Robust H_∞ Consensus for Homogeneous Multi-agent Systems with Parametric Uncertainties

Alexandre Capone[1], Junjie Jiao[1], Mostafa Zarei[2], Shiqi Zhang[3], and Sandra Hirche[1]

Abstract— **This paper addresses the problem of robust consensus of an undirected network of homogeneous multi-agent systems with uncertain agent dynamics and system noise. We consider uncertain time-varying input matrices that are arbitrary up to a known bound for the singular values. We also assume that each agent's controller is able to access the neighbors' relative states. We focus on the design of a linear controller gain that is to be identical across all agents. We provide sufficient conditions for the control gains to achieve both consensus in the noiseless setting and a transfer function with a given bounded H_∞ norm in the setting with noise. More specifically, we show that this is achieved if a set of linear matrix inequalities containing the non-zero eigenvalues of the Laplacian are satisfied. In a numerical simulation, we illustrate these theoretical results and show that our method outperforms a consensus region-based approach.**

I. Introduction

Over the last two decades, the design of distributed controllers for achieving the synchronization of multi-agent systems has received much attention from the control community, owing to their broad scope of applications. These include satellite formation flying [1], sensor networks [2], and smart grids [3]. However, many practical challenges still exist, particularly due to the presence of uncertainties that are typically not accounted for in the control design. To this date, a few different approaches have been developed that aim to obtain robust synchronization in the presence of uncertainties. In [4] and [5], robust synchronization methods were developed for systems with coprime factor uncertainties and additive uncertainties on the agent dynamics, respectively. The works of [6] and [7] addressed robust synchronization for uncertain networks. A significant drawback of the aforementioned works is that they only consider uncertainties in the frequency domain, i.e., perturbations in the transfer matrices of each agent. This is restrictive, as it assumes that the number of unstable poles of the open loop system is fixed [8], which might not apply, e.g., in the case of additive or multiplicative perturbations. A more general class of system uncertainties are those that are described in the time domain. These types of uncertainties, often referred to as parametric uncertainties, are widely encountered whenever the system matrices of the plants are known up to a bounded time-dependent uncertainty with known bound [9]. For multi-agent systems under parametric uncertainties, robust synchronization is still an open problem. Another factor that makes the synchronization problem difficult is the presence of process noise, which typically leads to non-zero synchronization errors. This is often addressed by designing the controller in such a way that the synchronization errors stay below a predefined value in the presence of process noise. The paper [10] considered the problem of synchronization of a multi-agent system in the presence of parametric uncertainties and input noise by considering its H_2 norm. In spite of the aforementioned advances, designing a controller that achieves a bounded H_∞ norm while guaranteeing synchronization in the noiseless setting remains an open problem.

In this paper, we address the synchronization of a multi-agent system that exhibits bounded parametric uncertainties in the input matrices as well as process noise. We show that, provided that a set of linear matrix inequalities is satisfied, our approach reaches consensus in the noiseless setting while also achieving a bounded H_∞ norm in the noisy setting.

The remainder of this paper is organized as follows: In Section II, we state the notation and some preliminaries on graph theory. In Section III we formally formulate the problem considered in this paper. The main result of this paper is then stated in Section IV. In Section V, we then illustrate our results and compare our approach to a state-of-the-art method using a numerical simulation.

II. Preliminaries

A. Notation

Throughout this paper, we assume $\mathbb{R}^{n\times m}$ to be the set of $n \times m$ real matrices. The n-dimensional vector with all entries equal to one is denoted by $\mathbf{1}_n \in \mathbb{R}^n$. The identity matrix with dimension n is denoted by I_n. The superscript $\top$ represents the transpose of a real matrix. For two symmetric matrices P and P', we employ the notation $P > (\geq) P'$ to denote that $P - P'$ is positive (semi-)definite. The maximum singular value of a matrix G is denoted by $\bar{\sigma}(G)$. A matrix is called Hurwitz if all its eigenvalues have negative real parts. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. We employ $\text{diag}\{a_1, a_2, \ldots, a_n\}$ to denote the $n \times n$ diagonal matrix with $a_1, a_2, \ldots, a_n$ as diagonal

This work was supported by the Federal Ministry of Education and Research of Germany (BMBF) as part of the project 6G-life (project identification number 16KISK002), and by the European Union's Horizon 2020 Research and Innovation Programme through the Marie Skłodowska-Curie under Grant 899987.

[1]A. Capone, J. Jiao, and S. Hirche are with the Chair of Information-oriented Control, TUM School of Computation, Information and Technology, Technical University of Munich, Munich, Germany. Email: alexandre.capone@tum.de; junjie.jiao@tum.de; hirche@tum.de

[2]M. Zarei is with the Department of Mechanical Engineering, Politecnico di Milano, Milan, Italy mostafa.zarei@mail.polimi.it

[3]S. Zhang is with the Department of Electrical Engineering, City University of Hong Kong, Hong Kong SAR, China zsqpkuedu@pku.edu.cn

entries. Similarly, for quadratic matrices $A_1,\dots,A_n$, we use $\text{blkdiag}(A_1,\dots,A_n)$ to denote the block diagonal matrix with $A_1,\dots,A_n$ on the diagonal and zeros everywhere else. We employ Im(A) and Ker(A) to denote the image and kernel of a matrix A, respectively. For a stable proper transfer matrix $G(s)\in\mathbb{R}^{n\times m}$, its infinite norm is denoted by $\|G(s)\|_\infty = \sup_{\omega}\{\bar{\sigma}(G(j\omega))\}$.

B. Graph

A graph consists of a pair $\mathscr{G}=(\mathscr{V},\mathscr{E})$, where $\mathscr{V}=\{1,2,\dots,N\}$ is the set of nodes and $\mathscr{E}\subset\mathscr{V}\times\mathscr{V}$ is the set of edges. For any two nodes $i,j\in\mathscr{V}$ with $i\neq j$, an edge from i to j is represented by the pair $(i,j)\in\mathscr{E}$. A graph with the property that $(i,j)\in\mathscr{E}$ implies $(j,i)\in\mathscr{E}$ is called undirected. The adjacency matrix A of a graph has entries $a_{ij}=1$ if there exists an edge from ith node to the jth node, and $a_{ij}=0$ otherwise. In addition, the Laplacian matrix L of a graph has entries $l_{ii}=\sum_{j=1}^{N}a_{ij}$ and $l_{ij}=-a_{ij}$ for $i\neq j$. If the graph is undirected, L is a positive semi-definite real symmetric matrix, i.e., all its eigenvalues are non-negative and real. An undirected graph is called connected if and only if its Laplacian L has rank $N-1$. In this case the zero eigenvalue of L has multiplicity one, i.e., all eigenvalues except one are strictly positive.The matrix of eigenvectors U of the Laplacian L of a connected undirected graph can be partitioned as $U=[\frac{1}{N}\mathbf{1}_N,V]$, where the eigenvector $\frac{1}{N}\mathbf{1}_N$ corresponds to the zero eigenvalue of L. Furthermore, L can be diagonalized as $\Lambda=U^TLU$ where $\Lambda=\text{diag}\{0,\lambda_2,\dots,\lambda_N\}$ and $0<\lambda_2\leq\cdots\leq\lambda_N$.

III. Problem Formulation

We consider a homogeneous linear multi-agent system consisting of N identical agents, whose input matrices are subjected to time-dependent parametric uncertainties. The dynamics of the i-th agent are given by

$$\dot{x}_i = Ax_i + (B+B_{\Delta_i})u_i + B_w w_i, \tag{1}$$

where $x_i\in\mathbb{R}^n$ is the agent state, $u_i\in\mathbb{R}^m$ is the control input, $w_i\in\mathbb{R}^q$ is an external disturbance, and the matrices A, B, B_{Δ_i}, and B_w are of suitable dimensions. The matrix B_{Δ_i} is an unknown matrix with bounded time-varying entries representing the parametric uncertainty associated with the input matrix B of the i-th agent. We assume that B_{Δ_i} is of the form of

$$B_{\Delta_i} = D\Delta_i E \tag{2}$$

where $D\in\mathbb{R}^{n\times r}$ and $E\in\mathbb{R}^{k\times m}$ are known constant matrices that characterize the structure of the uncertainties, and the entries of $\Delta_i\in\mathbb{R}^{r\times k}$ are time-dependent and Lebesgue measurable [11]. We assume the time-varying matrices Δ_i to be bounded in the sense that

$$\Delta_i^\top\Delta_i \leq \delta^2 I_k \tag{3}$$

holds at all times for some known and constant $\delta>0$. This assumption is not very restrictive, since it corresponds to a very rich space of matrices, and model uncertainties are often bounded in practice.

As is common in the context of distributed control of multi-agent systems, we are interested in the differences between the states of the agents [12]. Hence, for the i-th agent, we consider the performance output

$$z_i = \frac{1}{N}\sum_{j=1}^{N} C(x_i - x_j) \tag{4}$$

where $C\in\mathbb{R}^{p\times n}$ is a given constant matrix.

In this paper, we assume that the states of the agents can be directly measured and always available for control. We then consider a distributed state-feedback control law

$$u_i = K\sum_{j=1}^{N} a_{ij}(x_i - x_j), \tag{5}$$

where $K\in\mathbb{R}^{m\times n}$ is the feedback gain matrix to be designed and a_{ij} is the ij-th entry of the adjacency matrix $\mathscr{A}$ of the communication graph between the agents. Throughout this paper, we assume that the communication graph is a connected undirected graph and use L to refer to the corresponding Laplacian matrix.

For simplicity of exposition, we introduce the notation $\mathbf{x}:=(x_1^\top,x_2^\top,...,x_N^\top)^\top$, $\mathbf{w}:=(x_1^\top,x_2^\top,...,x_N^\top)^\top$, $\mathbf{z}:=(z_1^\top,z_2^\top,...,z_N^\top)^\top$, $\Delta:=\text{blkdiag}(\Delta_1,\dots,\Delta_N)$. By substituting the distributed controller (5) into the system (1), we obtain the compact form for the full closed-loop multi-agent system equations

$$\begin{aligned}\dot{\mathbf{x}} &= A_o\mathbf{x} + B_o\mathbf{w}\\ \mathbf{z} &= C_o\mathbf{x},\end{aligned} \tag{6}$$

where

$$\begin{aligned}&A_o := I_N\otimes A + L\otimes BK + (I_N\otimes D)\Delta(L\otimes EK),\\ &B_o := I_N\otimes B_w, \qquad C_o := M\otimes C,\end{aligned}$$

and

$$M := I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top.$$

The transfer function matrix of the controlled network (6) from $\mathbf{w}$ to $\mathbf{z}$ is then equal to

$$T_{\mathbf{wz}} = C_o(sI - A_o)^{-1}B_o. \tag{7}$$

The goal of this paper is to design a distributed controller of the form (5) such that the agents (1) reach robust H_∞ consensus, as defined in the following.

Definition 1: The distributed controller (5) is said to achieve robust H_∞ consensus with tolerance γ for the multi-agent system (1) if

i) provided that $w=0$ for all $t>0$, the states of the agents (1) achieve consensus for all parametric uncertainties satisfying (3), i.e., $(x_i-x_j)\to 0$ as $t\to\infty$, for $i=1,2,\dots,N$.
ii) the H_∞ norm of the transfer function (7) is less than γ, i.e., $||T_{\mathbf{wz}}||_\infty<\gamma$.

The problem that we address in this paper is then the following.

Problem 1: Let $\gamma>0$ be a given tolerance. Design a local gain K such that the distributed controller (5) achieves robust H_∞ consensus with tolerance γ for the multi-agent system (1).

IV. Robust Synchronization

In this section, we derive our main result, which provides sufficient conditions for solving the robust H_∞ consensus problem given a pre-defined tolerance parameter γ. To this end, we begin by presenting some preliminary results that will be required in the following. Our main result can be found at the end of this section.

We begin by showing that solving Problem 1 corresponds to solving a robust H_∞ stabilization problem for a $n(N-1)$-dimensional system, obtained after applying a transformation to the original system (1). To this end, we will employ the following result.

Lemma 1: Let the communication graph of system (1) be connected and undirected, and consider the the system

$$\begin{aligned}\dot{\bar{\mathbf{x}}} &= \Big(I_{N-1}\otimes A + \bar{\Lambda}\otimes BK + (I_{N-1}\otimes D)\bar{\Delta}(\bar{\Lambda}\otimes EK)\Big)\bar{\mathbf{x}} \\ &\quad + (V^\top \otimes B_w)\mathbf{w}, \\ \mathbf{z} &= (V\otimes C)\bar{\mathbf{x}},\end{aligned} \tag{8}$$

Then local controllers (5) with control gain K achieve robust H_∞ consensus with tolerance γ if and only if the following conditions hold.

- The system (8) is robustly stable, i.e., $\lim_{t\to\infty}\|\bar{\mathbf{x}}\|_2 = 0$, for $\mathbf{w}=\mathbf{0}$ and all $\bar{\Delta}$ with $\bar{\Delta}^\top\bar{\Delta}\le\delta^2 I_{(N-1)k}$.
- The H_∞ norm of the transfer function from $\mathbf{w}$ to $\mathbf{z}$ of (8) is smaller or equal to γ for all $\bar{\Delta}$ with $\bar{\Delta}^\top\bar{\Delta}\le\delta^2 I_{(N-1)k}$.

To prove Lemma 1, we will employ the following result.

Lemma 2: Let $\Delta\in\mathbb{R}^{M\times N}$, where $M,N\in\mathbb{N}$, be a quadratic matrix with $\Delta^\top\Delta\le\delta^2 I_N$, and let $\tilde{\Delta}$ denote the submatrix of Δ obtained by deleting the first column and the first row of $\tilde{\Delta}$. Then $\tilde{\Delta}^\top\tilde{\Delta}\le\delta^2 I_{N-1}$ holds.

Proof: Consider the decomposition

$$\Delta = \begin{pmatrix}\Delta_{11} & \Delta_{1,2:N}\\ \Delta_{2:M,1} & \tilde{\Delta}\end{pmatrix},$$

where Δ_{11} denotes the entry in the first column and first row of Δ, $\Delta_{1,2:N}$ the vector corresponding to the first row and columns 2 through N, and $\Delta_{2:M,1}$ the vector corresponding to the first column and rows 2 through M. Furthermore, define $x_{2:N} := (x_2,\dots,x_N)$. We then have

$$\begin{aligned} x_{2:N}^\top \begin{pmatrix}\Delta_{1,2:N}\\ \tilde{\Delta}\end{pmatrix}^\top \begin{pmatrix}\Delta_{1,2:N}\\ \tilde{\Delta}\end{pmatrix} x_{2:N} &= \begin{pmatrix}0\\ x_{2:N}\end{pmatrix}^\top \Delta^\top\Delta \begin{pmatrix}0\\ x_{2:N}\end{pmatrix} \\ \le \begin{pmatrix}0\\ x_{2:N}\end{pmatrix}^\top \delta^2 I_N \begin{pmatrix}0\\ x_{2:N}\end{pmatrix} &= \delta^2 x_{2:N}^\top x_{2:N} = x_{2:N}^\top \delta^2 I_{N-1} x_{2:N}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned} x_{2:N}^\top\tilde{\Delta}^\top\tilde{\Delta}x_{2:N} &\le x_{2:N}^\top\tilde{\Delta}^\top\tilde{\Delta}x_{2:N} + x_{2:N}^\top\Delta_{1,2:N}^\top\Delta_{1,2:N}x_{2:N} \\ &= x_{2:N}^\top \begin{pmatrix}\Delta_{1,2:N}\\ \tilde{\Delta}\end{pmatrix}^\top \begin{pmatrix}\Delta_{1,2:N}\\ \tilde{\Delta}\end{pmatrix} x_{2:N}.\end{aligned}$$

Hence, $x_{2:N}^\top\tilde{\Delta}^\top\tilde{\Delta}x_{2:N}\le x_{2:N}^\top\delta^2 I_{N-1}x_{2:N}$ holds. Since x can be chosen arbitrarily, this implies the desired result. ■

Proof of Lemma 1: The following proof is adapted partially from the proof of [5, Lemma 3.2]. Recall that, since the network communication graph is connected and undirected, there exists an orthonormal transformation U that diagonalizes the Laplacian matrix L, i.e., $L=U\Lambda U^\top$ where $\Lambda=\mathrm{diag}\{0,\lambda_2,\dots,\lambda_N\}$. Define the vector and matrix

$$\tilde{\mathbf{x}} := (U^\top\otimes I_n)\mathbf{x}, \tag{9}$$

$$\begin{aligned}\tilde{\Delta} &:= (U^\top\otimes I_r)\Delta(U\otimes I_k)\\ &= \begin{pmatrix}\tilde{\Delta}_1^1 & \tilde{\Delta}_2^1 & \dots & \tilde{\Delta}_N^1\\ \tilde{\Delta}_1^2 & \tilde{\Delta}_2^2 & \dots & \tilde{\Delta}_N^2\\ \vdots & \vdots & & \vdots\\ \tilde{\Delta}_1^N & \tilde{\Delta}_2^N & \dots & \tilde{\Delta}_N^N\end{pmatrix} = \begin{pmatrix}\tilde{\Delta}_1^1 & \tilde{\Delta}^1\\ \tilde{\Delta}_1 & \bar{\Delta}\end{pmatrix}.\end{aligned} \tag{10}$$

By employing (9) and (10), we can rewrite the closed-loop dynamics equations (6) as

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \Big(I_N\otimes A + \Lambda\otimes BK + (I_N\otimes D)\tilde{\Delta}(\Lambda\otimes EK)\Big)\tilde{\mathbf{x}}\\ &\quad + (U^\top\otimes B_w)\mathbf{w},\\ \mathbf{z} &= (MU\otimes C)\tilde{\mathbf{x}}.\end{aligned} \tag{11}$$

Now, consider the term

$$(I_N\otimes D)\tilde{\Delta}(\Lambda\otimes EK) = \begin{pmatrix}\mathbf{0} & D\tilde{\Delta}^1(\bar{\Lambda}\otimes EK)\\ \mathbf{0} & (I_{N-1}\otimes D)\bar{\Delta}(\bar{\Lambda}\otimes EK)\end{pmatrix}, \tag{12}$$

and partition the matrix of eigenvectors as $U=[\frac{1}{N}\mathbf{1}_N, V]$ and note that $MU=[\mathbf{0},V]$. As a result, (11) can be decomposed into

$$\dot{\tilde{x}}_1 = A\tilde{x}_1 + D\tilde{\Delta}^1(\bar{\Lambda}\otimes EK)\bar{\mathbf{x}} + \left(\frac{1}{N}\mathbf{1}_N^\top\otimes B_w\right)\mathbf{w}, \tag{13}$$

and

$$\begin{aligned}\dot{\bar{\mathbf{x}}} &= \Big(I_{N-1}\otimes A + \bar{\Lambda}\otimes BK + (I_{N-1}\otimes D)\bar{\Delta}(\bar{\Lambda}\otimes EK)\Big)\bar{\mathbf{x}}\\ &\quad + (V^\top\otimes B_w)\mathbf{w},\\ \mathbf{z} &= (V\otimes C)\bar{\mathbf{x}},\end{aligned} \tag{14}$$

where $\bar{\mathbf{x}} := (\tilde{x}_2^\top,\dots,\tilde{x}_N^\top)^\top$, $\bar{\Lambda} := \mathrm{diag}\{\lambda_2,\dots,\lambda_N\}$. Due to Lemma 2, $\Delta^\top\Delta\le\delta^2 I_{Nk}$ implies $\bar{\Delta}^\top\bar{\Delta}\le\delta^2 I_{(N-1)k}$. Since $\mathbf{z}$ only depends on $\bar{\mathbf{x}}$, as can be seen from (8), this implies that $\|T_{\mathbf{wz}}\|\le\gamma$ holds for (6) and all admissible disturbances if and only if the transfer function from $\mathbf{w}$ to $\mathbf{z}$ of (8) is smaller or equal to γ for all disturbances with $\bar{\Delta}^\top\bar{\Delta}\le\delta^2 I_{(N-1)k}$. Consider now $\mathbf{w}=\mathbf{0}$. The states of the agents (1) reach consensus, i.e., $\lim_{t\to\infty} x_i - x_j = 0$ for all $i,j=1,\dots,N$, if and only if $\lim_{t\to\infty}\mathbf{x} = \mathrm{Im}(\mathbf{1}_N\otimes I_n)$. Note that, since $\mathrm{Im}(\mathbf{1_N}) = \mathrm{Ker}(L)$, $\lim_{t\to\infty}\mathbf{x} = \mathrm{Im}(\mathbf{1}_N\otimes I_n)$ holds if and only if $\lim_{t\to\infty}(L\otimes I_n)\mathbf{x}=0$. Hence, $\lim_{t\to\infty}(L\otimes I_n)\mathbf{x}(t)=0$ holds if and only if $\lim_{t\to\infty}(LU\otimes I_n)\tilde{\mathbf{x}}=0$. Recall that $LU=U\Lambda$ and U is not singular. Consequently, $\lim_{t\to\infty}(U\Lambda\otimes I_n)\tilde{\mathbf{x}}(t)=0$ holds if and only if $(\Lambda\otimes I_n)\tilde{\mathbf{x}}(t)\to 0$. As a result, since $\Lambda=\mathrm{diag}\{0,\lambda_2,\dots,\lambda_N\}$, consensus is achieved if and only if $\lim_{t\to\infty}\tilde{x}_i=0$ holds for $i=2,3,\dots,N$.

■

In particular, Lemma 1 implies that we can solve Problem 1 by designing a controller gain K such that (8) is stabilized and its H_∞ norm is smaller or equal to γ for all time-varying and Lebesgue measurable $\bar{\Delta}$ with $\bar{\Delta}^\top\bar{\Delta}\le\delta^2 I_{(N-1)k}$.

The following states that a quadratically stable system with parametric uncertainties is equivalent to a system with bounded H_∞ norm.

Lemma 3: Consider an uncertain system of the form

$$\dot{x} = (\tilde{A} + \Delta_{\tilde{A}})x \tag{15}$$

with $\Delta_{\tilde{A}} = F\Delta G$, the entries of Δ being time-dependent and Lebesgue measurable, and $\Delta^\top \Delta \leq \delta^2 I$. Then, there exists a symmetric positive-definite matrix $\tilde{P}$, such that

$$\left(\tilde{A} + \Delta_{\tilde{A}}\right)^\top \tilde{P} + \tilde{P}\left(\tilde{A} + \Delta_{\tilde{A}}\right) < 0 \tag{16}$$

holds for all admissible uncertainties $\Delta_{\tilde{A}}$, if and only if $\tilde{A}$ is Hurwitz and $\|G(sI - \tilde{A})^{-1}F\|_\infty \leq \frac{1}{\delta}$ holds.

Proof: The result follows directly from [13, Theorem 3.2]. ■

Lemma 3 is particularly useful, since it implies that a system with bounded H_∞ norm is also quadratically stable. This can be shown easily using the quadratic Lyapunov function $V(x) = x^\top \tilde{P} x$. Quadratic stability then follows from

$$\dot{V}(x) = x^\top \left(\tilde{A} + \Delta_{\tilde{A}}\right)^\top \tilde{P}x + x^\top \tilde{P}\left(\tilde{A} + \Delta_{\tilde{A}}\right)x < 0.$$

Hence, in order to solve Problem 1, it is sufficient to show that (6) has bounded H_∞ norm.

We now state our main result, a sufficient condition for solving the roust H_∞ consensus problem for the multi-agent system (1).

Theorem 1 (Main result): Assume that the communication graph $\mathscr{G}$ of the multi-agent system (1) is connected and undirected, and that the parametric uncertainties in (1) satisfy $B_{\Delta_i} = D\Delta_i E$, where Δ_i is time-dependent and Lebesgue measurable with $\Delta_i^\top \Delta_i \leq \delta^2 I_k$ for all times and δ a positive scalar. Furthermore, let P and Q be matrices that satisfy the following requirements:

- P is symmetric positive definite.
- The linear matrix inequality (LMI)

$$\begin{pmatrix} A_c & \lambda_i Q^\top E^\top & PC^\top \\ \lambda_i EQ & -\varepsilon I & 0 \\ CP & 0 & -I \end{pmatrix} < 0 \tag{17}$$

  is satisfied for an arbitrary scalar value $\varepsilon > 0$,

$$A_c := AP + PA^\top + \lambda_i BQ + \lambda_i Q^\top B^\top + \frac{1}{\gamma^2} B_w B_w^\top + \varepsilon\delta^2 DD^\top,$$

and $i = 2, N$, i.e., the smallest and largest non-zero eigenvalues of the Laplacian matrix L. Then the agents (1) achieve robust H_∞ consensus with tolerance $\gamma > 0$ using the state feedback controller (5) and feedback gain matrix $K = QP^{-1}$.

In particular, Theorem 1 states that, in order to obtain the desired guarantees, we only need to solve two separate $(n+k+p)$-dimensional LMIs, which can be done easily in many practical scenarios.

In order to prove Theorem 1, we employ the two following Lemmas, which are adapted directly from well-known results.

Lemma 4: Let $\gamma > 0$ and $T_{\mathbf{wz}}(s) = C_o(sI - A_o)^{-1}B_o$. Then, the following two statements are equivalent:

- The matrix A_o is Hurwitz and $\|T_{\mathbf{wz}}(s)\|_\infty < \gamma$.
- There exists symmetric positive definite matrix $\tilde{P} > 0$, such that

$$\tilde{P}A_o^\top + A_o\tilde{P} + \gamma^{-2}B_oB_o^\top + \tilde{P}C_o^\top C_o\tilde{P} < 0. \tag{18}$$

Proof: This follows directly from Lemma 2.1 in [13]. ■

Lemma 5: Given matrices S, Q_1, and Q_2 of appropriate dimensions and S symmetric,

$$S + Q_1\Delta Q_2 + Q_2^\top \Delta^\top Q_1^\top < 0 \tag{19}$$

holds for all matrices Δ satisfying $\Delta^\top \Delta \leq \delta^2 I$ if and only if there exists a scalar $\varepsilon > 0$ such that

$$S + \varepsilon\delta^2 Q_1 Q_1^\top + \varepsilon^{-1} Q_2^\top Q_2 < 0. \tag{20}$$

Proof: This follows directly from Lemma 2.4 in [15]. ■

We are now ready to prove Theorem 1.

Proof of Theorem 1: We prove Theorem 1 by showing that (8) satisfies the requirements of Lemma 4. More specifically, we aim to show that (18) holds for the system matrices of (8) and a tolerance parameter γ. By Lemma 1 and 3, this implies that the closed-loop system (6) achieves robust H_∞ consensus with tolerance γ. To this end, we need to show that there exists a symmetric positive definite matrix $\tilde{P}$, such that

$$\begin{aligned} &\tilde{P}\Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes (BK) + (I_{N-1} \otimes D)\bar{\Delta}(\bar{\Lambda} \otimes (EK))\Big)^\top \\ &+ \Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes (BK) + (I_{N-1} \otimes D)\bar{\Delta}(\bar{\Lambda} \otimes (EK))\Big)\tilde{P} \\ &+ \frac{1}{\gamma^2}(V^\top \otimes B_w)(V^\top \otimes B_w)^\top + \tilde{P}(V \otimes C)^\top (V \otimes C)\tilde{P} < 0 \end{aligned} \tag{21}$$

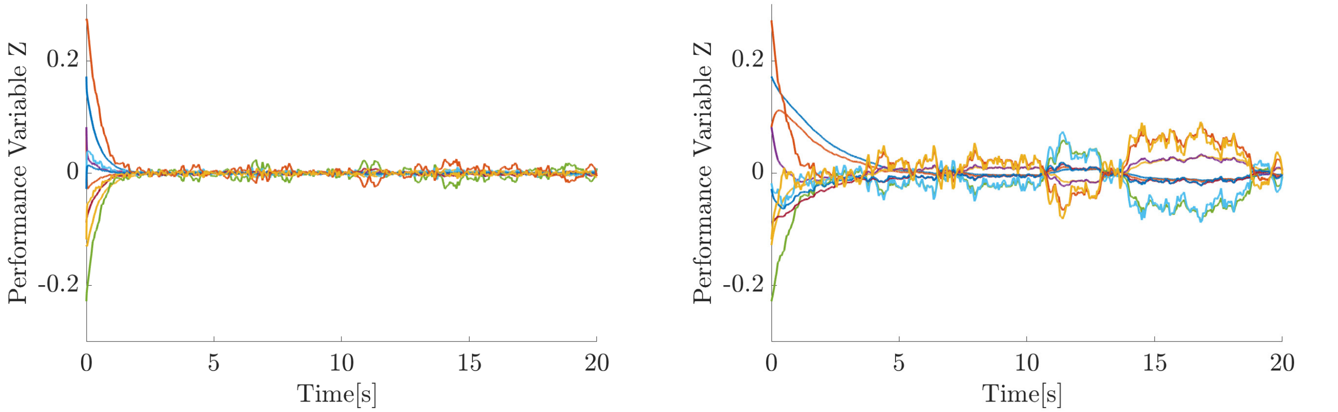
holds. By applying Lemma 5 with

$$\begin{aligned} S =& \tilde{P}\Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes (BK)\Big)^\top + \Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes (BK)\Big)\tilde{P} \\ &+ \frac{1}{\gamma^2}(V^\top \otimes B_w)(V^\top \otimes B_w)^\top + \tilde{P}(V \otimes C)^\top (V \otimes C)\tilde{P}, \\ Q_1 =& I_{N-1} \otimes D, \qquad Q_2 = (\bar{\Lambda} \otimes (EK))\tilde{P}, \end{aligned}$$

and $\bar{\Delta}^\top \bar{\Delta} < \delta^2 I_{k-1}$, we obtain that (21) holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$\begin{aligned} &\tilde{P}\Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes BK\Big)^\top + \Big(I_{N-1} \otimes A + \bar{\Lambda} \otimes BK\Big)\tilde{P} \\ &+ \frac{1}{\gamma^2}(I_{N-1} \otimes (B_w B_w^\top)) + \tilde{P}(I_{N-1} \otimes (C^\top C))\tilde{P} \\ &+ \varepsilon\delta^2(I_{N-1} \otimes (DD^\top)) + \frac{1}{\varepsilon}\tilde{P}((\bar{\Lambda}^2) \otimes (K^\top E^\top EK)\tilde{P} < 0 \end{aligned} \tag{22}$$

holds. Here we employ the identities $(V \otimes C)^\top (V \otimes C) = I_{N-1} \otimes (C^\top C)$, $(V^\top \otimes B_w)(V^\top \otimes B_w)^\top = I_{N-1} \otimes (B_w B_w^\top)$ $(I_{N-1} \otimes D)(I_{N-1} \otimes D)^\top = I_{N-1} \otimes (DD^\top)$ and $(\bar{\Lambda} \otimes EK)^\top (\bar{\Lambda} \otimes EK) = (\bar{\Lambda}^2) \otimes ((EK)^\top (EK)) = ((\bar{\Lambda}^2) \otimes (K^\top E^\top EK)$, which hold due V being composed of orthogonal vectors, and I_{N-1} and $\bar{\Lambda}$ being diagonal matrices. Now, by limiting the choice of symmetric matrix $\tilde{P}$ such that it has block diagonal form



(a) Performance variable $\mathbf{z}$ obtained when employing the proposed approach.

(b) Performance variable $\mathbf{z}$ obtained with the consensus region-based approach [14].

Fig. 1: Performance variable $\mathbf{z}$ obtained with the proposed approach Figure 1a and the consensus region-based approach from [14] Figure 1b. The H_∞ norm associated with our approach is $\|T_{\mathbf{wz}}\|_\infty = 0.2838$, that of the consensus region-based method is $\|T_{\mathbf{wz}}\|_\infty = 0.6076$. Our approach outperforms the consensus region-based one in terms of H_∞ norm. This discrepancy in performance occurs because our approach takes uncertainties into account during control design, whereas the consensus region-based one does not. Furthermore, the H_∞ norm obtained with our approach is below the desired tolerance $\gamma = 0.3$, which is to be expected due to Theorem 1.

$\tilde{P} = I_{N-1} \otimes P$ for some symmetric positive definite matrix P, we can rewrite (22) as

$$\begin{aligned} &I_{N-1} \otimes (PA^\top) + \bar{\Lambda} \otimes (P(BK)^\top) + I_{N-1} \otimes (AP) + \bar{\Lambda} \otimes (BKP) \\ &+ \frac{1}{\gamma^2}(I_{N-1} \otimes (B_w B_w^\top)) + (I_{N-1} \otimes (PC^\top CP)) \\ &+ \varepsilon\delta^2 (I_{N-1} \otimes (DD^\top)) + \frac{1}{\varepsilon}((\bar{\Lambda}^2) \otimes (PK^\top E^\top EKP)) < 0. \end{aligned} \tag{23}$$

It is easy to show that the left-hand side of (23) corresponds to a block diagonal matrix. Since a block diagonal matrix is positive definite if and only if its diagonal blocks correspond to symmetric positive definite matrices, we obtain that (23) holds if and only if

$$\begin{aligned} &PA^\top + \lambda_i PK^\top B^\top + AP + \lambda_i BKP + \frac{1}{\gamma^2} B_w B_w^\top \\ &+ PC^\top CP + \varepsilon\delta^2 DD^\top + \frac{1}{\varepsilon}\lambda_i^2 PK^\top E^\top EKP < 0, \end{aligned} \tag{24}$$

holds for $i = 2, \ldots, N$. Now, note that, for any symmetric matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$, symmetric positive definite matrix $S_3 \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$, the equation

$$x^\top S_1 x + \lambda_i x^\top S_2 x + \lambda_i^2 x^\top S_3 x$$

corresponds to a convex function of λ_i. Hence, (24) holds for all $i = 2, \ldots N$ if and only if it holds for $i = 2$ and $i = N$, i.e., the smallest and largest non-zero eigenvalues of L. Define $Q := KP$ and $A_c := PA^\top + \lambda_i Q^\top B^\top + AP + \lambda_i BQ + \frac{1}{\gamma^2} B_w B_w^\top + \varepsilon\delta^2 DD^\top$. Due to the properties of the Schur complement [16], we then have that (24) holds for $i = 2, \ldots, N$ if the matrices

$$\begin{pmatrix} A_c & \lambda_i Q^\top E^\top & PC^\top \\ \lambda_i EQ & -\varepsilon I & 0 \\ CP & 0 & -I \end{pmatrix} \tag{25}$$

are negative definite for $i = 2$ and $i = N$. ■

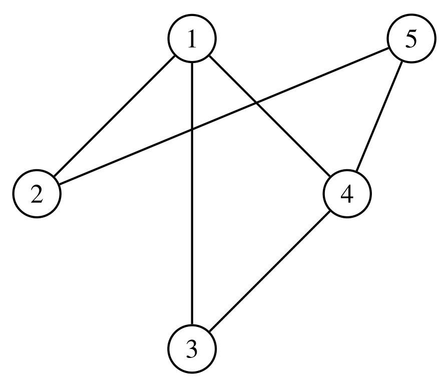


Fig. 2: Communication graph of system used for numerical experiments.

V. Numerical Validation

In this section, we illustrate our theoretical results using a numerical simulation. Furthermore, we compare our results to those obtained with the consensus region-based approach proposed in [14].

We employ a setting similar to the one considered in [10]. It consists of a multi-agent system with 5 agents and a communication graph as shown in Figure 2. The dynamics of each agent are described by (1), where the nominal dynamic matrices are given by

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_w = \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix},$$

and the constant matrices associated with the uncertainty

$B_{\Delta_i} = D\Delta_i E$ are

$$D = \begin{bmatrix} 0.5 & 0.8 \\ 0.6 & 0.7 \end{bmatrix}, \quad E = \begin{bmatrix} 0.3190 \\ 0.9478 \end{bmatrix}.$$

The time-varying matrices Δ_i are given by

$$\Delta_1 = \begin{bmatrix} 0.37\sin(15t) & 0 \\ 0 & 0.34 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0.28\sin(18t) & 0 \\ 0 & 0.35 \end{bmatrix},$$
$$\Delta_3 = \begin{bmatrix} 0.36 & 0 \\ 0 & 0.35\cos(15t) \end{bmatrix}, \quad \Delta_4 = \begin{bmatrix} 0.33 & 0 \\ 0 & 0.36\cos(20t) \end{bmatrix},$$
$$\Delta_5 = \begin{bmatrix} 0.36 & 0 \\ 0 & 0.38\cos(5t) \end{bmatrix}.$$

The upper bound for the norms of the uncertainties is set to $\delta = 0.4$, and we aim to keep the H_∞ norm of the transfer function $T_{\mathbf{wz}}$ below $\gamma = 0.3$, i.e., $\|T_{\mathbf{wz}}\|_\infty < 0.3$. We then choose $\varepsilon = 20$ and employTheorem 1, which yields the control gain matrix

$$K = QP^{-1} = \begin{bmatrix} -14.5819 & -48.1630 \end{bmatrix}.$$

The corresponding symmetric positive definite matrix is

$$P = \begin{bmatrix} 2.8487 & -0.8531 \\ -0.8531 & 0.3678 \end{bmatrix}.$$

The external disturbances w_i correspond to zero mean Gaussian distributed random variables that are sampled and held at a frequency of $20Hz$ for all agents, with different variances for each individual agent. The controller gain obtained via the consensus region-based method in [14] is $K = \begin{bmatrix} 0.5922 & -0.9237 \end{bmatrix}$, with $c = 0.75$. The evolution of the performance variable $\mathbf{z}$ using our approach and the consensus region-based approach are illustrated in Figure 1a and Figure 1b, respectively. Our approach yields superior performance in terms of the H_∞ norm of the transfer function, achieving a norm of $\|T_{\mathbf{wz}}\|_\infty = 0.2838$, whereas the consensus region-based method results in a norm of $\|T_{\mathbf{wz}}\|_\infty = 0.6076$. This is is to be expected, since our approach explicitly takes the system uncertainty into account, whereas the consensus region-based approach does not. This also means that the performance tolerance of $\gamma = 0.3$ was achieved, i.e., $\|T_{\mathbf{wz}}\|_\infty \leq 0.3$, which is also to be expected due to Theorem 1.

VI. Conclusion

We have addressed the problem of achieving robust H_∞ consensus for an undirected network of homogeneous multi-agent systems with uncertain agent dynamics and input noise. We have provided sufficient guarantees for achieving this, which amounts to choosing the control gain matrix such that a set of linear matrix inequalities is satisfied. In a numerical experiment, our approach outperformed a consensus region-based method that is commonly used for disturbance-free systems.

References

[1] D. Lee, S. Prabhakaran, L. Holguin, S. Amit, and E. Butcher, "Decentralized guidance and control for spacecraft formation flying using virtual leader configuration," pp. 4826–4831, 06 2013.

[2] J. Cortés and F. Bullo, "Coordination and geometric optimization via distributed dynamical systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 5, pp. 1543–1574, 2005.

[3] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proceedings of the National Academy of Sciences*, vol. 110, no. 6, pp. 2005–2010, 2013.

[4] H. Jongsma, H. Trentelman, and M. Camlibel, "Robust synchronization of coprime factor perturbed multi-agent systems," in *52nd IEEE Conference on Decision and Control*, pp. 1337–1342, 2013.

[5] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1511–1523, 2013.

[6] Z. Li and J. Chen, "Robust consensus of linear feedback protocols over uncertain network graphs," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 4251–4258, 2017.

[7] D. Zelazo and M. Bürger, "On the robustness of uncertain consensus networks," *IEEE Transactions on Control of Network Systems*, vol. 4, no. 2, pp. 170–178, 2017.

[8] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control Theory for Llinear Systems*, pp. 270–278. Springer, first ed., 2002.

[9] K. Z. Liu and Y. Yao, *Robust Control: Theory and Applications*. Singapore: Wiley, 2016.

[10] S. Zhang, Z. Li, and X. Wang, "Robust H_2 consensus for multi-agent systems with parametric uncertainties," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 68, no. 7, pp. 2473–2477, 2021.

[11] Z. Li, Z. Duan, L. Xie, and X. Liu, "Distributed robust control of linear multi-agent systems with parameter uncertainties," *International Journal of Control*, vol. 85, no. 8, pp. 1039–1050, 2012.

[12] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative control of multi-agent systems: optimal and adaptive design approaches*. Springer Science & Business Media, 2013.

[13] L. Xie, M. Fu, and C. de Souza, "H_∞ control and quadratic stabilization of systems with parameter uncertainty via output feedback," *IEEE Transactions on Automatic Control*, vol. 37, no. 8, pp. 1253–1256, 1992.

[14] Z. Li and Z. Duan, *Cooperative Control of Multi-Agent Systems: A Consensus Region Approach*, pp. 31–33. CRC Press, first ed., 2015.

[15] L. Xie, "Output feedback H_∞ control of systems with parameter uncertainty," *International Journal of Control*, vol. 63, no. 4, pp. 741–750, 1996.

[16] F. Zhang, *The Schur complement and its applications*, vol. 4. Springer Science & Business Media, 2006.