Bounded Consensus of Linear Multi-Agent Systems with External Disturbances Through a Reduced-Order Adaptive Feedback Protocol

Xianwei Li and Sandra Hirche
Chair of Information-oriented Control, Technical University of Munich, 80333 Munich, Germany
E-mail: lixianwei1985@gmail.com; hirche@tum.de

Abstract: In this paper, we study the bounded consensus control problem of linear multi-agent systems with external disturbances. Provided that the external disturbances are bounded, bounded consensus means the consensus errors and other system parameters are also guaranteed to be bounded. A novel reduced-order adaptive output-feedback protocol is proposed for ensuring bounded consensus of the closed-loop system. Then a sufficient condition is established for the existence of the proposed protocol and an estimation of the domain to which the consensus errors and the adaptive gains finally converge is presented. Compared with the existing results, the proposed protocol provides a tractable characterization for both full- and reduced-order adaptive dynamic output-feedback protocols. Moreover, since designing the protocol gains does not require to know the communication graph, the proposed protocol can be designed and implemented in a fully distributed way. Finally, a numerical example clearly illustrates the effectiveness of the proposed protocol.

Key Words: Multi-agent systems; Bounded consensus; Adaptive control; Output feedback; Reduced order.

1 Introduction

The past decade has witnessed the fast development of control theory of multi-agent systems (MASs) and its application to various areas such as distributed optimization, unmanned vehicles, smart grids and sensor networks [1–8]. The idea of consensus control about MASs is to force the states of all agents to track a common dynamic trajectory, which is one of the basic and widely investigated control problems about MASs (see the aforementioned literature as well as [9–13]). At a relative early stage, much attention was paid to special linear MASs of first- and second-order integrators [14], from which it has been well understood that the existence of a spanning tree of the underlying communication graph is necessary to reach consensus. In the past ten years, many efforts have been made to explore effective methods for designing distributed protocols of general linear MASs, especially those results about observer-type output-feedback protocols [1, 10–13, 15, 16]. In traditional linear control theory, the beauty of an observer-type output-feedback controller lies in the satisfaction of Separation Principle such that an observer gain and a state-feedback gain can be separately effectively designed. However, for consensus of MASs, the way to assign the communication topology between controllers is very crucial to keep the tractability of designing distributed protocols, which is a non-trivial step from the linear traditional control theory. Please refer to [10, 11, 15, 16] about the design of various dynamic output-feedback protocols with/without controller interaction for general linear MASs.

For output-feedback protocols, except for the challenges resulting from the availability of output information only, another important issue is how to design the protocol gains without knowing the communication graph. When the network size is large, it is usually difficult to exactly obtain the eigenvalues of the Laplacian matrix of the communication graph, which, however, are necessary for determining the graph-related gains in most of the aforementioned results. To cope with this issue, a popular idea is to explore an adaptive mechanism that makes use of relative information between agents to adjust the graph-related gains. Early results about adaptive strategies for complex networks synchronization can be found in [17] and the references therein. In [18–20], adaptive state-feedback protocols are developed for general linear MASs and second-order MASs, while in [21, 22], various adaptive dynamic output-feedback protocols are proposed for general linear systems with or without external disturbances. Since designing the protocol gains is independent of determining the graph-related gains, these protocols can be designed and implemented in a fully distributed manner. Note, however, that each controller of the protocols in [21, 22] has the same order as that of agents, while it is known that lower-order controller laws are less complex than full-order ones and thus are more attractive in practice. To the best of our knowledge, no reduced-order adaptive output-feedback protocol that is tractable to design as the full-order results, has been reported.

In this paper, we are interesting in the bounded consensus problem of linear MASs with external disturbances. Bounded consensus means that the consensus error is kept to be bounded, provided that external disturbances are bounded. Due to the existence of external disturbances, it is in general impossible for an MAS to reach exact consensus. In view of this fact, bounded consensus is the more practical notion. In this paper, we will propose a novel reduced-order adaptive output-feedback protocol for general linear MASs over undirected graphs. It will be shown that bounded consensus can be reached by the proposed protocol if the protocol gains are properly parameterized. Particularly, in virtue of the results in [16], we can show that there exist some reduced-order or full-order output-feedback protocols that satisfy our requirements and can be easily designed. Compared with the existing results about output-feedback

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consensus control with external disturbances, the proposed adaptive protocol has the following novelties:

1) The proposed protocol also characterizes a kind of reduced-order dynamic output-feedback protocols that are computationally *less demanding* than the full-order ones in [21, 22]. Actually, the problem in [22] is about leader-follower bounded consensus and the related one in [21] is about state-feedback protocols, which are different from the leaderless bounded consensus problem through an adaptive output-feedback protocol in this paper.

2) The proposed protocol makes use of relative output information between agents for bounded consensus. Although some results about reduced-order consensus protocols have been reported [23, 24], they either require absolute output information or relative input information about neighbouring agents. On the contrary, our protocol fuses the obtained relative output information in a more straightforward way. What's more, the proposed reduced-order adaptive protocol can be designed and implemented in a fully distributed way.

Notation: Let the set of all $m \times n$ real matrices denoted by $\mathbb{R}^{m\times n}$ and an $n \times n$ identity matrix by $\mathbf{I}_n$. Represent a square, positive definite (semi-definite) matrix by $P > 0$ (≥ 0). $A \otimes B$ and $A \circ B$ stand for Kronecker product and Hadamard product, respectively. Let $||\cdot||_2$ denote the Euclidean norm of a vector, and let $\lambda_{\max}(\cdot)$ denote the maximum eigenvalue of a symmetric matrix. Denote $\mathcal{V} = \{1, \ldots, N\}$ as the set of N nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ as the edge set, then an undirected graph is represented by $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Define the adjacency matrix associated with $\mathcal{G}$ as $\mathcal{A} = [a_{ij}]_{N\times N}$, where $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. In this paper, if a graph is undirected, we assume $a_{ij} = a_{ji}$ for $i, j = 1, \ldots, N$. We say node j is a neighbouring node of node i if $(j, i) \in \mathcal{E}$ or $a_{ij} > 0$. Denote the set of all the neighbouring node of node i by $\mathcal{N}_i$. Accordingly, the Laplacian matrix associated with $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}]_{N\times N}$ with $l_{ii} = \sum_{k\in\mathcal{N}_i} a_{ik}$ and $l_{ij} = -a_{ij}$ for $i, j = 1, 2, \ldots, N$ and $i \neq j$. A path of the graph is a sequence of edges connecting two nodes. An undirected graph is said to be connected if every node can be reached from every other node over any path.

2 Main Results

2.1 Problem Statement

Consider a group of N $(N \geq 2)$ homogeneous linear agents that are represented by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + B_u u_i(t) + w_i(t), \\ y_i(t) &= C_y x_i(t),\ i = 1, \ldots, N, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_x}$ is the state of agent i, and $u_i \in \mathbb{R}^{n_u}$, $y_i \in \mathbb{R}^{n_y}$ and $w_i \in \mathbb{R}^{n_x}$ are the corresponding control input, local output and external disturbance, respectively. System matrices A, B_u and C_y are real and appropriately-dimensioned. For each agent, the matrix triple (A, B_u, C_y) is assumed to be stabilizable and detectable. The external disturbances $w_i(t)$ are assumed to be bounded in a way as $\|w_i(t)\|_2 < \bar{w}_i$, $i = 1, \ldots, N$, for all $t \geq 0$, where $\bar{w}_i$ is a known positive constant.

Consensus of MASs means that the states of all the agents are synchronized to some common trajectories. Since we consider a linear MAS with external disturbances, it is in general impossible to reach exact consensus. In this paper, we consider the bounded consensus problem where the objective is to design a distributed protocol such that the consensus state errors of all the agents are bounded. To this end, suppose that the communication graph denoted by $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is undirected, where $\mathcal{A} = [a_{ij}]_{N\times N}$ and $\mathcal{L} = [l_{ij}]_{N\times N}$ are the adjacency matrix and Laplacian matrix. We propose the following distributed adaptive output-feedback protocol,

$$\begin{aligned} \dot{r}_i &= Hr_i + F_r \sum_{j\in\mathcal{N}_i} c_{ij} a_{ij} \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right), \\ u_i &= Gr_i + F_u \sum_{j\in\mathcal{N}_i} c_{ij} a_{ij} \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right), \end{aligned} \tag{2}$$

where the adaption law is given by

$$\begin{aligned} &\dot{c}_{ij} = \alpha_{ij} \left[\left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right)^{\mathrm{T}} \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right) - \phi_{ij} c_{ij}\right], \\ &c_{ij}(0) = c_{ji}(0),\ \alpha_{ij} = \alpha_{ji} > 0,\ \phi_{ij} > 0, \\ &\quad i = 1, \ldots, N;\ j \in \mathcal{N}_i. \end{aligned} \tag{3}$$

with $\tilde{y}_{ij} \triangleq y_i - y_j$ and $\tilde{r}_{ij} = r_i - r_j$, and H, G, B_r, F_r and F_u are properly-dimensioned real matrices to be designed.

In the above equation, the feedback signals $\tilde{y}_{ij}$ and $\tilde{r}_{ij}$ are some relative information between agents i and j. Especially, note that $\tilde{y}_{ij}$ is the relative output between agents i and j. Thus, rather than using the relative state, the proposed protocol is of the output-feedback kind, which is more practical than the state-feedback protocols in [18, 21]. What's more, the order of each controller in (2) (that is, the dimension of r_i), denoted by n_r, is allowed to be smaller than that of every agent, n_x. Thus, as a reduced-order counterpart of the full-order dynamic output-feedback protocols in [22], the above protocol can also characterize a kind of *reduced-order* dynamic output-feedback protocols, which are computationally less demanding than full-order ones. It will be seen that the above protocol actually covers both full-order and reduced-order ones as special cases.

2.2 Consensus Analysis

Let $c_{ii} = \frac{\sum_{j\in\mathcal{N}_i} c_{ij} a_{ij}}{l_{ii}}$ for $i = 1, \ldots, N$ and $c_{ij} = 0$ for $i = 1, \ldots, N$ and $j \notin \mathcal{N}_i$. Then the protocol (2) can be re-written as

$$\begin{aligned} \dot{r}_i &= Hr_i + F_r \sum_{j=1}^{N} c_{ij} l_{ij} \left(y_j + C_y B_r r_j\right), \\ u_i &= Gr_i + F_u \sum_{j=1}^{N} c_{ij} l_{ij} \left(y_j + C_y B_r r_j\right). \end{aligned} \tag{4}$$

Let $s_i \triangleq \mathrm{col}\{x_i, r_i\}$ and $s \triangleq \mathrm{col}\{s_1, \ldots, s_N\}$. From (1) and (4), we can obtain the closed-loop system that is given by

$$\dot{s} = \left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C} \circ \mathcal{L}) \otimes \tilde{B}\right] s(t) + \left(\mathbf{I} \otimes \tilde{B}_w\right) w, \tag{5}$$

where $w = \mathrm{col}\{w_1, \ldots, w_N\}$, $\mathcal{C} \triangleq [c_{ij}]_{N\times N}$, $\tilde{B}_w \triangleq [\mathbf{I}; 0]$ and

$$\tilde{A} \triangleq \begin{bmatrix} A & B_u G \\ 0 & H \end{bmatrix},\ \tilde{B} \triangleq \begin{bmatrix} B_u F_u C_y & B_u F_u C_y B_r \\ F_r C_y & F_r C_y B_r \end{bmatrix}.$$

To analyze the consensus error, let $s_{ei} \triangleq s_i - \frac{1}{N}\sum_{j=1}^{N} s_j$, $i=1,\ldots,N$, and $s_e \triangleq \mathrm{col}\{s_{e1},\ldots,s_{eN}\}$. Then s_e can be expressed as $s_e = (\mathcal{L}_e \otimes \mathbf{I})\, s$, where $\mathcal{L}_e \triangleq \mathbf{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^{\mathrm{T}}$. Since the graph is undirected, that is, $a_{ij} = a_{ji}$, and moreover it follows from $c_{ij}(0) = c_{ji}(0)$ and $\alpha_{ij} = \alpha_{ji}$ that $c_{ij}(t) = c_{ji}(t)$ for all $t \geq 0$, we have $\mathcal{L}_e\left(\mathcal{C}(t) \circ \mathcal{L}\right) = \mathcal{C}(t) \circ \mathcal{L} = (\mathcal{C}(t) \circ \mathcal{L})\mathcal{L}_e$. Consequently, we have the following system:

$$\begin{aligned}\dot{s}_e &= (\mathcal{L}_e \otimes \mathbf{I})\left\{\left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C} \circ \mathcal{L}) \otimes \tilde{B}\right] s + \left(\mathbf{I} \otimes \bar{B}_w\right) w\right\} \\ &= \left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C} \circ \mathcal{L}) \otimes \tilde{B}\right] s_e + \left(\mathcal{L}_e \otimes \bar{B}_w\right) w. \qquad (6)\end{aligned}$$

Note that $s_i = s_j$ for $i,j = 1,\ldots,N$ if and only if $s_e = 0$. Thus, s_e can be defined as the consensus error indicating whether consensus is reached or not. The following theorem is the main result of this paper, which provides a sufficient condition under which the consensus error in terms of s_e as well as the adaptive gains c_{ij} are guaranteed to be bounded.

Theorem 1 *Consider the MAS (1) and the protocol (2), and suppose that the communication graph $\mathcal{G}$ is undirected and connected. Then the consensus error $s_e(t)$ and the adaptive gains $c_{ij}(t)$, $i,j = 1,\ldots,N$, are uniformly ultimately bounded, if the following statements hold:*

1) *Matrices H, G and B_r are such that*

$$\begin{gathered} H \text{ is Hurwitz},\ AB_r - B_r H = B_u G, \\ \mathrm{rank}(B_r) = n_r \text{ and } \mathrm{rank}([B_u, B_r]) = n_x. \qquad (7)\end{gathered}$$

2) *Matrices F_u and F_r are given by*

$$\begin{bmatrix} F_u \\ F_r \end{bmatrix} = -\begin{bmatrix} R_u \\ R_r \end{bmatrix} PC_y^{\mathrm{T}}, \qquad (8)$$

where R_u and R_r are general real matrices such that $R_r B_r = \mathbf{I}_{n_r}$ and $B_u R_u + B_r R_r = \mathbf{I}_{n_x}$, and P is a positive definite matrix solving the following Riccati equation for any positive definite matrix Q:

$$PA^{\mathrm{T}} + AP - PC_y^{\mathrm{T}} C_y P + Q = 0. \qquad (9)$$

Proof. Since $B_u R_u + B_r R_r = \mathbf{I}_{n_x}$ and $R_r B_r = \mathbf{I}_{n_r}$, we have $R_r = R_r\left(B_u R_u + B_r R_r\right) = R_r B_u R_u + R_r$, which implies $R_r B_u R_u = 0$. Since B_r is of full column rank and $R_r B_r = \mathbf{I}_{n_r}$, R_r has full row rank and $\mathrm{rank}(R_r) = \mathrm{rank}(B_r) = n_r$. Thus, the following matrix T and its inverse T^{-1} are well-defined:

$$T = \begin{bmatrix} \mathbf{I}_{n_x} & B_r \\ -R_r & 0_{n_r \times n_r} \end{bmatrix},\ T^{-1} = \begin{bmatrix} B_u R_u & -B_r \\ R_r & \mathbf{I}_{n_r} \end{bmatrix}.$$

Using the above relations, $AB_r - B_r H = B_u G$, $B_u R_u + B_r R_r = \mathbf{I}$ and $R_r B_r = \mathbf{I}$, we have

$$\begin{aligned} & AB_u R_u + B_u G R_r + B_r H R_r = AB_u R_u + AB_r R_r = A, \\ & R_r A B_u R_u + R_r B_u G R_r \\ & \quad = R_r A B_u R_u + R_r\left(AB_r - B_r H\right) R_r \\ & \quad = R_r A\left(B_u R_u + B_r R_r\right) - R_r B_r H R_r = R_r A - H R_r, \\ & R_r A B_r - R_r B_u G \\ & \quad = R_r A B_r - R_r\left(AB_r - B_r H\right) = R_r B_r H = H. \end{aligned}$$

Consequently, by substituting the above equations and the expression of F_u and F_r in (8), we obtain

$$\begin{aligned} \bar{A} &\triangleq T\tilde{A}T^{-1} \\ &= \begin{bmatrix} \mathbf{I} & B_r \\ -R_r & 0 \end{bmatrix}\begin{bmatrix} A & B_u G \\ 0 & H \end{bmatrix}\begin{bmatrix} B_u R_u & -B_r \\ R_r & \mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ HR_r - R_r A & H \end{bmatrix}, \\ \bar{B} &\triangleq T\tilde{B}T^{-1} \\ &= \begin{bmatrix} \mathbf{I} & B_r \\ -R_r & 0 \end{bmatrix}\begin{bmatrix} B_u F_u C_y & B_u F_u C_y B_r \\ F_r C_y & F_r C_y B_r \end{bmatrix} \\ &\quad \times \begin{bmatrix} B_u R_u & -B_r \\ R_r & \mathbf{I} \end{bmatrix} \\ &= -\begin{bmatrix} B_u & B_r \\ -R_r B_u & 0 \end{bmatrix}\begin{bmatrix} R_u \\ R_r \end{bmatrix} PC_y^{\mathrm{T}} C_y \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix} \\ &= -\begin{bmatrix} \mathbf{I} \\ 0 \end{bmatrix} PC_y^{\mathrm{T}} C_y \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}. \end{aligned}$$

Let $\bar{s} \triangleq (\mathbf{I} \otimes T)\, s_e$, which, combined with (6), satisfies

$$\begin{aligned} \dot{\bar{s}} &= (\mathbf{I} \otimes T)\left\{\left[\mathbf{I} \otimes \tilde{A} + (\mathcal{C} \circ \mathcal{L}) \otimes \tilde{B}\right] s_e + (\mathcal{L}_e \otimes B_w)\, w\right\} \\ &= \left[\mathbf{I} \otimes \bar{A} + (\mathcal{C} \circ \mathcal{L}) \otimes \bar{B}\right]\bar{s} + (\mathcal{L}_e \otimes B_w)\, w. \end{aligned}$$

Obviously, the boundedness of $s_e(t)$ is equivalent to that of $\bar{s}(t)$. For convenience, we can further write the state equation of $\bar{s}(t)$ as

$$\begin{aligned} \dot{\bar{x}} &= \left[\mathbf{I} \otimes A - (\mathcal{C} \circ \mathcal{L}) \otimes PC_y^{\mathrm{T}} C_y\right]\bar{x} + (\mathcal{L}_e \otimes \mathbf{I})\, w, \qquad (10) \\ \dot{\bar{r}} &= (\mathbf{I} \otimes H)\,\bar{r} + \left[\mathbf{I} \otimes (HR_r - R_r A)\right]\bar{x}, \end{aligned}$$

where

$$\begin{bmatrix} \bar{x} \\ \bar{r} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \otimes [\mathbf{I}_{n_x}, 0_{n_x \times n_r}] \\ \mathbf{I} \otimes [0_{n_r \times n_x}, \mathbf{I}_{n_r}] \end{bmatrix}\bar{s}.$$

Thus, the boundedness of $s_e(t)$ is equivalent to that of $\bar{x}(t)$ and $\bar{r}(t)$, which will be proved next.

Construct a candidate Lyapunov function as

$$\begin{aligned} V(t) &= \bar{x}^{\mathrm{T}}(t)\left(\mathbf{I} \otimes P^{-1}\right)\bar{x}(t) + \bar{r}^{\mathrm{T}}(t)\left(\mathbf{I} \otimes \varepsilon M\right)\bar{r}(t) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{a_{ij}}{2\alpha_{ij}}\left(c_{ij}(t) - \bar{c}\right)^2, \end{aligned}$$

where $\bar{c}$ and ε are positive constants to be determined, P is the symmetric, positive definite matrix satisfying (9) and M is any symmetric, positive definite matrix satisfying $MH + H^{\mathrm{T}}M < 0$ (since H is Hurwitz, this is possible). Taking the derivative of $V(t)$ along the solution of $\bar{x}(t)$ in (10) and $c_{ij}(t)$ in (3) gives rise to

$$\begin{aligned} \dot{V} &= 2\bar{x}^{\mathrm{T}}\left(\mathbf{I} \otimes P^{-1}\right)\dot{\bar{x}} + 2\bar{r}^{\mathrm{T}}(t)\left(\mathbf{I} \otimes \varepsilon M\right)\dot{\bar{r}}(t) \\ &\quad + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{a_{ij}}{\alpha_{ij}}\left(c_{ij} - \bar{c}\right)\dot{c}_{ij} \\ &= 2\bar{x}^{\mathrm{T}}\left(\mathbf{I} \otimes P^{-1}\right) \\ &\quad \times \left\{\left[\mathbf{I} \otimes A - (\mathcal{C} \circ \mathcal{L}) \otimes PC_y^{\mathrm{T}} C_y\right]\bar{x} + (\mathcal{L}_e \otimes \mathbf{I})\, w\right\} \\ &\quad + 2\bar{r}^{\mathrm{T}}\left(\mathbf{I} \otimes \varepsilon M\right)\left\{(\mathbf{I} \otimes H)\,\bar{r} + \left[\mathbf{I} \otimes (HR_r - R_r A)\right]\bar{x}\right\} \\ &\quad + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} a_{ij}\left(c_{ij} - \bar{c}\right)\left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right)^{\mathrm{T}} \\ &\quad \times \left(\tilde{y}_{ij} + C_y B_r \tilde{r}_{ij}\right) - \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} a_{ij}\left(c_{ij} - \bar{c}\right)\phi_{ij} c_{ij}. \end{aligned}$$

One can verify

$$
\begin{aligned}
&\tilde{y}_{ij}+C_yB_r\tilde{r}_{ij}\\
&=C_y\left[\begin{array}{cc}\mathbf{I} & B_r\end{array}\right](s_i-s_j)\\
&=C_y\left[\begin{array}{cc}\mathbf{I} & B_r\end{array}\right](s_{ei}-s_{ej})\\
&=C_y\left[\begin{array}{cc}\mathbf{I} & B_r\end{array}\right]T^{-1}(\bar{s}_i-\bar{s}_j)\\
&=C_y\left[\begin{array}{cc}\mathbf{I} & 0\end{array}\right](\bar{s}_i-\bar{s}_j)=C_y(\bar{x}_i-\bar{x}_j).
\end{aligned}
$$

Moreover, it is easy to see that $c_{ij}(t)=c_{ji}(t)$ for all $t\geq 0$. Thus, it follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}(c_{ij}-\bar{c})(\tilde{y}_{ij}+C_yB_r\tilde{r}_{ij})^{\mathrm{T}}(\tilde{y}_{ij}+C_yB_r\tilde{r}_{ij})\\
&=\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}(c_{ij}-\bar{c})(\bar{x}_i-\bar{x}_j)^{\mathrm{T}}C_y^{\mathrm{T}}C_y(\bar{x}_i-\bar{x}_j)\\
&=2\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}(c_{ij}-\bar{c})\bar{x}_i^{\mathrm{T}}C_y^{\mathrm{T}}C_y(\bar{x}_i-\bar{x}_j)\\
&=2\bar{x}^{\mathrm{T}}\left[(\mathcal{C}\circ\mathcal{L}-\bar{c}\mathcal{L})\otimes C_y^{\mathrm{T}}C_y\right]\bar{x}.
\end{aligned}\tag{11}
$$

Since $\lambda_{\max}(\mathcal{L}_e)=1$ can be directly obtained from the definition of $\mathcal{L}_e$, for any positive scalar θ, we have

$$
\begin{aligned}
&2\bar{x}^{\mathrm{T}}\left(\mathbf{I}\otimes P^{-1}\right)(\mathcal{L}_e\otimes\mathbf{I})w\\
&\leq\theta\bar{x}^{\mathrm{T}}\left(\mathcal{L}_e\otimes P^{-1}\right)\bar{x}+\theta^{-1}w^{\mathrm{T}}\left(\mathcal{L}_e\otimes P^{-1}\right)w\\
&\leq\theta\bar{x}^{\mathrm{T}}\left(\mathbf{I}\otimes P^{-1}\right)\bar{x}+\theta^{-1}\lambda_{\max}\left(P^{-1}\right)w^{\mathrm{T}}w\\
&=\theta\bar{x}^{\mathrm{T}}\left(\mathbf{I}\otimes P^{-1}\right)\bar{x}+\theta^{-1}\lambda_{\min}^{-1}(P)\sum_{i=1}^{N}\bar{w}_i^2.
\end{aligned}\tag{12}
$$

Since $\left(\mathbf{1}^{\mathrm{T}}\otimes\mathbf{I}\right)(\mathbf{I}\otimes C_y)\bar{x}=\left(\mathbf{1}^{\mathrm{T}}\mathcal{L}_e\otimes C_y[\mathbf{I},0]T\right)s=0$, by using [25, Lemma 1], it follows that

$$
\bar{x}^{\mathrm{T}}\left(\mathcal{L}\otimes C_y^{\mathrm{T}}C_y\right)\bar{x}\geq\lambda_2(\mathcal{L})\bar{x}^{\mathrm{T}}\left(\mathbf{I}\otimes C_y^{\mathrm{T}}C_y\right)\bar{x}.\tag{13}
$$

In addition, one can also verify

$$
\begin{aligned}
-(c_{ij}-\bar{c})c_{ij}&=-(c_{ij}-\bar{c})^2-(c_{ij}-\bar{c})\bar{c}\\
&\leq-(c_{ij}-\bar{c})^2+\frac{1}{2}(c_{ij}-\bar{c})^2+\frac{1}{2}\bar{c}^2\\
&=-\frac{1}{2}(c_{ij}-\bar{c})^2+\frac{1}{2}\bar{c}^2.
\end{aligned}\tag{14}
$$

By using the equations (11)–(14) and introducing a positive scalar δ, $\dot{V}$ can be computed as

$$
\begin{aligned}
\dot{V}&=-\delta V+\delta V+2\bar{x}^{\mathrm{T}}\left[\mathbf{I}\otimes P^{-1}A-(\mathcal{C}\circ\mathcal{L})\otimes C_y^{\mathrm{T}}C_y\right]\bar{x}\\
&\quad+\theta\bar{x}^{\mathrm{T}}\left(\mathbf{I}\otimes P^{-1}\right)\bar{x}+\theta^{-1}\lambda_{\min}^{-1}(P)\sum_{i=1}^{N}\bar{w}_i^2\\
&\quad+2\bar{r}^{\mathrm{T}}(\mathbf{I}\otimes\varepsilon MH)\bar{r}+2\bar{r}^{\mathrm{T}}\left[\mathbf{I}\otimes\varepsilon(MHR_r-MR_rA)\right]\bar{x}\\
&\quad+2\bar{x}^{\mathrm{T}}\left[(\mathcal{C}\circ\mathcal{L}-\bar{c}\mathcal{L})\otimes C_y^{\mathrm{T}}C_y\right]\bar{x}\\
&\quad-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}\phi_{ij}(c_{ij}-\bar{c})^2+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}\phi_{ij}\bar{c}^2\\
&\leq-\delta V+\left[\begin{array}{c}\bar{x}\\ \bar{r}\end{array}\right]^{\mathrm{T}}\left[\begin{array}{cc}\mathbf{I}\otimes\Phi_{11} & \mathbf{I}\otimes\varepsilon\Phi_{21}^{\mathrm{T}}\\ \mathbf{I}\otimes\varepsilon\Phi_{21} & \mathbf{I}\otimes\varepsilon\Phi_{22}\end{array}\right]\left[\begin{array}{c}\bar{x}\\ \bar{r}\end{array}\right]\\
&\quad-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}\left(\phi_{ij}-\frac{\delta}{\alpha_{ij}}\right)(c_{ij}-\bar{c})^2\\
&\quad+\theta^{-1}\lambda_{\min}^{-1}(P)\sum_{i=1}^{N}\bar{w}_i^2+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}\phi_{ij}\bar{c}^2.
\end{aligned}
$$

where

$$
\begin{aligned}
&\Phi_{11}\triangleq P^{-1}A+A^{\mathrm{T}}P^{-1}+(\theta+\delta)P^{-1}-2\bar{c}\lambda_2(\mathcal{L})C_y^{\mathrm{T}}C_y,\\
&\Phi_{21}\triangleq MHR_r-MR_rA,\\
&\Phi_{22}\triangleq MH+H^{\mathrm{T}}M+\delta M.
\end{aligned}\tag{15}
$$

For the above inequality about $\dot{V}$, firstly δ can be chosen such that $0<\delta\leq\alpha_{ij}\phi_{ij}$ for $i,j=1,\ldots,N$, which implies $\phi_{ij}-\frac{\delta}{\alpha_{ij}}\geq 0$. Secondly, the Riccati equation (9) implies $P^{-1}A+A^{\mathrm{T}}P^{-1}-C_y^{\mathrm{T}}C_y=-P^{-1}QP^{-1}<0$. Thus, for any $\bar{c}\geq\frac{1}{2\lambda_2(\mathcal{L})}$, one can take sufficiently small $\theta>0$ and $\delta>0$ such that $\Phi_{11}\leq P^{-1}A+A^{\mathrm{T}}P^{-1}+(\theta+\delta)P^{-1}-C_y^{\mathrm{T}}C_y<0$. Thirdly, it follows from $MH+H^{\mathrm{T}}M<0$ that $\Phi_{22}<0$ for some sufficiently small δ. Consequently, there always exist positive, sufficiently small scalars θ and δ and scalar $\bar{c}\geq\frac{\lambda_2(\mathcal{L})}{2}$ such that $\phi_{ij}-\frac{\delta}{\alpha_{ij}}\geq 0$, $\Phi_{11}<0$ and $\Phi_{22}<0$. Moreover, with ε taken sufficiently small, it is easy to see that $\left[\begin{array}{cc}\mathbf{I}\otimes\Phi_{11} & \mathbf{I}\otimes\varepsilon\Phi_{21}^{\mathrm{T}}\\ \mathbf{I}\otimes\varepsilon\Phi_{21} & \mathbf{I}\otimes\varepsilon\Phi_{22}\end{array}\right]<0$ still holds. With these facts in mind, we conclude that it is always possible to find some proper scalars θ, δ, $\bar{c}$ and ε such that $\dot{V}\leq-\delta V+\bar{V}$, where $\bar{V}$ is a positive constant defined as

$$
\bar{V}\triangleq\theta^{-1}\lambda_{\min}^{-1}(P)\sum_{i=1}^{N}\bar{w}_i^2+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}a_{ij}\phi_{ij}\bar{c}^2\tag{16}
$$

By the Comparison Lemma ([26, Lemma 3.4]), we obtain

$$
V(t)\leq\mathrm{e}^{-\delta t}V(0)+\bar{V}\frac{1}{\delta}\left(1-\mathrm{e}^{-\delta t}\right)=\mathrm{e}^{-\delta t}\left(V(0)-\frac{\bar{V}}{\delta}\right)+\frac{\bar{V}}{\delta},
$$

which implies that $\bar{x}$, $\bar{r}$ and c_{ij} will exponentially converge to the domain $\bar{\mathbb{D}}=\{\bar{x},\bar{r},c_{ij}|0<V\leq\frac{\bar{V}}{\delta}\}$. Consequently, $\bar{x}$, $\bar{r}$ and c_{ij} are uniformly ultimately bounded and so is the consensus error s_e. The proof is completed. ■

Theorem 1 reveals that the proposed distributed adaptive output-feedback protocol (2) always solves the bounded consensus problem, provided that the protocol gains are properly designed. The following corollary is a straightforward result that can be obtained from the proof of Theorem 1.

Corollary 1 *Consider the MAS (1) and the protocol (2), and suppose that bounded consensus is affirmed by Theorem 1. Then the consensus error $s_e(t)$ and the adaptive gains $c_{ij}(t)$, $i,j=1,\ldots,N$, exponentially converge to the domain*

$$
\mathbb{D}_e\triangleq\left\{s_e,c_{ij}|0<V_e\leq\frac{\bar{V}}{\delta}\right\},
$$

where $\bar{V}$ is defined in (16),

$$
V_e\triangleq s_e^{\mathrm{T}}\left(\mathbf{I}\otimes\left[\begin{array}{cc}P^{-1}+\varepsilon R_r^{\mathrm{T}}MR_r & P^{-1}B_r\\ B_r^{\mathrm{T}}P^{-1} & B_r^{\mathrm{T}}P^{-1}B_r\end{array}\right]\right)s_e
$$

$$+\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\frac{a_{ij}}{2\alpha_{ij}}\left(c_{ij}-\bar{c}\right)^2,$$

and positive scalars δ, ε, θ and $\bar{c}$ are such that

$$\begin{cases} 0<\delta\leq\alpha_{ij}\phi_{ij}, \\ \bar{c}\geq\frac{1}{2\lambda_2(\mathcal{L})}, \\ \begin{bmatrix} \Phi_{11} & \varepsilon\Phi_{21}^{\mathrm{T}} \\ \varepsilon\Phi_{21} & \varepsilon\Phi_{22}\end{bmatrix}<0, \end{cases}$$

with matrices Φ_{11}, Φ_{21} and Φ_{22} defined in (15).

Proof. The proof is straightforward by following that of Theorem 1. Note that $V_e(t) = V(t)$ is implied by $\bar{x} = (\mathbf{I}\otimes[\mathbf{I},0])\,\bar{s}$, $\bar{r} = (\mathbf{I}\otimes[0,\mathbf{I}])\,\bar{s}$ and $\bar{s} = (\mathbf{I}\otimes T)\,s_e$. ■

Remark 1 According to Theorem 7, the proposed adaptive output-feedback protocol (2) solves the bounded consensus problem if the protocol gains satisfy (7)–(9). It is well-known that the Riccati equation (9) is feasible if the matrix pair (A, C_y) is detectable as assumed here. The only question regarding the existence of desired protocol gains is the feasibility of the conditions in (7), $R_rB_r = \mathbf{I}$ and $B_uR_u + B_rR_r = \mathbf{I}$. It is in general challenging to answer, because the protocol (2) is a kind of fixed-order controllers which are known to be inherently difficult to design. Fortunately, with the proposed characterization as in (2), this question can be answered at least for two special, but general enough cases. *First*, if the controller order is identical to that of agents, that is, $n_r = n_x$, it is easy to see that $B_r = -\mathbf{I}$, $R_r = -\mathbf{I}$ and $R_u = 0$ satisfy these specifications. As such, we have $H = A + B_uG$, and since the pair (A, B_u) is stabilizable, finding a matrix G such that H is Hurwitz is feasible and tractable. *Second*, by following the results in [16], it is also tractable to find protocol gains with $n_r = n_x - n_u$ that satisfy the specifications, only requiring the pair (A, B_u) is stabilizable. In view of the well-known results in [10, 11, 15], the protocol (2) is of the *full-order* kind for the first case and of the *reduced-order* kind for the second case, respectively.

Remark 2 When no disturbance is considered, one can set $\phi_{ij} = 0$. Then the protocol (2) of the full-order kind (see Remark 1) reduces to the one in [22]. Thus, for this case, exact consensus can be reached. Bounded leader-follower consensus with the leader of possibly nonzero input is also studied in [22], while we investigate bounded leaderless consensus with external disturbances. Thus, the results in [22] cannot be directly applied to the problem in this paper.

Remark 3 Bounded consensus of linear MASs is also addressed in [21], but the corresponding protocol is of the state-feedback class. In this paper, we consider the design of output-feedback protocols, which are more practical than state-feedback ones. Especially, as aforementioned, our results can deal with both full- and reduced-order protocols. Moreover, note that the existence condition, [21, Theorem 12], requires to solve $AQ + QA^{\mathrm{T}} + \varepsilon Q - 2B_uB_u^{\mathrm{T}} < 0$ with $\varepsilon > 1$ for some $Q > 0$. Since this inequality is not always feasible, even when (A, B_u) is stabilizable, the design method therein cannot always solve the bounded consensus problem. On the contrary, as pointed out previously, Theorem 1 in this paper always admits some feasible solution for $n_r = n_x - n_u$ or $n_r = n_x$. Thus, the proposed method is advantageous over the one in [21, Theorem 12].

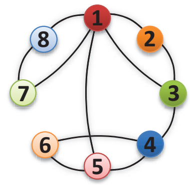

Fig. 1: The undirected communication graph used in the example

Next, a numerical example is presented to demonstrate the effectiveness of the proposed design method.

Example 1 The considered MAS consists of 8 agents which are connected according to the undirected graph shown in Fig. 1 and satisfy the system description in (1) with

$$A=\begin{bmatrix}0 & 0.2\\ -0.2 & 0\end{bmatrix},\ B_u=\begin{bmatrix}1\\0\end{bmatrix},\ C_y=\begin{bmatrix}1 & 0\end{bmatrix}.$$

The edge weights of the graph are all 1. By the design method in [16], we can obtain the following protocol gains that satisfy the specifications in Theorem 1:

$$H=-0.6325,\ B_r=\begin{bmatrix}3.1623\\1\end{bmatrix},\ G=2.2,$$
$$F_u=6.0428,\ F_r=-2.9686.$$

Fig. 2 displays the simulation results of the example with the designed reduced-order adaptive output-feedback protocol, where the disturbances w_i are some sine or cosine signals (which are not presented for saving space). It is easy to found that the consensus state errors of the agents and the adaptive gains c_{ij} are bounded. Note that the protocol states (and thus the corresponding errors) are also bounded. Thus, the effectiveness of the proposed method is obvious. Moreover, the Lyapunov function $V_e(t)$ clearly converges to a range with upper bound determined by $\bar{V}/\delta = 12.3839$.

3 Conclusion

In this paper, we have investigated the bounded consensus problem of linear MASs with external bounded disturbances. A novel reduced-order adaptive output-feedback protocol has been proposed to ensure that the consensus errors and the graph-related gains are bounded. A sufficient condition has been derived for the existence of desired protocol gains, which needs to solve some matrices equations. Compared with the existing adaptive output-feedback protocols, the proposed method can characterize both reduced- and full-order adaptive dynamic output-feedback protocols. Compared with the existing reduced-order protocols, the proposed protocol makes use of relative output information between agents in a more straightforward way. Moreover, the proposed protocol can be designed and implemented in a fully distributed way. An estimation of the attractive domain of the consensus errors and the graph-related gains has also been obtained. The effectiveness of the proposed protocol has been clearly demonstrated by a numerical example.

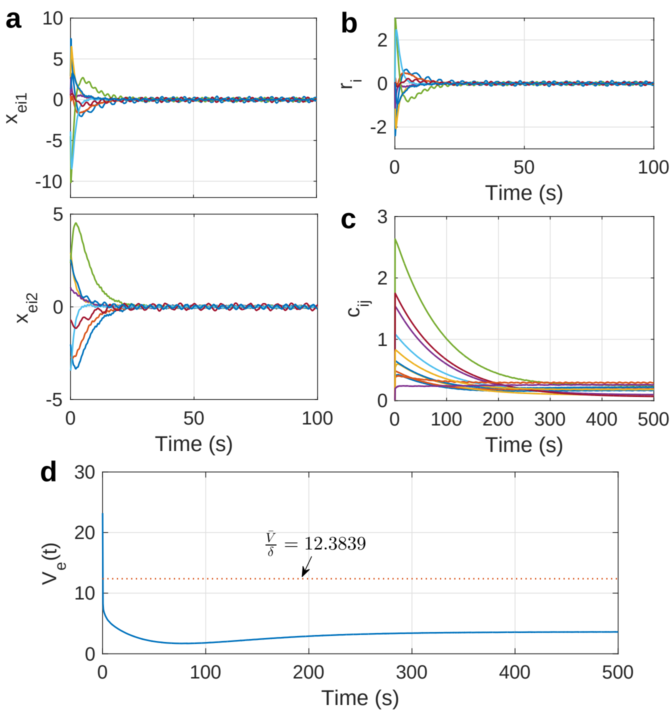

Fig. 2: Simulation results. (a) Consensus state errors of agents, $x_{ei} = \sum_{j=1}^{N} l_{eij} x_j$; (b) Protocol states r_i; (c) Adaptive gains c_{ij}; (d) The Lyapunov function $V_e(t)$

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