)\|_1}{2\|u\|_1}$
eigenvalue problem	$-\Delta^D u = \lambda u;\ u\cdot n \equiv 0 \text{ on } \partial M.$	$-\Delta^D u = \lambda u;\ u \equiv 0 \text{ on } \partial M.$

Table 1.1: An overview of variations on the Cheeger problem. The subset Γ is assumed to be a $(d-1)$-dimensional submanifold partitioning M into two parts. The set $D \subset M$ is supposed to be compactly contained in the interior $\mathring{M}$ and n denotes the outward pointing unit normal field on the boundary ∂M. In this thesis, we only handle the Dirichlet case.

1.2.4 The transfer and the Koopman operator

We recall the definition of the transfer operator (also *pushforward* or *Perron-Frobenius* operator) and the Koopman (or *pullback*) operator. They are linear operators that act on function spaces on the domain of a dynamical system and carry information about the dynamics by capturing how densities evolve with it: for a measure space $(X, \mathcal{A}, \mu)$ and a measurable map $T : X \to X$, the transfer operator (denoted by T_* here) is the map from $L^1(X)$ to $L^1(X)$ that for every $f \in L^1(X)$ and measurable $A \subset X$ fulfills

$$\int_A T_* f d\mu = \int_{T^{-1}(A)} f d\mu$$

(see e.g. [LM98, Definition 3.2.3]).

If we are given some probability density $f \in L^1(M), f \geq 0, \|f\|_1 = 1$ describing the distribution of a random point $x \in X$, then $T_* f$ will be another probability density describing the distribution of $T(x)$. The pullback operator $T^* : L^\infty(X) \to L^\infty(X)$ can then be introduced as the dual of the transfer operator, i.e. the operator fulfilling

$$\langle T^* f, v\rangle = \langle f, T_* v\rangle \tag{1.12}$$

for all $f \in L^\infty, v \in L^1(M)$, where $\langle f, v\rangle := \int_X fv \; d\mu$ denotes the duality pairing on $L^\infty(X) \times L^1(X)$. Under this implicit identification of the dual space $(L^1(M))^*$ with $L^\infty(M)$, the definition above is equivalent to setting $T^* f = f \circ T$, which is another common way of defining T^* [see e.g LM98, Definition 3.1].

For our purposes, X is a subset $M \subset \mathbb{R}^d$ and T is a diffeomorphism on M. In this setting, the transfer operator reduces to

$$T_* f(x) = f \circ T^{-1}(x) \det(DT^{-1}(x)),$$

where $DT(x)$ is the Jacobian of T at x [see LM98, Corollary 3.2.1]. If T is volume-preserving, this simplifies even further to

$$T_* f = f \circ T^{-1}.$$

We will use this characterization to define T_* on any space B of functions on M and define the Koopman operator as the dual of T_*.

Definition 1.2.1. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Define the* transfer operator *T_* by*

$$T_*(u) = (u \circ T^{-1}) \cdot \det(DT^{-1}) = u \circ T^{-1}$$

for any function $u : M \to \mathbb{R}$.

As said, above, the dual T^* of T_* is commonly introduced by $T^* f := f \circ T$. We will use a more abstract definition that behaves more naturally in the context of variational calculus and will play more nicely together with the characterization of the dynamic p-Laplacian as a Gâteaux derivative.

Definition 1.2.2. *For some diffeomorphism T, $p > 1$ and some space B of functions we define the* pullback operator *or* Koopman operator *$T_* : B^* \to B^*$ as the dual of T^*, i.e., for some $v \in B^*$, i.e., we define $T^* v$ to be the unique element in B^* fulfilling*

$$\langle u, T^* v\rangle = \langle T_* u, v\rangle,$$

which is equivalent to setting

$$T^* v := v \circ T_*$$

for a functional $v : B \to \mathbb{R}$.

We will mostly set B to be the Sobolev space $W_0^{1,p}(M)$. Then B^* is its dual $W^{-1,q}(M)$ (see Appendix B).

1.3 The (classical) Cheeger problem

We now formally introduce the classical version of the geometric problem that we are interested in. It goes back to Cheeger, who used it in [Che70] to get a lower bound on the smallest eigenvalue of Δ. For a manifold M with nonempty boundary, he studies the infimum of the quantity

$$\frac{A(S)}{V(D_S)} \tag{1.13}$$

over smooth compact $(d-1)$-dimensional submanifolds $S \subset M$ with $S \cap \partial M = \emptyset$, where D_S denotes the (unique) submanifold of M with boundary S (note that this implies that $\partial M \cap D_S = \emptyset$). Here, $A(\cdot)$ denotes the $(d-1)$ dimensional volume of a manifold and $V(\cdot)$ the d-dimensional volume of an open set. We adopt the notation of [Fro15] and use the d-dimensional and $(d-1)$-dimensional Hausdorff measures ℓ_{d-1}, ℓ_d instead.

We closely follow the introduction by Parini [Par11] in the presentation here. Like there, we start with a different but equivalent definition. This definition needs less regularity assumptions on the subset $D \subset M$ over which one optimizes and uses its *perimeter*

$P(D, \mathbb{R}^d)$. The perimeter of a measurable set D is defined as the variation of its characteristic function χ_D in the space $BV(\mathbb{R}^d)$ of functions of bounded variation:

$$P(D, \mathbb{R}^d) = |D(\chi_D)|(\mathbb{R}^d).$$

See Appendix C for a short introduction to functions of bounded variation and sets of finite perimeter.

Definition 1.3.1. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. The* Cheeger constant *of M is defined as*

$$h(M) \coloneqq \inf_{D \subset M} \frac{P(D, \mathbb{R}^d)}{\ell_d(D)},$$

where the infimum ranges over all Borel subsets of M and the value of the fraction is assumed to be ∞ if $P(D, \mathbb{R}^d) = \infty$ or $\ell_d(D) = 0$. For a subset $D \subset M$, we call the quantity

$$\frac{P(D, \mathbb{R}^d)}{\ell_d(D)}$$

the Cheeger ratio *of D. If a set has a Cheeger ratio of $h(M)$, we call it a* Cheeger set *of M.*

Remark 1.3.2. *Note that from that definition, it follows that every measurable subset $D \subset M$ (be it of finite perimeter or not) fulfills*

$$P(D, \mathbb{R}^d) \geq h(M)\ell_d(D),$$

and if this is an equality, then D is either a null set or a Cheeger set.

There are other equivalent characterizations of $h(M)$, which we will use interchangeably:

$$h(M, T) = \inf_{\substack{D \subset \mathring{M} \\ \partial D \text{ smooth}}} \frac{\ell_{d-1}(\partial D)}{\ell_d(D)} \tag{1.14}$$

$$h(M, T) = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\|\nabla u\|_1}{\|u\|_1} \tag{1.15}$$

$$h(M, T) = \inf_{u \in BV(M) \setminus \{0\}} \frac{|Du|(\mathbb{R}^d)}{\|u\|_1} \tag{1.16}$$

- Equation (1.14) uses the same quantity as (1.13), just with different notation. The subset $D \subset \mathring{M}$ is assumed to be compactly contained in the interior of M, i.e., its closure $\overline{D}$ does not touch the boundary ∂M. The equivalence is proven in [Par11, Proposition 3.3].
- Equation (1.15) is known as the *Federer-Fleming theorem*, referring to [FF60]. The results there are very general. For a proof of this speific case, see, e.g., [Leo15, Remark 2.1]. The right hand side is known as the *Sobolev constant* of M.
- in (1.16), $|Du|(\mathbb{R}^d)$ denotes the variation of $u \in BV(M)$ in $\mathbb{R}^d$. For a short introduction to the notation of functions of bounded variation, see Appendix C. The claim is shown in the proof of [Par11, Proposition 3.1].

The definitions that require higher regularity in the domain of the infimum tend to be easier to work with but come at the disadvantage that the infimum might not be attained. For example, on a star-shaped domain, it is easy to see that there is no minimizer in (1.14): if a set is separated from the boundary, it can be slightly scaled up, which decreases its Cheeger ratio (the volume of the boundary scales with the order $d-1$ while the volume scales with order d [KF03, Theorem 8]). Thus, one has to take another characterization to ensure existence of a minimizer like in Theorem 1.3.3 below.

For us, an important aspect of these equivalent definitions is the question of how solutions of the geometric characterizations are related to solutions of the variational characterization. As the former are subsets of M and the latter are functions on M, this is not immediately apparent.

The main idea that connects the functional quantity $\|\nabla u\|_1/\|u\|_1$ with the geometrical quantity $\ell_{d-1}(\partial D)/\ell_d(D)$ originates from an identity known as the *coarea formula* (see Appendix E). To first give an informal account of how the argument works, we start with a nonnegative smooth scalar function u for which the coarea formula implies that $\|\nabla u\|_1$ can be calculated if one knows the perimeter of the superlevel sets $A_t \coloneqq \{x \in M \mid u(x) > t\}$:

$$\|\nabla u\|_1 = \int_0^\infty \ell_{d-1}(\partial A_t)dt.$$

Now, by Remark 1.3.2, every Borel $D \subset M$ fulfills $\ell_{d-1}(\partial D) \geq h(M)\ell_d(D)$, so

$$\int_0^\infty \ell_{d-1}(\partial A_t)dt \geq h(M) \int_0^\infty \ell_d(A_t)dt.$$

Finally, the last integral is just $\|u\|_1$ by Cavalieri's principle (Theorem 1.5.1), so we have obtained the chain

$$\|\nabla u\|_1 = \int_0^\infty \ell_{d-1}(\partial A_t)dt \geq \int_0^\infty h(M)\ell_d(A_t)dt = h(M)\|u\|_1. \tag{1.17}$$

The immediate use of (1.17) is twofold: first, it directly yields

$$\frac{\|\nabla u\|_1}{\|u\|_1} \geq h(M),$$

which – save technical details – already constitutes the first half of a proof of Equation (1.15). Second, if for some u, the quotient $\|\nabla u\|_1/\|u\|_1$ attains the minimal possible value $h(M)$, then the inequality in the middle of (1.17) is forced to be an equality, meaning that the integrands of both integrals must coincide for almost all t. This can only happen if almost all superlevel sets are either null sets or Cheeger sets. If there is a unique Cheeger set, then this means that u has to be a suitably scaled characteristic function of the Cheeger set [KF03, Remark 10].

Obviously, this contradicts smoothness of u, but the same argument can be made in the context of the BV-characterization (1.16). With u just being required to be of bounded variation, the existence of a minimizer can be shown with variational techniques, and we obtain existence of a Cheeger set and an analogous strong constraint on the superlevel sets of minimizers in (1.16).

Theorem 1.3.3. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. Then:*

(a) The infimum in (1.16) is attained by some nonnegative $u \in BV(M)$.

(b) If for some nonnegative $u \in BV(M)$, the ratio $|Du|(\mathbb{R}^d)/\|u\|_1$ attains $h(M)$, then for almost all $t > 0$ the superlevel sets

$$A_t := \{x \in M \mid u(x) > t\}$$

are either null sets or Cheeger sets of M.

Remark 1.3.4. *On a convex subset of $\mathbb{R}^d$, 1.3.3(b) implies [KH16] that u is a multiple of the characteristic function of the unique[AC09] Cheeger set. Without uniqueness of the Cheeger set, however, this is not possible, and there are indeed examples of domains with nonunique Cheeger sets. In [Leo15, Example 5.6], for example, a domain M and a one-parameter family of nested subsets D_t is constructed such that all sets D_t are Cheeger sets of M.*

Proof of Theorem 1.3.3. For existence, we use the direct method of the calculus of variations, as in [Par11, Proposition 3.1]: Let $(u_k)_k \subset BV(M)$ be an infimizing sequence for the infimum in Equation (1.16). We may assume without loss of generality that $\|u_k\|_1 = 1$ for all k, as $|D(u_k)|(\mathbb{R}^d)/\|u_k\|_1$ is scaling invariant. By convergence of $|D(u_k)|(\mathbb{R}^d)/\|u_k\|_1$, this normalization implies in particular that $|D(u_k)|(\mathbb{R}^d)$ is bounded, so the sequence u_k is bounded $BV(M)$. Thus, by the compactness property shown in [Par11, Proposition 2.2] there is a subsequence of u_k converging in $L^1(M)$ to some $u \in BV(M)$. We pass to that subsequence without loss of generality. Now

$$h(M) \leq \frac{|Du|(\mathbb{R}^d)}{\|u\|_1} = |Du|(\mathbb{R}^d) \overset{(*)}{\leq} \liminf_{k\to\infty} |Du_k|(\mathbb{R}^d) = \lim_{k\to\infty} \frac{|Du_k|(\mathbb{R}^d)}{\|u_k|_1} = h(M),$$

where $(*)$ is the lower semi-continuity of the variation with respect to L^1-convergence [Amb00, Proposition 3.6]. This implies

$$h(M) = |Du|(\mathbb{R}^d).$$

Nonnegativity of u can be assumed by the inequality $|D|u||(\mathbb{R}^d) \leq |Du|(\mathbb{R}^d)$, as done in [Par11, Proposition 3.1], so we are finished with (a). To show (b), let $u \in BV(M)$ be nonnegative and assume

$$\frac{|Du|(\mathbb{R}^d)}{\|u\|_1} = h(M).$$

Define $\tilde{u} \in BV(\mathbb{R})^d$ by extending it by zero and $\tilde{A}_t := \{x \in \mathbb{R}^d \mid \tilde{u}(x) > t\}$. Then we have

$$\begin{aligned} h(M) = |D\tilde{u}|(\mathbb{R}^d) &\overset{(a)}{=} \int_{-\infty}^{\infty} P(\tilde{A}_t, \mathbb{R}^d)dt && (1.18)\\ &\overset{(b)}{=} \int_0^{\infty} P(A_t, \mathbb{R}^d)dt && (1.19)\\ &\geq h(M) \int_0^{\infty} \ell_d(A_t)dt && (1.20)\\ &\overset{(c)}{=} h(M)\|u\|_1 && (1.21)\\ &= h(M), && (1.22) \end{aligned}$$

where we have used the following:

(a) the coarea formula for functions of bounded variations (Theorem E.3).

(b) the integration limits can be changed because $\tilde{u} \geq 0$, and thus for $t < 0$, we have $\tilde{A}_t = \mathbb{R}^d$ which implies $P(\tilde{A}_t, \mathbb{R}^d) = 0$. We can the replace $\tilde{A}_t$ by A_t because, for $t > 0$, we have $\tilde{A}_t \subseteq M$, and thus $\tilde{A}_t = A_t$, in particular $P(\tilde{A}_t, \mathbb{R}^d) = P(A_t, \mathbb{R}^d)$.

(c) Cavalieri's principle (see Theorem 1.5.1).

Now this chain starts and ends with $h(M)$, so inequality (1.20) is actually an equality and hence for almost all $t > 0$

$$P(A_t, \mathbb{R}^d) = h(M)\ell_d(A_t).$$

This can only hold if A_t is a null set or a Cheeger set, proving claim (b). $\square$

Corollary 1.3.5. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. Then there exists a Cheeger set for M.*

Proof. By Theorem 1.3.3(a), there exists a nonnegative function $u \in BV(M) \setminus \{0\}$ such that

$$\frac{|Du|(\mathbb{R}^d)}{\|u\|_1} = h(M).$$

As $\|u\|_1 \neq 0$, Cavalieri's principle tells us that its superlevel sets cannot be all null sets, so at least one of them must be a Cheeger set. $\square$

There is much more to say about the Cheeger problem and the more general isoperimetric problems. For a broader introduction, we refer the reader to the expositions [Par11; Leo15; Cha01].

We close with an example where the Cheeger constant and Cheeger set are known analytically:

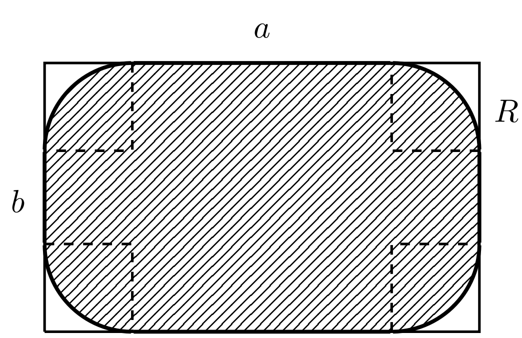


Figure 1.5: The Cheeger set of a rectangle $[0, a] \times [0, b]$ is a rectangle with rounded corners of radius R (hatched area). The volume of such a set is $ab - 4R^2 + \pi R^2$ and its perimeter is $2(a - 2R) + 2(b - 2R) + 2\pi R$.

Example 1.3.6 (Cheeger set of a rectangle)**.** *It can be shown that the Cheeger set of a convex subset of $\mathbb{R}^2$ is unique and can be constructed as the union of all balls of some radius $R = \frac{1}{h(M)}$ with centers in the set of points that have distance at most $\frac{1}{h(M)}$ from* *the boundary (see, e.g. [KL06, Theorem 1] and [Par11, Proposition 5.1]). The Cheeger set of a rectangle* $[0,a] \times [0,b]$ *is therefore another rectangle whose corners are rounded (see Figure 1.5). The radius* R *of the corners can be calculated by optimizing the Cheeger ratio of such a rounded rectangle:*

$$\frac{\overbrace{2(a-2R)+2(b-2R)+2\pi R}^{\textit{Perimeter}}}{\underbrace{ab-4R^2+\pi R^2}_{\textit{volume}}} = \frac{2(\pi-4)R+2(a+b)}{(\pi-4)R^2+ab} =: \frac{2c_1R+c_2}{c_1R^2+c_3}$$

over R. *The derivative in* R *is a fraction whose numerator is*

$$2c_1(c_1R^2+c_3) - 2c_1R(2c_1R+c_2) = -2c_1^2R^2 - 2c_1c_2R + 2c_1c_3.$$

Necessary optimality conditions yield

$$R_{1,2} = \frac{2c_1c \pm \sqrt{4c_1^2c_2^2 + 16c_1^3c_3}}{-4c_1^2} = \frac{\frac{1}{2}c_2 \pm \sqrt{\frac{1}{4}c_2^2 + c_1c_3}}{-c_1}$$

plugging the definitions of c_1, c_2, c_3 *and using that the minimal* R *we look for is smaller than* $\min\{a,b\}$, *we end up with:*

$$R = \frac{(a+b) - \sqrt{(a-b)^2 + \pi ab}}{(4-\pi)}. \tag{1.23}$$

And thus

$$h([0,a]\times[0,b]) = \frac{(4-\pi)}{(a+b) - \sqrt{(a-b)^2+\pi ab}}.$$

This formula also appears in [Hor11, eq. 38] and [KL06, Remark after Theorem 3]. [7] *In the case of a square* $[0,a]\times[0,a]$, *the formula reduces to*

$$h([0,a]^2) = \frac{4-\pi}{2a - a\sqrt{\pi}} = \frac{2+\sqrt{\pi}}{a}.$$

1.4 The p-Laplacian Δ_p

The classical p-Laplacian Δ_p is a nonlinear differential operator commonly introduced[8] as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

If we use this definition classically, it requires nontrivial regularity conditions on u for the divergence to be defined. Further – as it is not uncommon with differential operators – we are mostly interested in Δ_p in conjunction with a variational problem, where it appears as a Gâteaux derivative. Hence, we will always use Δ_p in its *weak* (or *distributional*) form, accepting arguments u that are only required to be in the Sobolev space $W_0^{1,p}(M)$ and attaining values $\Delta_p u$ in its dual space $W^{-1,q}(M)$. For a short introduction of these spaces, see Appendix B. We restrict the domain to functions vanishing on the boundary for reasons explained in Section 1.2.3.

[7] In [KL06] the domain is said to be $[-a,a]\times[-b,b]$, but we believe $[0,a]\times[0,b]$ to be the right domain. In [Hor11] the domain coincides with ours.

[8] we are liberally following [Lin08; PK09; Hor11] in the definitions of Δ_p and the eigenvalue problem.

Definition 1.4.1. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2 and $p \in (1, \infty)$. Define the* p-Laplacian $\Delta_p : W_0^{1,p}(M) \to W^{-1,q}(M)$ *by*

$$\Delta_p u =: \left(v \mapsto -\int_M |\nabla u|^{p-2} \nabla u \nabla v \right)$$

or, writing $\langle \cdot, \cdot \rangle$ for the duality pairing on $W^{-1,q}(M) \times W_0^{1,p}(M)$

$$\langle \Delta_p u, v \rangle = -\int_M |\nabla u|^{p-2} \nabla u \nabla v.$$

for all $v \in W_0^{1,p}(M)$. For a proof that this indeed defines an element in $W^{-1,q}(M)$, see Theorem 1.4.7.

Remark 1.4.2. *If $|\nabla u|^{p-2} \nabla u$ is differentiable and $\operatorname{div}(|\nabla u|^{p-2} \nabla u) \in L^q(M)$ then by partial integration*

$$\langle \Delta_p u, v \rangle = -\int_M |\nabla u|^{p-2} \nabla u \nabla v = \int_M \operatorname{div}(|\nabla u|^{p-2} \nabla u) v$$

for every $v \in W_0^{1,p}(M)$, so if we identify any kernel $f \in L^q$ with the functional $v \mapsto \int_M f v$ then we may say

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

in the above case. If $|\nabla u|^{p-2} \nabla u$ lacks the necessary regularity, then $\Delta_p u$ might not allow a kernel from $L^q(M)$ in this sense. For example, if we set $M := [-1, 1] \subset \mathbb{R}$ and $u(x) = 1 - |x|$, we have that $\Delta_p u$ is the delta distribution $\delta_0 \in W^{-1,q}(M)$. Note also that differentiability of $|\nabla u|^{p-2} \nabla u$ is not a "linear" condition that is preserved by sums (see also Figure 3.11).

Remark 1.4.3. *For $p = 2$, we get the weak form of the standard Laplacian Δ.*

The partial differential equation that will be in the center of our attention concerning Δ_p is a nonlinear eigenvalue problem associated with it:

Definition 1.4.4. *We say that $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(M)$ is a (weak Dirichlet)* eigenpair *of $-\Delta_p$ if*

$$-\Delta_p u = \lambda |u|^{p-2} u \tag{1.24}$$

as elements of $W^{-1,q}(M)$ (see Theorem 1.4.7 for a proof that the right-hand side is in $W^{-1,q}(M)$). If (u, λ) is an eigenpair of $-\Delta_p$ then we call u an eigenfunction *and λ an* eigenvalue *of $-\Delta_p$. We denote the infimum of the eigenvalues of $-\Delta_p$ by*

$$\lambda_p := \inf\{\lambda \in \mathbb{R} \mid \lambda \text{ is eigenvalue of } -\Delta_p\}.$$

We always mean Dirichlet eigenvalues and eigenfunctions in this thesis and thus often omit the "Dirichlet".

Remark 1.4.5. *The term "eigenvalue problem" is used liberally here: Δ_p is not a linear operator, and sums of solutions of (1.24) need not be solutions themselves. Scalar multiples of "eigenfunctions", however, are again eigenfunctions with respect to the same eigenvalue λ, as both sides of the equation scale with the same order $(p - 1)$.*

Remark 1.4.6. *From the definition of λ_p it is not clear whether λ_p itself is an eigenvalue of Δ_p. We will show that in Corollary 1.4.9.*

As alluded to above, we now introduce the functionals needed to define the variational problem in which the eigenvalue problem (1.24) will appear as the Euler-Lagrange equation. We will also deliver the postponed proofs that for $u \in W_0^{1,p}(M)$, the functionals $v \mapsto (\Delta_p u)v$ and $v \mapsto \int_M |u|^{p-2}uv$ are elements of $W^{-1,q}(M)$:

Theorem 1.4.7. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2 and $p \in (1,\infty)$. Define the functionals $F, G : W_0^{1,p}(M) \to \mathbb{R}$ by*

$$F(u) \coloneqq \|\nabla u\|_p^p \coloneqq \int_M |\nabla u|^p \tag{1.25}$$

$$G(u) \coloneqq \|u\|_p^p \coloneqq \int_M |u|^p \tag{1.26}$$

Then

(a) F and G are Gâteaux differentiable on $W_0^{1,p}(M)$ with

$$F'(u) = -p\Delta_p u \in W^{-1,q}(M)$$

and

$$G'(u) = p|u|^{p-2}u \in W^{-1,q}(M)$$

in the sense that $G'(u)v = p\int_M |u|^{p-2}uv$ for all $v \in W_0^{1,p}(M)$.

(b) F is and weakly lower semicontinuous. In particular, for any sequence of functions $u_k \in W_0^{1,p}(M)$, weak convergence $u_k \rightharpoonup u$ implies $\liminf_k F(u_k) \le F(u)$.

(c) F is coercive, i.e., if $\|u_k\|_{W_0^{1,p}(M)} \to \infty$ then $F(u_k) \to \infty$.

Proof. (a) For differentiability of the functionals and the formulas for the Gâteaux derivatives F', G' see [PK09, Remark 4.3.40] (where $\frac{1}{p}F$ and $\frac{1}{p}G$ are called ξ and η). Here we just prove that the right-hand sides are elements of $W^{-1,q}(M)$: Indeed, by Hölder, we have

$$|G'(u)v| \le p\int_M |u|^{p-1}|v| \le \left(\int_M |u|^p\right)^{\frac{p-1}{p}} \left(\int_M v^p\right)^{\frac{1}{p}} \tag{1.27}$$

$$= \|u\|_p^{p-1}\|v\|_p \le \|u\|_{W_0^{1,p}(M)}\|v\|_{W_0^{1,p}(M)} \tag{1.28}$$

and by Cauchy-Schwartz and Hölder:

$$|F'(u)v| \le \int_M \left||\nabla u|^{p-2}\nabla u \nabla v\right| \le \int_M |\nabla u|^{p-1}|\nabla v| \tag{1.29}$$

$$\le \left(\int_M |\nabla u|^p\right)^{\frac{p-1}{p}} \left(\int_M |\nabla v|^p\right)^{\frac{1}{p}} \le C\|u\|_{W_0^{1,p}(M)}^{p-1}\|v\|_{W_0^{1,p}(M)} \tag{1.30}$$

which implies boundedness of $F'(u)$.

(b) We point out that F is convex and strongly continuous: Convexity follows directly from convexity of the Euclidean norm and convexity of the function $x \mapsto |x|^p$. Continuity follows from (strong) continuity of the norm (together with the remark before Theorem B.4) and from continuity of the function $x \mapsto |x|^p$. This implies weak lower semicontinuity by [Bre11, Corollary 3.9. and Remark 6].

(c) This follows directly from the Poincaré inequality (see Theorem B.4).

$\square$

Now we are ready to state the theorem that will connect the eigenproblem Equation (1.24) to a variational problem. We formulate it in a more abstractly than necessary here, as this will make it possible to apply it to our dynamic version of Δ_p in Chapter 2.

Theorem 1.4.8. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2, $p \in (1, \infty)$and $F : W_0^{1,p}(M) \to \mathbb{R}$ a functional that is nonnegative, Gâteaux differentiable, coercive, and weakly lower semicontiuous. Assume further that $F(tu) = t^p F(u)$ and $F'(tu) = t^{p-1}F'(u)$ for all $t > 0$ and $u \in W_0^{1,p}(M)$. Define the functional $G : W_0^{1,p}(M) \to \mathbb{R}$ by $G(u) = \|u\|_p^p$ and $J : W_0^{1,p}(M)\backslash\{0\} \to \mathbb{R}$ by*

$$J(u) = \frac{F(u)}{G(u)}. \tag{1.31}$$

Then:

(a) J attains its infimum

$$\lambda^* := \inf_{\substack{u \in W_0^{1,p}(M) \\ u \neq 0}} J(u)$$

in some function u^* with $\|u\|_p = 1$, which fulfills the equation

$$F'(u^*) = \lambda^* G'(u^*). \tag{1.32}$$

(b) If some $u \in W_0^{1,p}(M)\backslash\{0\}$ and $\lambda \in \mathbb{R}$ fulfill

$$F'(u) = \lambda G'(u), \tag{1.33}$$

then $J(u) = \lambda$.

Proof. We use the direct method of the calculus of variations. The blueprint for our particular proof was taken from [BS11, Theorem 2.6.11], where it is used for a weaker statement. A similar proof also appears in [DFJK23]. By nonnegativity of J, we have $\lambda^* \geq 0$; in particular, $\lambda^* > -\infty$. Let u_k be a minimizing sequence for J. By the homogeneity assumption on F, we have $J(tu) = J(u)$ for all $t > 0$; hence, we may assume that $\|u_k\|_p = 1$ and thus $J(u_k) = F(u_k)$. Then, by coercivity of F, the sequence is bounded in $W_0^{1,p}(M)$, and by Banach-Alaoglou we may pass to a weakly convergent subsequence and assume $u_k \rightharpoonup u^*$ for some $u^* \in W_0^{1,p}(M)$. The Rellich-Kondrachov theorem (see Theorem B.6) now implies that $u_k \to u^*$ in L^p, as compact operators map weakly convergent sequences to strongly convergent ones. Thus, $\|u^*\|_p = 1$ and $J(u^*) = F(u^*)$.

Now, by definition of λ^* and weak lower semicontinuity of F, we have

$$\lambda^* \leq F(u^*) \leq \liminf_{k\to\infty} F(u_k) = \lambda^*,$$

showing $F(u^*) = \lambda^*$, which is the first part of (a). For the second part of (a), note that in an optimum, the Gâteaux derivative vanishes, so $J'(u^*) = 0$. The Gâteaux derivative of J at u^* is

$$J'(u^*) = \frac{F'(u^*)G(u^*) - F(u^*)G'(u^*)}{G(u^*)^2} = F'(u^*) - \lambda^* G'(u^*),$$

which finishes the proof of (a).

For (b) (we follow the proof of [YZ07, Lemma 2.1]), assume that u fulfills (1.32). Then, by $F'(tu) = t^{p-1}F'(u)$ and $G(tu) = t^{p-1}G(u)$, the scalar multiples tu fulfill (1.32) as well (for $t > 0$). Thus,

$$\frac{d}{dt}F(tu) = F'(tu) = \lambda G'(tu) = \lambda \frac{d}{dt}G(tu)$$

Integrating this yields that $F(tu)$ and $\lambda G(tu)$ only differ by a constant that does not depend on t. As $\lim_{t\to 0} F(tu) = \lim_{t\to 0} t^p F(u) = 0$ and $G(0) = 0$, this constant is 0, and we have $F(tu) = \lambda G(tu)$ for all $t \geq 0$. Setting $t = 1$ finishes the proof of $J(u) = \lambda$. □

With the above Theorem 1.4.8 we have done the majority of work to prove an existence result of a smallest eigenvalue of Δ_p:

Corollary 1.4.9. *The infimum λ_p of the eigenvalues of Δ_p is itself an eigenvalue and can be written as*

$$\lambda_p = \inf_{\substack{u\in W_0^{1,p}(M)\\ u\neq 0}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}.$$

Further, λ_p is positive, and u_p can be chosen to be nonnegative.

Proof. By Theorem 1.4.7 the functional

$$F(u) = \|\nabla u\|_p^p$$

is Gâteaux differentiable with

$$F'(u) = -p\Delta_p,$$

and it is nonnegative, coercive and weakly lower semicontinuous. The homogeneity conditions on F and F' are also easily checked. Hence, we can apply Theorem 1.4.8(a) to it and get some u^* that realizes the infimum λ^* of J and

$$-\Delta_p u_p^* = \frac{1}{p}F'(u^*) = \frac{1}{p}\lambda^* G'(u^*) = \lambda^* |u^*|^{p-2}u^*,$$

which means that (u^*, λ^*) is an eigenpair of $-\Delta_p$. Further, for every other eigenpair (u, λ) of $-\Delta_p$, we must have $J(u) = \lambda$ by Theorem 1.4.8(b), which means that $J(u^*) \leq J(u)$. Hence, $\lambda^* = \lambda_p$, which shows that λ_p is indeed an eigenvalue and $u_p := u^*$ is a corresponding eigenfunction. By the Poincaré inequality, we have $\lambda_p > 0$. Finally, as $\|\nabla |u|\|_p = \|\nabla u\|_p$ (see remarks at the beginning of [BS11, Section 2.6]), we can choose u_p to be nonnegative without changing the fact that it is an infimizer. □

Like with the Cheeger problem, we have only scratched the surface of the theory about the p-Laplacian. Some examples of further results are

- The first eigenvalue is isolated [Lin08, Theorem 9] and simple ([Lin08, Theorem 6]).
- A first eigenfunction does not change sign [Lin08].
- Regularity results for the first eigenfunction [DiB82].
- the first eigenvalue λ_p depends continuously on p [Hua97].
- variational characterizations of a subset of all eigenvalues [PK09, section 4.3].

We refer the reader to [Lin08] for a concise introduction and more references.

1.5 Connections between Δ_p and the Cheeger problem

We finish this chapter with an overview over the known connections between Δ_p and the Cheeger problem that we wish to generalize to the dynamic case. The main result we present here is that in the limit $p \to 1$, the first eigenvalue λ_p approaches the Cheeger constant $h(M)$. This can be found e.g. in [Leo15; Par11; KF03] and is not too surprising: after all, for $p \to 1$, the expression $\|\cdot\|_p^p$ approaches $\|\cdot\|_1$. Still, we will have to work for this result, proving it in two steps, namely

(a) $\liminf_{p\to 1} \lambda_p \geq h(M)$ and

(b) $\limsup_{p\to 1} \lambda_p \leq h(M)$.

The first step will be done via a generalization of the Cheeger inequality for general p, which states that

$$\lambda_p \geq \left(\frac{h(M)}{p}\right)^p$$

(note that the inequality $h(M) \geq 2\sqrt{\lambda_2}$ is a special case of this). The main reason that this holds is, like in the beginning of Section 1.3, the coarea formula (see Appendix E), together with Cavalieri's principle. The coarea formula connects $\|\nabla u\|_1$ to the perimeters of superlevel sets and Cavalieri's principle connects the volumes of the superlevel sets to $\|u\|_1$. In Equation (1.17) we have already seen how this produces the case $p = 1$ of the Cheeger inequality. In Theorem 2.3.1 we will see precisely how applying this to $u = |v|^p$ yields the general Cheeger inequality.

The second step (b) is done with a geometric argument: if one takes a smooth approximation u_ε of the characteristic function of a Cheeger set D, then $\|u_\varepsilon\|_1 \approx \ell_d(D)$ and $\|\nabla u_\varepsilon\|_1 \approx \ell_{d-1}(\partial D)$, as the gradient is concentrated in a sharp ramp around the boundary ∂D. This means, that from a Cheeger set we can produce smooth functions with Rayleigh quotient $\|\nabla u_\varepsilon\|_1/\|u_\varepsilon\|_1$ approaching $h(M)$. In the limit, this implies (b). Theorem 1.5.3 will make this more precise.

Finally, we state without proof a convergence result of (nonnegative) first eigenfunctions of u_p to a (nonnegative) minimizer of the variational characterization of $h(M)$ in

$BV(M)$. As we have seen in Theorem 1.3.3, such a minimizer is a lot like[9] the characteristic functionof the Cheeger set: almost all of its nontrivial superlevel sets are Cheeger sets.

The most important result for us (which we generalize for the dynamic p-Laplacian Δ_p^D in Section 2.3) is that the first eigenvalue λ_p of Δ_p converges to the Cheeger constant $h(M)$ as $p \to 1$. Another, less formal connection is that the eigenfunctions become "flatter" and "plateau-like". Under the right conditions (uniqueness of the Cheeger set is, e.g., sufficient), they converge to a characteristic function of a Cheeger set (after possibly passing to a subsequence), as shown in [KF03].

We proceed to the proof of Cheeger's inequality, which yields the first step (a) in the convergence proof. As said above, one main tool for showing it is the coarea formula (see Appendix E). For the reader's convenience we state the second main tool, Cavalieri's principle, that connects the volumes of the superlevel sets of a nonnegative function u to the norm $\|u\|_1$.

Theorem 1.5.1 (Cavalieri's principle)**.** *Let $(M, \mathcal{A}, \mu)$ be a measure space and $f : M \to \mathbb{R}$ be nonnegative and $\mathcal{A}$-measurable. If we define $A_t \coloneqq \{x \in M \mid f(x) > t\}$ then*

$$\int_M f d\mu = \int_0^\infty \mu(A_t) dt$$

Proof. Set ν to the standard Lebesgue measure on $\mathbb{R}$ in [Cha01, Proposition I.3.3]. □

Equipped with this, we state the Cheeger inequality for general p, which gives a lower bound for λ_p.

Theorem 1.5.2 (Cheeger's inequality for general p)**.** *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2 and let $1 < p < \infty$. If λ_p is the first eigenvalue of $-\Delta_p$ from Definition 1.4.4, and $h(M)$ is the Cheeger constant of M from Definition 1.3.1 then*

$$\lambda_p \geq \left(\frac{h(M)}{p}\right)^p.$$

Proof. In the case $p = 2$, this was shown first by Cheeger in [Che70]. The proof here follows the proofs in e.g. [KF03, Theorem 3] and in the appendix of [LW97].

First, let $w \in C_0^\infty(M)$ be nonnegative and define the sets $A_t \coloneqq \{x \in M \mid w(x) > t\}$. Then we get a chain like in Equation (1.17), namely

$$\int_M |\nabla w| \overset{(a)}{=} \int_0^\infty \ell_{d-1}(\partial A_t) dt \overset{(b)}{\geq} h(M) \int_0^\infty \ell_d(A_t) dt \overset{(c)}{=} h(M) \int_M |w|, \tag{1.34}$$

where

(a) is the Coarea formula in the form of Corollary E.2. The lower integration limit can be changed to 0 because by nonnegativity of u, we have $A_t = M$ for $t < 0$ and hence $\partial A_t = \emptyset$ in M.

[9] by [KF03] the limit function actually coincides with a scaled characteristic function for convex M

(b) is applying the definition of $h(M)$ (see also Remark 1.3.2 and note that by Theorem C.5 and Sard's theorem we have $P(A_t, \mathbb{R}^d) = \ell_{d-1}(\partial A_t)$ for almost all t).

(c) holds by the Cavalieri principle (see Theorem 1.5.1).

Inequality (1.34) in fact holds for all $w \in C_0^\infty(M)$, as $\|\nabla w\| = \|\nabla |w|\|$ by Proposition F.1 and we can approximate $|w|$ in $W_0^{1,1}(M)$ by a series of nonnegative functions in $C_0^\infty(M)$ by Proposition F.2. As $C_0^\infty(M)$ is dense in $W_0^{1,1}(M)$, we have thus shown

$$\int_M |\nabla w| \geq h(M) \int_M |w| \tag{1.35}$$

for every $w \in W_0^{1,1}(M)$. Now for $v \in W_0^{1,p}(M)$ one can define $\Phi(v) := \varphi_p(v) := |v|^{p-1}v$. Then $\nabla\Phi(v) = \varphi_p'(v)\nabla v = p|v|^{p-1}\nabla v$ and by the Hölder inequality

$$\int_M |\nabla\Phi(v)| = p\int_M |v|^{p-1}|\nabla v| \leq p\||v|^{p-1}\|_q\|\nabla v\|_p = p\|v\|_p^{p-1}\|\nabla v\|_p. \tag{1.36}$$

This means that $\Phi(v) \in W_0^{1,1}(M)$, and thus (1.35) applies and

$$h(M) \overset{(1.35)}{\leq} \frac{\int_M |\nabla\Phi(v)|}{\int_M |\Phi(v)|} \overset{(1.36)}{\leq} \frac{p\|v\|_p^{p-1}\|\nabla v\|_p}{\|v\|_p^p} = p\frac{\|\nabla v\|_p}{\|v\|_p}.$$

Hence for every $v \in W_0^{1,p}(M)$

$$\frac{\|\nabla v\|_p^p}{\|v\|_p^p} \geq \left(\frac{h(M)}{p}\right)^p,$$

which shows the claim after passing to the infimum over $v \in W_0^{1,p}(M) \setminus \{0\}$. □

The other part of the convergence proof constructs smooth approximations to the characteristic function of a Cheeger set (or, more precisely smooth sets close to the Cheeger set).

Theorem 1.5.3. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. If λ_p is the first eigenvalue of $-\Delta_p$ from Definition 1.4.4, and $h(M)$ is the Cheeger constant of M from Definition 1.3.1, then*

$$\limsup_{p\to 1} \lambda_p \leq h(M).$$

Proof. The argument can be also found in, e.g., [KF03, Corollary 6]. Consider a sequence of subdomains $D_k \subset M$ with smooth boundaries not touching ∂M and

$$\frac{\ell_{d-1}(\partial D_k)}{\ell_d(D_k)} \xrightarrow{k\to\infty} h(M).$$

Now for fixed k one can define smooth functions $f_{\varepsilon,k} := \chi_{D_k} * \rho_\varepsilon$ with a convolution kernel ρ_ε like in [Amb00, Proposition 3.7]. For ε smaller than some ε_k^*, these vanish on ∂M and so we get get smooth functions $(f_{\varepsilon,k})_\varepsilon \subset C_0^\infty(M), \varepsilon > 0$ converging strictly to χ_{D_k} in

$BV(M)$ for $\varepsilon \to 0$ By taking the Rayleigh quotients of $f_{\varepsilon,k}$ we obtain upper bounds for λ_p:

$$\lambda_p \leq \frac{\|\nabla f_{\varepsilon,k}\|_p^p}{\|f_{\varepsilon,k}\|_p^p} =: C_{p,\varepsilon,k} \qquad \forall k \in \mathbb{N},\ \varepsilon \in (0, \varepsilon_k^*),\ p \in (1, \infty). \tag{1.37}$$

We then proceed by showing

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} \lim_{p\to 1} C_{p,\varepsilon,k} = h(M) \tag{1.38}$$

in the following way: first observe that we have

$$\lim_{p\to 1} C_{p,\varepsilon,k} = \frac{\|\nabla f_{\varepsilon,k}\|_1}{\|f_{\varepsilon,k}\|_1} \qquad \forall k \in \mathbb{N},\ \varepsilon \in (0, \varepsilon_k^*), \tag{1.39}$$

by dominated convergence ($|f_{\varepsilon,k}|$ and $|\nabla f_{\varepsilon,k}|$ are both bounded by compactness of M). Second, by convergence of $f_{\varepsilon,k}$ to χ_{D_k} in $BV(M)$ we have the limits $\|f_{\varepsilon,k}\|_1 \to \ell_d(D_k)$ and $\|\nabla f_{\varepsilon,k}\|_1 \to P(D, \mathbb{R}^d) = \ell_{d-1}(\partial D_k)$ for $\varepsilon \to 0$ (where the last equality follows from Theorem C.5). Hence

$$\lim_{\varepsilon\to 0} \frac{\|\nabla f_{\varepsilon,k}\|_1}{\|f_{\varepsilon,k}\|_1} = \frac{\ell_{d-1}(\partial D_k)}{\ell_d(D_k)} \qquad \forall k \in \mathbb{N}. \tag{1.40}$$

The last expression converges to $h(M)$ for $k \to \infty$ by choice of the D_k, which completes the proof of (1.38). Combinig this with (1.37), we arrive at

$$\limsup_{p\to 1} \lambda_p \leq \lim_{k\to\infty} \lim_{\varepsilon\to 0} \lim_{p\to 1} C_{p,\varepsilon,k} = h(M),$$

which proves the claim. ☐

Theorem 1.5.4. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. If λ_p is the first eigenvalue of $-\Delta_p$ from Definition 1.4.4, and $h(M)$ is the Cheeger constant of M from Definition 1.3.1, then*

$$\lim_{p\to 1} \lambda_p = h(M).$$

Proof. From Theorem 1.5.2 we get

$$\liminf_{p\to 1} \lambda_p \geq \lim_{p\to 1} \left(\frac{h(M)}{p}\right)^p = h(M).$$

Together with $\limsup_{p\to 1} \lambda_p \leq h(M)$ from Theorem 1.5.3 this proves the claim. ☐

Theorem 1.5.5. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. Let further p_k be a sequence of real numbers with $p_k \geq 1$ and $p_k \to 1$ for $k \to \infty$ and u_{p_k} corresponding nonnegative first eigenfunctions of Δ_{p_k} with $\|u_{p_k}\|_{p_k} = 1$.*

Then there is a subsequence of u_{p_k} that converges in $L^1(M)$ to a nonnegative minimizer $u \in BV(M)$ of (2.20). In particular, if we define the superlevel sets

$$A_t := \{x \in M \mid u(x) > 0\}$$

then a non-null superlevel set A_t of u is a Cheeger set for almost all $t > 0$.

Proof. A similar claim is shown in [KF03] and [Fri03]. The generalization we will show in Theorem 2.3.4 includes this case as the special case $T = id$, so we do not state a proof here. ☐

Chapter 2

A dynamic p-Laplacian Δ_p^D

In this section, we will introduce and study the dynamic p-Laplacian Δ_p^D as it is also done in [DFJK23]. We will formally state the dynamic Cheeger problem associated to it and prove properties that are analogous to the properties presented in Chapter 1. In particular, we prove:

($*$) **Theorem 2.1.3**: The dynamic p-Laplacian arises as a Gâteaux derivative (this generalizes Theorem 1.4.7(a)).

($*$) **Corollary 2.1.4**: There exists a smallest eigenvalue λ_p^D of the associated eigenvalue problem (this generalizes Corollary 1.4.9)

Corollary 2.2.4: The dynamic Cheeger problem has a solution (this generalizes Corollary 1.3.5).

($*$) **Theorem 2.3.1**: A dynamic Cheeger inequality analogous to the one in Theorem 1.5.2 holds

($*$) **Theorem 2.3.3**: As $p \to 1$, the first eigenvalue λ_p^D converges to the dynamic Cheeger constant $h^D(M,T)$ (this generalizes Theorem 1.5.4)

Theorem 2.3.4: As $p \to 1$, a sequence of first eigenfunctions u_p^D converges, on a subsequence, to a solution of the BV-characterization of the dynamic Cheeger constant analogous to (1.16) (this generalizes Theorem 1.5.5).

The results marked with ($*$) are also published in [DFJK23]. Often, the proofs are close to the static case. The main property allowing this is the linearity and continuity of T_* as an operator on the function spaces $W_0^{1,p}(M)$ and $BV(M)$ (see Theorem D.1). This has the effect that the dynamic objects inherit relevant properties of their static counterparts.

2.1 Definitions and basic properties

Froyland's dynamic Laplacian is defined by

$$\Delta^D := \frac{1}{2}(\Delta + T^*\Delta T_*),$$

where T_* and T^* are the *transfer operator* and its dual, the *Koopman operator*. For a volume-preserving T and some function $f \in C^\infty(M)$, they are commonly written as $T_*f = f \circ T^{-1}$ and $T^*f = f \circ T$.

We can introduce the dynamic p-Laplacian analogously, but we have to be careful with the definition of T^* because, in the way we have introduced it here, Δ_p attains values in $W^{-1,q}(M)$, which contains distributions [AF03, Theorem 3.12]. We thus define T^* in a purely linear algebraic way, mapping some $v \in W^{-1,q}(M)$ to the unique element $T^*v \in W^{-1,q}(M)$ satisfying

$$\langle u, T^*v\rangle = \langle T_*u, v\rangle \qquad \text{for all } v \in W_0^{1,p}(M),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $W^{-1,q}(M) \times W_0^{1,p}(M)$ (see also Definition 1.2.2). This may also be written as $T^*v = v \circ T_*$ and coincides with the more common definition $T^*f = f \circ T$ under the usual identifications of dual spaces like $(L^p(M))^*$ with function spaces like $L^q(M)$ (see Theorem D.2). With this setup, we are able to proceed to the definition of the dynamic p-Laplacian.

Definition 2.1.1. *Let $M \subset \mathbb{R}^d$ be a compact, d-dimensional submanifold with Lipschitz boundary and $T : M \to M$ a volume-preserving diffeomorphism on M with the properties from Setup 1.1.2. Define the* dynamic p-Laplacian Δ_p^D *by*

$$\Delta_p^D = \frac{1}{2}\left(\Delta_p + T^*\Delta_p T_*\right).$$

We omit the dependence on T in the notation Δ_p^D.

Definition 2.1.2. *We say that $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(M)$ is a (weak Dirichlet)* eigenpair *of $-\Delta_p^D$ if*

$$-\Delta_p^D u = \lambda |u|^{p-2}u \tag{2.1}$$

as elements of $W^{-1,q}(M)$. If (u, λ) is an eigenpair of $-\Delta_p^D$, then we call u an eigenfunction *and λ an* eigenvalue *of $-\Delta_p^D$. We denote the infimum of the eigenvalues of $-\Delta_p^D$ by*

$$\lambda_p^D := \inf\{\lambda \in \mathbb{R} \mid \lambda \textit{ is eigenvalue of } -\Delta_p^D\}.$$

In view of Corollary 2.1.4 we will call λ_p^D the first eigenvalue *of $-\Delta_p^D$. Note, however, that it is not immediate from the definition that λ_p^D is itself an eigenvalue of $-\Delta_p^D$. As in the static case, we always only handle the Dirichlet case in this thesis and thus often only speak of "eigenvalues" and "eigenfunctions".*

The following theorem says that, as in the static case, Δ_p^D appears as the derivative of a functional that has the properties we need for the direct method of the calculus of variations:

Theorem 2.1.3. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2, $p \in (1, \infty)$, and F be defined as in Theorem 1.4.7. Define the functional $F^D : W_0^{1,p}(M) \to \mathbb{R}$ by*

$$F^D(u) := \frac{1}{2}(F(u) + F(T_*u)) = \frac{1}{2}\left(\|\nabla u\|_p^p + \|\nabla T_* u\|_p^p\right). \tag{2.2}$$

Then F^D is weakly lower semicontinuous, coercive, and Gâteaux differentiable with

$$(F^D)' = -p\Delta_p^D. \tag{2.3}$$

Proof. We know by Theorem 1.4.7 that F is weakly lower semicontinuous, coercive and Gâteaux differentiable. By continuity and linearity of T_*, the functional $F \circ T_*$ must also have these properties: coercivity is immediate, as $\|u_k\|_{W_0^{1,p}(M)} \to \infty$ implies $\|T_*u_k\|_{W_0^{1,p}(M)} \to \infty$ and thus $F(T_*u_k) \to \infty$ by coercivity of F. Semicontinuity follows like in the static case from convexity and strong continuity, which are both preserved by concatenation with a continuous linear operator. Also, T_* is linear and bounded, so it is Gâteaux differentiable with $(T_*)' = T_*$, and we get Gâteaux differentiability of $F \circ T_*$ by the chain rule [PK09, Proposition 1.1.14], which also yields:

$$(F \circ T_*)'(u) = F'(T_*u) \circ T_*.$$

If we plug in some $v \in W_0^{1,p}(M)$, we can see that

$$\langle F'(T_*u) \circ T_*, v\rangle = \langle F'(T_*u), T_*v\rangle = \langle T^*F'(T_*u), v\rangle = -p\langle T^*\Delta_p(T_*u), v\rangle,$$

and thus $(F \circ T_*)' = -pT^*\Delta_pT_*$.

The desired properties are preserved under linear combinations (for coercivity, we need nonnegativity; for semicontinuity, see Remark 3 after the definition of lower semicontinuity in [Bre11]), so all in all, the operator $F^D = \frac{1}{2}(F + F \circ T_*)$ is weakly lower semicontinuous, coercive, and Gâteaux differentiable with

$$(F^D)' = \frac{1}{2}(F' + (F \circ T_*)') = -p(\Delta_p + T^*\Delta_pT_*) = -p\Delta_p^D,$$

as claimed. ☐

Analogously to the static case in Corollary 1.4.9, we can now show the existence of a minimal eigenvalue.

Corollary 2.1.4. *The infimum λ_p^D of the eigenvalues of $-\Delta_p^D$ is itself an eigenvalue and can be written as*

$$\lambda_p^D = \inf_{\substack{u \in W_0^{1,p}(M) \\ u \neq 0}} \frac{\|\nabla u\|_p^p + \|\nabla T_*u\|_p^p}{2\|u\|_p^p}. \tag{2.4}$$

Any corresponding eigenfunction u_p^D attains the infimum on the right-hand side of (2.4). Further, λ_p^D is positive, and u_p^D can be chosen to be nonnegative.

Proof. This result is also published in [DFJK23], where it is proved directly. Here, we exploit that we have stated Theorem 1.4.8 in a form abstract enough to apply it to F^D, which is nonnegative, coercive, weakly lower semicontinuous, and Gâteaux differentiable with $(F^D)'(u) = -p\Delta_p^D$ by Theorem 2.1.3. The homogeneity conditions on F^D and $(F^D)'$ are again easily checked. Hence, we can apply Theorem 1.4.8(a) to F^D and get some u^* that realizes the infimum λ^* of $J^D := F^D/G$ and

$$-\Delta_p^D u_p^* = \frac{1}{p}(F^D)'(u^*) = \frac{1}{p}\lambda^* G'(u^*) = \lambda^* |u^*|^{p-2}u^*,$$

which means that (u^*, λ^*) is an eigenpair of $-\Delta_p^D$. Further, for every other eigenpair (u, λ) of $-\Delta_p^D$, we must have $J^D(u) = \lambda$ by Theorem 1.4.8(b), which means that $J(u^*) \leq J(u)$. Hence, $\lambda^* = \lambda_p^D$, which shows that λ_p^D is indeed an eigenvalue and $u_p^D := u^*$ is a corresponding eigenfunction. Positivity of $\lambda_p^D = \lambda^*$ follows by the Poincaré inequality: there is a $C > 0$ such that $\|u\|_p \leq C\|\nabla u\|_p$ and hence

$$\frac{F^D(u)}{G(u)} \geq \frac{\|\nabla u\|_p^p + \|\nabla(T_* u)\|_p^p}{2\|u\|_p^p} \geq \frac{\|\nabla u\|_p^p}{\|u\|_p^p} \geq \frac{1}{2C^p} > 0,$$

which implies $\lambda_p^D > 0$. There are nonnegative infimizers of F^D/G because of the equality $\|\nabla u\|_p = \|\nabla |u|\|_p$ [see BS11, Remarks in Section 2.6], so u_p^D can be chosen to be nonnegative. □

Remark 2.1.5. *The eigenfunction u_p^D is not unique: scalar multiples of eigenfunctions are again eigenfunctions corresponding to the same eigenvalue. We do not show further properties here that would uniquely single out one of the eigenfunctions. We also do not show simpleness of the first eigenvalue (i.e., that all first eigenfunctions are scalar multiples of each other) as it can be show in the classical case [Lin08].*

We have written Δ_p^D only in the concise form $\frac{1}{2}(\Delta_p + T^*\Delta_p T_*)$ so far, but at some point, sooner or later, one needs explicit representations. In the following theorem, we give expanded expressions for Δ_p^D.

Theorem 2.1.6. *For $u, v \in W_0^{1,p}(M)$, we can express $\langle \Delta_p^D u, v\rangle$ by*

$$\frac{1}{2}\int_M \left(|\nabla u|^{p-2}\nabla u \nabla v + |\nabla(T_* u)|^{p-2}\nabla(T_* u))(\nabla(T_* v))\right)$$

and

$$\frac{1}{2}\int_M \left(|\nabla u|^{p-2}\nabla u \nabla v + |DT^{-T}\nabla u|^{p-2}(DT^{-T}\nabla u)(DT^{-T}\nabla v)\right).$$

Proof. We get the first equality by plugging

$$\langle \Delta_p u, v\rangle = \int_M |\nabla u|^{p-2}\nabla u \nabla v$$

into

$$\begin{aligned}\langle \Delta_p^D u, v\rangle &= \frac{1}{2}\langle \Delta_p u + T^*\Delta_p T_*, v\rangle \\ &= \frac{1}{2}\left(\langle \Delta_p u, v\rangle + \langle T^*\Delta_p T_* u, v\rangle\right) \\ &= \frac{1}{2}\left(\langle \Delta_p u, v\rangle + \langle \Delta_p T_* u, T_* v\rangle\right).\end{aligned}$$

The second equality arises from a substitution: by the transformation rule for the gradient, we have

$$(\nabla(T_*u))(T(x)) = (DT(x))^{-T} \cdot \nabla u(x),$$

and we have $\det DT \equiv 1$ by volume preservation of T, so under substitution with T, the integral

$$\int_M |\nabla(T_*u)|^{p-2}\nabla(T_*u))(\nabla(T_*v))$$

becomes

$$\int_{T^{-1}M} |DT^{-T}\nabla u|^{p-2}(DT^{-T}\nabla u)(DT^{-T}\nabla v)$$

which, after using $T^{-1}M = M$, becomes the second claim.

□

We conclude with a technical result after [Lin08, Lemma 4]. We will use this in the proof of Theorem 2.3.4.

Lemma 2.1.7. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2 , $p \in (1,2)$, $d \geq 2$ and let u_p^D be a nonnegative first eigenfunction of Δ_p^D. Then there is a constant C_p depending only on p such that*

$$\|u_p^D\|_\infty < (C_p)^d(\lambda_p^D)^{\frac{d}{p}}\|u_p^D\|_1,$$

and $\limsup_{p\to 1} C_p < \infty$.

Proof. Lindqvist's bound in [Lin08, Lemma 4] for some eigenpair (u, λ) is

$$\|u\|_\infty < 4^d\lambda^{\frac{d}{p}}\|u\|_1,$$

We expand on the proof here and add modifications for the dynamic case. For ease of notation, set $u \coloneqq u_p^D, \lambda \coloneqq \lambda_p^D$.

We start by defining the function

$$\eta(x) \coloneqq \max\{u(x) - k, 0\}.$$

Plugging in η as a test function in the eigenvalue equation yields

$$\int_{A_k} \frac{1}{2}\left(|\nabla u|^p + |\nabla T_*u|^p\right) = \lambda \int_{A_k} |u|^{p-2}u(u-k), \tag{2.5}$$

where

$$A_k \coloneqq \{x \in M \mid u(x) > k\}. \tag{2.6}$$

One has $k \cdot \ell_d(A_k) \leq \|u\|_1$ and $\ell_d(A_k) \to 0$ as $k \to \infty$. The elementary inequality $a^{p-1} \leq 2^{p-1}(a-k)^{p-1} + 2^{p-1}k^{p-1}$ follows from monotonicity of $x \mapsto x^{p-1}$ and implies

$$\int_{A_k} u^{p-1}(u-k) \leq 2^{p-1}\int_{A_k}(u-k)^p + 2^{p-1}k^{p-1}\int_{A_k}(u-k). \tag{2.7}$$

Further, with $p^* = \frac{dp}{d-p}$, one can obtain existence of a p-dependent constant K_p such that

$$\int_{A_k}(u-k)^p \leq \ell_d(A_k)^{\frac{p}{d}}\|u\|_{p^*}^p \leq \ell_d(A)^{\frac{p}{d}}K_p^p\int_{A_k}|\nabla u|^p, \tag{2.8}$$

by first using the Hölder inequality and then the Sobolev inequality. The optimal constant K_p is known to be less than $\frac{p(d-1)}{d-p}$ (see, e.g., the proof of the Sobolev inequality in [Eva22], or footnote 9 in [Bre11, Theorem 9.9]), and hence, $\limsup_{p\to 1} K_p < \infty$.

We proceed like in the proof of Lindqvist, carrying the additional constant K_p along. We also add the additional estimate

$$\int_{A_k} |\nabla u|^p \leq 2 \int_{A_k} \frac{1}{2} \left(|\nabla u|^p + |\nabla T_* u|^p\right). \tag{2.9}$$

Setting $C \coloneqq 2^2 K_p^p$ and combining (2.9) with (2.8) yields

$$\int_{A_k} (u-k)^p \leq 2^{-1} C \ell_d(A_k)^{\frac{p}{d}} \int_{A_k} \frac{1}{2} \left(|\nabla u|^p + |\nabla T_* u|^p\right). \tag{2.10}$$

Now we can combine equations (2.5), (2.7) and (2.10) like Lindqvist, but with our modified expressions we get

$$\int_{A_k} \frac{1}{2} \left(|\nabla u|^p + |\nabla T_* u|^p\right) \overset{(2.5)}{=} \lambda \int_{A_k} |u|^{p-2} u(u-k) \tag{2.11}$$

$$\overset{(2.7)}{\Rightarrow} \int_{A_k} \frac{1}{2} \left(|\nabla u|^p + |\nabla T_* u|^p\right) \leq 2^{p-1}\lambda \int_{A_k} (u-k)^p + 2^{p-1}\lambda k^{p-1} \int_{A_k} (u-k) \tag{2.12}$$

$$\overset{(2.10)}{\Rightarrow} \int_{A_k} 2C^{-1} \ell_d(A_k)^{-\frac{p}{d}} (u-k)^p \leq 2^{p-1}\lambda \int_{A_k} (u-k)^p + 2^{p-1}\lambda k^{p-1} \int_{A_k} (u-k) \tag{2.13}$$

$$\Rightarrow \left[C^{-1} - 2^{p-2}\lambda \ell_d(A_k)^{\frac{p}{d}}\right] \int_{A_k} (u-k)^p \leq 2^{p-2}\lambda k^{p-1} \ell_d(A_k)^{\frac{p}{d}} \int_{A_k} (u-k) \tag{2.14}$$

We can then force $2^{p-2}\lambda \ell_d(A_k)^{\frac{p}{d}} \leq \frac{1}{2} C^{-1}$ by assuming

$$k \geq k_1 \coloneqq \left(2^{(p-1)}\lambda C\right)^{\frac{d}{p}} \|u\|_1$$

because $\ell_d(A_k) \leq \|u\|_1 / k$. We thus obtain

$$\int_{A_k} (u-k)^p \leq 2^{p-1} C \lambda k^{p-1} \ell_d(A_k)^{\frac{p}{d}} \int_{A_k} (u-k) \tag{2.15}$$

for $k \geq k_1$. By Hölder's inequality, the left-hand side can be made smaller to

$$\int_{A_k} (u-k)^p \geq \left(\frac{\int_{A_k} (u-k)}{\left(\int_{A_k} 1^q\right)^{\frac{p-1}{p}}} \right)^p = \ell_d(A_k)^{-(p-1)} \left(\int_{A_k} (u-k)\right)^p.$$

Plugging this into (2.15) and then dividing by $\int_{A_k} (u-k)$, as well as raising to the power of $\frac{1}{p-1}$, yields

$$\int_{A_k} (u-k) \leq 2 \left(C\lambda\right)^{\frac{1}{p-1}} k \ell_d(A_k)^{1+\frac{p}{(p-1)d}}, \tag{2.16}$$

This yields the claim in the following way: define

$$f(k) := \int_{A_k} (u-k).$$

By Cavalieri's principle, we can write $f(k) = \int_k^\infty \ell_d(A_k)$ and hence $f'(k) = -\ell_d(A_k)$. Thus, (2.16) yields the differential inequality

$$f(k) \leq \underbrace{2\,(C\lambda)^{\frac{1}{p-1}}}_{=:C_2} k(-f'(k))^{1+\overbrace{\frac{p}{(p-1)d}}^{=:\varepsilon}} = C_2 k(-f'(k))^{1+\varepsilon} \tag{2.17}$$

for f. One can deduce from (2.17) that $f(k)$ must vanish eventually: if f is positive on the interval $[k_1, k_2]$ for some k_2, then, in this interval, (2.17) can be rewritten to

$$\left(k^{\frac{\varepsilon}{1+\varepsilon}}\right)' \leq -C_2^{\frac{1}{1+\varepsilon}} \left(f(k)^{\frac{\varepsilon}{1+\varepsilon}}\right)',$$

where the derivatives are taken in k. Integrating this on $[k_1, k_2]$ yields

$$(k_2^{\frac{\varepsilon}{1+\varepsilon}} - k_1^{\frac{\varepsilon}{1+\varepsilon}}) \leq C_2^{\frac{1}{1+\varepsilon}} (f(k_1)^{\frac{\varepsilon}{1+\varepsilon}} - f(k_2)^{\frac{\varepsilon}{1+\varepsilon}}).$$

Applying $f(k_1) \leq f(0) \leq \|u\|_1$ and $f(k_2) \geq 0$, this reduces to a bound on k_2:

$$\begin{aligned} k_2^{\frac{\varepsilon}{1+\varepsilon}} &\leq C_2^{\frac{1}{1+\varepsilon}} \|u\|_1^{\frac{\varepsilon}{1+\varepsilon}} + k_1^{\frac{\varepsilon}{1+\varepsilon}} \\ &= \left(\left(2\,(C\lambda)^{\frac{1}{p-1}}\right)^{\frac{1}{1+\varepsilon}} + \left(2^{(p-1)}\lambda C\right)^{\frac{d}{p}\cdot\frac{\varepsilon}{1+\varepsilon}}\right) \|u\|_1^{\frac{\varepsilon}{1+\varepsilon}} \\ &= \left(\left(2^{p-1}C\lambda\right)^{\frac{1}{p-1}\cdot\frac{1}{1+\varepsilon}} + \left(2^{p-1}\lambda C\right)^{\frac{1}{p-1}\cdot\frac{1}{1+\varepsilon}}\right) \|u\|_1^{\frac{\varepsilon}{1+\varepsilon}} \\ &= 2\left(2^{p-1}C\lambda\right)^{\frac{1}{p-1}\cdot\frac{1}{1+\varepsilon}} \|u\|_1^{\frac{\varepsilon}{1+\varepsilon}} \end{aligned}$$

and thus

$$k_2 \leq 2\left(2^{p-1}C\lambda\right)^{\frac{1}{p-1}\cdot\frac{1}{\varepsilon}} \|u\|_1 = 2^{1+\frac{d(p-1)}{p}} (C\lambda)^{\frac{d}{p}} \|u\|_1.$$

All in all this means that $f(k)$ can only be positive if k is smaller than the right-hand side (recall that f is decreasing monotonically). This bounds the essential supremum of u_p and hence

$$\|u_p\|_\infty \leq 2^{1+\frac{d(p-1)}{p}} (C\lambda)^{\frac{d}{p}} \|u\|_1 = 2^{1+\frac{d(p+1)}{p}} \lambda^{\frac{d}{p}} K_p^d \|u\|_1 \leq 2^{3d}\lambda^{\frac{d}{p}} K_p^d \|u\|_1,$$

which yields the claim after setting $C_p := 2^{3d}K_p$. As $\limsup_{p\to 1} K_p < \infty$, the same is true for C_p. □

2.2 The dynamic Cheeger problem

We introduce the dynamic version of the Cheeger problem, which, in this variation, can be first found in [FJ18, Section 2.2]. As in the classical case in Section 1.3, we will state the dynamic Cheeger problem using the perimeter $P(D, \mathbb{R}^d)$ of a set D (see Appendix C for details), which allows arbitrary measurable sets as candidates for Cheeger sets. This will allow us to use the same variational techniques to prove the existence of a minimizer as in the classical case.

Concerning notation, we will follow Froyland in the convention to annotate the dynamical versions of objects with a superscript "D". We start with the generalization of Definition 1.3.1.

Definition 2.2.1. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Define the* dynamic Cheeger constant *of M to be*

$$h^D(M,T) \coloneqq \inf_{D \subset M} \frac{P(D,\mathbb{R}^d) + P(T(D),\mathbb{R}^d)}{2\ell_d(D)}$$

where the infimum ranges over all Borel subsets of M and the value of the fraction is assumed to be ∞ if $P(D, \mathbb{R}^d) = \infty$ or $\ell_d(D) = 0$. Like in the static case, we call the quantity in the infimum the dynamic Cheeger ratio of D. *If a set $D \subseteq M$ has a dynamic Cheeger ratio of $h^D(M,T)$, we call D a* dynamic Cheeger set.

Remark 2.2.2. *As in Remark 1.3.2, we have that for every measurable subset $D \subset M$ that*

$$\frac{1}{2}(P(D,\mathbb{R}^d) + P(T(D),\mathbb{R}^d)) \geq h^D(M,T)\ell_d(D),$$

and if this is an equality, then D is either a null set or a dynamic Cheeger set.

Analogous to the reformulations in Equations (1.14) to (1.16), we have some equivalent expressions for $h^D(M,T)$:

$$h^D(M,T) = \inf_{\substack{D \subset M \\ \partial D \text{ smooth}}} \frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)} \tag{2.18}$$

$$h^D(M,T) = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1} \tag{2.19}$$

$$h^D(M,T) = \inf_{u \in BV(M)\setminus\{0\}} \frac{|Du|(\mathbb{R}^d) + |D(T_* u)|(\mathbb{R}^d)}{2\|u\|_1}. \tag{2.20}$$

The equalities (2.18) and (2.19) are shown in Appendix F. We will call (2.20) the *BV-characterization* of $h^D(M,T)$. We state the proof below in Theorem 2.2.3, as it is closely tied to an existence proof of a minimizer of the BV-characterization (i.e. some u that realizes the infimum on the right hand side of (2.20)). See also Appendix C for an introduction to $BV(M)$ and the definition of $|Du|(\mathbb{R}^d)$.

Theorem 2.2.3. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Then*

(a) Equation (2.20) holds and the infimum is is attained by some nonnegative function $u \in BV(M)$.

(b) *If some nonnegative $u \in BV(M)$ attains $h^D(M,T)$, then for almost all $t > 0$ the superlevel sets*

$$A_t \coloneqq \{x \in M \mid u(x) > t\}$$

are either null sets or dynamic Cheeger sets.

Proof. We stay close to the techniques and structure that Parini uses in the proof of [Par11, Proposition 3.1]. The incorporation of dynamics is done similarly to the proof of [Fro15, Theorem 3.1].

We first give a name to the right-hand side of the equality in (2.20):

$$h^D_{BV}(M,T) \coloneqq \inf_{u \in BV(M)\setminus\{0\}} \frac{|Du|(\mathbb{R}^d) + |D(T_*u)|(\mathbb{R}^d)}{2\|u\|_1}.$$

Then, before showing $h^D_{BV}(M,T) = h^D(M,T)$, we start by showing that $h^D_{BV}(M,T)$ is attained. This follows by applying the direct method of the Calculus of Variations in $BV(M)$. Let u_k be a minimizing, L^1-normalised sequence for $h^D_{BV}(M,T)$. By the appearance of $|Du_k|(\mathbb{R}^d)$ it must be bounded in $BV(M)$. Therefore, by the compactness property from [Par11, Proposition 2.2], there is a subsequence on which it u_k converges in L^1 to some $u \in BV(M)$. We pass to that subsequence. Now by the lower semicontinuity property from [Par11, Proposition 2.1] we have

$$|Du|(\mathbb{R}) \leq \liminf_{k\to\infty} |Du_k|(\mathbb{R}).$$

As $T_* : BV(M) \to BV(M)$ is continous (see Theorem D.1), the functions T_*u_k converge to T_*u in $L_1(M)$ as well, so by the same argument we have

$$|D(T_*u)|(\mathbb{R}) \leq \liminf_{k\to\infty} |D(T_*u_k)|(\mathbb{R}).$$

Thus, with

$$F^D(u) \coloneqq \frac{1}{2}(|Du|(\mathbb{R}) + |D(T_*u)|(\mathbb{R}))$$

we have that

$$h^D_{BV}(M,T) \leq \frac{F^D(u)}{\|u\|_1} = F^D(u) \leq \liminf_{i\to\infty} F^D(u_k) = \lim_{i\to\infty} \frac{F^D(u_k)}{\|u_k\|_1} = h^D_{BV}(M,T), \quad (2.21)$$

This implies

$$h^D_{BV}(M,T) = F^D(u).$$

And we have shown that $h^D_{BV}(M,T)$ is attained. We assume nonnegativity of u, which is possible by the estimate $|D|u||(\mathbb{R}^d) \leq |Du|(\mathbb{R}^d)$ (as used in the proof of [Par11, Proposition 2.1]). Now let $u \in BV(M)$ be arbitrary and define $\tilde{u} \in BV(\mathbb{R}^d)$ by extending it by zero and $\tilde{T}$ by extending T to the identity on $\mathbb{R}^d$, i.e,

$$\tilde{u}|_M \equiv u, \quad \tilde{u}|_{\mathbb{R}^d\setminus M} \equiv 0$$
$$\tilde{T}|_M \equiv T, \quad \tilde{T}|_{\mathbb{R}^d\setminus M} \equiv id$$

Further, define $\tilde{A}_t \coloneqq \{x \in \mathbb{R}^d \mid \tilde{u} > t\}$. Then we have

$$\begin{aligned}
h_{BV}^D(M,T) &= \frac{1}{2}\Big(|Du|(\mathbb{R}^d) + |D(T_* u)|(\mathbb{R}^d)\Big) && (2.22)\\
&\overset{(a)}{=} \int_{-\infty}^{\infty} \frac{1}{2}\Big(P(\tilde{A}_t, \mathbb{R}^d)dt + P(\tilde{T}(\tilde{A}_t), \mathbb{R}^d)dt\Big) && (2.23)\\
&\overset{(b)}{=} \int_{0}^{\infty} \frac{1}{2}\Big(P(A_t, \mathbb{R}^d)dt + P(T(A_t), \mathbb{R}^d)dt\Big) && (2.24)\\
&\geq h^D(M,T) \int_0^\infty \ell_d(A_t)dt && (2.25)\\
&\overset{(c)}{=} h^D(M,T)\|u\|_1 && (2.26)\\
&= h^D(M,T), && (2.27)
\end{aligned}$$

where we have used the following:

(a) the coarea formula in the variations of Theorem E.3 and Theorem E.4.

(b) the integration limits can be changed because for $t < 0$ we have $\tilde{A}_t = \mathbb{R}^d$ and hence $P(\tilde{A}_t, \mathbb{R}^d) = P(\tilde{T}(\tilde{A}_t), \mathbb{R}^d) = 0$. The sets $\tilde{A}_t, \tilde{T}(\tilde{A}_t)$ can be changed to $A_t, T(A_t)$, because, for $t > 0$, we have $\tilde{A}_t$, so $\tilde{A}_t = A_t$. and $T(\tilde{A}_t) = T(A_t)$.

(c) Cavalieri's principle (see Theorem 1.5.1).

This chain of inequalities shows two things: first, it shows $h_{BV}^D(M,T) \geq h^D(M,T)$ and as $h_{BV}^D(M,T) \leq h^D(M,T)$ by definition, this implies the equality claimed in (a). Second, this shows that whenever some u is a minimizer of the infimum in Equation (2.20), then the chain of inequalities above starts with $h^D(M,T)$ and ends with it, so that inequality (2.25) is actually an equality and hence for almost all $t > 0$

$$\frac{1}{2}(P(A_t, \mathbb{R}^d) + P(T(A_t), \mathbb{R}^d)) = h^D(M,T)\ell_d(A_t).$$

This can only hold if A_t is a null set or a dynamic Cheeger set, which proves the second claim. □

Corollary 2.2.4. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. Then there exists a dynamic Cheeger set.*

Proof. By Theorem 2.2.3 there is a nonnegative function $u \in BV(M)$ that attains $h^D(M,T)$ and by Theorem 2.2.3(b) almost all of its nonnull superlevel sets are dynamic Cheeger sets. As u is not zero, it has a superlevel set of nonzero measure, which must be a dynamic Cheeger set. □

2.3 Connections between Δ_p^D and the dynamic Cheeger problem

We now generalize the results from Section 1.5 to the dynamic case, connecting the dynamic p-Laplacian Δ_p^D from Section 2.1 to the dynamic Cheeger problem from Section 2.2. This comprises a generalization of the Cheeger inequality from Theorem 1.5.2 to the dynamic case, a convergence result of λ_p^D for $p \to 1$, and L^1-convergence of u_p^D to a minimizer of the BV-characterization on a subsequence for $p \to 1$. Results that also appear in [DFJK23] are indicated in the proofs.

The convergence of the first eigenvalue λ_p^D will be done analogously to Section 1.5 in two steps. The first step is to show a generalization of the general Cheeger inequality to the dynamic case.

Theorem 2.3.1 (Dynamic Cheeger inequality)**.** *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2 and let $1 < p < \infty$. If λ_p^D is the first eigenvalue of $-\Delta_p^D$ from Definition 2.1.2 and $h^D(M,T)$ is the dynamic Cheeger constant of M from Definition 2.2.1 then*

$$\lambda_p^D \geq \left(\frac{h^D(M,T)}{p}\right)^p.$$

Proof. For $p = 2$, this has been done by Froyland in [FJ18]. The structure of the proof follows the statuc case Theorem 1.5.2. A similar proof has also been published in [DFJK23]. First, let $w \in C_0^\infty(M)$ be nonnegative and define

$$\begin{aligned} A_t &:= \{x \in M \mid w(x) > t\}, \\ B_t &:= \{x \in M \mid (T_* w)(x) > t\}. \end{aligned}$$

Then, the coarea formula in Corollary E.2 can be applied to w and to $T_* w$, yielding:

$$\int_M |\nabla w| = \int_0^\infty \ell_{d-1}(\partial A_t) dt \tag{2.28}$$
$$\int_M |\nabla T_* w| = \int_0^\infty \ell_{d-1}(\partial B_t) = \int_0^\infty \ell_{d-1}(T(\partial A_t)) dt. \tag{2.29}$$

(the lower integration limit can be changed to 0 because, for $t < 0$, the boundary of $A_t = M$ in M is empty). Note that by continuity of T, we have $T(\partial A_t) = \partial(T(A_t))$. We can then continue as in the static case with

$$\begin{aligned} \int_M \frac{1}{2}(|\nabla w| + |\nabla T_* w|) &= \int_0^\infty \frac{1}{2}(\ell_{d-1}(\partial A_t) + \ell_{d-1}(\partial(T(A_t))) dt && (2.30) \\ &\geq h^D(M,T) \int_0^\infty \ell_d(A_t) dt && (2.31) \\ &= h^D(M,T) \int_M |w|, && (2.32) \end{aligned}$$

where step (2.31) follows from Remark 2.2.2, Theorem C.5(b), and Sard's theorem, and step (2.32) is Cavalieri's principle (Theorem 1.5.1). As argued in Theorem 1.5.2, this inequality in fact holds for all $w \in W_0^{1,1}(M)$ by density of $C_0^\infty(M)$. Thus, we have shown

$$\int_M \frac{1}{2}(|\nabla w| + |\nabla(T_* w)|) \geq h^D(M,T) \int_M |w|. \tag{2.33}$$

for every $w \in W_0^{1,1}(M)$. With the definition $\Phi(v) \coloneqq \varphi_p(v) := v|v|^{p-1}$ for $v \in W_0^{1,p}(M)$ we have

$$\nabla\Phi(v) = \varphi_p'(v)\nabla v = p|v|^{p-1}\nabla v \tag{2.34}$$
$$\nabla\Phi(T_*v) = \varphi_p'(T_*v)\nabla(T_*v) = p|T_*v|^{p-1}\nabla(T_*v), \tag{2.35}$$

so by the Hölder inequality

$$\int_M |\nabla\Phi(v)| = p\int_M |v|^{p-1}|\nabla v| \leq p\||v|^{p-1}\|_q\|\nabla v\|_p = p\|v\|_p^{p-1}\|\nabla v\|_p \tag{2.36}$$

and

$$\begin{aligned}\int_M |\nabla\Phi(T_*v)| &= p\int_M |T_*v|^{p-1}|\nabla(T_*v)| \\ &\leq p\|T_*v\|_p^{p-1}\|\nabla(T_*v)\|_p = p\|v\|_p^{p-1}\|\nabla(T_*v)\|_p \end{aligned}\tag{2.37}$$

(where the equality $\|v\|_p = \|T_*v\|_p$ follows from the fact that T is volume-preserving). This means that $\Phi(v), \Phi(T_*v) \in W_0^{1,1}(M)$ and thus (2.33) applies, yielding

$$h^D(M,T) \overset{(2.33)}{\leq} \frac{\int_M |\nabla\Phi(v)| + |\nabla\Phi(v)|}{2\int_M |\Phi(v)|} \tag{2.38}$$
$$\overset{(2.36),(2.37)}{\leq} \frac{p\|v\|_p^{p-1}(\|\nabla v\|_p + \|\nabla T_*v\|_p)}{2\|v\|_p^p} \tag{2.39}$$
$$= p\frac{\|\nabla v\|_p + \|\nabla T_*v\|_p}{2\|v\|_p}. \tag{2.40}$$

Hence, for every $v \in W_0^{1,p}(M)$ (using convexity of $x \mapsto |x|^p$):

$$\frac{\|\nabla v\|_p^p + \|\nabla T_*v\|_p^p}{2\|v\|_p^p} \geq \left(\frac{\|\nabla v\|_p + \|\nabla T_*v\|_p}{2\|v\|_p}\right)^p \geq \left(\frac{h^D(M,T)}{p}\right)^p,$$

which shows the claim after passing to the infimum over v. □

Theorem 2.3.2. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. . Then*

$$\limsup_{p\to 1} \lambda_p^D \leq h^D(M,T).$$

Proof. Like the proof of Theorem 2.3.1, this proof has the same structure as its static counterpart Theorem 1.5.3 and appears also in [DFJK23]. Key ideas for this proof appear in [KF03, Corollary 6] for the static case and in [Fro15, Proof of Theorem 3.1] for the incorporation of dynamics in the case $p = 2$.

Like in Theorem 1.5.3, we start with a sequence of subdomains $D_k \subset M$ with smooth boundaries not touching ∂M, fulfilling the limit

$$\frac{\ell_{d-1}(\partial D_k) + \ell_{d-1}(\partial(T(D_k)))}{2\ell_d(D_k)} \overset{k\to\infty}{\longrightarrow} h^D(M,T),$$

and define $f_{\varepsilon,k} \coloneqq \chi_{D_k} * \rho_\varepsilon$ with a convolution kernel ρ_ε, such that we get smooth functions vanishing at ∂M if ε smaller than some ε_k^*. Again, we get upper bounds on λ_p^D from

$$\lambda_p^D \leq \frac{\|\nabla f_{\varepsilon,k}\|_p^p + \|\nabla(T_* f_{\varepsilon,k})\|_p^p}{2\|f_{\varepsilon,k}\|_p^p} =: C_{p,\varepsilon,k}^D \quad \forall k \in \mathbb{N},\ \varepsilon \in (0, \varepsilon_k^*),\ p \in (1,\infty). \tag{2.41}$$

The identity

$$\lim_{k\to\infty} \lim_{\varepsilon\to 0} \lim_{p\to 1} C_{p,\varepsilon,k}^D = h^D(M,T) \tag{2.42}$$

can be shown in the same way as in Theorem 1.5.3: The first limit

$$\lim_{p\to 1} C_{p,\varepsilon,k}^D = \frac{\|\nabla f_{\varepsilon,k}\|_1 + \|\nabla(T_* f_{\varepsilon,k})\|_1}{2\|f_{\varepsilon,k}\|_1} \tag{2.43}$$

follows by dominated convergence ($|f_{\varepsilon,k}|$, $|\nabla f_{\varepsilon,k}|$ and $|\nabla T_* f_{\varepsilon,k}|$ are pointwise bounded by compactness of M).

The functions $f_{\varepsilon,k}$ converge in $BV(M)$ to χ_{D_k} for $\varepsilon \to 0$ by [Amb00, Proposition 3.7]. By continuity of T_* on $BV(M)$, this implies that also $T_* f_{\varepsilon,k}$ converges to $T_*(\chi_{D_k}) = \chi_{T(D_k)}$ in $BV(M)$. Together, we get $\|\nabla f_{\varepsilon,k}\|_1 \to \ell_{d-1}(\partial D_k)$ and $\|f_{\varepsilon,k}\| \to \ell_d(D_k)$, like in Theorem 1.5.3, and, additionally, $\|\nabla(T_* f_{\varepsilon,k})\|_1 \to \ell_{d-1}(\partial(T(D_k)))$ by Theorem C.5. This implies the second limit

$$\lim_{\varepsilon\to 0} \frac{\|\nabla f_{\varepsilon,k}\|_1 + \|\nabla(T_* f_{\varepsilon,k})\|_1}{2\|f_{\varepsilon,k}\|_1} = \frac{\ell_{d-1}(\partial D_k) + \ell_{d-1}(\partial(T(D_k)))}{2\ell_d(D_k)}. \tag{2.44}$$

The last expression in (2.44) converges to $h^D(M,T)$ for $k \to \infty$ by definition of the D_k, which completes the proof of (2.42). We can now combine (2.42) with (2.41) to arrive at

$$\limsup_{p\to 1} \lambda_p^D \leq \lim_{k\to\infty} \lim_{\varepsilon\to 0} \lim_{p\to 1} C_{p,\varepsilon,k}^D = h^D(M,T), \tag{2.45}$$

which proves the claim. □

Theorem 2.3.3. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Then*

$$\lambda_p^D \to h^D(M,T)$$

for $p \to 1$.

Proof. From Theorem 2.3.1 we get

$$\liminf_{p\to 1} \lambda_p^D \geq \lim_{p\to 1} \left(\frac{h^D(M,T)}{p}\right)^p = h^D(M,T).$$

Together with $\limsup_{p\to 1} \lambda_p^D \leq h^D(M,T)$ from Theorem 2.3.2 this proves the claim. □

Theorem 2.3.4. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2, $d \geq 2$. Let further p_k be a sequence of real numbers with $p_k \geq 1$ and $p_k \to 1$ for $k \to \infty$ and $u_{p_k}^D$ corresponding nonnegative first eigenfunctions of $\Delta_{p_k}^D$ with $\|u_{p_k}^D\|_{p_k} = 1$.*

Then there is a subsequence of $u_{p_k}^D$ that converges in $L^1(M)$ to a nonnegative minimizer $u \in BV(M)$ of (2.20). In particular, if we define the superlevel sets

$$A_t \coloneqq \{x \in M \mid u(x) > 0\}$$

then a non-null superlevel set A_t of u is a dynamic Cheeger set for almost all $t > 0$.

Proof. We roughly use the ideas from [Par09, Theorem 2.33]. For clarity of notation, we just write p instead of p_k and say $p \to 1$ when we mean $k \to \infty$.

As $W_0^{1,p}(M) \subset BV(M)$, we have that $u_p^D \in BV(M)$. For L^1-convergence on a subsequence, boundedness of u_p^D in $BV(M)$ is sufficient by the compactness property in [Amb00, Theorem 3.23]. To apply this, we need boundedness of u_p^D in $BV(M)$, i.e., boundedness of $\|u_p^D\|_1$ and $|Du_p^D|(\mathbb{R}^d)$. The former can be estimated using the Hölder inequality

$$\|u_p^D\|_1 \leq \|u_p^D\|_p \|1\|_q = \ell_d(M)^{\frac{1}{q}} \xrightarrow{p\to 1} 1. \tag{2.46}$$

For the latter, we use Lemma C.2, the Hölder inequality, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ and Theorem 2.3.3 in the calculation

$$\begin{aligned}
|Du_p^D|(\mathbb{R}^d) &= \|\nabla u_p^D\|_1 && (2.47)\\
&\leq \|\nabla u_p^D\|_1 + \|\nabla(T_* u_p^D)\|_1 && (2.48)\\
&\leq (\|\nabla u_p^D\|_p + \|\nabla(T_* u_p^D)\|_p)\|1\|_q && (2.49)\\
&\leq \left(2^{p-1}(\|\nabla u_p^D\|_p^p + \|\nabla(T_* u_p^D)\|_p^p)\right)^{\frac{1}{p}} \|1\|_q && (2.50)\\
&= 2\lambda_p^D(M,T)^{\frac{1}{p}} \ell_d(M)^{\frac{1}{q}} \xrightarrow{p\to 1} 2h^D(M,T). && (2.51)
\end{aligned}$$

This completes the proof of boundedness of u_p^D in $BV(M)$ and thus of convergence on a subsequence. By Lemma 2.1.7, there is some C_p depending on p such that

$$\|u_p^D\|_\infty \leq C_p^d (\lambda_p^D)^{\frac{d}{p}} \|u_p^D\|_1.$$

By Theorem 2.3.3 and (2.46), we have that

$$\limsup_{p\to 1}\ C_p^d (\lambda_p^D)^{\frac{d}{p}} \|u_p^D\|_1 \leq (\limsup_{p\to 1} C_p)^d h^D(M,T)^d < \infty,$$

and thus, the necessary conditions on boundedness of $\|u_p^D\|_\infty$ to apply Lemma F.3 are fulfilled. We obtain that the limit function u must fulfill

$$\|u\|_1 = \lim_{p\to 1} \|u_p^D\|_p^p = 1.$$

The function u is nonnegative because the u_p^D are nonnegative (nonnegativity is equivalent to being a fixed point of the L^1-continuous map $f \mapsto |f|$).

We now show that u is a minimizer of the variational problem from Theorem 2.2.3, which will yield the claim about the superlevel sets. First note that by Theorem 2.2.3 we have

$$\frac{1}{2}(|Du|(\mathbb{R}^d) + |D(T_* u)|(\mathbb{R}^d)) \geq h^D(M,T)$$

(using $\|u\|_1 = 1$). Thus, for equality we just have to show the converse inequality. To do so, first use lower semicontinuity [Par11, Proposition 2.1] and continuity of T^* on $BV(M)$ to get

$$\begin{aligned}
\frac{1}{2}(|Du|(\mathbb{R}^d) + |D(T_* u)|(\mathbb{R}^d)) &\leq \liminf_{p\to 1} \frac{1}{2}(|Du_p^D|(\mathbb{R}^d) + |D(T_* u_p^D)|(\mathbb{R}^d)) && (2.52)\\
&= \liminf_{p\to 1} \frac{1}{2}(\|\nabla u_p^D\|_1 + \|\nabla(T_* u_p^D)\|_1). && (2.53)
\end{aligned}$$

With the Hölder inequality we can estimate

$$\frac{1}{2}(\|\nabla u_p^D\|_1 + \|\nabla(T_* u_p^D)\|_1 \leq \frac{1}{2}\left(\|\nabla u_p^D\|_p + \|\nabla(T_* u_p^D)\|_p\right)\ell_d(M)^{\frac{1}{q}} \tag{2.54}$$

$$\leq \frac{1}{2}\left(2^{p-1}(\|\nabla u_p^D\|_p^p + \|\nabla(T_* u_p^D)\|_p^p)\right)^{\frac{1}{p}}\ell_d(M)^{\frac{1}{q}} \tag{2.55}$$

$$= \left(\lambda_p^D\right)^{\frac{1}{p}}\ell_d(M)^{\frac{1}{q}} \xrightarrow{p\to 1} h^D(M,T). \tag{2.56}$$

Therefore

$$\frac{1}{2}(|Du|(\mathbb{R}^d) + |D(T_*u)|(\mathbb{R}^d) \leq h^D(M,T)$$

and thus, u is a minimizer of the BV-characterization of $h^D(M,T)$, which, by Theorem 2.2.3, means that, almost surely in $t > 0$, its superlevel sets A_t are either Cheeger sets or null sets. $\square$

Chapter 3

Numerical treatment of the eigenvalue problem

In contrast to the case $p = 2$, where Δ_p and Δ_p^D are linear, for $p \neq 2$, the operator Δ_p is nonlinear. This makes its numerical treatment much harder. Finite element approximations of the problem $\Delta_p u = f$ have already been studied in 1993 by Liu and Barret in [BL93]. The literature on the numerical treatment of the eigenvalue problem value is more recent. For the first eigenfunction, where the problem is a minimization problem, Lefton and Wei [LW97] used the Levenberg-Marquardt algorithm for the optimization. In [YZ07], Yao & Zhou give a min-max algorithm for higher eigenvalues of a general class of nonlinear eigenvalue problems that include the p-Laplacian. (the latter approach is the one we will adapt and use in our experiments). Further work includes a descent algorithm for the first eigenfunction and a mountain pass algorithm for the second one [Hor11]. More recently a method based on radial basis functions was used in [Ant19].

For our purposes, we first tried a homotopy approach: whenever there is a parameter-dependent problem which is easy to solve for some values of the parameter, it can be a good idea to first solve the problem for the easy parameters and then gradually change the parameter so that it approaches the desired value. For the eigenvalue problem of Δ_p, the case $p = 2$ is the easy case, as Δ_2 is linear. It is thus sensible to try solving the linear case and then apply a predictor-corrector method where one decreases the value of p by some decrement and uses the old solution as an initial guess for the new value of p in a corrector step. If the corrector step fails, one decreases the step one does in p.

We found that for p approaching 1, this happens very often, and the step sizes get small very fast. The properties in Section 3.1 and the example in Section 3.2.4 are our attempts to explain these difficulties, which indicate that local convergence in the corrector step is hard to achieve if one formulates the problem directly in the "operational" form $-\Delta_p u = \lambda |u|^{p-2} u$.

For these reasons, we ultimately used a variational descent method for the experiments in Chapter 4. In Section 3.2 we will give a short introduction to this method by Yao and Zhou [YZ07]. We also derive the modifications that we apply for the special case of Δ_p^D.

Finally, we present some ideas and observations about possible alternative approaches to the eigenvalue problem.

3.1 Properties posing challenges

We point out two properties of the eigenvalue problem that make the numerical treatment especially hard. We do this only for the classical p-Laplacian Δ_p, as it is a special case of the dynamic p-Laplacian. As a general method has to work also for this special case, difficulties for Δ_p apply to Δ_p^D as well.

3.1.1 Nondifferentiability

A possible approach for finding eigenvalues of Δ_p is the "operational" approach [SFG74], which consists of searching for solutions (u, λ) of the equation $\Delta_p u - \lambda |u|^{p-2}u = 0$ after some discretization (e.g. on some finite element spaces). Root finding in the nonlinear setting is generally much easier if we have the derivative of the objective at hand, i.e., using the Newton method.

The appearance of the term $|\nabla u|^{p-2}u$ in the p-Laplacian, however, hints at a nondifferentiability: the term $|t|^{p-2}t$ is not differentiable in $t = 0$ for $p < 2$. One can show that indeed, the p-Laplacian is not differentiable in some circumstances.

Theorem 3.1.1. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2and let $1 < p < 2$. Then the (nonlinear) operator*

$$\Delta_p : W_0^{1,p}(M) \to W^{-1,q}(M) \tag{3.1}$$
$$u \mapsto \left(v \mapsto \int_M |\nabla u|^{p-2}\nabla u \nabla v\right) \tag{3.2}$$

is not Gâteaux-differentiable in u if $\nabla u \equiv 0$ on some open subset of M.

Proof. Assume Δ_p is Gâteaux-differentiable in u. For any fixed $v \in W_0^{1,p}(M)$, the map

$$\iota_v : W^{-1,q} \mapsto \mathbb{R}, \xi \mapsto \xi(v)$$

is linear and bounded and hence continuous. It follows that $\iota_v \circ \Delta_p$ is also Gâteaux-differentiable in u, as for any $w \in W^{1,p}(M)$:

$$\lim_{h\to 0} \frac{\iota_v(\Delta_p(u+hw)) - \iota_v(\Delta_p(u))}{h} = \lim_{h\to 0} \iota_v\left(\frac{\Delta_p(u+hw) - \Delta_p(u)}{h}\right) \tag{3.3}$$
$$= \iota_v\left(\lim_{h\to 0} \frac{\Delta_p(u+hw) - \Delta_p(u)}{h}\right). \tag{3.4}$$

Now, since $\{\nabla u = 0\}$ has an open subset, we can choose a function $v \in W^{1,p}(M)$ such that $\operatorname{supp} v \subseteq \{\nabla u = 0\}$ and $\|\nabla v\|_p \neq 0$. Then, as we just saw, $\iota_v \circ \Delta_p$ is also Gâteaux-differentiable; in particular, the expression

$$(\iota_v \circ \Delta_p)(u + tv) \tag{3.5}$$

should be differentiable in t for $t = 0$. As ∇u vanishes on the support of v, this evaluates to

$$(\iota_v \circ \Delta_p)(u + tv) = \int_M |\nabla(u + tv)|\nabla(u + tv)\nabla v \tag{3.6}$$

$$= \int_{\mathrm{supp}(v)} |\nabla(u + tv)|\nabla(u + tv)\nabla v \tag{3.7}$$

$$= \int_{\mathrm{supp}(v)} |\nabla(tv)|^{p-2}\nabla(tv)\nabla v \tag{3.8}$$

$$= |t|^{p-2}t\|\nabla v\|_p^p. \tag{3.9}$$

But $t \mapsto |t|^{p-2}t$ is not differentiable in $t = 0$ for $p < 2$ and hence we have a contradiction. □

Theorem 3.1.2. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Let $1 < p < 2$ and $N \subset W_0^{1,p}(M)$ be the set of all u in which the mapping $\Delta_p : W_0^{1,p}(M) \to W^{-1,q}(M)$ is not Gâteaux-differentiable. Then N is dense in $W^{1,p}(M)$.*

Proof. We use Theorem 3.1.1 and prove that for $u \in C_0^1(M)$, there are $\bar{u}$ arbitrarily close to u such that $\nabla\bar{u} \equiv 0$ on an open subset of M. This shows the claim, as $C_0^1(M)$ is dense in $W_0^{1,p}(M)$. To do so, first choose some $\psi \in C_0^\infty(\mathbb{R}^n)$ to vanish outside the ball $B_2(0)$ of radius 2 while having the constant value 1 on the ball $B_1(0)$ of radius 1. Let x_0 be an arbitrary point in the interior of M and define $\psi_\varepsilon(x) := \psi(\varepsilon^{-1}(x - x_0))$ for $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(x_0, \partial M)$. With

$$\bar{u} := \psi_\varepsilon u(x_0) + (1 - \psi_\varepsilon)u$$

we have $u - \bar{u} = (u - u(x_0))\psi_\varepsilon$. Defining $f := u - u(x_0)$ we will now show $\bar{u} \to u$ in $W_0^{1,p}(M)$ by showing $\lim_{\varepsilon\to 0} \|f\psi_\varepsilon\|_{W_0^{1,p}(M)} = 0$. By the Poincaré inequality, we just have to consider the norm $\|\nabla(f\psi_\varepsilon)\|_p$. It can be bound by the sum of $\|(\nabla f)\psi_\varepsilon\|_p$ and $\|f\nabla\psi_\varepsilon\|_p$ with the triangle inequality, where for the first term we can estimate

$$\int_M |(\nabla f)\psi_\varepsilon|^p \leq \underbrace{\ell_n(\mathrm{supp}\,\psi_\varepsilon)}_{=\mathcal{O}(\varepsilon^n)} \underbrace{\|\nabla f\|_\infty^p \|\psi_\varepsilon\|_\infty^p}_{=\mathcal{O}(1)} \xrightarrow{\varepsilon\to 0} 0.$$

For the second term we first note that $\|\nabla\psi_\varepsilon\|_\infty = \varepsilon^{-1}\|\nabla\psi\|_\infty = \mathcal{O}(\varepsilon^{-1})$, as well as $f(x) = \mathcal{O}((x - x_0))$ for $x \to x_0$, as $f(x_0) = 0$ and $f \in C^1(M)$, so

$$\int_M |f\nabla\psi_\varepsilon|^p \leq \underbrace{\ell_n(\mathrm{supp}\,\psi_\varepsilon)}_{=\mathcal{O}(\varepsilon^n)} \underbrace{\Big(\sup_{|x-x_0|<2\varepsilon} |f(x)|^p\Big)}_{=\mathcal{O}(\varepsilon^p)} \underbrace{\|\nabla\psi_\varepsilon\|_\infty^p}_{=\mathcal{O}(\varepsilon^{-1})} \xrightarrow{\varepsilon\to 0} 0.$$

This completes the proof that $\|\nabla(u - \bar{u})\|_p \to 0$ for $\varepsilon \to 0$. But as $\bar{u}$ is constant on $B_\varepsilon(x_0)$ we get $\bar{u} \in N$ by Theorem 3.1.1 and we have shown the claim. □

Remark 3.1.3. *In a numerical context, a situation similar to Theorem 3.1.1 can appear when the integration is approximated by a quadrature rule and one of the quadrature points lies exactly on an extremum of u (e.g. by symmetry). The set $\{\nabla f = 0\}$ is then numerically "amplified" and a nondifferentiability can occur in a discretization of Δ_p, even though $\{\nabla f = 0\}$ may be a null set. Trying to avoid this by ensuring that no quadrature point lies too close to an extremum of the eigenfunction leads to unwieldy constraints on the discretization of the domain.*

3.1.2 Flatness of eigenfunctions

As seen in Theorem 1.3.3(b), a function that minimizes $|Du|(\mathbb{R}^d)/\|u\|_1$ is a characteristic function if the Cheeger set is unique. The eigenfunctions of Δ_p partially inherit the plateau-like quality of a characteristic function as p approaches 1: around extrema, the deviation from the extremal value of an eigenfunction scales at most with the q-th order of the distance to the extremum, where q is the conjugate exponent of p (this is made more precise in Theorem 3.1.4). While the approaching of characteristic functions is one of the motivations for Δ_p^D, it leads to problems in the numerical setting, inhibiting one from calculating $\Delta_p(u_p)$ accurately.

Theorem 3.1.4. *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2 and $u \in W_0^{1,p}(M)$ the first eigenfunction of Δ_p on M. If u attains its maximum in $x_0 \in M$, then*

$$\limsup_{x \to x_0} \frac{u(x_0) - u(x)}{|x|^q} < \infty.$$

Proof. The proof of this claim from [Gar03, Theorem 1, Remark a)] is splitted into several parts across the paper, so we bundle the arguments here for this special case. Assume without loss of generality that $x_0 = 0$. The main work is done by [Gar03, Lemma 5], which states that if one defines

$$u^\alpha := \frac{u(0) - u(\alpha x)}{\alpha^q},$$

then every sequence $\alpha_n \to 0$ has a subsequence such that $u^{\alpha_n} \to \bar{u}$ in $\bar{u} \in C^1_{loc}(\mathbb{R}^d)$ and

$$\Delta_p \bar{u} \equiv \text{const}.$$

The proof only uses local properties of u and thus does not depend on regularity, symmetry or convexity of the domain.

Now assume that there is a sequence $x_n \to 0$ with

$$\lim_{n \to \infty} \frac{u(0) - u(x_n)}{|x_n|^q} = \infty.$$

Proceeding along the lines of the proof of [Gar03, Theorem 7], we define $\alpha_n := |x_n|$ and $y_n := x_n/|x_n|$, which implies

$$\frac{u(0) - u(x_n)}{|x_n|^q} = u^{\alpha_n}(y_n).$$

By aforementioned [Gar03, Lemma 5] we can assume without loss of generality that $u^{\alpha_n} \to \bar{u}$. We may also assume $y_n \to y_0$ by passing to another subsequence. Together, this yields the contradiction

$$\infty = \lim_{n \to \infty} \frac{u(0) - u(x_n)}{|x_n|^q} = \lim_{n \to \infty} u^{\alpha_n}(y_n) = \bar{u}(y_0) < \infty$$

which proves the claim. $\square$

Remark 3.1.5. *In the same way, one can also show*

$$\liminf_{x \to x_0} \frac{u(x_0) - u(x)}{|x|^q} > 0,$$

meaning that the "order" of the extremum is exactly q, but for our purposes, we will only need the upper bound.

This flatness of eigenfunctions means that they come close to the situation in Theorem 3.1.1 of having constant parts, and numerically, we have to expect the problems that come with nondifferentiability.

Relatedly, precise calculation of $\Delta_p(u)$ is inhibited. In the analytical setting, ∇u is of extremely small order $\mathcal{O}(|x - x_0|^{q-1})$, and this extreme cancels with the other extreme $|V|^{p-2}V$, which is of order $\mathcal{O}(|V|^{p-1})$ for small V (recall that $(p-1)(q-1) = 1$). In a numerical setting, we only have finite precision available, and tiny deviations in $u_p(x)$ quickly drown in numerical noise.

To illustrate the scale of this problem, we exemplarily consider $u(x) = 1 - |x|^q$ in points around 0. For some small h, the relative difference

$$\delta_{\text{rel}}^h := \frac{u(h) - u(0)}{u(0)} = |h|^q$$

becomes smaller than $\varepsilon_{\text{mach}}$ when $h < \sqrt[q]{\varepsilon_{\text{mach}} u_0}$. For, e.g. $p = 1.1$, $u_0 = 1$, and a machine precision of $\varepsilon_{\text{mach}} = 10^{-16}$, this means that any function value at a point closer than $h \approx 0.035$ to 0 will be completely dominated by numerical noise. If we want to keep the relative difference in function values smaller than 10^{-8}, this worsens to $h \approx 0.18$.

This implies practical limitations on the combinations of grid sizes h and exponents p, which are visualized in Figure 3.1.

For the eigenfunctions of the p-Laplacian, similar restrictions are to be expected by Theorem 3.1.4. One can even be more precise by using a power series expansion of the eigenfunction on $[-1, 1]$ around zero that approaches $1 - |x|^q$ for $p \to 1$.

The power series is derived in [Lin95], for the first eigenfunction of Δ_p on the interval $[-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$, where

$$\pi_p := \frac{2\sqrt[p]{p-1}}{p \sin(\frac{\pi}{p})}\pi.$$

The series expansion is

$$\sqrt[p]{p-1}\left(1 - \frac{x^q}{q} + \frac{x^{2q}}{2q^2(q+1)} - \dots\right).$$

If we scale this vertically to 1 at $x = 0$ and horizontally to the interval $[-1, 1]$, we get

$$\left(1 - \frac{\pi_p^q x^q}{2^q q} + \frac{\pi_p^{2q} x^{2q}}{2^{2q} 2q^2(q+1)} - \dots\right).$$

The coefficient $\frac{\pi_p^q}{2^q q}$ now approaches 1 as $p \to 1$: we can use the identity $\pi_p = \pi_q$ from [Lin95] and write

$$\frac{\pi_q^q}{2^q q} = \frac{(q-1)}{q^{q+1}\sin(\frac{\pi}{q})^q}\pi^q \tag{3.10}$$

$$= \left(\frac{\pi}{q\sin(\frac{\pi}{q})}\right)^q \frac{(q-1)}{q} \tag{3.11}$$

$$= \left(\frac{1}{1+\mathcal{O}(q^{-3})}\right)^q \frac{(q-1)}{q} \to 1 \quad (q \to \infty). \tag{3.12}$$

Hence for p close to 1, the eigenfunction behaves like $1 - |x|^q$ up to order $2q$.

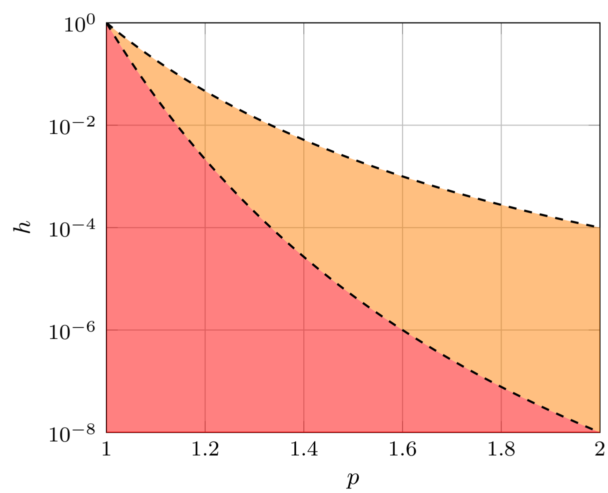


Figure 3.1: Combinations of grid sizes h and exponents p where $|h|^q < 10^{-8}$ (orange) and $|h|^q < 10^{-16}$ (red). In the red area, the function $u(x) = 1 - |x|^q$, is numerically constant on $[-h, h]$, so, numerically, $\Delta_p u = 0$ around $x = 0$.

3.2 The local min-max method of Yao & Zhou

Here we present the method used for the numerical experiments in Chapter 4. A similar exposition is done in [DFJK23]. We use a variational method by Yao and Zhou [YZ07], which treats a general nonlinear eigenvalue problem

$$F'(u) = \lambda G'(u)$$

for "iso-homogeneous" functionals F and G on a Banach space, meaning that there is some $r > 0$ such that $F'(tu) = t^r F'(u)$ and $G'(tu) = t^r G'(u)$ for all $t > 0$ and all u. The method uses a min-max algorithm on

$$J \coloneqq \frac{F}{G},$$

which is able to find not just the first eigenfunction but also higher eigenfunctions. We will mainly be interested in the first eigenfunction so we first restrict the presentation to this case, where the algorithm simplifies substantially. For a sketch of the method for higher eigenfunctions see Section 3.2.3.

We will also not work with general F and G but with the specific ones in Theorem 1.4.7. This is also the example that Yao and Zhou themselves use to showcase their method in [YZ07]. We incorporate the specializations and modifications that they develop in [YZ07, Section 4.1] for this case. Later, in Section 3.2.1, we will replace F with F^D in order to treat Δ_p^D.

For now, consider the classical case $F(u) = \|\nabla u\|_p^p$ and $G(u) = \|u\|_p^p$. By Theorem 1.4.8, the first eigenfunction is a minimizer of J. The method of Yao and Zhou boils down to a descent algorithm in this special case.

Descent direction The essential ingredient is the choice of a descent direction, i.e., for some $u \in W_0^{1,p}(M)$ we want a $w \in W_0^{1,p}(M)$ such that

$$\frac{d}{dt} J(u + tw)\bigg|_{t=0} < 0. \tag{3.13}$$

In [YZ07, Section 4.1], Yao and Zhou propose using the direction $w := -\operatorname{grad} J$, where $\operatorname{grad} J$ is defined in the following way:

Definition 3.2.1. *With F, G and J as above and $u \in W_0^{1,p}(M)$, define* $\operatorname{grad} J(u)$ *to be the (unique) weak solution $d \in W_0^{1,q}(M) \subset W_0^{1,p}(M)$ for*

$$-\Delta d = J'(u),$$

i.e., the d fulfilling

$$\int_M \nabla d \nabla v = \langle J'(u), v\rangle \tag{3.14}$$

for all $v \in W_0^{1,p}(M)$. The right-hand side, in expanded form, is

$$\frac{p}{G(u)^2} \int_M (G(u)|\nabla u|^{p-2} \nabla u \nabla v - F(u)|u|^{p-2} uv)$$

(see [YZ07, Section 4.1]). For the existence of a unique solution on C^1 domains see Theorem F.10.

Remark 3.2.2. *Because M is bounded, the inclusion $W_0^{1,q}(M) \subset W_0^{1,p}(M)$ holds, so a solution can be used as an ascent direction in $W_0^{1,p}(M)$. The notation* $\operatorname{grad} J$ *is inspired by [YZ07], where it is denoted by ∇J.*

The direction $w := -\operatorname{grad} J(u)$ is indeed a descent direction in the sense of (3.13): if d solves (3.14) then

$$\frac{d}{dt} J(u - td)\bigg|_{t=0} = -\langle J'(u), d\rangle \overset{(3.14)}{=} -\int_M \nabla d \nabla d = -\|\nabla d\|_2^2 < 0.$$

A detail we have omitted here is that Yao and Zhou require a stronger property of w than just being a descent direction, namely that $-w$ is a *pseudogradient* of J. This requires[1] $\|\nabla w\|_p \leq 1$ and the existence of some $0 < \vartheta \leq 1$ with

$$\langle J'(u), -w(u)\rangle \geq \vartheta \|J'(u)\|_{W^{-1,q}(M)} \tag{3.15}$$

for all $u \in W_0^{1,p}(M)$. Yao and Zhou claim numerical evidence that $\frac{d}{\|\nabla d\|_p}$ is a pseudogradient of J but do not prove this formally.

Initial guess The algorithm needs a first guess for u_p. Yao and Zhou propose taking a function with the "simplest nodal line structure". We use eigenfunctions of Δ_2 and Δ_2^D, respectively.

Step size control In [YZ07, Lemma 2.5], Yao and Zhou prove that if $\mathcal{G}$ is a pseudogradient with respect to ϑ of J as defined in (3.15), then one can expect a certain decrease in $J(u - t\mathcal{G}(u))$, namely

$$J(u + tw) - J(u) < \frac{1}{4} s\vartheta \|J'(u)\|$$

(we have simplified the expression to the special case of the first eigenfunction). This can be used in an Armijo-like step size control. For the special case of the p-Laplacian and the pseudogradient proposed by Yao and Zhou, the norm $\|J'(u)\|$ is replaced by $\|\nabla(\operatorname{grad} J(u))\|_2^2$ following [YZ07, Section 4.1].

Interpretation of $\operatorname{grad} J$ as a gradient The notation alludes to thinking of d as a gradient. To have an informal justification on this perspective, recall that the gradient $\nabla^A f$ of a scalar function f with respect to some scalar product "$\cdot_A$" is connected to the derivative Df by the equation

$$(\nabla^A f(x)) \cdot_A v = (Df(x))v \quad \text{ for all } v \in \mathbb{R}^d.$$

The equation (3.14) has the same structure if we think of the left-hand side as the H_0^1 scalar product $(d, v) \mapsto \int_M \nabla u \nabla v$. Strictly speaking, this is inaccurate, as the left-hand side of (3.14) is not a scalar product but a bilinear pairing $W_0^{1,q}(M) \times W_0^{1,p}(M) \to \mathbb{R}$. However, as M is bounded, we have the inclusions

$$W_0^{1,q}(M) \subset H_0^1(M) \subset W_0^{1,p}(M),$$

and if we allow d and restrict v to be in $H_0^1(M)$, the two objects coincide. The resulting equation is different from (3.14) (we have made it weaker), but we still take the above observations as enough to justify thinking of d as a gradient.

[1] note that Yao and Zhou are working with the norm $\|u\|_{W_0^{1,p}(M)} \coloneqq \|\nabla u\|_p$.

Interpretation of $\operatorname{grad} J(u)$ **as a regularized** $J'(u)$ We want to elaborate informally on a remark in [YZ07] saying that replacing $J'(u)$ with $\operatorname{grad} J(u)$ "increases its smoothness".

The main reason we do this is to try to understand why a certain modification of equation (3.14) defining "grad" leads to better results in the dynamic case in Section 3.2.1 below. This discussion, however, remains informal.

The operator Δ decreases regularity, mapping e.g. k-times differentiable functions to $(k-2)$-times differentiable functions, so it is intuitive that Δ^{-1} should increase regularity. This can be made a little bit more precise by the connection of the regularity of a function with the growth behavior of its Fourier transform. Roughly, the slower it grows for high frequencies, the more regular the function (see e.g. [Gra+08, Table 3.1] for periodic functions). "In frequency space", $-\Delta$ is diagonal, mapping $\exp(2\pi kx)$ to $|k|^2 \exp(2\pi kx)$ for some $k \in \mathbb{R}^d$. This enhances growth of the Fourier transform for high frequencies and thus decreases regularity. Conversely, Δ^{-1} should also be diagonal, mapping $\exp(2\pi kx)$ to $|k|^{-2} \exp(2\pi kx)$ and thus attenuating growth at high frequencies. We will not make this more precise here.

Another way of looking at the regularizing behavior is to focus on the variational problem associated with the Poisson equation $-\Delta u = f$, which is

$$\text{minimize } \left(\frac{1}{2}\|\nabla u\|_2^2 - \int_M fu\right),$$

and to point out that the term $\|\nabla u\|_2^2$ penalizes irregularity of u. Formally, it is less clear here that $\Delta^{-1} f$ should be *more regular than* f, but this formulation better highlights the changes that we will make in Section 3.2.1.

3.2.1 The dynamic case

In order to apply Algorithm 1 to Δ_p^D, we replace F with

$$F^D \coloneqq \frac{1}{2}(F + F \circ T_*) = \frac{1}{2}\left(\|\nabla(\cdot)\|_p^p + \|\nabla T_*(\cdot)\|_p^p\right)$$

from Theorem 2.1.3. It still has the iso-homogeneous conditions and differentiability, so the method is applicable with $J^D \coloneqq F^D/G$ in place of J.

When using Algorithm 1 in a straightforward way, however, a problem appears: the step sizes that the algorithm chooses get unusably tiny. In this section we present a modification of the choice of descent direction that, in our experiments, solved this problem. Instead of using $\operatorname{grad} J(u) = \Delta^{-1}(J^D)'(u)$, we use the dynamic Laplacian Δ^D, yielding the following definition.

Definition 3.2.3. *With* F^D, G *and* J^D *as above and* $u \in W_0^{1,p}(M)$*, define* $\operatorname{grad}^D J^D(u)$ *to be the weak solution* $d^D \in W_0^{1,q}(M) \subset W_0^{1,p}(M)$ *of*

$$-\Delta^D d^D = (J^D)'(u), \tag{3.16}$$

i.e., the d^D *fulfilling*

$$\frac{1}{2}\left(\int_M \nabla d^D \nabla v + \nabla(T_* d^D)\nabla(T_* v)\right) = \langle (J^D)'(u), v\rangle \quad \text{for all } v \in W_0^{1,p}(M).$$

After substitution, the left-hand side becomes

$$\int_M \frac{1}{2}(I + DT^{-1}DT^{-T})\nabla d^D \nabla v,$$

where $DT^{-1}DT^{-T}$ is the inverse of the Cauchy-Green tensor. The right-hand side, in expanded form, is

$$\frac{p}{2G(u)^2}\int_M (G(u)\left(|\nabla u|^{p-2}\nabla u \nabla v + |\nabla(T_* u)|^{p-2}\nabla(T_* u)\nabla(T_* v)\right) - F(u)|u|^{p-2}uv).$$

For the existence of a unique solution on C^1 domains see Theorem F.10.

While we haven't found a formal argument for the improvement, we give a heuristic reasoning for it by drawing parallels to convergence problems of gradient descent and their mitigation via preconditioning.

The algorithm that we are using is a descent method that is structurally the same as gradient descent. It is well known that a badly conditioned Hessian of the objective can cause the direction of "steepest descent", i.e., the direction of unit length with the lowest directional derivative, to be far away from the direction that would move the argument fastest to the optimum.

A possible remedy is measuring "unit length" with respect to some other scalar product, i.e., taking

$$\nabla^g f(x) = \underset{|v|_g = 1}{\operatorname{argmin}}\ Df(x)v,$$

where, ideally, the norm $|\cdot|_g$ induced by g penalizes eigenvectors of the Hessian corresponding to big eigenvalues more strongly than eigenvectors with small eigenvalue, counteracting the anisotropy.

Recall that in finite dimensions we may calculate $v^T H(x) v$ by

$$\left.\frac{d^2}{dt^2} f(x + tv)\right|_{t=0}.$$

We get the eigenvalues of the Hessian if we plug in normalized eigenvectors for v but even if we don't know the eigenvectors, we can expect a badly conditioned Hessian if $v^T H(x) v$ exhibits big differences in magnitude for different v.

This we apply to J^D: ignoring differentiability issues, we try to get a feeling on how the expression

$$\left.\frac{d^2}{dt^2}\frac{\|\nabla(u+tv)\|_p^p + \|\nabla(T_*(u+tv))\|_p^p}{2\|u+tv\|_p^p}\right|_{t=0} \tag{3.17}$$

behaves. Think of v as a function that has small support (e.g. the basis functions from some finite element discretization). If v has support in a part of the domain where dynamics exhibit a lot of stretching, then the term $\|\nabla(T_*(u+tv))\|_p$ will be very sensitive to t and by the identity $(|f|^p)'' = p(p-1)|f|^{p-2}(f')^2 + pf''|f|^{p-2}f$, the term $(f')^2$ is likely to introduce high curvature in the expression $\|\nabla(T_*(u+tv))\|_p^p$. Conversely, if u has support only in areas of little stretching of T, this does not happen and we may expect $\|\nabla(T_* u)\|_p^p$ to have moderate curvature in comparison.

We suspect that this is the anisotropy in the "Hessian" of J^D that leads to problems. In the same way as in gradient descent, it is advantageous to move a lot into the directions of high yield and little into the directions of low yield. This is supported by the numerical examples, which show spikes in $\operatorname{grad} J^D$ in areas of high stretching of T, indicating that a hat function in this area gave disproportionately high yield in J^D (see Figure 3.2, left).

We argue that the choice of $\operatorname{grad}^D J(u) := (\Delta^D)^{-1} J'(u)$ is a counteraction of this anisotropy: the expression in (3.17) gets big if there is high stretching around the support of v. The norm corresponding to the scalar product $g(u,v) = \int_M \nabla u \nabla v + \nabla(T_* u)\nabla(T_* v)$ penalizes exactly these directions stronger, as $\|v\|_g^2$ also contains the summand $\|\nabla(T_* v)\|_2^2$.

All in all, we still lack a formal justification for this choice of d^D. In practice however, d^D is much smoother than d (see Figure 3.2, right), and while the algorithm chooses unusably small stepsizes for the descent direction $-\operatorname{grad} J^D$, this doesn't happen with $-\operatorname{grad}^D J^D$.

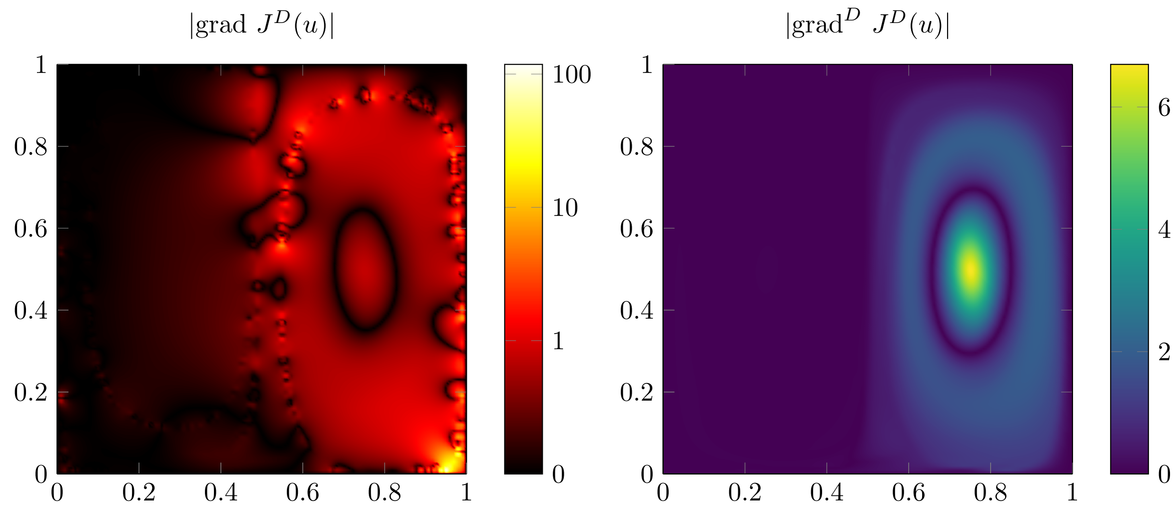


Figure 3.2: The descent directions $\operatorname{grad} J^D(u)$ and $\operatorname{grad}^D J^D(u)$, where J^D arises from the transitory double gyre described in Section 4.1.2. The value of p is 1.8 and u is the first eigenfunction of $\Delta^D = \Delta_2^D$. Note the spike in the right lower corner of the left plot. The color scaling in the left plot arises from the transformation $x \mapsto \log(|x| + 0.1)$. This is done to make different orders of magnitude visible.

3.2.2 The full algorithm for the first eigenfunction

Algorithm 1 shows pseudocode for the algorithm that approximates the first eigenfunction. In practice, one works on finite element approximations of $W_0^{1,p}(M)$ and numerical solutions to (3.16) for $\operatorname{grad}^D J^D$ and the first eigenfunction of Δ^D. In Chapter 4 we use the Cauchy-Green approach described in [FJ18, section 3.1]. Function norms are calculated by quadrature of the corresponding integrals. We also add maximal iteration counts to the loops in Line 3 and Line 6, even though they should both terminate eventually by results of [YZ07, Theorem 5.1] and [YZ07, Lemma 2.5], respectively.

Algorithm 1 Computation of the first eigenfunction (after [YZ07])

function FIRST-EIGENFUNCTION($\varepsilon_{\text{tol}} > 0$)
  $u \leftarrow$ INITIAL-GUESS() # initial guess
  while $\|\text{grad}^D J^D(u)\|_p > \varepsilon_{\text{tol}}$ **do** # stopping criterion
    $w \leftarrow -\text{grad}^D J^D(u)$ # descent direction
    $s^* \leftarrow 2/\max(1, \|w\|_p)$ # initial step size
    repeat # step size control
      $s^* \leftarrow s^*/2$
      $u^* \leftarrow (u + s^* w)/\|u + s^* w\|_p$ # normalized candidate
    until $J(u^*) \leq J(u) - \frac{1}{4} s^* \|\nabla w\|_2^2$ # check decrease in J
    $u \leftarrow u^*$ # accept candidate
  end while
  return u
end function

function INITIAL-GUESS( )
  return the first eigenfunction of Δ^D
end function

3.2.3 Higher eigenfunctions

Similar to the Courant-Fischer principle in the linear case [Zei13], higher eigenfunctions of Δ_p can be described by a minimum-maximum principle. There is a characterization based on the Krasnoselskij genus of symmetric subsets $\mathcal{A} \subset W_0^{1,p}(M), -\mathcal{A} = \mathcal{A}$, which is defined as the smallest natural number k such that there exists a continuous, odd mapping $\gamma : \mathcal{A} \to \mathbb{R}^k, \gamma(-u) = -\gamma(u)$. The variational characterization is stated, e.g., in [Lin08] and [YZ07] and reads

$$\lambda_p^{(k)} := \inf_{\mathcal{A} \in \Sigma_k} \sup_{u \in \mathcal{A}} J(u),$$

where Σ_k denotes the family of compact subsets of $W_0^{1,p}(M)$ with Krasnoselskij genus k (note that, according to [Lin08], it is not known whether these exhaust the spectrum of Δ_p). In [YZ07], they state a different, equivalent characterization that iteratively characterizes a $(k+1)$-th eigenfunction u_{k+1} from k lower eigenfunctions $u_1, \dots, u_k$. Choosing a complement space L' of $L := \text{span}\,(u_1, \dots, u_k)$, i.e. one that fulfills $L \oplus L' = W_0^{1,p}(M)$, they define for $v_i \in W_0^{1,p}(M)$ the subset

$$[v_1, \dots, v_{k+1}]_S := \left\{ \sum_{i=1}^{k} t_i v_i \mid \sum_{i=1}^{k} t_i^2 = 1 \right\} \subset W_0^{1,p}(M),$$

and characterize $\lambda_p^{(k)}$ as

$$\min_{v \in S_{L'}} \max_{u \in [u_1, \dots, u_k, v]_S} J(u), \tag{3.18}$$

where $S_{L'} := \{u \in L' \mid \|u\|_p = 1\}$. The numerical advantage of this formulation is that the inner optimization can be solved as a low dimensional constrained optimization on the t_i. In [YZ07], Yao and Zhou define a *peak selection of J with respect to L* to be a mapping $u^+ : S_{L'} \to W_0^{1,p}(M)$ such that

$$u^+(v) \in \operatorname*{arg\,max}_{u \in [u_1, \ldots, u_k, v]} J(u).$$

Using this definition, the characterization in (3.18) becomes

$$\min_{v \in S_{L'}} J(u^+(v)).$$

This minimization problem can then be solved with a descent on v: Let $w = -\operatorname{sign}(t_{k+1})d$ with $d \coloneqq \operatorname{grad} J(u^+(v))$ and t_{k+1} being defined as the coefficient in the representation $u^+(v) = \sum_{i=1}^{k+1} t_i u_i$. From [YZ07, Lemma 2.5], it arises that w is a descent direction for $v \mapsto J(u^+(v))$. Note that this is subtly different from a direct application of what we already showed about d, namely

$$\frac{d}{dt} J(u^+(v) - td) < 0,$$

as opposed to the claim we are using here:

$$\frac{d}{dt} J(u^+(v + tw)) < 0.$$

We have already incorporated the simplifications from the remarks in [YZ07, Section 4.1], namely leaving out a projection of w onto L' and using a specific choice of pseudogradient for w. The statement in [YZ07, Lemma 2.5] is stronger than the directional derivative just being negative and allows for an Armijo-like stepsize control of a descent algorithm on $v \mapsto J(u^+(v))$.

Algorithm 2 shows pseudocode for calculating a next eigenfunction $u^{(n)}$ from already computed eigenfunctions $u^{(1)}, \ldots, u^{(n-1)}$. This is also the algorithm that is implemented in the accompanying package `DynamicPLaplacian.jl`.

For $n = 1$, there is no optimization to be done in the function PEAK-SELECTION, as $[v]_S = \{\pm v\}$. Thus, PEAK-SELECTION just returns its argument, and the algorithm degrades to Algorithm 1.

We use the Algorithm 2 for $n = 2$ in the exploratory example in Section 4.3, where we also elaborate more on the specific numerical methods used.

For $n > 2$, the convergence is very slow in the presence of dynamics and we have not done successful experiments except for $n = 1$ and $n = 2$.

Algorithm 2 Computation of a higher eigenfunction (after [YZ07])

function NTH-EIGENFUNCTION($u^{(1)}, \dots u^{(n-1)} \in W_0^{1,p}(M)$, $\varepsilon_{\text{tol}} > 0$)
  $v \leftarrow$ INITIAL-GUESS(n) # initial guess
  $t \leftarrow [0, \dots, 0, 1]$
  while $\|\text{grad}^D J^D(u)\|_p > \varepsilon_{\text{tol}}$ **do** # stopping criterion
    $u, t \leftarrow$ PEAK-SELECTION($[u^{(1)}, \dots, u^{(n-1)}, v], t$)
    $w \leftarrow -\text{sign}(t_n)\text{grad}^D J^D(u)$ # descent direction
    $s^* \leftarrow 2/\max(1, \|w\|_p)$ # initial step size
    repeat # step size control
      $s^* \leftarrow s^*/2$
      $v^* \leftarrow (v + s^* w)/\|v + s^* w\|_p$ # normalized candidate
      $u^*, t^* \leftarrow$ PEAK-SELECTION($[u^{(1)}, \dots, u^{(n-1)}, v^*], t$)
    until $J(u^*) \leq J(u) - \frac{1}{4} s^* |t_n^*| \|\nabla w\|_2^2$ # check decrease in J
    $v \leftarrow v^*$ # accept candidate
  end while
  return u
end function

function INITIAL-GUESS(k)
  return the k-th eigenfunction of Δ^D
end function

function PEAK-SELECTION($[v_1 \dots, v_n]$, $[t_1, \dots, t_n]$)
  Find a local maximum $\tilde{u} = \sum_i \tilde{t}_i v_i$ of J^D closest to $u = \sum_i t_i v_i$.
  return $\tilde{u}, \tilde{t}$
end function

3.2.4 A numerical example in one dimension

In order to illustrate some of the aspects discussed so far, we apply the method of Yao and Zhou for the first eigenfunction to the one-dimensional case $d = 1$ on the interval $I = [-1, 1]$. The eigenvalues of Δ_p are known analytically in this case[2]:

$$(p-1)\left(\frac{\pi}{p\sin(\frac{\pi}{p})}\right)^p.$$

The (strong) eigenvalue equation becomes an ordinary differential equation

$$\left(|u'|^{p-2}u'\right)' = -\lambda|u|^{p-2}u.$$

In Section 3.3.1, we will show how to write this as a Hamiltonian system. We approximate a solution to high accuracy with the Dormand-Prince solver from the Julia package `DifferentialEquations.jl`[RN17] to get reference solutions (see Figure 3.3).

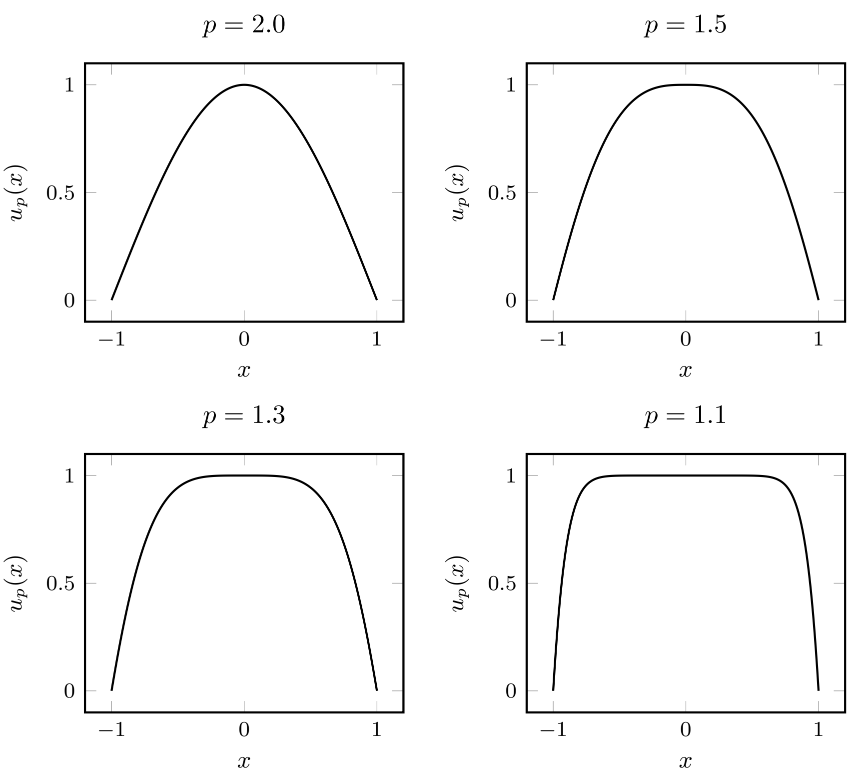


Figure 3.3: The first eigenfunction of Δ_p on the unit interval $[0, 1]$ for the exponent $p \in \{2.0, 1.5, 1.3, 1.1\}$, approximated to high accuracy with a numerical ODE solver.

The first thing to note is the flatness of the eigenfunction around its extremum, which is consistent with the theoretical prediction in Section 3.1.2, where the behavior of u_p was determined to be $1 - |x|^q$ for p close to 1.

[2]derived from [Lin95, section 2] by scaling the domain

We try to approximate a density of $J'(u_p)$ with finite differences by the straightforward finite difference scheme

$$\Delta_p^h(u)(x) = \frac{1}{h}\left(\varphi_{p-1}\left(\frac{u(x+h)-u(x)}{h}\right) - \varphi_{p-1}\left(\frac{u(x)-u(x-h)}{h}\right)\right),$$

where $\varphi_{p-1}(x) = |x|^{p-2}x$. We evaluate this scheme for $h = \frac{1}{100}$ on a numerical reference solution u_p and subtract $\lambda_p|u_p|^{p-2}u_p$. In theory, this should result in 0 everywhere, but as can be seen in Figure 3.4, we deviate from this: first, we see a horizontal line that appears for all p around the extremum. This is actually due to the finite difference scheme not being fully consistent (see Theorem F.4).

The second deviation is an area of numerical noise around the extremum at $x = 0$ that gets bigger for p approaching 1. This is because of the increasing flatness of the eigenfunctions as described at the end of Section 3.1.2. We remind the reader of the areas of the combinations of p and h that were heuristically selected to be problematic in Figure 3.1 because $|h|^q$ is smaller than 10^{-8} and 10^{-16}, respectively. In Figure 3.4, we indicate where (p, h) enters these areas with the chosen grid size $h = \frac{1}{100}$.

Next, we calculate the pseudogradient with a linear FEM approximation of the weak equation $\Delta d = J'(u_p)$. Again, we should get a vanishing function, but numerical noise appears for small p (see Figures 3.5 and 3.6).

Finally, we look at the convergence speed of Algorithm 1. With a numerical reference solution of the ODE calculated at high precision, we check the L^p-distance of the iterates of Algorithm 1. One can observe (see Figure 3.7) that the convergence speed decreases substantially for p approaching 1. Also note that iterates with exponentially increasing iteration numbers appear equally spaced vertically in the logarithmic plot, which indicates that the convergence speed is sublinear.

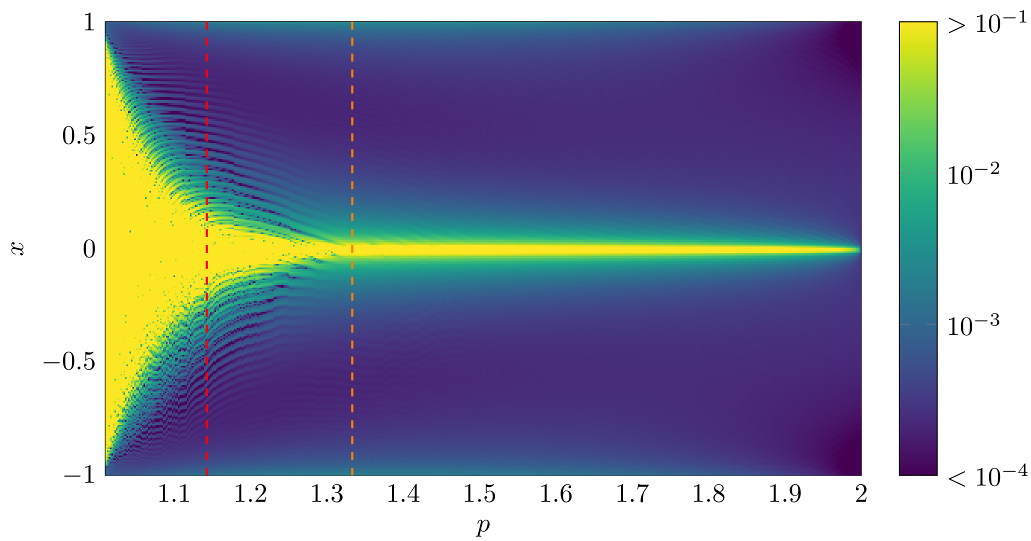


Figure 3.4: Absolute value of finite difference approximations of $(|u'|^{p-2}u')' - \lambda|u|^{p-2}u$ for a numericial reference solution u and different p. The finite difference grid size is $h = \frac{1}{100}$. The red and orange lines indicate where $h^q \in \{10^{-16}, 10^{-8}\}$, respectively (compare to Figure 3.1).

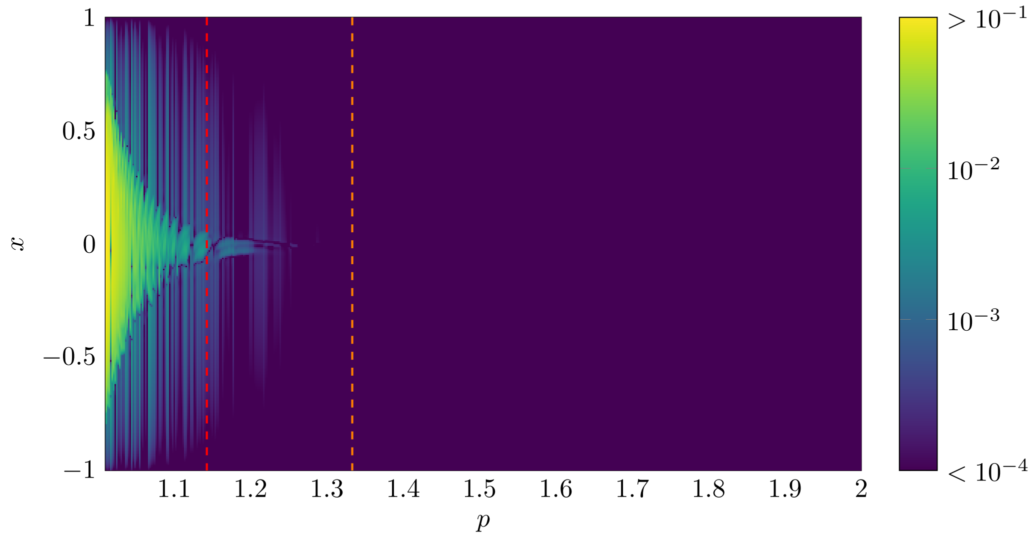


Figure 3.5: Absolute value of the pseudogradient grad $J(u_p)$ (calculated with linear FEM on uniform grid of step lenght $\frac{1}{100}$) for a numerical reference solution u_p. The red and orange lines indicate where $h^q \in \{10^{-16}, 10^{-8}\}$, respectively (compare to Figure 3.1).

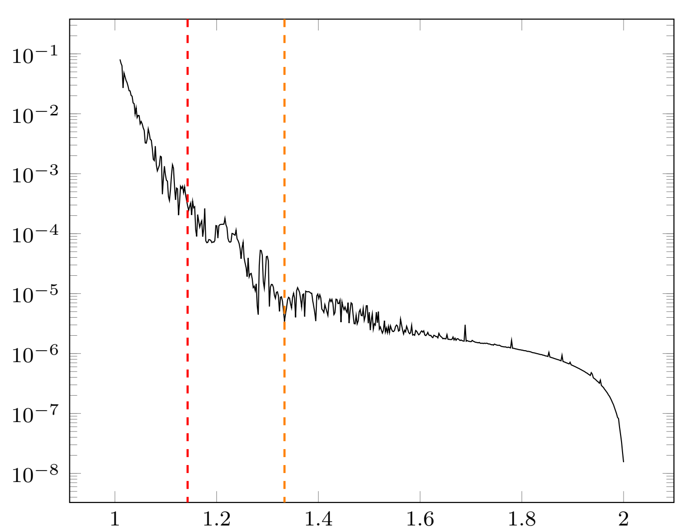


Figure 3.6: The norm of grad $J(u_{p,h})$ on a numerical reference solution $u_{p,h}$, calculated with linear finite elements on a grid of size $h = \frac{1}{100}$ The red and orange lines indicate where $h^q \in \{10^{-16}, 10^{-8}\}$, respectively (compare to Figure 3.1).

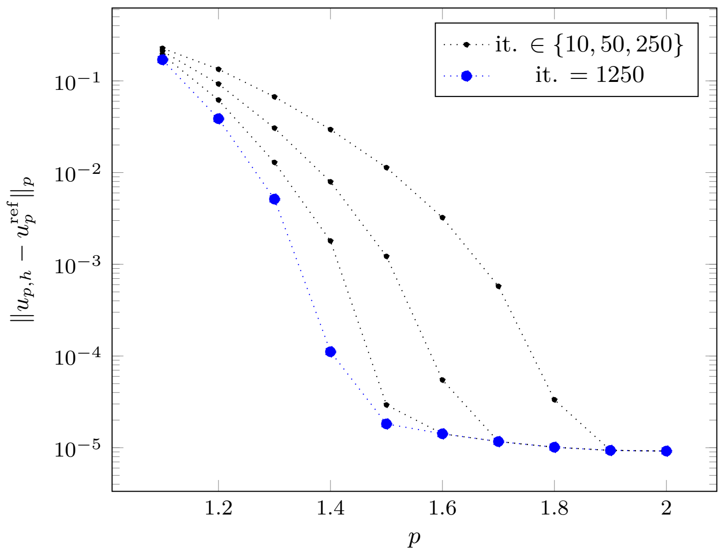


Figure 3.7: The L^p-residual of $u_{h,p}$ to the reference solution after different numbers of iterations of the descent algorithm of Yao & Zhou. The limiting accuracy of $\mathcal{O}(10^{-5})$ for p close to 2 is dominated by the interpolation error with piecewise linear functions of grid size $h = \frac{1}{100}$.

3.3 Other approaches

Even though we exclusively use the method of Yao & Zhou [YZ07] from Section 3.2 for the experiments in Chapter 4, a few other approaches were investigated. We present them here shortly, together with some related insights.

3.3.1 The one-dimensional eigenvalue equation as a Hamiltonian system

We start with an observation about the one-dimensional case, namely that the eigenvalue problem can be reduced to a Hamiltonian system. This can be used for numerical approximation but we will also use it to compare the eigenvalue equation to a regularized equation in Section 3.3.2. In order to be able to do this, we formulate our findings generally for differential equations of the form

$$-(g_l(u'))' = \lambda g_r(u) \tag{3.19}$$

for two functions g_l, g_r. We will specify properties that g_l and g_r have to fulfill in Theorem 3.3.3. We first recall the definition of a Hamiltonian system:

Definition 3.3.1. *A first-order ordinary differential equation* $(x', y') = F(x, y, x', y', t)$ *on* $\mathbb{R}^{2d}$ *is called* Hamiltonian *if there is a differentiable function* $H : \mathbb{R}^{2d} \to \mathbb{R}$ *such that* *it is equivalent to the system*

$$x_i' = \frac{\partial H}{\partial y_i} \tag{3.20}$$

$$y_i' = -\frac{\partial H}{\partial x_i}. \tag{3.21}$$

One then calls H the Hamiltonian *of the system. The equation can be written equivalently as*

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla H(x, y). \tag{3.22}$$

Remark 3.3.2. *For Hamiltonian systems the notation (q, p) instead of (x, y) is more common. We chose (x, y) to avoid ambiguity of notation with the exponents p and q.*

Hamiltonian systems lie at the heart of classical mechanics and are well studied (see e.g. [Der17] for an introduction). We will show the following:

Theorem 3.3.3. *Assume that $g_r : \mathbb{R} \to \mathbb{R}$ is continuous, $g_l : \mathbb{R} \to \mathbb{R}$ is injective, and the inverse $g_l^{-1} : g_l(\mathbb{R}) \to \mathbb{R}$ is continuously differentiable. Then for fixed λ, the ordinary differential equation*

$$-(g_l(u'))' = \lambda g_r(u) \tag{3.23}$$

is equivalent to the Hamiltonian system with Hamiltonian

$$H_\lambda(x, y) = \lambda \int_{x_0}^{x} g_r(t)dt + \int_{y_0}^{y} g_l^{-1}(t)dt \tag{3.24}$$

for arbitrary $x_0 \in \mathbb{R}, y_0 \in g_l(\mathbb{R})$, namely

$$x' = \frac{\partial H}{\partial y} = g_l^{-1}(y) \tag{3.25}$$

$$y' = -\frac{\partial H}{\partial x} = -\lambda g_r(x). \tag{3.26}$$

The equivalence is realized via the substitution

$$x(t) = u(t) \tag{3.27}$$

$$y(t) = g_l(u'(t)) \tag{3.28}$$

i.e., every solution (x, y) of (3.24) yields a solution of (3.19) by setting $u(t) := x(t)$ and every solution u of (3.19) yields a solution of (3.24) by setting $x(t) = u(t)$ and $y(t) = g_l(u'(t))$.

Remark 3.3.4. *This does not say yet how to solve the nonlinear eigenvalue problem with boundary conditions. We will come to that in Corollary 3.3.9.*

Proof. A solution (x, y) of the Hamiltonian system defined by (3.24) satisfies

$$x' = \frac{\partial H}{\partial y} = g_l^{-1}(y) \tag{3.29}$$

$$y' = -\frac{\partial H}{\partial x} = -\lambda g_r(x) \tag{3.30}$$

by definition. Plugging this into (3.19) yields

$$\big(g_l(x')\big)' = g_l'(x')x'' = g_l'(g_l^{-1}(y))\big(g_l^{-1}(y)\big)' = \tag{3.31}$$

$$= \underbrace{g_l'(g_l^{-1}(y))\left(\left(g_l^{-1}\right)'(y)\right)}_{=(g_l(g_l^{-1}(y))'=1} y' = y' = \tag{3.32}$$

$$\overset{(3.30)}{=} -\lambda g_r(x), \tag{3.33}$$

showing that $u = x$ fulfills (3.19). On the other hand, if x is a solution to (3.19) and we set $x \coloneqq u$ and $y \coloneqq g_l(x')$, then by definition $x' = g_l^{-1}(y)$, and

$$y' = (g_l(x'))' = (g_l(u'))' \overset{(3.19)}{=} -\lambda g_r(u) = -\lambda g_r(x),$$

showing that $(x, y) = (u, g_l(u'))$ is a solution to the system with Hamiltonian (3.24). □

Example 3.3.5. *For $r > 0$ define*

$$\varphi_r(x) := |x|^{r-1}x.$$

Then, for fixed $\lambda \in \mathbb{R}$ and $p \in (1, 2)$ the differential equation

$$-\left(\varphi_{p-1}(u')\right)' = \lambda\varphi_{p-1}(u) \tag{3.34}$$

fulfills the assumptions of Theorem 3.3.3 with $g_r = g_l = \varphi_{p-1}$, leading to the Hamiltonian

$$H_\lambda(x, y) = \lambda \int_0^x \varphi_{p-1}(t)dt + \int_0^y \varphi_{q-1}(t)dt = \frac{\lambda}{p}|x|^p + \frac{1}{q}|y|^q, \tag{3.35}$$

with the associated system of differential equations being

$$\begin{aligned} x' &= \quad \varphi_{q-1}(y) \\ y' &= -\lambda\varphi_{p-1}(x). \end{aligned} \tag{3.36}$$

A contour plot of the Hamiltonian for $p = 1.3$ and $\lambda = 1$ is depicted in Figure 3.8. Note that despite the level sets appearing to have kinks at the crossing with the y-Axis, they are differentiable everywhere away from the origin (the Hamiltonian is differentiable, and ∇H_λ only vanishes in the origin). A graph of $|x|^p$ for $1 < p < 2$ exhibits the same apparent kinks despite being differentiable.

Example 3.3.6. *Another example that arises from*

$$g_l(x) = g_r(x) = \frac{x}{\sqrt{x^2 + \varepsilon^2}}$$

for some $\varepsilon > 0$ is shown in Section 3.3.2. It leads to the Hamiltonian

$$H_{\lambda,\varepsilon}(x, y) = \lambda\sqrt{x^2 + \varepsilon^2} - \varepsilon\sqrt{1 - y^2}$$

and shows different behavior of solutions.

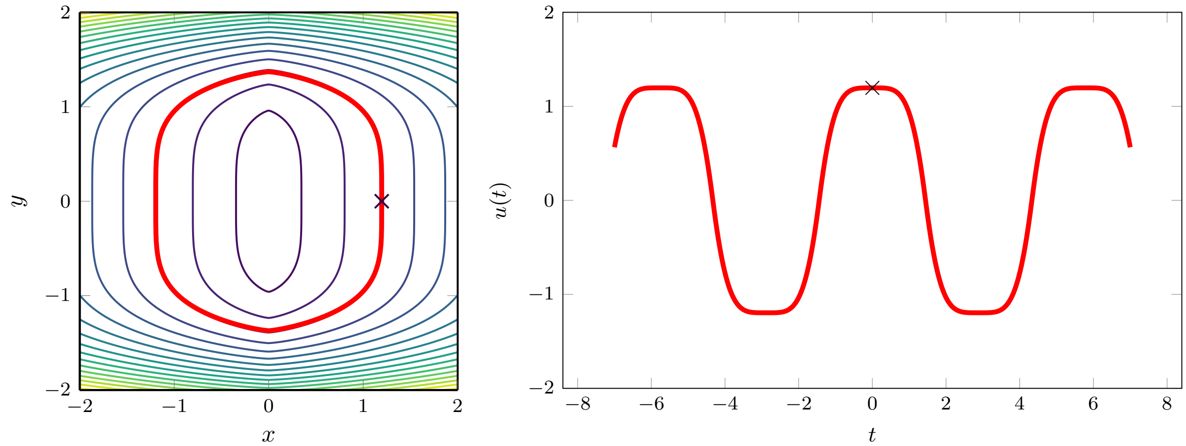


Figure 3.8: Left: a plot of the Hamiltonian $H_\lambda(x, y) = \frac{\lambda}{p}|x|^p + \frac{1}{q}|y|^q$ for $p = 1.3$ and $\lambda = 1$. The red line shows a trajectory of the flow. The plot on the right shows the corresponding solution of (3.34).

We will use the remainder of this section to prove some elementary properties of the system (3.36) and to draw conclusions about the p-Laplacian eigenvalue problem in one dimension.

Lemma 3.3.7. *Let $z(t) = (x(t), y(t))$ be a solution of (3.36). Then:*

(a) The value $H_\lambda(z)$ is constant.

(b) The norm $|z|$ is bounded, and hence the solution can be continued to all $t \in \mathbb{R}$.

(c) If $H_\lambda(z) \neq 0$ then $\inf_{t\in\mathbb{R}} |z'| > 0$.

(d) If $\bar{z}$ is another solution with $\bar{z}(t_0) = z(t_0)$ for some $t_0 \in \mathbb{R}$ then $\bar{z} = z$.

Proof. (a) This is a well-known fact about Hamiltonian systems. It follows from

$$(H_\lambda(x, y))' = x'\partial_1 H_\lambda(x, y) + y'\partial_2 H_\lambda(x, y) = x'y' - y'x' = 0$$

(b) This follows from (a) and from $H_\lambda(x, y) \geq \max(\frac{\lambda}{p}|x|^p, \frac{1}{q}|y|^q)$. As there are no blowups, there exists a solution for all $t \in \mathbb{R}$.

(c) Assume there are t_n such that $|z'(t_n)| \to 0$. The set $M := H_\lambda^{-1}(\{H(z)\})$ is closed and bounded (see (b)), and hence we may assume $z(t_n) \to z^* \in M_C$. By continuity of ∇H_λ, this implies $\nabla H_\lambda(z^*) = 0$, as $|\nabla H_\lambda(z)| = |z'|$. Inspecting ∇H_λ, one sees that this implies $z^* = 0$. But then $H_\lambda(z^*) = 0$ which contradicts the assumption as $z^* \in M$.

(d) This needs a bit of care, as the right-hand side of the differential equation is not locally Lipschitz-continuous for $x = 0$. As the solutions are restricted to level sets of H_λ, we may reduce the system to a one-dimensional differential equation: First, we may assume that $z \neq 0$, as H_λ only vanishes in the origin, so uniqueness of the solution is ensured by (a). Now at a point $z_0 \neq 0$ there exists an immersion $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ of a neighborhood of $H_\lambda^{-1}(H_\lambda(w))$ by the implicit function theorem. If we are now given two solutions $z, \bar{z} : I \to \mathbb{R}^2$ on some interval $I \subseteq \mathbb{R}$ that agree up to some $L \in I$, then we can set $z_0 := z(L) = \bar{z}(L)$ and define two functions $s, \bar{s} : I \to (-\varepsilon, \varepsilon)$ by $z(t) = \gamma(s(t))$ and $\bar{z}(t) = \gamma(\bar{s}(t))$, as the solutions have to stay on the level set. Differentiating the latter equations we see that s and $\bar{s}$ fulfill the equation

$$s'(t) = \frac{|z'(t)|^2}{\langle \gamma'(t), z'(t)\rangle} = \frac{|X(\gamma(s(t)))|^2}{\langle \gamma'(s(t)), X(\gamma(s(t)))\rangle} = \frac{|X(\gamma(s(t)))|}{\pm|\gamma'(s(t))|},$$

where $X(z) := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \nabla H_\lambda(z)$ is the vector field that corresponds to Equation (3.36), and the sign on the right-hand side is fixed for small t. We may assume it to be positive by flipping the parametrization γ if necessary. Now, the right-hand side of this ordinary differential equation satisfies the assumptions of Lemma F.5: it is bounded from below by (c), continuous as X is continuous, and γ is continuously differentiable. Further, the points where it is not locally Lipschitz are isolated, as the level sets of H_λ cross the y-Axis orthogonally ($\partial_1 H(0, y) = 0$), and X is differentiable and hence locally Lipschitz wherever $x \neq 0$ (remember $1 < p < 2$ and $2 < q = \frac{p}{p-1}$).

We can hence apply Lemma F.5 around $t = L$ and conclude that s and $\bar{s}$ also agree on t slightly bigger than L.

□

For ease of notation, define for $r > 0$ the function

$$\varphi_r(x) := |x|^{r-1}x,$$

as in Example 3.3.5.

Theorem 3.3.8. *Let $\lambda \in \mathbb{R}$ be positive. Then all nonconstant solutions $z : \mathbb{R} \to \mathbb{R}^2$ of (3.36) are periodic with the same period $L_p(\lambda) \in \mathbb{R}$.*

Proof. Note that the only constant solution is the one in the origin. For periodicity, we observe that the level sets of the Hamiltonian are homeomorphic to S^1 via the continuous mapping

$$f : (x, y) \mapsto \left(\left(\frac{p}{\lambda}\right)^{\frac{1}{p}} \varphi_{\frac{2}{p}}(x), \varphi_{\frac{2}{q}}(y) \right)$$

that maps the set $\{(x, y) \mid x^2 + y^2 = C\}$ bijectively to the set $\{(x, y) \mid \frac{\lambda}{p}|x|^p + |y|^p = C\}$ for any $C > 0$, which can be seen by direct calculation and by giving the continuous inverse

$$f^{-1}(x, y) \mapsto \left(\left(\frac{\lambda}{p}\right)^{\frac{1}{2}} \varphi_{\frac{p}{2}}(x), \varphi_{\frac{q}{2}}(y) \right).$$

Now let z be a solution of (3.36) and define $\gamma := f^{-1} \circ z : \mathbb{R} \to S^1$. We show noninjectivity of γ, from which periodicity follows by uniqueness of solutions. We don this with a topological argument: by Lemma 3.3.7(c), we have $|z'| > c > 0$ for some c if z is a nonconstant solution. Such a z cannot converge for $t \to \infty$. But Lemma F.6 says that an injective function $\mathbb{R} \to S^1$ would converge, so this means that z cannot be injective. Hence there must be $t_1 \neq t_2$ with $z(t_1) = z(t_2)$, i.e., z is periodic.

To show that period of all orbits is the same, observe that if $z(t) = (x(t), y(t))$ is a solution of (3.36) then $(\alpha x, \alpha^{p-1} y)$ is one as well, as

$$(\alpha x)' = \alpha x' = \alpha \varphi_{q-1}(y) = \varphi_{q-1}(\alpha^{p-1} y),$$

and

$$(\alpha^{p-1} y)' = \alpha^{p-1} y' = -\lambda \alpha^{p-1} \varphi_{p-1}(x) = -\lambda \varphi_{p-1}(\alpha x).$$

The new solution trivially has the same period. The level sets of H_λ are uniquely determined by their two intersection points with the x-axis, which are symmetrical around the origin. As we can map any such pair onto any other such pair by multiplying with some $\alpha > 0$, we have shown that for every two solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$, there is some α such that $(\alpha x_2, \alpha^{p-1} y_2)$ is in the same level set of H_λ as (x_1, y_1), and hence it coincides with (x_1, y_1) up to a shift in t. Thus, the periods of (x_1, y_1) and (x_2, y_2) coincide. □

We are now ready to state a rather concise characterization of the Dirichlet eigenvalues of Δ_p in terms of the Hamiltonian system.

Corollary 3.3.9. *Let $p \in (1, 2)$ be fixed. Some $\lambda > 0$ is a (strong) Dirichlet eigenvalue of Δ_p on $[0, a]$ if and only if $L_p(\lambda) = \frac{2a}{k}$ for some $k \in \mathbb{N}$*

Proof. A Dirichlet eigenfunction corresponding to an eigenvalue λ can be extended to an eigenfunction on $[0, 2a]$ by setting $\tilde{u}(x) = -u(2a - x)$. This yields an orbit with (not necessarily minimal) period $2a$ by means of the substitution in Theorem 3.3.3. This period is an integer multiple of the minimal period, and hence $L_p(\lambda) = \frac{2a}{k}$ for some k. Conversely, a periodic solution of the Hamiltonian system with period $\frac{2a}{k}$ yields a Dirichlet eigenfunction of $-\Delta_p$ by setting $u(x) := \gamma(x)$. The Dirichlet boundary conditions are fulfilled at half the period because of symmetry. □

3.3.2 Regularization

In the limit $p \to 1$, the function φ_{p-1} converges to $\frac{x}{|x|}$, which is not continuous in 0. One could try to replace this with

$$\sigma_\varepsilon(x) = \frac{x}{\sqrt{x^2 + \varepsilon^2}}$$

and let $\varepsilon \to 0$. This is done, for example, in [FP03] for the inverse p-Laplace problem. The function σ_ε has the same asymptotics as $x/|x|$ for $x \to \pm\infty$ but is smooth in $x = 0$ (see Figure 3.9). Now, one can look at the equation

$$-\operatorname{div}(\sigma_\varepsilon(\nabla u)) = \lambda \sigma_\varepsilon(u) \tag{3.37}$$

in the hope that in the limit $\varepsilon \to 0$, it behaves similarly to the case “$p = 1$”

$$-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \lambda \operatorname{sgn}(u).$$

We do not handle the formal subtleties of setting p exactly to 1 here and leave the above equation informal.

In one dimension, Equation (3.37) is of the form (3.19) with

$$g_r(x) = g_l(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}},$$

leading to the Hamiltonian

$$H_{\lambda,\varepsilon}(x,y) = \lambda\sqrt{x^2+\varepsilon^2} - \varepsilon\sqrt{1-y^2}.$$

(see Figure 3.10). We informally report on some behavior that we found:

- most trajectories exhibit a blowup in the derivative and are not periodic. In Figure 3.10, these are the trajectories ending at the boundary of the horizontal stripe $\mathbb{R}\times[-1,1]$.
- the periods of the periodic orbits for some fixed λ and ε do not coincide anymore (observed in numerical experiments).
- for some (λ, x_0) where the orbit starting at x_0 is periodic, one can change the orbit period by either changing x_0 or λ. Assuming that this dependence is differentiable, the implicit function theorem then implies that one can change λ and x_0 simultaneously such that the period is kept constant. This means that the "spectrum" contains open intervals (if one defines the spectrum on $[0,T]$ as the set of values λ for which there is a periodic solution of period T in the Hamiltonian system $H_{\lambda,\varepsilon}$). In particular, it contains points that are not isolated.

Ultimately, we did not pursue this approach further.

3.3.3 Reparametrized finite elements

When discretizing the eigenvalue equation with the finite element method, we solve the corresponding optimization problem on a finite-dimensional "trial space". Now, as seen in Section 3.1.2, the eigenfunctions of Δ_p exhibit a property that is not present in the standard finite element spaces: close to a local extremum x_0, an eigenfunction u fulfills $u(x_0+h) = u(x_0) + \mathcal{O}(|h|^q)$, where $\frac{1}{p}+\frac{1}{q}=1$. Now, Δ_p is very sensitive to this order q of convergence around the extrema. If the order is smaller, then Δ_p will have a pole at the extremum. In order to find eigenfunctions of Δ_p, one might try to constrain the trial functions to only have extrema of the appropriate order.

This approach is hindered by the fact, that the set of such functions does not form a vector space. Even if two functions f_1, and f_2 both only have extrema of order q, their sum f_1+f_2 might have extrema that are not of the right order. Take, for example, $f_1(x) = |x-1|^q$ and $f_2(x) = |x+1|^q$. Then f_1+f_2 has an extremum of order 2 in $x=0$ due to symmetry, see Figure 3.11). We may thus not hope to be able to choose a vector space U that contains only functions whose extrema have order q.

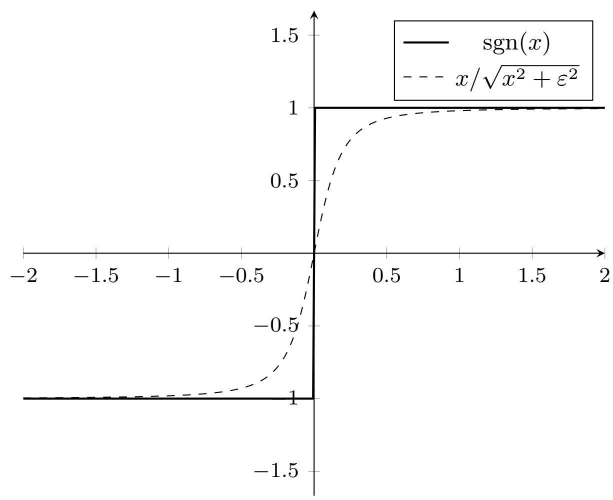


Figure 3.9: The sign function and the regularization $\frac{x}{\sqrt{x^2+\varepsilon^2}}$ for $\varepsilon = 0.2$.

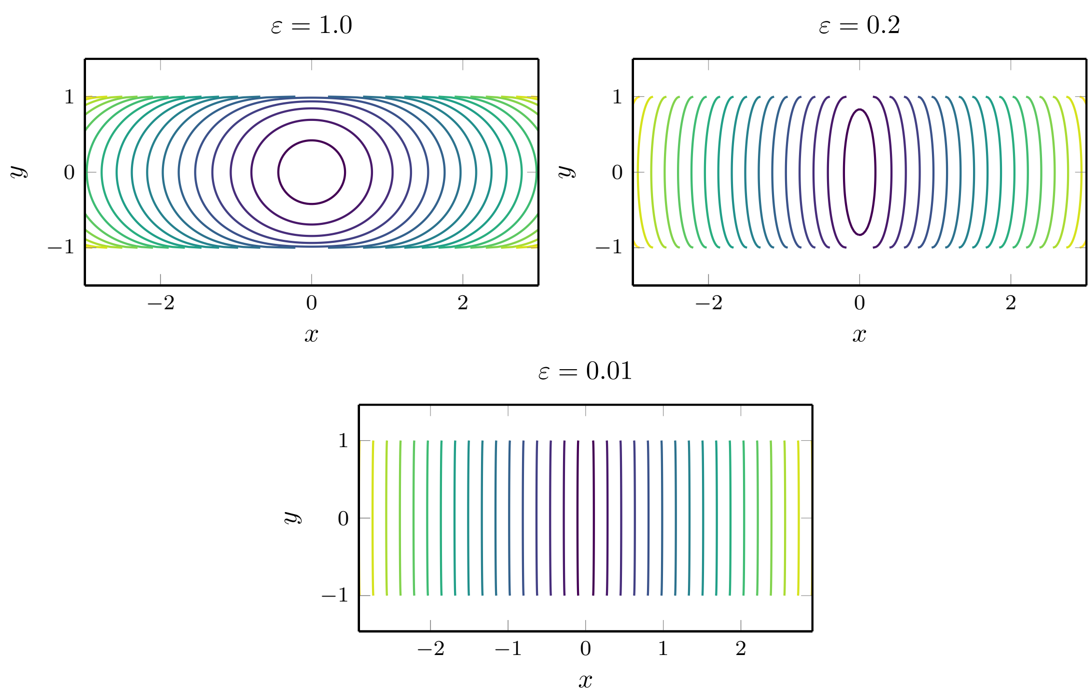


Figure 3.10: Contour lines of the Hamiltonian $H_\lambda(x, y) = \lambda\sqrt{x^2 + y^2} - \varepsilon\sqrt{1 - y^2}$ for $\lambda = 1$ and different ε. Compare these to Figure 3.8

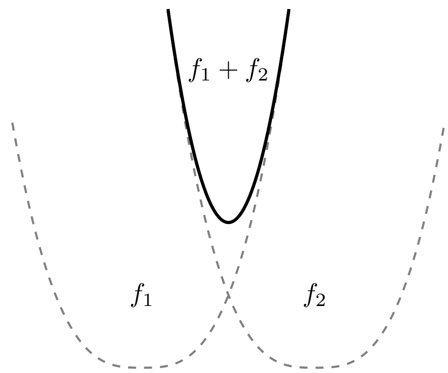


Figure 3.11: The functions that only have extrema of order q do not form a vector space.

3.3.4 Substitutions

The p-Laplacian contains the expression $|\nabla u|^{p-2}\nabla u$, which is very sensitive around small values of ∇u for $1 < p < 2$ due to the derivative of $|x|^{p-2}x$ diverging around 0. At the same time, the gradient of an eigenfunction becomes very small in a neighborhood of an extremum (see Section 3.1.2). This means that computing ∇u from function values of u becomes numerically unstable.

Then, a natural question to ask is, whether a substitution alleviates this situation. A possible substitution that adresses the nondifferentiability is to state the problem for $W := |\nabla u|^{p-2}\nabla u$ instead. Note that from $a = |b|^{p-2}b$ follows $b = |a|^{q-2}a$, so the inverse substitution is $\nabla u = |W|^{q-2}W$. In one dimension, one can then express $u(x)$ as an integral over $|W|^{q-2}W$, so that the (strong) eigenvalue equation takes the form

$$-W'(x) = \lambda\varphi_{p-1}\left(\int_0^x |W(t)|^{q-2}W(x)\mathrm{d}t\right),$$

where $\varphi_{p-1}(x) := |x|^{p-2}x$. In higher dimensions, however, this approach becomes less feasible for two reasons:

- It is harder to calculate the value of u from values of w, as there is no canonical path from the boundary to a given point in the interior.

- The Jacobian of a discretized functional is likely to be not sparse anymore because the effects of changing W have global effects on u. This already happens in one dimension, where it is lower triangular (values of w at some x affect values of u at all bigger x), but computationally, this has a larger impact in high dimension.

- Not every vector field is a gradient, so one must add adittional constraints on W.

In the general case, one can deduce the following reformulation of the eigenvalue problem (seen classically here). The substitution leads to

$$-\operatorname{div}(W) = \lambda|u|^{p-2}u \tag{3.38}$$
$$\nabla u = |W|^{q-2}W. \tag{3.39}$$

One can obtain $u = -|\operatorname{div}(W/\lambda)|^{q-2}\operatorname{div}(W/\lambda)$ from (3.38), which, plugged into (3.39) yields

$$-\nabla\left(|\operatorname{div}(W)|^{q-2}\operatorname{div}(W)\right) = \mu|W|^{q-2}W,$$

where $\mu \coloneqq |\lambda|^{q-2}\lambda$. This gets rid of the nondifferentiability stemming from the term $|x|^{p-2}x$. However, the root of $|x|^{q-2}x$ has vanishing derivatives up to the $\lfloor q-2 \rfloor$-th derivative, which poses other problems for root-finding algorithms. Also, the substitution defining W might amplify error terms. We did not pursue this approach further.

Chapter 4

Numerical experiments

We investigate the properties of the first eigenfunction of Δ_p^D on a series of examples in two dimensions. These experiments have also been published in [DFJK23]. Related code can be found in the accompanying Julia package `DynamicPLaplacian.jl`.

Given the results in Section 3.1, we let p range down to 1.3 only. We found that lower p lead to increasingly slow convergence (see Figure 4.1)

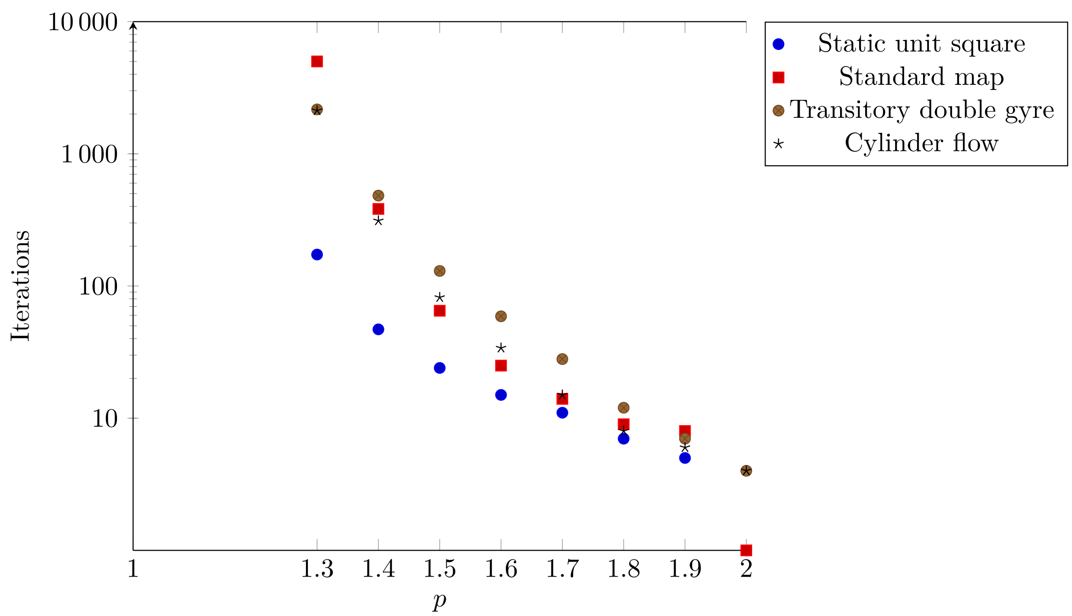


Figure 4.1: The number of iterations to reach the stopping criterion $\|\mathrm{grad}^D J^D\|_p \leq 10^{-3}$ in Algorithm 1 and Algorithm 2 for the examples in Sections 4.1.1 to 4.1.3 and 4.3. Figure also appears in [DFJK23].

Finite element discretization For all examples, we use a triangulation over a uniform Cartesian grid and the subspace $V^h \subset W_0^{1,p}(M)$ of piecewise linear functions on that grid as a finite element discretization. The basic finite element routines are taken from the finite element toolbox `Gridap.jl`[BV20].

The descent direction For calculating the descent direction $\operatorname{grad}^D J^D$, the weak equation for the approximation $d_h^D \in V^h$ is

$$\int_M \frac{1}{2}(I + DT^{-1}DT^{-T})\nabla d_h^D \nabla v = \frac{p}{2G(u)^2} \int_M \Big(G(u)\big(|\nabla u|^{p-2}\nabla u \nabla v + \\ + |\nabla(T_* u)|^{p-2}\nabla(T_* u)\nabla(T_* v)\big) \\ - F^D(u)|u|^{p-2}uv \Big) \tag{4.1}$$

for all $v \in V^h$, where DT is the Jacobian of T. After substituting with T on the right-hand side, (4.1) changes to

$$\int_M \frac{1}{2}(I + DT^{-1}DT^{-T})\nabla d_h^D \nabla v = \frac{p}{2G(u)^2} \int_M \Big(G(u)\big(|\nabla u|^{p-2}\nabla u \nabla v + \\ + |DT^{-T}\nabla u|^{p-2}(DT^{-T}\nabla u)(DT^{-T}\nabla v)\big) \\ - F^D(u)|u|^{p-2}uv \Big).$$

This approach of calculating the integrals on the right-hand side using the Cauchy-Green Tensor $DT^T DT$ is analogous to the "Cauchy-Green approach" from [FJ18]. A sketch of the part of the code implementing this can be seen in Figure 4.2.

Initial guess As an initial guess, we approximate the first eigenfunction of Δ^D. We also do this with the Cauchy-Green approach from [FJ18], calculating the matrices $A = (a_{ij})_{i,j}$ and $M = (m_{ij})_{i,j}$ from a basis $\varphi_1, \ldots, \varphi_N$ of V^h by

$$a_{ij} = \int_M \left(\frac{1}{2}(I + DT^{-T}DT^{-1})\nabla\varphi_i\right)\nabla\varphi_j$$
$$m_{ij} = \int_M \varphi_i\varphi_j,$$

and solving the generalized eigenvalue problem $Ax = \lambda Mx$. We do so using the Julia wrapper `Arpack.jl` of the *arpack-ng* library [Sha; SLYM].

Cheeger ratio The superlevel sets of a function are calculated using the marching squares algorithm implemented in the Julia package `Contour.jl`[DL]. The area of a polygon enclosed by points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n) \in \mathbb{R}^2$ with $(x_1, y_1) = (x_n, y_n)$ can be calculated by the formula $\left|\frac{1}{2}\sum_{k=1}^{n-1} x_k y_{k+1} - x_{k+1} y_k\right|$ [e.g. RW04, Section 3.5, 6.]. This is only possible on the domains in Sections 4.1.1 and 4.1.2, as the domains in Sections 4.1.3 and 4.3 have periodicity. In the latter case, we count the number of grid points within the superlevel set (indicated by the function value) to estimate its volume.

ODE integration In the examples that contain dynamics induced by a time-dependent vector field, we use the Tsitouras 5/4 Runge-Kutta method implemented in the Julia package `DifferentialEquations.jl` [RN17] with an absolute tolerance of 10^{-7}. The Jacobian DT is approximated with automatic differentiation provided by the Julia package `ForwardDiff.jl` [RLP16].

```
using Gridap
using ForwardDiff

function T(x)
    # insert dynamics
end

# derivatives of T
const DTinv = x -> inv(transpose(jacobian(T, x)))
const mCG = x -> (I + DTinv(x)'⋅DTinv(x))/2

# set up Gridap context here:
# const U, V, dΩ = ...

# lazy gridap wrappers for pointwise functions
const normp = Operation(x -> norm(x)^p)
const absp = Operation(x -> abs(x)^p)
const η1d = Operation(x -> abs(x)^(p-1) * sign(x))
const η2d = Operation(x -> norm(x)^(p-1) * normalize2(x))


function gradJ(u, p)
    G = sum(integrate(absp(u))*dΩ ))
    F = sum(integrate(normp(∇(u)) + normp(DTinv ⋅∇(u)))*dΩ ))

    # left-hand side: dynamic Laplacian of d
    lhs(d, v) = integrate(∇(d) ⋅ (mCG ⋅ ∇(v))) * dΩ

    # right-hand side: (J^D)'(u)
    rhs(v) = integrate((p / (2G^2)) *
                (  G * ( η2d(        ∇(u)) ⋅          ∇(v))
                 + G * ( η2d(DTinv ⋅ ∇(u)) ⋅ (DTinv ⋅∇(v)) )
                 - F * η1d(u) * v)
                 ) * dΩ

    # solve with Gridap
    op = Gridap.AffineFEOperator(lhs, rhs, U, V)
    ls = Gridap.LUSolver()
    solver = Gridap.LinearFESolver(ls)
    d =  Gridap.solve(solver, op)
    return d
 end
```

Figure 4.2: Simplified Julia code for calculating $\text{grad}^D J^D(u)$ using the FEM toolbox `Gridap.jl`[BV20] and the package `ForwardDiff.jl`[RLP16].

4.1 The examples

4.1.1 Unit square (static)

As a first example, we consider the unit square $[0,1]^2$ without dynamics, i.e., we set $T = id$ and obtain $\Delta_p^D = \Delta_p$. As seen in Example 1.3.6, the Cheeger ratio and Cheeger set can be determined explicitly: the Cheeger ratio is

$$h([0,1]^2) = 2 + \sqrt{\pi} \approx 3.7724\ldots,$$

and the Cheeger set is a square with rounded corners of radius

$$R = \frac{1}{2+\sqrt{\pi}} \approx 0.265\ldots\ .$$

The domain and its Cheeger set are depicted in Figure 4.3.

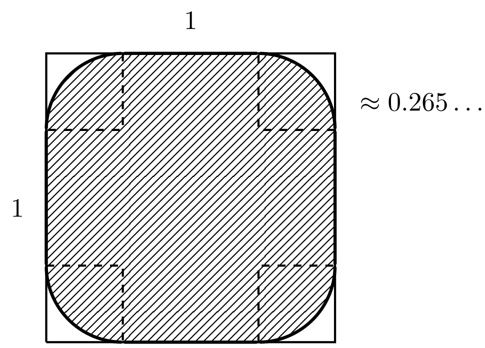


Figure 4.3: The Cheeger set of the unit square $[0,1]^2$.

4.1.2 Transitory double gyre

The first example with dynamics we consider is the transitory double gyre introduced in [MM11]. It consists of the time-one flow map of the nonautonomous differential equation $(\dot{x}, \dot{y}) = (\partial_y \psi, -\partial_x \psi)$ on $M = [0,1]^2$ defined by the stream function

$$\psi(x,y,t) = (1 - s(t))\psi_P(x,y) + s(t)\psi_F(x,y)$$

with $\psi_P(x,y) = \sin(2\pi x)\sin(\pi y)$, $\psi_F(x,y) = \sin(\pi x)\sin(2\pi y)$, and

$$s(t) = \begin{cases} 0 & \text{for } t < 0 \\ t^2(3-2t) & \text{for } t \in [0,1] \\ 1 & \text{for } t > 1. \end{cases}$$

It features two vortices spinning in opposite directions. The vortices change position from being side-to-side at time 0 to being on top of each other at time 1 (see, e.g., [FJ18]). In Figure 4.4, the flow is visualized.

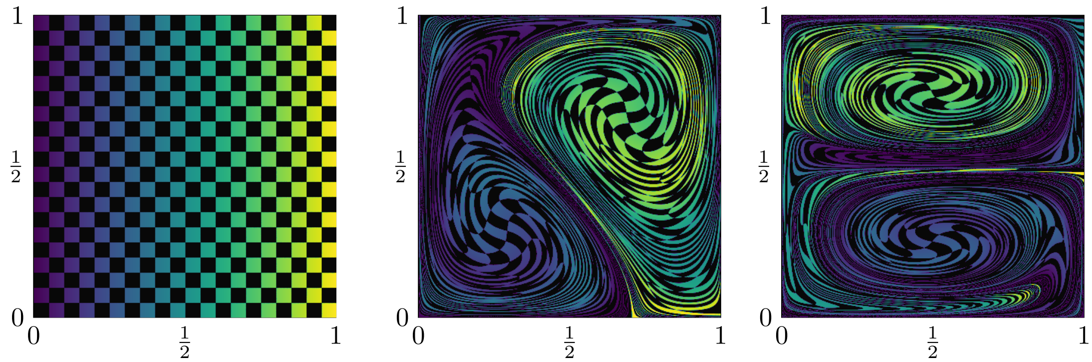


Figure 4.4: The advection of a colored grid under the transitory double gyre flow for the flow times $t = 0$ (left), $t = 0.5$ (middlea,) and $t = 1.0$ (right).

4.1.3 Cylinder flow

As a more complicated example, we take the "cylinder flow", a nonautonomous system on a cylinder [FLS10; FJ15]. It is defined by

$$\begin{aligned}\dot{x}(t) &= c - A(t)\sin(x - \nu t)\cos(y) + \varepsilon\Gamma(g(x,y,t))\sin(t/2)\\ \dot{y}(t) &= A(t)\cos(x - \nu t)\sin(y)\end{aligned}$$

on the domain $M = 2\pi S^1 \times [0,\pi]$, where $A(t) = 1 + \sin(2\sqrt{5}t)/8$, $\Gamma(\psi) = 1/(\psi^2+1)^2$, $g(x,y,t) = \sin(x-\nu t)\sin(y) + y/2 - \pi/4$, $c = 0.5$, $\nu = 0.5$, and $\varepsilon = 0.25$.

We choose $T = 40$ for the flow time; the time integration is performed as in the transitory double gyre example. In Figure 4.5, the flow is visualized.

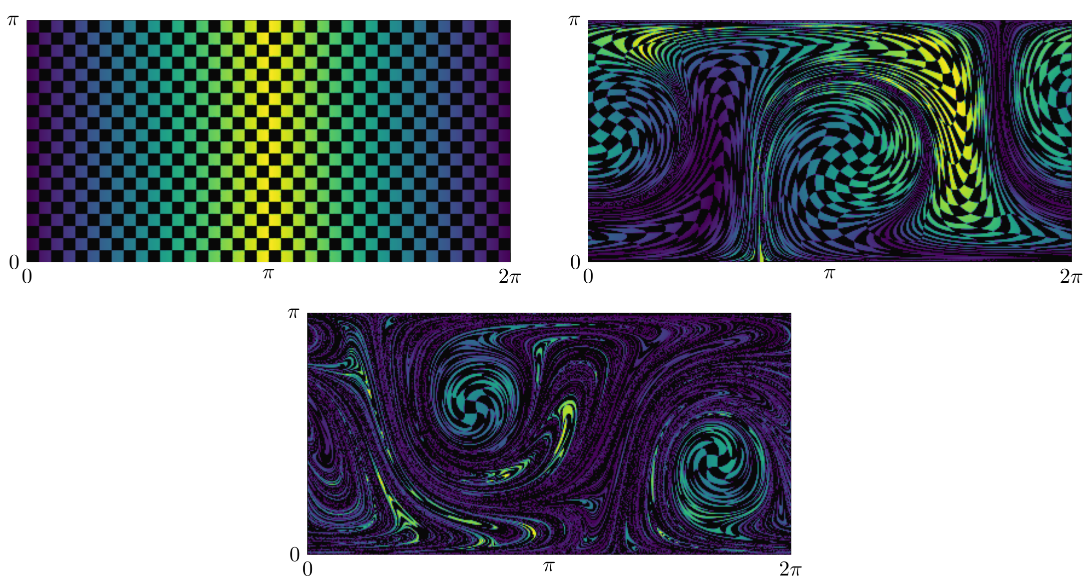


Figure 4.5: The advection of a colored grid under the cylinder flow for the flow times $t = 0$ (top left), $t = 0.2$ (top right), and $t = 1.0$ (bottom).

4.2 Experiments

4.2.1 Visual comparison of the first eigenfunctions

We compare the first eigenfunction, eigenvalue, and the level sets visually for p approaching 1. Figure 4.6 shows the approximated eigenvalues. In the case of the static unit square, the convergence of λ_p to p that is predicted by Theorem 1.5.4 is plausibly achieved. For the transitory double gyre and the cylinder flow, the graphs of the eigenvalue looks similar. Note that the transitory double gyre and the cylinder flow live in different scales: for the former, the domain is $[0,1]^2$ and for the latter $[0,2\pi] \times [0,\pi]$. Scaling the domain by $1/\pi$ increases λ_p^D by the factor π^p, which explains the big difference in magnitude of λ_p^D for the two examples with dynamics.

Figures 4.7 to 4.9 show the first eigenfunction next to its level sets. The first thing to note about the eigenfunctions is the "plateaus" forming around the extremum, which is predicted by Theorem 3.1.4. In the cylinder flow example (Figure 4.9), this happens around both extrema. This also means that one of the initial hopes is met: in the dynamic case, like in the static case, the eigenfunctions start to "look" more like characteristic functions. Second, one can observe that the level sets move closer to each other, indicating that the convergence predicted by Theorem 2.3.4 might also imply convergence to a function whose level sets all coincide (a priori, Theorem 2.3.4 only shows that they must almost all be Cheeger sets). In Section 4.2.2 and Section 4.2.3, we analyze the level sets and their (dynamic) Cheeger ratio more precisely.

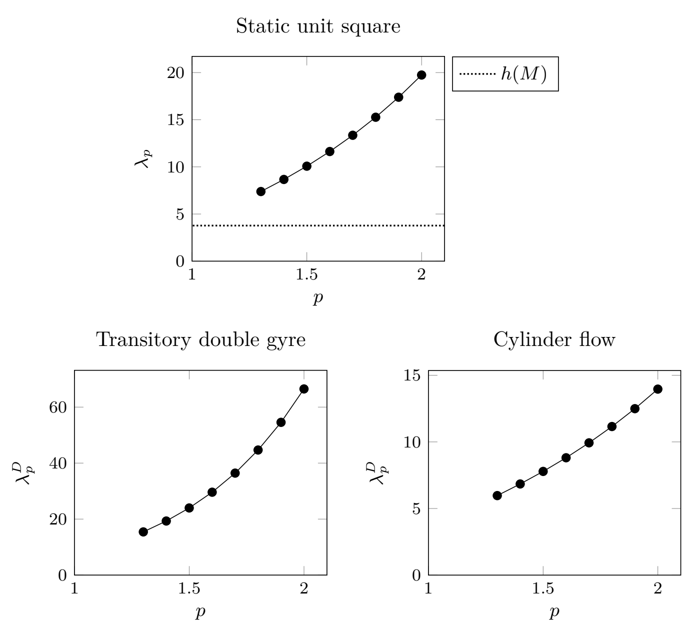


Figure 4.6: The numerical approximation to λ_p^D, calculated by $J^D(u_{h,p})$ at the numerical approximation $u_{p,h}$ of the first eigenfunction.

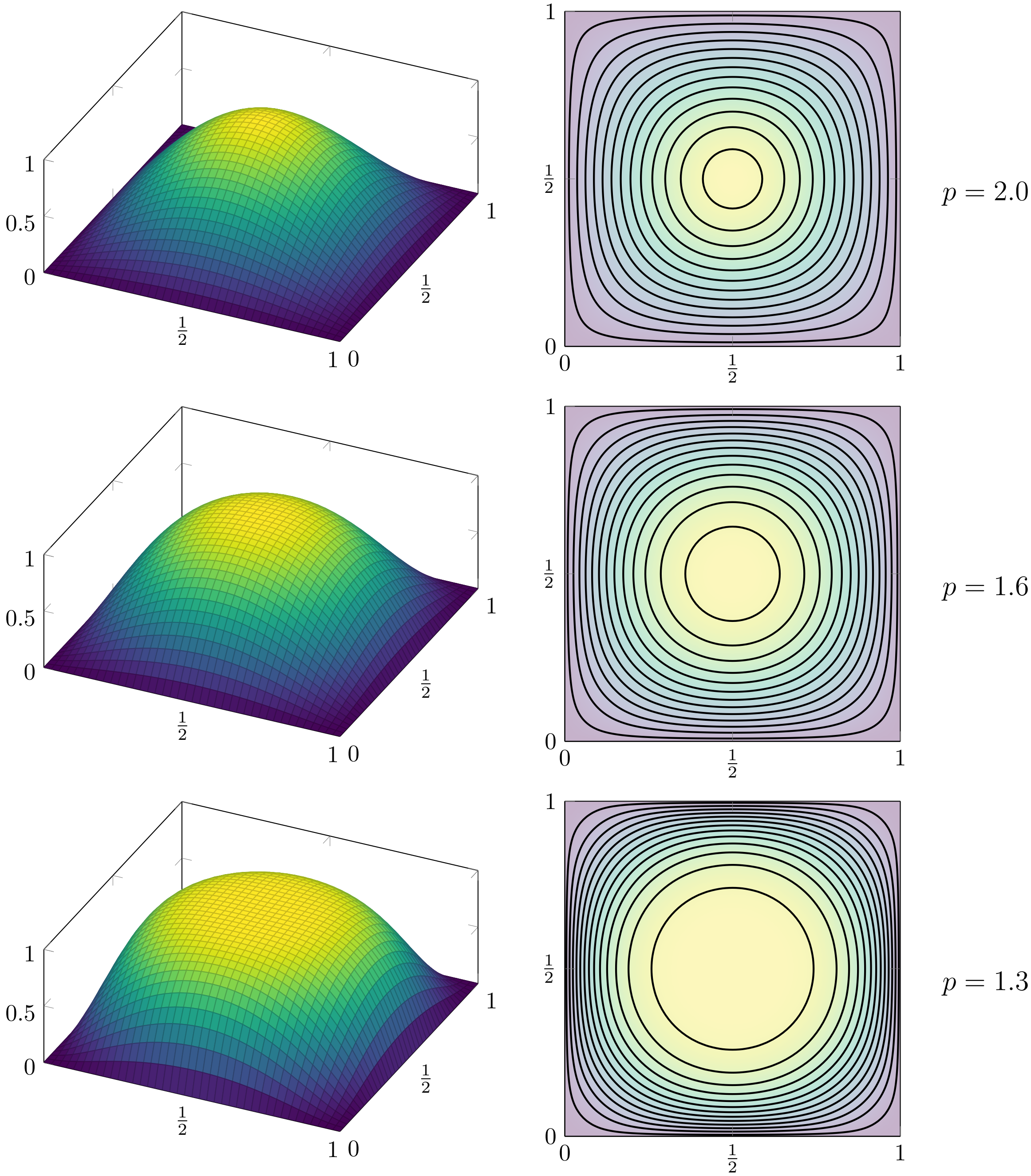


Figure 4.7: The first eigenfunction u_p of Δ_p on the unit square for the exponents $p \in \{2.0, 1.6, 1.3\}$ (from top to bottom).

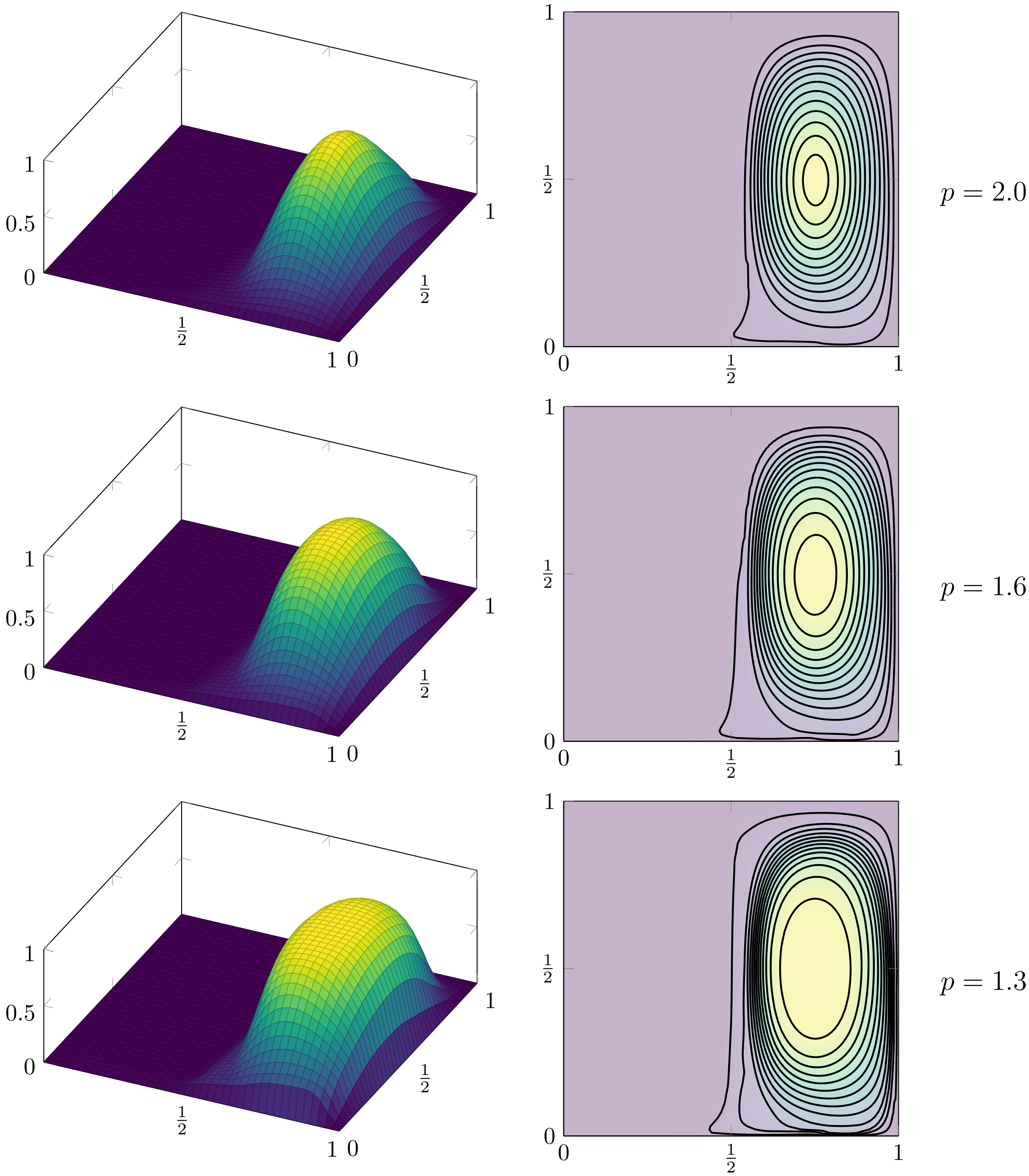


Figure 4.8: The first eigenfunction of Δ_p^D for the transitory double gyre and the exponents $p \in \{2.0, 1.6, 1.3\}$ from top to bottom.

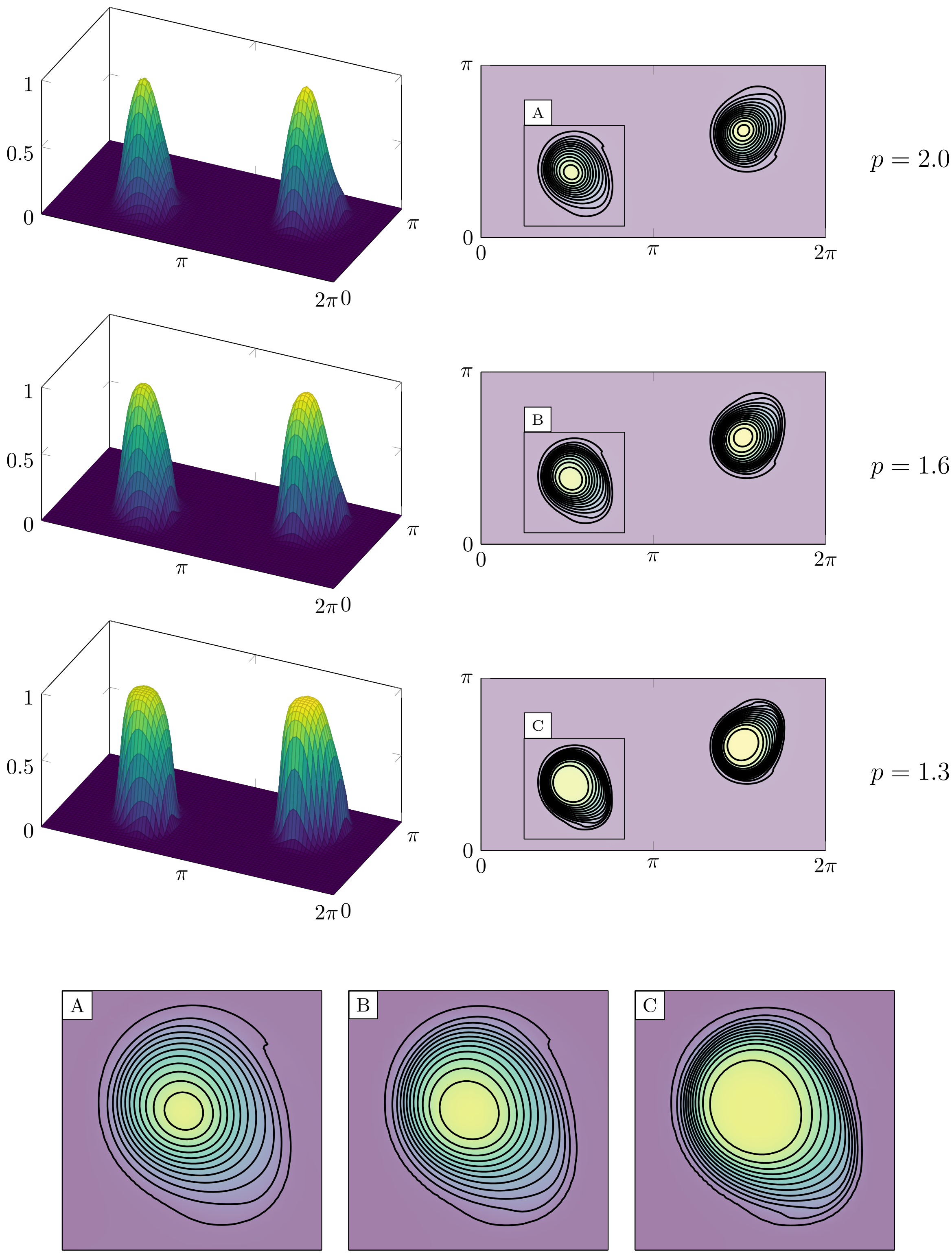


Figure 4.9: The first eigenfunction of Δ_p^D for the cylinder flow and $p \in \{2.0, 1.6, 1.3\}$

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4.2.2 Statistics of the dynamic Cheeger ratio

We investigate how the dynamic Cheeger ratio of the superlevel sets

$$A_t := \{x \in M \mid u_p(x) > t\}$$

changes overall for decreasing p. To do so, we normalize the first eigenfunction by scaling to $\|u_p\|_\infty = 1$ and take superlevel sets for levels between 0 and 1. Figure 4.10 shows the dynamic Cheeger ratio of the superlevel sets for the examples in Sections 4.1.1 to 4.1.3 and 4.3.

One can observe an overall improvement: for most t, the dynamic Cheeger ratio gets smaller for p approaching 1. We quantify the improvement by determining statistical quantities associated to a level t chosen randomly from the uniform distribution on $[0, 1]$. Note that there is a pole in $t = 1$ that appears because for t close to 1, the superlevel set is tiny, and the volume of the surface scales with a different order than the volume of the interior (in $\mathbb{R}^2$ with orders 1 and 2, respectively).

Because of the presence of this pole, there can be extreme outliers in an approximation of the mean by samples. Thus, we also plot the median, which is more robust to outliers (Note that due to non-monotonicity of A_t with respect to t, the median is not just the value of $A_{\frac{1}{2}}$) Both quantities improve significantly with lower p, as can be seen in Figure 4.11. The third quantity we plot is the minimum of the (dynamic) Cheeger ratio. In contrast to the mean and the median, the minimum changes very little: The relative deviation in the minimal (dynamic) Cheeger ratio is smaller than two percent in all examples. Within this error margin, the minimum actually increases slightly in some cases.

4.2.3 The level set with smallest dynamic Cheeger ratio

We now turn our attention to the level set with the best dynamic Cheeger ratio. We choose it from a set of 2000 equally spaced t between 0 and 1. In Section 4.2.2, we already observed that the best dynamic Cheeger ratio does not decrease substantially for p approaching 1. We see, in turn, that the corresponding level sets are also very close to each other. Figure 4.12 shows them for different p. In Figure 4.13, they are shown after the application of T.

The little variation in the level set indicates that even though there is no sharp bound on the dynamic Cheeger ratio of superlevel sets of the dynamic Laplacian, in practice, they can have dynamic Cheeger ratios that are nearly optimal. Another possibility is that a value of $p = 1.3$ is not close enough to 1 in order for the convergence of the eigenfunction to have a significant effect on the level sets. Either way, if one is only interested in the best level set, then it is clear that the decrease of p within the bounds that we found possible here is not enough to get improved results.

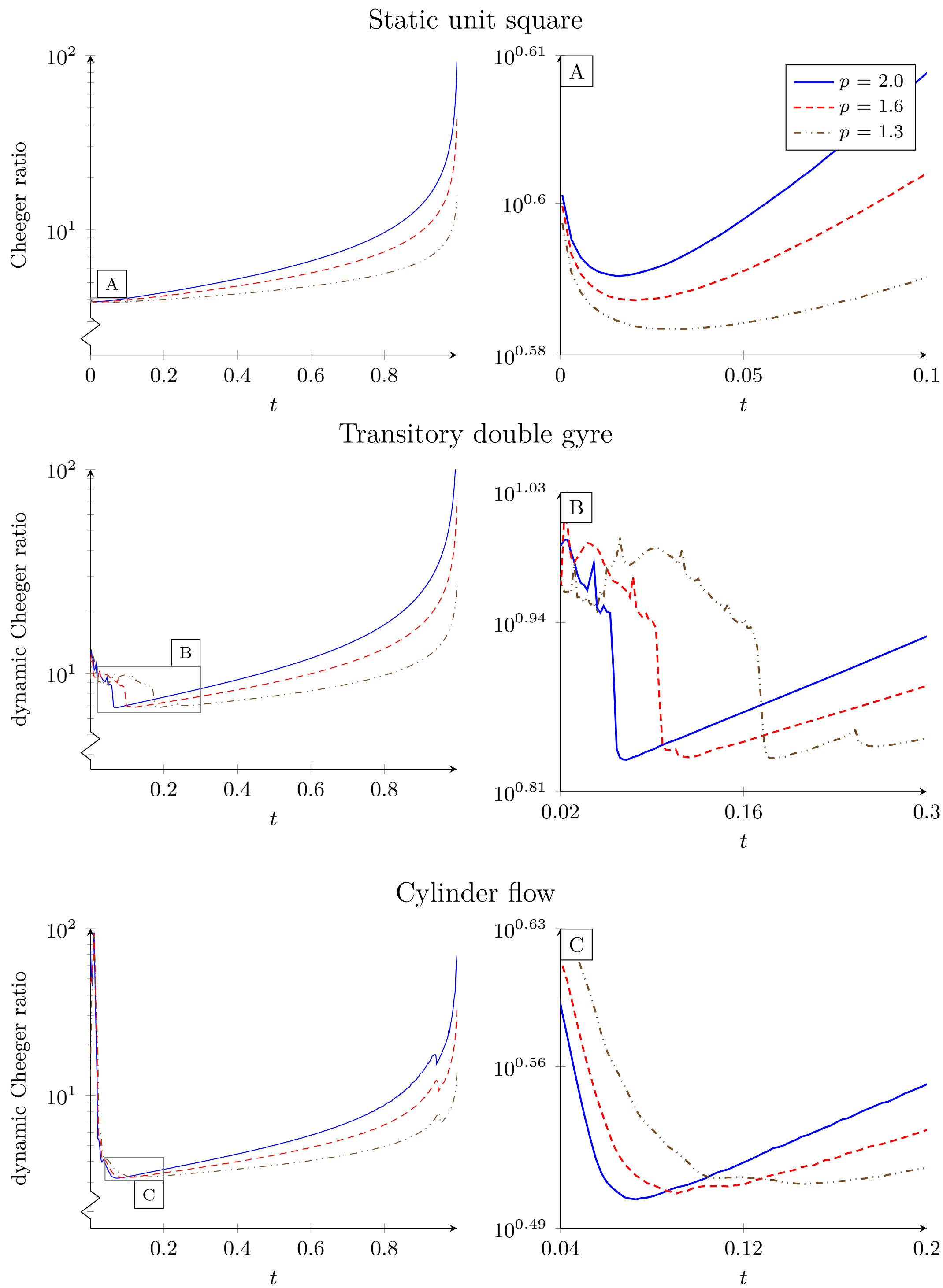

Figure 4.10: The Cheeger ratio of a superlevel set $\{u_p^D > t\}$ in terms of the level t for the static unit square (top row), the transitory double gyre (middle row), and the cylinder flow (bottom row). The exponents are $p = 2.0$ (blue, solid), $p = 1.6$ (red, dashed), $p = 1.3$ (brown, dashdotted). Closeups of indicated areas are shown on the right.

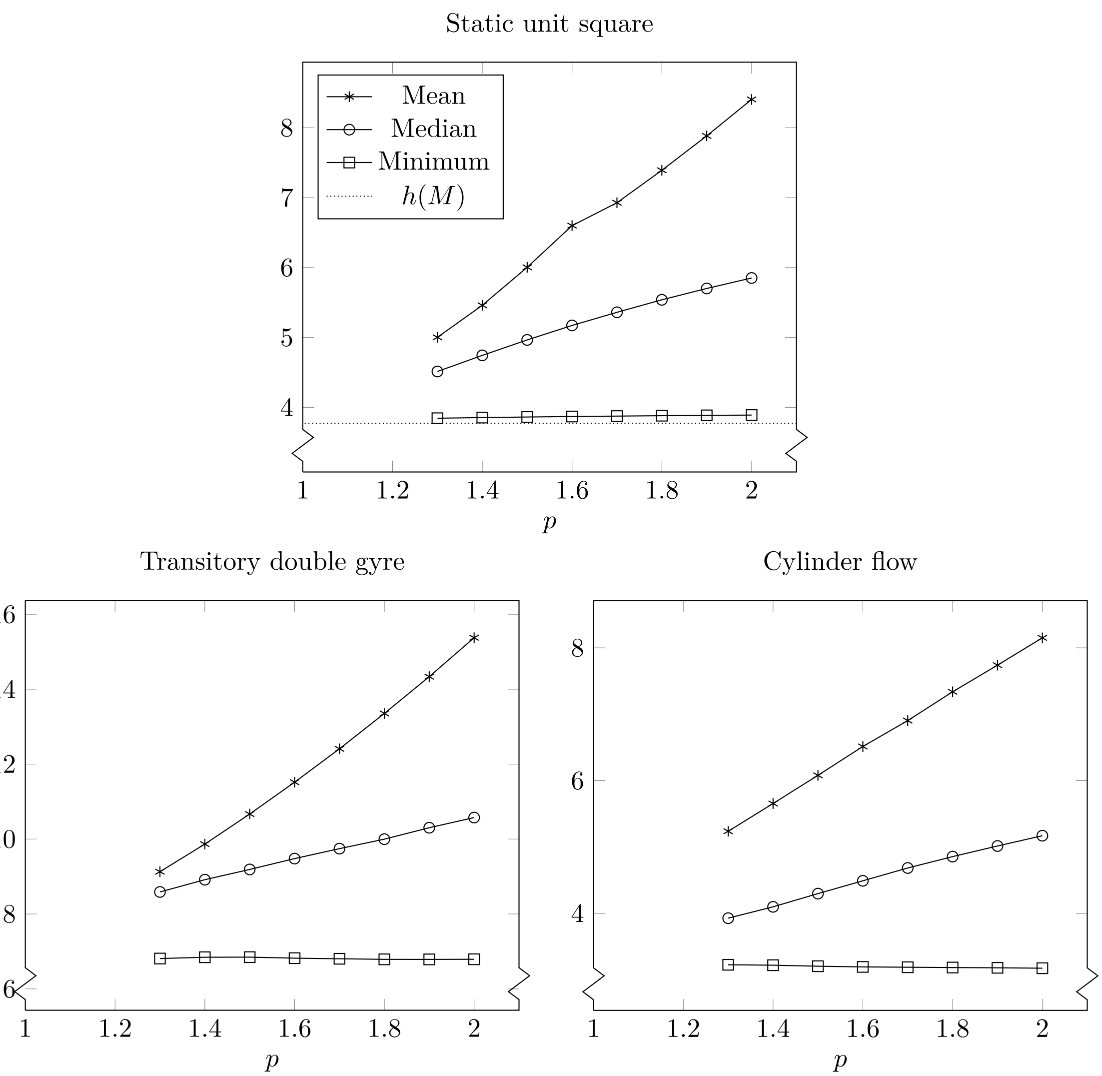

Figure 4.11: Mean, median, and minimum of the dynamic Cheeger ratio of the superlevel set $A_t := \{x \in M \mid u_p(x) > t\}$ for a random level $t \in [0, 1]$. The actual Cheeger ratio is indicated with a hatched line in the case of the static unit square.

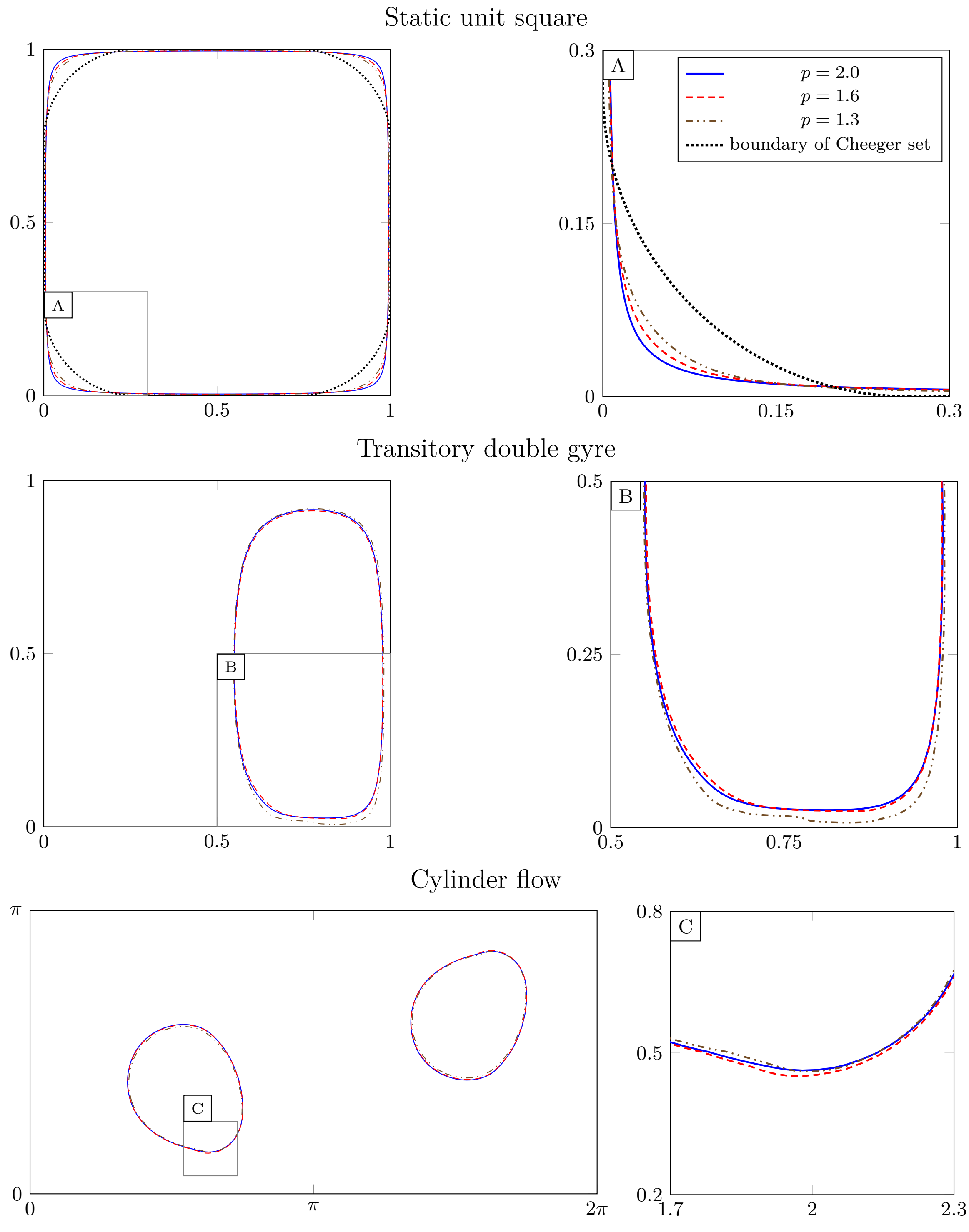

Figure 4.12: The boundary of the superlevel set of u_p^D with the best dynamic Cheeger ratio for the static unit square (top row), the transitory double gyre (middle row) and the cylinder flow (bottom row). Closeups of the areas indicated in the left picture are shown on the right. The values of p are $p = 2.0$ (blue, solid), $p = 1.6$ (red, dashed), and $p = 1.3$ (brown, dashdotted). For the static unit square, the boundary of the exact Cheeger set is also shown.

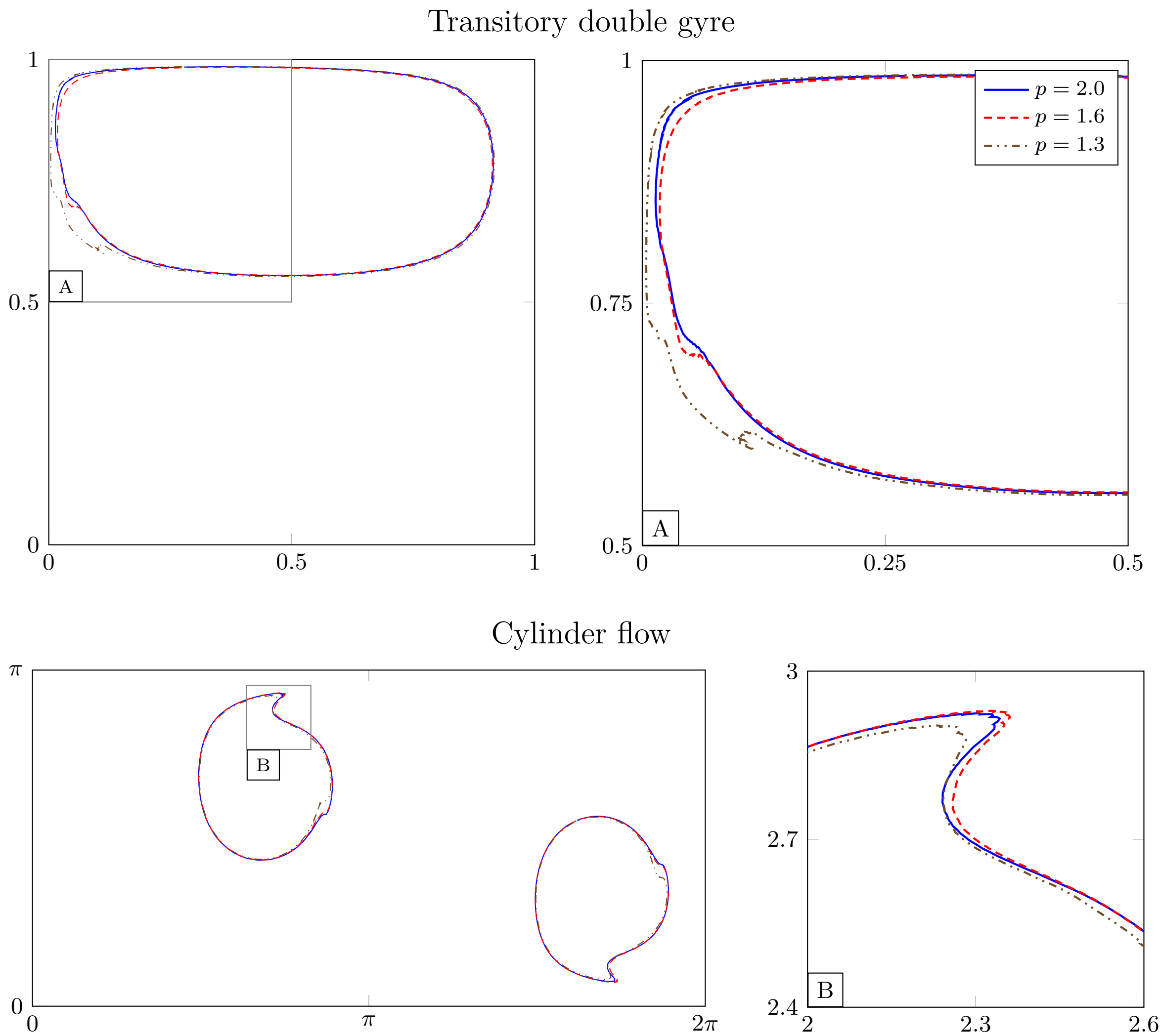

Figure 4.13: The transported boundary $T(\partial A_t)$ of the superlevel set A_t of u_p^D with the best dynamic Cheeger ratio for the transitory double gyre (top row), and the cylinder flow (bottom row). Closeups of the areas indicated in the left picture are shown on the right. The values of p are $p = 2.0$ (blue, solid), $p = 1.6$ (red, dashed), and $p = 1.3$ (brown, dashdotted).

4.3 An exploratory example: the standard map

We add an exploratory example on the two-dimensional torus $M = \mathbb{T}^2$, the standard map

$$T(x, y) = (x + y + a\sin(x), y + a\sin(x)) \pmod{2\pi}$$

with $a = 0.971635$, as in [FJ15]. This domain has empty boundary; hence the Dirichlet boundary conditions do not restrict functions and the first eigenvalue vanishes (the first eigenfunction is constant). We thus apply Algorithm 2 with $n = 2$. A few additions have to be made: first, the bilinear form $\int_M \nabla u \nabla v$ does not fulfill the conditions of the Fredholm alternative in [Sim72, Theorem 7.5] because a pair of constant functions maps to zero. We add zero-mean conditions $\int_M d = 0$ and $\int_M v = 0$ in the definition of d in Equation (3.14) to mitigate the resulting lack of uniqueness and existence of solutions. The finite-dimensional optimization in line 23 of Algorithm 2 is done using an implementation of the BFGS method on the unit sphere from the Julia package `Optim.jl`[MR18]. The method only finds a local optimum. Due to the vanishing boundary we use the quantity

$$\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\min\{\ell_d(D), \ell_d(M \setminus D)\}}$$

as the dynamic Cheeger ratio of D (this is in fact the original quantity from [Fro15]). Without the modified denominator, one could make the ratio arbitrarily small by setting D to the complement of a tiny ball. Figures 4.14 and 4.15 show the eigenfunction, the dynamic Cheeger ratio, the statistics of the dynamic Cheeger ratio, and the level set with the best dynamic Cheeger ratio for varying p.

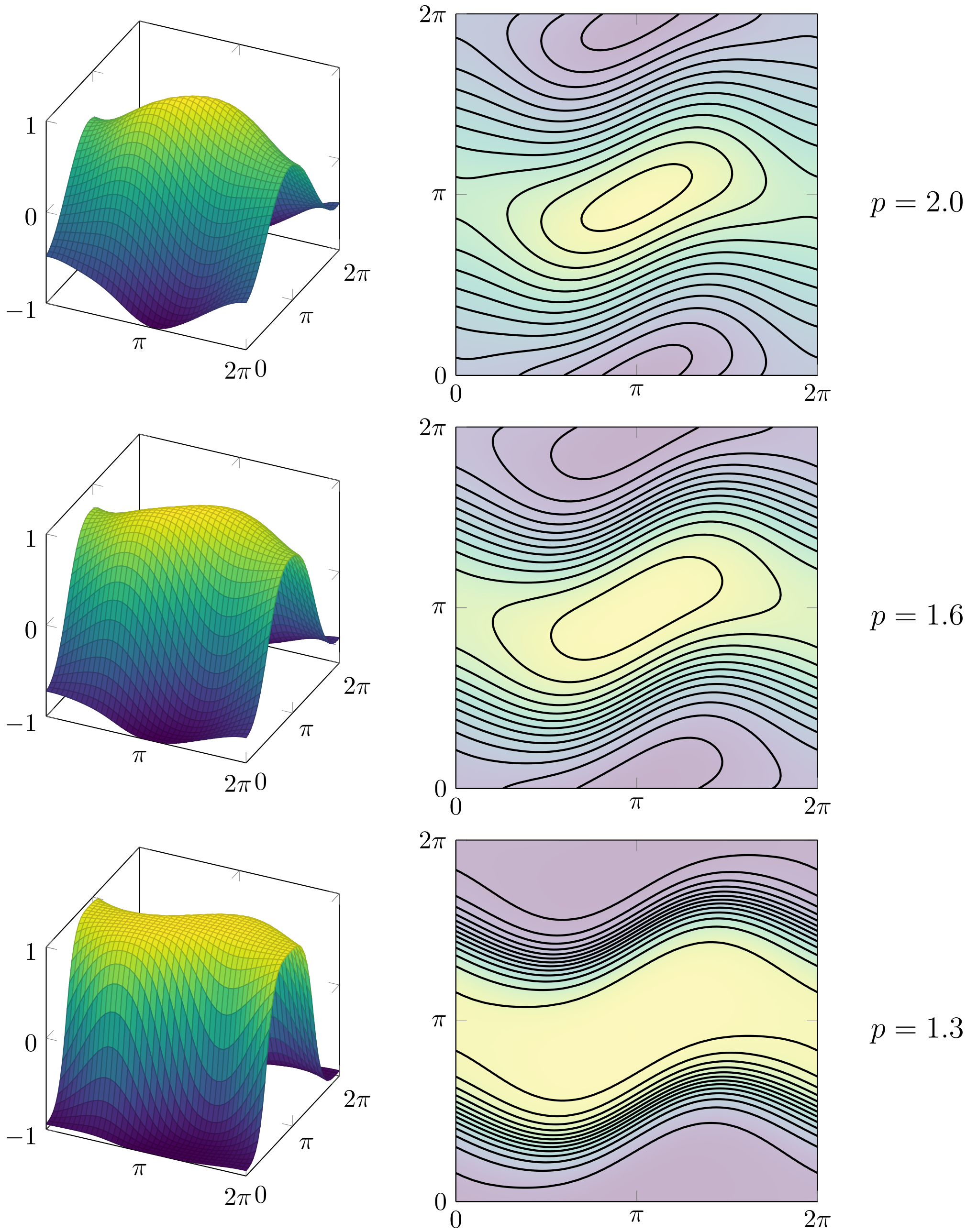

Figure 4.14: The first eigenfunction of Δ_p^D for the standard map and for the exponent $p \in \{2.0, 1.6, 1.3\}$ (from top to bottom).

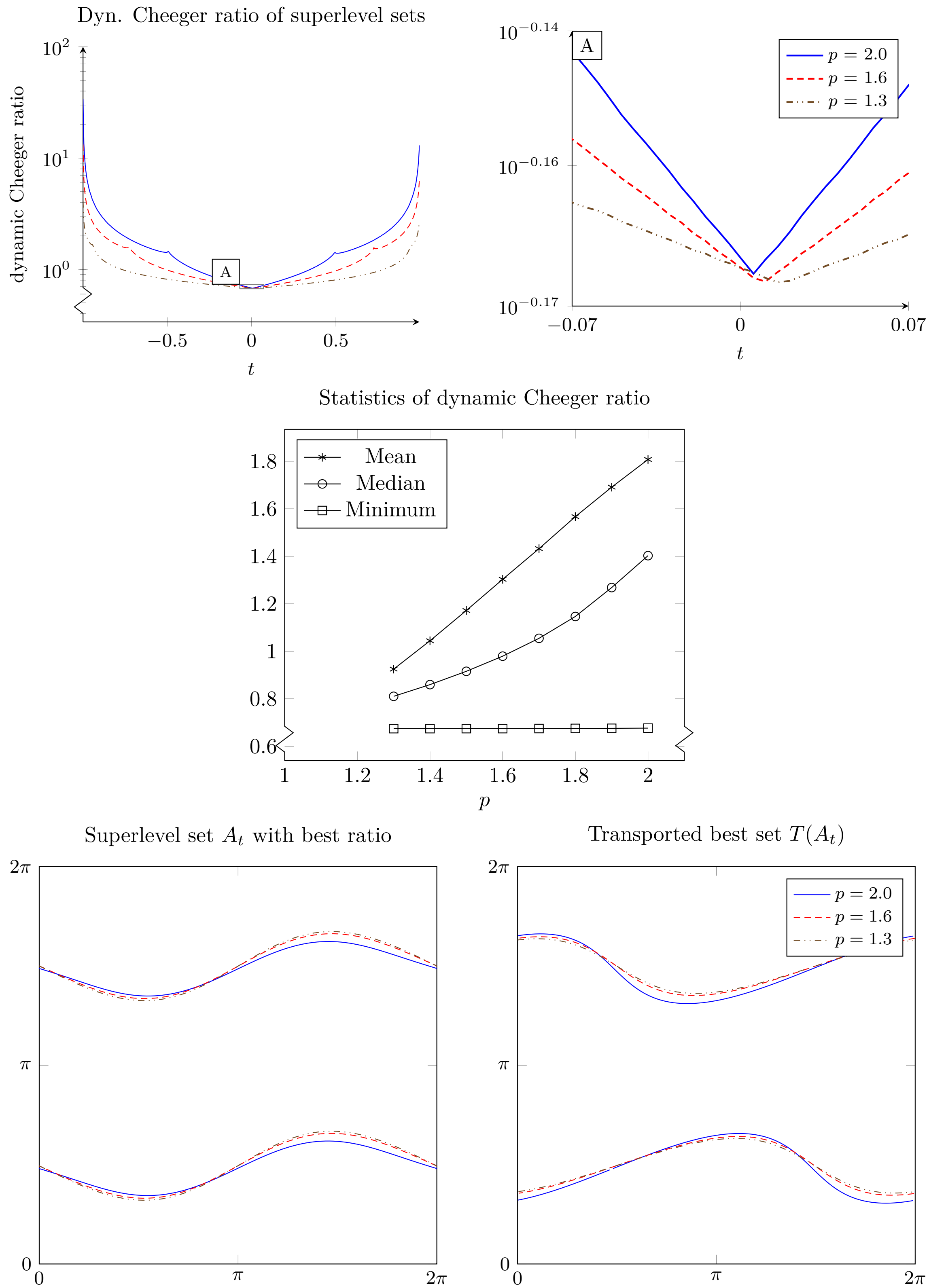

Figure 4.15: The dynamic Cheeger ratio (top row), its statistics (middle row) and boundary of the superlevel set A_t with the best dynamic Cheeger ratio, as well as $T(A_t)$ (bottom row) for the standard map.

Appendix A

Notation

M	compact d-dimensional submanifold of $\mathbb{R}^d$ with Lipschitz boundary
T	volume-preserving diffeomorphism on M (see also Setup 1.1.2)
p	exponent in $(1, 2) \subset \mathbb{R}$
q	conjugate exponent to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$
$C_0^\infty(M)$	smooth functions with compact support in M
$BV(\Omega)$	functions of bounded variation in Ω (see Appendix C)
$W_0^{1,p}(M)$	Sobolev space with vanishing trace (see Appendix B)
$W^{-1,q}(M)$	the dual of $W_0^{1,p}(M)$
$\langle \cdot, \cdot \rangle$	duality pairing on $W^{-1,q}(M) \times W_0^{1,p}(M)$
$\lvert x \rvert$	Euclidean norm of some $x \in \mathbb{R}^d$
$\lVert V \rVert_p$	L^p-norm of some $V : M \to \mathbb{R}^d$, i.e., $(\int_M \lvert V \rvert^p)^{\frac{1}{p}}$
$\lvert Du \rvert(\Omega)$	variation of a function u in Ω (see Definition C.1)
$P(D, \Omega)$	perimeter of a subset D in Ω (see Definition C.3
T_*	the transfer operator defined by $T_* f = f \circ T^{-1}$ (see Definition 1.2.1)
T^*	the dual of the transfer operator (see Definition 1.2.2)
Δ	the Laplacian
Δ^D	the dynamic Laplacian $\frac{1}{2}(\Delta + T^* \Delta T_*)$
Δ_p	the p-Laplacian defined by $\Delta_p u = \operatorname{div}(\lvert \nabla u \rvert^{p-2} \nabla u)$ (distributionally)
Δ_p^D	the dynamic p-Laplacian $\frac{1}{2}(\Delta_p + T^* \Delta_p T_*)$
$h(M)$	Cheeger constant of M
$h^D(M, T)$	dynamic Cheeger constant of M under T
λ_p, λ_p^D	first eigenvalue of the (dynamic) Laplacian
u_p, u_p^D	first eigenfunction of the (dynamic) Laplacian
$F(u)$	functional on $W_0^{1,p}(M)$ defined by $F(u) := \lVert \nabla u \rVert_p^p$
$F^D(u)$	functional on $W_0^{1,p}(M)$ defined by $F^D(u) := \frac{1}{2}(F(u) + F(T_* u))$
$G(u)$	functional on $W_0^{1,p}(M)$ defined by $G(u) := \lVert u \rVert_p^p$

Appendix B

Sobolev spaces

We introduce some notation and well-known results about Sobolev spaces. For further reading on the topic see e.g. [AF03; Bre11].

For any function $U : M \to \mathbb{R}^k$ we define

$$\|U\|_p \coloneqq \left(\int_M |U|^p \right)^{\frac{1}{p}},$$

where $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^k$. The space $L^p(M, \mathbb{R}^d)$ is the usual space of functions $U : M \to \mathbb{R}^k$ with $\|U\|_p < \infty$. For $k = 1$ we just write $L^p(M)$.

If $p > 1$, we denote by $q = \frac{p}{p-1}$ the conjugate exponent that fulfills $\frac{1}{p} + \frac{1}{q} = 1$.

Definition B.1. *For $p \geq 1$, the* Sobolev space $W^{1,p}(M)$ *is defined to be the space consisting of functions in $L^p(M)$ with weak partial first derivatives in $L^p(M)$, endowed with the norm*

$$\|u\|_{W^{1,p}(M)} \coloneqq \|u\|_p + \sum_{k=1}^{d} \|\partial_k u\|_p.$$

The space $W_0^{1,p}(M)$ is defined as the closure of the subspace $C_0^\infty(M) \subset W^{1,p}(M)$ with respect to this norm.

Definition B.2. *The dual space of $W_0^{1,p}(M)$ (i.e., the space of linear bounded functionals $W_0^{1,p}(M) \to \mathbb{R}$) is denoted by $W^{-1,q}(M)$. The duality pairing on $W^{-1,q}(M) \times W_0^{1,p}(M)$ is denoted by $\langle \cdot, \cdot \rangle$.*

Remark B.3. *In view of Theorem B.5 we identify a function $f \in L^q(M)$ with the functional $(v \mapsto \int_M fv) \in W^{-1,q}(M)$ and just write $f \in W^{-1,q}(M)$. Note that with this identification $L^q(M) \subsetneq W^{-1,q}(M)$.*

The norm in Definition B.1 is only one of many equivalent norms we can choose on $W_0^{1,p}(M)$. Another equivalent norm is, for example, the norm $\|u\|_p + \|\nabla u\|_p$. It is also well known that the Dirichlet boundary conditions in $W_0^{1,p}(M)$ allow us to simplify the norm even further and just use $\|\nabla u\|_p$. The reason for this is the *Poincaré inequality*:

Theorem B.4 (Poincare inequality). *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. Then there is a $C > 0$ such that*

$$\|u\|_p \leq C \|\nabla u\|_p$$

for all $u \in W_0^{1,p}(M)$.

Proof. See e.g. [Eva22]. Note that this only holds in $W_0^{1,p}(M)$, not in $W^{1,p}(M)$. $\square$

Theorem B.5. *Let $\varphi \in W^{-1,q}(M)$. Then there are $f_0, f_1, \dots, f_d \in L^q(M)$ such that*

$$\langle \varphi, v \rangle = \int_M f_0 v + \sum_{i=1}^d \int_M f_i \partial_i v$$

for all $v \in W_0^{1,p}(M)$. Conversely, if $f_0, f_1, \dots, f_d \in L^q(M)$ then

$$v \mapsto \int_M f_0 v + \sum_{i=1}^d \int_M f_i \partial_i v$$

defines an element in $W^{-1,q}(M)$.

Proof. For the first part see [Bre11, Proposition 9.20]. The second part is a consequence of the Hölder inequality. $\square$

We close with stating the well-known Rellich-Kondrachov theorem, which we use in the proof of existence of the first eigenvalue in Theorem 1.4.8.

Theorem B.6 (Rellich-Kondrachov)**.** *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2 and $p \in (1, \infty)$. Then the imbedding*

$$W_0^{1,p}(M) \to L^p(M)$$

is compact.

Proof. This is proven in a more general form form e.g. in [AF03, Theorem 6.3]. There the domain Ω may be any domain satisfying the cone condition. As M has Lipschitz boundary, this is the case (see remarks after [AF03, 4.9 and 4.11]). If for some $k \leq d$, M^k is the intersection of M with a k-dimensional plane and $j \geq 0$, $m \geq 1$, then [AF03, Theorem 6.3 Part I+II+IV] together with [AF03, Remark 6.4.1] states that the imbedding

$$W_0^{j+m,p}(M) \to W^{j,p'}(M^k)$$

is compact in the following cases:

$$\begin{aligned}
\text{if } mp < d \text{ and}& \qquad (B.1)\\
0 < d - mp < k& \qquad (B.2)\\
1 \leq p' < kp/(d - mp)& \qquad (B.3)\\
\text{or if } mp = d \text{ and}& \qquad (B.4)\\
1 \leq p' < \infty& \qquad (B.5)\\
\text{or if } mp > d \text{ and}& \qquad (B.6)\\
1 \leq p' < \infty.& \qquad (B.7)
\end{aligned}$$

If we set $p' := p$, $k := d$, $m := 1$ and $j := 0$ then we see that (B.2) and (B.3) become $0 < d - p < d$ and $1 \leq p \leq dp/(d - p)$, which are both fulfilled in the case $p < d$. As for the other two cases ($p = d$ and $p > d$) the necessary inequalities are also fulfilled, this means that the imbedding

$$W_0^{1,p}(M) \to L^p(M)$$

is compact for all cases of $1 \leq p < \infty$. $\square$

Appendix C

Functions of bounded variation

In Sections 1.3 and 2.2, we use the space $BV(M)$ of functions of bounded variation as a space of functions with low regularity that still allow for some notion of derivative. We give the needed defintions here. For a more detailed introduction (which contains the definitions below) see e.g [Par11, Section 2] or [Amb00, Chapter 3].

Definition C.1. *For an open subset $\Omega \subseteq \mathbb{R}^d$ and a function $u \in L^1(\Omega)$, define its* variation in Ω

$$|Du|(\Omega) \coloneqq \sup \left\{ \int_\Omega u \operatorname{div} \varphi \mid \varphi \in C_0^1(\Omega, \mathbb{R}^d), \|\varphi\|_\infty \leq 1 \right\}.$$

The space of functions $u \in L^1$for which $|Du|(\Omega) < \infty$ is called the space $BV(\Omega)$ of functions of bounded variation *(see e.g. [Par11, Section 2] or [Amb00, Definition 3.1 and Theorem 3.6] for an equivalent definition using measures). We endow $BV(\Omega)$ with the norm*

$$\|u\|_{BV(\Omega)} \coloneqq \|u\|_1 + |Du|(\Omega).$$

If M is as in Setup 1.1.2 we just write $BV(M)$ for $BV(\mathring{M})$ and for some $u \in BV(M)$, we denote by $|Du|(\mathbb{R}^d)$ the variation $|D\tilde{u}|(\mathbb{R}^d)$ of the zero-extension $\tilde{u}$ of u, i.e., the function $\tilde{u}$ with $\tilde{u}|_M = u$ and $\tilde{u}|_{\mathbb{R}^d \setminus M} \equiv 0$ (see also Theorem C.5 below).

Lemma C.2. *If $u \in W_0^{1,1}(\Omega)$ then $u \in BV(\Omega)$ and $\|\nabla u\|_1 = |Du|(\Omega)$.*

Proof. See remarks after [Amb00, Definition 3.1]. □

Definition C.3. *For some $D \subseteq \Omega \subseteq \mathbb{R}^d$ define the* perimeter of D in Ω *as*

$$P(D, \Omega) \coloneqq |D(\chi_D)|(\Omega),$$

where χ_D is the characteristic function on D. A set D is said to have finite perimeter in Ω *if $P(D, \Omega) < \infty$ (see e.g. [Par11, Section 2]).*

Remark C.4. *The perimeter of a set D is invariant by modifications of D by a set of measure zero [Mag12, Remark 12.4]. Thus, for a function $u \in L^1(M)$, the superlevel sets $A_t \coloneqq \{x \in M \mid u(x) > t\}$, which are only well defined up to a set of measure zero, still have a well defined perimeter.*

Theorem C.5. *If $M \subseteq \mathbb{R}^d$ is a d-dimensional compact manifold with Lipschitz boundary, then the following holds:*

(a) M is of finite perimeter.

(b) $P(M, \Omega) = \ell_{d-1}(\Omega \cap \partial M)$.

(c) If ∂M is C^1, then $P(M, \Omega)$ also coincides with the classical volume of ∂M.

(d) There is a bounded inclusion $BV(M) \to BV(\mathbb{R}^d)$ by extending some $u \in BV(M)$ to zero outside of M.

Proof. For (a) and (b), see [Mag12, Example 12.6]. For (c), apply [Mag12, Theorem 8.1] to a finite covering of ∂M and corresponding local parametrizations. For (d), apply [Eva18, Theorem 5.8]. □

Theorem C.6. *Let $d \geq 2$ and $D \subseteq M \subseteq \mathbb{R}^d$, where D is of finite perimeter and ∂M is Lipschitz. Then there exists a sequence D_k of open sets compactly contained in M with smooth boundaries of dimension $(d-1)$ such that*

$$\chi_{D_k} \xrightarrow{L^1} \chi_D$$

and

$$P(D_k, \mathbb{R}^d) \to P(D, \mathbb{R}^d)$$

for $k \to \infty$.

Proof. This is constructed in the proof of [Par11, Proposition 3.3]. □

Appendix D

Some properties of the transfer and Koopman operator

The transfer operator and its dual, the Koopman operator are introduced in Section 1.2.4. Here, we prove additional properties that we use in the main text. In particular, boundedness of the transfer operator in $BV(M)$ allows us to transfer techniques from the classical case to the dynamic case.

Theorem D.1. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2 and $p \geq 1$. Then the spaces $L^p(M)$, $W^{1,p}$ and $BV(M)$ are invariant under T_*, and the restriction of T_* to any of them is a bounded linear operator. On $L^p(M)$ it is even an isometry.*

Proof. Linearity is immediate. For boundedness in L^p, let $u \in L^p(M)$ and use volume preservation of T to transform the integral in

$$\|u \circ T^{-1}\|_p^p = \int_M |u \circ T^{-1}|^p = \int_M |u|^p = \|u\|_p^p,$$

showing that T_* is an isometry on $L^p(M)$. For $W^{1,p}$ use that T^{-1} has, by definition, bounded derivatives. Hence, if we use the matrix operator norm $|\cdot|_{op}$ to define

$$\psi(x) \coloneqq |DT^{-T}(x)|_{op}$$

then $\psi \in L^\infty(M)$ and

$$|\nabla(u \circ T^{-1})| \leq \psi|\nabla u|.$$

We can then use Hölder's inequality to get

$$\|\nabla(u \circ T^{-1})\|_p \leq \|\psi \nabla u\|_p \leq \|\psi\|_\infty \|\nabla u\|_p$$

and have proven that T_* is also bounded on $W_0^{1,p}(M)$. We finish by showing that T_* is bounded on $BV(M)$. This can be done directly with the defintion of $|Du|$ from Definition C.1 by substitution with T and using boundedness of the scalar field $|DT^{-1}|_{op}$ on the domain M.

Alternatively, one can use [Amb00, Theorem 3.9], which says that a function u is in $BV(M)$ if and only if there exists a sequence $(u_k)_k \subset\in C^\infty(M)$ that converges in $L^1(M)$ to u and the constant

$$L_{(u_k)} \coloneqq \lim_{k\to\infty} \|\nabla u_k\|_1$$

associated to that sequence is finite. It also says that by taking the infimum of $L_{(u_k)}$ over all such sequences, we get the variation of u:

$$|Du|(M) = \inf_{\substack{(u_k)_k \subset C^\infty(M) \\ \|u_k - u\|_1 \to 0}} L_{(u_k)}. \tag{D.1}$$

This can be used to show boundedness of T_* on $BV(M)$ in the following way: assume that $u \in BV(M)$, and apply the above characterization to get existence of a sequence $(u_k)_k \subset C^\infty(M)$ with $u_k \to u$ in $L^1(M)$ and $L_{(u_k)} \leq \infty$. We have already shown that T_* is continuous on $L^1(M)$ and $W^{1,1}(M)$, so $T_* u_k \to T_* u$ in $L^1(M)$, and hence for some positive C we get

$$L_{(T_* u_k)} = \lim_{k\to\infty} \|\nabla(T_* u_k)\|_1 \leq C \lim_{k\to\infty} \|\nabla u_k\|_1 = C L_{(u_k)} < \infty.$$

Thus, by [Amb00, Theorem 3.9], we get $T_* u \in BV(M)$. Additionally, this means that $T_* u_k$ is an admissable sequence in the infimum in (D.1) for $T_* u$, which gives us the estimate

$$|D(T_* u)|(M) \leq L_{(T_* u_k)} \leq C L_{(u_k)}.$$

As we may choose u_k such that $L_{(u_k)}$ is arbitrarily close to $|Du|(M)$, this shows

$$|D(T_* u)|(M) \leq C|Du|(M),$$

and we are done showing boundedness of T_* on $BV(M)$, as $\|\cdot\|_{BV(M)} = \|\cdot\|_1 + |D(\cdot)|_{BV(M)}$ and $\|\cdot\|_1$ was already shown to be preserved by T_*. □

Definition 1.2.2 directly defines the Koopman operator as the linear algebraic dual of the transfer operator. We show that this definition is equivalent to the more common definition $T^* f \coloneqq f \circ T$ under the usual identification of $(L^p(M))^*$ with $L^q(M)$.

Theorem D.2. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2 and let $(L^p(M))^*$ be the space of bounded linear functionals on $L^p(M)$. Let $T^*_{(1)} : (L^p(M))^* \to (L^p(M))^*$ be defined by*

$$T^*_{(1)} v = v \circ T_*$$

and $T^*_{(2)} : L^q(M) \to L^q(M)$ defined by

$$T^*_{(2)} u = v \circ T.$$

Let further

$$i : L^q \to (L^p(M))^* \tag{D.2}$$

$$v \mapsto (u \mapsto \int_M uv) \tag{D.3}$$

be the usual identification of L^q with the dual space $(L^p(M))^*$ of bounded linear forms on $L^p(M)$. Then the following diagram commutes:

$$\begin{tikzcd}
L^q(M) \arrow[r, "T^*_{(2)}"] \arrow[d, "i"'] & L^q(M) \arrow[d, "i"'] \\
(L^p(M))^* \arrow[r, "T^*_{(1)}"'] & (L^p(M))^*
\end{tikzcd}$$

i.e., $i \circ T^{(1)} = T^{(2)} \circ i$.

Proof. The function i is well-defined by the Hölder inequality. The claim follows by volume preservation of T and the substitution in

$$\left(u \mapsto \int_M u \cdot f\right) \circ T_* = \left(u \mapsto \int_M (u \circ T^{-1}) \cdot f\right) = \left(u \mapsto \int_M u \cdot (f \circ T)\right).$$

for $f \in W_0^{1,p}(M)$. □

We finish with a statement about the extention $\tilde{T}$ of T by the identity outside of M. Even though $\tilde{T}$ is not differentiable, $\tilde{T}$ induces a bounded operator $\tilde{T}_*$ on $BV(M)$.

Theorem D.3. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Define the extension*

$$\tilde{T}(x) := \begin{cases} T(x) & x \in M \\ x & else \end{cases}. \tag{D.4}$$

Then the associated transfer operator $\tilde{T}_* f := f \circ \tilde{T}^{-1}$ (see also Definition 1.2.1) is a bounded operator on $BV(\mathbb{R}^d)$.

Proof. Let $u \in BV(\mathbb{R}^d)$. We use [Eva18, Theorem 5.8], which says that for a set of finite perimeter and two functions $f_1 \in BV(M)$ and $f_2 \in BV(\mathbb{R}^d \backslash M)$, the combination

$$f(x) := \begin{cases} f_1(x) & x \in M \\ f_2(x) & x \notin M \end{cases}$$

is in $BV(\mathbb{R}^d)$. Our domain M is of finite perimeter by Theorem C.5. We set $f_1 := \tilde{T}_* u|_M$ and $f_2 := \tilde{T}_* u|_{\mathbb{R}^d \backslash M}$. These are indeed of bounded variation in their domains: first, $f_1 = T_*(u|_M)$ is in $BV(M)$ by Theorem D.1. Second, $\tilde{T} \equiv id$ outside of M, so f_2 coincides with $u|_{\mathbb{R}^d \backslash M} \in BV(\mathbb{R}^d \setminus M)$. Finally the second part of [Eva18, Theorem 5.8] says that we have

$$|D(T_* u)|(\mathbb{R}^d) = |Df_1|(M) + |Df_2|(\mathbb{R}^d \backslash M) + \int_{\partial M} |Tr(f_1) - Tr(f_2)| d\mathcal{H}^{d-1}, \tag{D.5}$$

where $Tr : BV(M) \to L^1(\partial M, \mathcal{H}^{d-1})$ is the (continuous) trace operator (see, for example, [Eva18, Definition 5.3]). Now the terms on the right-hand side of (D.5) are estimated in the following way:

- By boundedness of T_* on $BV(M)$ there is a $C_1 > 0$ such that

$$|Df_1|(M) \leq \|f_1\|_{BV(M)} \leq C_1 \|u|_M\|_{BV(M)} \leq C_1 \|u\|_{BV(\mathbb{R}^d)}.$$

- The second term reduces to

$$|Df_2|(\mathbb{R}\backslash M) \leq \|f_2\|_{BV(M)} = \|u|_{(\mathbb{R}^d\backslash M)}\|_{BV(\mathbb{R}^d\backslash M)} \leq \|u\|_{BV(\mathbb{R}^d)}.$$

- Recall that by Setup 1.1.2, T maps ∂M to ∂M, so $Tr(f_1 - f_2) = Tr(T_* u)$. Then by boundedness of the trace, there is some $C_2 > 0$ such that

$$\begin{aligned}\int_{\partial M} |Tr(f_1) - Tr(f_2)| d\mathcal{H}^{d-1} &= \int_{\partial M} |Tr(T_*(u|_M))| d\mathcal{H}^{d-1} \\ &\leq C_2 \|T_*(u|_M)\|_{BV(M)} \leq C_2 C_1 \|u|_M\|_{BV(M)} \leq C_2 C_1 \|u\|_{BV(\mathbb{R}^d)}.\end{aligned}$$

These three estimates, combined with (D.5), show boundedness of $\tilde{T}_*$ on $BV(\mathbb{R}^d)$. $\square$

Appendix E

The coarea formula

The coarea formula connects the functional quantity $\|\nabla u\|_1$ with the geometric quantity $\ell_{d-1}(\partial A_t)$, where $A_t = \{x \in M \mid u(x) > 0\}$. For us, it is one of the main tools connecting the variational characterizations of the (dynamic) Cheeger constant like (1.16) and (2.20) with the geometric characterizations like (1.14) and (2.18). For the reader's convenience, we state the formula here in a few different variations that we use.

Theorem E.1 (Coarea formula for scalar functions)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz. Then for every measurable $M \subseteq \mathbb{R}^d$.*

$$\int_M |\nabla f| = \int_{-\infty}^{\infty} \ell_{d-1}(M \cap f^{-1}(y))dy,$$

where ℓ_{d-1} is the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

Proof. For a proof of the general case of maps $f : \mathbb{R}^m \to \mathbb{R}^n$, see [Fed14, Theorem 3.2.11]. The notation here is adapted to the rest of the thesis. □

Corollary E.2. *Let $M \subseteq \mathbb{R}^d$ be compact and measurable, $f \in C_0^\infty(M)$, and, for $t \in \mathbb{R}$, define the set $A_t \coloneqq \{x \in M \mid f(x) > t\}$. Then*

$$\int_M |\nabla f| = \int_{-\infty}^{\infty} \ell_{d-1}(\partial A_t)dt,$$

where the boundary ∂A_t is taken within M.

Proof. This is also published in the appendix of [DFJK23]. One can show that for almost all $t \in \mathbb{R}$ we have

$$M \cap f^{-1}(y) = \partial A_t(M).$$

To see this, first note that by continuity of f, the inclusion $\partial A_t \subseteq M \cap f^{-1}(t)$ holds for all $t \in \mathbb{R}$. On the other hand, if $x \in M$ and $f(x) \neq 0 \neq \nabla f(x)$, then $x \in \mathring{M}$ and $f(x + s\nabla f)$ yields an element of $M \cap A_t$ for small positive s and an element of $M \cap (\mathbb{R}^d \backslash A_t)$ for small negative s. Hence $x \in \partial A_t$.

Now Sard's theorem guarantees if $X \subset M$ is the set of points in which ∇f vanishes, then $f(X)$ has measure zero. Hence the above argument can be applied to almost all $t \in \mathbb{R}$ (as $\{0\}$ is also a null set) and we can conclude that

$$\int_M |\nabla f| = \int_{-\infty}^{\infty} \ell_{d-1}(\partial A_t) = \int_{-\infty}^{\infty} \ell_{d-1}(M \cap f^{-1}(y))dy,$$

which shows the claim. ☐

Theorem E.3 (Coarea formula for functions of bounded variation)**.** *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in BV(\Omega)$ and*

$$A_t \coloneqq \{x \in M \mid u(x) > t\},$$

then for almost all $t \in \mathbb{R}$ the set A_t has finite perimeter in Ω and

$$|Du|(\Omega) = \int_{-\infty}^{\infty} P(A_t, \Omega) dt.$$

Proof. See e.g. [Amb00, Theorem 3.40]) ☐

Theorem E.4. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2, and let $\tilde{T} : \mathbb{R}^d \to \mathbb{R}^d$ be the extension of T to the identity outside of M like in Theorem D.3. Let further $u \in BV(M)$ and define $\tilde{u}$ to be the zero-extension of u to $\mathbb{R}^d$, i.e., $\tilde{u}|_M = u$ and $\tilde{u}|_{\mathbb{R}\setminus M} \equiv 0$. Define further*

$$\tilde{A}_t \coloneqq \{x \in \mathbb{R} \mid \tilde{u}(x) > t\}$$

for every $t \in \mathbb{R}$. Then $T(\tilde{A}_t)$ is of finite perimeter in $\mathbb{R}^d$ for almost all $t \in \mathbb{R}^d$, and

$$|D(T_* u)|(\mathbb{R}^d) = \int_{-\infty}^{\infty} P(\tilde{T}(\tilde{A}_t), \mathbb{R}^d).$$

Proof. The function $\tilde{u}$ is a zero-extension of $u \in BV(M)$, and because M has Lipschitz boundary, this means that $\tilde{u}$ is in $BV(M)$ by [Eva18, Theorem 5.8]. By Theorem D.3 the function $\tilde{T}_* \tilde{u}$ is in $BV(M)$, too. We may thus apply the coarea formula from Theorem E.3 above to $\tilde{T}_* \tilde{u}$ on $\Omega = \mathbb{R}^d$, which shows that for almost all t, the set

$$\{x \in \mathbb{R}^d \mid (\tilde{T}_* u)(x) > t\}$$

is of finite perimeter. This set can be rewritten to

$$\{x \mid x \in \mathbb{R}^d, u(\tilde{T}^{-1}(x)) > t\} = \{\tilde{T}(x) \mid x \in \mathbb{R}^d, u(x) > t\}, = \tilde{T}(\tilde{A}_t)$$

and thus the coarea formula yields

$$|D(T_* u)|(\mathbb{R}^d) = \int_{-\infty}^{\infty} P(\tilde{T}(\tilde{A}_t), \mathbb{R}^d),$$

which was the claim. ☐

Appendix F

Technical proofs

This appendix contains postponed proofs from the rest of the document.

Proposition F.1. *Let $u \in C_0^\infty(M)$. Then $|u| \in W_0^{1,1}(M)$ and*

$$\|\nabla u\|_1 = \|\nabla |u|\|_1.$$

Proof. We expand the proof of [Cha01, Lemma II.2.1]: Define $V \in L^1(M, \mathbb{R}^d)$ by

$$V(x) := \begin{cases} \operatorname{sgn}(u(x))\nabla u(x) & u(x) \neq 0 \\ 0 & u(x) = 0 \end{cases}$$

and for $\varepsilon > 0$ let $u_\varepsilon := \sqrt{u^2 + \varepsilon^2}$. Then $\nabla u_\varepsilon(x) = 0$ if $u(x) = 0$ and for $u(x) \neq 0$ and $\varepsilon \to 0$:

$$\nabla u_\varepsilon(x) = \nabla u \frac{u(x)}{\sqrt{u(x)^2 + \varepsilon^2}} = \nabla u(x) \frac{\operatorname{sgn}(u(x))}{\sqrt{1 + \frac{\varepsilon^2}{u(x)^2}}} \xrightarrow{\leq} V(x),$$

hence $\nabla u_\varepsilon \to V$ in $L^1(M, \mathbb{R}^d)$ by dominated convergence. But $u_\varepsilon \to |u|$ in L^1 as well and thus if $v \in C_0^\infty(M, \mathbb{R}^d)$ we can exchange limits and integrals twice in:

$$\int_M V \cdot v = \int_M \left(\lim_{\varepsilon \to 0} \nabla u_\varepsilon\right) \cdot v = \lim_{\varepsilon \to 0} \int_M (\nabla u_\varepsilon) \cdot v \tag{F.1}$$

$$= -\lim_{\varepsilon \to 0} \int_M u_\varepsilon \operatorname{div}(v) = -\int_M (\lim_{\varepsilon \to 0} u_\varepsilon) \operatorname{div}(v) \tag{F.2}$$

$$= -\int_M |u| \operatorname{div}(v). \tag{F.3}$$

Hence V is a weak derivative of $|u|$. Finally, note that $|V(x)| = |\nabla u(x)|$ if $u(x) \neq 0$ or $\nabla u(x) = 0$. Additionally, the set where $u(x) = 0$ and $\nabla u(x) \neq 0$ is is a $(d-1)$-dimensional submanifold and thereby a null set, so altogether $\|V\|_1 = \|\nabla u\|_1$. ☐

Proposition F.2. *Let $f \in C_0^\infty(M)$. Then there is a sequence $f_n \subset C_0^\infty(M)$ of nonnegative functions converging to $|f|$ in $W_0^{1,1}(M)$.*

Proof. By mollifiying one can get a sequence in $C^\infty(M)$ converging to $|f|$. This sequence eventually ends up in $C_0^\infty(M)$, as $|f|$ has compact support in M. ☐

Lemma F.3. *Let $p_n \to 1$ be a sequence in $\mathbb{R}$ and assume that $u_n \subset L^1(M)$ converges in $L^1(M)$ to $u \in L^1(M)$ for $n \to \infty$. If there is a $c > 0$ such that $\|u_n\|_\infty < c$ for all n, then*

$$\|u_n\|_{p_n}^{p_n} \to \|u\|_1.$$

Proof. See [Par09, Lemma 2.31] $\square$

Theorem F.4. *Let $u \in C^2(\mathbb{R})$ and $\Delta_{p,h}$ defined as*

$$(\Delta_{p,h}u)(x) \coloneqq \frac{1}{h}\left(\varphi_{p-1}\left(\frac{u(x+h)-u(x)}{h}\right) - \varphi_{p-1}\left(\frac{u(x)-u(x-h)}{h}\right)\right),$$

where $\varphi_{p-1}(x) = |x|^{p-2}x$. Then the following holds:

(i) For fixed $x \in \mathbb{R}$ with $u'(x) \neq 0$ we have

$$\Delta_{p,h}u(x) = \Delta_p u(x) + \mathcal{O}(h)$$

for $h \overset{>}{\to} 0$.

(ii) If $u(x+h) = u(x) + C|x|^q + \mathcal{O}(h^{2q})$ then

$$\Delta_p u(x) = 2\varphi_{p-1}(C) + \mathcal{O}(h^q).$$

In particular, if $\Delta_p u(0) \neq 2\varphi_{p-1}(C)$, then the scheme is not consistent.

Proof. By Taylor expansion for $u(x \pm h)$, the expression for $\Delta_{p,h}u(x)$ expands to

$$\frac{1}{h}\left(\varphi_{p-1}\left(u'(x) + \frac{1}{2}u''(x)h + \mathcal{O}(h^2)\right) - \varphi_{p-1}\left(u'(x) - \frac{1}{2}u''(x)h + \mathcal{O}(h^2)\right)\right), \tag{F.4}$$

which, for $u'(x) \neq 0$ can be further expanded using two Taylor expansions of φ_{p-1} around $u'(x)$:

$$\begin{aligned}\frac{1}{h}\Big(&\varphi_{p-1}(u'(x)) + \varphi'_{p-1}(u'(x))\Big(\frac{1}{2}u''(x)h + \mathcal{O}(h^2)\Big) + \mathcal{O}(h^2) - \\ &\varphi_{p-1}(u'(x)) - \varphi'_{p-1}(u'(x))\Big(-\frac{1}{2}u''(x)h + \mathcal{O}(h^2)\Big) + \mathcal{O}(h^2)\Big).\end{aligned} \tag{F.5}$$

Simplifying leaves us with

$$\Delta_{p,h}u(x) = \varphi_{p-1}(u'(x))u''(x) + \mathcal{O}(h).$$

Now note that if $u'(x) \neq 0$ then $\varphi_p(u')$ is differentiable in x and $\Delta_p u(x) = (\varphi_{p-1}(u'))'(x) = u''\varphi_{p-1}(u'(x))$, as φ_{p-1} is differentiable for nonzero arguments. This shows the claim (i).

For claim (ii) we plug in the expansion for u and get:

$$\Delta_{p,h}u(x) = \frac{1}{h}\left(\varphi_{p-1}\left(Ch^{q-1} + \mathcal{O}(h^{2q-1})\right) - \varphi_{p-1}(-Ch^{q-1} + \mathcal{O}(h^{2q-1}))\right) \tag{F.6}$$

$$= \frac{1}{h}\left(\varphi_{p-1}\left(Ch^{q-1}(1 + \mathcal{O}(h^{q}))\right) + \varphi_{p-1}(Ch^{q-1}(1 + \mathcal{O}(h^{q})))\right) \tag{F.7}$$

$$= \frac{2}{h}\varphi_{p-1}(C)h^{(p-1)(q-1)}(1 + \mathcal{O}(h^{q})) \tag{F.8}$$

$$= 2\varphi_{p-1}(C) + \mathcal{O}(h^{q}) \tag{F.9}$$

$$\tag{F.10}$$

which proofs the second case. □

Lemma F.5. *Let $f : \mathbb{R} \supseteq I \to \mathbb{R}$ be continuous, bounded from below by some $C > 0$ and Lipschitz outside of a closed set N of isolated points. then solutions of the initial value problem*

$$w' = f(w) \tag{F.11}$$

$$w(0) = w_0 \tag{F.12}$$

are unique.

Proof. Figure F.1 shows a sketch of the proof. We show that solutions coincide for $t > 0$. The claim for $t < 0$ is obtained analogously. Let first $T \coloneqq \inf\{t \geq 0 : w(t) \neq \bar{w}(t)\}$. If $T = \infty$ we are done. If $T < \infty$ then by continuity $w(T) = \bar{w}(T)$ and thus there is a $t_1 > T$ such that $w(t_1) \neq \bar{w}(t_1)$. Assume without loss of generality that $w(t_1) < \bar{w}(t_1)$. Now we must have $T \in N$, as otherwise Picard-Lindelöf would give us uniqueness for a small neighbourhood of T, contradicting the definition of T. Thus, by taking t_1 small enough, we can also ensure that there are no points of N in the trajectories $w((T, t_1)$ and $\bar{w}((T, t_1))$, as $\mathbb{R}\backslash N$ is assumed to be open.

By continuity of $\bar{w}$ there now is a $t_2 < t_1$ with $\bar{w}(t_2) = w(t_1)$. The functions $\bar{w}$ and $\hat{w}(t) \coloneqq w(t + (t_1 - t_2))$ both fulfill Equation (F.11) while evaluating to $w(t_1)$ at $t = t_2$. As there are no points of N on $\bar{w}((T, t_2)) \subseteq \bar{w}((T, t_1))$, the solutions to this initial value problem are unique at least back to $t = T$ by Picard-Lindelöf, so by continuity it must thus hold that $\hat{w}(T) = \bar{w}(T)$. But this is a contradiction, as it must also hold $\hat{w}(T) > w(T)$:

$$\hat{w}(T) = w(T + (t_1 - t_2)) = w(T) + \underbrace{\int_{T}^{T+t_1-t_2} w'(t)dt}_{\geq C(t_1-t_2)>0} > w(T)$$

This shows that $T < \infty$ cannot occur and hence $w(t) = \bar{w}(t)$ for all $t > 0$.

□

Lemma F.6. *Let $\gamma : \mathbb{R} \to S^1$ be injective. Then $\gamma(t)$ converges for $t \to \infty$.*

Proof. Let $\gamma^* : \mathbb{R} \to \mathbb{R}$ be a lift of γ to the universal covering of S^1. Then γ^* must be monotonous due to local injectivity of γ. By injectivity of γ, γ^* is also bounded and thus it converges for $t \to \infty$. This means that $\gamma(t)$ also converges, as the covering is a local homeomorphism. □

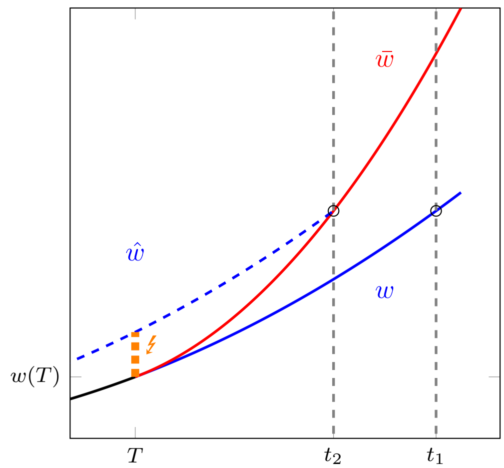

Figure F.1: Sketch for the proof of Lemma F.5 From $f > C$ it follows that $\hat{w}(T) > w(T)$ But as we may assume that there are no points of N between T and t_1, $\hat{w}$ must also coincide with $\bar{w}$ between T and t_2, which is a contradiction.

Definition F.7 (After [Sim72]). *Let $M \subset \mathbb{R}^d$ be like in Setup 1.1.2. For some matrix-valued function $A : M \to R^{d\times d}$, $A(x) = (a_{i,j}(x))_{i,j}$, a bilinear form of the form*

$$B(\varphi, \psi) := \int_M (A\nabla\varphi)\nabla\psi$$

is called a uniformly strongly elliptic Dirichlet bilinear form in M of order 2 *if*

(i) *There exists a constant $E > 0$ such that*

$$l^T A(x) l \geq E|l|^2$$

for all $x \in M$ and $l \in \mathbb{R}^d$.

(ii) *It fullfills the* root condition, *i.e., for every $l' \in \mathbb{R}^{d-1} \setminus \{0\}$ and $x \in M$ the following polynomial in τ:*

$$P(\tau, l', x) := \begin{pmatrix} l' \\ \tau \end{pmatrix}^T A(x) \begin{pmatrix} l' \\ \tau \end{pmatrix}$$

has exactly one root with positive imaginary part and one root with negative imaginary part.

Remark F.8. *Differing from [Sim72, Definitions 1.3, 1.4], we have restricted the definition to order 2 and real coefficients. We also adapted the notation and joined the prerequisites of ellipticity and strong ellipticity, which simplify to (i) in the real case. We have also added the condition $l' \neq 0$ in the root condition (otherwise the root condition can never be satisfied, so we believe the condition $l' \neq 0$ is missing in [Sim72]).*

Proposition F.9. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Then the bilinear form*

$$B(\varphi, \psi) := -\int_M \left(\frac{1}{2}\left(I + DT^{-1}DT^{-T}\right)\nabla\varphi\right)\nabla\psi$$

is a uniformly strongly elliptic Dirichlet bilinear form in M of order 2.

Proof. The given B is a bilinear form in the form of Definition F.7 with

$$A(x) = \frac{1}{2}\left(I + DT^{-1}DT^{-T}\right).$$

For (i) observe that

$$l^T A(x) l = \frac{1}{2}(|l|^2 + |DT^{-T}l|^2) > \frac{1}{2}|l|^2.$$

For (ii) note that for $l \in 0$ we have $(l, \tau) \neq 0$ for all τ. As $\frac{1}{2}(I + DT^{-1}DT^{-T})$ is symmetric and positive definite, this means that

$$(l', \tau)^T A(x) \begin{pmatrix} l' \\ \tau \end{pmatrix} \neq 0$$

for all $\tau \in \mathbb{R}$. As a quadratic polynomial with no real roots has exactly one root in the upper complex plane and one in the lower complex plane, the root condition is fulfilled. □

Theorem F.10. *Let $M \subset \mathbb{R}^d$ be a compact, d-dimensional submanifold with C^1 boundary and $T : M \to M$ a volume-preserving diffeomorphism on the interior $\mathring{M}$ such that and T as well as T^{-1} are be Lipschitz continuous on M. Define $J^D := F^D/G$ with $F^D(u) := \frac{1}{2}\left(\|\nabla u\|_p^p + \|\nabla T_* u\|_p^p\right)$, as in Theorem 2.1.3 and $G(u) := \|u\|_p^p$. If $\Delta^D := \frac{1}{2}(\Delta + T^* \Delta T_*)$ is the dynamic Laplacian, then the equation*

$$-\Delta^D d^D = (J^D)'(u)$$

has a unique weak solution in $W_0^{1,q}(M)$ in the following sense: there exists a unique $d^D \in W_0^{1,q}(M)$ such that for all $v \in W_0^{1,p}(M)$.

$$\int_M \frac{1}{2}\left(\left(I + DT^{-1}DT^{-T}\right)\nabla d^D\right)\nabla v = \langle (J^D)'(u), v\rangle.$$

This includes the case $\Delta^D = \Delta$, which arises from $T = id$.

Proof. This is a direct application of the Fredholm Alternative in [Sim72, Theorem 7.5] (note that we flip p and q compared to the notation there). The theorem assumes a uniformly strongly elliptic Dirichlet bilinear form B on $W_0^{1,q}(M) \times W_0^{1,p}(M)$ (see Definition F.7) and some $F \in W^{-1,q}(M)$ that vanishes on the space N_p defined by

$$N_p := \{v \in W_0^{1,q}(M) \mid B(\varphi, \psi) = 0 \text{ for all } \psi \in W_0^{1,p}(M)\}.$$

The theorem says that with this setup, there is a $\varphi \in W_0^{1,q}(M)$ such that

$$B(\varphi, \psi) = F(\psi) \quad \forall \psi \in W_0^{1,p}(M) \tag{F.13}$$

is fulfilled. It also says that φ is unique if $\dim N_p = 0$.

To apply the theorem, we set $F := J'(u)$ and

$$B(\varphi, \psi) := \int_M \frac{1}{2}\left(\left(I + DT^{-1}DT^{-T}\right)\nabla\varphi\right)\nabla\psi,$$

which is a uniformly strongly elliptic Dirichlet bilinear form by Proposition F.9. As the matrix field $\frac{1}{2}(I + DT^{-1}DT^{-T})$ is positive definite everywhere in M, we have that $\nabla\psi \equiv 0$ if $\psi \in N_p$. By the Poincaré inequality this means $\psi = 0$, so $\dim N_p = 0$. Hence F trivially vanishes on N_p and there exists a unique φ fulfilling Equation (F.13), which proves the claim. □

Theorem F.11. *Let $M \subset \mathbb{R}^d$ and $T : M \to M$ be as in Setup 1.1.2. Then*

$$h^D(M,T) = \inf_{D \subset M} \frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)},$$

where the infimum is taken over d-dimensional submanifolds of M that are compactly contained in M and have smooth boundary. Secondly,

$$h^D(M,T) = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1}.$$

Proof. The equality of the two right-hand sides has been shown in [FJ18, Theorem 1]. Here, we connect both quantities to the formulation in Definition 2.2.1, using techniques similar to [Par11, proof of Proposition 3.3] and, for the incorporation of dynamics, to [Fro15, proof of Theorem 3.1].

For the first equality, first note that the left-hand side is smaller or equal than the right-hand side, since the domain of the infimum is made smaller and $\ell_{d-1}(\partial D) = P(D, \mathbb{R}^d)$ for sets D with smooth boundary (see Theorem C.5). To prove "$\geq$", let $D \subseteq M$ be of finite perimeter. By Theorem C.6 there exists a sequence of D_k with smooth boundary such that

$$\chi_{D_k} \xrightarrow{L^1} \chi_D$$

and

$$P(D_k, \mathbb{R}^d) \to P(D, \mathbb{R}^d)$$

for $k \to \infty$. We define the operator $\tilde{T}_* : BV(\mathbb{R}^d)$ as in Theorem D.3 and note that because of its boundedness

$$\tilde{T}_* \chi_{D_k} \to \tilde{T}_* \chi_D$$

in $BV(\mathbb{R}^d)$ and hence

$$|D(\tilde{T}_* \chi_{D_k})|_{\mathbb{R}^d} \to |D(\tilde{T}_* \chi_D)|_{\mathbb{R}^d}$$

As $\tilde{T}_* \chi_{D_k} = \chi_{T(D_k)}$ and $\tilde{T}_* \chi_D = \chi_{T(D)}$ we get

$$P(D_k, \mathbb{R}^d) = |D(\chi_{T(D_k)})|_{\mathbb{R}^d} \to |D(\chi_{T(D)})|_{\mathbb{R}^d} = P(D, \mathbb{R}^d).$$

This shows

$$h^D(M,T) \geq \inf_{D \subset M} \frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)}.$$

and we are done with the first equality.

For the second equality we want to show

$$h^D(M,T) = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1},$$

where the infimum is taken over d-dimensional submanifolds of M that do not touch its boundary. Let u in $C_0^\infty(M)$ be nonnegative. By the coarea formula applied to u and $T_* u$ we have

$$\begin{aligned} \frac{1}{2}\left(\|\nabla u\|_1 + \|\nabla T_* u\|_1\right) &= \frac{1}{2}\int_0^\infty \left(\ell_{d-1}(\partial A_t) + \ell_{d-1}(\partial(T(A_t)))\right) dt && \text{(F.14)} \\ &\geq h^D(M,T) \int_0^\infty \ell_d(A_t) dt && \text{(F.15)} \\ &= h^D(M,T)\|u\|_1, && \text{(F.16)} \end{aligned}$$

where the last step is Cavalieris principle. This holds also for arbitrary $u \in C_0^\infty(M)$, as we can approach $|u|$ in $W_0^{1,1}(M)$ by smooth nonnegative functions by Proposition F.2 and $\|\nabla|u|\|_1 = \|\nabla u\|_1$ by Proposition F.1. Thus, we have shown that

$$h^D(M,T) \leq \inf_{\substack{u \in C_0^\infty(M) \\ u \neq 0}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1}.$$

For the other direction "$\geq$", let $\varepsilon > 0$ and $D \subseteq M$ be a d-dimensional submanifold with boundary not touching ∂M and $\frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)} - h^D(M,T) < \varepsilon$. By [Amb00, Theorem 3.9] and continuity of T_* on $BV(M)$ there is a sequence $u_k \subset C_0^\infty(M)$ such that

$$\frac{\|\nabla u_k\|_1 + \|\nabla(T_* u_k)\|_1}{2\|u_k\|_1} \overset{k\to\infty}{\longrightarrow} \frac{\ell_{d-1}(\partial D) + \ell_{d-1}(\partial(T(D)))}{2\ell_d(D)}.$$

By letting $\varepsilon \to 0$ and using that for smooth domains not touching the boundary we have $\ell_{d-1}(\partial D) = P(D, \mathbb{R}^d)$ (see Theorem C.5), this implies

$$h^D(M,T) \geq \inf_{\substack{u \in C_0^\infty(M) \\ u \neq 0}} \frac{\|\nabla u\|_1 + \|\nabla(T_* u)\|_1}{2\|u\|_1},$$

which completes the proof of the second equality.

$\square$

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