Event-Triggered Consensus of Multi-Agent Systems on Strongly Connected Graphs

Xianwei Li[1] and Sandra Hirche[1]

Abstract— **This paper studies consensus control of linear multi-agent systems (MASs) over directed graphs. The communication topology is assumed to be strongly connected. A distributed event-triggered state-feedback protocol is proposed for achieving consensus, which monitors the relative state information between neighboring agents and triggers a sampling event when some state-related inequalities are violated. An existence condition for the protocol is derived, which needs to solve an algebraic Riccati equation and find some scalar parameters related to the graph Laplacian. It is proved that the protocol excludes both singular triggering behavior and Zeno behavior if some scalars are properly assigned. Compared with the existing results, the proposed method can deal with general linear MASs on strongly connected graphs but does not require the absolute state information of agents. The effectiveness of the proposed method is illustrated by a numerical example.**

I. INTRODUCTION

The last decade has witnessed the fast development of coordination control theory of networked multi-agent systems (MASs) [1]–[3]. Especially, considerable attention has been paid to the fundamental *consensus* problem, that is, to find a distributed control protocol such that a group of autonomous dynamic agents reach an agreement of states in some sense [4]–[6]. Both state- and output-feedback protocols have been investigated for general linear MASs.

For most of aforementioned existing results, the control input is updated with continuous- and real-time information about agents, which is not economical for control of MASs. In particular, for applications such as sensor networks, mobile robots or unmanned aerial vehicles, each agent usually has limited communication resources. Thus, for such resource-limited applications, how to save communication resources is very important to achieve control objectives. To this end, a nature idea is to design protocols which update the control input at discrete-time instants, so that each agent does not need to continuously occupy the communication resource. In this way, a traditional method is to design periodic sampled-data control protocols with a fixed sampling frequency [7]. To further flexibly schedule the sampling frequency, a recent trend is to employ event-triggered sampling schemes for consensus control of MASs [8]–[17]. In [8]–[10], [14]–[16], various event-triggered state-feedback protocols are proposed for single- or double-integrator MASs over undirected or directed graphs. Event-triggered consensus control of general linear MASs can be found in [11]–[13], [15], [17]. Note that the sampled error in [8]–[12] is defined in terms of absolute state/output information of agents, thus these results still require the absolute information about agents, which is restrictive for some applications where only relative information between agents is available. In [13]–[17], events are triggered when inequalities related to relative state information is violated. As a result, the protocols therein only rely on relative state information between agents.

In this paper, we investigate event-triggered consensus control of linear MASs on directed graphs. The goal is to design an event-triggered control protocol such that a group of linear agents reach state consensus. It is assumed that the communication graph is strongly connected. A distributed event-triggered state-feedback protocol is proposed, which compares the sampling error of the relative state information with a time-dependent term. Once an event is triggered at one agent, the protocol updates the control input for that agent. A sufficient condition for the existence of the protocol is derived, and triggering behavior analysis shows that no singular triggering and Zeno behavior occurs in the protocol if some scalars are properly assigned. A numerical example is finally presented to illustrate the effectiveness of the proposed protocol.

Compared with the existing related results in [13]–[19] which aim at special or different cases of event-triggered consensus, the contribution of this paper is twofold: *1) this paper deals with event-triggered leaderless consensus of general linear MASs on strongly connected graphs;* and *2) Zeno behaviors are analyzed in a more systematic way, particularly distinguishing when the intervals of two successive events have strictly positive infimums.*

Notation: $\mathbb{R}^{m\times n}$ represents the set of all $m \times n$ real matrices and $\mathbf{I}_n$ stands for an $n\times n$ identity matrix. A square, positive-definite matrix is denoted by $P > 0$. Kronecker product for two matrices A and B is represented by $A \otimes B$. $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ denote the maximum and minimal eigenvalue for a square, symmetric matrix A, respectively. $\|(\cdot)\|$ denotes the Euclidean norm of a vector $(\cdot)$ or the spectral norm of a matrix $(\cdot)$. $\mathrm{diag}\{\cdots\}$ means a (block) diagonal matrix with "$\cdots$" on the diagonal.

Let $\mathcal{G}(\mathcal{V},\mathcal{E})$ be a directed graph with $\mathcal{V} = \{1,\dots,N\}$ the set of N nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ the edge set. Define the adjacency matrix associated with a directed graph as $\mathcal{A} = [a_{ij}]_{N\times N}$, where $a_{ij} > 0$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We say node j is a neighbor of node i, if $(j,i) \in \mathcal{E}$ or

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[1]The authors are with the Chair of Information-oriented Control, TUM Department of Electrical and Computer Engineering, Technical University of Munich, 80333 Munich, Germany {lixianwei1985@gmail.com, hiche@tum.de}.

$a_{ij} > 0$, and represent the set of all the neighboring node of node i by $\mathcal{N}_i$. Thus, we can denote $\mathcal{L} = [l_{ij}]_{N\times N}$ with $l_{ii} = \sum_{k\in\mathcal{N}_i} a_{ik}$ and $l_{ij} = -a_{ij}$ for $i, j = 1, 2, \dots, N$ and $i \neq j$, as the associated Laplacian matrix. A directed path of the graph is an ordered sequence of edges connecting two nodes. A directed graph is said to be strongly connected if every node can be reached from every other node through any directed path.

Lemma 1: Consider a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and suppose it is strongly connected. Then zero is a simple eigenvalue of the Laplacian matrix $\mathcal{L}$ [20]. Moreover, there is a positive vector $\boldsymbol{h} \triangleq \mathrm{col}\{h_1, \dots, h_N\}$ such that $\boldsymbol{h}^{\mathrm{T}}\mathcal{L} = 0$, and $0 < \alpha(\mathcal{L}) \triangleq \min_{\boldsymbol{h}^{\mathrm{T}}x=0,\ x\neq 0} \frac{x^{\mathrm{T}}(\mathcal{H}\mathcal{L}+\mathcal{L}^{\mathrm{T}}\mathcal{H})x}{2x^{\mathrm{T}}\mathcal{H}x}$, where $\mathcal{H} \triangleq \mathrm{diag}\{h_1, \dots, h_N\}$ [21].

II. Problem Statement

Consider a group of N ($N \geq 2$) linear dynamic agents that are described by

$$\dot{x}_i(t) = Ax_i(t) + B_u u_i(t),\ i = 1, \dots, N, \tag{1}$$

where $x_i \in \mathbb{R}^{n_x}$ and $u_i \in \mathbb{R}^{n_u}$ are the state and control input of agent i, respectively, and A and B_u are system matrices which are real and appropriately-dimensioned. It is assumed that the matrix pair (A, B_u) is stabilizable.

To achieve consensus, we aim at designing a distributed even-triggered state-feedback control protocol, where each controller can make use of the relative information between neighbours. Suppose that the overall communication topology is determined by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The relative state of agent i with regard to its neighbors is defined as

$$\tilde{x}_i(t) \triangleq \sum_{j\in\mathcal{N}_i} a_{ij}\left(x_i(t) - x_j(t)\right),$$

which can be accessed by agent i. However, instead of directly using the above relative signal, an event-based state-feedback protocol applying the sampled data of the above signal at discrete-time instants is of our interest. Specifically, consider the following control protocol:

$$u_i(t) = cK\tilde{x}_i(t_k^i),\ t \in [t_k^i, t_{k+1}^i),\ i = 1, \dots, N,$$
$$k = 0, 1, \dots, \tag{2}$$

where t_k^i is the kth sampling instant for the controller of agent i, K is a constant matrix with appropriate dimensions and c is a positive constant to be determined. Define T_k^i as

$$T_k^i \triangleq t_{k+1}^i - t_k^i, i = 1, \dots, N; k = 0, 1, \dots. \tag{3}$$

The following assumption is needed for later use.

Assumption 1: Graph $\mathcal{G}$ is strongly connected.

Under the above assumption, the consensus problem is to design a control protocol (2) such that the resulting closed-loop system satisfies $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0$, $i, j = 1, \dots, N$. In this paper, the triggering functions for $i = 1, \dots, N$ are chosen as

$$f_i(e_i(t), t) \triangleq \|Ke_i(t)\|^2 - \theta_i \mathrm{e}^{-\delta_i t},$$

where δ_i and θ_i are positive constants to be determined, and

$$e_i(t) \triangleq \tilde{x}_i(t_k^i) - \tilde{x}_i(t),$$

which is the error between the real-time relative state $\tilde{x}_i(t)$ and its sampled one $\tilde{x}_i(t_k^i)$ for agent i. The triggering instant t_{k+1}^i is determined as

$$t_{k+1}^i = \sup\{t | f_i(e_i(t), t) < 0, t \geq t_k^i\}.$$

At the triggering instant t_0^i, we have $e_i(t_0^i) = 0$. Thus, $f_i(e_i(t_0^i), t_0^i) = -\theta_i \mathrm{e}^{-\delta_i t_0^i} < 0$ holds. As t increases from t_0^i, $f_i(e_i(t), t) < 0$ maintains for a time interval until the change of $e_i(t)$ is such that $f_i(e_i(t), t) < 0$ is violated, which triggers the event $k = 1$. This process repeats and so on.

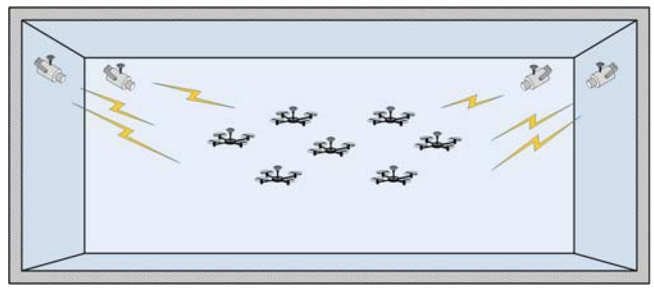

Fig. 1. An illustrative example on formation flying of indoor unmanned flying robots. The flying state of robots is monitored by external cameras, which transmit the obtained information to the robots for flight control.

Remark 1: Compared with the usual continuous-time state-feedback protocols using $\tilde{x}_i(t)$, the event-triggered state-feedback protocol (2) only updates the control input at the triggering instants. This is beneficial if an external monitoring layer (e.g., consisting of wireless sensors) is implemented separately to monitor the state of the physical layer (i.e., agents), and communicates with the physical layer through certain communication medium (see Fig. 1 for an illustrative example). However, to check the triggering condition, the external monitoring layer still needs to continuously monitor the states of neighboring agents because of $e_i(t)$. Interested readers can refer to the method in [18, Section 5] on how to generate $\tilde{x}_i(t)$ as well as $e_i(t)$ from sampled information, thus circumventing the above drawback, while in this paper, we will not specifically treat this issue.

Another kind of event-triggered protocols are like $u_i(t) = cK\sum_{j\in\mathcal{N}_i} a_{ij}\left(g(x_i(t_k^i), t) - g(x_j(t^j_{\bar{k}^j_t}), t)\right)$, where $\bar{k}_t^j \triangleq \arg\min_{l\in\{1,2,\dots\};t\geq t_l^j}\{t - t_l^j\}$, and g is a vector function, see [8]–[12]. Most of the existing results for this type merit in the fact that no continuous monitoring of neighboring agents is needed. However, these protocols usually require absolute state information about agents (in terms of $x_i(t_k^i)$ and/or $x_j(t_k^j), j \in \mathcal{N}_i$), which is unavailable in some applications where only relative information can be accurately accessed.

III. Main Results

A. Consensus Analysis

Let $x(t) = \mathrm{col}\{x_1(t), \dots, x_N(t)\}$, $e(t) = \mathrm{col}\{e_1(t), \dots, e_N(t)\}$ and $\tilde{x}(t) = \mathrm{col}\{\tilde{x}_1(t), \dots, \tilde{x}_N(t)\}$.

By combining (1) and (2), we can obtain the closed-loop system given by

$$\dot{x}(t) = (\mathbf{I} \otimes A)\, x(t) + (c\mathbf{I} \otimes B_u K)\, (e(t) + \tilde{x}(t)).$$

Since $\tilde{x}(t) = (\mathcal{L} \otimes \mathbf{I})\, x(t)$, we have

$$\dot{\tilde{x}}(t) = (\mathbf{I} \otimes A)\, \tilde{x}(t) + (c\mathcal{L} \otimes B_u K)\, (e(t) + \tilde{x}(t)). \quad (4)$$

Under Assumption 1, it is known from Lemma 1 that 0 is a single eigenvalue of the Laplacian $\mathcal{L}$ with $\mathbf{1}$ as the right eigenvector. Thus, it is easy to see that $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ if and only if $\tilde{x}(t) \to 0$. As a result, the consensus problem is solved if and only if the designed protocol guarantees the convergence of $\tilde{x}(t)$. With this fact in mind, we have the following theorem on the existence of the desired event-triggered state-feedback protocol.

Theorem 1: Consider the MAS (1) and the protocol (2) under Assumption 1. Then consensus is reached, if $K = -B_u^{\mathrm{T}} P$, where P is the positive-definite matrix solving the following equation for a given positive-definite matrix Q:

$$A^{\mathrm{T}} P + PA - PB_u B_u^{\mathrm{T}} P + Q = 0, \quad (5)$$

and θ_i and δ_i, $i = 1, \ldots, N$, are any positive constants, c is any positive constant satisfying

$$c \geq \frac{1}{2\alpha - \beta\gamma}, \; \beta \triangleq \bar{\lambda}(\mathcal{H}^{\frac{1}{2}} \mathcal{L}\mathcal{L}^{\mathrm{T}} \mathcal{H}^{\frac{1}{2}}), \quad (6)$$

where α and $\mathcal{H}$ are defined in Lemma 1, and γ is any constant satisfying $0 < \gamma < \frac{2\alpha}{\beta}$.

Proof: Consider a candidate Lyapunov function as

$$V(t) = \tilde{x}^{\mathrm{T}}(t)\, (\mathcal{H} \otimes P)\, \tilde{x}(t),$$

where $\mathcal{H}$ is defined in Lemma 1 and P is the positive definite matrix given in the theorem. From Lemma 1, it is known that $\mathcal{H}$ is positive definite. Thus, $V(t) \geq 0$, and $V(t) = 0$ if and only if $\tilde{x}(t) = 0$. Taking the derivative of $V(t)$ along the solution of $\tilde{x}(t)$ in (4) and substituting $K = -B_u^{\mathrm{T}} P$ into the derivative give rise to

$$\begin{aligned}\dot{V}(t) &= 2\tilde{x}^{\mathrm{T}}(t)\, (\mathcal{H} \otimes P)\, \dot{\tilde{x}}(t) \\ &= 2\tilde{x}^{\mathrm{T}}(t)\, (\mathcal{H} \otimes P)\, (\mathbf{I} \otimes A)\, \tilde{x}(t) \\ &\quad + 2\tilde{x}^{\mathrm{T}}(t)\, (\mathcal{H} \otimes P)\, (c\mathcal{L} \otimes B_u K)\, (e(t) + \tilde{x}(t)) \\ &= \tilde{x}^{\mathrm{T}}(t) \left[\mathcal{H} \otimes \left(A^{\mathrm{T}} P + PA\right)\right] \tilde{x}(t) \\ &\quad - \tilde{x}^{\mathrm{T}}(t) \left[c\left(\mathcal{L}^{\mathrm{T}}\mathcal{H} + \mathcal{H}\mathcal{L}\right) \otimes PB_u B_u^{\mathrm{T}} P\right] \tilde{x}(t) \\ &\quad - 2\tilde{x}^{\mathrm{T}}(t) \left(c\mathcal{H}\mathcal{L} \otimes PB_u B_u^{\mathrm{T}} P\right) e(t).\end{aligned}$$

Since $(\boldsymbol{h}^{\mathrm{T}} \otimes \mathbf{I})\tilde{x}(t) = (\boldsymbol{h}^{\mathrm{T}} \otimes \mathbf{I})\, (\mathcal{L} \otimes \mathbf{I})\, x(t)$, it follows from Lemma 1 that

$$\begin{aligned}&\tilde{x}^{\mathrm{T}}(t) \left[c\left(\mathcal{L}^{\mathrm{T}}\mathcal{H} + \mathcal{H}\mathcal{L}\right) \otimes PB_u B_u^{\mathrm{T}} P\right] \tilde{x}(t) \\ &\geq \tilde{x}^{\mathrm{T}}(t) \left(2c\alpha\mathcal{H} \otimes PB_u B_u^{\mathrm{T}} P\right) \tilde{x}(t).\end{aligned}$$

Moreover, it is easy to verify that

$$\begin{aligned}&- 2\tilde{x}^{\mathrm{T}}(t) \left(c\mathcal{H}\mathcal{L} \otimes PB_u B_u^{\mathrm{T}} P\right) e(t) \\ &\leq \tilde{x}^{\mathrm{T}}(t) \left(c\gamma\mathcal{H}\mathcal{L}\mathcal{L}^{\mathrm{T}}\mathcal{H} \otimes PB_u B_u^{\mathrm{T}} P\right) \tilde{x}(t) \\ &\quad + e^{\mathrm{T}}(t) \left(\frac{c}{\gamma}\mathbf{I} \otimes PB_u B_u^{\mathrm{T}} P\right) e(t) \\ &\leq \tilde{x}^{\mathrm{T}}(t) \left(c\gamma\beta\mathcal{H} \otimes PB_u B_u^{\mathrm{T}} P\right) \tilde{x}(t) \\ &\quad + e^{\mathrm{T}}(t) \left(\frac{c}{\gamma}\mathbf{I} \otimes PB_u B_u^{\mathrm{T}} P\right) e(t)\end{aligned}$$

holds for all $\gamma > 0$. It follows from the equation (5) and $c > \frac{1}{2\alpha - \beta\gamma} > 0$ (which is specified in (6)) that

$$\begin{aligned}\dot{V}(t) &\leq \tilde{x}^{\mathrm{T}}(t) \left\{\mathcal{H} \otimes \left[A^{\mathrm{T}} P + PA - c\,(2\alpha - \beta\gamma)\, PB_u B_u^{\mathrm{T}} P\right]\right\} \\ &\quad \times \tilde{x}(t) + e^{\mathrm{T}}(t) \left(\frac{c}{\gamma}\mathbf{I} \otimes PB_u B_u^{\mathrm{T}} P\right) e(t) \\ &\leq -\tilde{x}^{\mathrm{T}}(t)\, (\mathcal{H} \otimes Q)\, \tilde{x}(t) + e^{\mathrm{T}}(t) \left(\frac{c}{\gamma}\mathbf{I} \otimes K^{\mathrm{T}} K\right) e(t) \\ &= -\sum_{i=1}^{N} h_i \tilde{x}_i^{\mathrm{T}}(t) Q \tilde{x}_i(t) + \frac{c}{\gamma} \sum_{i=1}^{N} \|Ke_i(t)\|^2 .\end{aligned}$$

Noting $f_i(e_i(t), t) = \|Ke_i(t)\|^2 - \theta_i \mathrm{e}^{-\delta_i t} \leq 0$ and $Q \geq \underline{\lambda}(Q)\, \mathbf{I} \geq \frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)} P$, we have

$$\begin{aligned}\dot{V}(t) &\leq -\sum_{i=1}^{N} \frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)} h_i \tilde{x}_i^{\mathrm{T}}(t) P \tilde{x}_i(t) + \frac{c}{\gamma} \sum_{i=1}^{N} \|Ke_i(t)\|^2 \\ &\leq -\frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)} V(t) + \frac{cN\bar{\theta}}{\gamma} \mathrm{e}^{-\underline{\delta} t},\end{aligned}$$

where $\bar{\theta} = \max_i\{\theta_i\}$ and $\underline{\delta} = \min_i\{\delta_i\}$. Thus, it follows from the Comparison Principle (see [22, Lemma 3.4]) that

$$V(t) \leq V(0)\mathrm{e}^{-\lambda_{PQ} t} + \frac{cN\bar{\theta}}{\gamma\,(\underline{\delta} - \lambda_{PQ})} \left(\mathrm{e}^{-\lambda_{PQ} t} - \mathrm{e}^{-\underline{\delta} t}\right) \quad (7)$$

if $\underline{\delta} \neq \lambda_{PQ}$, or

$$V(t) \leq V(0)\mathrm{e}^{-\lambda_{PQ} t} + \frac{cN\bar{\theta}}{\gamma} t\mathrm{e}^{-\lambda_{PQ} t} \quad (8)$$

if $\underline{\delta} = \lambda_{PQ}$, where $\lambda_{PQ} \triangleq \frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)}$. Thus, $V(t) \to 0$ as $t \to \infty$, which implies $\tilde{x}(t) \to 0$ as $t \to \infty$, that is, consensus is reached. The proof is completed. ■

Theorem 1 shows that the consensus problem can be solved if the gain K is obtained by solving an algebraic Riccati equation and the scalar parameters satisfy a few constraints related to the graph Laplacian. Since (A, B_u) is stabilizable, it is well-known that the equation (5) always admits a solution $P > 0$ for any $Q > 0$. Moreover, obviously the constraint in (6) is feasible. Consequently, it is seen that Theorem 1 provides feasible solutions to the event-triggered consensus control problem on strongly connected graphs. Note that no specification has been imposed on θ_i and δ_i (except that they are positive). In the next subsection, we will show that these assigned parameters also guarantee the non-existence of singular triggering behavior or Zeno behavior.

Remark 2: Similar event-triggered protocols have also been studied for MASs in [13]–[17], which, however, are limited to some special cases or aim at different control problems. Double-integrator MASs over undirected and directed communication graphs are discussed in [14], [16], respectively. The results in [13], [17], though addressing general linear MASs, also deal with undirected graphs only.

Although the graph is allowed to be directed in [15], the concerned consensus problem therein is of the leader-follower kind, which does not cover the leaderless consensus problem considered in this paper. On the contrary, the MAS in this paper is not required to satisfy this specification. Consequently, these existing results *cannot* be straightforwardly applied to solve the leaderless consensus problem of general linear MASs considered in this paper.

B. Triggering Behavior Analysis

Singular triggering and Zeno behavior are two factors preventing the practical usefulness of event-triggering controllers. Singular triggering means no event will be triggered after a specific event, while Zeno behavior means there exists at least one finite time interval during which an infinite number of events are triggered. Thus, it is important to exclude these two kinds of behavior from the triggering process. The following theorem reveals the choice of the parameters in Theorem 1 excludes these undesirable behaviors, and further illuminates when strictly positive infimums are ensured for intervals of two successive events.

Theorem 2: Consider the MAS (1) and the protocol (2) under Assumption 1. With the protocol parameters specified in Theorem 1, for *every* positive θ_i and δ_i, $i=1,\ldots,N$, both of singular triggering behavior and Zeno behavior are excluded from the triggering process. Moreover, if $\underline{\delta}<\lambda_{PQ}$, then $\inf_{k=0,1,\ldots} T_k^i>0$ for those $\delta_i=\underline{\delta}$, $i=1,\ldots,N$.

Proof: We first prove the avoidance of singular triggering, that is, for any triggering instant t_k^i, the next triggering instant $t_{k+1}^i\in(t_k^i,\infty)$ must exist. Note that the triggering process implies $\|Ke_i(t)\|^2\le\theta_i\mathrm{e}^{-\delta_i t}$ for $t\in[t_k^i,t_{k+1}^i]$ and the equality holds for $t=t_{k+1}^i$. Since $\left\|Ke_i(t_k^i)\right\|^2=0<\theta_i\mathrm{e}^{-\delta_i t_k^i}$ and $\|Ke_i(t)\|^2$ is continuous with respect to $t\in[t_k^i,t_{k+1}^i)$, the inequality $\|Ke_i(t)\|^2<\theta_i\mathrm{e}^{-\delta_i t}$ can be maintained for a while or forever for $t>t_k^i$. In the proof of Theorem 1, we have shown $\tilde{x}(t)\to 0$ as $t\to\infty$, which implies $\left\|K\tilde{x}_i(t_k^i)-K\tilde{x}_i(t)\right\|\to\left\|K\tilde{x}_i(t_k^i)\right\|>0$ at $t\to\infty$. Moreover, $\theta_i\mathrm{e}^{-\delta_i t}\to 0$ as $t\to\infty$. In view of $e_i(t)=\tilde{x}_i(t_k^i)-\tilde{x}_i(t)$, it is easy to see that $\|Ke_i(t)\|^2=\left\|K\tilde{x}_i(t_k^i)-K\tilde{x}_i(t)\right\|^2=\theta_i\mathrm{e}^{-\delta_i t}>0$ eventually holds at some finite instant t_{k+1}^i, which triggers the event $k+1$. Thus, there exists an instant t_{k+1}^i such that $t_k^i<t_{k+1}^i<\infty$ for $k=0,1,\ldots$. That is, no singular triggering behavior occurs.

Next, we show the exclusion of Zeno behavior. To this end, we need to prove that T_k^i is strictly positive for all finite k and $t_k^i\to\infty$ as $k\to\infty$. Three cases are discussed in the following, respectively.

1) $\underline{\delta}>\lambda_{PQ}$ (Case I). Obviously, $-\frac{cN\bar{\theta}}{\gamma(\underline{\delta}-\lambda_{PQ})}\mathrm{e}^{-\underline{\delta}t}<0$. Thus, it follows from (7) that

$$V(t)\le\mu_1\mathrm{e}^{-\lambda_{PQ}t},\ \mu_1\triangleq V(0)+\frac{cN\bar{\theta}}{\gamma\left(\underline{\delta}-\lambda_{PQ}\right)},\tag{9}$$

which implies

$$\begin{aligned}\|\tilde{x}_i(t)\|&\le\frac{1}{\sqrt{\underline{\lambda}(P)h_i}}\sqrt{\sum_{j=1}^N\underline{\lambda}(P)h_j\|\tilde{x}_j(t)\|^2}\\&\le\sqrt{\frac{V(t)}{\underline{\lambda}(P)h_i}}\le\sqrt{\frac{\mu_1}{\underline{\lambda}(P)h_i}}\mathrm{e}^{-\frac{\lambda_{PQ}}{2}t},\\&\forall t\ge 0,\ i=1,\ldots,N.\end{aligned}\tag{10}$$

Noting $e_i(t)=\tilde{x}_i(t_k^i)-\tilde{x}_i(t)$ for $t\in[t_k^i,t_{k+1}^i)$, we have

$$\dot{e}_i(t)=-A\tilde{x}_i(t)-cB_uK\sum_{j\in\mathcal{N}_i}a_{ij}\left(\tilde{x}_i(t_k^i)-\tilde{x}_j(t_{\bar{k}_t^j}^j)\right),\tag{11}$$

where $t_{\bar{k}_t^j}^j$ is defined in Remark 1. Using (10) and (11) yields

$$\begin{aligned}\frac{\mathrm{d}\,\|Ke_i(t)\|}{\mathrm{d}t}&=\frac{e_i^{\mathrm{T}}(t)K^{\mathrm{T}}K\dot{e}_i(t)}{\|Ke_i(t)\|}\le\|K\dot{e}_i(t)\|\\&\le\|KA\|\,\|\tilde{x}_i(t)\|+\sum_{j=1}^N c\,\|l_{ij}KB_uK\|\left\|\tilde{x}(t_{\bar{k}_t^j}^j)\right\|\\&\le\kappa_i\mathrm{e}^{-\frac{\lambda_{PQ}}{2}t_k^i},\forall t\in[t_k^i,t_{k+1}^i),\end{aligned}$$

where

$$\kappa_i\triangleq\|KA\|\sqrt{\frac{\mu_1}{\underline{\lambda}(P)h_i}}+\sum_{j=1}^N c\,\|l_{ij}KB_uK\|\sqrt{\frac{\mu_1}{\underline{\lambda}(P)h_j}}.\tag{12}$$

Thus, according to the Comparison Lemma (see [22, Lemma 3.4]) and $\left\|Ke_i(t_k^i)\right\|=0$, we have

$$\|Ke_i(t)\|\le\kappa_i\mathrm{e}^{-\frac{\lambda_{PQ}}{2}t_k^i}(t-t_k^i),\ t\in[t_k^i,t_{k+1}^i).\tag{13}$$

Since $\left\|Ke_i(t_{k+1}^i)\right\|^2=\theta_i\mathrm{e}^{-\delta_i t_{k+1}^i}$, it is seen from (13) that

$$\left\|Ke_i(t_{k+1}^i)\right\|=\sqrt{\theta_i}\mathrm{e}^{-\frac{\delta_i}{2}t_{k+1}^i}\le\kappa_i\mathrm{e}^{-\frac{\lambda_{PQ}}{2}t_k^i}(t_{k+1}^i-t_k^i).$$

Further noting $\delta_i\ge\underline{\delta}>\lambda_{PQ}$, we obtain

$$\sqrt{\theta_i}\mathrm{e}^{\frac{\lambda_{PQ}-\delta_i}{2}t_k^i}\le\kappa_i\mathrm{e}^{\frac{\delta_i}{2}(t_{k+1}^i-t_k^i)}(t_{k+1}^i-t_k^i),$$

or equivalently

$$\frac{\sqrt{\theta_i}}{\kappa_i}\mathrm{e}^{\frac{\lambda_{PQ}-\delta_i}{2}t_k^i}\le\mathrm{e}^{\frac{\delta_i}{2}T_k^i}T_k^i.\tag{14}$$

If $\lim_{k\to\infty}t_k^i=t_\infty^i<\infty$, since $\lim_{k\to\infty}T_k^i=0$, (14) implies $0<\frac{\sqrt{\theta_i}}{\kappa_i}\mathrm{e}^{\frac{\lambda_{PQ}-\delta_i}{2}t_\infty^i}\le 0$. This cannot be true, and thus contrarily implies that $t_k^i\to\infty$ as $k\to\infty$. Moreover, it obviously follows from (14) that T_k^i is strictly positive for any finite horizon t_k^i. Consequently, no Zeno behavior exists.

2) $\underline{\delta}<\lambda_{PQ}$ (Case II). For this case, $\frac{cN\bar{\theta}}{\gamma(\underline{\delta}-\lambda_{PQ})}\mathrm{e}^{-\lambda_{PQ}t}<0$ and $V(0)\mathrm{e}^{-\lambda_{PQ}t}\le V(0)\mathrm{e}^{-\underline{\delta}t}$. Thus, it follows from (7) that

$$V(t)\le\mu_2\mathrm{e}^{-\underline{\delta}t},\ \mu_2\triangleq V(0)+\frac{cN\bar{\theta}}{\gamma\left(\lambda_{PQ}-\underline{\delta}\right)}.\tag{15}$$

By using similar arguments as those for Case I, we can show that the instant t_{k+1}^i satisfies

$$\sqrt{\theta_i}\mathrm{e}^{-\frac{\delta_i}{2}t_{k+1}^i}\le\kappa_i\mathrm{e}^{-\frac{\underline{\delta}}{2}t_k^i}(t_{k+1}^i-t_k^i),$$

where κ_i is similarly defined in (12) but with μ_1 replaced with μ_2. As a result, we have $\frac{\sqrt{\theta_i}}{\kappa_i}\mathrm{e}^{\frac{\underline{\delta}-\delta_i}{2}t_k^i}\le\mathrm{e}^{\frac{\delta_i}{2}T_k^i}T_k^i$, which again implies that $t_k^i\to\infty$ as $k\to\infty$, and moreover T_k^i is strictly positive for any finite horizon t_k^i. Thus, Zeno behavior is excluded for this case.

3) $\underline{\delta} = \lambda_{PQ}$ (Case III). For saving space, we omit the details for this case. Basically, it follows from similar arguments as those for the previous two cases.

At last, for Case II above, the inequality $\frac{\sqrt{\theta_i}}{\kappa_i}\mathrm{e}^{\frac{\underline{\delta}-\delta_i}{2}t_k^i} \leq \mathrm{e}^{\frac{\delta_i}{2}T_k^i}T_k^i$, when $\delta_i = \underline{\delta}$, reduces to $\frac{\sqrt{\theta_i}}{\kappa_i} \leq \mathrm{e}^{\frac{\delta_i}{2}T_k^i}T_k^i$, which always admits a strictly positive T_k^i even for $k \to \infty$, that is, $\inf_{k=0,1,\ldots} T_k^i > 0$ for those $\delta_i = \underline{\delta}$, $i = 1, \ldots, N$. ■

Remark 3: The results in [13], [14] cannot guarantee the exclusion of Zeno behavior. As is pointed out in [19], Zeno behavior for the protocols in [13], [14] will take place when the relative state $\tilde{x}_i(t)$ crosses zero. The problem arises from the fact that the triggering analysis therein does not prove $t_k^i \to \infty$ as $k \to \infty$. If $t_k^i \to t_\infty^i < \infty$ as $k \to \infty$, then an infinite number of events will be triggered before the finite horizon t_∞^i, that is, Zeno behavior is exhibited. Similar to [19], we employ the triggering function f_i containing an exponentially time-dependent term for this purpose, which, as is shown by Theorem 2, is important for Zeno-freeness.

Remark 4: Theorem 2 contains two main contributions. First, it shows that scalars θ_i and δ_i in the protocol can take any positive scalars so that Zeno behavior is excluded. Second, it further clarifies when the event intervals have strictly positive infimums. It is readily to see that the latter is stronger than the former. However, most of the existing results usually just illuminate one of the two aspects, for instance, [19] only discusses Zeno-freeness while [15] only finds positive infimums of event intervals. In addition, to achieve Zeno-freeness, the protocol in this paper ensures consensus, while that in [18] only guarantees bounded consensus.

C. Numerical Example

We employ a numerical example to demonstrate the effectiveness of the proposed protocol.

Example 1: Consider the MAS (1) with

$$A = \begin{bmatrix} 0.09 & -1.002 & 0.2 \\ 1 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}, \; B_u = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix},$$

and suppose that 8 agents communicate with each other according to the graph shown in Fig. 2, where the edge weights all are 1. Obviously, the graph is strongly connected.

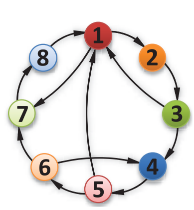

Fig. 2. The communication graph used in the example.

The eigenvalues of the Laplacian are 0, 2, 2.7172 ± 0.8220i, 1.6511 ± 0.8537i and 0.6317 ± 0.3221i, and the vector $\boldsymbol{h}$ defined in Lemma 1 is $\boldsymbol{h} = [0.1826,\ 0.4564,\ 0.4564,\ 0.2739,\ 0.5477,\ 0.3651,\ 0.0913,\ 0.1826]^{\mathrm{T}}$, which is directly output by the function `eig` of MATLAB. To design the protocol gain K, let $Q = \mathbf{I}$. Then the solution to the equation (5) is

$$P = \begin{bmatrix} 0.7106 & 0.4460 & 0.5525 \\ 0.4460 & 1.8014 & 1.3786 \\ 0.5525 & 1.3786 & 5.8000 \end{bmatrix},$$

which leads to the protocol gain

$$K = \begin{bmatrix} -1.4212 & -0.8919 & -1.1050 \end{bmatrix}.$$

Direct calculation gives $\alpha = 0.3647$, $\beta = 2.9131$, $\underline{\lambda}(Q) = 1$ and $\bar{\lambda}(P) = 6.3072$. Choose $\gamma = \frac{\alpha}{\beta} = 0.1252 \leq \frac{2\alpha}{\beta}$ and $c = 5.4834 = \frac{1}{2\alpha - \beta\gamma}$. Moreover, for the scalars δ_i, we can take $\delta_i = \frac{\underline{\lambda}(Q)}{\bar{\lambda}(P)} = 0.0793$, $i = 1, \ldots, N$.

To demonstrate the effectiveness of the protocol parameters found above, simulation results are presented in Figs. 3–6. For simulation, the scalars θ_i in the triggering condition all take 1. Fig. 3 displays the agent states, Figure 4 shows the control input, Fig. 5 demonstrates the event instants and Fig. 6 depicts the evolution of $\|Ke_i(t)\|^2$ and $\theta_i\mathrm{e}^{-\delta_i t}$ (right), and event intervals T_k^i (right). It is clear from Figs. 3 and 4 that the closed-loop MAS with the designed event-triggered protocol reaches state consensus. Fig. 5 demonstrates that each agent determines its own event-triggered instants, thus the triggering mechanism works in an asynchronous manner. Fig. 6 illustrates that event intervals are strictly positive, implying that no Zeno behavior of the protocol is exhibited. Moreover, it is seen that relatively dense events are triggered at the beginning for most agents in this example.

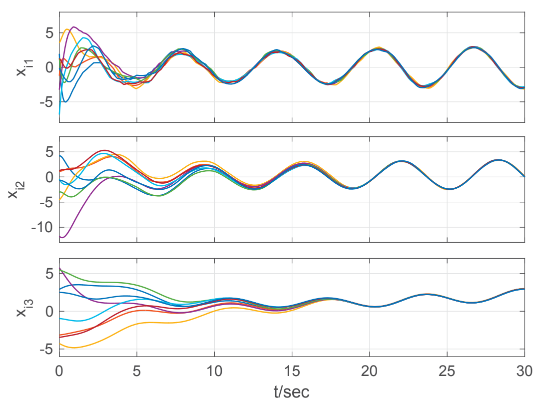

Fig. 3. States of agents.

IV. Conclusion

In this paper, the leaderless consensus problem of general linear MASs has been investigated under a distributed event-triggered state-feedback protocol over directed graphs. Under the assumption that the communication graph is strongly connected, a sufficient condition has been established for the existence of the protocol, which needs to solve an algebraic Riccati equation and some scalar parameters related to the graph Laplacian. Theoretical analysis has shown that no singular triggering behavior or Zeno behavior is exhibited

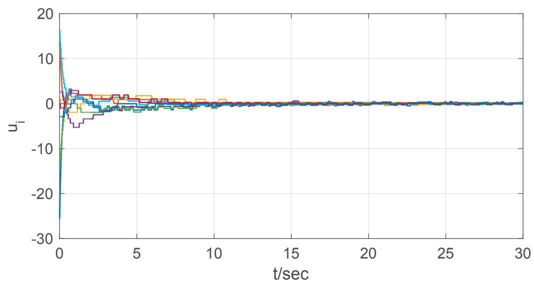

Fig. 4. Control input.

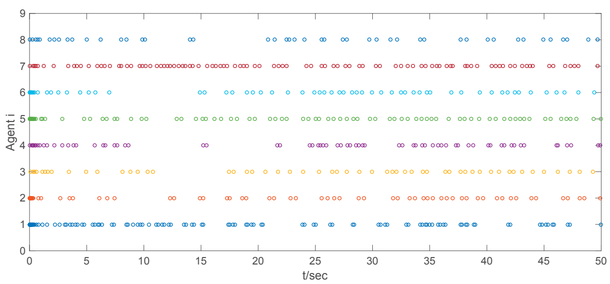

Fig. 5. Event instants.

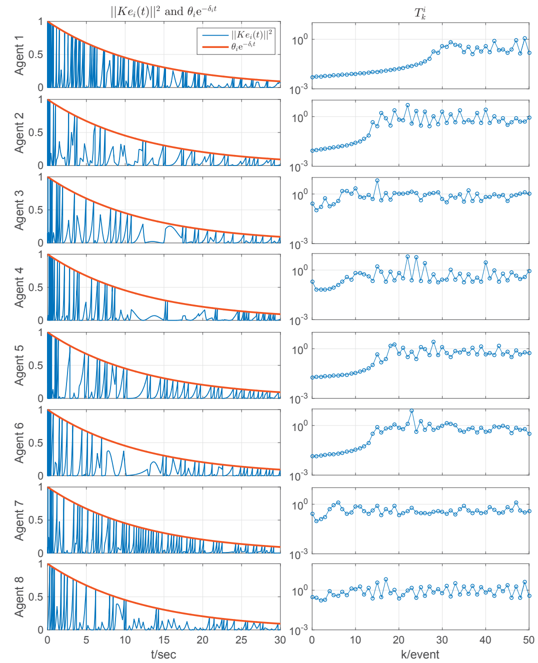

Fig. 6. Triggering process. *Left:* $\|Ke_i(t)\|^2$ and $\theta_i \mathrm{e}^{-\delta_i t}$; *Right:* Event intervals T_k^i for $k \leq 50$.

in the triggering process of the protocol. The effectiveness of the proposed protocol has been clearly illustrated by a numerical example.

This paper is only concerned with leaderless consensus of general linear MASs on strongly connected graphs. More systematic and comprehensive results on general directed graphs are still undergoing investigation.

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