H_2 suboptimal containment control of multi-agent systems

Yuan Gao, Junjie Jiao, and Sandra Hirche

Abstract— **This paper deals with the distributed H_2 suboptimal containment control problem by static state feedback for linear multi-agent systems. Given multiple autonomous leaders, a number of followers, and an H_2 cost functional, we aim to design a distributed protocol that achieves containment control while the associated H_2 cost is smaller than an a priori given upper bound. To that end, we first show that the H_2 suboptimal containment control problem can be equivalently recast into the H_2 suboptimal control problem of a set of independent systems. Based on this, a design method is provided to compute such a distributed protocol. The computation of the feedback gain involves a single Riccati inequality whose dimension is equal to the dimension of the states of the agents. The performance of the proposed protocol is illustrated by a simulation example.**

I. Introduction

In the past two decades, distributed control for multi-agent systems has received much attention due to its wide range of applications, e.g., flocking [1], formation control [2], and intelligent transportation systems [3]. One of the fundamental research problems of multi-agent systems is consensus [4]. Depending on the number of leaders in the network, consensus problems can be classified into leaderless consensus [5], leader-follower consensus (one leader) [6], and containment control problem (multiple leaders) [7].

In the scenario of multiple leaders, the problem of containment control arises, where the states of followers converge into the convex hull formed by the states of leaders. In the existing literature, containment control has been studied for single-integrators under fixed and switching directed network topologies [8], for double-integrators under stationary and dynamic leaders [9], for time-delayed high-order linear systems with directed interaction topologies [10] and for general linear systems under directed fixed topology [11], [12].

In the operation of multi-agent systems, it should be noted that the agent dynamics might be affected by external disturbances. Therefore, researchers have focused on seeking performance requirements for containment control of multi-agent systems. To this end, there are works dealing with performance guarantees. In [13], over switching topologies with communication time delay, an observer-based containment control protocol for linear multi-agent systems was established to guarantee certain H_∞ performance. In [14], with a prescribed L_2 - L_∞ performance, L_2 - L_∞ containment control was investigated for single-integrators of multi-agent systems with Markovian switching topologies and non-uniform time-varying delays. In [15], with a prescribed H_∞ performance, a distributed static state protocol achieves containment control for second-order heterogeneous nonlinear multi-agent systems under directed topologies. We note that the works mentioned above only focused on the H_∞ performance index, which is a criterion that indicates the robustness of a system to the worst case of external disturbances.

Meanwhile, it is worth mentioning that quite some efforts have been devoted to addressing H_2 performance in the leaderless consensus problem. The H_2 consensus control problems of general linear multi-agent systems with undirected [16] and directed graphs [17] were studied by considering performance regions. Suboptimal distributed protocols based on static relative state feedback [18] and dynamic output feedback [19] were established to minimize a given H_2 cost criterion while achieving consensus for linear multi-agent systems. However, the above works regarding H_2 performance did not consider the case of multi-agent systems with multiple leaders, i.e., the H_2 containment control problem.

Motivated by the above, this paper deals with the H_2 optimal containment control problem for linear multi-agent systems. To this end, we will first introduce a suitable performance output and, subsequently, an associated H_2 cost functional. The objective is to design a distributed static protocol that minimizes the given H_2 cost functional while achieving containment control. However, due to the communication constraints among the agents, this problem is non-convex, and, so far, no closed-form solution has been given in the literature. Therefore, this paper considers an alternative form of this problem that requires only suboptimality.

The outline of this paper is as follows. In Section II, basic notations and graph theories are reviewed. In Section III, we formulate the distributed H_2 suboptimal containment control problem. After that, we establish a design method for computing a suboptimal static state feedback protocol in Section IV. A simulation example is provided in Section V to illustrate the performance of our proposed protocol. Finally, Section VI concludes this paper.

II. Preliminaries

A. Notation

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ be the space of n dimensional real vectors, and $\mathbb{R}^{m\times n}$ be the space of $m\times n$ real matrices. We denote by I_n the identity matrix of dimension $n\times n$. The superscript $\top$ means transpose for the real vector and matrix. We denote by $\mathrm{tr}(A)$ the trace of a square matrix A.

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A matrix is Hurwitz (or stable) if it has negative real parts for all eigenvalues. For a symmetric matrix P, if P is positive definite we denote $P>0$, and if P is negative definite we denote $P<0$. The $n\times n$ diagonal matrix with $d_1,\dots,d_n$ on the diagonal is denoted by $diag(d_1,\dots,d_n)$. For matrices $M_1,\dots,M_m$, the block diagonal matrix with diagonal blocks M_i is denoted by $blockdiag(M_1,\dots,M_m)$. We use $A\otimes B$ to denote the Kronecker product of matrices A and B.

B. Graph theory

A directed graph is denoted by $\mathcal{G}=(\mathcal{V},\mathcal{E})$, which consists of the node set $\mathcal{V}=\{1,\dots,N\}$ and edge set $\mathcal{E}=\{e_1,\dots,e_M\}$ satisfying $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$. The edge from node i to node j is represented by the pair $(i,j)\in\mathcal{E}$. We say a graph is undirected, if $(i,j)\in\mathcal{E}$ implies $(j,i)\in\mathcal{E}$. A graph is called simple if $(i,i)\notin\mathcal{E}$, which means no self-loops. The adjacency matrix $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ with nonnegative elements a_{ij} is defined as $a_{ii}=0$, $a_{ij}=1$ if $(j,i)\in\mathcal{E}$, and $a_{ij}=0$ otherwise. Subsequently, the Laplacian matrix $L=[L_{ij}]\in\mathbb{R}^{N\times N}$ of $\mathcal{G}$ is defined as $L_{ii}=\sum_{j=1}^N a_{ij}$ and $L_{ij}=-a_{ij}$. Note that $L=\mathcal{D}-\mathcal{A}$, where $\mathcal{D}=diag(d_1,\dots,d_N)$ is the degree matrix of $\mathcal{G}$ with $d_i=\sum_{j=1}^N a_{ij}$. For an undirected graph, its Laplacian matrix L is symmetric and only has real nonnegative eigenvalues.

III. Problem formulation

We consider a multi-agent system with N agents, consisting of M followers with external disturbances and $N-M$ autonomous leaders. Without loss of generality, we assume that the agents indexed by $1,...,M$ are followers and we denote the follower set as $\mathcal{F}\triangleq\{1,\dots,M\}$, while the agents indexed by $M+1,...,N$ are leaders and the leader set is denoted by $\mathcal{L}\triangleq\{M+1,...,N\}$. The dynamics of the ith leader is represented by

$$\dot{x}_i=Ax_i,\quad z_i=Cx_i,\qquad i\in\mathcal{L},\tag{1}$$

and the dynamics of the ith follower is represented by

$$\dot{x}_i=Ax_i+Bu_i+Ed_i,\quad z_i=Cx_i+Du_i,\qquad i\in\mathcal{F},\tag{2}$$

where $x_i\in\mathbb{R}^n$, $u_i\in\mathbb{R}^m$, $d_i\in\mathbb{R}^q$, and $z_i\in\mathbb{R}^p$ are, respectively, the state, the coupling input, the unknown external disturbance, and the output to be controlled. The matrices A, B, C, D and E are of compatible dimensions. Throughout this paper, we assume that the pair (A,B) is stabilizable.

In this paper, we consider the case that the relative states between neighboring agents are available for control. The leaders (1) and followers (2) can then be interconnected by a distributed static protocol of the form

$$u_i=K\sum_{j=1}^N a_{ij}(x_i-x_j),\quad i\in\mathcal{F},\tag{3}$$

where $K\in\mathbb{R}^{m\times n}$ is the feedback gain to be designed and a_{ij} is the ijth entry of the adjacency matrix $\mathcal{A}$ associated with graph $\mathcal{G}$ which satisfies the following assumption.

Assumption 1: The leaders receive no information from any followers. The leaders' states are available to at least one follower, and leaders have a directed path to that follower. The communication graph between the M followers is connected, simple, and undirected.

Since the leaders have no neighbors, the Laplacian matrix associated with graph $\mathcal{G}$ can be partitioned as

$$L=\begin{bmatrix} L_1 & L_2\\ 0_{(N-M)\times M} & 0_{(N-M)\times(N-M)}\end{bmatrix},\tag{4}$$

where $L_1\in\mathbb{R}^{M\times M}$ and $L_2\in\mathbb{R}^{M\times(N-M)}$.

Denote $\boldsymbol{x_f}=\left[x_1^\top,\dots,x_M^\top\right]^\top$, $\boldsymbol{x_l}=\left[x_{M+1}^\top,\dots,x_N^\top\right]^\top$, $\boldsymbol{u}=\left[u_1^\top,\dots,u_M^\top\right]^\top$, $\boldsymbol{d}=\left[d_1^\top,\dots,d_M^\top\right]^\top$, $\boldsymbol{z_f}=\left[z_1^\top,\dots,z_M^\top\right]^\top$ and $\boldsymbol{z_l}=\left[z_{M+1}^\top,\dots,z_N^\top\right]^\top$. We can then write the agents (1) and (2) in compact form as

$$\begin{aligned}&\dot{\boldsymbol{x}}_{\boldsymbol{l}}=(I_{N-M}\otimes A)\boldsymbol{x_l},\quad \boldsymbol{z_l}=(I_{N-M}\otimes C)\boldsymbol{x_l},\\ &\dot{\boldsymbol{x}}_{\boldsymbol{f}}=(I_M\otimes A)\boldsymbol{x_f}+(I_M\otimes B)\boldsymbol{u}+(I_M\otimes E)\boldsymbol{d},\\ &\boldsymbol{z_f}=(I_M\otimes C)\boldsymbol{x_f}+(I_M\otimes D)\boldsymbol{u}.\end{aligned}\tag{5}$$

Correspondingly, the protocol (3) can be written as

$$\boldsymbol{u}=(L_1\otimes K)\boldsymbol{x_f}+(L_2\otimes K)\boldsymbol{x_l}.\tag{6}$$

Foremost, we want the protocol (6) to solve the containment control problem without considering disturbances. Note that by containment control, we mean that the states of the followers converge to the convex hull formed by the states of the leaders [11]. The critical point of containment control is the convex hull formulated in the following definition.

Definition 1: For a set $X=\{x_1,\dots,x_n\}$ in $V\subseteq\mathbb{R}^p$, its convex hull $co(X)$ is defined as $co(x)=\left\{\sum_{i=1}^n\alpha_i x_i|x_i\in V,\alpha_i\geq0,\sum_{i=1}^n\alpha_i=1\right\}$.

For containment control of the multi-agent system (5), the convex hull $\boldsymbol{\omega_x}(t)$ spanned by leaders (1) is defined as

$$\boldsymbol{\omega_x}(t)\triangleq\left(-L_1^{-1}L_2\otimes e^{At}\right)\begin{bmatrix}x_{M+1}(0)\\ \vdots\\ x_N(0)\end{bmatrix},\tag{7}$$

where the sum of each row of $-L_1^{-1}L_2$ is equal to 1 which is derived from the following lemma and $x_i(0)$, $i=M+1,\dots,N$ are the initial states of the leaders. See e.g. [11], [20].

Lemma 1 ([20]): Under Assumption 1, L_1 is positive definite and each row of $-L_1^{-1}L_2$ has its sum equal to 1.

We now give a formal definition of containment control.

Definition 2: We say the protocol (6) achieves containment control for the multi-agent system (5) if, whenever $\boldsymbol{d}=0$, the states of followers $\boldsymbol{x_f}$ converge into the convex hull formed by the states of leaders $\boldsymbol{x_l}$, as $t\to\infty$, i.e., $\boldsymbol{x_f}(t)\to\boldsymbol{\omega_x}(t)$ as $t\to\infty$, where $\boldsymbol{\omega_x}(t)$ is defined in (7).

To proceed, we introduce a new state error for each follower as $\xi_i=\sum_{j=1}^N a_{ij}(x_i-x_j), i\in\mathcal{F}$. Denote $\boldsymbol{\xi}=\left[\xi_1^\top,\dots,\xi_M^\top\right]^\top$, we then have $\boldsymbol{\xi}=(L_1\otimes I_n)\boldsymbol{x_f}+(L_2\otimes I_n)\boldsymbol{x_l}$.

Note that, in the case $\boldsymbol{d}=0$, we have $\boldsymbol{x_f}$ tends to $\left(-L_1^{-1}L_2\otimes I_n\right)\boldsymbol{x_l}$, whenever $\boldsymbol{\xi}$ converges to 0, i.e., $\boldsymbol{x_f}(t)\to\boldsymbol{\omega_x}(t)$. Hence, containment control is achieved.

In the context of distributed H_2 optimal control for multi-agent systems, we are interested in the differences between the output values of leaders and followers. Therefore, we introduce the performance output $\varepsilon_i=\sum_{j=1}^N a_{ij}(z_i-z_j), i\in$

$\mathcal{F}$. Denote $\boldsymbol{\varepsilon} = \left[\varepsilon_1^\top, \ldots, \varepsilon_M^\top\right]^\top$, we have $\boldsymbol{\varepsilon} = (L_1 \otimes I_p)\boldsymbol{z_f} + (L_2 \otimes I_p)\boldsymbol{z_l}$. Thus, the performance output $\boldsymbol{\varepsilon}$ reflects the disagreements between the outputs of leaders and followers.

By interconnecting the dynamics of agents system (5) with the control protocol (6), the overall controlled error system satisfies the following dynamics

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= (I_M \otimes A + L_1 \otimes BK)\boldsymbol{\xi} + (L_1 \otimes E)\boldsymbol{d}, \\ \boldsymbol{\varepsilon} &= (I_M \otimes C + L_1 \otimes DK)\boldsymbol{\xi}. \end{aligned} \tag{8}$$

Denote $A_o := I_M \otimes A + L_1 \otimes BK,\ E_o := L_1 \otimes E,\ C_o := I_M \otimes C + L_1 \otimes DK$. The impulse response matrix for the error system (8) from the external disturbance $\boldsymbol{d}$ to the performance output $\boldsymbol{\varepsilon}$ is then equal to

$$T_K(t) = C_o e^{A_o t} E_o. \tag{9}$$

Subsequently, the associated H_2 cost functional is defined as

$$J(K) := \int_0^\infty \operatorname{tr}\left[T_K^\top(t) T_K(t)\right] dt, \tag{10}$$

which measures the performance of the system (8) as the square of the $\mathcal{L}_2$- norm of its impulse response. Since the protocol (3) has a particular structure due to the communication graph, the associated H_2 optimal control problem is a non-convex optimization problem. It is difficult to solve this problem and a closed-form solution is unknown to exist. As an alternative, in the present paper, a suboptimal version of this problem is solved.

The H_2 suboptimal containment control problem of linear multi-agent systems is then defined as follows.

Definition 3: The protocol (3) is said to solve the H_2 suboptimal distributed containment control problem of the multi-agent system (5) if,

- whenever the disturbances of all followers are equal to zero, i.e., $\boldsymbol{d} = 0$, we have $\boldsymbol{x_f}(t) \to \boldsymbol{\omega_x}(t)$ as $t \to \infty$.
- $J(K) < \gamma$, where γ is an a priori given upper bound.

The problem that we want to address is as follows.

Problem 1: Let $\gamma > 0$. Design a feedback gain $K \in \mathbb{R}^{m\times n}$ such that the distributed protocol (3) achieves containment control and $J(K) < \gamma$.

IV. H_2 SUBOPTIMAL DISTRIBUTED CONTROL OF MULTI-AGENT SYSTEMS BY STATIC STATE FEEDBACK

In this section, we address Problem 1 and provide a design approach to obtain an appropriate feedback matrix K.

From Assumption 1 and Lemma 1, we know that L_1 is positive definite, and therefore, L_1 is diagonalizable. Let $U \in \mathbb{R}^{M\times M}$ be an orthogonal matrix that diagonalizes L_1, i.e., $U^\top L_1 U = \Lambda = diag(\lambda_1, \ldots, \lambda_M)$, where $\lambda_i > 0, i = 1, \ldots, M$ are the eigenvalues of L_1. By applying the state transformation

$$\hat{\boldsymbol{\xi}} = (U^\top \otimes I_n)\boldsymbol{\xi}, \tag{11}$$

the error dynamics (8) becomes

$$\begin{aligned} \dot{\hat{\boldsymbol{\xi}}} &= (I_M \otimes A + \Lambda \otimes BK)\hat{\boldsymbol{\xi}} + (U^\top L_1 \otimes E)\boldsymbol{d}, \\ \boldsymbol{\varepsilon} &= (U \otimes C + L_1 U \otimes DK)\hat{\boldsymbol{\xi}}. \end{aligned} \tag{12}$$

Note that the impulse response matrix of (12) from the disturbance input $\boldsymbol{d}$ to the output $\boldsymbol{\varepsilon}$ is equal to (9).

To proceed, we present the following M auxiliary systems:

$$\dot{\tilde{\xi}}_i = A\tilde{\xi}_i + \lambda_i B\tilde{u}_i + \lambda_i E\tilde{d}_i,\ \tilde{\varepsilon}_i = C\tilde{\xi}_i + \lambda_i D\tilde{u}_i,\ i = 1, \ldots, M, \tag{13}$$

where $\lambda_i > 0, i = 1, \ldots, M$ are the eigenvalues of L_1, $\tilde{\xi} \in \mathbb{R}^n$, $\tilde{u}_i \in \mathbb{R}^m$, $\tilde{d}_i \in \mathbb{R}^q$, and $\tilde{\varepsilon}_i \in \mathbb{R}^p$ are, respectively, the state, the control input, the external disturbance, and the output of the ith auxiliary system. Meanwhile, we consider the following associated static state feedback controllers

$$\tilde{u}_i = K\tilde{\xi}_i, \qquad i = 1, \ldots, M, \tag{14}$$

with the same gain matrix K. By interconnecting (13) and (14), we obtain M independent closed-loop systems:

$$\begin{aligned} \dot{\tilde{\xi}}_i &= (A + \lambda_i BK)\tilde{\xi}_i + \lambda_i E\tilde{d}_i, \\ \tilde{\varepsilon}_i &= (C + \lambda_i DK)\tilde{\xi}_i, \end{aligned} \qquad i = 1, \ldots, M. \tag{15}$$

Denote $\tilde{A}_i := A + \lambda_i BK,\ \tilde{C}_i := C + \lambda_i DK,\ \tilde{E}_i := \lambda_i E$, the impulse response matrix from the disturbance $\tilde{d}_i$ to the output $\tilde{\varepsilon}_i$ is $\tilde{T}_{i,K}(t) = \tilde{C}_i e^{\tilde{A}_i t}\tilde{E}_i$. For each closed-loop system in (15), we introduce the associated H_2 cost functional $J_i(K) := \int_0^\infty \operatorname{tr}\left[\tilde{T}_{i,K}^\top(t)\tilde{T}_{i,K}(t)\right] dt, i = 1, \ldots, M$.

The following theorem holds.

Theorem 1: Assume that $D^\top C = 0$ and $D^\top D = I_m$. The static protocol (3) with feedback gain K achieves containment control for the agents (5) if and only if the controllers (14) with the same feedback gain K internally stabilize all M systems (13). Moreover, we have

$$J(K) := \sum_{i=1}^{M} J_i(K). \tag{16}$$

Proof: It follows from (11) that $\hat{\boldsymbol{\xi}} = 0$ if and only if $\xi_1 = \xi_2 = \cdots = \xi_M = 0$. Hence, the containment control problem is solved if and only if $\lim_{t\to\infty}\hat{\boldsymbol{\xi}}(t) = 0$. Recall that $U^\top L_1 U = \Lambda = diag(\lambda_1, \ldots, \lambda_M)$. By introducing two other transformations:

$$\hat{\boldsymbol{d}} = (U^\top \otimes I_n)\boldsymbol{d}, \quad \hat{\boldsymbol{\varepsilon}} = (U^\top \otimes I_n)\boldsymbol{\varepsilon}, \tag{17}$$

the error system (12) can be transformed into

$$\begin{aligned} \dot{\hat{\boldsymbol{\xi}}} &= (I_M \otimes A + \Lambda \otimes BK)\hat{\boldsymbol{\xi}} + (\Lambda \otimes E)\hat{\boldsymbol{d}}, \\ \hat{\boldsymbol{\varepsilon}} &= (I_M \otimes C + \Lambda \otimes DK)\hat{\boldsymbol{\xi}}. \end{aligned} \tag{18}$$

Denote $\hat{A}_o := I_M \otimes A + \Lambda \otimes BK$, $\hat{C}_o := I_M \otimes C + \Lambda \otimes DK$ and $\hat{E}_o := \Lambda \otimes E$. It can be easily seen that the system $(\hat{A}_o, \hat{E}_o, \hat{C}_o)$ is equivalent to the composition of the M independent closed-loop systems $(\tilde{A}_i, \tilde{E}_i, \tilde{C}_i)$ in (15), for $i = 1, \ldots, M$. So $\lim_{t\to\infty}\hat{\boldsymbol{\xi}}(t) = 0$ if and only if $\tilde{\xi}_1 = \cdots = \tilde{\xi}_M = 0$ as $t \to \infty$. Then, it follows from [11, Algorithm 2] that containment control is achieved if and only if the matrices $A + \lambda_i BK$ of the M closed-loop systems are stable.

Next, we prove (16). Let K be such that matrices $A + \lambda_i BK$ are Hurwitz. It can be seen from (11) and (17) that $(U^\top \otimes$

$I_n)A_o(U\otimes I_n)=\hat{A}_o$, $(U^\top\otimes I_n)C_o(U\otimes I_n)=\hat{C}_o$ and $(U^\top\otimes I_n)E_o(U\otimes I_n)=\hat{E}_o$. Then by substituting (9) in (10) we have

$$\begin{aligned}J(K) &:= \int_0^\infty \operatorname{tr}\left[T_K^\top(t)T_K(t)\right]dt\\ &= \int_0^\infty \operatorname{tr}\left[(C_oe^{A_ot}E_o)^\top(C_oe^{A_ot}E_o)\right]dt\\ &= \int_0^\infty \operatorname{tr}\left[(U\otimes I_n)(\hat{C}_oe^{\hat{A}_ot}\hat{E}_o)^\top(\hat{C}_oe^{\hat{A}_ot}\hat{E}_o)(U^\top\otimes I_n)\right]dt.\end{aligned}$$

Recall that $D^\top C=0$, $D^\top D=I_m$, and $\Lambda=diag(\lambda_1,\dots,\lambda_M)$, it is easy to see $\hat{C}_oe^{\hat{A}_ot}\hat{E}_o=blockdiag(\widetilde{T}_{1,K}(t),\widetilde{T}_{2,K}(t),\dots,\widetilde{T}_{M,K}(t))$. Subsequently,

$$\begin{aligned}J(K) &:= \int_0^\infty \sum_{i=1}^M \operatorname{tr}\left[U\widetilde{T}_{i,K}(t)^\top\widetilde{T}_{i,K}(t)U^\top\right]dt\\ &= \int_0^\infty \sum_{i=1}^M \operatorname{tr}\left[\widetilde{T}_{i,K}(t)^\top\widetilde{T}_{i,K}(t)\right]dt=\sum_{i=1}^M J_i(K).\end{aligned}$$

This completes the proof. ■

Note that the assumptions $D^\top C=0$ and $D^\top D=I_m$ are made here to simplify notation, and can be replaced by the regularity condition $D^\top D>0$ alone.

By applying Theorem 1, we have converted the distributed H_2 suboptimal containment control problem for linear multi-agent system (5) into a number of H_2 suboptimal control problems for M systems (15).

Next, the following lemma presents sufficiency and necessity for the existence of the controllers (14) with the given matrix $K\in\mathbb{R}^{m\times n}$ that solves the H_2 suboptimal control problems of the M systems (15), i.e., the closed-loop systems (15) are internally stabilized and $\sum_{i=1}^M J_i(K)<\ \gamma$.

Lemma 2: The static state feedback controllers (14) internally stabilize all systems (15) and $\sum_{i=1}^M J_i(K)<\gamma$ if and only if there exist positive definite matrices $P_i, i=1,\dots,M$ satisfying

$$(A+\lambda_iBK)^\top P_i+P_i(A+\lambda_iBK)+ (C+\lambda_iDK)^\top(C+\lambda_iDK)<0, \quad (19)$$

$$\sum_{i=1}^M \lambda_i^2\operatorname{tr}(E^\top P_iE)<\gamma. \quad (20)$$

Proof: (Sufficiency) By (20), for $\varepsilon_i>0$ sufficiently small, we have $\sum_{i=1}^M\gamma_i<\gamma$, where $\gamma_i:=\lambda_i^2\operatorname{tr}(E^\top P_iE)+\varepsilon_i$. It follows from [18, Theorem 3], inequalities (19) and (20) can be considered as the concrete form of the H_2 suboptimal control problem for linear systems, by taking $\overline{A}=\widetilde{A}_i=A+\lambda_iBK$, $\overline{E}=\widetilde{E}_i=\lambda_iE$ and $\overline{C}=\widetilde{C}_i=C+\lambda_iDK$. So there exist P_i such (19) and $\lambda_i^2\operatorname{tr}(E^\top P_iE)<\gamma_i$ hold for all $i=1,\dots,M$. Moreover, all systems (15) are internally stable and $J_i(K)<\gamma_i$. Therefore, $\sum_{i=1}^M J_i(K)<\gamma$.

(Necessity) Since all systems (15) are internally stable and $J_i(K)<\gamma_i$, for $i=1,\dots,M$, by taking $\overline{A}=\widetilde{A}_i=A+\lambda_iBK$, $\overline{E}=\widetilde{E}_i=\lambda_iE$, and $\overline{C}=\widetilde{C}_i=C+\lambda_iDK$. It follows from [18, Theorem 3] that there exist positive definite matrices P_i such that (19) and $\lambda_i^2\operatorname{tr}(E^\top P_iE)<\gamma_i$ hold for $i=1,\dots,M$. Since $\sum_{i=1}^M J_i(K)<\gamma$, there exist $\varepsilon_i>0$ sufficiently small such that $\sum_{i=1}^M\gamma_i<\gamma$, where $\gamma_i:=\lambda_i^2\operatorname{tr}(E^\top P_iE)+\varepsilon_i$, this implies that $\sum_{i=1}^M\lambda_i^2\operatorname{tr}(E^\top P_iE)<\gamma$. ■

Note that, in Lemma 2, a method to find such a gain matrix K has not yet been provided. In the following theorem, we provide a design method to compute one such K.

Theorem 2: Let $\gamma>0$ be a given upper bound. Assume that $D^\top C=0$ and $D^\top D=I_m$. Consider the multi-agent system (1) and (2) with the associated H_2 cost functional (10). Furthermore, consider the following two cases:

(i) if
$$0<c<\frac{2}{\lambda_1+\lambda_M}, \quad (21)$$
where λ_1 is the smallest eigenvalue and λ_M is the largest eigenvalue of L_1. Then there exists $P>0$ satisfying
$$A^\top P+PA+(c^2\lambda_1^2-2c\lambda_1)PBB^\top P+C^\top C<0. \quad (22)$$

(ii) if
$$\frac{2}{\lambda_1+\lambda_M}\le c<\frac{2}{\lambda_M}, \quad (23)$$
then there exists $P>0$ satisfying
$$A^\top P+PA+(c^2\lambda_M^2-2c\lambda_M)PBB^\top P+C^\top C<0. \quad (24)$$

Moreover, if in both cases P also satisfies
$$\operatorname{tr}(E^\top PE)<\frac{\gamma}{\lambda_M^2M}, \quad (25)$$
then the protocol (3) with $K:=-cB^\top P$ achieves containment control and the protocol is H_2 suboptimal, i.e., $J(K)<\gamma$.

Proof: For case (ii) above, using the upper and lower bound on c given by (23), $c^2\lambda_M^2-2c\lambda_M<0$ can be verified. Since the Riccati inequality (24) has positive definite solution P. For $i=1,\dots,M$, taking $P_i=P$ and $K=-cB^\top P$ in (19) immediately yields

$$\begin{aligned}&(A-c\lambda_iBB^\top P)^\top P+P(A-c\lambda_iBB^\top P)\\ &\quad+(C-c\lambda_iDB^\top P)^\top(C-c\lambda_iDB^\top P)<0.\end{aligned} \quad (26)$$

Recall the conditions $D^\top C=0$ and $D^\top D=I_m$, this yields

$$\begin{aligned}&(A-c\lambda_iBB^\top P)^\top P+P(A-c\lambda_iBB^\top P)\\ &\quad+c^2\lambda_i^2PBB^\top P+C^\top C<0.\end{aligned} \quad (27)$$

Since $c^2\lambda_1^2-2c\lambda_1\le c^2\lambda_i^2-2c\lambda_i\le c^2\lambda_M^2-2c\lambda_M<0$ and $\lambda_i\le\lambda_M$ for $i=1,\dots,M$, the positive definite matrix P of (24) also satisfies the M Riccati inequalities

$$A^\top P+PA+(c^2\lambda_i^2-2c\lambda_i)PBB^\top P+C^\top C<0. \quad (28)$$

For case (i) above, using the upper and lower bound on c given by (21), it can be verified that $c^2\lambda_M^2-2c\lambda_M\le c^2\lambda_i^2-2c\lambda_i\le c^2\lambda_1^2-2c\lambda_1<0$ and $\lambda_i\le\lambda_M$ for $i=1,\dots,M$. The proof of case (i) is similar to case (ii) and it is omitted here.

Next, it follows from (25) that also (20) holds. By Lemma 2 then, all M systems (15) are internally stabilized and $\sum_{i=1}^M J_i(K)<\gamma$. Consequently, it can be concluded from Theorem 1 and Lemma 2 that the protocol (3) achieves containment control for the network (5) and $J(K)<\gamma$. ■

Remark 1: Theorem 2 states that by choosing suitable c and P, the distributed static containment control protocol (3) with feedback matrix $K=-cB^\top P$ is H_2 suboptimal. Then the

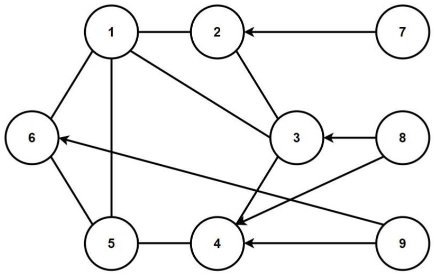

Fig. 1. The communication topology between the leaders and followers

question arises: how do we find the smallest upper bound γ such that $\mathrm{tr}(E^\top PE) < \gamma/\lambda_M^2 M$? Notice that a feasible given γ directly depends on $\mathrm{tr}(E^\top PE)$. Moreover, the smaller P lead to smaller $\mathrm{tr}(E^\top PE)$, therefore, smaller allowed γ. One way we could try is to find P as small as possible. With $\delta > 0$, we can establish two equalities from the two cases (i) and (ii) of Theorem 2 in the following to find the solution $P(c,\delta) > 0$.

$$A^\top P + PA - (-c^2\lambda_1^2 + 2c\lambda_1)PBB^\top P + C^\top C + \delta I_n = 0,$$

$$A^\top P + PA - (-c^2\lambda_M^2 + 2c\lambda_M)PBB^\top P + C^\top C + \delta I_n = 0.$$

Let $r_1 = (-c^2\lambda_1^2 + 2c\lambda_1)$ and $r_2 = (-c^2\lambda_M^2 + 2c\lambda_M)$. Obviously, the larger the coefficient r_1(or r_2) of the term $PBB^\top P$ and the smaller δ, the smaller P is. It can be shown that the maximum of r_1 is obtained when $c^* = \frac{1}{\lambda_1}$ and the maximum of r_2 is obtained when $c^* = \frac{1}{\lambda_M}$. Therefore, for both cases $0 \le c < \frac{2}{\lambda_1+\lambda_M}$ and $\frac{2}{\lambda_1+\lambda_M} \le c < \frac{2}{\lambda_M}$, if we choose $\delta > 0$ very close to 0 and choose $c = \frac{2}{\lambda_1+\lambda_M}$, we find the 'best' small solution to the Riccati inequalities (26) and (27) in the sense as elaborated above.

V. Simulation Example

In this section, we give a simulation example to illustrate the performance of our designed protocol. Consider a multi-agent system consisting of three leaders of the form (1) and six followers of the form (2), where $A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 1 & 0 & -1.5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1.2 \\ 1.5 \end{bmatrix}$, $E = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 1 & 1 \\ 4 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$. It is easy to check that the pair (A,B) is stabilizable. We also have that $D^\top C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $D^\top D = 1$.

For illustration, let the communication graph $\mathcal{G}$ between the agents be given by Figure 1, where nodes 7, 8, and 9 denote the leaders and the other nodes denote the followers. Correspondingly, due to the particular form of the Laplacian matrix associated with $\mathcal{G}$, the smallest and largest eigenvalue of L_1 of the Laplacian matrix (4) are computed to be $\lambda_1 = 0.6856$ and $\lambda_6 = 5.8245$. Now we will use the method proposed in Theorem 2 to design a protocol (3) to solve the containment problem while the H_2 cost functional satisfies $J(K) < \gamma$. We let the desired upper bound for the cost be $\gamma = 33$. We use case (ii) in Theorem 2 to compute a P solve

$$A^\top P + PA + (c^2\lambda_6^2 - 2c\lambda_6)PBB^\top P + C^\top C + \delta I_n = 0 \quad (29)$$

with $\delta = 0.001$, which is sufficiently small. In addition, we choose $c = \frac{2}{\lambda_1+\lambda_6} = 0.3072$, which is the 'best' choice to minimize the H_2 performance upper bound γ. Then, by solving (29) in Matlab with the command `icare`, we compute the feedback gain matrix $K = cB^\top P = (-2.8768, -0.7079, -0.3584)$. Moreover, we compute the associated H_2 cost to be $6\lambda_6^2 tr(E^\top PE) = 32.6304$, which is indeed smaller than the upper bound $\gamma = 33$. In Matlab, we use the command `norm(sys,2)` and compute the actual H_2 norm of the controlled system (8) as $||T_K||_{H_2} = 3.5608$, which is indeed smaller than $\sqrt{\gamma} = \sqrt{33} = 5.7446$.

Next, we compare the performance of our protocol with that of the proposed protocol in [12]. The corresponding feedback gain is computed as $\bar{K} = (-0.2455, -1.0000, -0.8898)$. The associated actual H_2 norm of the controlled system is computed to be $||\bar{T}_K||_{H_2} = 7.6992$. It can be seen that the performance of the static protocol in [12] is not comparable to our static protocol since its associated actual H_2 norm is much bigger than that of our protocol, i.e., $||\bar{T}_K||_{H_2} = 7.6992 > ||T_K||_{H_2} = 3.5608$.

As an example we choose, the initial states of the followers to be $x_{10} = \begin{bmatrix} 10 & -12 & -4 \end{bmatrix}^\top$, $x_{20} = \begin{bmatrix} -13 & -10 & 5 \end{bmatrix}^\top$, $x_{30} = \begin{bmatrix} 5 & 12 & 12 \end{bmatrix}^\top$, $x_{40} = \begin{bmatrix} -9 & -12 & 3 \end{bmatrix}^\top$, $x_{50} = \begin{bmatrix} -8 & -11 & -5 \end{bmatrix}^\top$, $x_{60} = \begin{bmatrix} -12 & -10 & -1 \end{bmatrix}^\top$ and the initial states of the leaders to be $x_{70} = \begin{bmatrix} 4 & 1.2 & -3 \end{bmatrix}^\top$, $x_{80} = \begin{bmatrix} 5.2 & 1.5 & 4 \end{bmatrix}^\top$, $x_{90} = \begin{bmatrix} 3 & 10 & 10 \end{bmatrix}^\top$. The same white noise $\boldsymbol{d}$ with an amplitude between -60 and 60 is applied to further compare the performance of our proposed protocol and the protocol proposed in [12]. We have plotted the trajectories of the states $\boldsymbol{x}$ and the performance output variable $\boldsymbol{\varepsilon}$ using our designed static protocol and the protocol in [12]. It can be seen in Figures 2 and 3 that our protocol has a better tolerance for external disturbance, this further shows that our proposed static protocol has a better performance than the static protocol in [12].

VI. Conclusions

In this paper, we have studied the distributed H_2 suboptimal containment control problem of linear multi-agent systems using static state feedback. Given a multi-agent system with N agents consisting of M followers and $N-M$ leaders, and an associated global H_2 cost functional with a desired upper bound, we have provided a design method for computing a suboptimal distributed protocol that achieves containment control, i.e., the states of followers can be driven into the convex hull spanned by the states of leaders and the associated H_2 cost functional is smaller than a given bound.

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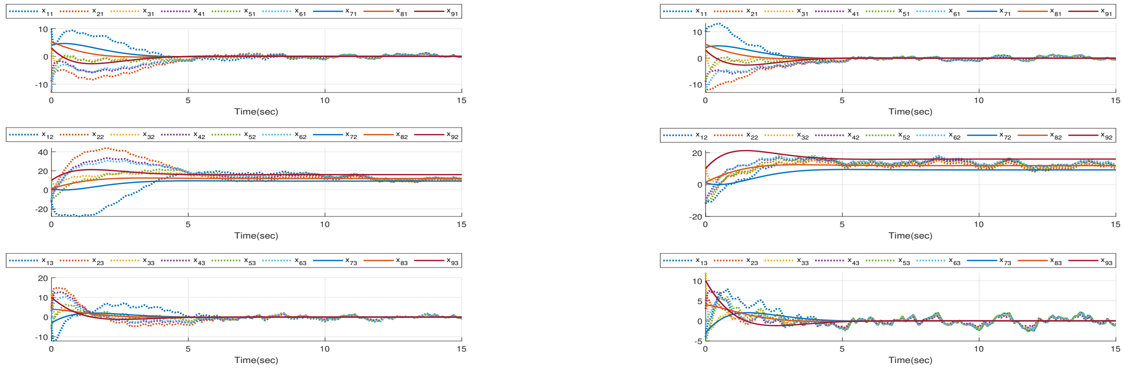

Fig. 2. Left plot: trajectories of the states x_{i1}, x_{i2}, x_{i3} of all agents using our proposed protocol (with disturbances). Right plot: trajectories of the states x_{i1}, x_{i2}, x_{i3} of all agents using the protocol in [12] (with disturbances)

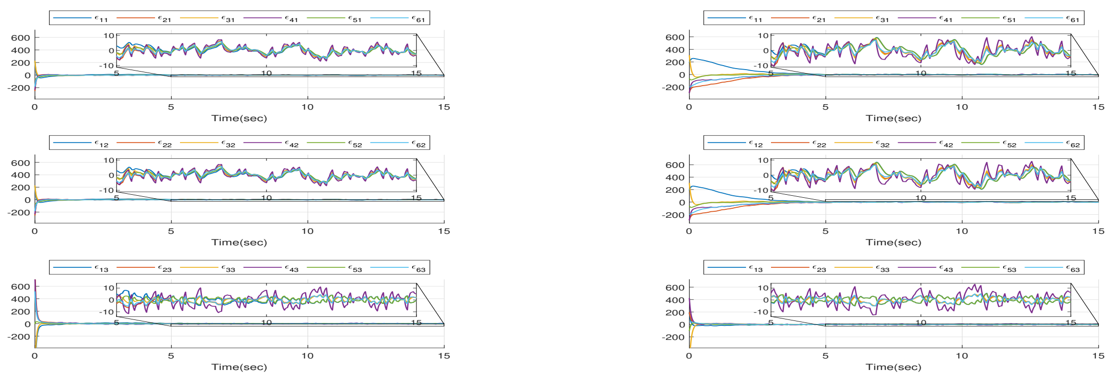

Fig. 3. Left plot: trajectories of the performance outputs $\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}$ for all agents of error system using our proposed protocol (with disturbances). Right plot: trajectories of the performance outputs $\varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}$ for all agents of error system using the protocol in [12] (with disturbances)

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