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Duality Methods for Dynamic Portfolio Optimization with Constraints

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Abstract

This thesis studies the dynamic portfolio optimization problem of an investor who seeks to maximize his expected utility from terminal wealth, trades continuously in time, and whose portfolio allocation is restricted by constraints. This thesis aims to provide new solution methods that lead to closed-form characterizations for the optimal terminal wealth and optimal portfolio process for realistic settings with sophisticated financial market models and constraints that have previously not been discussed extensively in the existing academic literature. In this spirit, we developed a dual methodology that provides several equivalent dual optimality conditions which can be used to solve the portfolio optimization problem under simultaneous constraints on the investor's relative portfolio allocation and terminal wealth for general financial market models. We illustrate the usefulness of this methodology in a Black-Scholes model with convex cone allocation constraints, pointwise bounds on terminal wealth, and simultaneous Value-at-Risk or expected shortfall constraints by characterizing the investor's optimal terminal wealth up to two deterministic constants. Furthermore, by viewing one of the dual optimality conditions from a stochastic control perspective, we provide a condition under which the exponentially affine separability structure of the investor's value function in stochastic factor models is retained when allocation constraints are introduced. By verifying this condition in financial market models with stochastic short rate or stochastic volatility, we can ensure that the optimal constrained portfolio process remains deterministic and can be determined by solving an associated Riccati differential equation. We verify this condition and derive a closed-form solution for the optimal allocation-constrained portfolio in Heston's stochastic volatility model and make a surprising observation: The optimal constrained portfolio in a Heston market is not always equal to the 'naïve' capped portfolio, which caps off the optimal unconstrained portfolio at the boundaries of the constraints. Thus, our results suggest that allocation constraints have a fundamentally different impact on the optimal portfolio in financial markets with stochastic volatility compared to markets with constant volatility. In extensive numerical experiments, we demonstrate that following the naïve capped portfolio causes high annual wealth-equivalent losses for risk-averse investors in volatile markets, which can be experienced during a financial crisis.

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Contents

1 Introduction

1.1 Motivation & Objective

Expected utility theory has been widely used within the realm of decision theory to determine and discuss how people should behave in uncertain environments. As such, it has been widely applied in mathematical finance and in particular in the context of dynamic portfolio optimization. Such applications can be roughly divided by their underlying model for the financial market into discrete time (see e.g. [68] and [79]) and continuous time (see e.g. [64], [65]). While discrete time models can in principle perfectly reflect how prices of exchange-traded assets change (after all, physical limitations ensure that order routing and processing at exchanges occur in discrete time), they lack analytical tractability and closed-form solutions for all but the most simple return distributions. For this reason, the analysis of sophisticated and realistic discrete time models may require complex or computation heavy numerical schemes. In contrast, applications of continuous time models yield tractable closed-form solutions in a variety of contexts and are widely employed in the mathematical finance literature and in the financial industry, despite the fact that they require the heuristic assumption of continuous trading.
In this spirit, this thesis considers several variations of a dynamic portfolio optimization problem for an investor who aims to maximize his expected utility and trades continuously in time. As suggested by the thesis title, we additionally consider an aspect which extremely relevant for investors trading in the financial markets: constraints, especially allocation constraints.
In most practical applications investors need to abide by allocation constraints either due to regulatory requirements or due to client preferences. [85] and [49] first presented a duality approach for portfolio optimization problems with constraints on short-selling and trading of individual assets. Their duality approach was later generalized for general convex allocation constraints in [17], who derived a dual optimal control problem which seeks the least favorable market coefficients among a suitable set of 'dual' stochastic processes. For CRRA-utility functions, [17] determined the optimal constrained portfolio process in closed-form up to a deterministic minimizer of a real convex optimization problem. [89] considered a similar setting in a one-dimensional Black Scholes market, but did not employ any duality techniques. Rather, the author characterized the value function as the unique viscosity solution of the associated HJB PDE and gave a semi-explicit expression of the optimal portfolio allocation in terms of the value function. From both [17] and [89], one can easily see that the optimal constrained portfolio allocation for an investor with a CRRA utility function in the Black-Scholes model is equal to the unconstrained optimal solution if the constraints are satisfied, and is otherwise capped at the boundary of the constraint. Ever since, allocation constraints have been integrated into portfolio optimization problems in a myriad of ways (see e.g. [16], [71], [60], [66], [6], [70], [23] and [24]). However, only in rare cases were closed-form formulae or solution methods presented which do not require extensive Monte-Carlo simulation. The most notable exceptions to this observation are financial market models and utility functions which result in a myopic optimal portfolio process. This includes the logarithmic utility function for general diffusion models as well as the power utility function for stochastic factor models where the stochastic factor only describes systematic risk (of which the Black-Scholes model is a special case), see [17] and [71]. In other cases, solution to the allocation constrained portfolio optimization problem need to be estimated by sophisticated numerical schemes (e.g. [19], [92]).

Even though the continuous time dynamic portfolio optimization problem under allocation constraints has been extensively studied in the existing literature, we make the argument that there is a lack of tractable closed-form solutions for many realistic settings and therefore, the main upside of continuous time over discrete time financial market models is hardly utilized in the existing literature. Hence, the objective of this thesis is to extend the existing literature on constrained portfolio optimization and provide new methodologies which yield new closed-form solutions to existing problems or provide otherwise new insights about them. Specifically, we

- extend the auxiliary framework of [17] to include simultaneous constraints on relative portfolio allocation and terminal wealth (Chapter 3),
- develop a novel solution approach for allocation constrained portfolio optimization problems in incomplete financial markets with market dynamics influenced by an external stochastic factor (Chapter 4),
- and apply the aforementioned approach to derive closed form solution formulae for the optimal portfolio in Heston's stochastic volatility model (Chapter 5).

1.2 Structure of the Thesis

The structure of this thesis is as follows: In Chapter 2 we begin by collecting a selection of useful results from different fields. As this thesis focusses on portfolio optimization problems with (allocation) constraints, we pay special attention to convex duality theory and some of its applications to constrained optimization in Section 2.1. Moreover, we formally define the overarching setting of this thesis and give an overview of two classic methods for (unconstrained) portfolio optimization in continuous time: The martingale method and the stochastic control approach.

In Chapter 3, we consider a portfolio optimization problem with simultaneous constraints on portfolio allocation and terminal wealth. We propose a generalized martingale method which is applicable under the presence of constraints on terminal wealth in complete financial markets. Afterwards, we proceed to integrate this methodology into the auxiliary market framework of [17] and derive a set of equivalent dual optimality conditions for portfolio optimization problems with simultaneous constraints on portfolio allocation and terminal wealth. The optimality conditions correspond to different duality approaches generated by the allocation constraints or the wealth constraints. If the allocation constraints are a convex cone and risky asset prices follow a geometric Brownian motion, then we can derive closed-form expressions for the optimal terminal wealth as a function of an adjusted pricing kernel and a deterministic Lagrange multiplier. This is demonstrated for constraints on Value-at-Risk and expected shortfall.

In Chapter 4, we study a portfolio optimization problem in an incomplete financial market where the risky asset dynamics depend on stochastic factors and the relative portfolio allocation is constrained to lie within a given convex set. We employ fundamental duality results from real constrained optimization to formally derive a dual representation of the associated HJB PDE. Using this representation, we provide a condition on the market dynamics and the allocation constraints, which ensures that the solution to the HJB PDE is exponentially affine and separable. This condition is used to derive an explicit expression for the optimal allocation-constrained portfolio up to a deterministic minimizer and the solution to a system of Riccati ODEs in a market with CIR volatility and in a market with multi-factor OU short rate.

In Chapter 5, we specifically consider a portfolio optimization problem with convex constraints in Heston's stochastic volatility model. Naturally, this setting is a special case of Chapter 5 and we may thus apply the previously developed duality methods developed to obtain a closed-form

expression for the optimal portfolio allocation. In doing so, we observe that allocation constraints impact the optimal constrained portfolio allocation in a fundamentally different way in Heston's stochastic volatility model than in the Black Scholes model. In particular, the optimal constrained portfolio may be different from the naive 'capped' portfolio, which caps off the optimal unconstrained portfolio at the boundaries of the constraints. Despite this difference, we illustrate by way of a numerical analysis that in most realistic scenarios the capped portfolio leads to slim annual wealth equivalent losses compared to the optimal constrained portfolio. During a financial crisis, however, a capped solution might lead to compelling annual wealth equivalent losses.
Chapter 6 summarizes the main findings of this thesis and gives an outlook on potential future research.
All detailed mathematical derivations and proofs have been moved to Appendix A to improve readability.

1.3 Contributions

This thesis is based on and parts of it have been quoted verbatim from the following research articles:

Chapter 3:

[34] Marcos Escobar-Anel, Michel Kschonnek and Rudi Zagst. 'Portfolio Optimization: Not Necessarily Concave Utility and Constraints on Wealth and Allocation'. In: *Mathematical Methods of Operations Research* 95 (2022), pp. 101–140.

Chapter 4:

[33] Marcos Escobar-Anel, Michel Kschonnek and Rudi Zagst. 'Portfolio Optimization with Allocation Constraints and Stochastic Factor Market Dynamics'. 2023. arXiv: `2303.09835 [q.fin.PM]`

Chapter 5:

[32] Marcos Escobar-Anel, Michel Kschonnek and Rudi Zagst. 'Mind the Cap! – Constrained Portfolio Optimisation in Heston's Stochastic Volatility Model'. 2023. arXiv: `2306.11158 [q.fin.PM]`

As part of the elite doctoral program 'TopMath', an earlier version of [34] has been accredited as M.Sc. thesis of the author of this thesis. However, the inclusion of expected value constraints and the discussion of the associated dual optimization problems in Chapter 3 constitute an innovation which has not been published as part of [34].
Further, the author of this thesis has co-authored the following publications, which are not part of this thesis:

[35] Marcos Escobar-Anel, Yevhen Havrylenko, Michel Kschonnek and Rudi Zagst. 'Decrease of capital guarantees in life insurance products: Can reinsurance stop it?'. In: *Insurance: Mathematics and Economics* 105 (2022), pp. 14–40.

[58] Michel Kschonnek, Iryna Dobrovolska, Ulrike Protzer and Rudi Zagst. 'COVIX–An Index Allowing for the Assessment of the Pandemic Situation Based on Infections and Hospitalisation Data'. In: *Applied Sciences* 13.7 (2023).

2 Mathematical Preliminaries

2.1 Convex Analysis

2.1.1 Basic Notions from Convex Analysis

The majority of concepts and results presented in this section can be found in [76] and [29]. Throughout this section we consider a general vector space V over $\mathbb{R}$ with a scalar product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ and induced norm $\| \cdot \| : V \to \mathbb{R}$. Note that all results that hold for V also hold for its natural extension with the real line $V_{\mathbb{R}} := V \times \mathbb{R}$ and the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{R}} : V_{\mathbb{R}}^2 \to \mathbb{R}$, $\langle (x, \alpha), (y, \beta) \rangle_{\mathbb{R}} = \langle x, y \rangle + \alpha\beta$. An easy illustrating example would be $\mathbb{R}^d$ for some $d \in \mathbb{N}$ with the standard scalar product $\langle x, y \rangle = x'y$ and the Euclidean distance $\|x\| = \sqrt{x'x}$ as norm. Further, we define the extended real numbers as $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$.

Definition 2.1.1 (Convex Sets)**.** *A set $K \subset V$ is called convex if for all $x, y \in K$ every point on the line segment connecting x and y is again in K, i.e., $\forall \lambda \in (0, 1)$ we have*

$$\lambda x + (1 - \lambda) y \in K.$$

The convex hull co K of a set $K \subset V$ is the smallest convex set containing K, i.e.,

$$co\ K = \bigcap_{\substack{M \text{ convex} \\ K \subset M}} M.$$

Definition 2.1.2 (Open and Closed Sets)**.** *A set $K \subset V$ is open, if for all $x \in K$, there exists $\epsilon > 0$ such that*

$$\|x - y\| \leq \epsilon \quad \Rightarrow y \in K, \quad \forall y \in V.$$

A set $K \subset V$ is closed, if its complement $V \backslash K$ is open.

The closure cl K of a set $K \subset V$ is the smallest closed set containing K, i.e.,

$$cl\ K = \bigcap_{\substack{M \text{ closed} \\ K \subset M}} M.$$

Clearly, any $K \subset V$ is closed if and only if *cl* $K = K$. Similarly, K is convex if and only if *co* $K = K$. Moreover, if K is indeed closed *and* convex, then there exists a more concise characterization of K in terms of closed half spaces.

Definition 2.1.3 (Closed Half Space)**.** *For given $(y, \beta) \in V_{\mathbb{R}}$, the set*

$$H(y, \beta) = \{x \in V \mid \langle x, y \rangle \leq \beta\}$$

is called a closed half space.

Lemma 2.1.4 (Closed Convex Hull as Intersection of Half Spaces)**.** *Let $K \subset V$ be arbitrary. Then,*

$$cl\ co\ K = \bigcap_{\substack{K \subset H(y,\beta) \\ (y,\beta) \in V_{\mathbb{R}}}} H(y, \beta).$$

In particular, K is closed and convex (closed convex) if and only if

$$K = \bigcap_{\substack{K \subset H(y,\beta) \\ (y,\beta) \in V_{\mathbb{R}}}} H(y, \beta).$$

Since we will at a later point almost exclusively be considering concave optimization problems (maximization) instead of convex optimization problems (minimization) we introduce the following classic tools for the analysis of concave and convex functions (epigraphs, hypographs, conjugates, etc.) for concave functions only.

Definition 2.1.5 (Domain in Concave Sense and Hypograph)**.** *For a function $f : V \to \bar{\mathbb{R}}$, we define its effective domain $dom_f \subset V$ and its hypograph $H_f \subset V_{\mathbb{R}}$ as*

$$dom_f = \{x \in V \mid f(x) > -\infty\}$$
$$H_f = \{(x, \alpha) \in V_{\mathbb{R}} \mid f(x) \geq \alpha\}.$$

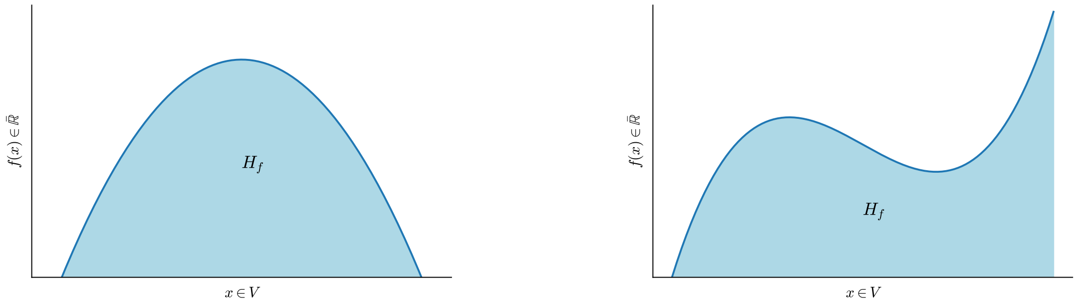
Geometrically, one may think of the hypograph of a function f as the closure of the area below the graph of a function f. It is easy to see that the hypograph of an affine function $h : V \to \mathbb{R}$ defines a half space in $V_{\mathbb{R}}$. Note that the hypograph characterizes a function uniquely and every function $f : V \to \bar{\mathbb{R}}$ can be reconstructed from its hypograph. We frequently switch between the classic representation of a function f and the representation of a function f through its hypograph. More precisely, given a hypograph $H_f \subset V_{\mathbb{R}}$, we can recover f by setting

$$f(x) := \sup \{\alpha \mid (x, \alpha) \in H_f\},$$

where we make use of the convention $\sup \emptyset = -\infty$ and $\sup \mathbb{R} = \infty$.

Definition 2.1.6 (Convex & Concave Functions)**.** *A function $f : V \to \bar{\mathbb{R}}$ is concave if its hypograph H_f is convex. If additionally there exists an $\hat{x} \in V$ such that $f(\hat{x}) > -\infty$ and $f(x) < \infty$ $\forall x \in V$, then f is called a proper concave function.*
If $-f$ is concave, then f is called a convex function.

Remark 2.1.7. *Analogously to Definition 2.1.6, we also refer to non-concave functions $f : V \to \bar{\mathbb{R}}$ as proper if $f(x) < \infty$ for all $x \in V$ and $f(\hat{x}) > -\infty$ for some $\hat{x} \in V$.*



(a) (b)

Figure 2.1: Illustration of the hypograph H_f for a function f which is concave (Figure 2.1(a)) and non-concave (Figure 2.1(b)).

Definition 2.1.6 is slightly more abstract than usual, but has the advantage that we can classify functions taking values only in $\{-\infty, \infty\}$ as concave (or non-concave) without encountering the undefined operations $-\infty + \infty$ or $\infty - \infty$. If we restrict ourselves to functions $f : V \to \bar{\mathbb{R}}$ with $f(x) < \infty \ \forall x \in V$, then we recover the usual definition of concavity.

Lemma 2.1.8. *Consider a proper function $f : V \to \bar{\mathbb{R}}$. Then, f is concave if and only if it satisfies*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \quad \forall x, y \in V, \lambda \in (0,1). \tag{2.1}$$

Definition 2.1.9 (Usc Functions)**.** *A function $f : V \to \bar{\mathbb{R}}$ is upper semi-continuous ('usc') if*

$$\textit{for any } \alpha \in \mathbb{R}, \quad M_f(\alpha) = \big\{x \in V \mid f(x) \geq \alpha\big\} \textit{ is closed in } V\ . \tag{2.2}$$

An equivalent characterization is

$$\limsup_{x' \to x} f(x') \leq f(x) \quad \forall x \in V. \tag{2.3}$$

Analogous to the convex hull *co* K and closure *cl* K of a set $K \subset V$, we may define similar notions for functions $f : V \to \bar{\mathbb{R}}$.

Definition 2.1.10 (Concave Hull, Usc Hull and Closure)**.** *Consider a function $f : V \to \bar{\mathbb{R}}$.*

(i) The concave hull co $f : V \to \bar{\mathbb{R}}$ is the smallest concave function greater or equal than f.

(ii) The usc hull usc $f : V \to \bar{\mathbb{R}}$ is the smallest usc function greater or equal than f.

(iii) The closure cl $f : V \to \bar{\mathbb{R}}$ is defined as

$$cl\ f(x) = \begin{cases} usc\ f(x), & \forall x \in V,\ if\ usc\ f(x) < \infty\ \forall x \in V \\ \infty, & \forall x \in V,\ if\ usc\ f(x) = \infty\ for\ some\ x \in V \end{cases}$$

If *usc* f (resp. *usc co* f) is proper, then *cl* $f \equiv$ *usc* f (resp. *cl co* $f \equiv$ *usc co* f). This technical construction allows us to characterize *usc co* f by the closed convex hull of its hypograph and *cl co* f as the minimum of affine functions h larger or equal than f. This illustrates the strong link between the concave hull, usc hull and closure of a function f and the convex hull and closure of its hypograph H_f.

Lemma 2.1.11. *Consider a proper function $f : V \to \bar{\mathbb{R}}$. Then the following holds:*

(i) f *is usc if and only if* H_f *is closed in* $V_{\mathbb{R}}$

(ii) $cl\ H_f = H_{usc\ f}$

(iii) $co\ H_f = H_{co\ f}$

(iv) $H_{usc\ co\ f} = cl\ co\ H_f = \Big(cl\ co\ dom_f \times \mathbb{R}\Big) \cap \Big(\bigcap_{\substack{h:V\to\mathbb{R}\ affine \\ f\le h}} H_h \Big)$

(v) $cl\ co\ f(x) = \inf_{z\in V} \Big(\sup_{\hat{x}\in V} \big(f(\hat{x}) - \langle z, \hat{x}\rangle\big) + \langle z, x\rangle \Big)$

Especially statement (iv) and (v) of Lemma 2.1.11 and their proof are quite remarkable. Not only is illustrated that it is equivalent to compute the usc hull and concave hull *usc co f* of a function f directly or computing the closure of the concave hull $cl\ co\ H_f$ of H_f and deriving *usc co f* from it, but it connects these two equivalent operations to a min-max-problem for f.

The final statements of Lemma 2.1.11 motivate the definition of the (concave) conjugate and bi-conjugate of a given function $f : V \to \bar{\mathbb{R}}$:

Definition 2.1.12 (Concave Conjugate)**.** *Consider a function $f : V \to \bar{\mathbb{R}}$. We define the conjugate $f^* : V \to \bar{\mathbb{R}}$ and the bi-conjugate $f^{**} : V \to \bar{\mathbb{R}}$ as*

- $f^*(y) = \sup_{x\in V} \big(f(x) - \langle y, x\rangle\big)$ *for* $y \in V$
- $f^{**}(x) = \inf_{y\in V} \big(\langle y, x\rangle + f^*(y)\big)$ *for* $x \in V$.

Going back to the proof of Lemma 2.1.11 (iv), for a given slope $z \in V$, the conjugate $f^*(z)$ can be identified as the minimal intercept β_z such that the affine function $h_z : V \to \mathbb{R}, \quad h_z(x) = \beta_z + \langle x, z\rangle$ is larger or equal than f, i.e., such that the hypograph H_h is contained in the half space H_{h_z}. Finding z_x such that $h_{z_x}(x) \ge f(x)$ is minimized, we take the bi-conjugate $f^{**}(x)$ of f. This is in turn equivalent to taking the intersection of all half spaces H_{h_z}, which contain H_f.

Remark 2.1.13. *Let $x \in V$ be arbitrary but fixed. Then, so-called "weak duality" holds, i.e.,*

$$\begin{aligned} f^{**}(x) &= \inf_{y\in V} \big(\langle y, x\rangle + \underbrace{f^*(y)}_{\ge f(x) - \langle y, x\rangle} \big) \\ &\ge \inf_{y\in V} \big(\langle y, x\rangle + f(x) - \langle y, x\rangle\big) = \inf_{y\in V} \big(f(x)\big) = f(x) \end{aligned}$$

Remark 2.1.14. *In the convex analysis literature (see e.g. equation (3.25) in [76]) it is customary to define the conjugate of a function $f : V \to \bar{\mathbb{R}}$ in the concave sense as*

$$f^*(y) = -\sup_{x\in V} \big(f(x) - \langle y, x\rangle)\big) = \inf_{x\in V} \big(\langle y, x\rangle - f(x)\big).$$

In contrast to our Definition 2.1.12, this has the satisfying consequence that the bi-conjugate of f is obtained from taking the conjugate of f^*, i.e., $f^{**} = (f^*)^*$.

According to our definition we have the (slightly) less elegant version $f^{**} = -(-f^*)^*$. However, in the mathematical finance literature, in particular in a portfolio optimization context, Definition 2.1.12 is more prevalent. Since, we are going to apply the above theory in a portfolio optimization context, we decided to stick to the corresponding convention.

We summarize some useful properties of the conjugate and bi-conjugate in Lemma 2.1.15:

Lemma 2.1.15. *Consider a proper function $f : V \to \bar{\mathbb{R}}$. The following holds:*

(i) $-f^*$ and f^{**} are usc and concave

(ii) f is usc and concave if and only if

$$f(x) = f^{**}(x) = cl\ co\ f(x) \quad \forall x \in V.$$

Example 2.1.16 (Indicator Functions of Convex Sets)**.** *Consider a non-empty closed convex set $K \subset V$ and define its indicator function $\mathcal{X}_K : V \to \bar{\mathbb{R}}$ as*

$$\mathcal{X}_K(x) = \begin{cases} 0, & \text{if } x \in K \\ -\infty, & \text{if } x \notin K. \end{cases}$$

This indicator function is intentionally chosen differently from the standard indicator function $\mathbb{1}_K(x)$, which takes values in $\{0, 1\}$, depending on whether $x \in K$ or not. Regardless, the above concept comes in handy, when considering concave optimization problems that are constrained over convex sets K. The concave conjugate of $\mathcal{X}_K$ is given as

$$\begin{aligned} \mathcal{X}_K^*(y) &= \sup_{x \in V} \big(\mathcal{X}_K(x) - \langle y, x\rangle\big) = \sup_{x \in K} \big(- \langle y, x\rangle\big) \\ &= -\inf_{x \in K} \big(\langle y, x\rangle\big) =: \delta_K(y). \end{aligned}$$

The function δ_K is called the support function of K and plays an important role in constrained concave optimization problems. The domain of δ_K is the barrier cone

$$X_K = dom_{\delta_K} = \big\{ y \in V \;\big|\; \delta_K(y) < \infty \big\}.$$

The support function δ_K offers a natural characterization of closed convex sets in a dual sense. As K is a closed convex set, the indicator function $\mathcal{X}_K$ is usc and concave. Thus, by Lemma 2.1.15, we have $\mathcal{X}_K = \mathcal{X}_K^{**}$. This yields

$$\begin{aligned} x \in K &\Leftrightarrow \mathcal{X}_K(x) \geq 0 \\ &\Leftrightarrow \mathcal{X}_K^{**}(x) \geq 0 \\ &\Leftrightarrow \inf_{y \in V} \big(\delta_K(y) + \langle x, y\rangle\big) \geq 0 \\ &\Leftrightarrow \delta_K(y) + \langle x, y\rangle \geq 0 \ \forall y \in V \\ &\Leftrightarrow \delta_K(y) + \langle x, y\rangle \geq 0 \ \forall y \in X_K \end{aligned} \tag{2.4}$$

Due to its significance for the remainder of the thesis, we summarize the key properties of the support function δ_K below.

Lemma 2.1.17. *Let $K \subset V$ be a non-empty closed, convex set. Then,*

(i) δ_K is positive homogeneous of order 1:

$$\delta_K(\alpha y) = \alpha \delta_K(y) \quad \forall \alpha \geq 0, \ \forall y \in V.$$

(ii) δ_K is sub-additive:

$$\delta_K(y_1 + y_2) \leq \delta_K(y_1) + \delta_K(y_2) \quad \forall y_1, y_2 \in V.$$

(iii)

$$\begin{aligned} x \in K \quad &\Leftrightarrow \quad 0 \leq \delta_K(y) + \langle x, y\rangle \quad \forall y \in X_K \\ &\Leftrightarrow \quad 0 \leq \delta_K(y) + \langle x, y\rangle \quad \forall y \in X_K \text{ with } \langle y, y\rangle \leq 1. \end{aligned}$$

(iv) $\delta_K(y) = 0 \quad \forall y \in X_K \;\Leftrightarrow\; K$ is a convex cone.

2.1.2 Concave Optimization

In the following, we consider a proper function $f : V \to \bar{\mathbb{R}}$ and the primal optimization problem $(\mathbf{P})$ of the form

$$(\mathbf{P}) \begin{cases} \Phi_P = \sup_{x \in V} f(x). \end{cases} \tag{2.5}$$

The representation (2.5) is sufficiently general so that classic examples of constraints can be incorporated into this formulation. Similar to a portfolio optimization context, we consider two different types of constraints:

(i) Constraints explicitly defined on $x \in V$, i.e., we require that $x \in K$ for a subset $K \subset V$.

(ii) Constraints implicitly defined through a function $g : V \to H$, i.e., we require that $g(x) \in K$ for a subset $K \subset H$ of another vector space H over $\mathbb{R}$.

Assuming we aim to maximize a proper function $f : V \to \bar{\mathbb{R}}$ subject to the constraint (i). Then, we can rewrite this constrained maximization over f as an unconstrained maximization problem over a suitable function $\tilde{f}$ by considering $\mathcal{X}_K : V \to \{-\infty, 0\}$ as defined in Example 2.1.16 and defining

$$\sup_{x \in K} f(x) = \sup_{x \in V} \big(\underbrace{f(x) + \mathcal{X}_K(x)}_{=:\tilde{f}(x)} \big) = \sup_{x \in V} \tilde{f}(x). \tag{2.6}$$

Note that $\mathcal{X}_K$ is usc and concave if and only if K is closed convex. Hence, if f is usc and concave, $\tilde{f}$ remains usc and concave if K is closed convex. Similarly, if we aim to maximize f subject to constraint (ii), then we may use the same notation as above, where $\mathcal{X}_K : H \to \{-\infty, 0\}$ now takes arguments $u \in H$, and write

$$\sup_{x \in V,\ g(x) \in K} f(x) = \sup_{x \in V} \big(\underbrace{f(x) + \mathcal{X}_K(g(x))}_{=:\tilde{f}(x)} \big) = \sup_{x \in V} \tilde{f}(x). \tag{2.7}$$

Clearly, we could have done the same transformation as in (2.6) by defining the set

$$\tilde{K} := \{x \in V \mid g(x) \in K\}$$

and maximizing over $x \in \tilde{K}$. However, we will later see that the distinction between (2.6) and (2.7) can still be useful. Again, $\tilde{f}$ is usc concave if f is usc concave and $\tilde{K}$ is closed convex.

We now follow the framework presented in [76] to define a dual optimization problem $(\mathbf{D})$ corresponding to the primal optimization problem $(\mathbf{P})$. For this purpose consider another vector space H over $\mathbb{R}$ with scalar product $\langle \cdot, \cdot \rangle$ and assume that there exists a function

$$F : V \times H \to \bar{\mathbb{R}}, \quad \text{with } F(x, \bar{u}) = f(x) \text{ for some } \bar{u} \in H.$$

One may think of F as a perturbation of f or as a function, which expresses the dependence of F on the parameter choice $\bar{u} \in H$. Clearly, the choice of F for a given f is not unique, but can be chosen freely, although there may exist a *natural* choice for F. Regardless, the following results hold for any choice of F, but their usefulness is heavily dependent on the specific choice of F.

Definition 2.1.18. *For any $u \in H$, we define the perturbed problem $(\mathbf{P_u})$ as*

$$(\mathbf{P_u}) \begin{cases} \Phi(u) = \sup_{x \in V} F(x, u). \end{cases} \tag{2.8}$$

In particular, we clearly have $\Phi(\bar{u}) = \Phi_P$.

Definition 2.1.19. *We define the Lagrangian of F as $L : V \times H \to \bar{\mathbb{R}}$ as*

$$L(x,\lambda) := \sup_{u\in H} \big(F(x,u) - \langle u,\lambda\rangle\big) + \langle \bar{u},\lambda\rangle.$$

Moreover, we define the dual problem (**D**) *as*

$$(\mathbf{D}) \begin{cases} \Psi_D &= \inf_{\lambda\in H} G(\lambda) \\ G(\lambda) &= \sup_{x\in V} L(x,\lambda). \end{cases} \tag{2.9}$$

Example 2.1.20. *Consider a proper function $f : V \to \mathbb{R}$, which we would like to optimize over the closed convex set $K \subset V$. We may equivalently optimize over the function $\tilde{f}(x) := f(x) + \mathcal{X}_K(x)$ instead (as seen in (2.6)). We assume that there exists at least one feasible $x \in V$ with $\tilde{f}(x) > -\infty$, i.e., $\tilde{f}$ is proper, too.*
Choosing $H = V$, we define a perturbation function $F : V \times V \to \mathbb{R}$ by setting

$$F(x,u) = f(x) + \mathcal{X}_K(x+u) = f(x) + \mathcal{X}_{K-u}(x).$$

As hinted by the last equality, this can be regarded as a perturbation to the constraint set K through an affine shift $K - u$. In particular for $\bar{u} := 0$, we clearly have $F(x,\bar{u}) = f(x)$ for $x \in K$. For fixed $(x,\lambda) \in V^2$, the Lagrangian L can be calculated as

$$\begin{aligned} L(x,\lambda) &= \sup_{u\in H} \big(F(x,u) - \langle u,\lambda\rangle\big) + \langle \bar{u},\lambda\rangle \\ &= \sup_{u\in V} \big(f(x) + \mathcal{X}_K(x+u) - \langle u,\lambda\rangle\big) + \underbrace{\langle 0,\lambda\rangle}_{=0} \\ &= f(x) + \sup_{u\in V} \big(\mathcal{X}_K(x+u) - \langle u,\lambda\rangle\big) \\ &= f(x) + \sup_{u\in V} \big(\mathcal{X}_K(\underbrace{x+u}_{=v\in V}) - \langle \underbrace{x+u}_{=v\in V},\lambda\rangle + \langle x,\lambda\rangle\big) \\ &= f(x) + \langle x,\lambda\rangle + \underbrace{\sup_{v\in V} \big(\mathcal{X}_K(v) - \langle v,\lambda\rangle\big)}_{\mathcal{X}_K^*(\lambda)=\delta_K(\lambda)} \\ &= f(x) + \langle x,\lambda\rangle + \delta_K(\lambda). \end{aligned}$$

The objective function G of the dual problem (**D**) *is thus given by*

$$G(\lambda) = \sup_{x\in V} f(x) + \langle x,\lambda\rangle + \delta_K(\lambda),$$

which is the value function of an unconstrained optimization over the objective function $f(x) + \langle x,\lambda\rangle + \delta_K(\lambda)$, where the additional term $\langle x,\lambda\rangle + \delta_K(\lambda)$ is non-negative whenever $x \in K$ (see Lemma 2.1.17). In this sense, this adjusted objective function implicitly rewards $x \in V$ which abide by the constraint $x \in K$. In particular, we have $G(\lambda) \geq \Phi_P$ for all $\lambda \in H = V$. Correspondingly, the dual problem (**D**) *in this context reads as*

$$(\mathbf{D}) \begin{cases} \Psi_D &= \inf_{\lambda\in V} G(\lambda) \\ G(\lambda) &= \sup_{x\in V} \big(f(x) + \langle x,\lambda\rangle\big) + \delta_K(\lambda), \end{cases} \tag{2.10}$$

i.e., the additional reward gained by the removal of the constraint K and the additional term $\langle x,\lambda\rangle + \delta_K(\lambda)$ is minimized.

Example 2.1.21. *Consider functions $f : V \to \mathbb{R}$, $g : V \to H$. We would like to maximize f over all $x \in V$ such that $g(x) \in K$, for a closed convex set $K \subset H$. By setting*

$$\tilde{f}(x) = f(x) + \mathcal{X}_K(g(x))$$

we obtain our usual representation of (**P**)*. By defining the perturbed objective function $F : V \times H \to \mathbb{R}$ as*

$$F(x, u) = f(x) + \mathcal{X}_K(g(x) + u),$$

we obtain the corresponding Lagrangian L (using the same calculations as in Example 2.1.20) as

$$L(x, \lambda) = f(x) + \langle g(x), \lambda \rangle + \delta_K(\lambda).$$

It is important to note that in this particular example it is possible that $H \neq V$ and hence $\lambda \in H$ may be of an entirely different form than $x \in V$. The dual optimization problem (**D**) *is thus defined as*

$$(\mathbf{D}) \begin{cases} \Psi_D &= \inf_{\lambda \in H} G(\lambda) \\ G(\lambda) &= \sup_{x \in V} \big(f(x) + \langle g(x), \lambda \rangle \big) + \delta_K(\lambda). \end{cases} \tag{2.11}$$

Lemma 2.1.22. *The Lagrangian $L : V \times H \to \bar{\mathbb{R}}$ has the following properties:*

(i) If $F(x, \cdot) : H \to \bar{\mathbb{R}}$ is proper for a given $x \in V$, then $-L(x, \cdot) : H \to \bar{\mathbb{R}}$ is usc and concave

(ii) If $F(x, \cdot) : H \to \bar{\mathbb{R}}$ is proper, usc and concave for a given $x \in V$, then

$$\inf_{\lambda \in H} L(x, \lambda) = F(x, \bar{u}) = f(x)$$

(iii) If $F(x, \cdot) : H \to \bar{\mathbb{R}}$ is proper, usc and concave $\forall x \in V$, then the primal problem (**P**) *can equivalently be written as*

$$(\mathbf{P}) \begin{cases} \Phi_P = \sup_{x \in V} \inf_{\lambda \in H} L(x, \lambda) \end{cases} \tag{2.12}$$

Theorem 2.1.23. *The following duality relation holds between the optimization problems* (**P**) *and* (**D**)*:*

$$\Phi(\bar{u}) = \sup_{x \in V} f(x) = \Phi_P \leq \Psi_D = \inf_{\lambda \in H} \sup_{x \in V} L(x, \lambda) = cl\ co\ \Phi(\bar{u}).$$

By virtue of Theorem 2.1.23, we have successfully derived a duality representation between (**P**) and (**D**). The optimal value of both optimization problems coincides if and only if the concave closure of Φ coincides with Φ at the parameter choice $u = \bar{u}$, i.e.,

$$\Phi_P = \Psi_D \quad \Leftrightarrow \quad \Phi(\bar{u}) = cl\ co\ \Phi(\bar{u}).$$

The difference $\Psi_D - \Phi_P$ is also referred to as duality gap. If $\Phi_P = \Psi_D$, we say the duality gap is zero. If the dependence of F on the perturbation (or parameter) u is u.s.c and concave, then according to *(iii)* in Lemma 2.1.22, we view (**P**) as a max-min problem and (**D**) as a min-max problem over the same Lagrangian L. When both (**P**) and (**D**) obtain the optimal value at an $(x^*, \lambda^*) \in V \times H$, we call (x^*, λ^*) a saddle-point.

Definition 2.1.24. *$(x^*, \lambda^*) \in V \times H$ is called a saddle-point of L, if*

$$L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda) \quad \forall (x, \lambda) \in V \times H.$$

Remark 2.1.25. *In the current abstract context, when $(x^*, \lambda^*) \in V \times H$ is a saddle-point we also say that (x^*, λ^*) satisfies the so-called Karush-Kuhn-Tucker condition ('KKT condition').*

Corollary 2.1.26. *If $(x^*, \lambda^*) \in V \times H$ is a saddle-point of L, then λ^* is optimal for the dual optimization problem $(\mathbf{D})$.*
If additionally $F(x^*, \cdot)$ is proper, usc and concave in u, then x^* is optimal for the primal optimization problem $(\mathbf{P})$.

In summary, we obtain the following equivalent statements:

Theorem 2.1.27. *Let $F(x^*, \cdot)$ be proper, usc and concave in u for a specific $x^* \in V$. Consider the following statements:*

(i) $\Phi_P = \Psi_D$

(ii) $\Phi(\bar{u}) = cl\ co\ \Phi(\bar{u})$

(iii) *x^* is optimal for $(\mathbf{P})$, λ^* is optimal for $(\mathbf{D})$ and $\Phi_P = \Psi_D$*

(iv) *(x^*, λ^*) is a saddle-point of the Lagrangian L.*

Then, (i) $\Leftrightarrow$ (ii) and (iii)$\Leftrightarrow$ (iv).

Theorem 2.1.27 does not provide direct properties of F, which ensure that either of the statements $(i) - (iv)$ are satisfied. However, it is possible to derive several properties for F, which guarantee that statement (ii) in Theorem 2.1.27 is satisfied and therefore the duality gap must be zero. The so-called Slater's condition is one prominent example.

Theorem 2.1.28 (Slater's condition)**.** *Assume F is concave jointly in (x, u) and there exists an $\hat{x} \in V$ such that $F(\hat{x}, \cdot)$ is bounded below on a neighbourhood of $\bar{u}$. Then, Φ is concave and continuous (hence usc) at $\bar{u}$. In particular, we then have*

$$\Phi(\bar{u}) = cl\ co\ \Phi(\bar{u}).$$

Throughout this thesis, we apply the duality theory developed in this section for two different applications: constrained optimization over $V = H = \mathbb{R}^d$ and constrained optimization over $V = H = L^2_Q$ on some complete probability space $(\Omega, \mathcal{F}, Q)$. We continue by developing useful duality results for these applications below.

Constrained Optimization over $\mathbb{R}^d$: Consider a real-valued usc function $f : \mathbb{R}^d \to \mathbb{R}$ and a non-empty closed convex set $K \subset \mathbb{R}^d$. Then, $V = H = \mathbb{R}^d$ and the standard scalar product $\langle x, y \rangle = x'y$ for $x, y \in \mathbb{R}^d$ form a Hilbert space on which we can apply the previously developed duality techniques. Consider the primal optimization problem

$$(\mathbf{P}) \begin{cases} \Phi_P & = \sup_{x \in \mathbb{R}^d} f(x) \\ \text{s.t.} & x \in K. \end{cases} \tag{2.13}$$

This setting is a special case of Example 2.1.20 throughout this Section. In particular, we can obtain the Lagrangian of $(\mathbf{P})$ directly from Example 2.1.20.

Lemma 2.1.29. *Consider $(\mathbf{P})$ as defined in (2.13) and define the associated Lagrangian as $L : \mathbb{R}^d \times \mathbb{R}^d \to \bar{\mathbb{R}}$ with*

$$L(x, \lambda) = f(x) + x'\lambda + \delta_K(\lambda).$$

If (x^*, λ^*) is a saddle point of L, then x^* is optimal for $(\mathbf{P})$.

Lemma 2.1.29 leads to the following dual optimization problem $(\mathbf{D})$ corresponding to $(\mathbf{P})$ from (2.13).

$$(\mathbf{D}) \left\{ \Psi_D = \inf_{\lambda \in \mathbb{R}^d} \left(\sup_{x \in \mathbb{R}^d} (f(x) + x'\lambda) + \delta_K(\lambda) \right) \right. .$$

Similarly, we can derive the KKT conditions for $(\mathbf{P})$ and $(\mathbf{D})$.

Corollary 2.1.30 (KKT Conditions)**.** *Consider $(\mathbf{P})$ as defined in (2.13). $(x^*, \lambda^*) \in \mathbb{R}^d \times \mathbb{R}^d$ is a saddle point of L if and only if*

$$\begin{aligned} &(i) && x^* \text{ maximizes } f(x) + \langle x, \lambda^* \rangle + \delta_K(\lambda^*) \text{ over } x \in \mathbb{R}^d, \\ &(ii) && (x^*)'\lambda^* + \delta_K(\lambda^*) = 0, \\ &(iii) && x^* \in K. \end{aligned} \tag{2.14}$$

Theorem 2.1.31. *Consider $(\mathbf{P})$ as defined in (2.13) and let $\lambda^* \in \mathbb{R}^d$ be optimal for $(\mathbf{D})$. If there exists a function $x^* : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\sup_{x \in \mathbb{R}^d} L(x, \lambda) = L(x^*(\lambda), \lambda) \quad \forall \lambda \in X_K$$

and $x^*(\lambda)'\Delta\lambda$ is usc at $\lambda = \lambda^*$ for all $\Delta\lambda \in X_K$, then $(x^*(\lambda^*), \lambda^*)$ satisfies (2.14). In particular, $x^*(\lambda^*)$ is optimal for $(\mathbf{P})$ and $\Phi_P = \Psi_D$.

Corollary 2.1.32. *Consider $(\mathbf{P})$ as defined in (2.13) and let*

$$f(x) = -x'Ax + b'x + c,$$

for a symmetric positive definite matrix $A \in \mathbb{R}^{d \times d}$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Then, there exists a unique optimal λ^* for $(\mathbf{D})$, the requirements of Theorem 2.1.31 are satisfied and in particular

$$\sup_{x \in K} f(x) = \inf_{\lambda \in \mathbb{R}^d} \sup_{x \in \mathbb{R}^d} L(x, \lambda) = \inf_{\lambda \in \mathbb{R}^d} \left(\sup_{x \in \mathbb{R}^d} \Big(f(x) + x'\lambda \Big) + \delta_K(\lambda) \right).$$

Constrained Optimization over L^2_Q: Consider a complete probability space $(\Omega, \mathcal{F}, Q)$ and let L^2_Q be the space of $\mathcal{F}$-measurable random variables X with $\mathbb{E}[X^2] < \infty$. Then, $V = L^2_Q$ with $\langle X, Y\rangle = \mathbb{E}[X \cdot Y]$ is a Hilbert space on which we can employ the previously developed duality theory. Further, consider a proper usc function $U : \mathbb{R} \to \bar{\mathbb{R}}$, a function $g = (g_1, ..., g_n) : \mathbb{R}^n \to \bar{\mathbb{R}}$ such that each g_i is usc and proper, and a non-empty closed convex set $K \subset \mathbb{R}^n$. Note that g is allowed to be random, as long as its values are $\mathcal{F}_T$-measurable. Then, we define the fully constrained optimization problem over L^2_Q as

$$(\mathbf{P}) \begin{cases} \Phi_P & = \sup_{D \in L^2_Q} \mathbb{E}\left[U(D)\right] \\ \text{s.t.} & \mathbb{E}\left[g(D)\right] \in K. \end{cases} \tag{2.15}$$

Lemma 2.1.33. *Consider* $(\mathbf{P})$ *as defined in (2.15). Define the associated Lagrangian* $L : L^2_Q \times \mathbb{R}^n \to \bar{\mathbb{R}}$ *as*

$$L(D, y) = \mathbb{E}\left[U(D)\right] + y'\mathbb{E}\left[g(D)\right] + \delta_K(y). \tag{2.16}$$

If (D^*, y^*) *is a saddle point of* L, *then* D^* *is optimal for* $(\mathbf{P})$.

Lemma 2.1.33 leads to the following dual optimization problem $(\mathbf{D})$ corresponding to $(\mathbf{P})$ from (2.15).

$$(\mathbf{D}) \left\{ \Psi_D = \inf_{y \in \mathbb{R}^n} \left(\sup_{D \in L^2_Q} \left(\mathbb{E}\left[U(D)\right] + y'\mathbb{E}\left[g(D)\right]\right) + \delta_K(y) \right) \right. . \tag{2.17}$$

Definition 2.1.34. *In the setting of* $(\mathbf{P})$ *as defined in (2.15), we define the generalized conjugate* U^*_g *and its maximizing argument* $\mathcal{I}_g$ *as*

$$\begin{aligned} U^*_g : \mathbb{R}^n \to \bar{\mathbb{R}}, \quad & U^*_g(y) = \sup_{x \in \mathbb{R}} \big(U(x) + y'g(x)\big) \\ \mathcal{I}_g : \mathbb{R}^n \to \bar{\mathbb{R}}, \quad & \mathcal{I}_g(y) = \inf\big\{x \in \mathbb{R} \bigm| U^*_g(y) \le U(x) + y'g(x)\big\} \end{aligned} \tag{2.18}$$

Lemma 2.1.35. *Consider* $(\mathbf{P})$ *as defined in (2.15) and* $(\mathbf{D})$ *as in (2.17). Then,*

$$\Psi_D \le \inf_{y \in \mathbb{R}^n} \left(\mathbb{E}\left[U^*_g(y)\right] + \delta_K(y)\right), \tag{2.19}$$

with equality if the infimum is attained at $y^* \in \mathbb{R}^n$ *such that* $\mathcal{I}_g(y^*) \in L^2_Q$.

Note that it is not totally unreasonable that the infimum in 2.19 is attained, as $U^*_g(y) + \delta_K(y)$ is convex in y (see Lemmas 2.1.17 and Lemma 2.1.38). We may use the expression from Lemma 2.1.35 to again derive the KKT conditions for $(\mathbf{P})$, respectively $(\mathbf{D})$.

Corollary 2.1.36 (KKT Conditions)**.** *Consider* $(\mathbf{P})$ *as in (2.15). Let* $y^* \in \mathbb{R}^n$ *be determined such that* $\mathcal{I}_g(y^*) \in L^2_Q$,

$$(y^*)'\, \mathbb{E}\left[g(\mathcal{I}_g(y^*))\right] + \delta_K\left(y^*\right) = 0 \quad \text{and} \quad \mathbb{E}\left[g(\mathcal{I}_g(y^*))\right] \in K. \tag{2.20}$$

Then, $(\mathcal{I}_g(y^*), y^*)$ *is a saddle-point of the Lagrangian* L. *In particular,* $\mathcal{I}_g(y^*)$ *is optimal for* $(\mathbf{P})$, y^* *is optimal for* $(\mathbf{D})$ *and* $\Phi_P = \Psi_D$.

Moreover, if the maximizing argument $\mathcal{I}_g$ satisfies an upper semi-continuity assumption, we can show that any optimal solution $y^* \in \mathbb{R}^n$ to (**D**) satisfies (2.20). Under this assumption, any $y^* \in \mathbb{R}^n$ which is optimal for (**D**) directly defines a saddle-point of L and an optimal solution to (**P**) as shown in Corollary 2.1.36. Here, the maximizing argument $\mathcal{I}_g$ takes the same role as x^* in Theorem 2.1.31.

Theorem 2.1.37. *Consider* (**P**) *as in (2.15) and let $y^* \in \mathbb{R}^n$ attain the infimum in (2.19). If $\mathcal{I}_g(y^*) \in L^2_Q$ and*

$$y \to \mathbb{E}\left[g(\mathcal{I}_g(y))\right]' \Delta y$$

is usc at $y = y^*$ for all $\Delta y \in X_K$, then y^* satisfies (2.20). In particular, $\mathcal{I}_g(y^*)$ is optimal for (**P**) *and $\Phi_P = \Psi_D$.*

We summarize a number of useful properties of the generalized conjugate and its maximizing argument for use in later chapters.

Lemma 2.1.38.

(i) If g_i is non-increasing and $g_i(x) \leq 0$ for all $x \in dom_U$, then $U^*_g(y)$ and $\mathcal{I}_g(y)$ are non-increasing in y_i.

(ii) Define $\hat{g} = (g_1, ..., g_{i-1}, \hat{g}_i, g_{i+1}, ..., g_n)'$ for a given function $\hat{g}_i : \mathbb{R} \to \bar{\mathbb{R}}$. If $g_i, \hat{g}_i$ are non-increasing, $g_i(x) \leq \hat{g}_i(x)$ for all $x \in dom_U$ and $y_i \geq 0$, then

$$U^*_g(y) \leq U^*_{\hat{g}}(y) \quad \text{and} \quad \mathcal{I}_g(y) \leq \mathcal{I}_{\hat{g}}(y).$$

(iii) U^*_g is a convex function.

2.2 Ordinary Differential Equations

In this section, selected results from the theory of ordinary differential equations are summarized.

Lemma 2.2.1.

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, for every $x_0 \in \mathbb{R}$, $\epsilon > 0$, there exists an $L > 0$ such that if $|x - x_0| < \epsilon$, $|y - x_0| < \epsilon$, then

$$|f(x) - f(y)| < L|x - y|.$$

In particular, f is locally Lipschitz continuous.

(ii) Consider an interval $I \subset \mathbb{R}$ and let $f_1, f_2 : I \to \mathbb{R}$ be Lipschitz continuous on I. Then, $f := min(f_1, f_2)$ is Lipschitz continuous on I.

Theorem 2.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous on $\mathbb{R}$ and $\tau_0, B_0 \in \mathbb{R}$ be constants. Consider the ordinary differential equation*

$$B'(\tau) = f(B(\tau)), \quad B(\tau_0) = B_0. \tag{2.21}$$

Then there exists an $\epsilon > 0$ such that the ODE (2.21) has a unique solution $B(\tau)$ for $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$.

Lemma 2.2.3. *Consider the setting of Theorem 2.2.2 and an open interval $I \subset \mathbb{R}$ with $\tau_0 \in I$, and let $B : I \to \mathbb{R}$ satisfy ODE (2.21). Then, B is a monotone function in τ. If $f(B_0) \neq 0$, then B is strictly monotone in τ. Moreover, if $f(B_0) = 0$, then B is constant in τ.*

Lemma 2.2.4. *Consider a terminal time point $T > 0$, real coefficients B_0, r_0, r_1, r_2 and the following Riccati-ODE*

$$B'(\tau) = -r_0 + r_1 B(\tau) + \frac{1}{2} r_2 B(\tau)^2, \quad B(0) = B_0, \tag{2.22}$$

where $r_2 \neq 0$, $r_1^2 + 2r_0 r_2 > 0$ and define $r_3 = \sqrt{r_1^2 + 2r_0 r_2}$.

(i) The function

$$B(\tau) = \frac{2r_2 r_3 B_0 + \left(e^{r_3\tau} - 1\right)\left(r_1 + r_3\right)\left(r_1 + r_2 B_0 - r_3\right)}{2r_2 r_3 - r_2\left(e^{r_3\tau} - 1\right)\left(r_1 + r_2 B_0 - r_3\right)} \tag{2.23}$$

is the unique solution of equation (2.22) on its maximal interval $[0, t_+(B_0))$ with life-time $t_+(B_0) > 0$. Moreover, for any $T \in [0, t_+(B_0))$ we have

$$\int_0^T B(\tau) d\tau = \frac{2}{r_2} \ln\left(\frac{2r_3 e^{\frac{r_3 - B_0}{2} T}}{r_3\left(e^{r_3 T} + 1\right) - r_1\left(e^{r_3 T} - 1\right) - r_2\left(e^{r_3 T} - 1\right) B_0} \right). \tag{2.24}$$

(ii) The life time $t_+(B_0)$ of B (in the sense of Lemma 10.1 in [38]) is given as

$$t_+(B_0) = \begin{cases} \frac{1}{r_3} \ln\left(\frac{r_1 + r_2 B_0 + r_3}{r_1 + r_2 B_0 - r_3}\right), & \text{if } r_1 + r_2 B_0 - r_3 > 0 \\ \infty, & \text{if } r_1 + r_2 B_0 - r_3 \leq 0. \end{cases}$$

Corollary 2.2.5. *Consider the setting of Lemma 2.2.4, let B be as in (2.23) and let $\hat{B} \in \mathbb{R}$ be given.*
If

$$\tau_{\hat{B}} := \frac{1}{r_3} \ln\left(\frac{2r_2 r_3\left(\hat{B} - B_0\right) + \left(r_1 + r_2 B_0 - r_3\right)\left(r_1 + \hat{B} r_2 + r_3\right)}{\left(r_1 + r_2 B_0 - r_3\right)\left(r_1 + \hat{B} r_2 + r_3\right)} \right) \leq t_+(B_0)$$

then $B(\tau_{\hat{B}}) = \hat{B}$.

2.3 The Continuous-Time Portfolio Optimization Problem

2.3.1 Problem Formulation

Throughout this thesis we consider different variations of a continuous-time portfolio optimization problem with finite investment horizon $T > 0$. The underlying setting assumes a complete, filtered probability space $(\Omega, \mathcal{F}_T, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, Q)$, where the filtration $\mathbb{F}$ is generated by the independent m-dimensional Wiener process $W^z = (W^z(t))_{t\in[0,T]}$ and d-dimensional Wiener process $\hat{W} = \left(\hat{W}(t)\right)_{t\in[0,T]}$. The set of all $\mathcal{F}_T$-measurable random variables will be denoted by $\mathbf{X}$ and $\mathbf{X}_+$ denotes the restriction of $\mathbf{X}$ to Q-a.s. non-negative random variables. Moreover, we consider an $\mathbb{R}^{m\times d}$-valued stochastic process ρ, which is assumed to be progressively measurable with respect to the natural filtration of W^z and whose columns ρ_i, $i = 1, ..., d$ satisfy

$$\|\rho_i(t,\omega)\| \leq 1 \quad \mathcal{L}[0,T] \otimes Q - \text{a.e..}^1$$

This construction allows us to define another d-dimensional $\mathbb{F}$-Wiener process $W = (W_1, ..., W_d)'$ through the equations

$$W_i(t) = \rho_i(t)' W^z(t) + \sqrt{1 - \|\rho_i(t)\|^2}\hat{W}_i(t), \quad i = 1, ..., d.$$

The Wiener process W is defined in such way that it correlates with W^z, since its covariation with W^z satisfies

$$d\langle W_i^z, W_j\rangle_t = \rho_{ij}(t)dt \quad \mathcal{L}[0,T] \otimes Q - \text{a.e..}$$

We employ this setting to define a market model $\mathcal{M}$ with d risky assets (e.g. stocks) $P = (P_1, ..., P_d)'$ and a risk-free asset (e.g. a bank account) P_0, which satisfy $P_0(0) = P_1(0) = ... = P_d(0) = 1$ and evolve according to the dynamics

$$\begin{aligned} dP(t) &= \operatorname{diag}(P(t))\big(\mu(t)dt + \Sigma(t)dW(t)\big) \\ dP_0(t) &= P_0(t)r(t)dt, \end{aligned} \tag{2.25}$$

where we used $\operatorname{diag}(P(t)) \in \mathbb{R}^{d\times d}$ to denote the diagonal matrix with diagonal entries $P(t) \in \mathbb{R}^d$. The volatility matrix is assumed to be $\mathcal{L}[0,T] \otimes Q$-a.e. non-singular and thereby has a well-defined inverse.[2] Moreover, the market coefficients, i.e., the risk-free interest rate r, the mean rate of return μ as well as the volatility matrix Σ and its inverse Σ^{-1} with columns $\Sigma_1^{-1}, ..., \Sigma_d^{-1}$, are assumed to be progressively measurable processes w.r.t. $\mathbb{F}$ such that

$$\sup_{t\in[0,T]} \Big\{ \max\big(|r(t)|, \|\mu(t)\|^2, \|\Sigma(t)\|^2, \|\Sigma(t)^{-1}\|^2\big)\Big\} < \infty \quad Q\text{-a.s.}, \tag{2.26}$$

where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^d$ and its induced matrix norm on $\mathbb{R}^{d\times d}$. This ensures that the SDEs in (2.25) have well-defined solutions. While the Wiener process W is the diffusion of the risky asset prices, the market coefficients may depend also on the Wiener process W^z. Unless W and W^z are perfectly correlated, this implies that not all randomness in the financial market $\mathcal{M}$ can be hedged by trading in the risk-free and risky assets.

Within the financial market $\mathcal{M}$, we consider a single investor with initial wealth $v_0 > 0$ at time $t = 0$, who trades continuously in time. The investor's wealth process is assumed to satisfy $V^{v_0,\pi}(0) = v_0$ and the SDE

$$dV^{v_0,\pi}(t) = V^{v_0,\pi}(t)\Big([r(t) + (\mu(t) - r(t)\mathbb{1})'\pi(t)]dt + \pi(t)'\Sigma(t)dW(t)\Big). \tag{2.27}$$

The d-dimensional portfolio process π is chosen by the investor and determines the fraction of wealth $\pi_i(t)$ that is allocated to the risky asset P_i at time t, while the remaining fraction $1 - \sum_{i=1}^d \pi_i(t)$ is allocated to the risk-free asset. Note that $1 - \sum_{i=1}^d \pi_i(t)$ may be negative, in which case the investor goes short the risk-free asset, or more intuitively, borrows from the bank account. To ensure that the investor allocates his wealth solely based on past price developments and to ensure that (2.27) is well-defined, we restrict the admissible portfolio processes π to

[1] $\mathcal{L}[0,T]$ denotes the Lebesgue-measure on $[0,T]$, $\omega \in \Omega$ is used to denote the elements of the sample space Ω and $\|\cdot\|$ denotes the standard Euclidean norm.

[2] For this assumption it is for example sufficient if $\Sigma(t)\Sigma(t)'$ is strongly positive definite $\mathcal{L}[0,T] \otimes Q - a.e.$, i.e., if

$$\|\Sigma(t)'x\|^2 \geq \xi\|x\|^2, \quad \forall x \in \mathbb{R}^d, \quad \mathcal{L}[0,T] \otimes Q - a.e.,$$

for some constant $\xi > 0$.

$$\Lambda = \Bigg\{ \pi = \left((\pi_1(t), ..., \pi_d(t))'\right)_{t\in[0,T]} \text{ progr. measurable } \Bigg| \ \mathbb{E}\left[\left(V^{v_0,\pi}(T)\right)^2\right] < \infty, \int_0^T \|\pi(t)\|^2 \, dt < \infty \ Q-a.s. \Bigg\} \tag{2.28}$$

In particular, if $\pi \in \Lambda$, then the investor's terminal wealth has finite variance and it is straightforward to show that the unique solution to (2.27) is given by

$$V^{v_0,\pi}(t) = v_0 \exp\left(\int_0^t r(s) + (\mu(s) - r(s)\mathbb{1})' \pi(s) - \frac{1}{2} \left\|\Sigma(s)'\pi(s)\right\|^2 ds + \int_0^t \pi(s)'\Sigma(s) dW(s) \right).$$

We assume that the investor aims to choose his portfolio process $\pi \in \Lambda$ to maximise his utility derived from terminal wealth $V^{v_0,\pi}(T)$ at time T. Thus, we choose an appropriate class of utility functions which can capture the investor's risk preferences in a flexible way. For this purpose we define the class of utility functions $\mathcal{U}$, which contains all functions $U : \mathbb{R} \to \mathbb{R} \cup \{\infty, -\infty\} =: \bar{\mathbb{R}}$ such that

$$\begin{aligned} &(i) \quad U \text{ is proper and usc and } dom_U = [c, \infty) \text{ for some } c \geq 0 \\ &(ii) \quad U \text{ is strictly increasing on } dom_U \\ &(iii) \quad U(x) = -\infty \quad \forall x < 0 \\ &(iv) \quad \lim_{x\to\infty} \frac{U(x)}{x} = 0. \end{aligned} \tag{2.29}$$

If U is finite on $[0, \infty)$, differentiable and strictly increasing (as in [17]), then (iv) is equivalent to the condition $\lim_{x\to\infty} U'(x) = 0$. We set $U(\infty) = \lim_{x\to\infty} U(x)$. It is fairly common to restrict this analysis to concave utility functions only ('less risk is preferred to more'), and we will do so throughout the ensuing chapters when appropriate. However, for a general formulation in the context of constraints on allocation and terminal wealth, such restrictions may hinder the exposition. We define the unconstrained portfolio optimization problem for our investor as

$$(\mathbf{P^{unc}}) \begin{cases} \Phi^{unc}(v_0) & = \sup_{\pi\in\Lambda} \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right]. \end{cases}$$

The unconstrained portfolio optimization problem ($\mathbf{P^{unc}}$) plays an important role in this thesis, as the addition of constraints on portfolio allocation or terminal wealth are a natural generalization of ($\mathbf{P^{unc}}$). In particular, any method used to solve a constrained portfolio optimization problem is to some extent a generalization of a method for solving ($\mathbf{P^{unc}}$). In this thesis we consider two different definitions of constraints:

(i) Constraints on the portfolio allocation π.

(ii) Constraints on the terminal wealth $V^{v_0,\pi}(T)$.

Naturally, any constraint on the portfolio allocation automatically induces a constraint on the terminal wealth and vice versa. However, during this thesis we aim to treat these constraints as separate, since they are defined in completely different ways. The allocation constraints considered in this thesis are defined by considering a convex set $K \subset \mathbb{R}^d$ and requiring that the investor's portfolio process π takes values only in K, i.e.,

$$\pi(t) \in K \quad \mathcal{L}[0,T] \otimes Q\text{-a.e..}$$

Classic examples for such constraints would be $K = [0, \infty)^d$ (no-shortselling), $K = 0^{d_1} \times \mathbb{R}^{d-d_1}$ for a $1 \leq d_1 \leq d$ (non-traded assets) or $K = \bigtimes_{i=1}^d [\alpha_i, \beta_i]$ for some constants $\alpha_i < \beta_i$, $i = 1, ..., d$

(interval constraints).
In contrast, terminal wealth constraints are formulated directly as restrictions on the admissible terminal wealths $V^{v_0,\pi}(T)$. A specific example would be pointwise bounds $B_1, B_2 \in \mathbb{R}$ with the requirement that

$$B_1 \leq V^{v_0,\pi}(T) \leq B_2 \quad Q-\text{a.s.}. \tag{2.30}$$

Such bounds could correspond to a minimum performance guarantee that the investor needs to fulfil or a maximal performance limit which is supposed to discourage excessive risk-taking. Alternatively, we consider expected-value constraints for a non-increasing real function $g : (0,\infty) \to \mathbb{R}$ such that

$$\mathbb{E}\left[g\left(V^{v_0,\pi}(T)\right)\right] \leq 0. \tag{2.31}$$

Letting $g(x) = \mathbb{1}_{[0,B_{VaR}]}(x) - \epsilon$ we obtain a Value-at-Risk ('VaR') constraint with lower bound B_{VaR} and tolerance level $\epsilon \geq 0$. Similarly, $g(x) = (B_{ES} - x)\mathbb{1}_{[0,B_{ES}]}(x) - \epsilon$ yields an expected shortfall ('ES') constraint with lower bound B_{ES} and tolerance level $\epsilon \geq 0$.
Both types of constraints lead to a restriction $\Lambda' \subset \Lambda$ of the set of admissible portfolio processes. For such a general constraint set $\Lambda' \subset \Lambda$, we define the general constrained portfolio optimization problem $(\mathbf{P})$ as

$$(\mathbf{P}) \begin{cases} \Phi(v_0) & = \sup_{\pi \in \Lambda'} \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right] \end{cases}$$

We impose the standing assumption throughout this thesis that both, the constrained portfolio optimization problem $(\mathbf{P})$ and the unconstrained portfolio optimization problem $(\mathbf{P^{unc}})$ are feasible and yield finite, well-defined expected utility:

Assumption 2.3.1 (**Standing Assumption**).

$$-\infty < \Phi(v_0) \leq \Phi^{unc}(v_0) < \infty \quad \forall v_0 > 0.$$

$(\mathbf{P})$ is the most general portfolio optimization problem considered in this thesis. Unsurprisingly, we are unable to find explicit solutions or useful characterizations of solutions for $(\mathbf{P})$ in this general form. However, if we impose more specific assumptions on the market parameters, the utility function and the constraints, then we can be more successful. In this sense, each of the following chapters works with specific assumptions which constitute a special case of $(\mathbf{P})$ as defined in this section. Each of the following chapters, aims to utilize different duality techniques from convex analysis, which we developed in Section 2.1 to obtain solutions to $(\mathbf{P})$.

2.3.2 Classic Methods for Unconstrained Portfolio Optimization

In this section we provide a short review of the two most common approaches which are used to solve the unconstrained portfolio optimization problem $(\mathbf{P^{unc}})$ for selected models and utility functions: The stochastic control approach and the martingale method. This review will help us to understand the limitation and challenges of each approach. Moreover, this review will allow us to highlight the extensions that are required in later Chapters, when we are facing constraints on the portfolio allocation and terminal wealth.

Stochastic Control Approach: The unconstrained portfolio optimization problem ($\mathbf{P^{unc}}$) can be regarded as a stochastic control problem, with control process π, controlled process $V^{v_0,\pi}$, and maximization objective $\mathbb{E}[U(V^{v_0,\pi}(T))]$. This is the approach followed in the seminal works [64] and [65], where the continuous time portfolio optimization problem was originally introduced.

If the wealth process (i.e., the 'state of the investor') and the market coefficients (i.e., the 'state of the market $\mathcal{M}$' can be jointly represented as Markov process $X = (X(t))_{t\in[0,T]}$, then principles from dynamic programming are applicable. This can be assured by restricting the analysis to stochastic factor models, i.e., assuming that

- there exists a stochastic factor $z = (z(t))_{t\in[0,T]}$, which takes values in $\mathbb{R}^m$ and satisfies the SDE
$$dz(t) = \mu^z(t, z(t))dt + \Sigma^z(t, z(t))dW^z(t), \quad z(0) = z_0 \in \mathbb{R}^m,$$
for some constant $z_0 \in \mathbb{R}^m$ and deterministic functions $\mu^z : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\Sigma^z : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$.
- the market coefficients $r(t)$, $\mu(t)$, $\Sigma(t)$, and $\rho(t)$ are deterministic function of the current time and value of the stochastic factor $(t, z(t))$.
- the current portfolio allocation is given as a deterministic function of time, current wealth and value of the stochastic factor $(t, V^{v_0,\pi}(t), z(t))$, i.e., there exists a deterministic function $\underline{\pi} : [0,T] \times (0,\infty) \times \mathbb{R}^m \to \mathbb{R}^d$ such that
$$\pi(t) = \underline{\pi}(t, V^{v_0,\pi}(t), z(t)).$$

In such cases, we say that the portfolio process π is defined in 'feedback-form' and $\underline{\pi}$ is the corresponding feedback control. In particular, we may define a wealth process directly through a feedback control. For a given feedback control $\underline{\pi}$, we may then define the state process $X^\pi = (V^{v_0,\pi}, z)'$ and one can show that X^π satisfies an SDE of the form

$$dX^\pi(t) = \mu^{X^\pi}(t, X^\pi(t))dt + \Sigma^{X^\pi}(t, X^\pi(t))dW(t) + \Sigma^{X^\pi,z}(t, X^\pi(t))dW^z(t),$$

and deterministic functions $\mu^{X^\pi}, \Sigma^{X^\pi}, \Sigma^{X^\pi,z}$. Then, we can define the dynamic version of the unconstrained portfolio optimization problem ($\mathbf{P^{unc}}$) as

$$(\mathbf{P^{unc}_{t,x}}) \begin{cases} \Phi^{unc}(t,x) & = \sup_{\pi\in\Lambda_f} \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right) \big| X^\pi(t) = x\right] \\ \Lambda_f & = \big\{\pi \in \Lambda \mid \pi(s) = \underline{\pi}(s, V^{v_0,\pi}(s), z(s)) = \underline{\pi}(s, X^\pi(s)) \quad \forall s \in [t,T] \\ & \qquad \text{and a feedback control } \underline{\pi}\big\}. \end{cases}$$

If the feedback control $\underline{\pi}^*$ defines an optimal portfolio process π^* for ($\mathbf{P^{unc}_{t,x}}$) and each $(t,x) \in [0,T] \times (0,\infty) \times \mathbb{R}^m$, then the optimality of π^* and Markovity of X^{π^*} ensure that

$$\begin{aligned} \Phi^{unc}(t, X^{\pi^*}(t)) &= \sup_{\pi\in\Lambda_f} \mathbb{E}\Big[U(V^{v_0,\pi}(T))\Big| X^\pi(t) = X^{\pi^*}(t)\Big] \\ &\overset{\pi^* \text{ optimal}}{=} \mathbb{E}\Big[U(V^{v_0,\pi^*}(T))\Big| X^{\pi^*}(t)\Big] \\ &\overset{\text{Markovity } X^{\pi^*}}{=} \mathbb{E}\Big[U(V^{v_0,\pi^*}(T))\Big| \mathcal{F}_t\Big]. \end{aligned}$$

Hence, it is straightforward to see that $\Phi^{unc}(t, X^{\pi^*}(t))$ is a martingale. Similarly, we have for any suboptimal portfolio process $\hat{\pi} \in \Lambda$[3] and $s \le t \le T$:

$$\mathbb{E}\Big[\Phi^{unc}(t, X^{\hat{\pi}}(t))\Big|\mathcal{F}_s\Big] = \mathbb{E}\Big[\mathbb{E}\Big[U(V^{v_0,\pi^*}(T))\Big| X^{\pi^*}(t) = X^{\hat{\pi}}(t)\Big]\Big|\mathcal{F}_s\Big]$$

[3] Note that the subsequent argument does not even require that the suboptimal portfolio process $\hat{\pi}$ is defined in feedback-form.

$$\begin{aligned}
&\overset{\text{Markovity } X^{\pi^*}}{=} \mathbb{E}\left[\mathbb{E}\left[U(V^{v_0,\pi^*}(T))\middle|X^{\pi^*}(t) = X^{\hat{\pi}}(t),\ X^{\pi^*}(s) = X^{\hat{\pi}}(s)\right]\middle|\mathcal{F}_s\right] \\
&\overset{\hat{\pi} \text{ suboptimal}}{\leq} \mathbb{E}\left[\mathbb{E}\left[U(V^{v_0,\pi^*}(T))\middle|X^{\pi^*}(s) = X^{\hat{\pi}}(s)\right]\middle|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\Phi(s, X^{\hat{\pi}}(s))\middle|\mathcal{F}_s\right] \\
&= \Phi^{unc}(s, X^{\hat{\pi}}(s)),
\end{aligned}$$

i.e., $\Phi^{unc}(t, X^{\hat{\pi}}(t))$ is a supermartingale. Therefore, under some regularity conditions (see Chapter III, Theorem 16 in [74]), the Doob-Meyer decomposition suggests that $\Phi^{unc}(t, X^{\hat{\pi}}(t))$ can be uniquely decomposed into the sum of a non-increasing, non-positive process $A(t)$ and a local martingale $M(t)$. If we additionally assume that Φ^{unc} is once continuously differentiable in $t \in [0,T]$ and twice continuously differentiable in $x \in (0,\infty) \times \mathbb{R}^m$, then we can define the operator $\mathcal{H}^\pi[\Phi^{unc}]$[4] as (omitting the arguments (t,x) for readability)

$$\mathcal{H}^\pi[\Phi^{unc}] = \Phi^{unc}_t + \left(\mu^{X^\pi}\right)' \nabla_x \Phi^{unc} + \frac{1}{2}\text{Trace}\left[\begin{pmatrix} \Sigma^{X^\pi} & \rho' \\ \rho & \Sigma^{X^\pi,z} \end{pmatrix}\begin{pmatrix} \Sigma^{X^\pi} & \rho' \\ \rho & \Sigma^{X^\pi,z} \end{pmatrix}' \nabla_x^2 \Phi^{unc}\right].$$

Then, an application of Itô's Lemma yields

$$d\Phi^{unc}(t, X^{\hat{\pi}}(t)) = \underbrace{\mathcal{H}^\pi[\Phi^{unc}](t, X^{\hat{\pi}}(t))dt}_{=dA(t)} + \underbrace{(...)dW(t) + (...)dW^z(t)}_{=dM(t)}.$$

As $A(t)$ is non-positive and non-increasing for any $\pi \in \Lambda$, the Doob-Meyer decomposition implies that $\mathcal{H}^\pi[\Phi^{unc}](t, X^{\hat{\pi}}(t)) \leq 0$ $Q \times \mathcal{L}[0,T]$-a.e., with equality for the optimal $\pi = \pi^* \in \Lambda_f$. Thus, we obtain a characterization of $\Phi^{unc}(t,x)$ and the optimal feedback control $\underline{\pi}^*(t,x)$ for all $(t,x) \in [0,T) \times (0,\infty) \times \mathbb{R}^m$ through a PDE, the so-called Hamilton-Jacobi-Bellman PDE ('HJB PDE'):

$$\mathcal{H}^\pi[\Phi^{unc}](t,x) = 0 \quad \text{and} \quad \underline{\pi}^*(t,x) \in \underset{\pi \in \mathbb{R}^d}{\text{argmax}}\ \mathcal{H}^\pi[\Phi^{unc}](t,x), \tag{2.32}$$

with terminal condition $\Phi^{unc}(T,x) = \Phi(t,v,z) = U(v)$ for all $x \in (0,\infty) \times \mathbb{R}^m$. We have arrived at the HJB PDE (2.32) under the assumption that an optimal feedback control exists and the corresponding value function Φ^{unc} is smooth. In contrast, the stochastic control approach begins by determining a solution to the HJB PDE (2.32) and then proves a 'verification theorem' which ensures that the obtained PDE-solution is indeed the value function Φ^{unc}. While this approach is valid for general (Markovian) market models $\mathcal{M}$ (i.e., $m > 0$ and deterministic functions $\mu^z, \Sigma^z, \rho, r, \mu$, and Σ), actually obtaining an explicit solution to the HJB PDE as well as proving a verification theorem is extremely challenging for complex models. In practice, this limits the usefulness of the stochastic control approach to models and utility functions where additional information about the functional form Φ^{unc} is known or can be guessed, such as CRRA utility functions or exponentially affine models.

Martingale Method: The martingale method goes back to [73], [47] and [15] and is closely linked to risk-neutral pricing in continuous time. The central idea underlying the martingale method is to split the portfolio optimization problem ($\mathbf{P^{unc}}$) from an optimization over all admissible portfolio processes into a 'static' optimization problem over all attainable terminal wealths and a subsequent 'hedging' problem, which seeks to determine the portfolio process that replicates the optimal terminal wealth for the static optimization problem.

[4] Throughout this thesis, this differential operator is mainly used within the context of Hamilton-Jacobi-Bellman PDEs and is therefore denoted as '$\mathcal{H}$'.

If the underlying financial market $\mathcal{M}$ is complete (i.e., all reasonable $\mathcal{F}_T$-measurable payoffs can be represented as a terminal wealth $V^{v_0,\pi}(T)$ for some $v_0 > 0$ and $\pi \in \Lambda$), then the static optimization problem will simplify considerably and allow a convenient characterization of the optimal terminal wealth for $(\mathbf{P^{unc}})$. To make the notion of market completeness more precise, we define the market price of risk γ and pricing kernel $\tilde{Z}$ as

$$\gamma(t) := \Sigma(t)^{-1}\big(\mu(t) - r(t)\mathbb{1}\big),$$
$$\tilde{Z}(t) := \exp\left(-\int_0^t r(s) + \frac{1}{2}\|\gamma(s)\|^2 ds - \int_0^t \gamma(s)' dW(s)\right)$$

and consider a Lemma which we will revisit in Chapter 3:

Lemma (Market Completeness, cf. Lemma 3.3.1).
Let $m = 0$ and consider $D \in \mathbf{X}_+$ with $\mathbb{E}[D^2] < \infty$ and $0 < \mathbb{E}[\tilde{Z}(T)D] =: v_0 < \infty$. Then, there exists a $\pi \in \Lambda$ so that $V^{v_0,\pi}(T) = D$ Q-a.s. and

$$V^{v_0,\pi}(t) = \mathbb{E}\left[D\frac{\tilde{Z}(T)}{\tilde{Z}(t)} \;\middle|\; \mathcal{F}_t\right]. \tag{2.33}$$

Therefore, if $m = 0$, i.e., all uncertainty in $\mathcal{M}$ is generated by the diffusion of the risky asset's log returns $(dW(t))$, then we can consider the following equivalent static representation of $(\mathbf{P^{unc}})$:

$$(\mathbf{P^{unc}})\begin{cases} \Phi^{unc}(v_0) &= \sup\limits_{D \in C(v_0)} \mathbb{E}\big[U(D)\big] \\ C(v_0) &= \big\{V^{v_0,\pi}(T) \;\big|\; \pi \in \Lambda\big\} \\ &\overset{(*)}{=} \big\{D \in L_Q^2 \;\big|\; \mathbb{E}[\tilde{Z}(T)D] = v_0\big\}, \end{cases}$$

where we used the market completeness and the fact that $U(x) = -\infty$ for all $x < 0$ in $(*)$. This is a constrained optimization problem over L_Q^2, as considered in Section 2.1.2 and it can be treated with the same duality techniques. In particular, if we set $g(x) = -\tilde{Z}(T)x$ and $K = \{-v_0\}$, then $\delta_K(y_0) = y_0 v_0$ and the associated dual optimization problem is

$$(\mathbf{D^{unc}})\Big\{\Psi_{D^{unc}} = \inf_{y_0 \in \mathbb{R}} \mathbb{E}[U_g^*(y)] + y_0 v_0.$$

Hence, if there exists a $y_0^* \in \mathbb{R}$ such that the KKT conditions of Corollary 2.1.36 are satisfied, i.e., $\mathcal{I}_g(y_0^*) \in L_Q^2$ and

$$0 = y_0^*\mathbb{E}\left[g\Big(\mathcal{I}_g(y_0^*)\Big)\right] + \underbrace{\delta_K(y_0^*)}_{=y_0^* v_0} = y_0^*\Big(\mathbb{E}[-\mathcal{I}_g(y_0^*)\tilde{Z}(T)] + v_0\Big) \;\;\&\;\; \mathbb{E}\left[g\Big(\mathcal{I}_g(y_0^*)\Big)\right] \in K$$
$$\Leftrightarrow \quad \mathbb{E}[\mathcal{I}_g(y_0^*)\tilde{Z}(T)] = v_0, \tag{2.34}$$

then $D^* = \mathcal{I}_g(y_0^*)$ is the optimal terminal wealth for $(\mathbf{P^{unc}})$ and thus a solution to the static optimization problem. Under mild regularity conditions on the utility function and the market coefficients, the existence of a y_0^* satisfying (2.34) can be guaranteed (see e.g. Theorem 7.4 in [17] or Lemma 3.1 in [34]) and can be determined numerically (as $\mathcal{I}_g$ is non-increasing in y_0 by Lemma 2.1.38, such numerical procedures are even reasonably reliable). Note that we arrived at this characterization of the optimal terminal wealth for $(\mathbf{P^{unc}})$ without any major structural assumptions on the utility function U, the market coefficients or the optimal portfolio process π^*.

In contrast, solving the hedging problem, i.e., determining an explicit representation for the optimal portfolio π^* such that $V^{v_0,\pi^*}(T) = \mathcal{I}_g(y_0^*)$, is no longer feasible in this general setting.

While the market completeness ensures the existence of a replicating portfolio for $\mathcal{I}_g(y_0^*)$, it does not offer a direct way of constructing this portfolio. However, we do obtain a characterization of the optimal wealth process through a conditional expectation by virtue of (2.33):

$$\tilde{Z}(t)V^{v_0,\pi^*}(t) = \mathbb{E}\left[\mathcal{I}_g(y_0^*)\tilde{Z}(T)\middle|\mathcal{F}_t\right]. \tag{2.35}$$

The right side of (2.35) is trivially a martingale and thus the left side is also a martingale. Hence, the left side is an Itô process with zero drift and SDE

$$d\Big(\tilde{Z}(t)V^{v_0,\pi^*}(t)\Big) \overset{\text{Itô's product rule}}{=} \tilde{Z}(t)V^{v_0,\pi^*}(t)\Big(\pi^*(t)'\Sigma(t) - \gamma(t)'\Big)dW(t).$$

Thus, if we can obtain an SDE for the right side of (2.35), then we can determine π^* by matching the diffusions on the left and right. For example, this can be achieved by using the Feynman-Kac representation, which establishes another link to partial differential equations:

Theorem 2.3.2. *Let $n \in \mathbb{N}$, $\bar{W}$ be an n-dimensional Wiener process and X be the unique solution to the stochastic differential equation $X(0) = x \in \mathbb{R}^n$,*

$$dX(t) = \mu^X(t, X(s))dt + \Sigma^X(t, X(t))d\bar{W}(t),$$

for deterministic functions $\mu^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Sigma^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$. For a function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable in $[0,T]$ and twice continuously differentiable in x, we define the operator $\mathcal{H}^X$ applied to u as (again omitting arguments (t,x) for readability)

$$\mathcal{H}^X[u] = u_t + (\mu^X)'\nabla_x u + \mathit{Trace}\left(\Sigma^X\left(\Sigma^X\right)'\nabla_x^2 u\right).$$

If $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous solution to the PDE

$$\mathcal{H}^X[u](t,x) = 0, \quad u(T,x) = f(x) \quad \forall t \in [0,T)\forall x \in \mathbb{R}^n, \tag{2.36}$$

for a continuous terminal condition $f : \mathbb{R}^m \to [0,\infty)$ and

- *there exists a constant $K > 0$ such that for all $x, y \in \mathbb{R}^n$*

$$\begin{aligned}\left\|\mu^X(t,x) - \mu^X(t,y)\right\| + \left\|\Sigma^X(t,x) - \Sigma^X(t,y)\right\| &\le K\left\|x-y\right\|,\\ \left\|\mu^X(t,x)\right\|^2 + \left\|\Sigma^X(t,x)\right\|^2 &\le K^2\big(1 + \|x\|^2\big).\end{aligned}$$

- *there exist constants $M > 0$, $\epsilon > 1$ such that for all $x \in \mathbb{R}^n$*

$$\max_{0\le t\le T}|u(t,x)| \le M\big(1 + \|x\|^{2\epsilon}\big),$$

then u satisfies the Feynman-Kac representation

$$u(t,x) = \mathbb{E}\Big[f(X(T))\Big|\, X(t) = x\Big].$$

The PDE (2.36) is also referred to as 'Cauchy problem' or 'Feynman-Kac PDE'. If now $\tilde{Z}$ is a Markov process[5], then we can aim to apply Theorem 2.3.2 with $X = \tilde{Z}$ and terminal condition

$$f(z) = \Big(\mathcal{I}_g(y_0^*)\tilde{Z}(T)\Big)\Big|_{\tilde{Z}(T)=z}$$

[5] The ensuing arguments can easily be generalized if $\tilde{Z}$ is a component of a multi-dimensional Markov process X (similar to the relation between $V^{v_0,\pi}$ and X^π in the stochastic control approach).

$$= \left(\inf \left\{ x \in \mathbb{R} \;\middle|\; \sup_{\hat{x} \in \mathbb{R}} \left(U(\hat{x}) - y_0^* z \hat{x} \right) \leq U(x) - y_0^* z x \right\} \right) z.$$

As the stochasticity of g (and U_g^*, $\mathcal{I}_g$) only depends on the value of $\tilde{Z}(t)$, the terminal condition f above is a deterministic function in z. If u is a solution to the Feynman-Kac PDE (2.36) with terminal condition f such that the requirements of Theorem 2.3.2 are satisfied, then (2.35) yields for any $t \in [0, T]$:

$$\begin{aligned}
v_0 + \int_0^t \tilde{Z}(s) V^{v_0,\pi^*}(s) \Big(\pi^*(s)'\Sigma(s) - \gamma(s)' \Big) dW(s) &= \tilde{Z}(t) V^{v_0,\pi^*}(t) \\
&\overset{(2.35)}{=} \mathbb{E}\left[\mathcal{I}_g(y_0^*)\tilde{Z}(T) \middle| \mathcal{F}_t \right] \\
&\overset{\text{Markovity of } \tilde{Z}}{=} \mathbb{E}\left[f(\tilde{Z}(T)) \middle| \tilde{Z}(t) \right] \\
&\overset{T.\ 2.3.2}{=} u(t, \tilde{Z}(t)) \\
&\overset{\text{Itô}}{=} \underbrace{u(0, \tilde{Z}(0))}_{v_0} + \underbrace{\int_0^t (...)dt}_{=0} - \int_0^t \tilde{Z}(s) u_z(s, \tilde{Z}(s))\gamma(s)' dW(s) \\
&= v_0 - \int_0^t \tilde{Z}(s) u_z(s, \tilde{Z}(s))\gamma(s)' dW(s).
\end{aligned}$$

As this equation holds for all $t \in [0, T]$, we conclude that

$$\begin{aligned}
\pi^*(t) &= \underbrace{\frac{1}{V^{v_0,\pi^*}(t)} \Big(V^{v_0,\pi^*}(t) - u_z(t, \tilde{Z}(t)) \Big)}_{=:\alpha^*(t, V^{v_0,\pi^*}(t), \tilde{Z}(t))} \underbrace{\big(\Sigma(t)\Sigma(t)'\big)^{-1} (\mu(t) - r(t)\mathbb{1})}_{=:\pi_M(t)} \\
&= \alpha^*(t, V^{v_0,\pi^*}(t), \tilde{Z}(t)) \pi_M(t)
\end{aligned}$$

Note that this expression for π^* demonstrates that the optimal portfolio process satisfies a version of a two fund theorem: For any utility function $U \in \mathcal{U}$, such that the optimal π^* can be determined using the Feynman-Kac-representation, we can decompose π^* as a linear combination of the risk-free portfolio $\pi_0 \equiv 0$ and the Merton portfolio π_M (for risk aversion 1) with stochastic weighting factor α^*. In particular, the optimal portfolio for all such utility functions have the same relative allocation between risky assets in $\mathcal{M}$ and only their exposure to the risky assets is influenced by the utility function.[6]

Discussion: The stochastic control approach is applicable for general Markovian market models, without requiring the completeness of the financial market. In particular, the stochastic control approach can be applied in incomplete markets, such as Heston's stochastic volatility model ([56]), whereas the martingale method cannot be applied directly. However, actually obtaining a closed-form expression for the optimal portfolio process requires the solution to the associated HJB PDE, for which explicit solutions are only known for selected combinations of market models and utility functions (e.g. for exponentially affine models and CRRA utility functions as considered in [61]).
The martingale method is only directly applicable in complete financial markets, i.e. when all uncertainty in $\mathcal{M}$ is generated by the Wiener process W, which is the diffusion of the risky asset's log returns. Therefore, it is not directly applicable to most stochastic volatility models

[6] A similar statement can be derived via the stochastic control approach if we explicitly write out μ^{X^π}, Σ^{X^π}, $\Sigma^{X^\pi,z}$ and determine an explicit expression for the (optimal) portfolio which maximizes the characteristic operator $\mathcal{H}^\pi[\Phi^{unc}]$.

unless the market is completed through the addition of (fictitious) volatility-dependent assets (as in [78]). However, unlike the stochastic control approach, the martingale method yields a characterization of the optimal terminal wealth $V^{v_0,\pi^*}(t) = \mathcal{I}_g(y_0^*)$ for $(\mathbf{P^{unc}})$, which is explicit up to $y_0^* \in \mathbb{R}$ – the solution to the budget equation (2.34). An explicit expression for the corresponding optimal portfolio π^* is more challenging to obtain and requires either the Markovity of the pricing kernel and a solution to the associated Feynman-Kac PDE or an explicit representation of the expectation in (2.35) as a stochastic integral. Therefore, explicit expressions for π^* have again only been derived for selected utility functions and market models.
Hence, the martingale method is ideally suited for characterizing the portfolio optimization problem and its associated optimal terminal wealth for a large, general class of utility functions and models - as long as the financial market is complete. Determining an explicit expression for the optimal portfolio process requires the solution to an associated partial differential equation in both the stochastic control approach and the martingale method. However, as the stochastic control approach can be directly applied to incomplete markets, it is the natural choice for this objective.
Following this logic, we extend the classic martingale method in Chapter 3, where we characterize the optimal terminal wealth of portfolio optimization problem $(\mathbf{P})$ with simultaneous constraints on terminal wealth and allocation for general utility functions and complete financial markets. In contrast, we extend the classic stochastic control approach in Chapter 4, where we determine (semi-)explicit expressions for the optimal portfolio process to a portfolio optimization problem $(\mathbf{P})$ with constraints on the portfolio allocation in (incomplete) stochastic factor models. A special case of these models, Heston's stochastic volatility model, is discussed in great detail in Chapter 5.

3 Constraints on Allocation and Wealth in Complete Financial Market Models

3.1 Introduction

In this chapter, we consider a finite-horizon portfolio optimization problem for an expected utility maximizing investor who trades in continuous time. The investor's portfolio choice is restricted by convex allocation constraints as well as pointwise bounds and expected value constraints on terminal wealth. These constraints are the main difference of our setting in comparison to the classic problem originally introduced by [64]. We give a short overview of the related literature on wealth-constrained portfolio optimization:

In a portfolio optimization context, the two most commonly considered constraints on terminal wealth are either pointwise bounds on terminal wealth or expected value constraints. As suggested by their name, pointwise bounds require that the terminal wealth at the end of the investment horizon lies within a (possibly random) real interval $[B_1, B_2] \subset \mathbb{R}$. The natural choice of a lower bound ($0 \leq B_1 < \infty$, $B_2 = \infty$), or 'minimum performance constraint', was first considered and analyzed by [82], who assumed classic Black-Scholes market dynamics and derived closed-form expressions for the optimal portfolio under the condition that the investor derives utility from a power utility function and the lower bound B_1 is deterministic. Later, [52] considered a similar problem and presented a two-step approach for portfolio optimization problems with lower bound on terminal wealth in complete financial market models. The approach is based on using a share of the investor's initial wealth to replicate the lower bound and then use the remaining share of the initial wealth to maximize the expected utility surplus over the lower bound. Despite being unconventional, [25] considered a pointwise upper bound on terminal wealth ($B_2 < \infty$) and illustrated that such upper bounds on terminal wealth increase the quantiles of the terminal wealth distribution, i.e., decrease the likelihood of poor performance of the portfolio. This is especially relevant for investments in a retirement plan.
On the other hand, expected value constraints impose a restriction on the expected value of a function $g(V^{v_0,\pi}(T))$ of the investors terminal wealth. As such, these type of constraints allow for 'softer' restrictions than strict pointwise bounds on terminal wealth. In [55], the authors propose a dual approach for general convex expected value constraints and showcase the developed approach on a mean-variance optimization problem. [4] uses a similar duality approach to solve portfolio optimization problems with Value-at-Risk ('VaR') or expected shortfall ('ES') constraints. The authors of [57] effectively consider the same problem setting as [4] with VaR and ES constraints, but use a novel methodology based on dynamic programming to solve these optimization problems. Expected value constraints based on more general risk measures were considered in [72], [67], [11] and [12].

Up until this point only few papers considered simultaneous constraints on portfolio allocation and terminal wealth. [3] extends the auxiliary market framework of [17] to include random lower bounds on terminal wealth and proves a set of analogue equivalent optimality conditions. However, the author failed to explicitly determine neither the optimal relative portfolio process nor the optimal terminal wealth for any example within his setting. [24] and [35] consider simultaneous VaR constraints and convex cone constraints in a Black-Scholes financial market, but their approaches do not directly extend to general constraints and models. A different type of simultaneous constraints was considered by [36]. The authors describe the Solvency II capital requirements as a wealth-dependent allocation constraint and transform the associated portfolio optimization problem into an equivalent problem with constant allocation constraints, which falls within the scope of [17]. None of the aforementioned papers, aside from [3], aimed to provide a general solution framework for wealth and allocation constrained portfolio optimization problems. However, [3] did not provide extensions going beyond strict lower bounds on terminal wealth, which is only one possible type of wealth constraints.

The contribution of this chapter to the existing literature is threefold:

- We present a generalized version of the martingale approach which is applicable for general expected value constraints and pointwise bounds on terminal wealth. The approach is based on a duality ansatz that can be applied in general complete financial markets.
- We integrate the generalized martingale approach into the auxiliary market framework of [17] to derive a set of equivalent optimality conditions, which can be used to solve portfolio optimization problems with simultaneous constraints on allocation and wealth. This is a generalization of the framework based on the 'capped Legendre-Fenchel transform', which was presented in [34].
- We illustrate the utility of our methodology in a Black-Scholes market by explicitly characterizing the optimal terminal wealth if the allocation constraints are a convex cone, the pointwise bounds are deterministic and the expected value constraint is either a VaR constraint or an ES constraint.

The remainder of this chapter is structured as follows: The financial market model, the portfolio optimization problem and the standing assumptions are introduced in Section 3.2. Afterwards, in Section 3.3, we generalize the well-known martingale approach to financial markets with constraints on terminal wealth. In a similar way, we generalize the auxiliary market framework of [17] by including constraints on terminal wealth in Section 3.4. Lastly, we propose a methodology for determining the optimal auxiliary market in a Black-Scholes setting in Section 3.5 and Section 3.6 concludes this chapter.

3.2 Setting

In this chapter we consider an investor who simultaneously faces constraints on his allocation and his terminal wealth. The constraints on terminal wealth can be either pointwise bounds (as considered in [34]), expected value constraints or both. In this regard, this chapter extends the work of [34]. To simplify the presentation we restrict the market setting defined in Section 2.3 by assuming that all randomness in $\mathcal{M}$ is generated by W, i.e., $m = 0$. This has the advantage that the considered financial market $\mathcal{M}$ is complete and we can derive a useful, equivalent characterization of $(\mathbf{P})$ as an optimization over random variables $D \in L^2_Q$. Market completeness is a key ingredient which will be frequently used in the duality arguments in the subsequent sections. In this regard, the restriction to $m = 0$ in this chapter, can also be seen as a restriction to complete financial markets $\mathcal{M}$. Note however, completeness of $\mathcal{M}$ is not a strict prerequisite for this chapter, but rather another instrument to facilitate the exposition. Just as in [49], in the case of an incomplete financial market $\mathcal{M}$, one can add additional fictitious assets to complete the market and use the theory developed in Section 3.4 to rule out investments into these assets through an allocation constraint.

For a non-empty closed convex set $K \subset \mathbb{R}^d$, $\mathcal{F}_T$-measurable random variables $0 \le B_1 < B_2$, and a non-increasing real function $g : (0, \infty) \to \mathbb{R}$, we define the set of admissible constrained portfolio processes as

$$\begin{aligned} \Lambda' &:= \Lambda(v_0, K, B_1, B_2, g) \\ &:= \left\{\pi \in \Lambda \;\middle|\; \pi(t) \in K, \quad B_1 \le V^{v_0,\pi}(T) \le B_2, \quad \mathbb{E}\left[g(V^{v_0,\pi}(T))\right] \le 0\right\}, \end{aligned} \tag{3.1}$$

where the requirement '$\pi \in K$' is meant to hold $\mathcal{L}[0,T] \otimes Q$-a.e. and '$B_1 \le V^{v_0,\pi}(T) \le B_2$' is meant to hold Q-a.s.. To ensure that the constraints are well-posed, i.e., $\Lambda' \neq \emptyset$, we need to make some additional technical assumptions which amount to requiring that the risk-free portfolio process $\pi \equiv 0$ is admissible for $(\mathbf{P})$, but not trivially optimal for $(\mathbf{P})$. Specifically, we assume that

$$0 \in K, \tag{3.2}$$

$$B_1 < v_0 P_0(T) < B_2, \tag{3.3}$$

$$\text{and} \quad \mathbb{E}\left[g(v_0 P_0(T))\right] \le 0. \tag{3.4}$$

Finally, this ensures that we can reasonably consider the following, fully constrained portfolio optimization problem

$$(\mathbf{P}) \begin{cases} \Phi(v_0) = \sup\limits_{\pi \in \Lambda(v_0, K, B_1, B_2, g)} \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right]. \end{cases}$$

Clearly, we can rewrite $(\mathbf{P})$ directly as a maximization over all attainable terminal wealths:

$$(\mathbf{P}) \begin{cases} \Phi(v_0) & = \sup\limits_{D \in C(v_0, K, B_1, B_2, g)} \mathbb{E}\big[U(D)\big] \\ C(v_0, K, B_1, B_2, g) & = \big\{V^{v_0,\pi}(T) \;\big|\; \pi \in \Lambda(v_0, K, B_1, B_2, g)\big\}. \end{cases}$$

This may seem trivial at first, but since $C(v_0, K, B_1, B_2, g)$ can later be simplified substantially (depending on the choice of K, B_1, B_2 and g), it is more convenient to write $(\mathbf{P})$ this way.

We attempt to solve $(\mathbf{P})$ by following a similar approach as the authors of [34]: First, we introduce a suitable class of dual processes λ which parametrize the 'auxiliary' financial markets $\mathcal{M}_\lambda$. The wealth process of an investor trading continuously in $\mathcal{M}_\lambda$ is adapted in such a way that it is advantageous to only choose portfolio processes which take values in K. In $\mathcal{M}_\lambda$, we consider a portfolio optimization problem $(\mathbf{P}_\lambda)$, without allocation constraint but with the same wealth constraints as in $\mathcal{M}$. Using a generalized martingale approach, which is based on the duality techniques introduced in Section 2.1.1, we can characterize the optimal terminal wealth for $(\mathbf{P}_\lambda)$ in $\mathcal{M}_\lambda$ and establish a duality relation between the optimal portfolio process π^* for the original portfolio optimization problem $(\mathbf{P})$ and a certain minimizing $\lambda^* \in \mathcal{D}$. A schematic overview of this approach is illustrated in Figure 3.1.

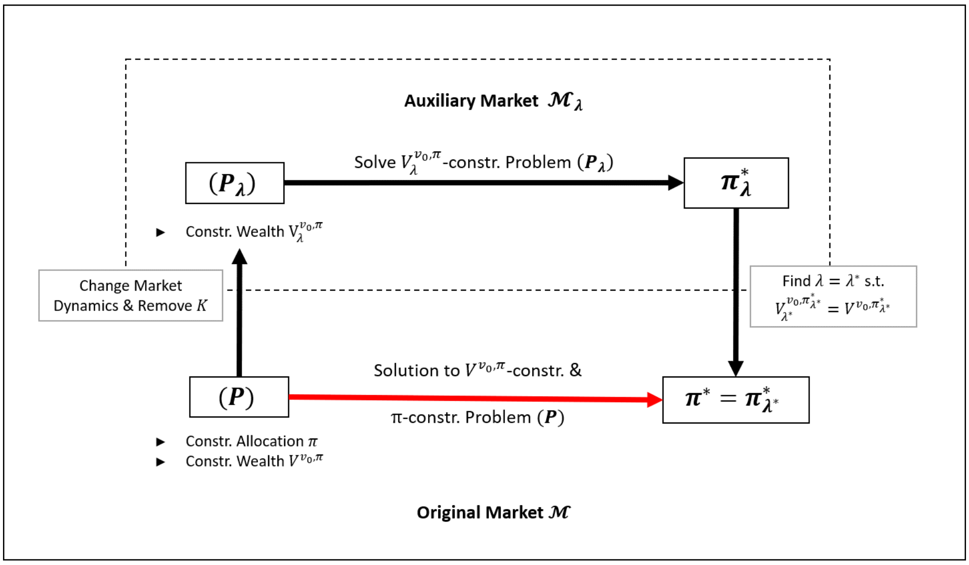


Figure 3.1: Schematic illustration of the solution approach employed throughout Chapter 3. V^{v_0,π_λ} denotes the wealth process of an investor trading in the auxiliary market $\mathcal{M}_\lambda$ and will be formally defined in Section 3.4.

The standing assumptions for this chapter are summarized below:

Assumption 3.2.1 (Standing Assumptions Chapter 3:). $m = 0$, *(3.1), (3.2), (3.3), and (3.4).*

3.3 Generalized Martingale Approach

In this section we present a generalized martingale approach, which can be applied in complete financial markets. This approach can be used to characterize the optimal terminal wealth for portfolio optimization problems with simultaneous pointwise and expected value constraints on terminal wealth. To this end, we consider the portfolio optimization problem $(\mathbf{P})$ under the assumption that no allocation constraints are present (denoted as '$(\mathbf{P_0})$'). Under this assumption, we can prove that the financial market $\mathcal{M}$ satisfies a notion of market completeness, which can be used to derive an equivalent representation of $(\mathbf{P_0})$ as a constrained optimization problem over L^2_Q. This representation falls within the setting of Section 2.1.2 and can be treated with the associated dual optimization problem $(\mathbf{D_0^V})$.

Let us for now consider the allocation-unconstrained analogue $(\mathbf{P_0})$ of the portfolio optimization problem $(\mathbf{P})$, i.e., we set $\Lambda(v_0, B_1, B_2, g) := \Lambda(v_0, \mathbb{R}^d, B_1, B_2, g)$ and define

$$(\mathbf{P_0}) \begin{cases} \Phi_0(v_0) & = \sup_{\pi \in \Lambda(v_0, \mathbb{R}^d, B_1, B_2, g)} \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right]. \end{cases}$$

As suggested earlier, we can equivalently write $(\mathbf{P}_0)$ as an optimization of $\mathbb{E}[U(D)]$ over all $D \in L^2_Q$ such that there exists a $\pi \in \Lambda(v_0, B_1, B_2, g)$ with $D = V^{v_0,\pi}(T)$. Although this may seem trivial at first glance, we can use the notion of market completeness to develop a useful equivalent representation of $(\mathbf{P_0})$. To make this notion more precise, recall the definition of the market price of risk γ, the corresponding exponential local martingale Z and the pricing Kernel $\tilde{Z}$ from Section 2.3.2:

$$\begin{aligned}
\gamma(t) &:= \Sigma(t)^{-1}\big(\mu(t) - r(t)\mathbb{1}\big),\\
Z(t) &:= \exp\left(-\frac{1}{2}\int_0^t \|\gamma(s)\|^2 ds - \int_0^t \gamma(s)' dW(s)\right),\\
\tilde{Z}(t) &:= \frac{Z(t)}{P_0(t)} = \exp\left(-\int_0^t r(s) + \frac{1}{2}\|\gamma(s)\|^2 ds - \int_0^t \gamma(s)' dW(s)\right),
\end{aligned}$$

for $t \in [0,T]$. Note that Z and $\tilde{Z}$ satisfy the SDEs

$$\begin{aligned}
dZ(t) &= -Z(t)\gamma(t)' dW(t)\\
d\tilde{Z}(t) &= -\tilde{Z}(t)\big(r(t)dt + \gamma(t)' dW(t)\big).
\end{aligned}$$

In particular, Z is a positive local martingale and thus a supermartingale. If Z satisfies Novikov's condition, then it is indeed a martingale. In our setting, $\mathcal{M}$ satisfies the following notion of market completeness.

Lemma 3.3.1 (Market Completeness)**.** *Consider* $D \in \mathbf{X}_+$ *with* $\mathbb{E}[D^2] < \infty$ *and* $0 < \mathbb{E}[\tilde{Z}(T)D] =: v_0 < \infty$. *Then, there exists a* $\pi \in \Lambda$ *so that* $V^{v_0,\pi}(T) = D$ *Q-a.s. and*

$$V^{v_0,\pi}(t) = \mathbb{E}\left[D\frac{\tilde{Z}(T)}{\tilde{Z}(t)} \;\middle|\; \mathcal{F}_t\right].$$

Lemma 3.3.1 implies that any non-negative payoff $D \in L_Q^2$ can be written as a payoff $V^{v_0,\pi}(T)$ for a $\pi \in \Lambda'$ if and only if D satisfies the so-called 'budget equation' $\mathbb{E}[\tilde{Z}(T)D] = v_0$. Noting that $U(x) = -\infty$ for all $x < 0$, this allows us to rewrite $(\mathbf{P_0})$ as an equivalent constrained optimization problem over $D \in L_Q^2$:

$$(\mathbf{P_0})\begin{cases}\Phi(v_0) &= \sup\limits_{D\in C(v_0,B_1,B_2,g)} \mathbb{E}\big[U(D)\big]\\ C(v_0,B_1,B_2,g) &= \big\{D \in L_Q^2 \;\big|\; \mathbb{E}[\tilde{Z}(T)D] = v_0,\ B_1 \le D \le B_2,\ \mathbb{E}[g(D)] \le 0\big\}.\end{cases}$$

Changing the optimization variable from admissible portfolio processes π to admissible terminal wealths $D = V^{v_0,\pi}(T)$ is a common technique in dynamic portfolio optimization (see e.g. [73], [47] and [17]) and is usually referred to as 'martingale approach'. If an optimal terminal payoff D^* for $(\mathbf{P_0})$ has been determined, then the completeness of $\mathcal{M}$ ensures that there exists an optimal portfolio process π^* such that $V^{v_0,\pi^*}(T) = D^*$ holds Q-a.s.. Despite this theoretical guarantee, as explained in Section 2.3.2, it is challenging to determine explicit expressions for π^*. For this reason, we exclusively focus on characterizing the optimal terminal wealth for $(\mathbf{P_0})$ and do not explicitly determine the corresponding optimal portfolio process π^*.

We proceed in a similar way as [34] and incorporate the pointwise constraint into an adjusted utility function and adjusted expected value constraint

$$U(x;B_1,B_2) := \begin{cases}-\infty, & x < B_1\\ U(x), & B_1 \le x \le B_2\\ U(B_2), & B_2 < x\end{cases}, \quad \text{and} \quad g(x;B_2) = \begin{cases} g(x), & x \le B_2\\ g(B_2), & B_2 < x\end{cases}.$$

Lemma 3.3.2. *Consider the optimization problem*

$$(\mathbf{\tilde{P}_0})\begin{cases}\tilde{\Phi}_0(v_0) &= \sup_{D\in C(v_0,g(\cdot;B_2))} \mathbb{E}\big[U(D;B_1,B_2)\big]\\ C(v_0,g) &= C(v_0,0,\infty,g(\cdot;B_2))\\ &= \big\{D\in L_Q^2 \;\big|\; \mathbb{E}[\tilde{Z}(T)D]=v_0,\ \mathbb{E}[g(D;B_2)]\le 0\big\}.\end{cases}$$

Then, $\tilde{\Phi}_0(v_0)=\Phi_0(v_0)$. In particular, D^* is optimal for $(\mathbf{\tilde{P}_0})$ if and only if D^* is optimal for $(\mathbf{P_0})$.

The optimization problem $(\mathbf{\tilde{P}_0})$ is a constrained optimization problem over L_Q^2, as considered in Section 2.1.2, with objective function $U(\cdot;B_1,B_2)$ and 2-dimensional expected value constraints

$$\Big(\mathbb{E}[\tilde{Z}(T)D],\mathbb{E}[g(D;B_2)]\Big)'\in\{v_0\}\times(-\infty,0] \quad\Leftrightarrow\quad \Big(\mathbb{E}[-\tilde{Z}(T)D],\mathbb{E}[-g(D;B_2)]\Big)'\in\{-v_0\}\times[0,\infty).$$

Hence, we may define

$$g_0:\mathbb{R}^2\to\mathbb{R},\quad g_0(x)=\begin{pmatrix}-\tilde{Z}(T)x\\-g(x;B_2)\end{pmatrix}\quad\text{and}\quad K_V:=\{-v_0\}\times[0,\infty)$$

and use the associated dual optimization problem and generalized conjugate

$$\begin{aligned}U_{g_0}^*(y;B_1,B_2) &\overset{y=(y_0,y_1)'}{=} \sup_{x\in\mathbb{R}}\Big(U(x;B_1,B_2)-y_0\tilde{Z}(T)x-y_1g(x;B_2)\Big)\\ &\overset{\text{if } y\in[0,\infty)^2}{=} \sup_{B_1\le x\le B_2}\Big(U(x)-y_0\tilde{Z}(T)x-y_1g(x)\Big)\end{aligned}$$

of $U(\cdot;B_1,B_2)$ to characterize the optimal terminal payoff D_0^* for $(\mathbf{\tilde{P}_0})$ (and $(\mathbf{P_0})$). We denote the maximizing argument corresponding to $U_{g_0}^*(y;B_1,B_2)$ as $\mathcal{I}_{g_0}(y;B_1,B_2)$. Note that we specifically used the 'negated' version of the constraints in $(\mathbf{\tilde{P}_0})$ (i.e., we consider g_0 instead of $-g_0$), as we would otherwise obtain a definition of the generalized conjugate and its associated dual optimization problem, which is uncommon for our portfolio optimization context (despite being perfectly equivalent).

Theorem 3.3.3. *Let $y^*=(y_0^*,y_1^*)'\in[0,\infty)^2$ be optimal for the dual minimization problem*

$$(\mathbf{D_0^V})\Big\{\Psi_{D_0^V}=\inf_{y=(y_0,y_1)'\in[0,\infty)^2}\Big(\mathbb{E}\left[U_{g_0}^*(y;B_1,B_2)\right]+y_0v_0\Big).$$

If $\mathcal{I}_{g_0}(y^*;B_1,B_2)\in L_Q^2$ and

$$y\to\mathbb{E}\left[g_0(\mathcal{I}_{g_0}(y;B_1,B_2))\right]'\Delta y$$

is usc at $y=y^*$ for all $\Delta y\in\mathbb{R}\times[0,\infty)$, then $\mathcal{I}_{g_0}(y^*;B_1,B_2)$ is optimal for $(\mathbf{P}_0)$ and $y_1^*\mathbb{E}\left[g\left(\mathcal{I}_{g_0}(y^*;B_1,B_2)\right)\right]=0$.

The minimization problem $(\mathbf{D_0^V})$ can be regarded as the dual optimization problem of $(\mathbf{P_0})$ in $\mathcal{M}$, where the duality is induced by the remaining wealth constraints (i.e., the budget equation and the expected-value constraints g.) This is reflected in the proof of Theorem 3.3.3, which heavily relies on the duality results from Section 2.1.2. A common way of proving the existence of an optimal $y^*=(y_0^*,y_1^*)'$ for $(\mathbf{D_0^V})$ is to show that there exists a y^* which satisfies the associated KKT-conditions, i.e., satisfies

$$\mathbb{E}[\mathcal{I}_{g_0}(y^*;B_1,B_2)\tilde{Z}(T)]=v_0\quad\text{and}\quad y_1^*\mathbb{E}\left[g\left(\mathcal{I}_{g_0}(y^*;B_1,B_2)\right)\right]=0.$$

One could easily generalize the present setting to consider n-dimensional expected-value constraints $g_i : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}[g_i(x)] \leq 0$ for all $i = 1, .., n$. This would in turn yield an $n+1$ dimensional function $g_0(x) = (\tilde{Z}(T)x, g_1(x; B_2), \ldots, g_n(x; B_2))'$ and is not investigated throughout this chapter to improve the presentation.

Remark 3.3.4. *The upper-semi continuity assumption on $\mathbb{E}\left[g_0(\mathcal{I}_{g_0}(y; B_1, B_2))\right]' \Delta y$ in 3.3.3 may seem quite abstract, but we refrain from providing general conditions under which it is satisfied. If there is no additional restriction on the admissible terminal payoffs due to g, e.g., if $g \equiv 0$, then we can disregard the argument y_1 and this assumption reduces to a continuity assumption on*

$$y_0 \to \mathbb{E}\left[\mathcal{I}_{g_0}(y_0; B_1, B_2)\tilde{Z}(T)\right].$$

As $\mathcal{I}_{g_0}$ is non-increasing in y_0 (Lemma 2.1.38), the above function mapping can only be continuous at $y_0 = y_0^* > 0$, if there exists a $0 < \hat{y}_0 < y_0^*$ such that

$$\mathbb{E}\left[\mathcal{I}_{g_0}(\hat{y}_0; B_1, B_2)\tilde{Z}(T)\right] < \infty,$$

as otherwise

$$\infty = \limsup_{y_0 \uparrow y_0^*} E\left[\mathcal{I}_{g_0}(y_0; B_1, B_2)\tilde{Z}(T)\right] > v_0 = \mathbb{E}\left[\mathcal{I}_{g_0}(y_0^*; B_1, B_2)\tilde{Z}(T)\right].$$

3.4 Auxiliary Markets with Bounds on Terminal Wealth

The generalized martingale approach from Section 3.3 is a dual optimization approach which can be used to solve wealth constrained portfolio optimization problems. In this section we develop a duality approach which can be used to solve portfolio optimization problems with simultaneous constraints on terminal wealth and portfolio allocation, and is thus applicable to (**P**). We proceed by first introducing the concept of auxiliary markets (see [17]) and then ensure the feasibility of the wealth constrained portfolio optimization problem by integrating the pointwise bounds on terminal wealth into the utility function (as in [34]). Further, we prove a set of equivalent optimality conditions, which yield two different dual optimization problems, ($\mathbf{D^K}$) and ($\mathbf{D^{K,V}}$), associated with (**P**).

Consider the following set of $\mathbb{R}^d$-valued dual processes $\mathcal{D}$:

$$\mathcal{D} := \Big\{ \lambda = \big((\lambda_1(t), ..., \lambda_d(t))'\big)_{t \in [0,T]} \text{ prog. measurable } \Big| \\ \int_0^T \|\lambda(t)\|^2 dt < \infty \ Q\text{-a.s.}, \int_0^T \delta_K(\lambda(t)) dt < \infty \ Q\text{-a.s.} \Big\}.$$

For any $\lambda \in \mathcal{D}$, the boundedness conditions guarantee that $\lambda(t)$ takes finite values $\mathcal{L}[0,T] \otimes Q$-a.e. and Q-a.s. only takes values in X_K. For a given $\lambda \in \mathcal{D}$, we define the auxiliary market $\mathcal{M}_\lambda$ as the asset universe with d risky assets $P^\lambda = (P_1^\lambda, ..., P_d^\lambda)$ and one risk-free asset P_0^λ, which satisfy $P_0^\lambda(0) = P_1^\lambda(0) = ... = P_d^\lambda(0) = 1$ and evolve according to the dynamics

$$dP^\lambda(t) = \operatorname{diag}\left(P^\lambda(t)\right) \big(\mu(t) + \lambda(t) + \delta_K(\lambda(t))\mathbb{1}dt + \Sigma(t)dW(t)\big)$$
$$dP_0^\lambda(t) = P_0^\lambda(t)\big(r(t) + \delta_K(\lambda(t))\big)dt.$$

The risk-free asset P_0^λ still represents the same risk-free asset and the assets P_i^λ represent the same risky assets from the original market $\mathcal{M}$, but with changed drift coefficients. As a consequence, the wealth process of an investor trading according to the relative portfolio process π in the market $\mathcal{M}_\lambda$ satisfies an SDE with adjusted drift coefficients:

$$\begin{aligned} dV_\lambda^{v_0,\pi}(t) &= V_\lambda^{v_0,\pi}(t)\big([r(t) + \delta_K(\lambda(t)) + (\mu(t) + \lambda(t) - r(t)\mathbb{1})'\pi(t)]dt + \pi(t)'\Sigma(t)dW(t)\big) \\ &= V_\lambda^{v_0,\pi}(t)\big([r(t) + (\mu(t) - r(t)\mathbb{1})'\pi(t)] + \pi(t)'\Sigma(t)dW(t)\big) \\ &\quad + \underbrace{V_\lambda^{v_0,\pi}(t)[\delta_K(\lambda(t)) + \lambda(t)'\pi(t)]}_{(*)}\, dt. \end{aligned} \tag{3.5}$$

The wealth process $V_{\lambda^*}^{v_0,\pi}$ satisfies the same SDE as in the original market $\mathcal{M}$ apart from the additional drift term $(*)$. Moreover, we recover $\mathcal{M} = \mathcal{M}_0$ for $\lambda \equiv 0$. Due to Lemma 2.1.17, the additional drift $(*)$ is non-negative as long $\pi(t) \in K$. Hence, any portfolio process $\pi \in \Lambda(v_0, K, B_1, B_2, g)$, which is admissible for $(\mathbf{P})$, yields a wealth process $V_\lambda^{v_0,\pi}$ in $\mathcal{M}_\lambda$ which is larger or equal than the corresponding wealth process $V^{v_0,\pi}$ in $\mathcal{M}$. This can be regarded as the central motivation behind the construction of $\mathcal{M}_\lambda$ and naturally leads to an optimization problem which is dual to $(\mathbf{P})$ with respect to the allocation constraints K. For this purpose, analogous to Section 3.3, we consider the allocation unconstrained, wealth-constrained portfolio optimization problem

$$(\mathbf{P}_\lambda) \begin{cases} \Phi_\lambda(v_0) &= \sup_{\pi \in \Lambda_\lambda(v_0, B_1, B_2, g)} \mathbb{E}\left[U\left(V_\lambda^{v_0,\pi}(T)\right)\right] \\ \Lambda_\lambda(v_0, B_1, B_2, g) &= \left\{\pi \in \Lambda \;\middle|\; B_1 \le V_\lambda^{v_0,\pi}(T) \le B_2,\ \mathbb{E}\left[g(V_\lambda^{v_0,\pi}(T))\right] \le 0\right\}. \end{cases}$$

$(\mathbf{P}_\lambda)$ is the same optimization problem as $(\mathbf{P_0})$, but formulated in the financial market $\mathcal{M}_\lambda$, rather than $\mathcal{M}$. In $\mathcal{M}_\lambda$, the market price of risk γ_λ, the corresponding exponential local martingale Z_λ and pricing kernel $\tilde{Z}_\lambda$ are given as

$$\begin{aligned} \gamma_\lambda(t) &= \Sigma(t)^{-1}(\mu(t) - r(t)\mathbb{1} + \lambda(t)) = \gamma(t) + \Sigma(t)^{-1}\lambda(t), \\ Z_\lambda(t) &:= \exp\left(-\frac{1}{2}\int_0^t \|\gamma_\lambda(s)\|^2 ds - \int_0^t \gamma_\lambda(s)'dW(s)\right), \\ \tilde{Z}_\lambda(t) &:= \frac{Z_\lambda(t)}{P_0^\lambda(t)} = \exp\left(-\int_0^t r_\lambda(s) + \frac{1}{2}\|\gamma_\lambda(s)\|^2 ds - \int_0^t \gamma_\lambda(s)'dW(s)\right), \end{aligned}$$

for $t \in [0,T]$. Again, Z_λ and $\tilde{Z}_\lambda$ satisfy the SDEs

$$\begin{aligned} dZ_\lambda(t) &= -Z_\lambda(t)\gamma_\lambda(t)'dW(t) \\ d\tilde{Z}_\lambda(t) &= -\tilde{Z}_\lambda(t)\big(r_\lambda(t)dt + \gamma_\lambda(t)'dW(t)\big). \end{aligned}$$

As before, the market coefficients in $\mathcal{M}_\lambda$ need not be uniformly bounded and thus it is not clear if the local martingale Z_λ is indeed a true martingale. However, since Z_λ is non-negative, it is a supermartingale. The adjusted drift coefficients and the remaining market coefficients in any auxiliary $\mathcal{M}_\lambda$ satisfy the same boundedness assumption (2.26) as in the original market $\mathcal{M}$. Thus, we can use the same notion of market completeness to rewrite $(\mathbf{P}_\lambda)$ as an optimization over terminal payoffs.

Lemma 3.4.1 (Market Completeness $\mathcal{M}_\lambda$)**.** *Consider $\lambda \in \mathcal{D}$, $D \in \mathbf{X}_+$ with $\mathbb{E}[D^2] < \infty$ and $0 < \mathbb{E}[\tilde{Z}_\lambda(T)D] =: v_0 < \infty$. Then, there exists a $\pi \in \Lambda$ so that $V_\lambda^{v_0,\pi}(T) = D$ Q-a.s. and*

$$V_\lambda^{v_0,\pi}(t) = \mathbb{E}\left[D\frac{\tilde{Z}_\lambda(T)}{\tilde{Z}_\lambda(t)} \;\middle|\; \mathcal{F}_t\right].$$

Lemma 3.4.1 directly leads to the equivalent formulation of $(\mathbf{P}_\lambda)$ as

$$(\mathbf{P}_\lambda)\begin{cases}\Phi_\lambda(v_0) &= \sup\limits_{D\in C_\lambda(v_0,B_1,B_2,g)} \mathbb{E}\big[U(D)\big]\\ C(v_0,B_1,B_2,g) &= \big\{D\in L^2_Q \;\big|\; \mathbb{E}[\tilde{Z}_\lambda(T)D]=v_0,\; B_1\le D\le B_2,\; \mathbb{E}[g(D)]\le 0\big\}.\end{cases}$$

As the bounds B_1 and B_2 not only constrain the downside of the portfolio value, but also its upside, an increase of the terminal wealth in an auxiliary market $\mathcal{M}_\lambda$ (due to the added positive drift in (3.5)) may lead to a violation of the terminal wealth constraints and may even lead to $(\mathbf{P}_\lambda)$ being infeasible. We circumvent this issue by incorporating the pointwise bounds on D into the utility function $U(\cdot;B_1,B_2)$, analogously to 3.3.

Lemma 3.4.2. *Let $\lambda\in\mathcal{D}$, consider the optimization problem*

$$(\tilde{\mathbf{P}}_\lambda)\begin{cases}\tilde{\Phi}_\lambda(v_0) &= \sup\limits_{D\in C_\lambda(v_0,g(\cdot;B_2))} \mathbb{E}\big[U(D;B_1,B_2)\big]\\ C_\lambda(v_0,g(\cdot;B_2)) &= C_\lambda(v_0,0,\infty,g(\cdot;B_2))\\ &= \big\{D\in L^2_Q \;\big|\; \mathbb{E}[\tilde{Z}_\lambda(T)D]=v_0,\; \mathbb{E}[g(D;B_2)]\le 0\big\}\end{cases}$$

and define $v_\lambda(B_2):=\mathbb{E}[B_2\tilde{Z}_\lambda(T)]$.

(i) If $v_0\le v_\lambda(B_2)$, then $\tilde{\Phi}_\lambda(v_0)=\Phi_\lambda(v_0)$ and D^*_λ is optimal for $(\tilde{\mathbf{P}}_\lambda)$ if and only if D^*_λ is optimal for $(\mathbf{P}_\lambda)$.

(ii) If $v_0>v_\lambda(B_2)$, then $D^*_\lambda=\frac{v_0}{v_\lambda(B_2)}B_2$ is optimal for $\tilde{\Phi}_\lambda(v_0)$ and $\tilde{\Phi}_\lambda(v_0)=\mathbb{E}[U(B_2)]$.

In contrast to $(\mathbf{P_0})$ and $(\tilde{\mathbf{P}}_\mathbf{0})$ from Section 3.3, the changed market coefficients in $\mathcal{M}_\lambda$ imply that $(\mathbf{P}_\lambda)$ and $(\tilde{\mathbf{P}}_\lambda)$ are only equivalent if $v_0\le v_\lambda(B_2)$. It is easy to see that $(\mathbf{P}_\lambda)$ is even infeasible if $v_0>v_\lambda(B_2)$ and thus we use $(\tilde{\mathbf{P}}_\lambda)$ to develop the duality relation with respect to the allocation constraints K. Due to the non-negativity of the drift $(*)$ in (3.5), we know that for any $\lambda\in\mathcal{D}$ and $\pi\in\Lambda(v_0,K,B_1,B_2,g)$, $V^{v_0,\pi}(T)\le V^{v_0,\pi}_\lambda(T)$ and $D=V^{v_0,\pi}_\lambda(T)\in C_\lambda(v_0,g)$. Hence, Lemma 3.4.2 yields the weak duality relation

$$\begin{aligned}\Phi(v_0) &= \sup_{\pi\in\Lambda(v_0,K,B_1,B_2,g)} \mathbb{E}[U(V^{v_0,\pi}(T))]\\ &\le \sup_{\pi\in\Lambda(v_0,K,B_1,B_2,g)} \mathbb{E}[U(V^{v_0,\pi}_\lambda(T))]\\ &\le \sup_{D\in C_\lambda(v_0,g)} \mathbb{E}[U(D;B_1,B_2)]=\tilde{\Phi}_\lambda(v_0)\quad \forall\lambda\in\mathcal{D}.\end{aligned}\tag{3.6}$$

To further develop this notion of duality, we aim to find a $\lambda^*\in\mathcal{D}$ which achieves equality in the above relation, i.e $\Phi(v_0)=\tilde{\Phi}_{\lambda^*}(v_0)$, and the optimal portfolio processes for $(\mathbf{P})$ and $(\tilde{\mathbf{P}}_{\lambda^*})$ coincide. This is the case if λ^* satisfies a similar slackness condition as Condition (B) from [17] (resp. $(\tilde{B})$ from [34]).

Lemma 3.4.3. *Let $\lambda^*\in\mathcal{D}$, $\pi_{\lambda^*}\in\Lambda_{\lambda^*}$ such that $D^*_\lambda=V^{v_0,\pi_{\lambda^*}}_{\lambda^*}$ is optimal for $(\tilde{\mathbf{P}}_{\lambda^*})$. If further*

$$v_0\le v_{\lambda^*}(B_2),\quad \pi_{\lambda^*}\in K\quad \textit{and}\ \delta_K(\lambda^*(t))+\pi_{\lambda^*}(t)'\lambda^*(t)=0\quad \mathcal{L}[0,T]\otimes Q-a.e.,\tag{3.7}$$

then π_{λ^*} is admissible and optimal for the primal problem $(\mathbf{P})$ and $\Phi(v_0)=\tilde{\Phi}_{\lambda^*}(v_0)$.

Equation (3.7) can be seen as an analogue of the KKT condition (Corollary 2.1.30) for allocation constrained portfolio optimization problems. In the same way, Lemma 3.4.3 only provides a sufficient condition under which the optimal portfolio for $(\mathbf{P})$ in $\mathcal{M}$ and $(\mathbf{P}_{\lambda^*})$ in $\mathcal{M}_{\lambda^*}$ coincide, but does not provide a constructive way of finding the 'correct' $\lambda^* \in \mathcal{D}$. However, we realize if $\lambda^* \in \mathcal{D}$ satisfies the requirements of Lemma 3.4.3, then it minimizes $\tilde{\Phi}_\lambda(v_0)$ over $\lambda \in \mathcal{D}$ and therefore can be regarded as the solution to a dual minimization problem

$$(\mathbf{D^K}) \left\{ \Psi_{D^K} = \inf_{\lambda \in \mathcal{D}} \tilde{\Phi}_\lambda(v_0), \right. \tag{3.8}$$

where the duality is now induced by the allocation constraints K. As a matter of fact, under some regularity conditions, the converse statement is also true and (3.7) is satisfied if $\lambda = \lambda^*$ is optimal for $(\mathbf{D^K})$. To prove this statement formally, we start by characterizing the optimal terminal wealth for $(\tilde{\mathbf{P}}_\lambda)$ by following the same methodology as in Section 3.3. For a given $\lambda \in \mathcal{D}$, we define

$$g_\lambda : \mathbb{R}^2 \to \mathbb{R}, \quad g_\lambda(x) = \begin{pmatrix} -\tilde{Z}_\lambda(T)x \\ -g(x; B_2) \end{pmatrix}$$

and again make use of the dual optimization problem associated with $(\tilde{\mathbf{P}}_\lambda)$ and its expected value constraints.

Theorem 3.4.4. *Let $\lambda \in \mathcal{D}$, $v_0 \le v_\lambda(B_2)$, $y^* = (y_0^*, y_1^*)' \in [0,\infty)^2$ be optimal for the dual minimization problem*

$$(\mathbf{D}_\lambda^\mathbf{V}) \left\{ \Psi_{D_\lambda^V} = \inf_{y=(y_0,y_1)' \in [0,\infty)^2} \Big(\mathbb{E}\left[U_{g_\lambda}^*(y; B_1, B_2) \right] + y_0 v_0 \Big). \right.$$

If $\mathcal{I}_{g_\lambda}(y^*; B_1, B_2) \in L_Q^2$ and

$$y \to \mathbb{E}\left[g_\lambda(\mathcal{I}_{g_\lambda}(y; B_1, B_2)) \right]' \Delta y$$

is usc at $y = y^*$ for all $\Delta y \in \mathbb{R} \times [0,\infty)$, then $\mathcal{I}_{g_\lambda}(y^*; B_1, B_2)$ is optimal for $(\mathbf{P}_\lambda)$ and $y_1^* \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y^*; B_1, B_2) \right] = 0$.

Combining the dual optimization problems $(\mathbf{D^K})$ and $(\mathbf{D}_\lambda^\mathbf{V})$ yields a joint duality relation between the allocation constraints *and* wealth constraints

$$\Phi(v_0) \overset{(\mathbf{D^K})}{\le} \inf_{\lambda \in \mathcal{D}} \tilde{\Phi}_\lambda(v_0) \overset{(\mathbf{D}_\lambda^\mathbf{V})}{\le} \inf_{\lambda \in \mathcal{D}} \left(\inf_{y=(y_0,y_1)' \in [0,\infty)^2} \Big(\mathbb{E}\left[U_{g_\lambda}^*(y; B_1, B_2) \right] + y_0 v_0 \Big) \right) \tag{3.9}$$

and naturally leads to the joint dual optimization problem

$$(\mathbf{D^{K,V}}) \begin{cases} \Psi_D^{K,V} & = \inf_{\lambda \in \mathcal{D}} \left(\inf_{y=(y_0,y_1)' \in [0,\infty)^2} \Big(\mathbb{E}\left[U_{g_\lambda}^*(y; B_1, B_2) \right] + y_0 v_0 \Big) \right) \\ & = \inf_{y=(y_0,y_1)' \in [0,\infty)^2} \left(\inf_{\lambda \in \mathcal{D}} \Big(\mathbb{E}\left[U_{g_\lambda}^*(y; B_1, B_2) \right] \Big) + y_0 v_0 \right) \end{cases}.$$

$(\mathbf{D^{K,V}})$ is a dual with respect to $(\mathbf{P})$, where the duality is induced jointly by the allocation constraints (K) *and* the wealth constraints (budget equation, B_1, B_2 and g). Moreover, as we can change the order of minimization in $(\mathbf{D^{K,V}})$, we know that if $(\lambda^*, y^*) \in \mathcal{D} \times [0,\infty)^2$ is optimal for $(\mathbf{D^{K,V}})$, then y^* is optimal for $(\mathbf{D}_{\lambda^*}^\mathbf{V})$ and λ^* minimizes $\mathbb{E}\left[U_{g_\lambda}^*(y^*; B_1, B_2) \right]$. The latter minimization plays an important role in Section 3.5.

To ensure that the optimal values (λ^*, y^*) for the joint dual $(\mathbf{D^{K,V}})$ actually yield an optimal solution to the primal jointly constrained portfolio optimization problem $(\mathbf{P})$ (i.e., $\Phi(v_0) = \Psi_{D^{K,V}}$), it is necessary that the allocation unconstrained optimization problem $(\tilde{\mathbf{P}}_{\lambda^*})$ in $\mathcal{M}_{\lambda^*}$ is solvable by the dual $(\mathbf{D}_{\lambda^*}^\mathbf{V})$ (i.e., (λ^*, y^*) satisfies the requirements of Theorem 3.4.4). To this end, we define the subset of dual processes $\mathcal{D}' \subset \mathcal{D}$ such that every $\lambda \in \mathcal{D}'$ satisfies

- $v_0 \le v_\lambda(B_2)$,
- there exist $(y_0^*, y_1^*)' \in [0,\infty)^2$ which is optimal for $(\mathbf{D}_\lambda^{\mathbf{V}})$,
- $\mathcal{I}_{g_\lambda}(y^*; B_1, B_2) \in L_Q^2$
- $y \to \mathbb{E}\left[g_\lambda(\mathcal{I}_{g_\lambda}(y; B_1, B_2))\right]' \Delta y$ is usc at $y = y^*$ for all $\Delta y \in \mathbb{R} \times [0,\infty)$.

For $\lambda^* \in \mathcal{D}'$, we can now define a set of equivalent conditions which ensure that the optimal portfolios for $(\mathbf{P}_{\lambda^*})$ and $(\mathbf{P})$ coincide.

Optimality Conditions for $(\mathbf{P})$: For $\pi^* \in \Lambda(v_0, K, B_1, B_2, g)$, $\lambda^* \in \mathcal{D}'$ with optimal $y^* \in [0,\infty)^2$ for $(\mathbf{D}_{\lambda^*}^{\mathbf{V}})$, we define the following conditions:

$(\tilde{A})$ $\forall \pi \in \Lambda(v_0, K, B_1, B_2, g)$ we have

$$\mathbb{E}[U(V^{v_0,\pi}(T))] \le \mathbb{E}[U(V^{v_0,\pi^*}(T))].$$

$(\tilde{B})$ The optimal portfolio process π_{λ^*} for $(\mathbf{P}_{\lambda^*})$ satisfies:

$$\pi_{\lambda^*} \in K \quad \text{and} \quad [\delta_K(\lambda^*(t)) + \pi_{\lambda^*}(t)'\lambda^*(t)] = 0 \quad \mathcal{L}[0,T] \otimes Q - a.e..$$

$(\tilde{C})$ $\forall \lambda \in \mathcal{D}$ we have

$$\tilde{\Phi}_{\lambda^*}(v_0) \le \tilde{\Phi}_\lambda(v_0).$$

$(\tilde{D})$ $\forall \lambda \in \mathcal{D}$ we have

$$\mathbb{E}\left[U_{g_{\lambda^*}}^*(y^*; B_1, B_2)\right] \le \mathbb{E}\left[U_{g_\lambda}^*(y^*; B_1, B_2)\right].$$

$(\tilde{E})$ $\forall \lambda \in \mathcal{D}$ we have

$$\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)\tilde{Z}_\lambda(T)\right] \le v_0.$$

Theorem 3.4.5. *Let $\lambda^* \in \mathcal{D}'$. Then, Conditions $(\tilde{B})$, $(\tilde{C})$, $(\tilde{D})$ and $(\tilde{E})$ are equivalent and imply $(\tilde{A})$ with $\pi^* := \pi_{\lambda^*}$.*

Due to Lemma 2.1.17, (iii), it is sufficient in the proof of $(\tilde{D}) \Rightarrow (\tilde{B})$ to consider only $\nu \in \mathcal{D}$ with $\|\nu(t)\| \le 1$ $\mathcal{L}[0,T] \otimes Q - a.e.$ and $\nu(t) = -\lambda^*(t)/\max(1, \|\lambda^*(t)\|)$. Even though this does not affect the proof of Theorem 3.4.5 in any meaningful way, it has the satisfying consequence that any local minimizer λ^* of Conditions $(\tilde{C})$ or $(\tilde{D})$ is indeed a global minimizer over the whole space $\mathcal{D}$ and satisfies Condition $(\tilde{B})$. This will be useful in a verification theorem in Section 3.5.

Corollary 3.4.6. *Let $\lambda^* \in \mathcal{D}'$. If Condition $(\tilde{B})$, $(\tilde{C})$, $(\tilde{D})$ or $(\tilde{E})$ is satisfied for all $\lambda \in \mathcal{D}$ with $\|\lambda^*(t) - \lambda(t)\| \le 1$ $\mathcal{L}[0,T] \otimes Q - a.e.$, then $(\tilde{B})$, $(\tilde{C})$, $(\tilde{D})$ and $(\tilde{E})$ are satisfied for all $\lambda \in \mathcal{D}$. In particular, $\pi^* := \pi_{\lambda^*}$ is optimal for $(\mathbf{P})$.*

The optimality conditions $(\tilde{B}) - (\tilde{E})$ offer alternative ways to find and verify the optimality of a portfolio process π^* for the fully constrained portfolio optimization problem $(\mathbf{P})$ in $\mathcal{M}$. The central underlying assumption is that we can find a different market with adjusted market coefficients $\mathcal{M}_{\lambda^*}$, where the optimal portfolio π_{λ^*} for the allocation unconstrained problem $(\tilde{\mathbf{P}}_{\lambda^*})$ coincides with π^*. According to Condition $(\tilde{B})$, π^* and π_{λ^*} coincide if the wealth processes $V^{v_0,\pi_{\lambda^*}}$ in $\mathcal{M}$ and $V_{\lambda^*}^{v_0,\pi_{\lambda^*}}$ in $\mathcal{M}_{\lambda^*}$ are equal. Hence, the change in market coefficients from the original $\mathcal{M}$ to $\mathcal{M}_{\lambda^*}$ must not have any impact on the portfolio performance of π^*. Following

Condition $(\tilde{C})$, we additionally know that $\mathcal{M}_{\lambda^*}$ yields the least expected utility under all $\mathcal{M}_\lambda$, $\lambda \in \mathcal{D}$, if the investor follows an optimal strategy. In this sense, $\mathcal{M}_{\lambda^*}$ has the least favourable market coefficients from the investor's perspective. Condition $(\tilde{D})$ is in fact just a dual reformulation of Condition $(\tilde{C})$, where the duality is now induced not by the allocation constraints K, but by the expectation constraint g on terminal wealth and the budget equation. As we will see in Section 3.5, Condition $(\tilde{D})$ proves to be particularly useful in explicitly determining λ^* and π^*. Lastly, Condition $(\tilde{E})$ states that there exists no market $\mathcal{M}_\lambda$, where hedging the optimal terminal wealth $D^*_{\lambda^*} := V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(T) = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)$ for $(\tilde{\mathbf{P}}_{\lambda^*})$ is more expensive than in $\mathcal{M}_{\lambda^*}$. Again, the market coefficients of $\mathcal{M}_{\lambda^*}$ can be regarded as least favourable for the investor. This is a special case of more general results about hedging contingent claims under allocation constraints, which is discussed in great detail in [18].

3.5 The Optimal Auxiliary Market

In this section we illustrate how one can make use of the equivalent optimality conditions derived in the previous section to solve the fully constrained portfolio optimization problem $(\mathbf{P})$ in a Black-Scholes market. We first aim to identify the minimizing λ^* for every $y^* \in [0,\infty)^2$ in Condition $(\tilde{D})$ in Section 3.4, by considering an associated HJB PDE. If $U^*_{g_\lambda}$ satisfies a polynomial growth condition for deterministic $\lambda \in \mathcal{D}$ and the allocation constraint K is a convex cone, then the minimizing $\lambda^* \in \mathcal{D}$ is deterministic, independent of y^* and can be determined as the optimizer of a deterministic convex optimization problem. This is shown in Section 3.5.2.

Throughout the whole of Section 3.5 we make the following additional assumptions about the market coefficients and bounds on terminal wealth:

Assumption 3.5.1 (Standing Assumption Section 3.5)**.** *The market coefficients r, μ and Σ as well as the bounds on terminal wealth $0 \le B_1 < v_0 e^{rT} < B_2 \le \infty$ are constants and δ_K is continuous on X_K.*

Note however that a generalization to deterministic and continuous $r(t)$, $\mu(t)$ and $\Sigma(t)$ is straightforward. To include the case $B_1 = 0$ or $B_2 = \infty$ one has to impose additional growth assumptions on $\mathcal{I}_{g_{\lambda^*}}$ for the (unknown!) $\lambda^* \in \mathcal{D}$ which satisfies condition $(\tilde{D})$. Assumption 3.5.1 allows the use of the ensuing dynamic programming techniques, which lead to closed-form solutions to the primal, fully constrained portfolio optimization problem $(\mathbf{P})$ for convex cone allocation constraints K.

3.5.1 Optimization Problem Associated with Condition D

By interchanging the order of minimization in $(\mathbf{D}^{\mathbf{K},\mathbf{V}})$ or by considering Condition $(\tilde{D})$, we have seen that the 'optimal' λ^*, for which the optimal portfolios for $(\mathbf{P})$ and $(\tilde{\mathbf{P}}_{\lambda^*})$ coincide, is the solution to the 'inner' optimization problem

$$(\mathbf{D_y}) \left\{ \Psi_{D_y} = \inf_{\lambda \in \mathcal{D}} \mathbb{E}\left[U^*_{g_\lambda}(y; B_1, B_2)\right], \right.$$

for the specific choice $y = y^*$, where y^* is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$. In particular, y^* already depends on λ^*. Hence, while the optimizer $\lambda^*(y^*)$ for $(\mathbf{D}_{\mathbf{y}^*})$ is optimal in the sense of Condition $(\tilde{D})$, this may not be the case for general $\lambda^*(y)$ and $y \in [0,\infty)^2$. Irrespectively, a characterization of the minimizer $\lambda^*(y)$ for $(\mathbf{D_y})$ for arbitrary $y \in [0,\infty)^2$ may still yield valuable insights for the solution of the primal portfolio optimization problem $(\mathbf{P})$. Recall that the generalized conjugate

$U^*_{g_\lambda}(y; B_1, B_2)$ only depends on $\lambda \in \mathcal{D}$ through the terminal value $\tilde{Z}_\lambda(T) \in (0, \infty)$. Hence, by conditioning on this value, we obtain a deterministic function

$$U^*_{g_\lambda}(y; B_1, B_2)\Big|_{\tilde{Z}_\lambda(T)=z} = \sup_{B_1 \le x \le B_2} \left(U(x) - y' \begin{pmatrix} xz \\ g(x) \end{pmatrix} \right) =: U^*_{\hat{g}}(y; B_1, B_2, z),$$

with $\hat{g}(x) = \hat{g}(x; z) := (-xz; g(x; B_2))'$. Therefore, we can view $(\mathbf{D_y})$ as a stochastic control problem with state process $\tilde{Z}_\lambda$, control process $\lambda \in \mathcal{D}$ and minimization objective $U^*_{\hat{g}}(y; B_1, B_2, \tilde{Z}_\lambda(T))$. The HJB equation associated with $(\mathbf{D_y})$ is then

$$\begin{aligned} 0 &= G_t(t, z; y) - rzG_z(t, z; y) \\ &\quad + \inf_{x \in X_K} \Big(-\delta_K(x) z G_z(t, z; y) + \frac{1}{2} \|\gamma + \Sigma^{-1} x\|^2 z^2 G_{zz}(t, z; y) \Big) \qquad (3.10) \\ G(T, z; y) &= U^*_{\hat{g}}(y; B_1, B_2, z). \end{aligned}$$

For the remainder of this section, we focus on the HJB equation (3.10) and show that its solution, provided it satisfies some regularity conditions, induces a solution to the (inner) dual optimization problem $(\mathbf{D_y})$. Assuming that G solves (3.10) and is strictly decreasing and convex in z, there exists a minimizer $\lambda^*(t, z; y)$, which attains the infimum in (3.10). By slightly rewriting the PDE, one can see that $\lambda^*(t, z; y)$ actually minimizes

$$\begin{aligned} \lambda^*(t, z; y) &= \operatorname*{argmin}_{x \in X_K} \left(\frac{z^2}{2} \|\gamma + \Sigma^{-1} x\|^2 G_{zz}(t, z; y) - \delta_K(x) z G_z(t, z; y) \right) \\ &\overset{G_z < 0}{=} \operatorname*{argmin}_{x \in X_K} \Big(\underbrace{-\frac{1}{2} \frac{z G_{zz}(t, z; y)}{G_z(t, z; y)}}_{=:\mathrm{RRA}(t,z;y) \ge 0} \|\gamma + \Sigma^{-1} x\|^2 + \delta_K(x) \Big) \qquad (3.11) \\ &= \operatorname*{argmin}_{x \in X_K} \Big(\mathrm{RRA}(t, z; y) \|\gamma + \Sigma^{-1} x\|^2 + \delta_K(x) \Big). \qquad (3.12) \end{aligned}$$

This means that the (non-negative) relative risk aversion $\mathrm{RRA}(t, z; y)$ of G with respect to z serves as a weighting factor in the minimization between the non-negative components $\|\gamma + \Sigma^{-1} x\|^2$ and $\delta_K(x)$.

Lemma 3.5.2. *Let $G = G(t, z; y)$ be continuously differentiable in $t \in [0, T)$, twice continuously differentiable in $z \in (0, \infty)$, be convex and strictly decreasing in z and a solution to the HJB equation (3.10). Then there exists a corresponding minimizing argument $\lambda^*(t, z; y)$ (as in (3.11)), which is uniformly bounded in $(t, z; y)$.*

Using this observation and assuming polynomial growth, convexity and monotonicity conditions in z, we are able to locally prove a verification theorem for the HJB equation (3.10). Having the minimization property locally is sufficient for our purposes as noted in Corollary 3.4.6.

Theorem 3.5.3 (Verification Theorem).
Let Assumption 3.5.1 hold, let $y \in [0, \infty)^2$ be fixed. Further, let $G = G(t, z; y)$ be continuously differentiable in $t \in [0, T)$, twice continuously differentiable in $z \in (0, \infty)$, be a solution to the HJB equation (3.10), be convex, strictly decreasing and satisfy the polynomial growth condition

$$G(t, z; y) \le C\big(z^{-\alpha} + z^{\alpha}\big), \qquad \text{for some } \alpha > 0,\ C > 0.$$

Further, let

$$\lambda^*(t,z;y) := \underset{x\in X_K}{argmin}\left(\frac{z^2}{2}\|\gamma+\Sigma^{-1}x\|^2 G_{zz}(t,z;y) - \delta_K(x) z G_z(t,z;y)\right), \tag{3.13}$$

be uniformly bounded in $(t,z;y)$*. Define the stochastic process* $\lambda^* = \left(\lambda^*(t)\right)_{t\in[0,T]}$ *in feedback-form as*

$$\lambda^*(t) = \lambda^*(t, \tilde{Z}_{\lambda^*}(t); y).$$

Then, $\forall \lambda \in \mathcal{D}$ *with* $\|\lambda(s)-\lambda^*(s)\| \le 1$ *and* $(t,z)\in[0,T]\times(0,\infty)$*:*

$$G(t,z;y) \le \mathbb{E}[U^*_{g_\lambda}(y;B_1,B_2)|\tilde{Z}_\lambda(t)=z]$$

and

$$G(t,z;y) = \mathbb{E}[U^*_{g_{\lambda^*}}(y;B_1,B_2)|\tilde{Z}_{\lambda^*}(t)=z].$$

Note that Theorem 3.5.3 does not provide verification for the fully constrained portfolio optimization problem ($\mathbf{P}$), but only for the inner dual optimization problem ($\mathbf{D_y}$). We still need to determine the corresponding 'outer' minimizer y^* and show that the process λ^* is indeed an element of $\mathcal{D}'$.

3.5.2 Convex Cone Constraints

In this section we solve the HJB equation (3.10) and determine the pointwise minimizer $\lambda^*(t,z;y)$ (3.13) under the assumption that K is a convex cone and the generalized conjugate satisfies a polynomial growth condition in z. If $|U(B_1)|$ and $|U(B_2)|$ are finite, then this growth condition is always satisfied. We then use the verification theorem from Section 3.5.1 to link the pointwise minimizer $\lambda^*(t,z;y)$ to optimality Condition $(\tilde{D})$ and finally characterize the optimal terminal wealth for ($\mathbf{P}$) in the original market $\mathcal{M}$ as the optimal terminal wealth for $\tilde{\mathbf{P}}_{\lambda^*}$ in the auxiliary market $\mathcal{M}_{\lambda^*}$.

As per Lemma 2.1.17, (iv), the allocation constraints K form a convex cone if and only if $\delta_K(x)=0$ for all $x\in X_K$. In this special case the HJB equation (3.10) simplifies to

$$0 = G_t(t,z;y) - rzG_z(t,z;y) + \frac{z^2}{2}\inf_{x\in X_K}\left(\|\gamma+\Sigma^{-1}x\|^2 G_{zz}(t,z;y)\right) \tag{3.14}$$
$$G(T,z;y) = U^*_{\tilde{g}}(y;B_1,B_2,z).$$

If G is convex, the infimum is attained by the pointwise minimizer

$$\lambda^*(t,z;y) := \lambda^* := \underset{x\in X_K}{\text{argmin}}\|\gamma+\Sigma^{-1}x\|^2,$$

which is independent of $(t,z;y)$. Further, the HJB PDE reduces to a linear PDE, which can be solved through a transformation to the well-studied heat equation (see e.g. [6]). For this purpose, recall the following result about the heat equation:

Lemma 3.5.4. *Consider a real function* $f:\mathbb{R}\to\mathbb{R}$ *for which exist constants* C_0*,* α_0 *such that*

$$|f(u)| \le C_0 e^{\alpha_0 u^2} \quad \forall u\in\mathbb{R}. \tag{3.15}$$

Then, for all $0<T<\frac{1}{4\alpha_0}$ *the function* $F:(0,T]\times\mathbb{R}\to\mathbb{R}$ *defined by*

$$F(\tau,u) = \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} e^{-\frac{(u-x)^2}{4\tau}} f(x)dx = \frac{1}{\sqrt{4\pi\tau}}\int_{\mathbb{R}} e^{-\frac{x^2}{4\tau}} f(u-x)dx,$$

is in $C^{(1,2)}((0,T]\times\mathbb{R})$ *and is a solution to the heat equation*

$$\begin{aligned} F_\tau(\tau,u) &= F_{uu}(\tau,u) \quad && \forall(\tau,u)\in(0,T]\times\mathbb{R} \\ F(0,u) := \lim_{\tau\downarrow 0} F(\tau,u) &= f(u) && \textit{for almost all } u\in\mathbb{R}. \end{aligned} \tag{3.16}$$

Lemma 3.5.5. *Let* $y\in[0,\infty)^2$ *and let* $U^*_{\hat{g}}(y;,B_1,B_2,z)$ *satisfy a polynomial growth condition in* z*:*

$$|U^*_{\hat{g}}(y;B_1,B_2,z)| \le C\big(z^{-\alpha}+z^{\alpha}\big), \quad \forall z\in(0,\infty) \tag{3.17}$$

and some constants $C,\ \alpha>0$*. If* K *is a convex cone, then*

$$G(t,z;y) := \frac{1}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}} e^{-\frac{x^2}{4(T-t)}}U^*_{\hat{g}}\left(y;B_1,B_2,zye^{-(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)-\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}\right)dx$$

is continuously differentiable in t*, twice continuously differentiable in* z*, convex in* z*, strictly decreasing in* z *and satisfies the HJB equation (3.10) with*

$$\lambda^*(t,z;y) := \lambda^* := \underset{x\in X_K}{argmin}\|\gamma+\Sigma^{-1}x\|^2 \quad \textit{and} \quad \gamma_{\lambda^*} := \gamma+\Sigma^{-1}\lambda^*.$$

Further, there exists a constant $\tilde{C}$ *such that* G *satisfies the polynomial growth condition*[1]

$$|G(t,z;y)| \le \tilde{C}\big(z^{-\alpha}+z^{\alpha}\big) \quad \forall z\in(0,\infty). \tag{3.18}$$

It is important to emphasize that the previous techniques heavily relied on K being a convex cone (hence $\delta_K(\lambda^*)=0$) as this simplifies the HJB equation (3.10) to a linear PDE. For more general allocation constraints with $\delta_K(\lambda^*)\neq 0$, the PDE may become non-linear and extremely difficult to solve.

The polynomial growth condition (3.17) is designed in such a way that it is satisfied if $U(x)-y_1g(x)$ is a power utility function. In particular, if U and $-g$ are bounded from above and below by a power utility function, then we can show that (3.17) is satisfied. Further, if B_2, $U(B_2)$ and $g(B_2)$ are finite, then the growth condition is satisfied trivially.

Lemma 3.5.6. *Consider* $U\in\mathcal{U}$*,* $(y_0,y_1)'\in[0,\infty)^2$ *and let Assumption 3.5.1 be satisfied. If one of the conditions*

(i) There exist constants $C_->0$*,* $C_+>0$ *and* $b_-<0$*,* $b_+>0$ *such that*

$$C_-\left(\frac{1}{b_-}x^{b_-}-1\right) \le \min\left(U(x),-g(x)\right) \ \textit{and} \ \max\left(U(x),-g(x)\right) \le C_+\left(\frac{1}{b_+}x^{b_+}+1\right).$$

(ii) $C=|U(B_2)|+y_0(B_1+B_2)+y_1|g(B_2)|<\infty$

is satisfied, then $U^*_{\hat{g}}$ *satisfies the growth condition (3.17).*

Corollary 3.5.7. *Let* K *be a convex cone, let Assumption 3.5.1 hold and either (i) or (ii) from Lemma 3.5.6 be satisfied. If*

$$\lambda^* := \underset{x\in X_K}{argmin}\ \|\gamma+\Sigma^{-1}x\|^2 \tag{3.19}$$

is an element of $\mathcal{D}'$*, then* λ^* *satisfies condition* $(\tilde{D})$ *and the optimal portfolio for the wealth-constrained portfolio optimization problem* $(\tilde{\mathbf{P}}_{\lambda^*})$ *is optimal for the fully constrained portfolio optimization problem* $(\mathbf{P})$*. In particular, if* y^* *is optimal for* $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$*, then the optimal terminal wealth for* $(\mathbf{P})$ *is* $D^*=\mathcal{I}_{g_{\lambda^*}}(y^*;B_1,B_2)$*.*

[1] Note that the constants C, $\tilde{C}$ and α are allowed to depend on y, B_1 and B_2.

Remark 3.5.8. *If K is a convex cone, we have $\delta_K(x) = 0$ for all $x \in X_K$ and thus*

$$V^{v_0,0}(T) = v_0 P_0(T) = v_0 P_0^{\lambda}(T) = V_{\lambda}^{v_0,0}(T) \quad \forall \lambda \in \mathcal{D}.$$

According to Assumption 3.2.1, this implies

$$v_\lambda(B_2) = \mathbb{E}[B_2 \tilde{Z}_\lambda(T)] > \mathbb{E}[V^{v_0,0}(T)\tilde{Z}_\lambda(T)] = \mathbb{E}[V_\lambda^{v_0,0}(T)\tilde{Z}_\lambda(T)] = v_0.$$

Moreover, if $B_2 < \infty$ under Assumption 3.5.1, the maximizing arguments $\mathcal{I}_{g_\lambda}(y; B_1, B_2)$ is bounded from above (by $B_2 < \infty$) Q-a.s. for all $\lambda \in \mathcal{D}$. Thus,

$$\mathbb{E}\left[\mathcal{I}_{g_\lambda}(y; B_1, B_2)^2\right] < \infty, \quad \textit{and} \quad \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y; B_1, B_2)\tilde{Z}_\lambda(T)\right] < \infty \quad \forall \lambda \in \mathcal{D}.$$

Therefore, λ^* (as in Corollary 3.5.7) is in $\mathcal{D}'$ if there exists a minimizer y^* for $(\mathbf{D}_{\lambda^*}^{\mathbf{V}})$ and $\mathbb{E}[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2))]'\Delta y$ is usc at $y = y^*$ for all $\Delta y \in \mathbb{R} \times [0, \infty)$.

3.5.3 Examples

In this section we consider examples where we can characterize the optimal terminal wealth for $(\mathbf{P})$ explicitly, i.e. up to the deterministic minimizers $(y_0^*, y_1^*) \in [0, \infty)^2$ and $\lambda^* \in X_K$. For this purpose we restrict our analysis to power utility functions of the form

$$U(x) = \begin{cases} \frac{1}{b}x^b, & x > 0 \\ \liminf\limits_{x \downarrow 0} \frac{1}{b}x^b, & x = 0 \\ -\infty, & x < 0 \end{cases} \quad \text{with } b \in (-\infty, 1)\backslash\{0\}. \tag{3.20}$$

This decision is motivated partially by (i) in Lemma 3.5.6, but it is also necessary to ensure the analytical tractability of the maximizing argument $\mathcal{I}_{g_{\lambda^*}}$. We begin by disregarding the expected value constraints, i.e., setting $g \equiv 0$, which leaves us in the setting of [34]. We then increase the complexity by additionally considering Value-at-Risk ('VaR') constraints and expected shortfall ('ES') constraints as natural and relevant practical examples. Further, for λ^* as in (3.19) and $y_0 > 0$, we define the expression

$$\mathcal{I}(y_0) = \left(y_0 \tilde{Z}_{\lambda^*}(T)\right)^{\frac{1}{b-1}} = \underset{x \geq 0}{\operatorname{argmax}} \left(\frac{1}{b}x^b - y_0 \tilde{Z}_{\lambda^*}(T)x\right).$$

If y_0 is chosen such that $\mathbb{E}[\mathcal{I}(y_0)\tilde{Z}_{\lambda^*}(T)] = v_0$, then $\mathcal{I}(y_0)$ is the optimal terminal wealth for $(\mathbf{P})$ if there are no wealth constraints (i.e., $B_1 = 0$, $B_2 = \infty$ and $g \equiv 0$). It will be useful to express $\mathcal{I}_{g_{\lambda^*}}$ in terms of $\mathcal{I}$, when we add wealth constraints in the subsequent analysis.

We summarize the additional technical requirements for this Section in Assumption 3.5.9.

Assumption 3.5.9. *$K \subset \mathbb{R}^d$ is a convex cone, U is a power utility function as in (3.20) and λ^* is defined as in (3.19).*

No Expected Value Constraints: We begin by assuming that there are no expected value constraints, i.e., that $g \equiv 0$. In this simple context, $U(x) - y_0 \tilde{Z}_{\lambda^*}(T)x - y_1 g(x)$ is a strictly concave function in x for all $(y_0, y_1)' \in [0, \infty)^2$ and therefore the maximizing argument $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)$ can be obtained by capping the global maximizing argument $\mathcal{I}(y_0)$ at the boundaries B_1 and B_2.

Example 3.5.10.
Let Assumptions 3.5.1 and 3.5.9 hold. If $g \equiv 0$ *and* $y^* = (y_0^*, 0)' \in (0,\infty) \times [0,\infty)$ *is optimal for* $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$*, then the optimal terminal wealth for* $(\mathbf{P})$ *is*

$$D^* = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2) = B_1 + \Big(\mathcal{I}(y_0^*) - B_1\Big)^+ - \Big(\mathcal{I}(y_0^*) - B_2\Big)^+ =: Cap\big(\mathcal{I}(y_0^*), B_1, B_2\big).$$

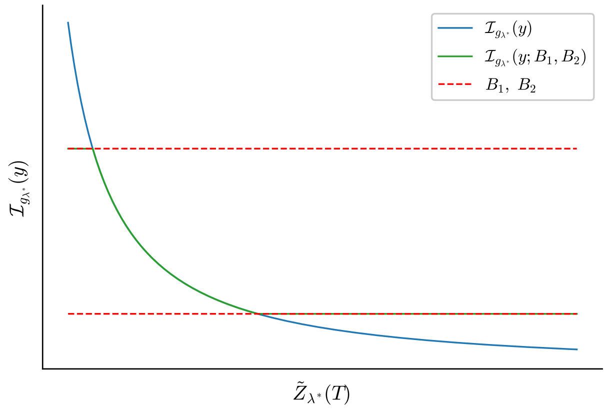


Figure 3.2: Illustration of the dependence of the optimal terminal wealth D^* on $\tilde{Z}_{\lambda^*}(T)$ without expected value constraints and for a given constant value of y_0^* (cf. Example 3.5.10).

Value-at-Risk Constraints: For constants $B_1 < B_{VaR} < B_2$ and $\epsilon > 0$ we define the VaR constraint as

$$g(x) = \mathbb{1}_{\{x < B_{VaR}\}} - \epsilon. \tag{3.21}$$

The VaR is a risk measure which is frequently used by practitioners in risk management and is deeply integrated into financial regulations (e.g. in Solvency II and Basel III). The VaR constraint restricts the probability that the investor's terminal wealth falls below B_{VaR} to a value of at most ϵ. Hence, the VaR constraint restricts the portfolio loss $L = -(V^{v_0,\pi}(T) - B)$ in comparison to a benchmark $B \in \mathbb{R}$ to at most $B - B_{VaR}$ in $(1-\epsilon)\%$ of all scenarios. However, conditioned on a loss of more than $B - B_{VaR}$ occurring, the VaR constraint does not impose a restriction on the magnitude of this loss. Further, the VaR is not a convex risk measure (i.e. g is not convex) and thus it is possible that the convex combination of two terminal payoffs violates the VaR constraint, even though the individual payoffs satisfy the VaR constraint. This can be regarded as a punishment for diversification and thus poses another weakness of VaR.

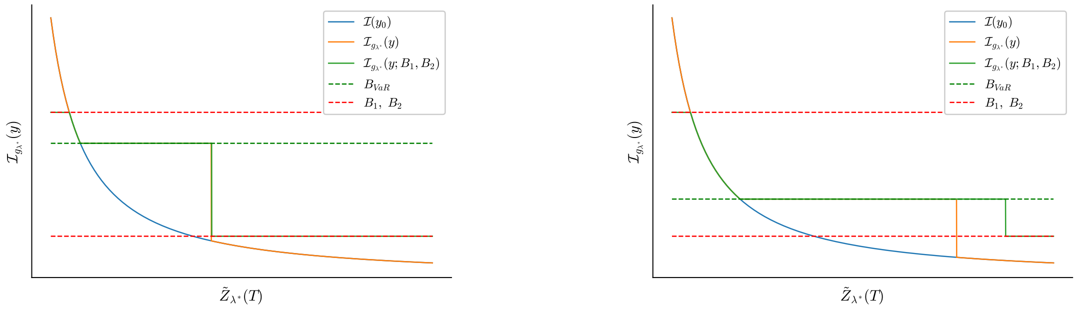
Example 3.5.11.
Let Assumptions 3.5.1 and 3.5.9 hold. If g *is a VaR constraint (as in (3.21)),* $B_1 = 0$*,* $B_2 = \infty$*, and* $y^* = (y_0^*, y_1^*)' \in (0,\infty) \times [0,\infty)$ *is optimal for* $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$*, then the optimal terminal wealth for* $(\mathbf{P})$ *is*

$$D^* = \mathcal{I}_{g_{\lambda^*}}(y^*) = \begin{cases} \mathcal{I}(y_0^*), & B_{VaR} < \mathcal{I}(y_0^*) \\ B_{VaR}, & \mathcal{I}(y_0^*) \le B_{VaR} \ \& \ \frac{1-b}{b}\mathcal{I}(y_0^*)^b + y_0^*\tilde{Z}_{\lambda^*}(T)B_{VaR} < \frac{1}{b}B_{VaR}^b + y_1^* \\ \mathcal{I}(y_0^*), & \mathcal{I}(y_0^*) \le B_{VaR} \ \& \ \frac{1-b}{b}\mathcal{I}(y_0^*)^b + y_0^*\tilde{Z}_{\lambda^*}(T)B_{VaR} \ge \frac{1}{b}B_{VaR}^b + y_1^* \end{cases}$$

As $U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1 g(x)$ is no longer concave in x (note the discontinuity of $g(x)$ at $x = B_{VaR}$), the maximizing argument $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)$ can no longer be obtained by capping $\mathcal{I}_{g_{\lambda^*}}(y)$ at the boundaries B_1 and B_2.

Example 3.5.12.
Let Assumptions 3.5.1 and 3.5.9 hold. If g is a VaR constraint (as in (3.21)) and $y^* = (y_0^*, y_1^*)' \in (0,\infty) \times [0,\infty)$ is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$, then the optimal terminal wealth for $(\mathbf{P})$ is

$$D^* = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2) = \begin{cases} B_1, & \mathcal{I}_{g_{\lambda^*}}(y^*) < B_1 \ \& \ \tilde{Z}_{\lambda^*}(T) \geq \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1^*}{y_0^*(B_{VaR} - B_1)} \\ B_{VaR}, & \mathcal{I}_{g_{\lambda^*}}(y^*) < B_1 \ \& \ \tilde{Z}_{\lambda^*}(T) < \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1^*}{y_0^*(B_{VaR} - B_1)} \\ \mathcal{I}_{g_{\lambda^*}}(y^*), & B_1 \leq \mathcal{I}_{g_{\lambda^*}}(y^*) \leq B_2 \\ B_2, & B_2 < \mathcal{I}_{g_{\lambda^*}}(y^*). \end{cases}$$



(a) (b)

Figure 3.3: Illustration of the dependence of the optimal terminal wealth D^* on $\tilde{Z}_{\lambda^*}(T)$ with VaR constraints and a given constant value of y^* (cf. Example 3.5.12). In Figure 3.3(a), B_{VaR} is large enough so that $\mathcal{I}_{g_{\lambda^*}}(y^*) < B_1$ already implies $\tilde{Z}_{\lambda^*}(T) \geq \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1^*}{y_0^*(B_{VaR} - B_1)}$ and thus the pointwise constraints B_1, B_2 just cap off $\mathcal{I}_{g_{\lambda^*}}(y^*)$. This is is no longer the case in Figure 3.3(b), as B_{VaR} is substantially smaller.

Expected Shortfall Constraints: For constants $B_1 < B_{ES} < B_2$ and $\epsilon > 0$ we define the ES constraint as

$$g(x) = (B_{ES} - x)\,\mathbb{1}_{\{x < B_{ES}\}} - \epsilon. \tag{3.22}$$

The ES is an alternative risk measure which mitigates some of the main weaknesses of VaR. The ES constraint restricts the average portfolio loss in comparison to a benchmark $B_{ES} > 0$, if the loss is positive, to at most ϵ. Therefore, in contrast to the VaR constraint, the ES constraint also takes the magnitude of the loss into account, if it occurs. Further, the ES is a convex risk measure and thus always encourages diversification of payoffs. For these reasons, ES has been proposed as an alternative tool for financial risk management and its advantages as well as some of its drawbacks have been discussed by e.g. [2] [44] and [86].

Example 3.5.13.
Let Assumptions 3.5.1 and 3.5.9 hold. If g is an ES constraint (as in (3.22)), $B_1 = 0$, $B_2 = \infty$, and $y^* = (y_0^*, y_1^*)' \in (0,\infty) \times [0,\infty)$ is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$, then the optimal terminal wealth for $(\mathbf{P})$ is

$$D^* = \mathcal{I}_{g_{\lambda^*}}(y^*) = \begin{cases} \mathcal{I}(y_0^*), & B_{ES} < \mathcal{I}(y_0^*), \\ B_{ES}, & \mathcal{I}(y_0^*) \leq B_{ES} \leq \mathcal{I}\left(y_0^* - \frac{y_1^*}{\tilde{Z}_{\lambda^*}(T)}\right), \\ \mathcal{I}\left(y_0^* - \frac{y_1^*}{\tilde{Z}_{\lambda^*}(T)}\right), & \mathcal{I}\left(y_0^* - \frac{y_1^*}{\tilde{Z}_{\lambda^*}(T)}\right) < B_{ES}. \end{cases}$$

In contrast to the VaR constraint, $U(x) - y_0 \tilde{Z}_{\lambda^*}(T)x - y_1 g(x)$ remains strictly concave in $x > 0$ for all $y = (y_0, y_1)' \in [0, \infty)^2$ and thus $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)$ can again be obtained by capping $\mathcal{I}_{g_{\lambda^*}}(y)$ at the boundaries B_1 and B_2.

Example 3.5.14.
Let Assumptions 3.5.1 and 3.5.9 hold. If g is an ES constraint (as in (3.22)) and $y^* = (y_0^*, y_1^*)' \in (0, \infty) \times [0, \infty)$ is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$, then the optimal terminal wealth for $(\mathbf{P})$ is

$$D^* = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2) = Cap(\mathcal{I}_{g_{\lambda^*}}(y^*), B_1, B_2).$$

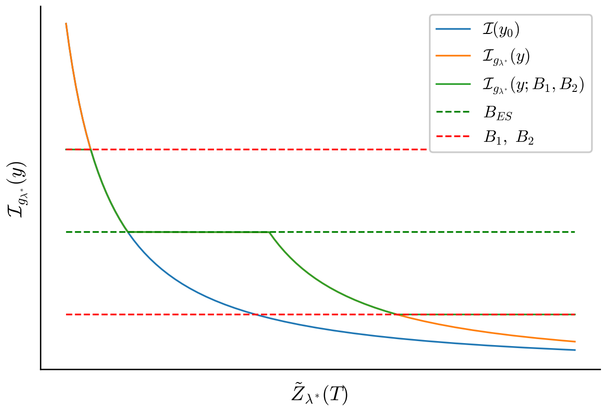


Figure 3.4: Illustration of the dependence of the optimal terminal wealth D^* on $\tilde{Z}_{\lambda^*}(T)$ with ES constraints and a given constant value of y^* (cf. Example 3.5.14).

3.6 Conclusion

In this chapter, we extended the duality framework of [17] and [34] to include simultaneous constraints on relative portfolio allocation and terminal wealth, where the constraints on wealth may be defined pointwise, in expected value or both. Similar to the previous work in [34], we were able to integrate the wealth constraints into the framework of [17] by defining a generalized concave conjugate of the utility function and using it to generalize the well-known martingale method for portfolio optimization in complete financial markets. Just as in [17], the developed framework allows us to disregard the allocation constraints if the drift coefficients in the financial market are adjusted suitably, i.e., if the 'optimal' auxiliary market $\mathcal{M}_{\lambda^*}$ is considered. The optimal auxiliary market can be determined by a set of equivalent optimality conditions, which are induced by two distinct dual optimization problems which correspond to the allocation constraints and the terminal wealth constraints. The optimal terminal wealth for this fully constrained portfolio optimization problem can then be expressed as a deterministic function of the pointwise bounds B_1, B_2 and the pricing kernel $\tilde{Z}_{\lambda^*}(T)$ in the optimal auxiliary market $\mathcal{M}_{\lambda^*}$. When assuming the dynamics of a Black-Scholes market, assuming that K is a convex cone and assuming some additional regularity conditions, then the adjustments to the market coefficients λ^* can be determined as the minimizer of a deterministic convex optimization problem. We illustrate our methodology for investors who face pointwise bounds on terminal wealth, Value-at-Risk constraints and expected shortfall constraints.

4 Constraints on Allocation in Stochastic Factor Models

4.1 Introduction

In this chapter, we consider a portfolio optimization problem of an investor who trades in continuous time and seeks to maximize his utility from terminal wealth at the end of a finite investment horizon. The investor is assumed to be risk-averse and his risk-preferences are modelled by a power utility function. Our problem setting differs from the classic problem formulated in [64] with respect to two main aspects: allocation constraints and market coefficients dependent on a stochastic factor. Since Section 1.1, already provided an overview over the relevant literature on allocation constraints, the following paragraph presents a brief overview of the relevant literature with respect to stochastic factor market dynamics:

Modelling stochastic market coefficients as a function of an additional stochastic factor is a natural extension to the classic Black-Scholes model which can capture some of the stylized facts observed in the financial market. One of the earliest discussions of such models in a portfolio optimization context was in [88], where the author was able to characterize the solution to the associated HJB equation for a power-utility function in terms of a linear parabolic PDE. Further, under the assumption of a global Lipschitz-condition on the market coefficients, a verification result was proven. However, explicit closed-form expressions for the optimal allocation were only given when the stochastic factor is completely uncorrelated with the financial market, i.e., when the optimal allocation is myopic. If the stochastic factor correlates with the financial market, closed-form expressions for the optimal allocation were recovered on individual occasions, e.g., in [20] and [53] for financial markets with stochastic short rate and in [56] for financial markets with stochastic volatility. These advances required the solvability of certain underlying Riccati ODEs. The seminal work of [61] unified these approaches by introducing a class of models where the asset returns have a quadratic dependence on the stochastic factor. Within such quadratic models, the author directly characterizes the HJB PDE as an exponentially quadratic function of the stochastic factor with coefficients determined by the solution to a system of Riccati ODEs. When reducing the framework of [61] to an affine dependence on the stochastic factor, the results are closely related to affine term structure models of [26]. This affine reduction proved to be particularly fruitful for portfolio optimization applications, see e.g. [46], [5] and [30]. In addition, an extensive overview of related literature is given in [90].

Both allocation constraints and stochastic factor market dynamics result in the investor not being able to replicate all measurable payoffs at the end of the investment horizon and therefore standard martingale techniques cannot be employed to characterize the value function of the optimization problem. Further, it is unclear if the Hamilton-Jacobi-Bellman PDE ('HJB' PDE) of the optimization problem admits a smooth solution due to pointwise constraints on the optimal relative portfolio allocation. Both aspects have only been studied simultaneously on rare occasions. [71] used a logarithmic transformation to characterize the solution to the constrained HJB PDE through the solution to a semi linear PDE. Assuming Lipschitz- and non-degeneracy conditions on the stochastic factor, the existence of a smooth solution to the transformed PDE can be guaranteed and closed-form expressions for the optimal portfolio can be given if the stochastic factor is uncorrelated with the financial market. In [21], semi-closed-form expressions are provided for the optimal allocation for a market with generalized Vasicek short rate and bounds on the portfolio allocation to a hedging instrument for interest-rate risk. [66] and [91] develop numerical schemes for allocation constrained portfolio optimization problems in jump-diffusion models, where the asset volatilities and jumps depend on an external stochastic factor.

Our contribution to this literature is threefold. First, we present an approach to constrained portfolio optimization that transforms the HJB PDE associated with the constrained portfolio optimization problem into an equivalent dual PDE, which is the HJBI PDE associated with a dual minimization problem akin to Condition (C) in [17] or Condition ($\tilde{C}$) from Chapter 3. However, unlike in [17], the validity of this method is not tied to the completeness of the underlying financial market and can thus be applied in a broader context. Secondly, in the spirit of [61], we derive a condition on the dynamics of the financial market and the allocation constraints, which ensures that the value function of the optimal investment problem is exponentially affine. Lastly, we provide expressions for the allocation constrained optimal allocation in a market with multi-factor stochastic volatility of CIR-Type and multi-factor short rate of OU-type. These expressions are explicit up to a deterministic minimizer and the solution of a system of Riccati ODEs, which leads to a non-myopic optimal allocation if the stochastic factor correlates with the financial market. In particular, the optimal allocation is generally non-myopic.

The remainder of this chapter is structured as follows: The financial market model, the portfolio optimization problem and the standing assumptions are introduced in Section 4.2. Afterwards, in Section 4.3, we use a result from real constrained optimization to show that the constrained HJB PDE associated with (**P**) is equivalent to the Hamilton-Jacobi-Bellman-Isaacs PDE ('HJBI PDE') associated with a dual min-max problem and derive a condition under which the solution to both PDEs is exponentially affine. The versatility and use of the derived condition is illustrated in examples with deterministic market coefficients, stochastic volatility and stochastic short rate in Section 4.4. Finally, Section 4.5 concludes the chapter.

4.2 Setting

In this chapter, we consider a Markovian stochastic factor model, which is a similar setting as in the introduction to the stochastic control approach in Section 2.3.2. Specifically, we assume that there exist deterministic functions

$$\begin{aligned}&\mu^z : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m,\ \Sigma^z : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{m\times m},\ \rho : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{m\times d} \\ &r : [0,T]\times\mathbb{R} \to \mathbb{R},\ \mu : [0,T]\times\mathbb{R}^m \to \mathbb{R}^d \text{ and } \Sigma : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{d\times d},\end{aligned} \tag{4.1}$$

and an m-dimensional stochastic factor z which satisfies the SDE

$$dz(t) = \mu^z(t,z(t))dt + \Sigma^z(t,z(t))'dW^z(t), \quad z(0) = z_0 \in \mathbb{R}. \tag{4.2}$$

such that the market coefficients in $\mathcal{M}$ are given as deterministic functions of $(t, z(t))$, i.e., we have $\mathcal{L}[0,T]\times Q$-a.e.

$$\rho(t) = \rho(t,z(t)),\ r(t) = r(t,z(t)),\ \mu(t) = \mu(t,z(t)),\ \text{and } \Sigma(t) = \Sigma(t,z(t)). \tag{4.3}$$

In particular, the risk-free asset P_0, the risky assets P and the Brownian motions $W,\ W^z$ then satisfy

$$dP_0(t) = P_0(t)\cdot r(t,z(t))dt, \qquad P_0(0) = 1,$$

$$dP(t) = \text{diag}(P(t)) \cdot [\mu(t,z(t))dt + \Sigma(t,z(t))dW(t)], \quad P(0) = \mathbb{1} \in \mathbb{R}^d,$$
$$d\langle W_i^z, W_j\rangle_t = \rho_{ij}(t,z(t))dt, \quad \forall 1 \le i \le m, \ 1 \le j \le d.$$

We assume that the deterministic functions (4.3) are sufficiently regular such that the SDEs for z, P_0 and P admit a unique strong solution and fall within the market model of $\mathcal{M}$ defined in Section 2.3. In particular, we require that Σ and the columns $\rho_1, ..., \rho_d$ of ρ satisfy

$$\det\left(\Sigma(t,z(t))\right) \neq 0 \text{ and } \|\rho_i(t,z(t))\| \le 1 \ \forall i = 1,..,d \quad \text{hold} \quad \mathcal{L}[0,T] \otimes Q - \text{a.e.}.$$

As working within this setting becomes very notation-heavy, we only write μ^z, Σ^z, ρ, r, μ, Σ instead of $\mu^z(t,z(t))$, $\Sigma^z(t,z(t))$, $\rho(t,z(t))$, $r(t,z(t))$, $\mu(t,z(t))$, $\Sigma(t,z(t))$ if the context is unambiguous to improve the clarity of presentation. In this sense, the wealth process $V^{v_0,\pi}$ of an investor with initial wealth $v_0 > 0$ and trading in $\mathcal{M}$ according to a d-dimensional relative portfolio process $\pi \in \Lambda$ satisfies the usual SDE

$$\begin{aligned} &V^{v_0,\pi}(0) = v_0 \\ &dV^{v_0,\pi}(t) = V^{v_0,\pi}(t)\left(\left[r + (\mu - r\mathbb{1})'\pi(t)\right]dt + \pi(t)'\Sigma dW(t)\right) \end{aligned}$$

and can be expressed in closed form as

$$V^{v_0,\pi}(t) = v_0 \exp\Big(\int_0^t r + (\mu - r\mathbb{1})'\pi(s) - \frac{1}{2}\|\Sigma'\pi(s)\|^2 ds + \int_0^t \pi(s)'\Sigma dW(s)\Big).$$

In this chapter we only consider allocation constraints , i.e., for a non-empty closed convex set $K \subset \mathbb{R}^d$, we define the set of admissible constrained portfolio processes as

$$\Lambda' := \Lambda(K) := \left\{\pi \in \Lambda \ \middle| \ \pi(t) \in K \ \mathcal{L}[0,T] \otimes Q - \text{a.e.}\right\}. \tag{4.4}$$

For a power utility function $U(v) = \frac{1}{b}v^b$ with $b < 1$ and $b \neq 0$ we then define the allocation constrained primal portfolio optimization problem as

$$(\mathbf{P}) \begin{cases} \Phi(v_0) & = \sup\limits_{\pi \in \Lambda} \mathbb{E}\left[U(V^{v_0,\pi}(T))\right], \\ \Lambda' & = \left\{\pi(t) \in K \ \mathcal{L}[0,T] \otimes Q - \text{a.e.} \ \middle| \ \pi \in \Lambda\right\}. \end{cases}$$

Since we consider a Markovian setting and did not assume that $m \neq 0$, we aim to solve $(\mathbf{P})$ using classic methods from stochastic optimal control and thus need to determine a solution to the HJB PDE associated with $(\mathbf{P})$. Even without the additional presence of allocation constraints, this is notoriously difficult in the general setting described above. For this reason, we devote the upcoming Section 4.3 to deriving an equivalent dual representation of the associated HJB PDE, which leads to a dual approach to solving the allocation constrained portfolio optimization problem $(\mathbf{P})$. The dual representation is closely related to Condition $(\tilde{C})$ from Section 3.4.

The standing assumptions for this chapter are summarized below:

Assumption 4.2.1 (**Standing Assumptions Chapter 4:**). *$\mathcal{M}$ is a stochastic factor model $\big($(4.2) and (4.3)$\big)$, (4.4) and U is a power utility function.*

4.3 The Dual HJBI PDE

We approach $(\mathbf{P})$ using classic methods from stochastic optimal control and frequently work with so-called feedback controls $\pi \in \Lambda$, which are defined in feedback-form

$$\pi(t) = \underline{\pi}(t, V^{v_0,\pi}(t), z(t)),$$

for a deterministic measurable function $\underline{\pi} : [0,T] \times (0,\infty) \times \mathbb{R}^m \to \mathbb{R}^d$ of the current state $(t, V^{v_0,\pi}(t), z(t))$ of the financial market $\mathcal{M}$. As these processes are uniquely defined through the function $\underline{\pi}$, we use the process π and the function $\underline{\pi}$ interchangeably. When a process is specifically defined in feedback-form, we follow the notation in [39] and denote this fact by a 'lower bar', i.e., $\underline{\pi}$. The set of admissible unconstrained Markovian controls is thus denoted by $\underline{\Lambda}$ and the wealth-process corresponding to a Markovian control $\underline{\pi}$ is denoted by $V^{v_0,\underline{\pi}}$. The remaining notation will carry over analogously. Further, we also introduce the generalized primal portfolio optimization problem $(\mathbf{P^{(t,v,z)}})$ as

$$(\mathbf{P^{(t,v,z)}}) \begin{cases} \Phi(t,v,z) & = \sup_{\pi \in \Lambda'(t)} \mathbb{E}\big[U(V^{v_0,\pi}(T)) \;\big|\; V^{v_0,\pi}(t) = v,\ z(t) = z\big] \\ \Lambda'(t) & = \big\{ \big(\pi(s)\big)_{s\in[t,T]} \;\big|\; \pi \in \Lambda_K \big\}. \end{cases}$$

Then, the Hamilton-Jacobi-Bellman equation ('HJB equation') associated with $(\mathbf{P^{(t,v,z)}})$ is given by

$$\begin{aligned} 0 &= \sup_{\pi \in K} \Big\{ G_t + v\left[r + (\mu - r\mathbb{1})'\pi\right] G_v + \frac{1}{2} v^2 \|\Sigma'\pi\|^2 G_{vv} + (\mu^z)'(\nabla_z G) \\ &\qquad + v\left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \Big\} \\ &= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\ &\qquad + v \sup_{\pi \in K} \Big\{ (\mu - r\mathbb{1})'\pi G_v + \left(\Sigma^z \rho \Sigma'\pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \Big\} \qquad (4.5) \\ G(T,v,z) &= U(v), \end{aligned}$$

Any (sufficiently regular) solution G to (4.5) yields a candidate optimal Markovian control through the maximizing argument

$$\underline{\pi}^*(t,v,z) = \underset{\pi \in K}{\mathrm{argmax}} \Big\{ (\mu - r\mathbb{1})'\pi G_v + \left(\Sigma^z \rho \Sigma'\pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \Big\}. \qquad (4.6)$$

4.3.1 Dual Approach to Allocation Constrained Portfolio Optimization

In this subsection, we use Corollary 2.1.32 to derive an equivalent dual representation of the HJB equation (4.5), which can be regarded as the Hamilton-Jacobi-Bellman-Isaacs equation ('HJBI equation') associated with a dual optimization problem over a certain class of stochastic processes. This dual optimization problem as well as our solution approach to constrained portfolio optimization will closely resemble optimality Condition $(\tilde{C})$ from Section 3.4 (resp. Condition (C) from [17]). However, unlike in Chapter 3, we arrive at this approach applying duality arguments directly to the pointwise optimization on the level of the HJB PDE (4.5) rather than on the level of stochastic processes. This reduces the level of technicality involved and removes the necessity for market completeness and the solvability of the wealth-constrained dual problem $(\mathbf{D^V_{\lambda^*}})$ as central underlying assumptions.

Lemma 4.3.1. *(Dual HJBI PDE)*
Let $G \in C^{(1,2,2)}([0,T] \times (0,\infty) \times \mathbb{R}^m)$ be strictly concave and strictly increasing in the second component v. Then, G is a solution to (4.5) if and only if $G(T,v,z) = U(v)$ and

$$0 = G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathit{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right]$$

$$+ v \inf_{\lambda \in \mathbb{R}^d} \sup_{\pi \in \mathbb{R}^d} \Big\{ \left[\delta_K(\lambda) + (\mu + \lambda - r\mathbb{1})'\pi\right] G_v + \left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \Big\} \tag{4.7}$$

$$= G_t + vrG_v + (\mu^z)' (\nabla_z G) + \frac{1}{2} \mathit{Trace} \left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] + v \inf_{\lambda \in \mathbb{R}^d} \Big\{ \delta_K(\lambda) G_v - \frac{1}{2} \frac{1}{v G_{vv}} \|\Sigma^{-1} \left[\mu + \lambda - r\mathbb{1}\right] G_v + (\Sigma^z \rho)' \nabla_z (G_v) \|^2 \Big\}.$$

The dual PDE (4.7) is the HJBI PDE (see e.g. Section 4.2 in [45]) associated with the min-max stochastic control problem

$$\inf_{\lambda \in \mathcal{D}} \sup_{\pi \in \Lambda} \mathbb{E}\Big[U\Big(\underbrace{V^{v_0,\pi}(T) \cdot \exp\Big(\int_0^T \lambda(t)'\pi(t) + \delta_K(\lambda(t)) dt\Big)}_{=: V_\lambda^{v_0,\pi}(T)}\Big)\Big], \tag{4.8}$$

where the dual process $\lambda = (\lambda(t))_{0 \le t \le T}$ is taken from the set of progressively measurable processes (cf. Section 3.4)

$$\mathcal{D} = \Big\{ \lambda = \left((\lambda_1(t), ..., \lambda_d(t))'\right)_{t \in [0,T]} \text{ prog. measurable } \Big| \sup_{0 \le t \le T} \|\lambda(t)\|^2 < \infty \ Q\text{-a.s.}, \ \sup_{0 \le t \le T} \delta_K(\lambda(t)) < \infty \ Q\text{-a.s.} \Big\}.$$

Just as with portfolio processes, we refer to dual processes defined in feedback form $\lambda(t) = \underline{\lambda}(t, V_\lambda^{v_0,\pi}(t), z(t))$, for a deterministic measurable function $\underline{\lambda}$, as Markovian dual controls. Analogously, when specifically referring to Markovian dual controls, we write '$\underline{\lambda}$' and collect all admissible Markovian controls in $\underline{\mathcal{D}}$.

Although intuitively appealing, the relationship between HJB(I) PDEs and the associated optimization problems still requires formal mathematical justification via verification theorems. Due to the generality of the setting considered in this chapter, we can only provide general verification theorems under additional assumptions on the candidate optimal controls $\underline{\pi}^*$ (and $\underline{\lambda}^*$), the solution G to the HJB(I) PDE and the financial market $\mathcal{M}$. Verifying such conditions is typically only feasible in more narrowly focussed settings (see e.g. Corollary 4.4.3 and 4.4.10 in Section 4.4).

Here, we make the relation between the PDE (4.7) and the dual control problem (4.8) more precise, by proving a verification theorem, which relies on an additional uniform integrability condition (UI$_\lambda$) (compare to e.g. Definition 4.2 in [56]).

Condition (UI$_\lambda$). *For given $n \in \mathbb{N}$, $t \in [0,T]$, $G \in C^{(1,2,2)}([0,T] \times (0,\infty) \times \mathbb{R}^m)$, $\lambda \in \mathcal{D}$, $\pi \in \Lambda$ we define the stopping time $\tau_{n,t}^\lambda = \min(T, \hat{\tau}_{n,t}^\lambda)$, with*

$$\hat{\tau}_{n,t}^\lambda = \inf \Big\{ t \le u \le T \ \Big| \ \int_t^u \left(V_\lambda^{v_0,\pi}(s) \cdot \|\Sigma(s,z(s))'\pi(s)\| \cdot G_v(s, V_\lambda^{v_0,\pi}(s), z(s))\right)^2 ds \ge n, \int_t^u \| \left(\Sigma^z(s,z(s))'\nabla_z (G)\left(s, V_\lambda^{v_0,\pi}(s), z(s)\right)\|^2 ds \ge n \Big\}.$$

We say that G, π, and λ satisfy condition (UI$_\lambda$) if for every $t \in [0,T]$, the sequence

$$\left(G(\tau_{n,t}^\lambda, V_\lambda^{v_0,\pi}(\tau_{n,t}^\lambda), z(\tau_{n,t}^\lambda))\right)_{n \in \mathbb{N}}$$

is uniformly integrable.

Remark 4.3.2. *According to Theorem 4.5.4 in [8], if G, π, and λ satisfy Condition (UI_λ), then we have for every $t \in [0,T]$ that $\tau_{n,t} \to T$ Q-a.s., as $n \to \infty$ and*

$$\mathbb{E}\Big[G(T, V_\lambda^{v_0,\pi}(T), z(T)) \;\Big|\mathcal{F}_t\Big] = \mathbb{E}\Big[\lim_{n\to\infty} G(\tau_{n,t}, V_\lambda^{v_0,\pi}(\tau_{n,t}), z(\tau_{n,t})) \;\Big|\mathcal{F}_t\Big]$$
$$= \lim_{n\to\infty} \mathbb{E}\Big[G(\tau_{n,t}, V_\lambda^{v_0,\pi}(\tau_{n,t}), z(\tau_{n,t})) \;\Big|\mathcal{F}_t\Big]. \qquad (4.9)$$

Lemma 4.3.3 (Verification Theorem Dual Control Problem)**.**
Let $G \in C^{(1,2,2)}([0,T] \times (0,\infty) \times \mathbb{R}^m)$ be a solution to the dual HJB equation (4.7), be non-negative, strictly concave and increasing in v. Let the feedback controls $\underline{\lambda}^*(t,v,z)$, $\underline{\pi}^*(t,v,z)$ be such that for all $(t,v,z) \in [0,T] \times (0,\infty) \times \mathbb{R}^m$

$$\inf_{\lambda\in\mathbb{R}^d} \sup_{\pi\in\mathbb{R}^d} \Big\{ \left[\delta_K(\lambda) + (\mu + \lambda - r\mathbb{1})'\pi\right] G_v + \left(\Sigma^z\rho\Sigma'\pi\right)' \nabla_z\left(G_v\right) + \frac{1}{2} v\|\Sigma'\pi\|^2 G_{vv} \Big\}$$
$$= \left[\delta_K(\underline{\lambda}^*(t,v,z)) + (\mu + \underline{\lambda}^*(t,v,z) - r\mathbb{1})'\underline{\pi}^*(t,v,z)\right] G_v$$
$$+ \left(\Sigma^z\rho\Sigma'\underline{\pi}^*(t,v,z)\right)' \nabla_z\left(G_v\right) + \frac{1}{2} v\|\Sigma'\underline{\pi}^*(t,v,z)\|^2 G_{vv}.$$

Then the following holds $\forall (t,v,z) \in [0,T] \times (0,\infty) \times \mathbb{R}^m$:

(i) If $(\underline{\lambda}^*, \pi) \in \underline{\mathcal{D}} \times \Lambda$ satisfy condition (UI_λ), then

$$G(t,v,z) \geq \mathbb{E}\left[U(V_{\underline{\lambda}^*}^{v_0,\pi}(T)) \mid V_{\underline{\lambda}^*}^{v_0,\pi}(t) = v,\ z(t) = z\right]. \qquad (4.10)$$

(ii) If $(\lambda, \underline{\pi}^*) \in \mathcal{D} \times \underline{\Lambda}$ satisfy condition (UI_λ), then

$$G(t,v,z) \leq \mathbb{E}\left[U(V_\lambda^{v_0,\underline{\pi}^*}(T)) \mid V_\lambda^{v_0,\underline{\pi}^*}(t) = v,\ z(t) = z\right]. \qquad (4.11)$$

(iii) If $(\underline{\lambda}^*, \underline{\pi}^*) \in \underline{\mathcal{D}} \times \underline{\Lambda}$ satisfy condition (UI_λ), then

$$G(t,v,z) = \mathbb{E}\left[U(V_{\underline{\lambda}^*}^{v_0,\underline{\pi}^*}(T)) \mid V_{\underline{\lambda}^*}^{v_0,\underline{\pi}^*}(t) = v,\ z(t) = z\right]. \qquad (4.12)$$

Remark 4.3.4. *If we restrict the minimization and maximization in the min-max optimization only to such $\lambda \in \mathcal{D}_{UI} \subset \mathcal{D}$, $\pi \in \Lambda_{UI} \subset \Lambda$ so that every pair (λ, π), $(\underline{\lambda}^*, \pi)$, $(\lambda^*, \underline{\pi}^*)$ and $(\underline{\lambda}^*, \underline{\pi}^*)$ and the solution G to the dual HJB PDE (4.7) satisfy Condition (UI_λ), then we directly obtain from Lemma 4.3.3*

$$G(t,v,z) = \mathbb{E}\left[U(V_{\underline{\lambda}^*}^{v_0,\underline{\pi}^*}(T)) \mid V_{\underline{\lambda}^*}^{v_0,\underline{\pi}^*}(t) = v,\ z(t) = z\right]$$
$$= \inf_{\lambda\in\mathcal{D}_{UI}} \sup_{\pi\in\Lambda_{UI}} \mathbb{E}\left[U(V_\lambda^{v_0,\pi}(T)) \mid V_\lambda^{v_0,\pi}(t) = v,\ z(t) = z\right].$$

(4.8) is the min-max control problem associated with Condition ($\tilde{C}$) from Section 3.4 in the absence of wealth constraints (resp. Condition (C) from [17]). However, we arrived at the same optimization problems by applying convex duality results from real constraints directly to the pointwise optimization at the level of the HJB PDE, whereas we applied martingale methods to the underlying stochastic processes in Chapter 3. For the setting of Chapter 3, were able to to prove that the optimal controls for the dual control problem (4.8) lead to an optimal portfolio process for the allocation constrained portfolio optimization problem (**P**). In doing so, we used the so-called (generalized) concave conjugate of U to transform the dual control problem (4.8) to 'another' dual representation of (**P**). These arguments heavily relied on the completeness of the underlying financial market (Assumption 3.2.1) as well as the solvability of the dual optimization problem ($\mathbf{D}_{\lambda^*}^{\mathbf{V}}$) associated with the wealth constraints and are thus not directly available to us in this chapter.

4.3.2 Exponential Affine Separability

In this section, we derive a condition under which the solution G to the dual HJB PDE (4.7) is of an exponentially affine and separable form, i.e.,

$$G(t,v,z)=\frac{1}{b}v^{b}\exp\left(A(T-t)+B(T-t)'z\right), \tag{4.13}$$

for some functions $A:[0,T]\to\mathbb{R}$, and $B:[0,T]\to\mathbb{R}^m$ with $A(0)=0$ and $B(0)=0$. In a setting without the presence of allocation constraints and time-independent market coefficients, [61] provides such a condition which can be directly verified for any given market coefficients (see equations (9)-(11) and (13)-(17) in [61]).[1] Under the presence of additional constraints on allocation, we need to adapt this condition suitably.
To this end, for any $(t,z,B)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^m$, we define $\hat{\lambda}^*(t,z,B)$ as the minimizing argument

$$\begin{aligned}\hat{\lambda}^*(t,z,B)&=\underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\left\{2(1-b)\delta_K(\lambda)+\left\|\Sigma^{-1}\left(\mu-r\mathbb{1}+\lambda\right)+\left(\Sigma^z\rho\right)'B\right\|^2\right\} \\ &=\underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\left\{2(1-b)\delta_K(\lambda)+2\lambda'\left(\Sigma\cdot\Sigma'\right)^{-1}\left[\mu-r\mathbb{1}+(\Sigma^z\rho\Sigma')'B\right]+\left\|\Sigma^{-1}\lambda\right\|^2\right\}.\end{aligned} \tag{4.14}$$

Given $\hat{\lambda}^*$, we provide a condition which ensures that (4.13) holds. This can be achieved by considering the corresponding condition from [61] and augmenting the market coefficients by $\hat{\lambda}^*$.

Condition (EAS).
We say that Condition (EAS) is satisfied if for any $(t,z,B)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}^m$ the market coefficients and the minimizer $\hat{\lambda}^*$ satisfy

$$\begin{aligned}&\mu^z(t,z)=k_0(t)+k_1(t)z\\ &\Sigma^z(t,z)\Sigma^z(t,z)'=h_0(t)+h_1(t)[z]\\ &\Sigma^z(t,z)\rho(t,z)\left(\Sigma^z(t,z)\rho(t,z)\right)'-\Sigma^z(t,z)\Sigma^z(t,z)'=l_0(t)+l_1(t)[z]\\ &r(t,z)+\delta_K(\hat{\lambda}^*(t,z,B))=p_0(t,B)+p_1(t,B)'z\\ &\left\|\Sigma^{-1}(t,z)\left(\mu(t,z)+\hat{\lambda}^*(t,z,B)-r(t,z)\mathbb{1}\right)\right\|^2=q_0(t,B)+q_1(t,B)'z\\ &\Sigma^z(t,z)\rho(t,z)\Sigma^{-1}(t,z)\left(\mu(t,z)+\hat{\lambda}^*(t,z,B)-r(t,z)\mathbb{1}\right)=g_0(t,B)+g_1(t,B)z,\end{aligned}$$

for some functions such that $p_0(t,B)$, $q_0(t,B)\in\mathbb{R}$, $k_0(t)$, $p_1(t,B)$, $q_1(t,B)$, $g_0(t,B)\in\mathbb{R}^m$ as well as $k_1(t)$, $h_0(t)$, $l_0(t)$, $g_1(t,B)\in\mathbb{R}^{m\times m}$ and the functions $h_1(t)[\cdot]$, $l_1(t)[\cdot]:\mathbb{R}^m\to\mathbb{R}^{m\times m}$ are linear[2] for every fixed $(t,B)\in[0,T]\times\mathbb{R}^m$.

[1] In fact, the result of [61] even includes the more general case of an exponentially quadratic separation. The approach we present below can be extended to include quadratic separation in a natural manner. However, such an extension would complicate the involved notation and thus diminish the presentation of the core concepts involved. Moreover, we were not able to construct realistic working examples that require quadratic separation in an allocation constrained setting. Hence, we restrict our analysis in this chapter to exponentially affine separation.

[2] The functions $h_1[\cdot]$ and $l_1[\cdot]$ are three-dimensional tensors, which are a generalization of vectors and matrices to higher dimensions. In our context, we may think of h_1 and l_1 as being represented by matrices, whose entries $(h_1)_{ij}$ and $(l_1)_{ij}$ are $\mathbb{R}^m$-valued. Upon being evaluated at a $z\in\mathbb{R}^m$, each entry of $h_1[z]$ and $l_1[z]$ is obtained by computing the scalar product $z'(h_1)_{ij}$ and $z'(l_1)_{ij}$. Hence, for any $x,y\in\mathbb{R}^m$ and applying the rules of ordinary vector-matrix multiplication, the products $x'h_1[\cdot]y$ and $x'l_1[\cdot]y$ are vectors in $\mathbb{R}^m$. In particular, $x'h_1[z]y=z'\left(x'h_1[\cdot]y\right)$ and $x'l_1[z]y=z'\left(x'l_1[\cdot]y\right)$ for any $x,y,z\in\mathbb{R}^m$.

Provided that Condition (EAS) is satisfied, we can characterize the exponents A and B in (4.13) through the system of ODEs[3]

$$\begin{aligned} A_\tau(\tau) = {} & bp_0\left(T-\tau, B(\tau)\right) + k_0(T-\tau)'B(\tau) + \frac{1}{2}B(\tau)h_0(T-\tau)B(\tau) \\ & + \frac{1}{2}\frac{b}{1-b}\Big[q_0\left(T-\tau, B(\tau)\right) + 2g_0\left(T-\tau, B(\tau)\right)' B(\tau) \\ & \qquad + B(\tau)\left(l_0(T-\tau) + h_0(T-\tau)\right)B(\tau)\Big] \end{aligned} \tag{4.15}$$

$$\begin{aligned} B_\tau(\tau) = {} & bp_1\left(T-\tau, B(\tau)\right) + k_1'(T-\tau)B(\tau) + \frac{1}{2}B(\tau)'h_1(T-\tau)[\cdot]B(\tau) \\ & + \frac{1}{2}\frac{b}{1-b}\Big[q_1\left(T-\tau, B(\tau)\right) + 2g_1\left(T-\tau, B(\tau)\right)B(\tau) \\ & \qquad + B(\tau)'\left(l_1(T-\tau)[\cdot] + h_1(T-\tau)[\cdot]\right)B(\tau)\Big]. \end{aligned} \tag{4.16}$$

Transferred to the context of Chapter 3, while disregarding the wealth constraints, Condition (EAS) effectively ensures that the allocation unconstrained optimization problem ($\mathbf{P}_{\lambda^*}$) in $\mathcal{M}_{\lambda^*}$ for the optimal dual process $\lambda^* = \hat{\lambda}^*(t, z(t), B(T-t))$ can be solved via the methodology presented in [61]. This idea is depicted in Table 4.1. From this perspective, it is not surprising that we can obtain a solution to the primal HJB PDE (4.5) under Condition (EAS).

Theorem 4.3.5. *Let Condition (EAS) be satisfied and let A, B be solutions to the ODEs (4.15) and (4.16) with initial condition $A(0) = 0$, $B(0) = 0$. Then,*

$$G(t, v, z) = \frac{1}{b}v^b \exp\left(A(T-t) + B(T-t)'z\right)$$

is a solution to the primal HJB PDE (4.5) and the dual HJBI PDE (4.7).

Remark 4.3.6. *Although we can by no means provide explicit solutions to the ODEs (4.15) and (4.16) in general, at least the local existence of a solution is guaranteed by the existence theorems of Peano (and Picard-Lindelöf) if their respective right-hand sides are continuous (Lipschitz-continuous). In particular, we can then obtain an approximate solution to the dual HJBI PDE (4.7) for small $\tau = T - t$ by approximating A and B, by e.g. the Euler method.*

	No Allocation Constraints ($K = \mathbb{R}^d$)	**Allocation Constraints** ($K \subset \mathbb{R}^d$)
Black-Scholes Model	**Merton, 1971** ([65]): $\underline{\pi}^* = \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbb{1}\right)$	**Cvitanic & Karatzas, 1992** ([17]) $\underline{\pi}^* = \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbb{1} + \lambda^*\right)$, where λ^* is obtained from dual problem $\Rightarrow \underline{\pi}^*$ optimal for $K = \mathbb{R}^d$ in $\mathcal{M}_{\lambda^*}$
Stochastic Factor Model	**Liu, 2006** ([61]): $\underline{\pi}^* = \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbb{1} + \left(\Sigma^z\rho\Sigma'\right)' B(T-t)\right)$, if Condition (EAS) is satisfied with $\lambda^* = 0$ & B solves associated ODE (4.16)	**Theorem 4.3.5**: $\underline{\pi}^* = \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbb{1} + \lambda^* + \left(\Sigma^z\rho\Sigma'\right)' B(T-t)\right)$, where λ^* is obtained from dual problem, market coefficients in $\mathcal{M}_{\lambda^*}$ satisfy Condition (EAS) & B solves associated ODE (4.16)

Table 4.1: A schematic illustration of how Theorem 4.3.5 relates the existing results of [65], [61] and [17].

[3] Here, A_τ and B_τ denote the derivatives of A and B with respect to $\tau \in [0, T]$.

If Condition (EAS) is satisfied, then we can extend the verification approach used in [13] to formally verify the optimality of the obtained candidate optimal portfolio $\underline{\pi}^*$. Unlike in Lemma 4.3.3, we only need to assume that $\underline{\pi}^*$ satisfies a uniform integrability condition and not make any assumption about other portfolios $\pi \in \Lambda_K$. This is possible because we can exploit the additional knowledge that G is exponentially affine due to Condition (EAS).

Theorem 4.3.7 (Verification Theorem Primal Problem)**.**
Let Condition (EAS) be satisfied, let A and B be solutions to the ODEs (4.15) and (4.16) with initial condition $A(0) = 0$ and $B(0) = 0$ and let G be defined as in (4.13). Define $\underline{\lambda}^*(t, v, z) := \hat{\lambda}^*(t, z, B(T-t))$ and

$$\underline{\pi}^*(t, v, z) := \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left[\mu + \underline{\lambda}^*(t, v, z) - r\mathbb{1} + \left(\Sigma^z\rho\Sigma'\right)' B(T-t)\right]. \tag{4.17}$$

If G, $\underline{\pi}^*$, $\lambda \equiv 0$ satisfy Condition (UI_λ), then

$$G(t, v, z) = \mathbb{E}\left[U(V^{v_0,\underline{\pi}^*}(T)) \;\middle|\; V^{v_0,\underline{\pi}^*}(t) = v,\ z(t) = z\right] \tag{4.18}$$

$$\geq \mathbb{E}\left[U(V^{v_0,\pi}(T)) \;\middle|\; V^{v_0,\pi}(t) = v,\ z(t) = z\right] \quad \forall \pi \in \Lambda_K(t). \tag{4.19}$$

In particular, $G(t, v, z) = \Phi(t, v, z)$, for all $(t, v, z) = [0, T] \times (0, \infty) \times \mathbb{R}^m$ and $\underline{\pi}^*$ is optimal for (**P**).

Remark 4.3.8. *If Condition (EAS) is satisfied, then we can alternatively use Theorem 4.3.5, compute the derivative of G and insert them into equation (4.6) to rewrite $\underline{\pi}^*$ as*

$$\underline{\pi}^*(t, v, z) = \underset{\pi \in K}{\mathit{argmax}}\left([\mu - r\mathbb{1}]'\pi + (1-b)\frac{1}{2}\pi'\Sigma\Sigma'\pi + B(T-t)'\Sigma^z\rho\Sigma'\pi\right).$$

In other words, we may equivalently write the optimal constrained feedback control $\underline{\pi}^*$ from (4.17) as the solution to a constrained pointwise optimization problem.

4.4 Examples

We consider three different choices of models for which Condition (EAS) can be verified and an explicit expression for the ODEs (4.15) and (4.16) can be derived. Throughout the examples, we always follow the same steps in chronological order:

(i) Define the underlying financial market model, by choosing the market coefficients μ^z, Σ^z, ρ, r, μ and Σ.

(ii) Derive an explicit representation of the minimizer $\hat{\lambda}^*$ of (4.14) in the given market.

(iii) Verify that Condition (EAS) is satisfied for the given market.

(iv) Derive an explicit representation for the ODEs (4.15), (4.16) and the candidate optimal portfolio $\underline{\pi}^*$ in terms of the market coefficients and $\hat{\lambda}^*$.

(v) *If possible:*
Formally verify the optimality of $\underline{\pi}^*$ for (**P**) by proving that $\underline{\pi}^*$, G (as in (4.13)) and $\lambda \equiv 0$ satisfy Condition (UI_λ).

4.4.1 Black-Scholes Model

First, we consider a d-dimensional Black-Scholes model $\mathcal{M}_{BS}$ with time-dependent coefficients, which is exactly the setting of Section 15 in [17] (and a canonical generalization of Section 3.5.3 with $g \equiv 0$, $B_1 = 0$ and $B_2 = \infty$).

Definition 4.4.1 ($\mathcal{M}_{BS}$)**.**
Let $m = 1$ and $d \in \mathbb{N}$.[4] *Consider continuous functions $r : [0,T] \to \mathbb{R}$, $\mu : [0,T] \to \mathbb{R}^d$, and $\Sigma : [0,T] \to \mathbb{R}^{d\times d}$ such that the inverse $\Sigma(t)^{-1}$ exists for all $t \in [0,T]$. Then, the d-dimensional Black-Scholes market $\mathcal{M}_{BS}$ is defined by the market coefficients*

$$\begin{aligned} z_0 = \mu^z(t,z) = \Sigma^z(t,z) = \rho(t,z) = 0, \\ \textit{and} \quad r(t,z) = r(t), \quad \mu(t,z) = \mu(t), \quad \Sigma(t,z) = \Sigma(t). \end{aligned}$$

We can directly apply the duality theory developed in Section 4.3 to verify Condition (EAS) and obtain a solution to the HJBI PDE (4.7) in $\mathcal{M}_{BS}$.

Lemma 4.4.2 (Dual ODEs in $\mathcal{M}_{BS}$)**.**
Consider the financial market $\mathcal{M}_{BS}$. Then Condition (EAS) is satisfied.
Let

$$\lambda^*(t) = \underset{\lambda\in\mathbb{R}^d}{\textit{argmin}} \left\{ 2(1-b)\delta_K(\lambda) + \left\| \Sigma(t)^{-1}(\mu(t) - r(t)\mathbb{1} + \lambda) \right\|^2 \right\} \tag{4.20}$$

and $A : [0,T] \to \mathbb{R}$ satisfy $A(0) = 0$ and

$$A_\tau(\tau) = br(T-\tau) + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda\in\mathbb{R}^d} \left\{ 2(1-b)\delta_K(\lambda) + \left\| \Sigma(T-\tau)^{-1}(\mu(T-\tau) - r(T-\tau)\mathbb{1} + \lambda) \right\|^2 \right\}.$$

Then,

$$G(t,v,z) = \frac{1}{b} v^b \exp(A(T-t))$$

is a solution to the dual HJBI PDE (4.7) and the corresponding candidate optimal portfolio is

$$\underline{\pi}^*(t,v,z) = \frac{1}{1-b} \left(\Sigma(t)\Sigma(t)' \right)^{-1} (\mu(t) - r(t)\mathbb{1} + \lambda^*(t)).$$

Unsurprisingly, the candidate optimal portfolio process $\underline{\pi}^*$ proposed by Lemma 4.4.2 is the same as that obtained in Example 15.2 by [17] via the auxiliary markets methodology. Moreover, due to the simplicity of this set-up, we can even formally verify the optimality of $\underline{\pi}^*$ by showing that Condition (UI_λ) is satisfied.

Corollary 4.4.3. *Consider the financial market $\mathcal{M}_{BS}$. Then, G, $\underline{\pi}^*$ as in Lemma 4.4.2 and $\lambda \equiv 0$ satisfy Condition (UI_λ). In particular, $\underline{\pi}^*$ is optimal for* ($\mathbf{P}$).

Remark 4.4.4. *The optimal unconstrained portfolio in $\mathcal{M}_{BS}$ is given by the Merton portfolio*

$$\pi_M(t) = \frac{1}{1-b} \left(\Sigma(t)\Sigma(t)' \right)^{-1} (\mu(t) - r(t)\mathbb{1}).$$

[4] We could equivalently consider $m = 0$ and completely disregard the stochastic factor z and its drift, diffusion and correlation coefficients in the definition of the Black-Scholes model $\mathcal{M}_{BS}$.

Following the argument in Remark 4.3.8, we realize that the candidate optimal constrained portfolio $\underline{\pi}^*$ *in* $\mathcal{M}_{BS}$ *can be obtained for arbitrary constraints* $K \subset \mathbb{R}^d$ *through the application of the (pointwise) projection* $\mathcal{P}_K^{BS} : \Lambda \to \Lambda'$ *defined through*

$$\begin{aligned}\underline{\pi}^*(t,v,z) = \underline{\pi}^* &= \underset{\pi \in K}{\mathit{argmax}} \Big(\underbrace{[\mu - r\mathbb{1}]'}_{=(1-b)\pi_M'\Sigma\Sigma'} \pi + (1-b)\frac{1}{2}\pi'\Sigma\Sigma'\pi + \underbrace{B(T-t)'}_{=0}\Sigma^z\rho\Sigma'\pi\Big) \\ &= \underset{\pi \in K}{\mathit{argmax}} \Big((1-b)\pi_M'\Sigma\Sigma'\pi - (1-b)\frac{1}{2}\pi'\Sigma\Sigma'\pi \Big) \\ &= \underset{\pi \in K}{\mathit{argmin}} \Big\| \Sigma'(\pi - \pi_M) \Big\|^2 \\ &= \underset{\pi \in K}{\mathit{argmin}} \Big\| \Sigma(t)'(\pi(t) - \pi_M(t)) \Big\|^2 =: \mathcal{P}_K^{BS}[\pi_M](t).\end{aligned}$$

4.4.2 Multi-Factor Stochastic Covariance of CIR-Type

Next, we consider a financial market model with a stochastic covariance matrix, which depends on m independent CIR processes. More specifically, we assume that the covariance matrix $\Sigma(t,z)$ is a block-diagonal matrix, whose diagonal blocks Σ_i are scaled proportionally to the i-th CIR process z_i. One may think of the underlying financial market $\mathcal{M}_{CIR}$ as consisting of risky assets from m unrelated asset classes, where the covariances within each asset class are driven by one of m independent stochastic (CIR) risk factors. Special cases of this model are the Heston model ([42]) for $m = d = 1$ and the PCSV model with independent assets ([30]) for $m = d \in \mathbb{N}$ and $d_i = 1$ for all $i = 1, ..., m$.[5]

For notational convenience in the following discussion, we introduce the element-wise product between any two real vectors x, y of identical dimension as $x \odot y$.

Definition 4.4.5 ($\mathcal{M}_{CIR}$)**.**
Let $m, d, d_1, ..., d_m \in \mathbb{N}$ *such that* $m \leq d$ *and* $\sum_{i=1}^m d_i = d$*. Consider constants* $\kappa, \theta, \sigma \in (0,\infty)^d$ *such that*

$$2\kappa_i\theta_i > \sigma_i^2 \quad \forall i = 1, ..., m. \tag{4.21}$$

Moreover, let $r \in \mathbb{R}$ *and* $\rho_i \in (-1,1)^{d_i}$*,* $\eta_i \in \mathbb{R}^{d_i}$*, and non-singular* $\Sigma_i \in \mathbb{R}^{d_i \times d_i}$ *be given for* $i = 1, ..., m$*. Then, the d-dimensional market* $\mathcal{M}_{CIR}$ *with m-factor volatility of CIR-type is defined by the market coefficients*

$$\mu^z(t,z) = \kappa \odot (\theta - z)\,,\ \Sigma^z(t,z) = \begin{pmatrix} \sigma_1\sqrt{z_1} & & 0 \\ & \ddots & \\ 0 & & \sigma_m\sqrt{z_m} \end{pmatrix},\ \rho(t,z) = \begin{pmatrix} \rho_1' & & 0 \\ & \ddots & \\ 0 & & \rho_m' \end{pmatrix} \in \mathbb{R}^{m\times d},$$

$$r(t,z) = r,\ \mu(t,z) = r(t,z)\mathbb{1} + \begin{pmatrix} \eta_1 z_1 \\ \vdots \\ \eta_m z_m \end{pmatrix},\ \Sigma(t,z) = \begin{pmatrix} \Sigma_1\sqrt{z_1} & & 0 \\ & \ddots & \\ 0 & & \Sigma_m\sqrt{z_m} \end{pmatrix}.$$

[5] We will later see in Lemma 4.4.6 that Condition (EAS) is only satisfied in $\mathcal{M}_{CIR}$ if the structure of the allocation constraints allows for a convenient separability in (4.14). In the definition of $\mathcal{M}_{CIR}$, we have intentionally limited the covariance Σ of risky assets to be of block-diagonal structure to facilitate the presentation of this fact. However, we can in principle also choose more complex models for Σ, such as the general PCSV model, and adjust the allocation constraints accordingly without changing the underlying theory in a significant way.

In $\mathcal{M}_{CIR}$, the minimization (4.14) can be equivalently rewritten as[6]

$$\underset{\substack{\lambda=(\lambda_1,\ldots,\lambda_m)' \\ \lambda_i\in\mathbb{R}^{d_i}}}{\operatorname{argmin}} \left\{ 2(1-b)\delta_K(\lambda) + \sum_{i=1}^{m} \left(2\left(\Sigma_i^{-1}\lambda_i\right)' \left(\Sigma_i^{-1}\eta_i + \sigma_i B_i \rho_i\right) + \left\|\Sigma_i^{-1}\lambda_i\right\|^2 z_i \right) \right\}.$$

However, as the underlying financial market model $\mathcal{M}_{CIR}$ consists of m independent asset classes, it is natural to assume a certain independence with respect to the allocation constraints, too. This independence can be expressed in mathematical terms by assuming that K can be written as the Cartesian product of m constraints $K_1, ..., K_m$ on the individual asset classes.

Lemma 4.4.6 (Dual ODEs in $\mathcal{M}_{CIR}$).
Consider the financial market $\mathcal{M}_{CIR}$. If $K = \times_{i=1}^{m} K_i$ with $K_i \subset \mathbb{R}^{d_i}$ closed convex and non-empty interior for every $i = 1, .., m$, then Condition (EAS) is satisfied.
Let

$$\lambda^*(t,z,B) := \begin{pmatrix} \lambda_1^*(B_1)z_1 \\ \vdots \\ \lambda_m^*(B_m)z_m \end{pmatrix},$$

where

$$\lambda_i^*(B_i) = \underset{\lambda_i\in\mathbb{R}^{d_i}}{\mathit{argmin}} \left\{ 2(1-b)\delta_{K_i}(\lambda_i) + \left\|\Sigma_i^{-1}(\eta_i+\lambda_i) + \sigma_i B_i \rho_i\right\|^2 \right\}$$

and $A : [0,T] \to \mathbb{R}$, $B : [0,T] \to \mathbb{R}^m$ satisfy $A(0) = 0$, $B(0) = 0$ and

$$\begin{aligned} A_\tau(\tau) &= br + (\kappa \odot \theta)' B(\tau) \\ (B_\tau)_i(\tau) &= -\kappa_i B_i(\tau) + \frac{1}{2}\sigma_i^2 (B_i(\tau))^2 \\ &\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda_i\in\mathbb{R}^{d_i}} \left\{ 2(1-b)\delta_{K_i}(\lambda_i) + \left\|\Sigma_i^{-1}(\eta_i+\lambda_i) + \sigma_i B_i \rho_i\right\|^2 \right\}. \end{aligned}$$

Then,

$$G(t,v,z) = \frac{1}{b} v^b \exp(A(T-t) + B(T-t)'z)$$

is a solution to the dual HJBI PDE (4.7) and the corresponding candidate optimal portfolio is

$$\underline{\pi}^*(t,v,z) = \begin{pmatrix} \pi_1^*(B_1(T-t)) \\ \vdots \\ \pi_m^*(B_m(T-t)) \end{pmatrix},$$

with

$$\pi_i^*(B_i(T-t)) = \frac{1}{1-b}\left(\Sigma_i\Sigma_i'\right)^{-1} \Big(\eta_i + \lambda_i^*(B_i(T-t)) + \sigma_i B_i(T-t)\Sigma_i\rho_i \Big).$$

The ODEs for B_i in Lemma 4.4.6 do not admit a general closed-form solution, as the right-hand side of the ODE still depends on a convex minimization problem. Note however, as long as all minimizer $\lambda_i^*(B_i)$ are continuous in B_i (e.g. if each K_i are compact sets), then the right-hand side of each ODE for B_i is continuous in B_i and therefore admits at least a local solution.
Due to this lack of an explicit representation for B, we also lack an explicit representation for $\underline{\pi}^*$. In addition, the quadratic variations of $\ln\left(V^{v_0,\underline{\pi}^*}\right)$ and z are stochastic and we can thus no longer follow the approach from Corollary 4.4.3 to formally verify the optimality of $\underline{\pi}^*$. This topic will be investigated thoroughly in Chapter 5.

[6] Compare to the derivation of (A.53) in the proof of the subsequent Lemma 4.4.6 for details.

4.4.3 Multi-Factor Short Rate of OU-Type

Lastly, we consider a financial market $\mathcal{M}_{OU}$ with a stochastic short rate r, which is driven by an m-dimensional Ornstein-Uhlenbeck process and $d = m$ zero-coupon bonds with maturities $T_1, ..., T_m > T$ as primary traded assets (similar models were discussed e.g. in Section 7.3 in [59] for derivatives pricing, in [87] for economic scenario generation as well as in [61] and [80] in a portfolio optimization context). For this purpose, we define μ^z and Σ^z as

$$\mu^z(t, z) = \kappa \odot [\theta - z], \quad \Sigma^z(t, z) = \sigma, \tag{4.22}$$

for arbitrary constants $\kappa \in (0, \infty)^m$, $\theta \in \mathbb{R}^m$, and a non-singular matrix $\sigma \in \mathbb{R}^{m \times m}$. For two weights $w_0 \in \mathbb{R}$, $w_1 \in \mathbb{R}^m$ we then define the short rate r through

$$r(t, z) = w_0 + w_1' z.$$

In particular, the Q-dynamics of the short rate are given as

$$dr(t, z(t)) = d(w_0 + w_1' z(t)) = w_1' dz(t) = w_1' (\kappa \odot [\theta - z(t)]) \, dt + w_1' \sigma dW^z(t)$$

To formally define $\mathcal{M}_{OU}$, we still need to explicitly determine the dynamics of the traded zero-coupon bonds. We determine these dynamics via risk-neutral pricing. Assuming a constant market price of risk $\eta \in \mathbb{R}^m$, we can define the equivalent martingale measure $\tilde{Q}$ through its Radon-Nikodym derivative

$$\frac{d\tilde{Q}}{dQ} = \exp\left(-\frac{1}{2} \|\eta\|^2 T - \int_0^T \eta' dW^z(t) \right).$$

Then, according to Girsanov's theorem, there exists a $\tilde{Q}$-Wiener process $\tilde{W}^z$ such that the $\tilde{Q}$-dynamics of the short rate are given as

$$dr(t, z(t)) = \left(\kappa \odot [\theta - z(t)] - w_1' \sigma \eta \right) dt + w_1' \sigma d\tilde{W}^z(t).$$

Moreover, we can now make use of risk-neutral pricing to determine the arbitrage-free prices of zero-coupon bonds with different maturities. The financial market $\mathcal{M}_{OU}$ belongs to the group of affine factor models (without stochastic volatility) studied in [26]. Hence, there exist suitable deterministic continuously differentiable functions $a : (0, \infty) \to \mathbb{R}$, $b : (0, \infty) \to \mathbb{R}^m$ such that the price of a zero-coupon bond with maturity T_i at time $t \in [0, T]$ is given by

$$P(t, T_i) = \mathbb{E}_{\tilde{Q}} \left[\exp\left(-\int_t^{T_i} r(s, z(s)) ds \right) \Big| \mathcal{F}_t \right] = \exp\left(a(T_i - t) + b(T_i - t)' z(t) \right). \tag{4.23}$$

By applying Itô's formula and noting that the discounted price processes $\left(P(t,T_i)/P_0(t) \right)_{t \in [0,T]}$ are martingales with respect to $\tilde{Q}$, we see that the Q-dynamics of $\left(P(t, T_i) \right)_{t \in [0,T]}$ are

$$dP(t, T_i) = P(t, T_i) \left(\left[r(t, z(t)) + b(T_i - t)' \sigma \eta \right] dt + b(T_i - t)' \sigma dW^z(t) \right).$$

These zero-coupon bonds $P(t, T_1), ..., P(t, T_m)$ constitute the primary traded assets of the financial market $\mathcal{M}_{OU}$ which is formally defined below.

Definition 4.4.7 ($\mathcal{M}_{OU}$).
Let $m = d \in \mathbb{N}$. Consider constants $w_0 \in \mathbb{R}$, $\kappa \in (0, \infty)^m$, $w_1, \theta, \eta \in \mathbb{R}^m$, a non-singular matrix

$\sigma \in \mathbb{R}^{m\times m}$, maturities $\hat{T} = (T_1, ..., T_m)' \in (T, \infty)^m$ and a continuously differentiable function $b : (0, \infty) \to \mathbb{R}^m$ such that the matrix

$$b(t; \hat{T}) = (b(T_1 - t), ..., b(T_m - t)) \in \mathbb{R}^{m\times m}$$

has an inverse $b(t; \hat{T})^{-1}$ for all $t \in [0, T]$. Then, the m-dimensional Bond market $\mathcal{M}_{OU}$ with OU short rate is defined by the market coefficients

$$\begin{aligned}
&\mu^z(t, z) = \kappa \odot (\theta - z)\,, \quad \Sigma^z(t, z) = \sigma, \quad \rho(t, z) = I_m\\
&r(t, z) = w_0 + w_1' z, \quad \mu(t, z) = r(t, z)\mathbb{1} + b(t; \hat{T})'\sigma\eta, \quad \Sigma(t, z) = b(t; \hat{T})'\sigma.
\end{aligned}$$

Despite the stochastic short rate, the market $\mathcal{M}_{OU}$ is surprisingly tractable. Specifically, none of the terms involved in the minimization (4.14) are dependent on the stochastic factor z, which results in a time-dependent but deterministic candidate optimal portfolio $\underline{\pi}^*$.

Lemma 4.4.8 (Dual ODEs in $\mathcal{M}_{OU}$)**.**
Consider the financial market $\mathcal{M}_{OU}$. Then Condition (EAS) is satisfied.
Let

$$\lambda^*(t, B) = \underset{\lambda\in\mathbb{R}^d}{\mathrm{argmin}} \left\{ 2(1-b)\delta_K(\lambda) + \left\| \eta + \left(b(t; \hat{T})'\sigma\right)^{-1} \lambda + \sigma' B \right\|^2 \right\}$$

and $A : [0, T] \to \mathbb{R}$, $B : [0, T] \to \mathbb{R}^m$ satisfy $A(0) = 0$, $B(0) = 0$ and

$$\begin{aligned}
A_\tau(\tau) &= bw_0 + (\kappa \odot \theta)'\, B(\tau) + \frac{1}{2}\|\sigma' B(\tau)\|^2\\
&\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda\in\mathbb{R}^d} \left\{ 2(1-b)\delta_K(\lambda) + \left\| \eta + \left(b(T-\tau; \hat{T})'\sigma\right)^{-1} \lambda + \sigma' B \right\|^2 \right\}\\
B_\tau(\tau) &= bw_1 - \kappa \odot B(\tau).
\end{aligned} \tag{4.24}$$

Then,

$$G(t, v, z) = \frac{1}{b} v^b \exp(A(T-t) + B(T-t)'z)$$

is a solution to the dual HJBI PDE (4.7) and the corresponding candidate optimal portfolio $\underline{\pi}^*$ (as in (4.17)) is given as

$$\underline{\pi}^*(t, v, z) = \frac{1}{1-b}\left(\sigma' b(t; \hat{T})\right)^{-1} \left(\eta + \left(b(t; \hat{T})'\sigma\right)^{-1} \lambda^*(t, B(T-t)) + \sigma' B(T-t)\right).$$

Remark 4.4.9. *The solution to the ODE (4.24) is known in closed-form (see e.g. equations (6) and (7) in Chapter 1, Section §2 of [83]) and is given by $B(\tau) = (B_1(\tau), ..., B_m(\tau))'$ with*

$$B_i(\tau) = (w_1)_i\, b e^{-\kappa_i \tau} \int_0^\tau e^{\kappa_i s} ds = \frac{(w_1)_i\, b}{\kappa_i}\left(1 - e^{-\kappa_i \tau}\right).$$

Unlike in $\mathcal{M}_{CIR}$, the quadratic variations of $\ln\left(V^{v_0, \underline{\pi}^*}\right)$ and z are deterministic and bounded in $\mathcal{M}_{OU}$. Therefore, it is straightforward to adapt the proof of Corollary 4.4.3 to formally verify the optimality of $\underline{\pi}^*$ for (**P**).

Corollary 4.4.10. *Consider the financial market $\mathcal{M}_{OU}$. Then, G, $\underline{\pi}^*$ as in Lemma 4.4.8 and $\lambda \equiv 0$ satisfy Condition (UI$_\lambda$). In particular, $\underline{\pi}^*$ is optimal for* (**P**).

Remark 4.4.11. *Other than in $\mathcal{M}_{CIR}$, the ODE for B in $\mathcal{M}_{OU}$ does not depend on the constraint K (cf. Lemma 4.4.6 and Lemma 4.4.8). Therefore, setting $K = \mathbb{R}^d$ and $\lambda^* \equiv 0$ yields the optimal unconstrained portfolio in $\mathcal{M}_{OU}$ as*

$$\pi_{OU}(t) = \frac{1}{1-b}\Big(\sigma' b(t;\hat{T})\Big)^{-1}\Big(\eta + \sigma' B(T-t)\Big).$$

Similarly, we can replicate the arguments in Remarks 4.3.8 and 4.4.4 to see that the optimal constrained portfolio in $\mathcal{M}_{OU}$ can be written as

$$\underline{\pi}^*(t,v,z) = \underset{\pi\in K}{\mathit{argmin}} \Big\| \underbrace{\sigma' b(t;\hat{T})}_{=:\Sigma(t)'}(\pi(t)-\pi_{OU}(t))\Big\|^2 = \underset{\pi\in K}{\mathit{argmin}} \Big\|\Sigma(t)'(\pi(t)-\pi_{OU}(t))\Big\|^2 = \mathcal{P}_K^{BS}[\pi_{OU}](t).$$

In other words, allocation constraints have exactly the same impact on the optimal allocation in $\mathcal{M}_{OU}$ as in $\mathcal{M}_{BS}$.

4.5 Conclusion

In this chapter, we examined a portfolio optimization problem in a financial market where asset dynamics depend on a stochastic factor. In the spirit of [61], we were able to derive Condition (EAS) which guarantees that the solution to the HJB PDE for an allocation constrained portfolio optimization problem is exponentially affine and separable in wealth and the stochastic factor. We were able to use Condition (EAS) to characterize the optimal allocation constrained portfolio up to the solution of a deterministic optimization problem and the solution of Riccati ODEs in a market with stochastic volatility of CIR-type and in a market with stochastic short rate of OU-type. Special examples of these models include the Heston model, the PCSV model and the Vasicek model. We derived a formal verification result for the market with stochastic short rate and a general verification result up to a uniform integrability condition for arbitrary markets satisfying Condition (EAS). The proposed methodology is general enough to derive and study the optimal allocation constrained portfolios for several financial markets with dynamics depending on a stochastic factor.

5 Constraints on Allocation in Heston's Stochastic Volatility Model

5.1 Introduction

Despite its widespread use in both academia and the financial industry, it has been well-documented in the mathematical finance literature that a variety of properties of financial time series, so-called stylized facts, are not captured by the Black-Scholes model (see e.g. [14], [63], [84]). A major point of criticism is the constant volatility of modelled log returns, whereas empirical returns appear to be time-dependent and random ([81]). The stochastic volatility model proposed by [42] aims to avoid this gap by modelling the volatility of log returns as a Cox-Ingersoll-Ross process (CIR process). While the Heston model was originally proposed in the context of option pricing, its analytical tractability has led to insightful applications in continuous-time portfolio optimization (see e.g. [56]), which is the subject of this chapter.
Specifically, we consider the portfolio optimization problem of an investor trading continuously in a financial market consisting of one risk-free asset and one risky asset with stochastic Heston volatility. The investor seeks to maximise his expected utility derived from terminal wealth at a finite time point $T > 0$ under the condition that his portfolio allocation π abides by given convex allocation constraints K. The considered optimization problem involves two major facets, which differ from the original portfolio optimization setting of [65]: stochasticity of the volatility of risky asset log returns ('stochastic volatility') and the presence of convex allocation constraints. Since Section 1.1, already provided an overview over the relevant literature on allocation constraints, the following paragraph presents a brief overview of the relevant literature with respect to stochastic volatility:

Our continuous time set-up can be traced back to the seminal work of [65], who used dynamic programming methods to derive explicit solutions to the unconstrained dynamic portfolio optimization problem for an investor with HARA utility function in a Black-Scholes model with constant volatility. Generalizations for different utility functions and more complex financial markets were achieved by employing martingale methods ([73] and [47]), which rely heavily on the assumption that all contingent claims in the financial market are replicable. However, this assumption is not generally satisfied for financial markets with stochastic volatility, which means that martingale methods are not directly applicable, unless the financial market is artificially completed by the addition of fictitious, volatility-dependent assets (see e.g. [62], [9], [28], [30], and [10]). Without such completion, the solvability of the portfolio optimization problem in financial markets with stochastic volatility is often directly linked to the solvability of the associated HJB PDE.[1] Solutions to such portfolio optimization problems were first characterized in terms of viscosity solutions to the associated HJB PDE for multi-factor models in [88], and explicit closed-form solutions for Heston's stochastic volatility model were first derived and formally verified in [56]. Subsequently, [61] derived explicit solution formulae for an optimal consumption and portfolio allocation problem in a general multi-factor model, where the factor is a quadratic diffusion. [46] used the notion of an opportunity process and semi-martingale

[1] Note that obtaining and formally verifying the optimality of a candidate portfolio process requires more than just a solution to the associated HJB PDE, as pointed out by [54].

characteristics to develop an approach which leads to closed-form solutions for the optimal portfolio process in a range of exponentially affine stochastic volatility models, including the Heston model and the jump model of [40]. The PCSV model and the Wishart process were considered as multi-dimensional extensions of the Heston model for multidimensional asset universes with stochastic correlation in [30] and [5]. An extensive overview of related papers in the field of dynamic portfolio optimization in stochastic factor models can be found in [90].

In this chapter, we make a threefold contribution to this literature:

- We complete a gap in Chapter 4 (resp. [33]) by deriving the first explicit, closed-form expression for the optimal portfolio allocation π^* in Heston's stochastic volatility model under the presence of convex allocation constraints. This optimality is verified formally in a verification theorem.
- We show that the optimal portfolio allocation π^* may be different from the capped optimal unconstrained portfolio allocation π_u for Heston's stochastic volatility model. In particular, we prove an equivalent characterization which describes when these two portfolios are different. We conduct a numerical study to show that this difference is slim for calm market scenarios, but can lead to significant annual wealth equivalent losses during turbulent market scenarios.
- We extend these results from the one-dimensional Heston's stochastic volatility model to a multi-dimensional PCSV model with constraints on the exposures to individual stochastic market factors and to generalized financial markets with inverse volatility constraints.

The remainder of this chapter is structured as follows: In Section 5.2, we introduce and solve the constrained portfolio optimization problem (**P**) in Heston's stochastic volatility model. Specifically, we derive a solution to the HJB PDE associated with (**P**) and the candidate optimal portfolio π^* in Section 5.2.1, verify its optimality formally by proving a verification theorem in Section 5.2.2 and discuss its relation to the optimal unconstrained portfolio π_u in Section 5.2.3. In Section 5.3, we consider a generalized financial market model which depends on a multi-dimensional CIR process and derive the optimal portfolio in the PCSV model under exposure constraints (Section 5.3.1) and in general financial markets with inverse volatility constraints (Section 5.3.2). In Section 5.4, we illustrate our theoretical results for Heston's stochastic volatility model in a numerical analysis, where we analyze the wealth equivalent loss of the optimal constrained portfolio for a Black Scholes model (Section 5.4.1) and the capped optimal unconstrained portfolio for Heston's stochastic volatility model (Section 5.4.2). Section 5.5 concludes this chapter.

5.2 Heston's Stochastic Volatility Model

In this chapter, we consider several special cases of the stochastic factor model from Chapter 5. In particular, the standing assumptions for this chapter are the same as for Chapter 5:

Assumption 5.2.1 (Standing Assumptions Chapter 5:). *$\mathcal{M}$ is a stochastic factor model $\big((4.2)$ and $(4.3)\big)$, (4.4) and U is a power utility function.*

The first such special case is Heston stochastic volatility model $\mathcal{M}_H$, which is defined by considering the stochastic factor model (4.3) and choosing $m = d = 1$ as well as the deterministic functions

$$\mu^z(t,z) = \kappa(\theta - z),\ \Sigma^z(t,z) = \sigma\sqrt{z},\ \rho(t,z) = \rho,$$

$$r(t,z) = r,\ \mu(t,z) = r(t,z) + \eta z,\ \Sigma(t,z) = \sqrt{z},$$

where the coefficients $r, \eta, \kappa, \theta, \sigma, z_0$ are assumed to be positive constants, and $\rho \in (-1, 1)$. Further, it is assumed that Feller's condition holds for the parameters of z, i.e., $2\kappa\theta > \sigma^2$, and, therefore, $z(t)$ is guaranteed to take only positive values with probability 1 (see [41]). Clearly, the financial market $\mathcal{M}_H$ is a special, one-dimensional case of $\mathcal{M}_{CIR}$ from Definition 4.4.5.

The resulting financial market $\mathcal{M}_H$ consists of one risk-free asset P_0 and one risky asset P_1, which satisfy $P_0(0) = P_1(0) = 1$ and follow the dynamics

$$\begin{aligned} dP_0(t) &= P_0(t) r dt, \\ dP_1(t) &= P_1(t)\big(r + \eta \cdot z(t)\big)dt + \sqrt{z(t)} dW(t), \end{aligned}$$

where $z(0) = z_0 > 0$ and

$$dz(t) = \kappa(\theta - z(t))dt + \sigma\sqrt{z(t)}dW^z(t).$$

As a consequence, the wealth process $V^{v_0,\pi}$ of an investor trading in $\mathcal{M}_H$ according to a relative portfolio process $\pi \in \Lambda$ and initial wealth $v_0 > 0$ satisfies the SDE

$$dV^{v_0,\pi}(t) = V^{v_0,\pi}(t)\Big([r + \eta z(t)\pi(t)]dt + \pi(t)\sqrt{z(t)}dW(t)\Big)$$

and it is straightforward to show that $V^{v_0,\pi}(t)$ can be expressed in closed-form as

$$V^{v_0,\pi}(t) = v_0 \exp\Big(\int_0^t r + \eta z(s)\pi(s) - \frac{1}{2} z(s)\pi(s)^2 ds + \int_0^t \pi(s)\sqrt{z(s)}dW(s)\Big). \tag{5.1}$$

For a non-empty closed convex set $K \subset \mathbb{R}$ and power utility function $U(v) = \frac{1}{b}v^b$ with $b < 1$ and $b \neq 0$, we consider the portfolio optimization problem

$$(\mathbf{P}) \begin{cases} \Phi(v_0) &= \sup_{\pi \in \Lambda'} \mathbb{E}\big[U(V^{v_0,\pi}(T))\big] \\ \Lambda' &= \big\{\pi \in \Lambda \;\big|\; \pi(t) \in K \ \mathcal{L}[0,T] \otimes Q - \text{a.e.}\big\}. \end{cases}$$

As the considered financial market contains only one risky asset, the set of allocation constraints K is a subset of the real numbers $\mathbb{R}$. However, in this one-dimensional setting, any such closed convex set $K \subset \bar{\mathbb{R}}$ with non-empty interior can be expressed as an interval of the form[2]

$$K = [\alpha, \beta], \quad \text{with} \quad -\infty \leq \alpha < \beta \leq \infty. \tag{5.2}$$

We make substantial use of this fact in the subsequent analysis.

5.2.1 Solution to HJB PDE

Following the arguments from Section 4.3, the HJB equation associated with $(\mathbf{P}^{(\mathbf{t},\mathbf{v},\mathbf{z})})$ is given by

$$0 = \sup_{\pi \in K}\Big(G_t + v(r + \eta z \pi)G_v + \kappa(\theta - z)G_z + \frac{1}{2}v^2 z \pi^2 G_{vv} + \rho v z \pi \sigma G_{vz} + \frac{1}{2}\sigma^2 z G_{zz}\Big) \tag{5.3}$$

$$G(T, v, z) = U(v),$$

[2] As any $\pi \in \Lambda$ can only take finite values $\mathcal{L}[0,T] \otimes Q$-a.e., we do not need to distinguish between $(-\infty, \beta]$ and $[-\infty, \beta]$ or $[\alpha, \infty)$ and $[\alpha, \infty]$ for any $-\infty \leq \alpha, \beta \leq \infty$.

Any (sufficiently regular) solution G to (5.3) naturally yields a candidate optimal portfolio π^* to $(\mathbf{P}^{(\mathbf{t},\mathbf{v},\mathbf{z})})$ (and therefore $(\mathbf{P})$) as the maximizing argument of (5.3). In Chapter 4 we characterized G as an exponentially affine function, whose exponents satisfy certain Riccati ODEs.

Lemma 5.2.2. *Let A and B be solutions to the system of ODEs*

$$A'(\tau) = br + \kappa\theta B(\tau), \tag{5.4}$$

$$B'(\tau) = -\kappa B(\tau) + \frac{1}{2}\sigma^2 (B(\tau))^2 + \frac{1}{2}\frac{b}{1-b}\inf_{\lambda\in\mathbb{R}}\Big(2(1-b)\delta_K(\lambda) + (\eta+\lambda+\sigma\rho B(\tau))^2\Big), \tag{5.5}$$

with initial condition $A(0) = B(0) = 0$. Then, $G(t,v,z) = \frac{1}{b}v^b\exp\left(A(T-t) + B(T-t)z\right)$ is a solution to (5.3).

Given the minimizing argument and the solution B from (5.5), a candidate optimal portfolio π^* for $(\mathbf{P})$ is known. If we define the sequence of stopping times $\tau_{n,t}$ as $\tau_{n,t} = \min(T, \hat{\tau}_{n,t})$, with

$$\hat{\tau}_{n,t} = \inf\Big\{t \le u \le T \;\Big|\; \int_t^u \Big(b\cdot\sqrt{z(s)}\cdot\pi(s)\cdot G(s, V^{v_0,\pi^*}(s), z(s))\Big)^2 ds \ge n,$$
$$\int_t^u \Big(\sigma\sqrt{z(s)}\cdot B(T-s)\cdot G(s, V^{v_0,\pi^*}(s), z(s))\Big)^2 ds \ge n\Big\},$$

then we can give a uniform integrability condition which guarantees that the candidate optimal portfolio π^* is indeed optimal for $(\mathbf{P})$.

Lemma 5.2.3. *Consider A, B and G from Lemma 5.2.2. Moreover, define*

$$\lambda^*(B) = \underset{\lambda\in\mathbb{R}}{argmin}\Big\{2(1-b)\delta_K(\lambda) + (\eta+\lambda+\sigma\rho B)^2\Big\} \tag{5.6}$$

$$\pi^*(t) = \frac{1}{1-b}\Big(\eta + \lambda^*(B(T-t)) + \sigma\rho B(T-t)\Big). \tag{5.7}$$

If $\left(G\left(\tau_{n,t}, V^{v_0,\pi^*}(\tau_{n,t}), z(\tau_{n,t})\right)\right)_{n\in\mathbb{N}}$ is uniformly integrable for every $t\in[0,T]$, then π^* is optimal for $(\mathbf{P})$ and $\Phi(t,z,v) = G(t,v,z)$ for all $(t,v,z)\in[0,T]\times(0,\infty)\times(0,\infty)$.

Lemma 5.2.2 and Lemma 5.2.3 naturally lead to a three-step procedure for finding the optimal portfolio π^* for $(\mathbf{P})$:

(i) Determine the minimizing argument λ^* in (5.6).

(ii) Determine the solution B to ODE (5.5) and thereby the candidate optimal portfolio π^* from (5.7).

(iii) Verify that π^* satisfies the uniform integrability condition from Lemma 5.2.3.

In the following, we complete these steps consecutively and complete the results from Section 4.4 by providing a fully closed-form solution for the optimal allocation-constrained portfolio in Heston's stochastic volatility model via steps (i) and (ii) and formally verifying its optimality in step (iii).

Since K is an interval, as specified in (5.2), the ODE (5.5) can be written as a composition of three Riccati ODEs - each with constant coefficients.

Lemma 5.2.4. *Define* $B_- = \frac{(1-b)\alpha-\eta}{\sigma}$ *and* $B_+ = \frac{(1-b)\beta-\eta}{\sigma}$. *Then, the minimizing argument* λ^*, *as in (5.6), is given as*

$$\lambda^*(B) = \big[(1-b)\alpha - (\eta+\sigma\rho B)\big]\mathbb{1}_{\{\rho B<B_-\}} + \big[(1-b)\beta - (\eta+\sigma\rho B)\big]\mathbb{1}_{\{\rho B>B_+\}}. \tag{5.8}$$

Moreover, $B(\tau)$ *is a solution to* (5.5) *if and only if* $B(0)=0$ *and*

$$\begin{aligned} B'(\tau) = \; & \Big(\underbrace{-\frac{1}{2}b\alpha\big((1-b)\alpha-2\eta\big)}_{=:r_0^-} + \underbrace{\big(b\sigma\rho\alpha-\kappa\big)}_{=:r_1^-}B(\tau) + \frac{1}{2}\underbrace{\sigma^2}_{=:r_2^-}\big(B(\tau)\big)^2\Big)\mathbb{1}_{\{\rho B(\tau)<B_-\}} \\ & +\Big(\underbrace{-\frac{-b}{2(1-b)}\eta^2}_{=:r_0} + \underbrace{\big(\frac{b}{1-b}\eta\sigma\rho-\kappa\big)}_{=:r_1}B(\tau) + \frac{1}{2}\underbrace{\sigma^2\big(1+\frac{b}{1-b}\rho^2\big)}_{=:r_2}\big(B(\tau)\big)^2\Big)\mathbb{1}_{\{B_-\le\rho B(\tau)\le B_+\}} \\ & +\Big(\underbrace{-\frac{1}{2}b\beta\big((1-b)\beta-2\eta\big)}_{=:r_0^+} + \underbrace{\big(b\sigma\rho\beta-\kappa\big)}_{=:r_1^+}B(\tau) + \frac{1}{2}\underbrace{\sigma^2}_{=:r_2^+}\big(B(\tau)\big)^2\Big)\mathbb{1}_{\{B_+<\rho B(\tau)\}} \qquad (5.9) \\ = \; & \Big(-r_0^- + r_1^-B(\tau) + \frac{1}{2}r_2^- + \big(B(\tau)\big)^2\Big)\mathbb{1}_{\{\rho B(\tau)<B_-\}} \\ & +\Big(-r_0 + r_1B(\tau) + \frac{1}{2}r_2\big(B(\tau)\big)^2\Big)\mathbb{1}_{\{B_-\le\rho B(\tau)\le B_+\}} \\ & +\Big(-r_0^+ + r_1^+B(\tau) + \frac{1}{2}r_2^+\big(B(\tau)\big)^2\Big)\mathbb{1}_{\{B_+<\rho B(\tau)\}}. \end{aligned}$$

Remark 5.2.5. *By restricting the minimization in ODE (5.5) from* $\lambda\in\mathbb{R}$ *to one of the three optimal values* $\lambda\in\{(1-b)\alpha-(\eta+\sigma\rho B),(1-b)\beta-(\eta+\sigma\rho B),0\}$ *(cf. (5.8)), we may use (5.9) to write*

$$\begin{aligned} B'(\tau) &= -\kappa B(\tau) + \frac{1}{2}\sigma^2\big(B(\tau)\big)^2 + \frac{1}{2}\frac{b}{1-b}\inf_{\lambda\in\mathbb{R}}\Big(2(1-b)\delta_K(\lambda) + (\eta+\lambda+\sigma\rho B(\tau))^2\Big) \\ &= \min\Big(-r_0^- + r_1^-B(\tau) + \frac{1}{2}r_2^-\big(B(\tau)\big)^2, -r_0 + r_1B(\tau) + \frac{1}{2}r_2\big(B(\tau)\big)^2, -r_0^+ + r_1^+B(\tau) + \frac{1}{2}r_2^+\big(B(\tau)\big)^2\Big) \\ &=: f(B(\tau)). \end{aligned}$$

The coefficients r_2^-, r_2 *and* r_2^+ *are non-negative, and therefore* f *is the minimum of three convex functions. As real convex functions are locally Lipschitz continuous and Lipschitz continuity is preserved when taking the minimum over a finite number of functions,* f *is locally Lipschitz continuous too. Hence, by the existence and uniqueness theorem of Picard-Lindelöf, there exists a unique solution* B *to (5.5) for small* $\tau>0$. *Moreover, as* f *does not depend on* τ, *the ODE for* B *is autonomous and its solution* B *is either constant (if* $f(0)=0$*) or strictly monotone in* τ *(if* $f(0)\neq 0$*). Analogous arguments can be used to conclude the (strict) monotonicity of* $B_u(\tau)$ *from Corollary 5.2.13 in Section 5.2.3.*

Remark 5.2.6. *Note that if* B *is a solution to (5.5), then Lemma 5.2.3 and Lemma 5.2.4 imply*

$$\pi^*(t) = \frac{\eta+\lambda^*(B(T-t))+\sigma\rho B(T-t)}{1-b} = \begin{cases} \alpha, & \rho B(T-t)<B_- \\ \frac{\eta+\sigma\rho B(T-t)}{1-b}, & B_-\le\rho B(T-t)\le B_+ \\ \beta, & B_+<\rho B(T-t). \end{cases}$$

Therefore, the zones $Z_-=(-\infty,B_-)$, $Z_0=[B_-,B_+]$ *and* $Z_+=(B_+,\infty)$ *determine whether the allocation constraint* $K=[\alpha,\beta]$ *is enforced for the candidate optimal portfolio process* π^*

from Lemma 5.2.3. Moreover, we may define $\hat{\pi}^*(t) := \frac{1}{1-b}\left(\eta + \sigma\rho B(T-t)\right)$ *and express* π^* *as a capped version of* $\hat{\pi}^*$*, i.e.,*

$$\pi^*(t) = Cap(\hat{\pi}^*(t), \alpha, \beta) := \begin{cases} \alpha, & \hat{\pi}^*(t) < \alpha \\ \hat{\pi}^*(t), & \alpha \leq \hat{\pi}^*(t) \leq \beta \\ \beta, & \beta < \hat{\pi}^*(t). \end{cases}$$

In a true constrained context, i.e., $K \neq \mathbb{R}$, we may either determine an approximation of B by using a suitable numerical ODE solver (such as an Euler method) to solve the ODE (5.9) or directly derive an explicit expression for B by individually solving each of the three Riccati ODEs in (5.9) and merging the solutions at the transition points between the zones Z_-, Z_0 and Z_+. To ensure that such solutions exist and do not explode before time T, we need to make the following assumption on $\mathcal{M}_H$ and the constraints $K = [\alpha, \beta]$.

Assumption 5.2.7.

(i) *Existence of Solution:*

$$\max \left\{ \begin{array}{l} \frac{b}{1-b}\eta\left(\frac{\kappa\rho}{\sigma} + \frac{\eta}{2}\right), \\ b\alpha\left(\eta - \frac{1}{2}\alpha + \frac{\kappa\rho}{\sigma} + \frac{1}{2}\alpha b(1-\rho^2)\right), \\ b\beta\left(\eta - \frac{1}{2}\beta + \frac{\kappa\rho}{\sigma} + \frac{1}{2}\beta b(1-\rho^2)\right) \end{array} \right\} < \frac{\kappa^2}{2\sigma^2}$$

(ii) *No Blow-Up:*

The coefficients of each of the three Riccati ODEs satisfy $t_+(B_0) > T$ *(cf. Lemma 2.2.4, (ii)) for each initial value*

$$B_0 \in \left\{ \left(\frac{B_-}{\rho}\right) \mathbb{1}_{\{\rho\neq 0\}}, \left(\frac{B_+}{\rho}\right) \mathbb{1}_{\{\rho\neq 0\}}, 0 \right\}.$$

Provided that Assumption 5.2.7 holds, the coefficients

$$r_3^- = \sqrt{(r_1^-)^2 + 2r_0^- r_2^-}, \quad r_3 = \sqrt{(r_1)^2 + 2r_0 r_2}, \quad r_3^+ = \sqrt{(r_1^+)^2 + 2r_0^+ r_2^+} \tag{5.10}$$

are well-defined and the solutions to each of the Riccati ODEs (5.9) do not blow up before time T when started at any of the transition points between the zones Z_-, Z_0 and Z_+.[3] For this reason, we define the following auxiliary functions:

- Let $\hat{B}^+$, $\hat{B}$, and $\hat{B}^-$ be the solution to Riccati ODE (2.22) with initial value 0 as well as coefficients r_0^+, r_1^+, r_2^+, r_0, r_1, r_2 and r_0^-, r_1^-, r_2^-, respectively.
- If $\rho \neq 0$, let $\hat{B}_+^+$, $\hat{B}_-^-$ be the solution to Riccati ODE (2.22) with initial value $\frac{B_+}{\rho}$, $\frac{B_-}{\rho}$ and coefficients r_0^+, r_1^+, r_2^+, r_0^-, r_1^-, r_2^-, respectively.
- If $\rho \neq 0$, let $\hat{B}_+$, $\hat{B}_-$ be the solution to Riccati ODE (2.22) with initial value $\frac{B_+}{\rho}$, $\frac{B_-}{\rho}$, respectively, and coefficients r_0, r_1, r_2.

[3] Technically, one can formulate this assumption less restrictively by expressing 'No Blow-Up' in terms of the time spent in each of the zones Z_-, Z_0 and Z_+. However, as this would significantly complicate the presentation without adding major additional insights, it is omitted here.

Moreover, if $\rho \neq 0$, we define the transition times[4]

$$\tau_1^+ = \inf\left\{\tau \mid \hat{B}^+(\tau) = \frac{B^+}{\rho}\right\}, \quad \tau_2^+ = \inf\left\{\tau \mid \hat{B}_+(\tau) = \frac{B^-}{\rho}\right\}, \quad \tau_1 = \inf\left\{\tau \mid \hat{B}(\tau) \in \left\{\frac{B^-}{\rho}, \frac{B^+}{\rho}\right\}\right\}$$
$$\tau_1^- = \inf\left\{\tau \mid \hat{B}^-(\tau) = \frac{B^-}{\rho}\right\} \quad \text{and} \quad \tau_2^- = \inf\left\{\tau \mid \hat{B}_-(\tau) = \frac{B^+}{\rho}\right\}.$$

Note that each of the above functions and transition times, if finite, admit a closed-form expression, which can be obtained via Lemma 2.2.4 and Corollary 2.2.5. Having introduced these auxiliary functions and transition times, we can finally express a closed-form solution for B in terms of these processes via a piecewise construction. Possible trajectories of B are displayed in Figures 5.2.1, 5.2.1 and 5.2.1, depending on whether $0 \in Z_-$, $0 \in Z_0$ or $0 \in Z_+$. The monotonicity of B (cf. Remark 5.2.5) ensures that $\rho B(\tau)$ can only pass through each zone at most once.

Theorem 5.2.8. *Let Assumption 5.2.7 hold. Then,*

$$B(\tau) = \begin{cases} \hat{B}^-(\tau)\mathbb{1}_{\{\tau \leq \tau_1^-\}} + \hat{B}_-(\tau - \tau_1^-)\mathbb{1}_{\{\tau_1^- < \tau \leq \tau_1^- + \tau_2^-\}} + \hat{B}_+^+(\tau - (\tau_1^- + \tau_2^-))\mathbb{1}_{\{\tau_1^- + \tau_2^- < \tau\}}, & \text{if } 0 \in Z_- \\ \hat{B}(\tau)\mathbb{1}_{\{\tau \leq \tau_1\}} + \hat{B}_-^-(\tau - \tau_1)\mathbb{1}_{\{\tau > \tau_1, \rho\hat{B}(\tau_1) = B_-\}} + \hat{B}_+^+(\tau - \tau_1)\mathbb{1}_{\{\tau > \tau_1, \rho\hat{B}(\tau_1) = B_+\}}, & \text{if } 0 \in Z_0 \\ \hat{B}^+(\tau)\mathbb{1}_{\{\tau \leq \tau_1^+\}} + \hat{B}_+(\tau - \tau_1^+)\mathbb{1}_{\{\tau_1^+ < \tau \leq \tau_1^+ + \tau_2^+\}} + \hat{B}_-^-(\tau - (\tau_1^+ + \tau_2^+))\mathbb{1}_{\{\tau_1^+ + \tau_2^+ < \tau\}}, & \text{if } 0 \in Z_+ \end{cases}$$

satisfies ODE (5.5) for $0 \leq \tau \leq T$.[5]

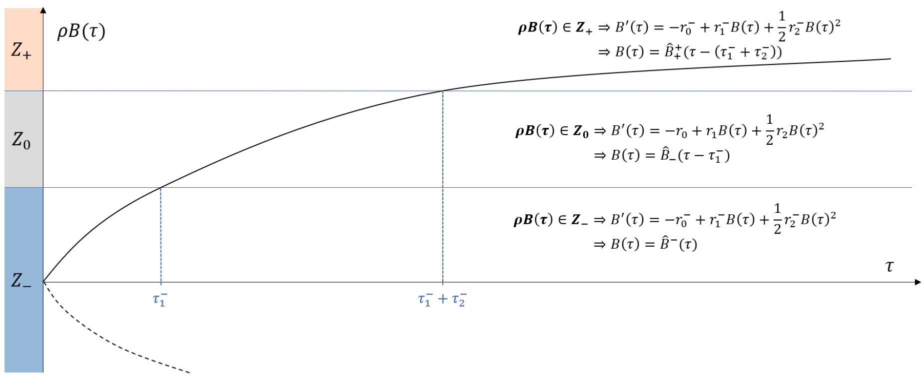


Figure 5.1: Illustration of possible trajectories of $\rho B(\tau)$, if B is a solution to ODE (5.5) and $0 \in Z_-$. The solid line represents an increasing trajectory that leaves Z_-, whereas the dashed line represents a decreasing trajectory that stays in Z_-.

[4]If $\rho = 0$ all of these transition times will be infinite.

[5]Using a similar separation with respect to the zones Z_-, Z_0, and Z_+ and equation (2.24), it is also possible to determine a closed-form expression for A from Lemma 5.2.2.

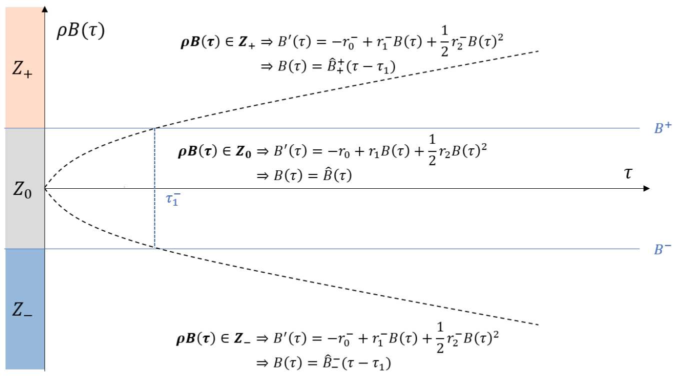


Figure 5.2: Illustration of possible trajectories of $\rho B(\tau)$, if B is a solution to ODE (5.5) and $0 \in Z_0$. Depending on the sign of $\rho B'(0) = -\rho r_0$, $\rho B(\tau)$ is increasing or decreasing. In particular, $\rho B(\tau)$ can only transition to Z_+ or Z_-, but not both.

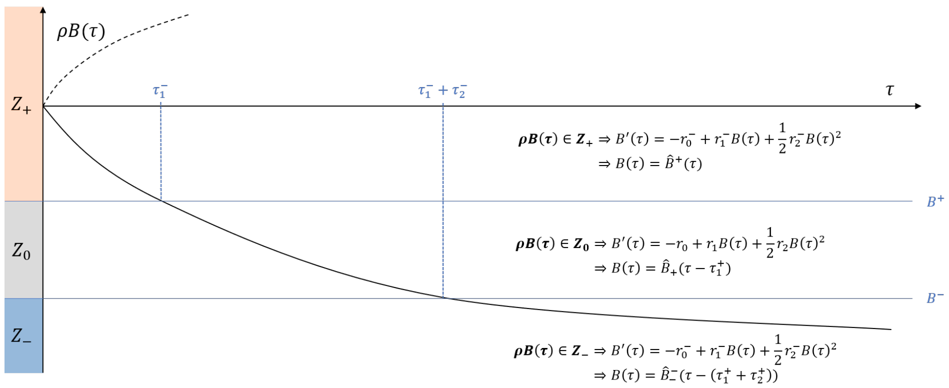


Figure 5.3: Illustration of possible trajectories of $\rho B(\tau)$, if B is a solution to ODE (5.5) and $0 \in Z_+$. The solid line represents a decreasing trajectory that leaves Z_+, whereas the dashed line represents an increasing trajectory that stays in Z_+.

5.2.2 Verification Theorem

Combining Remark 5.2.6 with Theorem 5.2.8 immediately yields a closed-form expression for the candidate optimal portfolio process π^*. It now just remains to prove a verification theorem which verifies that this candidate is indeed the optimal portfolio process corresponding to (**P**). This proof requires an additional assumption on the constraints $K = [\alpha, \beta]$, which ensures a certain boundedness of $\pi^*(t)$ for t close to maturity T as well as two auxiliary lemmas.

Assumption 5.2.9.

$$\max\left\{\frac{b\rho}{\kappa}\alpha,\ \frac{b\rho}{\kappa}\beta\right\} \leq \frac{\kappa}{\sigma^2}, \tag{5.11}$$

Lemma 5.2.10. *Let Assumptions 5.2.7 and 5.2.9 hold and let B be given as in Theorem 5.2.8. Then, the following inequality holds for all $t \in [0, T]$*

$$\frac{b\rho}{\sigma}\pi^*(t) + B(T-t) \leq \frac{\kappa}{\sigma^2}.$$

Lemma 5.2.11. *Let Assumptions 5.2.7 and 5.2.9 hold and let B be given as in Theorem 5.2.8. Then the following inequality holds for all $t \in [0,T]$*

$$\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-t)\right) + \sigma\rho B(T-t)\right)^2 - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2$$
$$+b\frac{\rho\kappa}{\sigma}\pi^*(t) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-t)\right) + \sigma\rho\right]B'(T-t) \quad < \frac{1}{2}\frac{\kappa^2}{\sigma^2}.$$

Theorem 5.2.12 (Verification Theorem in $\mathcal{M}_H$)**.**
Consider the financial market $\mathcal{M}_H$, let Assumptions 5.2.7 and 5.2.9 hold and let B be given as in Theorem 5.2.8. Then,

$$\pi^*(t) = \begin{cases} \alpha, & \rho B(T-t) < B_- \\ \frac{\eta + \sigma\rho B(T-t)}{1-b}, & B_- \leq \rho B(T-t) \leq B_+ \\ \beta, & B_+ < \rho B(T-t). \end{cases} \tag{5.12}$$

is optimal for $(\mathbf{P})$.

5.2.3 Comparison to Unconstrained Portfolio

Unsurprisingly, we can immediately recover the solution to the unconstrained optimization problem, as discussed in [56], from Lemma 5.2.3 and Lemma 5.2.4.

Corollary 5.2.13. *[Closed-form Unconstrained Optimal Portfolio as in [56]]*
Let $K = \mathbb{R}$ (i.e., $\alpha = -\infty$, $\beta = \infty$) and $B_u : [0,T] \to \mathbb{R}$ with $B_u(0) = 0$ satisfy

$$B_u'(\tau) = -r_0 + r_1 B_u(\tau) + \frac{1}{2}r_2 B_u(\tau)^2 \qquad \forall \tau \in [0,T]. \tag{5.13}$$

Then, $\lambda^*(B) = 0$ $\forall B \in \mathbb{R}$ and the candidate optimal portfolio π^* is given by

$$\pi_u(t) := \pi^*(t) = \frac{1}{1-b}\left(\eta + \sigma\rho B_u(T-t)\right).$$

Remark 5.2.14. *If the market parameters satisfy (cf. Assumption 5.2.7)*

$$\frac{b}{1-b}\eta\left(\frac{\kappa\rho}{\sigma} + \frac{\eta}{2}\right) < \frac{\kappa^2}{2\sigma^2}, \tag{5.14}$$

then

$$B_u(\tau) = \frac{2r_0(e^{r_3\tau} - 1)}{(r_1 - r_3)(e^{r_3\tau} - 1) - 2r_3} \tag{5.15}$$

and the optimality of π_u for the unconstrained portfolio optimization problem can be verified formally (see e.g. Theorem 5.3 in [56]).[6]

On an abstract level, when adding (allocation) constraints $K = [\alpha, \beta]$ to a portfolio optimization problem, the optimal constrained portfolio π^* for $(\mathbf{P})$ will be given by a projection $\mathcal{P}_K : \Lambda \to \Lambda_K$ which maps the optimal unconstrained portfolio π_u onto Λ_K, i.e.,

$$\pi^* = \pi_u + (\pi^* - \pi_u) =: \mathcal{P}_K(\pi_u).$$

[6] Equation (5.14) corresponds to part (i) of Assumption 5.2.7. In the setting of [56], part (ii) of Assumption 5.2.7 is also implied by (5.14) and so does not have to be mentioned explicitly.

In a Black-Scholes financial market $\mathcal{M}_{BS}$ with constant market coefficients (i.e., $\mathcal{M}_H$ with $\sigma = \kappa = \rho = 0$.), the optimal unconstrained portfolio is a constant-mix strategy $\pi_u(t) = \pi_M := \frac{1}{1-b}\eta$, the so-called 'Merton portfolio'. Setting $\sigma = \rho = 0$ and $B \equiv 0$ in Remark 5.2.6, one can easily see that the projection $\mathcal{P}_K = \mathcal{P}_K^{BS}$ in the Black-Scholes market simply caps off π_M at the boundaries if $\pi_M \notin K = [\alpha, \beta]$, i.e.,

$$\mathcal{P}_K^{BS}(\pi_M) = \text{Cap}\,(\pi_M, \alpha, \beta) = \begin{cases} \alpha, & \pi_M < \alpha \\ \pi_M, & \alpha \leq \pi_M \leq \beta \\ \beta, & \beta < \pi_M. \end{cases}$$

Given a solution B to (5.5) and considering Remark 5.2.6, it initially appears that the optimal constrained portfolio π^* in $\mathcal{M}_H$ can be obtained from the same projection. However, if $K \neq \mathbb{R}$, then B_u as in Corollary 5.2.13 and B as in Theorem 5.2.8 are solutions to possibly different ODEs. In particular, this implies that the portfolios π_u and $\hat{\pi}^*$ may not be identical, in which case the projection $\mathcal{P}_K^H$ for the Heston market does not necessarily coincide with the projection $\mathcal{P}_K^{BS}$ for the Black-Scholes market either. In other words, in a financial market with Heston stochastic volatility we in general have

$$\pi^* = \mathcal{P}_K^H(\pi_u) = \text{Cap}\big(\pi_u + \underbrace{(\hat{\pi}^* - \pi_u)}_{\neq 0}, \alpha, \beta\big) \neq \text{Cap}\big(\pi_u, \alpha, \beta\big) = \mathcal{P}_K^{BS}(\pi_u).$$

In the following, we render this observation more precise by providing both conditions under which $\mathcal{P}_K^H = \mathcal{P}_K^{BS}$ and conditions under which $\mathcal{P}_K^H \neq \mathcal{P}_K^{BS}$. The former case is true, whenever either $\rho = 0$ or $\pi_M \in K$.

Lemma 5.2.15. *Let π^* be as in Lemma 5.2.3, $\hat{\pi}^*$ be as in Remark 5.2.6 and π_u as in Corollary 5.2.13. If either*

$$\rho = 0 \qquad \textit{or} \qquad \pi_M \in K,$$

then

$$\pi^* = \mathcal{P}_K^H(\pi_u) = \textit{Cap}\,(\pi_u, \alpha, \beta) = \mathcal{P}_K^{BS}(\pi_u)\,.$$

If $\rho = 0$, the stochasticity of the volatility is completely unhedgeable in $\mathcal{M}_H$. As a consequence, the optimal unconstrained portfolio processes coincide in $\mathcal{M}_{BS}$ and in $\mathcal{M}_H$. Thus, the projections $\mathcal{P}_K^{BS}$ and $\mathcal{P}_K^H$ are identical if $\rho = 0$ too. In contrast, if $\rho \neq 0$, then the projections can only be different if the underlying ODE solutions B_u and B are different, specifically when π_u and $\hat{\pi}^*$ begin taking values inside K. This is the case if and only if π_u and $\hat{\pi}^*$ begin taking values inside K at different time points. This observation leads to an equivalent characterization of when the projections $\mathcal{P}_K^{BS}$ and $\mathcal{P}_K^H$ are different.

Lemma 5.2.16. *Let π^* be as in Lemma 5.2.3, $\hat{\pi}^*$ be as in Remark 5.2.6 and π_u as in Corollary 5.2.13. The following statements are equivalent:*

(i)

$$\pi^* = \mathcal{P}_K^H(\pi_u, \alpha, \beta) \neq \mathcal{P}_K^{BS}(\pi_u, \alpha, \beta) = \textit{Cap}(\pi_u, \alpha, \beta)$$

(ii)

$$\pi_M \notin [\alpha, \beta] \quad \textit{and} \quad \exists t \in (0, T) : \Big|\Big\{\hat{\pi}^*(t), \pi_u(t)\Big\} \cap (\alpha, \beta)\Big| = 1$$

We can construct an extreme case which satisfies the requirements of Lemma 5.2.16 by choosing α such that $B(\tau)$ is constant and choosing the market parameters such that π_u changes sufficiently during the investment horizon to ensure that $\pi_u(t^*) \in (\alpha, \beta)$ for some $t^* \in [0, T]$.

Corollary 5.2.17. *Let π^* be as in Lemma 5.2.3, $\hat{\pi}^*$ be as in Remark 5.2.6 and π_u as in Corollary 5.2.13. Let sign$(x) \in \{-1, 0, 1\}$ denote the sign of $x \in \mathbb{R}$.*

(i) If

$$0 < \pi_M = \frac{\alpha}{2} < \alpha \quad \text{and} \quad \alpha < \pi_u(t^*) < \beta \quad \text{for some } t^* \in [0, T],$$

then $B(\tau) = 0$ for all $\tau \in [0, T]$, $\pi^*(t) = \alpha$ for all $t \in [0, T]$ and $\pi^* = \mathcal{P}_K^H(\pi_u, \alpha, \beta) \neq \mathcal{P}_K^{BS}(\pi_u, \alpha, \beta) = Cap(\pi_u, \alpha, \beta)$.

(ii) If $\pi_M > \beta > 0$, then

$$sign\left(\frac{\partial}{\partial t}\hat{\pi}^*(t)\right) = sign\left(\frac{\partial}{\partial t}\pi_u(t)\right) = -sign(\rho b) \quad \forall t \in [0, T].$$

Hence, if in addition $b < 0$ and $\rho < 0$, then $\mathcal{P}_K^H = \mathcal{P}_K^{BS}$.

Clearly, the requirements on α in Corollary 5.2.17, (i) are quite restrictive, but they still provide a valuable insight into when we can expect to see a large difference between the projections $\mathcal{P}_K^{BS}$ and $\mathcal{P}_K^H$. Namely, if

- the optimal unconstrained portfolio π_u violates the constraint at maturity (i.e., $\pi_u(T) = \pi_M \notin K$) and there is sufficient change in $\pi_u(t)$ during the investment period such that $\pi_u(t^*) \in K$ for some $t^* \in K$.
- the derivatives of $B(\tau)$ and $B_u(\tau)$ are considerably different while $\pi_u \notin K$ (constant B being the extreme case).

As a matter of fact, we will later see in the numerical experiments in Section 5.4 that it is sufficient if $\alpha \approx 2\pi_M$ (i.e., $B(\tau)$ is nearly constant) to cause a considerable difference between the two projections.

As evidenced by the majority of empirical calibrations of Heston's stochastic volatility model to financial time series (see e.g. [31] for an overview), the parameter ρ is negative for most realistic applications. In the empirical study on risk preferences of mutual fund managers by [50], it is reported that the risk aversion parameter b has a median of $b = -1.43$ and a mean of $b = -4.8$. For more risk averse investors, such as insurance companies, reinsurance companies or pension funds, one can thus realistically assume negative values for b. Thus, for most realistic parameter configurations of $\mathcal{M}_H$ with $\pi_M > \beta$, the projections $\mathcal{P}_K^H$ and $\mathcal{P}_K^{BS}$ coincide for investors with a high degree of risk aversion (i.e., for a low value of b).

5.3 Implications for Related Models

In this section, we consider a generalized version of the financial market $\mathcal{M}_H$ with $d \in \mathbb{N}$ risky assets, $m = d$ independent CIR processes as risk drivers and a generalized dependence of market price of risk and risky asset volatility on these risk drivers. Assume that we are given functions $\gamma : (0, \infty)^d \to \mathbb{R}^d$, $\Sigma : (0, \infty)^d \to \mathbb{R}^{d \times d}$, and parameters κ, θ, σ, $z_0 \in (0, \infty)^d$ such that their components satisfy $2\kappa_i\theta_i > \sigma_i^2$ for $i = 1, ..., d$. Then, we define the financial market $\mathcal{M}_H^{\gamma,\Sigma}$, as the stochastic factor model (4.3) with functions

$$\mu^z(t, z) = \kappa \odot (\theta - z),\ \Sigma^z(t, z) = \begin{pmatrix} \sqrt{z_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{z_m} \end{pmatrix},\ \rho(t, z) = \rho,$$

$$r(t, z) = r,\ \mu(t, z) = r(t, z)\mathbb{1} + \Sigma(z)\gamma(z),\ \Sigma(t, z) = \Sigma(z).$$

Under these assumptions, the components of the stochastic factor $z = (z_1, ..., z_d)'$ are independent and satisfy the SDE

$$dz_i(t) = \kappa_i \left(\theta_i - z_i(t)\right) dt + \sigma_i \sqrt{z_i(t)} dW_i^z(t), \quad \text{for } i = 1, ..., d.$$

Further, $\mathcal{M}_H^{\gamma,\Sigma}$ consists of one risk-free asset P_0 and d risky assets $P = (P_1, \ldots, P_d)'$ with dynamics

$$\begin{aligned} dP_0(t) &= P_0(t) r dt \\ dP(t) &= P(t) \odot \left[\left(r\mathbb{1} + \Sigma(z(t))\gamma(z(t))\right) dt + \Sigma(z(t)) dW(t) \right]. \end{aligned}$$

Clearly, we can recover the financial market $\mathcal{M}_H$, as considered in Section 5.2, by assuming $d = 1$ and choosing $\gamma(z) = \eta\sqrt{z}$ and $\Sigma(z) = \sqrt{z}$ and the Black Scholes model $\mathcal{M}_{BS}$ if both γ and Σ are constants. Similar, but slightly more general financial market models than $\mathcal{M}_H^{\gamma,\Sigma}$ have been considered in [33] or [61], for example. In $\mathcal{M}_H^{\gamma,\Sigma}$, the wealth process $V^{v_0,\pi}$ of an investor with initial wealth v_0 who trades continuously in time with $\mathbb{R}^d$-valued relative portfolio process $\pi \in \Lambda$, satisfies the SDE

$$dV^{v_0,\pi}(t) = V^{v_0,\pi}(t) \left[\left(r + \gamma(z(t))'\Sigma(z(t))'\pi(t)\right) dt + \pi(t)'\Sigma(z(t)) dW(t) \right]$$

and the allocation constrained portfolio optimization problem $(\mathbf{P})$ in $\mathcal{M}_H^{\gamma,\Sigma}$ is then defined as

$$(\mathbf{P}) \begin{cases} \Phi(v_0) & = \sup_{\pi \in \Lambda'} \mathbb{E}\left[U(V^{v_0,\pi}(T))\right] \\ \Lambda' & = \left\{ \pi \in \Lambda \;\middle|\; \pi(t) \in K \; \mathcal{L}[0,T] \otimes Q - \text{a.e.} \right\}. \end{cases}$$

In the following two sections, we investigate the solvability of $(\mathbf{P})$ for given choices of γ, Σ, and K. In Section 5.3.1, we consider the PCSV Model, as discussed in [30], and in Section 5.3.2, we consider inverse volatility constraints K, which impose stronger restrictions on an investor's portfolio during periods of high volatility.

5.3.1 PCSV Model

We recover the Principal Component Stochastic Volatility model ('PCSV model') $\mathcal{M}^{PCSV}$, as proposed in [37], from $\mathcal{M}_H^{\gamma,\Sigma}$ by considering an orthogonal matrix $A = \left(a_1, ..., a_d\right) \in \mathbb{R}^{d \times d}$, with columns $a_1, ..., a_d$, and defining market price of risk and volatility as

$$\gamma(z) = \text{diag}(\sqrt{z})A'\eta = \Sigma(z)'\eta, \quad \Sigma(z) = A\text{diag}(\sqrt{z}) \quad \forall z \in (0, \infty)^d, \tag{5.16}$$

where $\text{diag}(x) \in \mathbb{R}^{d \times d}$ denotes the diagonal matrix with entries $x \in \mathbb{R}^d$, and $\eta \in \mathbb{R}^d$ is a constant.

If $A = I_d$, then $\mathcal{M}^{PCSV}$ can be regarded as a canonical generalization of Heston's stochastic volatility model $\mathcal{M}_H$ for a d-dimensional asset universe, where each risky asset's volatility is determined as the square root of one of the d independent CIR processes z_i. However, in its general form, the independent components of the d-dimensional CIR process z are not directly regarded as volatilities. Instead, the instantaneous covariance matrix of risky asset returns

$$\Sigma(z(t))\Sigma(z(t))' = A\text{diag}(z(t))A'$$

is decomposed into its principal components, i.e., the columns a_i of the matrix A represent its eigenvectors, and the independent CIR processes z_i represent their (stochastic) eigenvalues. This approach not only enables the modelling of stochastic covariances of asset returns because of additional degrees of freedom in A, but also allows for an interpretation of z as hidden risk

factors determining the volatility level in the financial market. Moreover, [30] demonstrated that several stylized facts are captured by the PCSV model, such as stochasticity of volatilities and correlation of risky asset returns, volatility and correlation leverage effect, volatility spillovers, and increasing correlation in periods of high market volatility.
Let Λ^{PCSV} be the set of admissible portfolios in $\mathcal{M}^{PCSV}$. Then, for any $\pi \in \Lambda^{PCSV}$, the wealth process $V^{v_0,\pi}$ satisfies the SDE

$$dV^{v_0,\pi}(t) = V^{v_0,\pi}(t)\left[\left(r + \eta' A\text{diag}(z(t))A'\pi(t)\right)dt + \pi(t)' A\text{diag}\left(\sqrt{z(t)}\right)dW(t)\right],$$

for $t \in [0,T]$. The instantaneous variance of $V^{v_0,\pi}(t)$ can therefore be decomposed into a weighted sum of the risk factors z, since

$$\left\|\text{diag}(\sqrt{z(t)})A'\pi(t)\right\| = \left\|\begin{pmatrix} a_1'\pi(t)\sqrt{z_1(t)} \\ \vdots \\ a_d'\pi(t)\sqrt{z_d(t)}\end{pmatrix}\right\|^2 = \sum_{i=1}^{d}\left(a_i'\pi(t)\right)^2 z_i(t).$$

In this sense, the portfolio weights determine a risk exposure $\left(a_i'\pi(t)\right)^2$ to the risk factor z_i. Hence, it is very natural to impose risk limits on these exposures, i.e., for given upper bounds $\beta_1, ..., \beta_d > 0$ we require that

$$\left(a_i'\pi(t)\right)^2 \le \beta_i \quad \forall i = 1, ..., d \quad \Leftrightarrow \quad A'\pi(t) \in \bigtimes_{i=1}^{d}[0, \sqrt{\beta_i}] \quad \Leftrightarrow \quad \pi(t) \in \underbrace{A \cdot \left(\bigtimes_{i=1}^{d}[0, \sqrt{\beta_i}]\right)}_{=:K_{PCSV}}.$$

We can reuse the ideas and results from Section 5.2 to obtain the optimal portfolio to the portfolio optimization problem $(\mathbf{P})$ in $\mathcal{M}_H^{\gamma,\Sigma} = \mathcal{M}_{PCSV}$ with constraints $K = K_{PCSV}$.

Theorem 5.3.1. *Consider the portfolio optimization problem* $(\mathbf{P})$ *in* $\mathcal{M}_{PCSV}$ *with constraints* $K = K_{PCSV}$*, let the parameters* b, $(\eta_A)_i := (A'\eta)_i$, κ_i, θ_i, σ_i, $\alpha_i = 0$, $\sqrt{\beta_i}$ *satisfy Assumptions 5.2.7 and 5.2.9 and* B_i *be defined as in Theorem 5.2.8. Define the portfolio* $\pi_A^*(t)$ *via*

$$\left(\pi_A^*(t)\right)_i = Cap\left(\frac{1}{1-b}\left(\eta_i + \sigma\rho B_i(T-t)\right), 0, \sqrt{\beta_i}\right).$$

Then the portfolio $\pi^*(t) = A \cdot \pi_A^*(t)$ *is optimal for* $(\mathbf{P})$.

The key argument in the proof of Theorem 5.3.1 lies in a change of control, which transforms the portfolio optimization problem $(\mathbf{P})$ into an equivalent portfolio optimization problem $(\mathbf{P_A})$ in a financial market, which consists of d risky assets with independent Heston volatilities and interval constraints. Thanks to this familiar structure and the independence of the risky asset volatilities, we can extend the ideas from Section 5.2 to solve $(\mathbf{P_A})$ and invert the change of control to obtain a solution to $(\mathbf{P})$.

5.3.2 Inverse Volatility Constraints

We now discuss a related problem, which we approach with a similar methodology to the one in Section 5.3.1. Consider again the one-dimensional setting with one risky asset, i.e., $d = 1$. In this section, we no longer assume that the convex constraints $K \subset \bar{\mathbb{R}}$ are static, but allow them to depend on the stochastic factor z. More specifically, we consider volatility-dependent constraints of the form

$$\pi(t) \in K(z(t)) \quad \mathcal{L}[0,T] \otimes Q - a.e.,$$

where $K : (0, \infty) \to \mathcal{B}(\mathbb{R})$ is a set-valued function, taking only closed-convex values in the Borel set $\mathcal{B}(\mathbb{R})$. The motivation for such constraints is quite clear: Depending on the current state of the financial market $\mathcal{M}_H^{\gamma,\Sigma}$, in particular the level of risky asset volatility $\Sigma(z(t))$ and the market price of risk $\gamma(z(t))$, investors may face different constraints on their portfolio, such as more relaxed bounds in periods of low volatility or stricter bounds in periods of high volatility. Further, in the spirit of mean-variance optimization, we can think of an investor seeking an optimal portfolio allocation subject to constraints on his instantaneous portfolio volatility

$$0 \leq \pi(t)\Sigma(z(t)) \leq \beta_z \quad \mathcal{L}[0,T] \otimes Q - a.e. \quad \Leftrightarrow \quad 0 \leq \pi(t) \leq \frac{\beta_z}{\Sigma(z(t))} \quad \mathcal{L}[0,T] \otimes Q - a.e.,$$

for a given volatility level $\beta_z > 0$.[7] Keeping this motivation in mind, we thus define the portfolio optimization problem with volatility-dependent constraints ($\mathbf{P^z}$) as

$$(\mathbf{P^z}) = \begin{cases} \Phi^z(v_0) & = \sup_{\pi \in \Lambda_{K(\cdot)}} \mathbb{E}\left[U(V^{v_0,\pi}(T))\right] \\ \Lambda' & = \left\{\pi \in \Lambda \;\middle|\; \pi(t) \in K(z(t))\ \mathcal{L}[0,T] \otimes Q - \text{a.e.}\right\}. \end{cases}$$

In its most general form, the portfolio optimization ($\mathbf{P^z}$) is highly non-trivial, since closed-form solutions for its optimal portfolio process π_z^* can rarely be determined for general γ and Σ, even in the absence of (stochastic) allocation constraints. In particular, the portfolio optimization ($\mathbf{P}$) is included as a special case in the definition of ($\mathbf{P^z}$). However, due to the results of [17] for the Black-Scholes model $\mathcal{M}_{BS}$ with constant volatility, as well as the results from Section 5.2 for Heston's stochastic volatility model $\mathcal{M}^H$, we know of at least two different models in which solutions to ($\mathbf{P}$) (respectively ($\mathbf{P^z}$)) with static constraints can be obtained in closed form. Using another change of control argument, we can therefore derive conditions on the market parameters γ, Σ and the constraints K, under which we can transform ($\mathbf{P^z}$) into an equivalent, solvable portfolio optimization problem ($\mathbf{P}$) in either $\mathcal{M}_{BS}$ or $\mathcal{M}^H$.

Theorem 5.3.2. *Consider the financial market $\mathcal{M}_H^{\gamma,\Sigma}$ and the portfolio optimization problem ($\mathbf{P^z}$) and constants $-\infty \leq \alpha < \beta \leq \infty$.*

(i) If $\gamma(z) = \eta$ for some $\eta > 0$ and $K(z) = \frac{1}{\Sigma(z)}[\alpha_z, \beta_z]$, then the portfolio process

$$\pi_z^*(t) = \frac{1}{\Sigma\left(z(t)\right)} Cap(\pi_M, \alpha, \beta)$$

is optimal for ($\mathbf{P^z}$).

(ii) If $\gamma(z) = \eta\sqrt{z(t)}$ for some constant $\eta > 0$, $K(z) = \frac{\sqrt{z}}{\Sigma(z)}[\alpha, \beta]$, and Assumptions 5.2.7 and 5.2.9 are satisfied, then for π^* defined as in Theorem 5.2.12, the portfolio process

$$\pi_z^*(t) = \frac{\sqrt{z(t)}}{\Sigma\left(z(t)\right)}\pi^*(t)$$

is optimal for ($\mathbf{P^z}$).

The statements of Theorem 5.3.2 can be easily generalized to financial markets with $d > 1$ risky assets by an analogous change of control argument. Using the results for constant volatility markets from Example 15.2 in [17], one can prove a multi-dimensional analogue to statement (i) and using the results for the PCSV model from Section 5.3.1, one can prove a multi-dimensional analogue to statement (ii). For ease of presentation, we refrain from a detailed discussion of this generalization.

[7] Note that this is different from classic mean-variance optimization, where the variance of the terminal portfolio wealth $V^{v_0,\pi}(T)$ is constrained.

5.4 Numerical Studies

In this section, we illustrate the properties of the optimal portfolio π^* for $(\mathbf{P})$ in Heston's stochastic volatility model $\mathcal{M}_H$, using a numerical example. In particular, we analyze the difference between π^* and two suboptimal 'naive' portfolio processes π, which either directly follow the optimal portfolio process in $\mathcal{M}_{BS}$ (i.e., $\pi = \mathrm{Cap}(\pi_M, \alpha, \beta)$) or apply the projection $\mathcal{P}_K^{BS}$ from $\mathcal{M}_{BS}$ to the optimal unconstrained portfolio π_u in $\mathcal{M}_H$ (i.e., $\pi = \mathrm{Cap}(\pi_u, \alpha, \beta)$). The suboptimality of these portfolios is quantified using the concept of wealth-equivalent loss.

Such an analysis is only meaningful if the differences between a financial market with stochastic (Heston) volatility $\mathcal{M}_H$ and a financial market with constant volatility $\mathcal{M}_{BS}$ are already reflected in the optimal unconstrained portfolios π_u for $\mathcal{M}_H$ and π_M for $\mathcal{M}_{BS}$. Since allocation constraints further restrict the set of admissible portfolio allocations, any existing differences between π_u and π_M tend to be diminished further when adding allocation constraints. From an investor's perspective, the distinction between $\mathcal{M}_{BS}$ and $\mathcal{M}_H$ is only relevant if the volatility of risky asset log returns $\sqrt{z(t)}$ changes significantly and these changes are partially hedgeable through trading in the risky asset. This is the case if the volatility of the volatility (σ) is large, the mean reversion speed (κ) is small, and the correlation between risky asset and volatility diffusion (ρ) is close to either 1 or -1 (i.e., $|\rho|$ is large).
Based on these requirements, we choose the market parameters (see Table 5.1) for our numerical example such that the resulting market dynamics resemble a financial crisis. The only volatility

Parameter		Value	Explanation
End of Investment-Horizon	T	1	Limited duration of financial crises
Risk Aversion Parameter	b	-2.5	Within ranges estimated in Table 1, [50]
Initial Wealth	v_0	1	For convenience
Risk-Free Interest Rate	r	0	For convenience
Market Price of Risk Driver	η	3.0071	Table 2, [13]
Mean Reversion Speed	κ	3.15	Table 3 '%MSE', [69]
Volatility of Volatility	σ	0.76	Table 3 '%MSE', [69]
Correlation	ρ	-0.81	Table 3 '%MSE', [69]
Long-Term Mean	θ	0.35	Feller's Condition
Initial Variance	z_0	0.35	Chosen equal to θ

Table 5.1: Base parameters for the financial market $\mathcal{M}_H$.

parameters which influence the optimal portfolio allocation are σ, κ, and ρ. Our choices for these parameters in Table 5.1, follow the calibration results of [69], who calibrated Heston's stochastic volatility model using option prices on the Eurostoxx 50 during the 2008 financial crisis. The relatively short investment horizon of $T = 1$ year is chosen to reflect the limited duration of most financial crises.[8]
We quantify the sub-optimality of both naive portfolio processes in comparison to the optimal constrained portfolio process using the concept of wealth-equivalent loss ('WEL'). For an arbitrary portfolio process $\pi \in \Lambda_K$, we define the expected utility functional $J^\pi : [0,T] \times (0,\infty) \times (0,\infty) \to \mathbb{R}$ as

$$J^\pi(t, v, z) = \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right) \mid V^{v_0,\pi}(t) = v,\ z(t) = z\right]. \tag{5.17}$$

When considering the optimal portfolio process π^*, the expected utility functional coincides with the value function of $(\mathbf{P})$, i.e., $J^{\pi^*}(t, v, z) = \Phi(t, v, z)$ for all $(t, v, z) \in [0,T] \times (0,\infty) \times (0,\infty)$.

[8] [75] reported that the average length of an S&P500 bear market (defined as a period with drawdown in excess of 20%) was 289 days.

The WEL $L^\pi = L^\pi(t,z)$ of π is then defined as the solution to the equation[9]

$$\Phi(t, v(1 - L^\pi(t,z)), z) = J^\pi(t,v,z).$$

An investor following the optimal portfolio allocation π^* only needs $(1 - L^\pi(t,z))$ as much capital to achieve the same average utility as an investor following the sub-optimal strategy π. In this sense, $L^\pi(t,z)$ can be interpreted as a relative loss incurred for investing sub-optimally. If π is deterministic and J^π is the unique solution to the associated Feynman-Kac PDE, one can use an exponentially affine ansatz to characterise J^π in terms of the solutions to a system of ODEs. If the ODE solutions are given, then the WEL $L^\pi(0, z_0)$ is known in closed form. We provide a description of this approach in Lemma 5.4.1 and Corollary 5.4.2 below. In our studies, we approximated the corresponding ODE solutions by an Euler method.

Lemma 5.4.1. *Consider a bounded deterministic relative portfolio process $\pi \in \Lambda_K$. Let $A_\pi,\ B_\pi : [0,T] \to \mathbb{R}$ with $A_\pi(0) = B_\pi(0) = 0$ be solutions to the system of ODEs*

$$A'_\pi(T-t) = rb + \kappa\theta B_\pi(T-t) \tag{5.18}$$

$$B'_\pi(T-t) = -\left[\frac{1}{2}\pi(t)^2 b(1-b) - b\eta\pi(t)\right] + \left[\sigma\rho b\pi(t) - \kappa\right] B_\pi(T-t) + \frac{1}{2}\sigma^2 B_\pi(T-t)^2. \tag{5.19}$$

If J^π (as defined in (5.17)) is the unique solution to the Feynman-Kac-PDE (omitting the argument (t,v,z) for readability)

$$0 = J^\pi_t + (r + \eta\pi(t)z)\, vJ^\pi_v + \kappa\,(\theta - z)\, J^\pi_z + \sigma\rho\pi(t)zvJ^\pi_{zv} + \frac{1}{2}v^2\pi(t)^2 zJ^\pi_{vv} + \frac{1}{2}\sigma^2 zJ^\pi_{zz}, \tag{5.20}$$

with boundary condition $J^\pi(T,v,z) = U(v) = \frac{1}{b}v^b$, then

$$J^\pi(t,v,z) = \frac{1}{b}v^b \exp\left(A_\pi(T-t) + B_\pi(T-t)z\right).$$

Corollary 5.4.2. *Let $\pi \in \Lambda_K$ be a deterministic portfolio process. Let $A_\pi,\ B_\pi$ be as in Lemma 5.4.1 and $A,\ B$ as in Lemma 5.2.2. Then, the wealth-equivalent loss L^π is given as*

$$L^\pi(t,z) = 1 - \exp\left(\frac{1}{b}\left(A_\pi(T-t) - A(T-t) + \left[B_\pi(T-t) - B(T-t)\right] z\right)\right)\ \forall (t,z) \in [0,T] \times (0,\infty).$$

5.4.1 Optimal Constrained Merton Portfolio $\pi = \mathbf{Cap}\,(\pi_M, \alpha, \beta)$

In this subsection, we compare π^* with the first naive portfolio process $\pi = \mathrm{Cap}(\pi_M, \alpha, \beta)$. Although π is static, it is at least known that π is optimal for an allocation constrained portfolio optimization problem in the Black-Scholes market $\mathcal{M}_{BS}$, whereas no theoretical guarantees were available for the corresponding optimization problem in $\mathcal{M}_H$ prior to this thesis. During this analysis, we consider the allocation constraint $K = [\alpha, \beta] = [0, 1]$, which corresponds to a no-borrowing constraint that prevents short-selling in the risk-free and risky asset.

[9] Since we exclusively work with power utility functions in this chapter, we may without loss of generality assume that the WEL is independent of wealth.

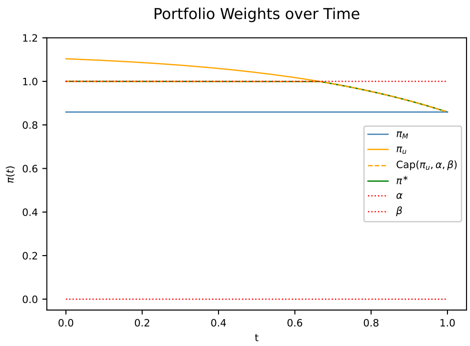
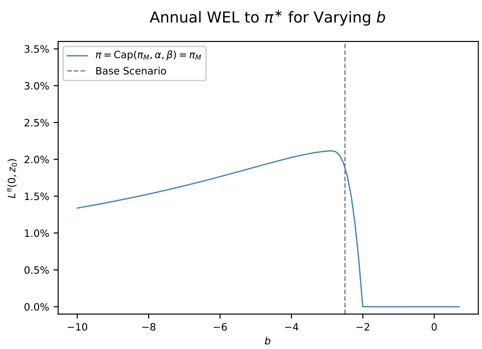


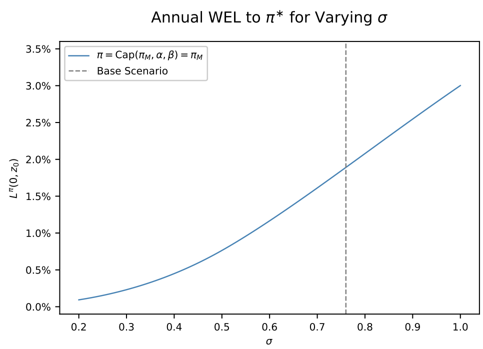
Figure 5.4: Portfolio weights $\pi(t)$ for $t \in [0, T]$, lower bound $\alpha = 0$, upper bound $\beta = 1$, and parameters as in Table 5.1.

Assuming the parameters in Table 5.1, the Merton portfolio satisfies the constraints, i.e., $\pi_M \in [\alpha, \beta]$ and thus $\text{Cap}(\pi_M, \alpha, \beta) = \pi_M$. In contrast to this constant allocation in the interior of $[\alpha, \beta]$, π^* initially takes a constant value at the upper bound β and later decreases towards π_M at the end of the investment horizon. Therefore, other than $\text{Cap}(\pi_M, \alpha, \beta)$, π^* is able to realise and benefit from a higher allocation to the risky asset. Note that $\pi_M \in [\alpha, \beta]$ implies $\pi^* = \text{Cap}(\pi_u, \alpha, \beta)$, as shown in Lemma 5.2.15.

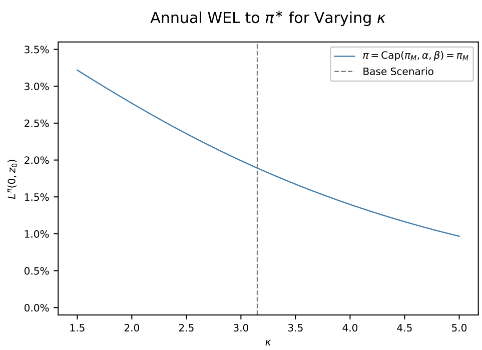
In the following, we quantify the impact of the suboptimal allocations $\pi = \text{Cap}(\pi_M, \alpha, \beta)$ by computing the annual WEL $L^\pi(0, z)$ at the beginning of the investment horizon and analyze its sensitivity with respect to the risk-aversion parameter b and the volatility drivers σ, κ and ρ. The ranges of the volatility parameter are chosen to be within the minimum and maximum parameter values obtained in individual calibrations in Table 5 of [69].



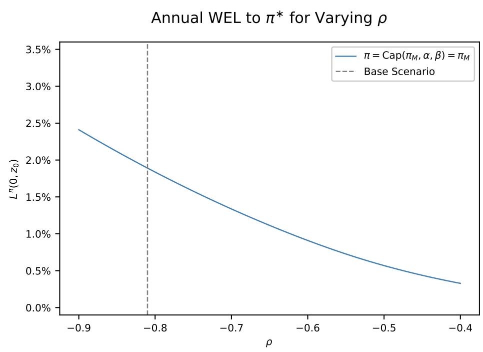
(a)



(b)



(c)



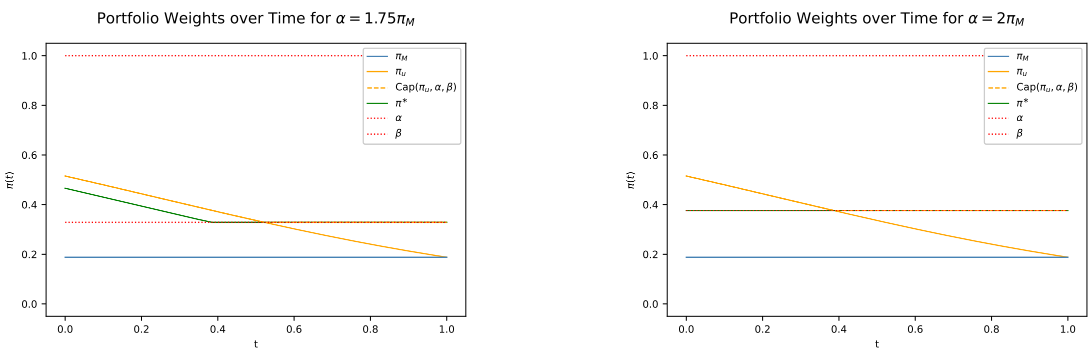
(d)

Figure 5.5: Dependence of the annual wealth-equivalent loss $L^{\pi}(0, z_0)$ on individual parameters for $\pi = \mathrm{Cap}(\pi_M, \alpha, \beta) = \pi_M$, lower bound $\alpha = 0$, upper bound $\beta = 1$, and parameters as in Table 5.1. Figure 5.5(a) illustrates the dependence on $b \in [-10, 0.7]\backslash\{0\}$, whereas Figures 5.5(b), 5.5(c) and 5.5(d) display the dependence on $\sigma \in [0.2, 1.0]$, $\kappa \in [1.5, 5.0]$ and $\rho \in [-0.9, -0.4]$.

For small values of b, where the allocation constraint $K = [0, 1]$ is largely satisfied by the unconstrained portfolios π_M and π_u, the WELs displayed in Figure 5.5(a) are increasing in b. However, as b increases past an inflection point of approximately $b = -3$, the allocation constraint K becomes active. From this point onwards, K forces π^* and $\mathrm{Cap}(\pi_M, \alpha, \beta)$ closer towards each other for increasing b and therefore leads to decreasing WELs. Ultimately, for $b \geq -2$, we have $\pi^*(t) = \mathrm{Cap}(\pi_M, \alpha, \beta) = \beta = 1$ for all $t \in [0, T]$ and thus the WEL is zero. Figures 5.5(b), 5.5(c) and 5.5(d) display WELs which are increasing in σ as well as decreasing in κ and ρ. This confirms the intuition voiced at the beginning of Section 5.4, in which we argued that $\mathcal{M}_H$ is 'more different' to $\mathcal{M}_{BS}$ under these circumstances. Within the chosen parameter ranges, we observe the largest annual WEL of 3.2% for small κ, whereas increasing σ and decreasing ρ leads to WELs of 3.0% and 2.5%, respectively. Changing any one of the volatility parameters σ, κ or ρ to less extreme levels, which are obtained during calibrations on long-term data sets[10] leads to significant decreases in annual WELs. Even if only $\sigma < 0.5$, while κ and ρ remain at crisis level, the annual WEL still drops to values of 0.75% or lower. Note that Feller's condition is satisfied for all parameter values that were considered in our analysis.

5.4.2 Capped Optimal Unconstrained Heston Portfolio $\pi = \mathrm{Cap}(\pi_u, \alpha, \beta)$

In this subsection, we compare π^* to the second naive portfolio process, the capped optimal unconstrained Heston portfolio $\mathrm{Cap}(\pi_u, \alpha, \beta)$. In particular, we aim to illustrate the phenomenon described in Section 5.2.3. Despite a theoretical guarantee that $\mathrm{Cap}(\pi_u, \alpha, \beta)$ is indeed different from π^* for certain parameter settings, these differences appear to be mostly meaningful in terms of WEL for extreme market scenarios in combination with specific, large lower bounds α. According to Lemma 5.2.16 and Corollary 5.2.17, we know that $\mathrm{Cap}(\pi_u, \alpha, \beta)$ and π^* are identical in the parameter setting of Table 5.1, unless $\pi_M < \alpha$. Further, Corollary 5.2.17 suggests that we should consider market parameters which ensure $\pi_u(t) > 2\pi_M$ for some $t \in [0, T]$. For these reasons, we adjust the previously considered parameter setting. In the following, we choose the most extreme volatility parameters from the sensitivity analysis in Figure 5.5, i.e., we set $\sigma = 1.0$, $\kappa = 1.5$ and $\rho = -0.9$, and increase the risk aversion coefficient to $b = -15$ to obtain realistic portfolio allocations.



(a) (b)

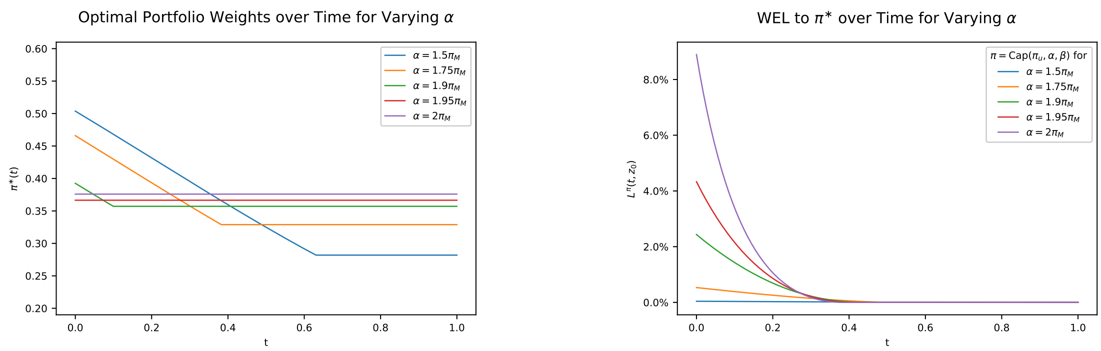
[10] For an overview, consider e.g. Table 4 in [31]. Here, the authors consider values of $\kappa = 3.5$, $\sigma = 0.3$, $\rho = -0.4$ as an 'Average Case' for their reviewed literature.

Figure 5.6: Portfolio weights $\pi(t)$ for $t \in [0, T]$, upper bound $\beta = 1$, and parameters as in Table 5.1, except for $b = 15$, $\sigma = 1.0$, $\kappa = 1.5$ and $\rho = -0.9$. Figure 5.6(a) considers a lower bound $\alpha = 1.75\pi_M$, and Figure 5.6(b) considers a lower bound $\alpha = 2\pi_M$.

In Figure 5.6, we compare the portfolio weights of $\pi^*(t)$ and $\mathrm{Cap}(\pi_u(t), \alpha, \beta)$ for lower bounds $\alpha \in \{1.75\pi_M, 2\pi_M\}$ such that the corresponding ODE solution B is (nearly) constant, as described in Corollary 5.2.17. If $\alpha = 1.75\pi_M$ then π^* is initially larger than the lower bound α, then decreases until α is reached, while π^* is constant for $\alpha = 2\pi_M$. Additionally, we observe that

$$\pi^*(t) \leq \mathrm{Cap}(\pi_u(t), \alpha, \beta) \quad \forall t \in [0, T],$$

with equality only if $\pi_u(t) \leq \alpha$. In both cases illustrated in Figure 5.6, π^* lowers the portfolio allocation early throughout the investment horizon, thus accounting for the fact that the lower bound forces the portfolio allocation to be larger than the optimal unconstrained allocation later during the investment horizon.



(a) (b)

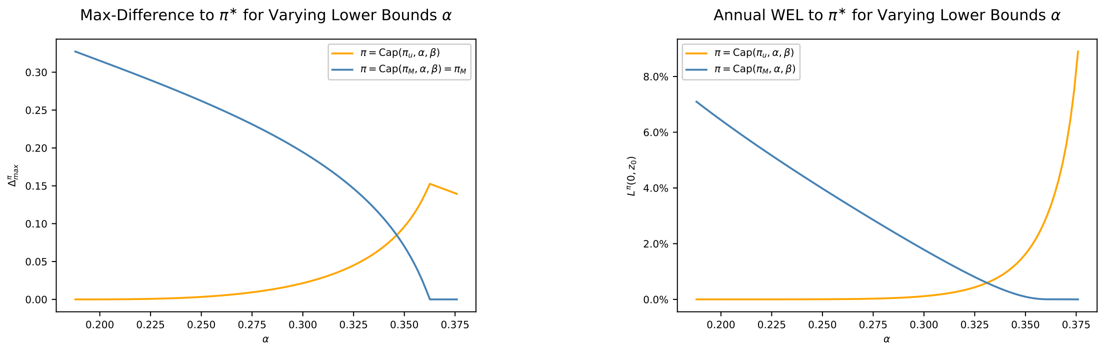
Figure 5.7: Lower bounds $\alpha \in \{1.5\pi_M, 1.75\pi_M, 1.9\pi_M, 1.95\pi_M, 2\pi_M\}$, upper bound $\beta = 1$, $t \in [0, T]$, and parameters as in Table 5.1, except for $b = 15$, $\sigma = 1.0$, $\kappa = 1.5$ and $\rho = -0.9$. Figure 5.7(a) displays the portfolio weights $\pi^*(t)$, and Figure 5.7(b) displays the WEL $L^{\pi}(t, z_0)$ for $\pi = \mathrm{Cap}(\pi_u, \alpha, \beta)$.

Figure 5.7(a) illustrates the behaviour of π^* for varying α. When increasing α, we observe that π^* decreases to the lower bound at earlier time points t. Despite π^* being constant for $\alpha = 1.95\pi_M$, the solution B to the ODE (5.9) for B is not stationary, but $\rho B(\tau)$ does not leave zone Z_- for $\tau \leq T$. Thus, we expect π^* to be constant for all $\alpha \in [1.95\pi_M, 2\pi_M]$ in our parameter setting. Figure 5.7(a) displays WELs $L^{\pi}(t, z_0)$ of $\pi = \mathrm{Cap}(\pi_u, \alpha, \beta)$, which are increasing in α at $t = 0$. However, this monotonicity does not hold throughout the entire investment horizon, as increasing the lower bound α implies that π^* and $\mathrm{Cap}(\pi_u, \alpha, \beta)$ coincide for longer parts of the investment horizon.

Clearly, Figures 5.6 and 5.7 suggest a strong link between the value of α and the difference between π^* and $\mathrm{Cap}(\pi_u, \alpha, \beta)$. Therefore, we quantify this difference not only using WELs, but additionally define the maximum absolute weight difference between π^* and a portfolio π as

$$\Delta^{\pi}_{\max} := \max_{t \in [0,T]} \Big| \pi(t) - \pi^*(t) \Big|. \tag{5.21}$$

The relationship between the lower bound α and the maximum absolute difference $\Delta^{\pi}_{\max}$ and the annual WEL $L^{\pi}(0, z_0)$ are analyzed in Figure 5.8.



(a) (b)

Figure 5.8: Lower bounds $\alpha \in [\pi_M, 2\pi_M]$, upper bound $\beta = 1$, $\pi = \text{Cap}\,(\pi_u, \alpha, \beta)$, and parameters as in Table 5.1, except for $b = 15$, $\sigma = 1.0$, $\kappa = 1.5$ and $\rho = -0.9$. For $\pi = \text{Cap}\,(\pi_u, \alpha, \beta)$, Figure 5.8(a) displays the maximum absolute difference Δ^{π} to the portfolio weights $\pi^*(t)$, and Figure 5.8(b) displays the WEL $\text{Ł}^{\pi}(0, z_0)$ to π^*.

For $\pi = \text{Cap}(\pi_u, \alpha, \beta)$, the maximum absolute difference $\Delta^{\pi}_{\max}$ is generally increasing with α except for larger lower bounds α. For large α, the optimal constrained portfolio π^* is constant throughout the investment horizon, as illustrated in Figure 5.7(a). Then, the monotonicity of π_u (see Remark 5.2.5) and $\pi^*(T) = \alpha = \text{Cap}(\pi_u(T), \alpha, \beta)$ ensure that the maximum in (5.21) is attained at $t = 0$. Further, $\pi_u(0) > 2\pi_M \geq \alpha$ does not depend on α. Therefore, if $\pi^*(t) = \alpha$ for all $t \in [0, T]$, then the difference

$$\Delta^{\text{Cap}(\pi_u,\alpha,\beta)}_{\max} = \max_{t\in[0,T]} \Big|\text{Cap}(\pi_u(t), \alpha, \beta) - \pi^*(t)\Big| = \Big|\pi_u(0) - \alpha\Big| = \pi_u(0) - \alpha$$

decreases linearly in α, which causes a slight kink in Figure 5.8(a) for large lower bounds α. Irrespectively, the annual WEL for $\pi = \text{Cap}(\pi_u, \alpha, \beta)$ is increasing with α. However, note that for all but very large lower bounds (e.g. $\alpha \geq 1.75\pi_M$), the annual WEL is still negligible.

5.5 Conclusion

In this chapter, we considered a portfolio optimization problem with allocation constraints in Heston's stochastic volatility model. We derived an explicit expression for the optimal portfolio and analyzed its properties. Surprisingly, this portfolio can be different from the naive constrained portfolio which caps off the optimal unconstrained portfolio at the boundaries of the constraint. In light of this fact, we have shown that the addition of allocation constraints can have a fundamentally different impact on the optimal portfolio in markets with stochastic volatility as compared to a Black-Scholes market with constant volatility - even in financial markets with only one risky asset. Irrespective of these theoretical certainties, we observed in a numerical study that the annual wealth equivalent loss incurred due to trading according to this naive portfolio is relatively small for the majority of realistic scenarios. In this sense, the naive 'capped' portfolio is nearly optimal for most applications. However, in turbulent financial markets, such as the financial crisis of 2008, investors with a high degree of risk aversion and lower bound on their portfolio allocation can suffer high wealth equivalent losses. For such scenarios, investors should be mindful of the difference between the optimal constrained portfolio and the naive capped portfolio.

6 Conclusion

To conclude this thesis, we briefly recall for the main Chapters 3, 4, and 5 what we have done, what we have learned and what could be investigated in future research.

What we have done: In Chapter 3, we began by generalizing the classic martingale approach in such a way that we can characterize the optimal terminal wealth to a portfolio optimization problem with constraints on terminal wealth in complete financial markets. The generalization is applicable for (simultaneous) pointwise constraints on terminal wealth and expected value constraints. Afterwards, we integrated this generalized martingale approach into the auxiliary market framework of [17] to derive equivalent dual optimality conditions which can be used to characterize the optimal terminal wealth and optimal portfolio process of an investor who faces simultaneous constraints on his relative portfolio allocation and his terminal wealth. If the allocation constraint $K \subset \mathbb{R}^d$ is a convex cone and the financial market is a Black-Scholes market, then we were able to show that the 'optimal' auxiliary market can be determined as the solution λ^* to a deterministic, convex optimization problem and does not depend on the employed utility function or the wealth constraint. For additional deterministic pointwise bounds on terminal wealth and VaR- or ES-constraints, this allowed us to derive a representation for the investor's optimal terminal wealth which is explicit up to a constant $(y^*, \lambda^*) \in [0,\infty)^2 \times \mathbb{R}^d$.
In the subsequent Chapter 4, we restricted our analysis to (incomplete) stochastic factor models, allocation constraints and power utility functions. As a consequence, the standard martingale approach is no longer applicable and we had to develop a different duality approach. Using a duality result from constrained real optimization, we were able to rewrite the associated HJB PDE as a dual HJBI PDE. The min-max problem associated with the dual HJBI PDE is the dual minimization problem in Condition $(\tilde{C})$ from Section 3.4, which establishes a connection to the classic auxiliary market framework. By exploiting this connection, we were able to derive Condition (EAS), which under some additional regularity conditions guarantees that the value function is of exponentially affine form and yields a closed-form expression for the optimal portfolio allocation. We verified Condition (EAS) for several examples of financial markets and constraints.
Lastly, in Chapter 5, we specifically focussed on allocation constrained portfolio optimization in a Heston market with one risky asset. We verified Condition (EAS), solved the associated HJB(I) PDE and thereby derived a closed-form expression for the optimal constrained portfolio process for the Heston market. We concluded the chapter by analyzing the sub-optimality of two 'naive' portfolio processes in comparison to the optimal constrained portfolio process in the Heston market in a numerical study.

What we have learned: In a complete financial market, the addition of pointwise constraints $(B_1 \leq V^{v_0,\pi}(T) \leq B_2)$ and expected value constraints $(\mathbb{E}\left[g(V^{v_0,\pi}(T))\right] \leq 0)$ on terminal wealth is, under some conditions, equivalent to adjusting the investor's utility function from $U \in \mathcal{U}$ to $U(\cdot; B_1, B_2) - y_1^* g(\cdot)$, where the adjustment y_1^* can be determined via the dual optimization problem $(\mathbf{D_0^V})$. In particular, we can obtain a closed-form characterization for the optimal terminal wealth (resp. portfolio process) under terminal wealth constraints, if the closed-form

characterization is known for the adjusted utility function. Similarly, [17] argued that, under some conditions, the introduction of convex allocation constraints is equivalent to adjusting the original financial market $\mathcal{M}$ to $\mathcal{M}_{\lambda^*}$, where the adjustment λ^* can be determined via the dual minimization problem ($\mathbf{D^K}$). What we have shown in Section 3.4 is that simultaneous constraints on terminal wealth and portfolio allocation can be regarded as a simultaneous adjustment to both utility function and financial market (y^*, λ^*), which can be computed via the joint dual minimization problem ($\mathbf{D^{K,V}}$).
Even without constraints, closed-form solutions for the optimal portfolio process in stochastic factor models are mainly available for the class of exponentially affine (respectively exponentially quadratic) models and power utility functions. Due to the lack of flexibility in the choice of utility function, we thus had to neglect wealth constraints in the ensuing analysis and only focussed on allocation constraints. Within this context, Condition (EAS) allows us to characterize the optimal constrained portfolio process in a stochastic factor model if the market parameters adjusted by a dual optimizer (i.e. in the auxiliary market $\mathcal{M}_{\lambda^*}$) leave the model exponentially affine. Surprisingly, even if the original model is exponentially affine, the adjusted model may no longer be exponentially affine, as illustrated in Section 4.4.2.
Since Condition (EAS) is satisfied in a one-dimensional Heston market, we were able to make another surprising observation: while interval constraints $K = [\alpha, \beta]$ in a one-dimensional Black-Scholes market lead to an optimal portfolio process which 'caps' the optimal unconstrained portfolio process, this is no longer the case in a Heston market. This suggests that allocation constraints have a different impact on the investor's portfolio process in financial markets with stochastic volatility. During numerical experiments we observed that this difference is particularly relevant for investors who trade during turbulent financial markets, exhibit a high degree of risk aversion and are constrained by a high lower bound on their relative portfolio allocation.

What could be investigated in future research: Naturally, one could extend Chapter 3 to incomplete financial markets ($m > 0$), which can be completed through the addition of fictitious assets, such as Heston's stochastic volatility model (see e.g., [62], [9] and [30]). If a trading restriction on these fictitious assets is imposed, then the framework of Chapter 3 can in principle be applied. It then remains to see for which financial market models, utility functions, allocation constraints and wealth constraints we can still derive useful characterizations of the optimal terminal wealth akin to the results in Section 3.5.3. A convenient choice for the fictitious assets will likely be crucial in this aspect.
The pointwise duality approach presented in Chapter 4 on the level of the HJB PDE is in principle also applicable for portfolio optimization problems with simultaneous constraints on allocation and wealth, as long as these constraints are time-consistent (i.e., we need to be able to formulate an HJB PDE). Similarly, our approach may also lead to new insights for portfolio optimization problems with time-inconsistent allocation and wealth constraints when following the equilibrium approach of [7]. Whether or not such an application can lead to new closed-form solutions for optimal portfolio processes will depend crucially on the right combination of market model, utility function and constraints. This is required to ensure that one can derive a semi-closed-form solution to the associated HJB(I) PDEs, which is possible only in rare cases – even without the presence of allocation constraints.
Lastly, it would be interesting to analyze the impact of allocation constraints on the optimal allocation via a similar projection argument as in Section 5.2.3 for multi-dimensional market models. The form of the projection $\mathcal{P}_K^{BS}$ in the Black-Scholes market $\mathcal{M}_{BS}$ is already suggested by Remark 4.4.11. It would be compelling to see if one can derive similar conditions as in Lemma 5.2.16, under which the projection in $\mathcal{M}_{CIR}$ and even more general markets coincide with $\mathcal{P}_K^{BS}$.

A Proofs

A.1 Proofs Chapter 2

Proof of Lemma 2.1.4. Noting that any vector space V with scalar product is a locally convex space, this statement is a special case of Chapter 1, Corollary 1.4 in [29]. □

Proof of Lemma 2.1.8. Let f satisfy (2.1). Further, let $(x, \alpha), (y, \beta) \in H_f$ and $\lambda \in (0, 1)$. Then, by the definition of H_f, we have $f(x) \geq \alpha$, $f(y) \geq \beta$. Moreover, we have

$$f(\lambda x + (1-\lambda)y) \overset{(2.1)}{\geq} \lambda f(x) + (1-\lambda) f(y) \overset{\text{Def. } H_f}{\geq} \lambda\alpha + (1-\lambda)\beta.$$

Thus,

$$\lambda(x, \alpha) + (1-\lambda)(y, \beta) = (\lambda x + (1-\lambda)y, \lambda\alpha + (1-\lambda)\beta) \in H_f.$$

Therefore, H_f is convex.

Let H_f be convex, $x, y \in V$ and $\lambda \in (0, 1)$. We have to differentiate between two cases:

1. $\underline{f(x) > -\infty, f(y) > -\infty}$: Then, $f(x), f(y)$ are finite and $(x, f(x))$, $(y, f(y)) \in H_f$. Since H_f is convex, we have

$$(\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda) f(y)) = \lambda(x, f(x)) + (1-\lambda)(y, f(y)) \in H_f.$$

   However, this implies (2.1) by the definition of H_f.

2. $\underline{f(x) = -\infty \text{ or } f(y) = -\infty}$: Then we immediately have

$$f(\lambda x + (1-\lambda)y) \geq -\infty = \lambda f(x) + (1-\lambda) f(y).$$

□

Equivalence of (2.2) and (2.3).
(2.2) '⇒' (2.3): Let $\hat{x} \in V$. If $f(\hat{x}) = \infty$, then (2.3) holds trivially. If $f(\hat{x}) < \infty$, assume that $\limsup_{x \to \hat{x}} f(x) > f(\hat{x})$. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ taking values in V with $\lim_{n\to\infty} x_n = \hat{x}$ and $\limsup_{n\to\infty} f(x_n) = \hat{f} > f(\hat{x})$. As $M_f(\alpha)$ is closed for arbitrary α, we know that for any $\epsilon > 0$ the complement

$$M_f(f(\hat{x}) + \epsilon)^c = \{x \in V \mid f(x) < f(\hat{x}) + \epsilon\}$$

is open. In particular, $\hat{x} \in M_f(f(\hat{x}) + \epsilon)^c$ and thus there exists $N \in \mathbb{N}$ such that $x_n \in M_f(f(\hat{x}) + \epsilon)^c$ for all $n > N$. Setting $\epsilon = \frac{1}{2}\big(\hat{f} - f(\hat{x})\big)$ leads to a contradiction. Hence,

$\limsup_{x\to\hat{x}} f(x) \le f(\hat{x})$.

(2.3) $\Rightarrow$ (2.2): Let $\alpha \in \mathbb{R}$ be arbitrary. If $M_f(\alpha) = \emptyset$, then $M_f(\alpha)$ is closed. Otherwise, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence taking values in $M_f(\alpha)$ with limit $\hat{x} \in V$. We need to show that $\hat{x} \in M_f(\alpha)$.
As $x_n \in M_f(\alpha)\ \forall n \in \mathbb{N}$, we have $f(x_n) \ge \alpha\ \forall n \in \mathbb{N}$. In combination with (2.3) we obtain

$$f(\hat{x}) \ge \limsup_{x\to\hat{x}} f(x) \ge \limsup_{n\to\infty} f(x_n) \ge \alpha$$

and therefore $\hat{x} \in M_f(\alpha)$ and $M_f(\alpha)$ is closed. □

Proof of Lemma 2.1.11.

(i) '$\Rightarrow$': Let f be usc and $M_f(\alpha)$ as in Definition 2.1.9. Then,

$$M_f(\alpha)^c \times (\alpha,\infty) = \big\{x\ \big| f(x) < \alpha\big\} \times (\alpha,\infty)$$

is open in $V_{\mathbb{R}}$ for all $\alpha \in \mathbb{R}$. Hence,

$$\begin{aligned} H_f^c &= \big\{(x,\alpha)\ \big|\ f(x) < \alpha\big\} \\ &= \bigcup_{\alpha\in\mathbb{R}} \big\{x\ \big|\ f(x) < \alpha\big\} \times (\alpha,\infty) \end{aligned}$$

is open in $V_{\mathbb{R}}$, i.e., H_f is closed in $V_{\mathbb{R}}$.
'$\Leftarrow$': Let H_f be closed in $V_{\mathbb{R}} = V \times \mathbb{R}$. For any $\alpha \in \mathbb{R}$ the set $V \times \{\alpha\}$ is closed in $V_{\mathbb{R}}$. Hence, $H_f \cap (V \times \{\alpha\}) = M_f(\alpha) \times \{\alpha\}$ is closed in $V_{\mathbb{R}}$. This implies that $M_f(\alpha)$ is closed in V. Hence, f is usc.

(ii) As $usc\ f \ge f$ is closed, we know by (i) that $H_{usc\ f}$ is closed and therefore,

$$H_f \subset H_{usc\ f} \Rightarrow cl\ H_f \subset cl\ H_{usc\ f} \overset{(i)}{\subset} H_{usc\ f}.$$

Further, $cl\ H_f$ is the hypograph H_g of a function $g : V \to \bar{\mathbb{R}}$

$$g(x) := \sup\big\{\alpha\ \big|\ (x,\alpha) \in cl\ H_f\big\}.$$

Since $H_f \subset cl\ H_f = H_g$, we have $f \le g$ and g is usc. In particular, this implies $usc\ f \le g$ and therefore

$$H_{usc\ f} \subset H_g = cl\ H_f.$$

(iii) As $f \le co\ f$ and $H_{co\ f}$ is convex, we must have $co\ H_f \subset H_{co\ f}$. Analogously to the proof of (ii), $co\ H_f$ defines the hypograph H_g of a concave function

$$g(x) := \sup\big\{\alpha\ \big|\ (x,\alpha) \in co\ H_f\big\}.$$

As g is concave and $H_f \subset co\ H_f = H_g$, we have $f \le co\ f \le g$ and thus $H_{co\ f} \subset H_g = co\ H_f$.

(iv) For $y = (z, \gamma) \in V_{\mathbb{R}}$, $\beta \in \mathbb{R}$, let $H(y, \beta)$ denote a closed half space in $V_{\mathbb{R}}$. Then, we may use Lemma 2.1.4 and the previous derivations to realize

$$\bigcap_{\substack{H_f \subset H(y,\beta) \\ (y,\beta) \in V_{\mathbb{R}} \times \mathbb{R}}} H(y, \beta) \overset{L.2.1.4}{=} cl\ co\ H_f \overset{(iv)}{=} cl\ H_{co\ f} \overset{(ii)}{=} H_{usc\ cl\ f}.$$

In particular, we have

$$usc\ cl\ f(x) = \sup \left\{ \alpha \,\middle|\, (x, \alpha) \in \bigcap_{\substack{H_f \subset H(y,\beta) \\ (y,\beta) \in V_{\mathbb{R}} \times \mathbb{R}}} H(y, \beta) \right\}$$

Our aim is to now characterize $H(y, \beta)$ and $H_{usc\ cl\ f}$ in such a way that (iv) follows immediately. We make a distinction between the three cases $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$, for $y = (z, \gamma)$.

For $\gamma < 0$, we have $(x, \alpha) \in H(y, \beta)$ if and only if

$$\langle x, z \rangle + \alpha\gamma \leq \beta \quad \overset{\gamma<0}{\Leftrightarrow} \quad \alpha \geq \frac{1}{\gamma} (\beta - \langle x, z \rangle)$$

The function f is proper and therefore there exists $\hat{x} \in V$ such that $-\infty < f(\hat{x})$. Hence, $(\hat{x}, \alpha) \in H_f$ for all $\alpha \leq f(\hat{x})$. However, when choosing

$$\hat{\alpha} := \min \left(f(\hat{x}), \frac{1}{\gamma} (\beta - \langle \hat{x}, z \rangle) \right) - 1,$$

we then have $(\hat{x}, \hat{\alpha}) \notin H(y, \beta)$ and $(\hat{x}, \hat{\alpha}) \in H_f$. Hence, $H_f \not\subset H(y, \beta)$ and

$$\bigcap_{\substack{H_f \subset H((z,\gamma),\beta) \\ (z,\gamma,\beta) \in V \times (-\infty,0) \times \mathbb{R}}} H(y, \beta) = V_{\mathbb{R}}. \tag{A.1}$$

For $\gamma = 0$, we have $(x, \alpha) \in H(y, \beta)$ if and only if

$$\langle x, z \rangle \leq \beta.$$

Therefore,

$$\begin{aligned} H_f \subset H(y, \beta) &\Leftrightarrow \forall (x, \alpha) \in H_f : (f(x) \geq \alpha \Rightarrow \langle x, z \rangle \leq \beta) \\ &\Leftrightarrow \forall (x, \alpha) \in H_f : (f(x) > -\infty \Rightarrow \langle x, z \rangle \leq \beta) \\ &\Leftrightarrow\ dom_f \subset H(z, \beta). \end{aligned}$$

As $(x, \alpha) \in H((z, 0), \beta)$ is true or false irrespective of the value of α, we thus obtain

$$\bigcap_{\substack{H_f \subset H((z,0),\beta) \\ (z,\beta) \in V \times \mathbb{R}}} H(y, \beta) = \Big(\bigcap_{\substack{dom_f \subset H(z,\beta) \\ (z,\beta) \in V \times \mathbb{R}}} H(z, \beta) \Big) \times \mathbb{R} \overset{L.2.1.4}{=} (cl\ co\ dom_f) \times \mathbb{R}. \tag{A.2}$$

For $\gamma > 0$, we have $(x, \alpha) \in H(y, \beta)$ if and only if

$$\langle x, z \rangle + \alpha\gamma \leq \beta \Leftrightarrow \underbrace{\frac{1}{\gamma} (\beta - \langle x, z \rangle}_{=:h(x):=h(x,z,\gamma,\beta)}) \geq \alpha.$$

$$\Rightarrow\ H(y, \beta) = \left\{ (x, \alpha) \,\middle|\, h(x; z, \gamma, \beta) \geq \alpha \right\} = H_h.$$

If we consider h as a function of x for fixed z, γ, β, then h is affine in x. Moreover, for an appropriate choice of parameters z, γ, β, we can obtain an arbitrary affine function $h : V \to \mathbb{R}$. Note that we do not need the degree of freedom in γ, as we can still obtain any affine function by setting $\gamma = 1$, changing the sign of $\langle z, x\rangle$ and varying z, β freely. Thus, we obtain

$$\bigcap_{\substack{H_f \subset H((z,\gamma),\beta) \\ (z,\gamma,\beta)\in V\times(0,\infty)\times\mathbb{R}}} H(y,\beta) = \bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ H_f\subset H_h}} H_h = \bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ f\le h}} H_h \tag{A.3}$$

In total, (A.1), (A.2) and (A.3) yield

$$\begin{aligned}\bigcap_{\substack{H_f \subset H(y,\beta) \\ (y,\beta)\in V_{\mathbb{R}}\times\mathbb{R}}} H(y,\beta) &= \Big(V_{\mathbb{R}}\Big) \cap \Big((cl\ co\ dom_f)\times\mathbb{R}\Big) \cap \Big(\bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ f\le h}} H_h\Big) \\ &= \Big((cl\ co\ dom_f)\times\mathbb{R}\Big) \cap \Big(\bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ f\le h}} H_h\Big)\end{aligned}$$

(v) We make a distinction between $cl\ co\ f \equiv \infty$ and $cl\ co\ f = usc\ co\ f$.
If $cl\ co\ f \equiv \infty$, then $usc\ co\ f$ is not proper and there does not exist an affine function $h : V \to \mathbb{R}$ such that $f \le h$. This implies that for any $(z, \beta) \in V_{\mathbb{R}}$, there exists an $\bar{x} = \bar{x}(z, \beta)$ such that

$$\begin{aligned} &\beta + \langle \bar{x}, z\rangle < f(\bar{x}) \Rightarrow \beta < f(\bar{x}) - \langle \bar{x}, z\rangle \le \sup_{\hat{x}\in V} \left(f(\hat{x}) - \langle \hat{x}, z\rangle\right) \\ \Rightarrow \quad & \infty = \sup_{\beta\in\mathbb{R}} \beta \le \sup_{\hat{x}\in V} \left(f(\hat{x}) - \langle \hat{x}, z\rangle\right). \end{aligned}$$

In particular, taking the infimum over $z \in V$ yields for any $x \in V$

$$\inf_{z\in V} \Big(\underbrace{\sup_{\hat{x}\in V} \big(f(\hat{x}) - \langle z, \hat{x}\rangle\big)}_{=\infty} + \langle z, x\rangle \Big) = \infty = cl\ co\ f(x).$$

If $cl\ co\ f = usc\ co\ f$, then $usc\ co\ f$ is proper and there exists an affine function $h : V \to \mathbb{R}$ such that $f \le h$. In particular, we have

$$\begin{aligned}\bigcap_{\substack{H_f \subset H(y,\beta) \\ (y,\beta)\in V_{\mathbb{R}}\times\mathbb{R}}} H(y,\beta) &= \Big((cl\ co\ dom_f)\times\mathbb{R}\Big) \cap \Big(\bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ f\le h}} H_h\Big) \\ &= \bigcap_{\substack{h:V\to\mathbb{R} \text{ affine} \\ f\le h}} H_h \\ &= \bigcap_{\substack{(z,\beta)\in V_{\mathbb{R}} \\ f(x)\le\beta+\langle z,x\rangle\ \forall x\in V}} \big\{(x,\alpha) \mid \beta + \langle z, x\rangle \ge \alpha\big\}. \end{aligned} \tag{A.4}$$

For any $z \in V$ we have,

$$f(\hat{x}) \le \beta + \langle z, \hat{x}\rangle\ \forall \hat{x} \in V \ \Leftrightarrow \sup_{\hat{x}\in V} \big(f(\hat{x}) - \langle z, \hat{x}\rangle\big) \le \beta$$

and therefore, by (A.4),

$$H_{usc\ co\ f} = \bigcap_{\substack{H_f \subset H(y,\beta) \\ (y,\beta)\in V_{\mathbb{R}}\times\mathbb{R}}} H(y,\beta) = \bigcap_{z\in V} \Big\{(x,\alpha) \mid \sup_{x\in V} \big(f(\hat{x}) - \langle z, \hat{x}\rangle\big) + \langle z, x\rangle \ge \alpha\Big\}$$

$$= \left\{ (x,\alpha) \;\Big|\; \inf_{z\in V} \left(\sup_{x\in V} \left(f(\hat{x}) - \langle z,\hat{x}\rangle \right) + \langle z,x\rangle \right) \geq \alpha \right\}$$

$$\Rightarrow \; cl\ co\ f(x) = usc\ co\ f(x) = \inf_{z\in V} \left(\sup_{x\in V} \left(f(\hat{x}) - \langle z,\hat{x}\rangle \right) + \langle z,x\rangle \right).$$

□

Proof of Lemma 2.1.15.

(i) For any $y \in V$, we have

$$-f^*(y) = -\sup_{x\in V} \left(f(x) - \langle y,x\rangle \right) = \inf_{x\in V} \left(-f(x) + \langle y,v\rangle \right).$$

Thus,

$$\begin{aligned} H_{-f^*} &= \left\{ (y,\alpha) \;\Big|\; \inf_{x\in V} \left(-f(x) + \langle y,x\rangle \right) \geq \alpha \right\} \\ &= \left\{ (y,\alpha) \;\Big|\; \underbrace{-f(x) + \langle y,x\rangle}_{=:h(y;x)} \geq \alpha \quad \forall x \in V \right\} \\ &= \left\{ (y,\alpha) \;\big|\; h(y;x) \geq \alpha \quad \forall x \in V \right\} \\ &= \bigcup_{x\in V} H_{h(\cdot;x)}. \end{aligned}$$

For fixed x, the function $h(y;x)$ is affine in y and thus usc concave in y. Hence, each hypograph $H_{h(\cdot;x)}$ is closed convex. Therefore, H_{-f^*} is closed convex, which is equivalent to $-f^*(y)$ being usc concave by Definition 2.1.6 and Lemma 2.1.11, (i). The statements about $f^{**} = cl\ co\ f$ follow directly from Lemma 2.1.11 (iv).

(ii) Let $f = f^{**}$. By Lemma 2.1.11, we have $f^{**} = cl\ co\ f$, which is usc and concave by construction. Hence, $f = f^{**} = cl\ co\ f$ is usc and concave, too.
Let f be usc and concave. Then,

$$f \overset{D.\ 2.1.10}{=} co\ f \text{ and } f \overset{D.\ 2.1.10}{=} usc\ f \overset{f \text{ proper}}{=} cl\ co\ f \overset{L.\ 2.1.11\ (iv)}{=} f^{**}.$$

□

Proof of Lemma 2.1.17.

(i) Let $y \in V,\ \alpha \geq 0$. Then,

$$\delta_K(\alpha y) = -\inf_{x\in K} \big(\langle x,\alpha y\rangle \big) \overset{\alpha\geq 0}{=} -\alpha \inf_{x\in K} \big(\langle x,y\rangle \big) = \alpha\delta_K(y).$$

(ii) Let $y_1, y_2 \in V$. Then,

$$\begin{aligned} \delta_K(y_1+y_2) &= -\inf_{x\in K} \big(\langle y_1+y_2,x\rangle \big) = -\inf_{x\in K} \big(\langle y_1,x\rangle + \langle y_2,x\rangle \big) \\ &\leq -\inf_{x\in K} \big(\langle y_1,x\rangle \big) - \inf_{x\in K} \big(\langle y_2,x\rangle \big) = \delta_K(y_1) + \delta_K(y_2). \end{aligned}$$

(iii) The first equivalence follows directly from (2.4). By scaling any non-zero $y \in X_K$ to

$$\hat{y} = \frac{1}{\sqrt{\langle y, y \rangle}} y$$

such that $\langle \hat{y}, \hat{y} \rangle \leq 1$, we can use the positive homogeneity of δ_K to see that

$$0 \leq \delta_K(y) + \langle x, y \rangle \ \forall y \in X_K \ \Leftrightarrow 0 \leq \delta_K(\hat{y}) + \langle x, \hat{y} \rangle \ \forall \hat{y} \in X_K \text{ with } \langle \hat{y}, \hat{y} \rangle \leq 1,$$

i.e., the second equivalence holds.

(iv) '⇒': Let $\delta_K(y) = 0 \ \forall y \in X_K$. Let $x \in K$, $\alpha \geq 0$. Since $x \in K$, we obtain from (iii):

$$\begin{aligned} & 0 \leq \underbrace{\delta_K(y)}_{=0} + \langle y, x \rangle = \langle y, x \rangle \quad \forall y \in X_K \\ \Rightarrow \ & 0 \leq \langle y, \alpha x \rangle = \delta_K(y) + \langle y, \alpha x \rangle \quad \forall y \in X_K \Rightarrow \alpha x \in K \quad \text{by } (iii). \end{aligned}$$

Hence, K is a convex cone.
'⇐': Let K be a convex cone. Then, $0 \in K$ and thus

$$\delta_K(y) = -\inf_{x \in K} (\langle x, y \rangle) \geq \langle 0, y \rangle = 0 \quad \forall y \in V.$$

Further, if $\delta_K(y) > 0$, then there exists $x^* \in K$ such that $\langle x^*, y \rangle < 0$. Since K is a convex cone, $\alpha x^* \in K$ for all $\alpha \geq 0$. Hence,

$$\delta_K(y) = -\inf_{x \in K} \big(\langle x, y \rangle \big) \geq -\inf_{\alpha \geq 0} \big(\langle \alpha x^*, y \rangle \big) = \sup_{\alpha \geq 0} \big(\underbrace{-\alpha}_{\leq 0} \underbrace{\langle x^*, y \rangle}_{<0} \big) = \infty.$$

Thus, $\delta_K(y) > 0 \Rightarrow y \notin X_K \overset{\delta \geq 0}{\Rightarrow} \delta_K(y) = 0 \ \forall y \in X_K$.

□

Proof of Lemma 2.1.22.

(i): For every fixed $x \in V$, $L(x, \lambda) - \langle \bar{u}, \lambda \rangle$ is the concave conjugate of $F(x, u)$ w.r.t. u. If $F(x, \cdot)$ is proper, then by Lemma 2.1.15, (i) this implies $-L(x, \cdot) + \langle \bar{u}, \lambda \rangle$ is usc and concave in u and therefore $-L(x, \cdot)$ is usc and concave in u, too.

(ii): $\inf_{\lambda \in H} L(x, \lambda)$ is the bi-conjugate of $F(x, \cdot)$ evaluated at $\bar{u}$. Hence, we have by Lemma 2.1.11 (iv)

$$\begin{aligned} \inf_{\lambda \in H} L(x, \lambda) &= \inf_{\lambda \in H} \Big(\sup_{u \in H} \big(F(x, u) - \langle u, \lambda \rangle \big) + \langle \bar{u}, \lambda \rangle \Big) \\ &= cl \ co \ F(x, \bar{u}) \\ &= f(x). \end{aligned}$$

(iii) By (ii), we now have

$$\begin{aligned} & f(x) = \inf_{\lambda \in H} L(x, \lambda) \ \forall x \in V \\ \Rightarrow \quad & \Phi_P = \sup_{x \in V} f(x) = \sup_{x \in V} \inf_{\lambda \in H} L(x, \lambda). \end{aligned}$$

□

Proof of Theorem 2.1.23. We have $\Phi(\bar{u}) = \sup_{x\in V} f(x) = \Phi_P$ by definition of Φ and Φ_P. Moreover, from Lemma 2.1.15 we obtain

$$\begin{aligned}
\Psi_D &= \inf_{\lambda\in H}\sup_{x\in V} L(x,\lambda)\\
&= \inf_{\lambda\in H}\sup_{x\in V}\sup_{u\in H}\Big(\big(F(x,u)-\langle u,\lambda\rangle\big)+\langle\bar{u},\lambda\rangle\Big)\\
&= \inf_{\lambda\in H}\sup_{u\in H}\Big(\big(\underbrace{\sup_{x\in V} F(x,u)}_{=\,\Phi(u)}-\langle u,\lambda\rangle\big)+\langle\bar{u},\lambda\rangle\Big)\\
&= \inf_{\lambda\in H}\sup_{u\in H}\Big(\big(\Phi(u)-\langle u,\lambda\rangle\big)+\langle\bar{u},\lambda\rangle\Big)\\
&= \inf_{\lambda\in H}\big(\Phi^*(\lambda)+\langle\bar{u},\lambda\rangle\big)\\
&= \Phi^{**}(\bar{u})\\
&= cl\ co\ \Phi(\bar{u}).
\end{aligned}$$

Finally, as noted in Remark 2.1.13, weak duality of the concave conjugate yields

$$\Phi_P = \Phi(\bar{u}) \le \Phi^{**}(\bar{u}) = cl\ co\ \Phi(\bar{u}) = \Psi_D.$$

□

Proof of Corollary 2.1.26. As (x^*,λ^*) is a saddle-point of L, we have

$$\sup_{x\in V} L(x,\lambda^*) = L(x^*,\lambda^*) \quad\text{and}\quad \inf_{\lambda\in H} L(x^*,\lambda) = L(x^*,\lambda^*).$$

However, as we additionally have

$$L(x^*,\lambda^*) = \sup_{x\in V} L(x,\lambda^*) \ge \sup_{x\in V}\inf_{\lambda\in H} L(x,\lambda) \ge \inf_{\lambda\in H} L(x^*,\lambda) = L(x^*,\lambda^*)$$

and

$$L(x^*,\lambda^*) = \inf_{\lambda\in H} L(x^*,\lambda) \le \inf_{\lambda\in H}\sup_{x\in V} L(x,\lambda) \le \sup_{x\in V} L(x,\lambda^*) = L(x^*,\lambda^*),$$

we obtain

$$\Psi_D = \inf_{\lambda\in H}\sup_{x\in V} L(x,\lambda) = \sup_{x\in V} L(x,\lambda^*),$$

i.e., λ^* is optimal for (**D**). Moreover, if $F(x^*,\cdot)$ is usc and concave in u, then we get

$$\sup_{x\in V} f(x) = \sup_{x\in V}\inf_{\lambda\in H} L(x,\lambda) = \inf_{\lambda\in H} L(x^*,\lambda) = f(x^*),$$

i.e., x^* is optimal for (**P**). □

Proof of Theorem 2.1.27. The equivalence of (i) and (ii) is immediate from Theorem 2.1.23 and the implication (iv) ⇒ (iii) holds true according to Corollary 2.1.26. It remains to show that (iii)⇒ (iv). Assume that (iii) holds. Then,

$$\Phi_P = \sup_{x\in V} f(x) \overset{x^*\text{ optimal for }(\mathbf{P})}{=} f(x^*) = F(x^*,\bar{u}) \overset{\text{L. 2.1.22}}{=} \inf_{\lambda\in H} L(x^*,\lambda)$$

$$\leq L(x^*,\lambda^*) \leq \sup_{x\in V} L(x,\lambda^*) \overset{\lambda^* \text{ optimal for } (\mathbf{D})}{=} \inf_{\lambda\in H}\sup_{x\in V} L(x,\lambda) = \Psi_D$$
$$= \Phi_P$$

This yields

$$\sup_{x\in V} L(x,\lambda^*) = L(x^*,\lambda^*) = \inf_{\lambda\in H} L(x^*,\lambda)$$

and therefore (x^*,λ^*) is a saddle-point. □

Proof of Theorem 2.1.28. Follows immediately by considering $-F$ in Theorem 18a) in [76]. □

Proof of Lemma 2.1.29. Recalling Example 2.1.20, we realize that L is the Lagrangian corresponding to the perturbed objective function

$$F(x,u) = f(x) + \mathcal{X}_K(x+u) = f(x) + \mathcal{X}_{K-u}(x).$$

We aim to apply Corollary 2.1.26 and thus need to verify that $F(x^*,\cdot)$ is proper, usc and concave in u. However, as $f(x^*)\in\mathbb{R}$, it is sufficient to show that the mapping

$$u \to \mathcal{X}_K(x^*+u)$$

proper, usc and concave in u. However, this is trivial because K is non-empty and closed convex and $u\to x^*+u$ is affine in u. Hence, by virtue of Corollary 2.1.26, x^* is optimal for $(\mathbf{P})$. □

Proof of Corollary 2.1.30.
'⇒': Assume that (x^*,λ^*) is a saddle-point of L. Then, we have

$$L(x,\lambda^*) \leq L(x^*,\lambda^*) \leq L(x^*,\lambda) \qquad \forall (x,\lambda)\in\mathbb{R}^d\times\mathbb{R}^d. \tag{A.5}$$

In particular, this implies for all $x\in\mathbb{R}^d$:

$$f(x) + \langle x,\lambda^*\rangle + \delta_K(\lambda^*) = L(x,\lambda^*) \leq L(x^*,\lambda^*) = f(x^*) + \langle x^*,\lambda^*\rangle + \delta_K(\lambda^*),$$

i.e., (i) is satisfied. Assume now that $\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) > 0$. But then we have

$$\begin{aligned} L(x^*,0) &= f(x^*) + \langle x^*,0\rangle + \delta_K(0) = f(x^*) \\ &< f(x^*) + \langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) = L(x^*,\lambda^*), \end{aligned}$$

which is a contradiction to (A.5). Hence, we must have $\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) \leq 0$. Similarly, assuming $\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) < 0$ yields

$$\begin{aligned} L(x^*,2\lambda^*) &= f(x^*) + \langle x^*,2\lambda^*\rangle + \delta_K(2\lambda^*) \\ &= f(x^*) + 2\big(\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*)\big) \\ &< f(x^*) + \langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) = L(x^*,\lambda^*), \end{aligned}$$

which is again a contradiction to (A.5). Hence, we must have $\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) \geq 0$. In total, this yields $\langle x^*,\lambda^*\rangle + \delta_K(\lambda^*) = 0$, i.e., (ii) is satisfied. This allows us to show that for any $\lambda\in\mathbb{R}^d$

$$\langle x^*,\lambda\rangle + \delta_K(\lambda) \;=\; -f(x^*) + f(x^*) + \langle x^*,\lambda\rangle + \delta_K(\lambda)$$

$$\begin{aligned}
&= -f(x^*) + L(x^*, \lambda) \\
&\overset{(A.5)}{\geq} -f(x^*) + L(x^*, \lambda^*) \\
&= \underbrace{-f(x^*) + f(x^*)}_{=0} + \underbrace{\langle x^*, \lambda^* \rangle + \delta_K(\lambda^*)}_{=0} \\
&\overset{(ii)}{=} 0.
\end{aligned}$$

According to (2.4), this implies $x^* \in K$ and thus (iii) is satisfied.
'$\Leftarrow$': Let (i), (ii), (iii) be satisfied for $(x^*, \lambda^*) \in V \times H$. From (i) we know that

$$L(x, \lambda^*) = f(x) + \langle x, \lambda^* \rangle + \delta_K(\lambda^*) \leq f(x^*) + \langle x^*, \lambda^* \rangle + \delta_K(\lambda^*) \leq L(x^*, \lambda^*) \quad \forall x \in V.$$

Moreover, using (ii) and (iii), we obtain

$$\begin{aligned}
L(x^*, \lambda^*) &= f(x^*) + \langle x^*, \lambda^* \rangle + \delta_K(\lambda^*) \\
&\overset{(ii)}{=} f(x^*) \\
&\overset{(iii)}{\leq} f(x^*) + \overbrace{\langle x^*, \lambda \rangle + \delta_K(\lambda)}^{\geq 0} \\
&= L(x^*, \lambda) \quad \forall \lambda \in H.
\end{aligned}$$

Thus, (x^*, λ^*) is a saddle-point of L. □

Proof of Theorem 2.1.31. Consider $0 < \epsilon < 1$, $\Delta\lambda \in X_K$ with $\|\Delta\lambda\| \leq 1$ and define $\lambda_\epsilon = \lambda^* + \epsilon\Delta\lambda$. Since λ^* is optimal for (**D**), we know that

$$\begin{aligned}
0 &\leq \frac{1}{\epsilon}\big(L(x^*(\lambda_\epsilon), \lambda_\epsilon) - \underbrace{L(x^*(\lambda^*), \lambda^*)}_{\geq L(x^*(\lambda_\epsilon), \lambda^*)}\big) \\
&\leq \frac{1}{\epsilon}\big(L(x^*(\lambda_\epsilon), \lambda_\epsilon) - L(x^*(\lambda_\epsilon), \lambda^*)\big) \\
&= \frac{1}{\epsilon}\big(f(x^*(\lambda_\epsilon)) + (x^*(\lambda_\epsilon))'\lambda_\epsilon + \delta_K(\lambda_\epsilon) - f(x^*(\lambda_\epsilon)) - (x^*(\lambda_\epsilon))'\lambda^* - \delta_K(\lambda^*)\big) \\
&= \frac{1}{\epsilon}\big((x^*(\lambda_\epsilon))' \underbrace{(\lambda_\epsilon - \lambda^*)}_{=\epsilon\Delta\lambda} + \delta_K(\lambda_\epsilon) - \delta_K(\lambda^*)\big) \\
&= (x^*(\lambda_\epsilon))'\Delta\lambda + \frac{1}{\epsilon}\left(\delta_K(\lambda_\epsilon) - \delta_K(\lambda^*)\right).
\end{aligned}$$

Taking the upper limit $\epsilon \downarrow 0$ and using the fact that $x^*(\lambda)'\Delta\lambda$ is usc at $\lambda = \lambda^*$, yields

$$\begin{aligned}
0 &\leq \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon}\big(L(x^*(\lambda_\epsilon), \lambda_\epsilon) - L(x^*(\lambda^*), \lambda^*)\big) \\
&\leq \limsup_{\epsilon \downarrow 0} \left((x^*(\lambda_\epsilon))'\Delta\lambda + \frac{1}{\epsilon}\left(\delta_K(\lambda_\epsilon) - \delta_K(\lambda^*)\right)\right) \\
&\leq x^*(\lambda^*)'\Delta\lambda + \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon}\left(\delta_K(\lambda_\epsilon) - \delta_K(\lambda^*)\right). \qquad \text{(A.6)}
\end{aligned}$$

Using the sub-additivity and positive homogeneity of δ_K (cf. Lemma 2.1.17), finally gives

$$\begin{aligned} 0 &\leq x^*(\lambda^*)'\Delta\lambda + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\big(\delta_K(\lambda^*) + \epsilon\delta_K(\Delta\lambda) - \delta_K(\lambda^*)\big) \\ &= x^*(\lambda^*)'\Delta\lambda + \delta_K(\Delta\lambda). \end{aligned}$$

Since $\Delta\lambda \in X_K$ was chosen arbitrarily, Lemma 2.1.17 implies that $x^*(\lambda^*) \in K$. On the other hand, repeating the same arguments with $\Delta\lambda = -\lambda^*$ up to (A.6) yields

$$\begin{aligned} & 0 \leq -x^*(\lambda^*)'\lambda^* + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\delta_K((1-\epsilon)\lambda^*) - \delta_K(\lambda^*)\right) \\ & \quad = -x^*(\lambda^*)'\lambda^* + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left((1-\epsilon)\delta_K(\lambda^*) - \delta_K(\lambda^*)\right) \\ & \quad = -x^*(\lambda^*)'\lambda^* - \delta_K(\lambda^*) \\ \Leftrightarrow \quad & 0 \geq x^*(\lambda^*)'\lambda^* + \delta_K(\lambda^*) \\ \overset{\lambda^*\in X_K}{\Rightarrow} \quad & 0 = x^*(\lambda^*)'\lambda^* + \delta_K(\lambda^*). \end{aligned}$$

Hence, $(x^*(\lambda^*), \lambda^*)$ satisfies all three KKT conditions from Corollary 2.1.30. Therefore, $(x^*(\lambda^*), \lambda^*)$ is a saddle point of L, $x^*(\lambda^*)$ is optimal for (**P**) and $\Phi_P = \Psi_D$. □

Proof of Corollary 2.1.32. For given $\lambda \in \mathbb{R}^d$, the Lagrangian L is continuously differentiable and concave in x. Hence, by the first order optimality condition and $A' = A$, the maximizer of the Lagrangian $L(\cdot, \lambda)$ is given as

$$x^*(\lambda) = \frac{1}{2}A^{-1}(b + \lambda).$$

Further,

$$\begin{aligned} L(x^*(\lambda), \lambda) &= -\frac{1}{4}(b+\lambda)' \underbrace{(A^{-1})'}_{=A'} AA^{-1}(b+\lambda) + \frac{1}{2}(b+\lambda)' \underbrace{(A^{-1})'}_{=A^{-1}}(b+\lambda) + c + \delta_K(\lambda) \\ &= \frac{1}{4}(b+\lambda)'A^{-1}(b+\lambda) + c + \delta_K(\lambda) \end{aligned}$$

A^{-1} is positive definite, since A is positive definite and δ_K is convex and positive homogeneous of order 1, by Lemma 2.1.17. Hence, $L(x^*(\lambda), \lambda)$ is strictly convex in λ. Moreover, $\delta_K(\lambda)$ grows at most linearly in $\|\lambda\|$ and thus $L(x^*(\lambda), \lambda) \to \infty$ as $\|\lambda\| \to \infty$. Therefore, there exists a unique minimizer $\lambda = \lambda^*$ of $L(x^*(\lambda), \lambda)$, i.e., a unique λ^* which is optimal for (**D**). As $x^*(\lambda)$ is continuous in $\lambda \in \mathbb{R}^d$, the requirements of Theorem 2.1.31 are satisfied. In particular, $(x^*(\lambda^*), \lambda^*)$ is a saddle-point of L and satisfies (2.14). □

Proof of Lemma 2.1.33. Following the line of argument in Example 2.1.21, L as in (2.16) defines the Lagrangian corresponding to the perturbation function $F : L^2_Q \times \mathbb{R}^n \to \bar{\mathbb{R}}$ defined by

$$F(D, u) = \mathbb{E}\left[U(D)\right] + \mathcal{X}_K\left(\mathbb{E}\left[g(D)\right] + u\right).$$

We aim to apply Corollary 2.1.26 and thus need to verify that $F(D^*, \cdot)$ is proper, usc and concave in u. It is sufficient to show that the mapping

$$u \to \mathcal{X}_K\left(\mathbb{E}\left[g(D^*)\right] + u\right). \tag{A.7}$$

is usc and concave in u. However, as K is non-empty and closed convex, $\mathcal{X}_K$ is usc concave. Moreover, for every $D \in L^2_Q$ the real function $u \to \mathbb{E}[g(D)] + u$ is affine and thus continuous. Hence, (A.7) is usc and concave in u and so F is usc and concave in u, too. Thus, by virtue of Corollary 2.1.26, D^* is optimal for $(\mathbf{P})$. □

Proof of Lemma 2.1.35. We obtain

$$\begin{aligned}
\Psi_D &= \inf_{y\in\mathbb{R}^n} \sup_{D\in L^2_Q} L(D,y) \\
&= \inf_{y\in\mathbb{R}^n} \left(\sup_{D\in L^2_Q} \Big(\mathbb{E}[U(D)] + y'\mathbb{E}[g(D)] \Big) + \delta_K(y) \right) \\
&= \inf_{y\in\mathbb{R}^n} \left(\sup_{D\in L^2_Q} \left(\mathbb{E}\Big[\underbrace{U(D) + y'g(D)}_{\leq \sup_{x\in\mathbb{R}} U(x)+y'g(x)} \Big] \right) + \delta_K(y) \right) \\
&\overset{(*)}{\leq} \inf_{y\in\mathbb{R}^n} \left(\mathbb{E}\left[\sup_{x\in\mathbb{R}} \big(U(x) + y'g(x) \big) \right] + \delta_K(y) \right) \\
&= \inf_{y\in\mathbb{R}^n} \big(\mathbb{E}\left[U^*_g(y) \right] + \delta_K(y) \big).
\end{aligned}$$

If y^* attains the infimum over $y \in \mathbb{R}^n$ and $\mathcal{I}_g(y^*) \in L^2_Q$, then we have equality in $(*)$. □

Proof of Corollary 2.1.36. For any $D \in L^2_Q$ we have

$$\begin{aligned}
L(D,y^*) &= \mathbb{E}[U(D)] + (y^*)'\mathbb{E}[g(D)] + \delta_K(y^*) \\
&\leq \sup_{D\in L^2_Q} \mathbb{E}\left[U(D) + (y^*)'g(D) \right] + \delta_K(y^*) \\
&\leq \mathbb{E}\left[\sup_{x\in\mathbb{R}} \big(U(x) + (y^*)'g(x) \big) \right] + \delta_K(y^*) \\
&= \mathbb{E}\left[U(\mathcal{I}_g(y^*)) + (y^*)'g(\mathcal{I}_g(y^*)) \right] + \delta_K(y^*) \\
&= L(\mathcal{I}_g(y^*), y^*).
\end{aligned}$$

On the other hand, as (2.20) holds, we can apply Lemma 2.1.17 to obtain for any $y \in \mathbb{R}^n$

$$\begin{aligned}
L(\mathcal{I}_g(y^*), y) &= \mathbb{E}[U(\mathcal{I}_g(y^*))] + \underbrace{y'\mathbb{E}[g(\mathcal{I}_g(y^*))] + \delta_K(y)}_{\geq 0} \\
&\geq \mathbb{E}[U(\mathcal{I}_g(y^*))] \\
&= \mathbb{E}[U(\mathcal{I}_g(y^*))] + \underbrace{(y^*)'\mathbb{E}[g(\mathcal{I}_g(y^*))] + \delta_K(y^*)}_{=0} \\
&= L(\mathcal{I}_g(y^*), y^*).
\end{aligned}$$

Therefore, $(\mathcal{I}_g(y^*), y^*)$ is a saddle-point of L. □

Proof of Theorem 2.1.37. Consider $0 < \epsilon < 1$, $\Delta y \in X_K$ with $\|\Delta y\| \leq 1$ and define $y_\epsilon = y^* + \epsilon \Delta y$. Since y^* attains the infimum in (2.19), we know that

$$\mathbb{E}\left[U_g^*\left(y^*\right)\right] + \delta_K\left(y^*\right) \leq \mathbb{E}\left[U_g^*\left(y_\epsilon\right)\right] + \delta_K\left(y_\epsilon\right). \tag{A.8}$$

Moreover, we also have the Q-a.s. inequality

$$\begin{aligned}
&\frac{1}{\epsilon}\big(U^*\left(y_\epsilon\right) + \delta_K\left(y_\epsilon\right) - U^*\left(y^*\right) - \delta_K\left(y^*\right)\big) \\
&\leq \frac{1}{\epsilon}\Big(U\left(\mathcal{I}_g(y_\epsilon)\right) + (y_\epsilon)' g(\mathcal{I}_g(y_\epsilon)) - \underbrace{\sup_{x\in\mathbb{R}}\big(U(x) - (y^*)' g(x)\big)}_{\geq U(\mathcal{I}_g(y_\epsilon)) + (y^*)' g(\mathcal{I}_g(y_\epsilon))} + \delta_K\left(y_\epsilon\right) - \delta_K\left(y^*\right)\Big) \\
&\leq \frac{1}{\epsilon}\left((y_\epsilon - y^*)' g\left(\mathcal{I}_g(y_\epsilon)\right) + \delta_K\left(y_\epsilon\right) - \delta_K\left(y^*\right)\right) \\
&= (\Delta y)' g\left(\mathcal{I}_g(y_\epsilon)\right) + \frac{1}{\epsilon}\left(\delta_K\left(y_\epsilon\right) - \delta_K\left(y^*\right)\right).
\end{aligned} \tag{A.9}$$

Thus, due to the usc assumption on $\mathbb{E}\left[g(\mathcal{I}_g(y))\right]' \Delta y$, we can combine equations (A.8), (A.9) and take the upper limit $\epsilon \downarrow 0$ to obtain

$$\begin{aligned}
0 &\leq \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\mathbb{E}\left[U^*\left(y_\epsilon\right)\right] + \delta_K\left(y_\epsilon\right) - \left[U^*\left(y^*\right)\right] - \delta_K\left(y^*\right)\right) \\
&\leq \limsup_{\epsilon\downarrow 0}\left((\Delta y)' \mathbb{E}\left[g\left(\mathcal{I}_g(y_\epsilon)\right)\right]\right) + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\delta_K\left(y_\epsilon\right) - \delta_K\left(y^*\right)\right) \\
&\leq (\Delta y)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] + \limsup_{\epsilon\downarrow 0}\left(\frac{1}{\epsilon}\left(\delta_K\left(y_\epsilon\right) - \delta_K\left(y^*\right)\right)\right)
\end{aligned}$$

As δ_K is sub-additive and positive-homogenous of order 1 (cf. Lemma 2.1.17), this implies

$$\begin{aligned}
0 &\leq (\Delta y)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\delta_K\left(y^*\right) + \epsilon \delta_K\left(\Delta y\right) - \delta_K\left(y^*\right)\right) \\
&= (\Delta y)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] + \delta_K\left(\Delta y\right).
\end{aligned}$$

This inequality holds for all $\Delta y \in X_K$ with $\Delta y \leq 1$ and therefore Lemma 2.1.17, (iii) implies that $\mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] \in K$. Following the same steps with $\Delta y = -y^*$ yields $\delta_K(y_\epsilon) = (1-\epsilon)\delta_K(y^*)$ and thus

$$\begin{aligned}
& 0 \leq -(y^*)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] + \limsup_{\epsilon\downarrow 0} \frac{1}{\epsilon}\big((1-\epsilon)\delta_K\left(y^*\right) - \delta_K\left(y^*\right)\big) \\
& \quad = -(y^*)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] - \delta_K\left(y^*\right) \\
\Leftrightarrow \quad & 0 \geq (y^*)' \mathbb{E}\left[g\left(I_g(y^*)\right)\right] + \delta_K\left(y^*\right),
\end{aligned}$$

which implies $0 = (y^*)' \mathbb{E}\left[g\left(\mathcal{I}_g(y^*)\right)\right] + \delta_K\left(y^*\right)$. According to Corollary 2.1.36, $(\mathcal{I}_g(y^*), y^*)$ is a saddle-point of L and $\mathcal{I}_g(y^*)$ is optimal for (**P**). □

Proof of Lemma 2.1.38. We begin by proving statements (i), (ii) and (iii) for the generalized conjugate U_g^*:

(i) By the definition of U_g^*, we have

$$U_g^*(y) = \sup_{x\in\mathbb{R}}\big(U(x) + y' g(x)\big) = \sup_{x\in\mathbb{R}}\left(U(x) + \sum_{i=1}^{n} y_i g_i(x)\right).$$

Hence, if $g_i(x) \leq 0$, then U_g^* is non-increasing in y_i.

(ii) Similarly, we have $y_i \geq 0$ and $g_i(x) \leq \hat{g}_i(x)$ for all $x \in dom_U$. Hence,

$$U_g^*(y) = \sup_{x\in\mathbb{R}} \big(U(x) + \underbrace{y'g(x)}_{\leq y'\hat{g}(x)}\big) \leq \sup_{x\in\mathbb{R}} \big(U(x) + y'\hat{g}(x)\big) = U_{\hat{g}}^*(y).$$

(iii) Let $\alpha \in (0,1)$, $y_1, y_2 \in \mathbb{R}^n$. Then,

$$\begin{aligned}
U_g^*\left(\alpha y_1 + (1-\alpha)y_2\right) &= \sup_{x\in\mathbb{R}} \Big(U(x) + (\alpha y_1 + (1-\alpha)y_2)'g(x)\Big) \\
&= \sup_{x\in\mathbb{R}} \Big(\alpha\big[U(x) + y_1'g(x)\big] + (1-\alpha)\big[U(x) + y_2'g(x)\big]\Big) \\
&\leq \alpha \sup_{x\in\mathbb{R}} \Big(U(x) + y_1'g(x)\Big) + (1-\alpha) \sup_{x\in\mathbb{R}} \Big(U(x) + y_2'g(x)\Big) \\
&= \alpha U_g^*(y_1) + (1-\alpha)U_g^*(y_1),
\end{aligned}$$

i.e., U_g^* is convex.

We continue by proving statements (i) and (ii) for the maximizing argument $\mathcal{I}_g$. For this purpose, consider an arbitrary $\hat{x} \geq 0$ such that $\mathcal{I}_g(y) \leq \hat{x}$. Then:

$$\begin{aligned}
U_g^*(y) &= U(\mathcal{I}_g(y)) + y'g(\mathcal{I}_g(y)) = U(\mathcal{I}_g(y)) + \sum_{j=1}^n y_j g_j(\mathcal{I}_g(y)) \\
&\geq U(\hat{x}) + \sum_{j=1}^n y_j g_j(\hat{x}) = U(\hat{x}) + y'g(\hat{x}).
\end{aligned} \tag{A.10}$$

(i) If now $\hat{y} = (y_1, ..., y_{i-1}, \hat{y}_i, y_{i+1}, ..., y_n)'$ and $\hat{y}_i \geq y_i$, then we can use the fact that $\mathcal{I}_g(y) \leq \hat{x}$ and g_i is non-increasing to see that

$$\begin{aligned}
U_g^*(\hat{y}) \geq U(\mathcal{I}_g(y)) + \hat{y}'g(\mathcal{I}_g(y)) &= \underbrace{U(\mathcal{I}_g(y)) + y'g(\mathcal{I}_g(y))}_{\overset{(A.10)}{\geq} U(\hat{x})+y'g(\hat{x})} + \underbrace{(\hat{y}_i - y_i)}_{\geq 0}\underbrace{g_i(\mathcal{I}_g(y))}_{\geq g_i(\hat{x})} \\
&\geq U(\hat{x}) + y'g(\hat{x}) + (\hat{y}_i - y_i)g_i(\hat{x}) \\
&= U(\hat{x}) + \hat{y}'g(\hat{x}).
\end{aligned} \tag{A.11}$$

Thus, by the definition of the maximizing argument $\mathcal{I}_g$ as an infimum, we obtain

$$\begin{aligned}
\mathcal{I}_g(\hat{y}) &= \inf\big\{x \in \mathbb{R} \;\big|\; U_g^*(\hat{y}) \leq U(x) + \hat{y}'g(x)\big\} \\
&\overset{(A.11)}{=} \inf\big\{x \leq \mathcal{I}_g(y) \;\big|\; U_g^*(\hat{y}) \leq U(x) + \hat{y}'g(x)\big\} \leq \mathcal{I}_g(y),
\end{aligned}$$

i.e., $\mathcal{I}_g$ is non-increasing in y_i.

(ii) If $\hat{g}_i(x) \geq g_i(x)$ for all $x \in dom_U$, if both functions are non-increasing and if $y_i \geq 0$, then

$$\begin{aligned}
U_{\hat{g}}^*(y) \geq U(\mathcal{I}_g(y)) + y'\hat{g}(\mathcal{I}_g(y)) &= \underbrace{U(\mathcal{I}_g(y)) + y'g(\mathcal{I}_g(y))}_{\overset{(A.10)}{\geq} U(\hat{x})+y'g(\hat{x})} + y_i\big(\underbrace{\hat{g}_i(x)}_{\geq \hat{g}_i(\hat{x})} - g_i(\mathcal{I}_g(y))\big) \\
&\geq U(\hat{x}) + y'g(\hat{x}) + y_i\big(\hat{g}_i(\hat{x}) - g_i(\mathcal{I}_g(y))\big) \\
&= U(\hat{x}) + y'\hat{g}(\hat{x}) + y_i\underbrace{\big(g_i(\hat{x}) - g_i(\mathcal{I}_g(y))\big)}_{\geq 0} \\
&\geq U(\hat{x}) + y'\hat{g}(\hat{x}).
\end{aligned} \tag{A.12}$$

Again, by the definition of the maximizing argument $\mathcal{I}_{\hat{g}}$ as an infimum, we obtain

$$\begin{aligned}
\mathcal{I}_{\hat{g}}(y) &= \inf\big\{x \in \mathbb{R} \;\big|\; U_{\hat{g}}^*(y) \leq U(x) + y'\hat{g}(x)\big\} \\
&\overset{(A.12)}{=} \inf\big\{x \leq \mathcal{I}_g(y) \;\big|\; U_{\hat{g}}^*(y) \leq U(x) + y'\hat{g}(x)\big\} \leq \mathcal{I}_g(y).
\end{aligned}$$

□

Proof of Lemma 2.2.1.
Proof of (i): Since f is convex on $\mathbb{R}$, f is continuous and thereby bounded on any compact subset of $\mathbb{R}$. In particular, there exists $m, M \in \mathbb{R}$ such that

$$m \leq f(x) \leq M \quad \text{for all } x \text{ with } |x - x_0| \leq \epsilon.$$

Hence, the statement follows from Section A, Lemma 3.1.1 in [43].
Proof of (ii): Let L_1, L_2 denote the Lipschitz constant of f_1, f_2 respectively. For any real numbers $x, y \in \mathbb{R}$ recall the identity

$$\min(x, y) = \frac{1}{2}\left(x + y - |x - y|\right). \tag{A.13}$$

Therefore, for any $x, y \in I$ we have

$$\begin{aligned}
|f(x) - f(y)| &= \Big|\min\left(f_1(x), f_2(x)\right) - \min\left(f_1(y), f_2(y)\right)\Big| \\
&\overset{(A.13)}{=} \frac{1}{2}\Big|f_1(x) + f_2(x) - |f_1(x) - f_2(x)| - f_1(y) - f_2(y) + |f_1(y) - f_2(y)|\Big| \\
&\leq \frac{1}{2}\Big(|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| + \Big||f_1(x) - f_2(x)| - |f_1(y) - f_2(y)|\Big|\Big) \\
&\leq \frac{1}{2}\big(|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| + |f_1(x) - f_2(x) - (f_1(y) - f_2(y))\,|\big) \\
&= \frac{1}{2}\big(|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| + \underbrace{|f_1(x) - f_2(x) - f_1(y) + f_2(y)|}_{\leq |f_1(x)-f_1(y)|+|f_2(x)-f_2(y)|}\big) \\
&\leq \underbrace{|f_1(x) - f_1(y)|}_{\leq L_1|x-y|} + \underbrace{|f_2(x) - f_2(y)|}_{\leq L_2|x-y|} \\
&\leq (L_1 + L_2)|x - y|.
\end{aligned}$$

Hence, f is Lipschitz continuous on I with Lipschitz constant $L = L_1 + L_2$. □

Proof of 2.2.2. This statement follows directly from Chapter §6, 'VII. Existence and Uniqueness Theorem' in [83]. □

Proof of Lemma 2.2.3. We first consider the case $f(B_0) \neq 0$. The following argument proceeds by contradiction. Assume that B is not a strictly monotone function of τ. Since f is locally Lipschitz continuous and therefore continuous, B is continuously differentiable and there must exist $\bar{\tau} \in I$ with $B(\bar{\tau}) = \bar{B}$ such that

$$B'(\bar{\tau}) = f(\bar{B}) = 0 \text{ and } B'(\tau) \neq 0 \text{ for all } \tau \text{ with } |\tau - \tau_0| < |\bar{\tau} - \tau_0|.$$

In particular, B is also a solution to the ODE

$$B'(\tau) = f(B(\tau)), \quad B(\bar{\tau}) = \bar{B}. \tag{A.14}$$

On the other hand, as $f(\bar{B}) = f(B(\bar{\tau})) = B'(\bar{\tau}) = 0$, the function $\hat{B}(\tau) := \bar{B}$ is also a solution to the ODE (A.14). However, for all τ with $|\tau - \tau_0| < |\bar{\tau} - \tau_0|$ we know that $f(B(\tau)) = B'(\tau) \neq 0$ and thus $B(\tau) \neq \hat{B}(\tau)$. This is a contradiction to Theorem 2.2.2. Therefore, B must be strictly monotone in τ.
If $f(B_0) = 0$, by an analogous argument, $B(\tau) = B_0$ is the unique solution to ODE (2.21) and therefore B is constant in τ. □

Proof of Lemma 2.2.4. Statement (i) follows directly from Lemma 10.12 in [38] with $A = \frac{1}{2}r_2$, $B = r_1$, $C = r_0$, $u = B_0$ and

$$
\begin{aligned}
&-\frac{2r_0\left(e^{r_3\tau}-1\right)-\left(r_3\left(e^{r_3\tau}+1\right)+r_1\left(e^{r_3\tau}-1\right)\right)B_0}{r_3\left(e^{r_3\tau}+1\right)-r_1\left(e^{r_3\tau}-1\right)-r_2\left(e^{r_3\tau}-1\right)B_0} \\
&= -\frac{-2r_3B_0+\left(e^{r_3\tau}-1\right)\left(2r_0-\left(r_1+r_3\right)B_0\right)}{2r_3-\left(e^{r_3\tau}-1\right)\left(r_1+r_2B_0-r_3\right)} \\
&= \frac{2r_2r_3B_0-\left(e^{r_3\tau}-1\right)\big(\overbrace{2r_0r_2}^{=r_3^2-r_1^2}-\left(r_1+r_3\right)r_2B_0\big)}{2r_2r_3-r_2\left(e^{r_3\tau}-1\right)\left(r_1+r_2B_0-r_3\right)} \\
&= \frac{2r_2r_3B_0-\left(e^{r_3\tau}-1\right)\left(\left(r_3-r_1\right)\left(r_3+r_1\right)-\left(r_1+r_3\right)r_2B_0\right)}{2r_2r_3-r_2\left(e^{r_3\tau}-1\right)\left(r_1+r_2B_0-r_3\right)} \\
&= \frac{2r_2r_3B_0+\left(e^{r_3\tau}-1\right)\left(r_1+r_3\right)\left(r_1+r_2B_0-r_3\right)}{2r_2r_3-r_2\left(e^{r_3\tau}-1\right)\left(r_1+r_2B_0-r_3\right)}.
\end{aligned}
$$

The function B as in (2.23) is well-defined and continuously differentiable as long as the denominator in (2.23) is non-zero. In particular, taking derivatives verifies that B satisfies (2.22) as long as its denominator is non-zero. This is true for $\tau = 0$, when the denominator is equal to $r_2r_3 > 0$, and the denominator stays positive if $r_1 + r_2B_0 - r_3 \leq 0$, i.e., $t_+(B_0) = \infty$. However, if $r_1 + r_2B_0 - r_3 > 0$, then

$$
\begin{aligned}
0 &\stackrel{!}{=} 2r_2r_3 - r_2\left(e^{r_3t_+(B_0)}-1\right)\left(r_1+r_2B_0-r_3\right) \\
\Leftrightarrow\quad 2r_3 &= \left(e^{r_3t_+(B_0)}-1\right)\left(r_1+r_2B_0-r_3\right) \\
\Leftrightarrow\quad t_+(B_0) &= \frac{1}{r_3}\ln\left(\frac{r_1+r_2B_0+r_3}{r_1+r_2B_0-r_3}\right).
\end{aligned}
$$

□

Proof of Corollary 2.2.5. If $\tau_{\hat{B}} < t_+(B_0)$, then

$$B(\tau_{\hat{B}}) = \frac{2r_2r_3B_0+\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_3\right)\left(r_1+r_2B_0-r_3\right)}{2r_2r_3-r_2\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_2B_0-r_3\right)}.$$

This in turn implies

$$
\begin{aligned}
&\hat{B} \stackrel{!}{=} B(\tau_{\hat{B}}) = \frac{2r_2r_3B_0+\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_3\right)\left(r_1+r_2B_0-r_3\right)}{2r_2r_3-r_2\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_2B_0-r_3\right)} \\
\Leftrightarrow\quad &\hat{B}\left(2r_2r_3-r_2\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_2B_0-r_3\right)\right) \\
&\quad = 2r_2r_3B_0+r_2\left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_3\right)\left(r_1+r_2B_0-r_3\right) \\
\Leftrightarrow\quad &2r_2r_3\left(\hat{B}-B_0\right) = \left(e^{r_3\tau_{\hat{B}}}-1\right)\left(r_1+r_2B_0-r_3\right)\left(r_1+\hat{B}r_2+r_3\right)
\end{aligned}
$$

$$\Leftrightarrow \quad (e^{r_3\tau_{\hat{B}}} - 1) = \frac{2r_2r_3\left(\hat{B} - B_0\right)}{(r_1 + r_2B_0 - r_3)\left(r_1 + \hat{B}r_2 + r_3\right)}$$

$$\Leftrightarrow \quad \tau_{\hat{B}} = \frac{1}{r_3}\ln\left(\frac{2r_2r_3\left(\hat{B} - B_0\right)}{(r_1 + r_2B_0 - r_3)\left(r_1 + \hat{B}r_2 + r_3\right)} + 1\right)$$

$$= \frac{1}{r_3}\ln\left(\frac{2r_2r_3\left(\hat{B} - B_0\right) + (r_1 + r_2B_0 - r_3)\left(r_1 + \hat{B}r_2 + r_3\right)}{(r_1 + r_2B_0 - r_3)\left(r_1 + \hat{B}r_2 + r_3\right)}\right)$$

□

Proof of Theorem 2.3.2. This is a special case of Theorem 3.41 in [51]. □

A.2 Proofs Chapter 3

Proof of Lemma 3.3.1. The proof follows along the lines of Proposition 7.3 in [17]. Consider the non-negative Q-martingale M defined as

$$M(t) = \mathbb{E}\left[D\tilde{Z}(T) \mid \mathcal{F}_t\right].$$

By a variation of the martingale representation theorem (see Theorem 4.15 in [48]), there exists an $\mathbb{R}^d$-valued process ψ with $\int_0^T \|\psi(s)\|^2 ds < \infty$ Q-a.s, such that M can be expressed as a stochastic integral

$$M(t) = v_0 + \int_0^t \psi(t)' dW(t).$$

Due to the non-negativity and martingale property of M, $M(t) = 0$ for one $t \in (0, T)$ implies $M(s) = 0$ $\forall s \in [t, T]$. We define the processes

$$X(t) = \frac{M(t)}{\tilde{Z}(t)}$$

$$\text{and} \quad \pi(t) := \left(\Sigma(t)'\right)^{-1}\left[\frac{\psi(t)}{M(t)} + \gamma(t)\right]\mathbb{1}_{\{M(t)>0\}}.$$

Applications of Itô's lemma and Itô's product rule yield

$$d\left(\frac{1}{\tilde{Z}(t)}\right) = -\frac{1}{(\tilde{Z}(t))^2}d\tilde{Z}(t) + \frac{1}{(\tilde{Z}(t))^3}d\langle\tilde{Z}\rangle_t = \frac{1}{\tilde{Z}(t)}\left(\left[r(t) + \|\gamma(t)\|^2 dt\right] + \gamma(t)'dW(t)\right)$$

and

$$\begin{aligned}
dX(t) &= \frac{1}{\tilde{Z}(t)}dM(t) + M(t)d\left(\frac{1}{\tilde{Z}(t)}\right)dt + \left\langle M, \frac{1}{\tilde{Z}}\right\rangle_t \\
&= \frac{1}{\tilde{Z}(t)}\left(\psi(t)'dW(t) + M(t)\left[r(t) + \|\gamma(t)\|^2\right]dt + M(t)\gamma(t)'dW(t) + \gamma(t)'\psi(t)dt\right) \\
&= \frac{1}{\tilde{Z}(t)}\left(\left[M(t)r(t) + \gamma(t)'\underbrace{(M(t)\gamma(t) + \psi(t))}_{=M(t)\Sigma(t)'\pi(t)}\right]dt + \underbrace{(M(t)\gamma(t) + \psi(t))}_{=M(t)\Sigma(t)'\pi(t)}{}'dW(t)\right) \\
&= X(t)\left(\left[r(t) + (\mu(t) - r(t)\mathbb{1})'\pi(t)\right]dt + \pi(t)'\Sigma(t)dW(t)\right).
\end{aligned}$$

Hence, $V^{v_0,\pi}$ and X satisfy the same SDE. Further, as $X(0) = M(0) = v_0$, the wealth process $V^{v_0,\pi}$ and X coincide. □

Proof of Lemma 3.3.2.
As $C(v_0,B_1,B_2,g)\subset C(v_0,g)$ and $U(D)=U(D;B_1,B_2)$ for all $D\in C(v_0,B_1,B_2,g)$, we clearly have $\Phi_0(v_0)\leq\tilde{\Phi}_0(v_0)$.
To show the opposite inequality, we take any $D\in C(v_0,g)$ with $Q(D>B_2)>0$, and show that there exists a $D'\in C(v_0,B_1,B_2,g)$ with $\mathbb{E}[U(D')]=\mathbb{E}[U(D';B_1,B_2)]>\mathbb{E}[U(D;B_1,B_2)]$. For this purpose, let $D\in C(v_0,g)$ with $Q(D>B_2)>0$.

- If $Q(D<B_1)>0$, then $\mathbb{E}[U(D;B_1,B_2)]=-\infty$ and by Assumption 3.2.1, we can choose $D'=v_0P_0(T)\in C(v_0,B_1,B_2,g)$ to obtain

$$E[U(D;B_1,B_2)]=-\infty<\mathbb{E}[U(v_0P_0(T))]=\mathbb{E}[U(D')]=\mathbb{E}[U(D';B_1,B_2)].$$

- If $Q(D<B_1)=0$, then we define for $\alpha\in[0,1]$:

$$D_\alpha=B_2\mathbb{1}_{\{D>B_2\}}+D\mathbb{1}_{\{D\leq B_2\}}+\alpha\left(B_2-D\right)\mathbb{1}_{\{D\leq B_2\}}.$$

By construction, we have $B_1\leq D<D_\alpha\leq B_2$ for all $\alpha\in(0,1]$. In particular, as U is strictly increasing on its domain and g is non-increasing, this implies for any $\alpha\in(0,1]$:

$$\mathbb{E}\left[U(D)\right]<\mathbb{E}\left[U(D_\alpha)\right]$$

and

$$\mathbb{E}\left[g(D_\alpha)\right]\leq\mathbb{E}\left[g(D)\right]\overset{D\in C(v_0,g)}{\leq}0.$$

Hence, we only need to find a specific $\hat{\alpha}\in(0,1]$ such that $v_0=\mathbb{E}[\tilde{Z}(T)D_{\hat{\alpha}}]$ and choose $D'=D_{\hat{\alpha}}$. Following Lemma 3.3.1 and standing assumption (3.3), the risk-free portfolio $\pi\equiv0$ satisfies

$$\mathbb{E}\left[\tilde{Z}(T)B_1\right]<\underbrace{\mathbb{E}\left[\tilde{Z}(T)V^{v_0,0}(T)\right]}_{=v_0}<\mathbb{E}\left[\tilde{Z}(T)B_2\right]. \tag{A.15}$$

We define the quantities

$$\begin{aligned}
\bar{v}&=\mathbb{E}\left[\tilde{Z}(T)D_0\right]<v_0\\
\hat{v}&=\mathbb{E}\left[\tilde{Z}(T)\left(D-B_2\right)\mathbb{1}_{\{D>B_2\}}\right]=v_0-\bar{v}>0,\\
\tilde{v}&=\mathbb{E}\left[\tilde{Z}(T)\left(B_2-D\right)\mathbb{1}_{\{D\leq B_2\}}\right]>0.
\end{aligned}$$

Due to $D\in C(v_0,g)$ and (A.15), we obtain

$$\hat{v}-\tilde{v}=\mathbb{E}\left[\tilde{Z}(T)\left(D-B_2\right)\right]=v_0-\mathbb{E}\left[\tilde{Z}(T)B_2\right]<0.$$

Hence, choosing $0<\hat{\alpha}=\frac{\hat{v}}{\tilde{v}}<1$ yields

$$\begin{aligned}
\mathbb{E}[\tilde{Z}(T)D_{\hat{\alpha}}]&=\mathbb{E}\left[\tilde{Z}(T)\Big(B_2\mathbb{1}_{\{D>B_2\}}+D\mathbb{1}_{\{D\leq B_2\}}+\hat{\alpha}\left(B_2-D\right)\mathbb{1}_{\{D\leq B_2\}}\Big)\right]\\
&=\mathbb{E}\left[\tilde{Z}(T)D_0\right]+\hat{\alpha}\mathbb{E}\left[\tilde{Z}(T)\left(B_2-D\right)\mathbb{1}_{\{D\leq B_2\}}\right]\\
&=\bar{v}+\frac{\hat{v}}{\tilde{v}}\tilde{v}=v_0.
\end{aligned}$$

In particular, $D_{\hat{\alpha}}\in C(v_0,B_1,B_2,g)$ and $\mathbb{E}\left[U(D)\right]<\mathbb{E}\left[U(D_{\hat{\alpha}})\right]$.

In total, this implies

$$\begin{aligned}\tilde{\Phi}_0(v_0) &= \sup_{D\in C(v_0,g)} \mathbb{E}[U(D;B_1,B_2)]\\ &= \sup_{D\in C(v_0,B_1,B_2,g)} \mathbb{E}[U(D;B_1,B_2)]\\ &= \sup_{D\in C(v_0,B_1,B_2,g)} \mathbb{E}[U(D)]\\ &= \Phi_0(v_0).\end{aligned}$$

□

Proof of Theorem 3.3.3. Following Lemma 2.1.35, we may write the dual optimization problem associated with $(\tilde{\mathbf{P}}_\mathbf{0})$ as

$$(\mathbf{D_0})\left\{\Psi_{D_0} = \inf_{y\in\mathbb{R}^2}\Big(\mathbb{E}\left[U^*_{g_0}(y;B_1,B_2)\right] + \delta_{K_V}(y)\Big).\right.$$

According to Theorem 2.1.37, if $y^* = (y_0^*, y_1^*) \in \mathbb{R}^2$ is optimal for $(\mathbf{D_0})$, $\mathcal{I}_{g_0}(y^*;B_1,B_2) \in L^2_Q$ and

$$y \to \mathbb{E}\left[g_0\big(\mathcal{I}_{g_0}(y;B_1,B_2)\big)\right]'\Delta y \tag{A.16}$$

is usc at $y = y^*$ for all $\Delta y \in X_{K_V}$, then $\mathcal{I}_{g_0}(y^*;B_1,B_2)$ is (admissible and) optimal for $(\tilde{\mathbf{P}}_\mathbf{0})$ and

$$(y^*)'\mathbb{E}\left[g_0(\mathcal{I}_{g_0}(y^*;B_1,B_2))\right] + \delta_{K_V}(y^*) = 0.$$

Thus, it remains to show that $\delta_{K_V}(y) = y_0v_0$ for all $y = (y_0, y_1) \in [0,\infty)^2$ and that the minimization in $(\mathbf{D_0})$ can be restricted to $[0,\infty)^2$.

For $y = (y_0, y_1)' \in \mathbb{R}^2$ the support function δ_{K_V} can be computed explicitly as

$$\delta_{K_V}(y) = -\inf_{x\in\{-v_0\}\times[0,\infty)}(y'x) = y_0v_0 - \inf_{x_1\in[0,\infty)}(y_1x_1) = \begin{cases} y_0v_0 & y_1 \ge 0\\ \infty & y_1 < 0.\end{cases} \tag{A.17}$$

In particular, $X_K = \mathbb{R}\times[0,\infty)$ and $\delta_K(y) = y_0v_0$ for all $y \in X_K$. Moreover, U is strictly increasing on its domain, $Q(\tilde{Z}(T) > 0) = 1$, g is non-increasing and $V^{v_0,0}(T) < B_2$ (Assumption 3.2.1). Thus, we have for any $y = (y_0, y_1)' \in (-\infty, 0)\times[0,\infty)$

$$\begin{aligned}U^*_{g_0}(y;B_1,B_2) + y_0v_0 &= \sup_{B_1\le x\le B_2}\big(\underbrace{U(x) - y_0\tilde{Z}(T)x - y_1g(x)}_{\text{strictly increasing in } x}\big) - y_0v_0\\ &= U(B_2) - y_0\left(\tilde{Z}(T)B_2 - v_0\right) - y_1g(B_2)\\ &> U(B_2) - y_0\left(\tilde{Z}(T)V^{v_0,0}(T) - v_0\right) - y_1g(V^{v_0,0}(T)).\end{aligned}$$

Further, by Assumption, $\mathbb{E}[g(B_2)] \le \mathbb{E}[g(V^{v_0,0}(T))] \le 0$. Hence,

$$\begin{aligned}\mathbb{E}[U^*_{g_0}(y;B_1,B_2)] - y_0v_0 &> \mathbb{E}\left[U(B_2)\right] - y_0\left(\mathbb{E}\left[\tilde{Z}(T)V^{v_0,0}(T)\right] - v_0\right) - y_1\mathbb{E}[g(V^{v_0,0}(T))]\\ &\overset{A.3.2.1}{\ge} \mathbb{E}[U(B_2)] - y_0\left(\mathbb{E}[\tilde{Z}(T)V^{v_0,0}(T)] - v_0\right)\\ &\overset{L.3.3.1}{=} \mathbb{E}[U(B_2)]\\ &= \mathbb{E}[U^*_{g_0}(0;B_1,B_2)] + 0\cdot v_0.\end{aligned} \tag{A.18}$$

Thus, we may restrict the minimization in ($\mathbf{D_0}$) to

$$\begin{aligned}\Psi_{\hat{D}_0} &= \inf_{y\in\mathbb{R}^2}\Big(\mathbb{E}\left[U^*_{g_0}(y;B_1,B_2)\right]+\delta_{K_V}(y)\Big)\\ &\overset{(A.17)}{=} \inf_{y=(y_0,y_1)'\in\mathbb{R}\times[0,\infty)}\Big(\mathbb{E}\left[U^*_{g_0}(y;B_1,B_2)\right]+y_0v_0\Big)\\ &\overset{(A.18)}{=} \inf_{y=(y_0,y_1)'\in[0,\infty)^2}\Big(\mathbb{E}\left[U^*_{g_0}(y;B_1,B_2)\right]+y_0v_0\Big).\end{aligned}$$

□

Proof of Lemma 3.4.1. The poof is completely analogous to the proof of Lemma 3.3.1. □

Proof of Lemma 3.4.2.

(i) This statement follows from the analogous steps as in the proof of Lemma 3.3.2.

(ii) By assumption, we know that $\frac{v_0}{v_\lambda(B_2)} > 1$, $D^* = \frac{v_0}{v_\lambda(B_2)}B_2 > B_2$,

$$\mathbb{E}[D^*\tilde{Z}_\lambda(T)] = \frac{v_0}{v_\lambda(B_2)}\mathbb{E}[B_2\tilde{Z}_\lambda(T)] = v_0.$$

and due to the monotonicity of g, (3.3) and (3.4)

$$\mathbb{E}[g(D^*)] \leq \mathbb{E}[g(B_2)] \leq \mathbb{E}[g(v_0P_0)] \leq 0.$$

Hence, $D^* \in C_\lambda(v_0,g)$ and

$$\tilde{\Phi}_\lambda(v_0) = \sup_{D\in C_\lambda(v_0,g)} \mathbb{E}\big[\underbrace{U(D;B_1,B_2)}_{\leq U(B_2)}\big] \leq \mathbb{E}[U(B_2)] \leq \mathbb{E}[U(D^*)] \leq \tilde{\Phi}_\lambda(v_0),$$

i.e., D^* is optimal for ($\tilde{\mathbf{P}}_\lambda$).

□

Proof of Lemma 3.4.3. The argument goes along the lines of [17], Proposition 8.3. As π_{λ^*} satisfies the slackness condition

$$\delta_K(\lambda^*(t)) + \pi_{\lambda^*}(t)'\lambda^*(t) = 0 \quad \mathcal{L}[0,T]\otimes Q - a.e.,$$

we have $V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(0) = V^{v_0,\pi_{\lambda^*}}(0) = v_0$ and

$$\begin{aligned}dV^{v_0,\pi_{\lambda^*}}_{\lambda^*}(t) &= V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(t)\big([r(t)+(\mu(t)-r(t)\mathbb{1})'\pi_{\lambda^*}(t)]+\pi_{\lambda^*}(t)'\Sigma(t)dW(t)\big)\\ &\quad+\underbrace{V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(t)[\delta_K(\lambda^*(t))+\pi_{\lambda^*}(t)'\lambda^*(t)]}_{=0}dt\\ &= V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(t)\big([r(t)+(\mu(t)-r(t)\mathbb{1})'\pi_{\lambda^*}(t)]+\pi_{\lambda^*}(t)'\Sigma(t)dW(t)\big).\end{aligned}$$

Therefore, $V^{v_0,\pi_{\lambda^*}}_{\lambda^*}$ and $V^{v_0,\pi_{\lambda^*}}$ are solutions to the same SDE and $V^{v_0,\pi_{\lambda^*}}_{\lambda^*}(T) = V^{v_0,\pi_{\lambda^*}}(T)$ holds Q-a.s.. Further,

- $\pi_{\lambda^*} \in K \; \mathcal{L}[0,T] \otimes Q - a.e.$.
- $V^{v_0,\pi_{\lambda^*}}(T) = V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T) \geq B_1$ Q-a.s., as $V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T)$ is optimal for $(\tilde{\mathbf{P}}_{\lambda^*})$.
- $V^{v_0,\pi_{\lambda^*}}(T) = V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T) \leq B_2$ Q-a.s., as $V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T)$ is optimal for $(\tilde{\mathbf{P}}_{\lambda^*})$ and $v_0 \leq v_{\lambda^*}(B_2)$ (Lemma 3.4.2).
- $\mathbb{E}\left[g(V^{v_0,\pi_{\lambda^*}}(T))\right] = \mathbb{E}\left[g(V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T))\right] \leq 0$, as $V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T)$ is admissible for $(\tilde{\mathbf{P}}_{\lambda^*})$.
- $\mathbb{E}\left[U(V^{v_0,\pi_{\lambda^*}}(T))\right] = \mathbb{E}\left[U(V^{v_0,\pi_{\lambda^*}}(T);B_1,B_2)\right] = \mathbb{E}\left[U(V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T);B_1,B_2)\right] = \tilde{\Phi}_{\lambda^*}(v_0)$.

Thus, $\pi_{\lambda^*} \in \Lambda(v_0, K, B_1, B_2, g)$ and

$$\mathbb{E}\left[U(V^{v_0,\pi_{\lambda^*}}(T))\right] \leq \Phi(v_0) \leq \tilde{\Phi}_{\lambda^*}(v_0) = \mathbb{E}\left[U(V^{v_0,\pi_{\lambda^*}}(T))\right],$$

i.e., π_{λ^*} is optimal for $(\mathbf{P})$. □

Proof of Theorem 3.4.4. The poof is completely analogous to the proof of Lemma 3.4.4. □

Lemma A.2.1. *Let $\lambda \in \mathcal{D}'$, $y^* = (y_0^*, y_1^*)' \in (0,\infty) \times [0,\infty)$. Then,*

(i)

$$Q\Big(y_0 \to \mathcal{I}_{g_\lambda}(y_0, y_1^*; B_1, B_2) \textit{ is continuous at } y_0 = y_0^*\Big) = 1.$$

(ii) There exists $y_0^+ < y_0^*$ such that

$$\mathcal{I}_{g_\lambda}(y_0^*, y_1^*; B_1, B_2) \leq \mathcal{I}_{g_\lambda}(y_0^+, y_1^*; B_1, B_2) \textit{ and } \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y_0^+, y_1^*; B_1, B_2)\tilde{Z}_\lambda(T)\right] < \infty.$$

Proof of Lemma A.2.1.

(i) Consider the function

$$\hat{g}(x) := \begin{pmatrix} -x \\ -g(x) \end{pmatrix}.$$

Then, $\mathcal{I}_{\hat{g}}(y_0, y^*; B_1, B_2)$ is non-increasing in y_0 by Lemma 2.1.38, (i). Hence, according to the Darboux-Froda Theorem (cf. Theorem 8-2 in [1]), the function

$$y_0 \to \mathcal{I}_{\hat{g}}(y_0, y_1^*; B_1, B_2) \tag{A.19}$$

has at most countably infinite points of discontinuity on $(0,\infty)$. Further,

$$\mathcal{I}_{g_\lambda}(y_0, y_1^*; B_1, B_2) = \mathcal{I}_{\hat{g}}(y_0\tilde{Z}_\lambda(T), y_1^*; B_1, B_2) \quad Q\text{-a.s.}$$

and the random variable $\tilde{Z}_\lambda(T)$ has a continuous distribution on $(0,\infty)$. As the points of discontinuity of (A.19) are a null set with respect to the Lebesgue measure $\mathcal{L}(0,\infty)$, this implies

$$Q\Big(y_0 \to \mathcal{I}_{g_\lambda}(y_0, y_1^*; B_1, B_2) \text{ is continuous at } y_0 = y_0^*\Big) = 1.$$

(ii) Since $\lambda \in \mathcal{D}'$, we know that

$$y \to \mathbb{E}\left[g_\lambda\big(\mathcal{I}_{g_\lambda}(y; B_1, B_2)\big)\right]' \Delta y$$

is usc at $y = y^* = (y_0^*, y_1^*)'$ for all $\Delta y \in \mathbb{R} \times [0, \infty)$. Choosing $\Delta y = (-1, 0)'$ implies

$$\begin{aligned}\limsup_{y \to y^*} \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y; B_1, B_2)\tilde{Z}_\lambda(T)\right] &= \limsup_{y \to y^*} \mathbb{E}\left[g_\lambda\big(\mathcal{I}_{g_\lambda}(y; B_1, B_2)\big)\right]' \Delta y \\ &\overset{usc}{\leq} \mathbb{E}\left[g_\lambda\big(\mathcal{I}_{g_\lambda}(y^*; B_1, B_2)\big)\right]' \Delta y = \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y^*; B_1, B_2)\tilde{Z}_\lambda(T)\right] \overset{\lambda \in \mathcal{D}'}{=} v_0 \end{aligned} \tag{A.20}$$

Hence, there must exist a $y_0^+ < y_0^*$, as described in the statement of this lemma, since otherwise

$$\limsup_{y \to y^*} \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y; B_1, B_2)\tilde{Z}_\lambda(T)\right] \geq \limsup_{y_0 \uparrow y_0^*} \mathbb{E}\left[\mathcal{I}_{g_\lambda}(y_0, y_1^*; B_1, B_2)\tilde{Z}_\lambda(T)\right] = \infty,$$

which contradicts (A.20).

□

Proof of Theorem 3.4.5.

$(\tilde{B}) \Rightarrow (\tilde{A})$:
This is the statement of Lemma 3.4.3.

$(\tilde{B}) \Rightarrow (\tilde{C})$:
By Lemma 3.4.3 and (3.6), we have for any $\lambda \in \mathcal{D}$:

$$\tilde{\Phi}_{\lambda^*}(v_0) \overset{L.3.4.3}{=} \Phi(v_0) \overset{(3.6)}{\leq} \tilde{\Phi}_\lambda(v_0).$$

$(\tilde{B}) \Rightarrow (\tilde{E})$:
Due to condition $(\tilde{B})$, the additional drift from the portfolio $\pi_{\lambda^*} \in \Lambda(v_0, K, B_1, B_2, g)$ in $\mathcal{M}_{\lambda^*}$ is zero $\mathcal{L}[0,T] \otimes Q$-a.e. and therefore, $V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(t) = V^{v_0,\pi_{\lambda^*}}(t)$ $\mathcal{L}[0,T] \otimes Q$-a.s. However, for any $\lambda \in \mathcal{D}$, we can use Itô's product rule to see

$$\begin{aligned} d(\tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t)) &= \tilde{Z}_\lambda(t)dV^{v_0,\pi_{\lambda^*}}(t) + V^{v_0,\pi_{\lambda^*}}(t)d\tilde{Z}_\lambda(t) + d\langle \tilde{Z}_\lambda, V^{v_0,\pi_{\lambda^*}}\rangle_t \\ &= -\tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t) \cdot \Big(\big(r(t) + \delta_K(\lambda(t))\big)dt + \gamma_\lambda(t)dW(t)\Big) \\ &\quad + \tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t) \cdot \Big(r(t) + [\mu(t) - r(t)\mathbb{1}]'\pi_{\lambda^*}(t)\Big)dt \\ &\quad + \tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t)\pi_{\lambda^*}(t)'\Sigma(t)dW(t) \\ &\quad - \tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t) \underbrace{\gamma_\lambda(t)'\Sigma(t)'\pi_{\lambda^*}(t)}_{= [\mu(t) - r(t)\mathbb{1} + \lambda(t)]'\pi_{\lambda^*}(t)} dt \\ &= -\tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t)\Big(\delta_K(\lambda(t)) + \lambda(t)'\pi_{\lambda^*}(t)\Big)dt \\ &\quad + \tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t)\Big(\pi_{\lambda^*}(t)'\Sigma(t) - \gamma_\lambda(t)'\Big)dW(t) \end{aligned}$$

Integrating this SDE then shows that

$$\tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t) + \int_0^t \tilde{Z}_\lambda(s)V^{v_0,\pi_{\lambda^*}}(s) \underbrace{\big(\delta_K(\lambda(s)) + \lambda(s)'\pi_{\lambda^*}(s)\big)}_{\geq 0 \text{ by L. 2.1.17, (iii)}} ds \tag{A.21}$$

$$= v_0 + \int_0^t \tilde{Z}_\lambda(s) V^{v_0,\pi_{\lambda^*}}(s)\big(\pi_{\lambda^*}(s)'\Sigma(s) - \gamma_\lambda(s)\big)' dW(s). \tag{A.22}$$

As (A.21) is non-negative, so is (A.22). Thus, both (A.21) and (A.22) are non-negative local martingales and therefore supermartingales. Hence, we obtain from the supermartingale property

$$\begin{aligned}
&\mathbb{E}[\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)\tilde{Z}_\lambda(T)] = \mathbb{E}[V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T)\tilde{Z}_\lambda(T)] = \mathbb{E}[V^{v_0,\pi_{\lambda^*}}(T)\tilde{Z}_\lambda(T)] \\
&\quad \le \mathbb{E}[V^{v_0,\pi_{\lambda^*}}(T)\tilde{Z}_\lambda(T) + \int_0^T \underbrace{\tilde{Z}_\lambda(t)V^{v_0,\pi_{\lambda^*}}(t)}_{\ge 0} \underbrace{\delta_K(\lambda(t)) + \lambda(t)'\pi(t)}_{\ge 0 \text{ by L. 2.1.17, (iii)}} dt] \le v_0.
\end{aligned}$$

$(\tilde{E}) \Rightarrow (\tilde{B})$:
Follows from exactly the same arguments as Theorem 9.1 in [17] with the choice

$$B = V_{\lambda^*}^{v_0,\pi_{\lambda^*}}(T) = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2).$$

$(\tilde{C}) \Rightarrow (\tilde{D})$:
Let λ^* satisfy Condition $(\tilde{C})$, let $\lambda \in \mathcal{D}$ and $D \in C_\lambda(v_0, g)$ be an admissible terminal wealth for $(\tilde{\mathbf{P}}_\lambda)$. Then,

$$\mathbb{E}\left[D\tilde{Z}_\lambda(T)\right] = v_0 \quad \text{and} \quad \mathbb{E}[g(D; B_2)] \le 0.$$

Thus, we obtain for any $y = (y_0, y_1)' \in [0, \infty)$

$$\begin{aligned}
&\mathbb{E}[U(D; B_1, B_2)] \\
&\quad = \mathbb{E}\Big[\underbrace{U(D; B_1, B_2) - y_0\tilde{Z}_\lambda(T)D - y_1 g(D; B_2)}_{\le U^*_{g_\lambda}(y; B_1, B_2)}\Big] + y_0 v_0 + \underbrace{y_0\left(\mathbb{E}\left[\tilde{Z}_\lambda(T)D\right] - v_0\right)}_{=0} + \underbrace{\mathbb{E}\left[y_1 g(D; B_2)\right]}_{\le 0} \\
&\quad \le \mathbb{E}[U^*_{g_\lambda}(y; B_1, B_2)] + y_0 v_0.
\end{aligned} \tag{A.23}$$

Noting that the right-hand side is independent of D, we can take the supremum over all terminal wealths $D \in C_\lambda(v_0, g(\cdot, B_2))$ admissible for $(\tilde{\mathbf{P}}_\lambda)$ and receive

$$\tilde{\Phi}_\lambda(v_0) \le \mathbb{E}[U^*_{g_\lambda}(y; B_1, B_2)] + y_0 v_0.$$

Due to Theorem 3.4.4, the choice of $\lambda = \lambda^*$, $y = y^*$ and $D = \mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)$ achieves equality in (A.23). Combining this fact with Condition $(\tilde{C})$ finally gives

$$\begin{aligned}
&\mathbb{E}[U^*_{g_\lambda}(y^*; B_1, B_2)] + y_0^* v_0 \ge \tilde{\Phi}_\lambda(v_0) \ge \tilde{\Phi}_{\lambda^*}(v_0) = \mathbb{E}[U^*_{g_{\lambda^*}}(y^*; B_1, B_2)] + y_0^* v_0 \\
&\Rightarrow \quad \mathbb{E}[U^*_{g_\lambda}(y^*; B_1, B_2)] \ge \mathbb{E}[U^*_{g_{\lambda^*}}(y^*; B_1, B_2)].
\end{aligned}$$

$(\tilde{D}) \Rightarrow (\tilde{B})$:
Let $\lambda^* \in \mathcal{D}'$ satisfy Condition $(\tilde{D})$. Let $\pi := \pi_{\lambda^*}$ be the portfolio process that attains $\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)$ in $\mathcal{M}_{\lambda^*}$. We now need to show two things:

$$\pi(t) \in K \quad \mathcal{L}[0,T] \otimes Q - a.e., \tag{A.24}$$

$$[\pi(t)'\lambda^*(t) + \delta_K(\lambda^*(t))] = 0 \quad \mathcal{L}[0,T] \otimes Q - a.e.. \tag{A.25}$$

Before doing this, we need to introduce a variety of different notations. Let $\nu \in \mathcal{D} \cup \{-\lambda^*\}$ be arbitrary but fixed. Let $\epsilon \in (0,1)$, $t \in [0,T]$ and $n \in \mathbb{N}$. We define:

- $$\hat{\delta}^{(\nu)}(\lambda^*(t)) := \begin{cases} -\delta_K(\lambda^*(t)), & \nu = -\lambda^* \\ \delta_K(\nu(t)), & \text{else} \end{cases}$$
- $L_t := L_t^{(\nu)} := \int_0^t \hat{\delta}^{(\nu)}(\lambda^*(s))ds$
- $W_{\lambda^*}(t) := W(t) + \int_0^t \gamma_{\lambda^*}(s)ds$
- $N_t := N_t^{(\nu)} := \int_0^t \Big(\Sigma(s)^{-1}\nu(s)\Big)' dW_{\lambda^*}(s)$
- $$\begin{aligned} \tau_n := \ & T \wedge \inf\Big\{t \in [0,T] \ \Big| \ |L_t| \geq n, \text{ or } |N_t| \geq n, \text{ or} \\ & \text{or } \int_0^t \|\Sigma(s)^{-1}\nu(s)\|^2 ds \geq n, \\ & \text{or } \int_0^t \|\gamma(t) + \Sigma(t)^{-1}\lambda^*(s)\|^2 ds \geq n, \\ & \text{or } \int_0^t \left(\frac{V_{\lambda^*}^{v_0,\pi}(s)}{P_0^{\lambda^*}(s)}\right)^2 \|\Sigma(s)^{-1}\nu(s) + (L_s + N_s)\Sigma(s)'\pi(s)\|^2 ds \geq n\Big\} \end{aligned}$$
- $\lambda_{\epsilon,n}(t) := \lambda^*(t) + \epsilon\nu(t)\mathbb{1}_{\{t \leq \tau_n\}} \Rightarrow \lambda_{\epsilon,n} \in \mathcal{D}$

The ensuing proof is split into three parts and is structured as follows. In Part 1 and Part 2 we prove (A.24) and (A.25) by showing the analogue of (9.14) and (9.15) from [17]:

$$\begin{aligned} 0 &\overset{(\tilde{D})}{\leq} \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon}\Big(\mathbb{E}\big[U^*_{g_{\lambda_{\epsilon,n}}}(y^*; B_1, B_2) - U^*_{g_{\lambda^*}}(y^*; B_1, B_2)\big]\Big) \\ &\overset{\text{Part 1}}{\leq} y_0^*\mathbb{E}\big[\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)\tilde{Z}_{\lambda^*}(T)(L_{\tau_n} + N_{\tau_n})\big] \\ &= y_0^*\mathbb{E}\big[V_{\lambda^*}^{v_0,\pi}(T)\tilde{Z}_{\lambda^*}(T)(L_{\tau_n} + N_{\tau_n})\big] \\ &\overset{\text{Part 2}}{\leq} y_0^*\mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{\tau_n} \underbrace{\frac{V_{\lambda^*}^{v_0,\pi}(t)}{P_0^{\lambda^*}(t)}}_{=:\tilde{V}_{\lambda^*}^{v_0,\pi}(t)}[\pi(t)'\nu(t) + \hat{\delta}^{(\nu)}(\lambda^*(t))]dt\Big] \end{aligned} \tag{A.26}$$

Part 1 closely follows the argument of the proof of Theorem 10.1 in [17]. In the end, the main objective of Part 1 is showing that we can apply Fatou's Lemma to interchange limsup and expectation by using properties of the conveniently stopped processes. Part 2 closely follows the argument in Step 5 of the proof of Theorem 9.1 in [17]. In Part 3, we choose arbitrary $\nu \in \mathcal{D}$ to show (A.24) and $\nu := -\lambda^*$ to show (A.25). This corresponds to Step 3 of the proof of Theorem 9.1 in [17].

We will elaborate further on the third inequality in (A.26), the last equality in (A.26) and the concluding argument mentioned previously in chronological order. This splits the proof into three parts.
Part 1: We start off by proving a useful set of inequalities that will form an integral part of this proof. First of all, in either case $\nu \in \mathcal{D}$ and $\nu = -\lambda^*$ we have for every $t \in [0, \tau_n]$:

$$\delta_K(\lambda_{\epsilon,n}(t)) - \delta_K(\lambda^*(t)) = \delta_K(\lambda^*(t) + \epsilon\nu(t)) - \delta_K(\lambda^*(t)) \leq \epsilon\hat{\delta}^{(\nu)}(\lambda^*(t)) \tag{A.27}$$

Case $\nu \in \mathcal{D}$:

$$\delta_K(\lambda^*(t) + \epsilon\nu(t)) - \delta_K(\lambda^*(t)) \leq \delta_K(\lambda^*(t)) + \epsilon\delta_K(\nu(t)) - \delta_K(\lambda^*(t)) = \epsilon\hat{\delta}^{(\nu)}(\lambda^*(t)),$$

by the positive homogeneity and subadditivity of δ_K.

Case $\nu = -\lambda^*$:

$$\delta_K(\lambda^*(t)+\epsilon\nu(t)) - \delta_K(\lambda^*(t)) = (1-\epsilon)\delta_K(\lambda^*(t)) - \delta_K(\lambda^*(t)) = -\epsilon\delta_K(\lambda^*(t)) = \epsilon\hat{\delta}^{(\nu)}(\lambda^*(t))$$

Furthermore, note that on $\{t \le \tau_n\}$

$$\begin{aligned} &\gamma_{\lambda_{\epsilon,n}}(t) - \gamma_{\lambda^*}(t) = \epsilon\Sigma(t)^{-1}\nu(t) \\ \Rightarrow\ &\|\gamma_{\lambda_{\epsilon,n}}(t)\|^2 - \|\gamma_{\lambda^*}(t)\|^2 = 2\epsilon\big(\Sigma(t)^{-1}\nu(t)\big)'\gamma_{\lambda^*}(t) + \epsilon^2\|\Sigma(t)^{-1}\nu(t)\|^2 \end{aligned} \tag{A.28}$$

and $\gamma_{\lambda_{\epsilon,n}}(t) = \gamma_{\lambda^*}(t)$ on $\{t > \tau_n\}$.
From these findings we can derive the following inequalities:

$$\frac{\tilde{Z}_{\lambda_{\epsilon,n}}(T)}{\tilde{Z}_{\lambda^*}(T)} \ge \underbrace{\exp\left(-\epsilon(L_{\tau_n}+N_{\tau_n}) - \frac{\epsilon^2}{2}\int_0^{\tau_n}\|\Sigma(t)^{-1}\nu(t)\|^2 dt\right)}_{=:\alpha_n^{(\nu)}(\epsilon)} \ge e^{-3\epsilon n} \ge e^{-3n} \tag{A.29}$$

and indeed, upon more detailed inspection of (A.29) we find that

$$\begin{aligned} \frac{\tilde{Z}_{\lambda_{\epsilon,n}}(T)}{\tilde{Z}_{\lambda^*}(T)} &= \exp\Bigg(-\int_0^{\tau_n} \underbrace{\delta_K(\lambda^*(t)+\epsilon\nu(t)) - \delta_K(\lambda^*(t))}_{\le\ \epsilon\hat{\delta}^{(\nu)}(\lambda^*(t))\text{ by }(A.27)} dt \\ &\qquad - \frac{1}{2}\int_0^{\tau_n}\|\gamma_{\lambda_{\epsilon,n}}(t)\|^2 - \|\gamma_{\lambda^*}(t)\|^2 dt - \int_0^{\tau_n}(\gamma_{\lambda_{\epsilon,n}}(t)-\gamma_{\lambda^*}(t))'dW(t)\Bigg) \\ &\overset{(A.28)}{\ge} \exp\Bigg(-\epsilon\int_0^{\tau_n}\hat{\delta}^{(\nu)}(\lambda^*(t))dt - \frac{\epsilon^2}{2}\int_0^{\tau_n}\|\Sigma(t)^{-1}\nu(t)\|^2 dt \\ &\qquad - \epsilon\int_0^{\tau_n}\big(\Sigma(t)^{-1}\nu(t)\big)'\gamma_{\lambda^*}(t)dt - \epsilon\int_0^{\tau_n}\big(\Sigma(t)^{-1}\nu(t)\big)'dW(t)\Bigg) \\ &= \exp\Bigg(-\epsilon\underbrace{\int_0^{\tau_n}\hat{\delta}^{(\nu)}(\lambda^*(t))dt}_{=L_{\tau_n}} - \frac{\epsilon^2}{2}\int_0^{\tau_n}\|\Sigma(t)^{-1}\nu(t)\|^2 dt \\ &\qquad - \epsilon\underbrace{\int_0^{\tau_n}\big(\Sigma(t)^{-1}\nu(t)\big)'dW_{\lambda^*}(t)}_{=\ N_{\tau_n}}\Bigg) \\ &= \exp\left(-\epsilon(L_{\tau_n}+N_{\tau_n}) - \frac{\epsilon^2}{2}\int_0^{\tau_n}\|\Sigma(t)^{-1}\nu(t)\|^2 dt\right) \\ &\ge \exp\big(-3\epsilon n\big) \ge \exp\big(-3n\big), \end{aligned}$$

where the second-to-last inequality follows from the choice of the stopping time τ_n. Note that the expression $\alpha_n^{(\nu)}(\cdot)$ is differentiable in ϵ with

$$\frac{\partial}{\partial\epsilon}\alpha_n^{(\nu)}(\epsilon)\Big|_{\epsilon=0} = \big(\alpha_n^{(\nu)}\big)'(0) = \lim_{\epsilon\to 0}\frac{\alpha_n^{(\nu)}(\epsilon)-1}{\epsilon} = -(L_{\tau_n}+N_{\tau_n}). \tag{A.30}$$

We go on to prove the second inequality in (A.26) by validating the following inequality:

$$\begin{aligned} &\limsup_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Big(\mathbb{E}\big[U^*_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2) - U^*_{g_{\lambda^*}}(y^*;B_1,B_2)\big]\Big) \\ &\qquad \le y_0^*\mathbb{E}\big[\mathcal{I}_{g_{\lambda^*}}(y^*;B_1,B_2)\tilde{Z}_{\lambda^*}(T)(L_{\tau_n}+N_{\tau_n})\big] \end{aligned} \tag{A.31}$$

We first derive a pointwise majorant with finite expectation to justify the application of Fatou's lemma:

$$\begin{aligned}
&\frac{1}{\epsilon}\big(U^*_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)-U^*_{g_{\lambda^*}}(y^*;B_1,B_2)\big)\\
=\;&\frac{1}{\epsilon}\Big(U(\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2))-y_0^*\tilde{Z}_{\lambda_{\epsilon,n}}(T)\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)-y_1^*g\left(\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)\right)\Big)\\
&-\frac{1}{\epsilon}\underbrace{\sup_{B_1\le x\le B_2}\Big(U(x)-y_0^*\tilde{Z}_{\lambda^*}(T)x-y_1^*g(x)\Big)}_{\ge U(x)-y_0^*\tilde{Z}_{\lambda^*}(T)x-y_1^*g(x)\text{ for }x=\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)}\\
\le\;&\underbrace{\frac{1}{\epsilon}y_0^*\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)}_{\ge 0}\Big(\tilde{Z}_{\lambda^*}(T)-\underbrace{\tilde{Z}_{\lambda_{\epsilon,n}}(T)}_{\ge\tilde{Z}_{\lambda^*}(T)\alpha_n^{(\nu)}(\epsilon)}\Big)\\
\overset{(A.29)}{\le}\;&y_0^*\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)\tilde{Z}_{\lambda^*}(T)\frac{1}{\epsilon}\Big(1-\alpha_n^{(\nu)}(\epsilon)\Big).
\end{aligned}\tag{A.32}$$

Again, recalling (A.29) yields for any $0\le B_1\le x\le B_2$

$$g_{\lambda_{\epsilon,n}}(x)=\begin{pmatrix}-\tilde{Z}_{\lambda_{\epsilon,n}}(T)x\\-g(x)\end{pmatrix}\le\begin{pmatrix}-e^{-3\epsilon n}\tilde{Z}_{\lambda^*}(T)x\\-g(x)\end{pmatrix}:=\hat{g}(x)\tag{A.33}$$

and thus by Lemma 2.1.38, (ii), $\mathcal{I}_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)\le\mathcal{I}_{\hat{g}}(y^*;B_1,B_2)$. However, we also have

$$\begin{aligned}
U^*_{\hat{g}}(y^*;B_1,B_2)&=\sup_{B_1\le x\le B_2}\Big(U(x)-y_0^*e^{-3\epsilon n}\tilde{Z}_{\lambda^*}(T)x-y_1^*g(x)\Big)\\
&=U^*_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon n},y_1^*;B_1,B_2\right)
\end{aligned}$$

and in particular $\mathcal{I}_{\hat{g}}(y^*;B_1,B_2)=\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon n},y_1^*;B_1,B_2\right)$. Combining these observations with (A.32), we have for any fixed $0<\epsilon^+$ and for all $0<\epsilon\le\epsilon^+$

$$\begin{aligned}
&\frac{1}{\epsilon}\big(U^*_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)-U^*_{g_{\lambda^*}}(y^*;B_1,B_2)\big)\\
&\overset{(A.33)}{\le}\quad y_0^*\mathcal{I}_{\hat{g}}(y^*;B_1,B_2)\tilde{Z}_{\lambda^*}(T)\frac{1}{\epsilon}\Big(1-\alpha_n^{(\nu)}(\epsilon)\Big)\\
&=\quad y_0^*\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon n},y_1^*;B_1,B_2\right)\tilde{Z}_{\lambda^*}(T)\frac{1}{\epsilon}\Big(1-\alpha_n^{(\nu)}(\epsilon)\Big)
\end{aligned}\tag{A.34}$$

$$\begin{aligned}
&\overset{L.\ 2.1.38,\ (i)}{\le}\quad y_0^*\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon^+n},y_1^*;B_1,B_2\right)\tilde{Z}_{\lambda^*}(T)\frac{1}{\epsilon}\Big(1-\underbrace{\alpha_n^{(\nu)}(\epsilon)}_{\ge e^{-3\epsilon n}}\Big)\\
&\le\quad y_0^*\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon^+n},y_1^*;B_1,B_2\right)\tilde{Z}_{\lambda^*}(T)\underbrace{\sup_{0<\epsilon<1}\left(\frac{1}{\epsilon}\Big(1-e^{-3\epsilon n}\Big)\right)}_{<\infty}
\end{aligned}\tag{A.35}$$

Since $\lambda^*\in\mathcal{D}'$, the auxiliary Lemma A.2.1, (ii) implies that we can choose ϵ^+ small enough such that

$$\mathbb{E}\left[y_0^*\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon^+n},y_1^*;B_1,B_2\right)\tilde{Z}_{\lambda^*}(T)\right]<\infty$$

Then (A.35) yields an integrable majorant for all $0<\epsilon\le\epsilon^+$ and we take expectations and apply Fatou's lemma to obtain

$$\begin{aligned}
&\limsup_{\epsilon\downarrow 0}\frac{1}{\epsilon}\Big(\mathbb{E}\big[U^*_{g_{\lambda_{\epsilon,n}}}(y^*;B_1,B_2)-U^*_{g_{\lambda^*}}(y^*;B_1,B_2)\big]\Big)\\
&\overset{(A.34)}{\le}\limsup_{\epsilon\downarrow 0}\mathbb{E}\Big[y_0^*\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon n},y_1^*;B_1,B_2\right)\tilde{Z}_{\lambda^*}(T)\frac{1}{\epsilon}\Big(1-\alpha_n^{(\nu)}(\epsilon)\Big)\Big]\\
&\overset{\text{Fatou}}{\le} y_0^*\mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)\limsup_{\epsilon\downarrow 0}\Big(\underbrace{\mathcal{I}_{g_{\lambda^*}}\left(y_0^*e^{-3\epsilon n},y_1^*;B_1,B_2\right)}_{\overset{\epsilon\to 0}{\longrightarrow}\mathcal{I}_{g_{\lambda^*}}(y^*;B_1,B_2)\text{ by } L.A.2.1,(i)}\cdot\underbrace{\frac{1}{\epsilon}\big(1-\alpha_n^{(\nu)}(\epsilon)\big)}_{\overset{\epsilon\to 0}{\longrightarrow}L_{\tau_n}+N_{\tau_n}\text{by }(A.30)}\Big)\Big]\\
&\overset{(A.30)}{=} y_0^*\mathbb{E}\Big[\tilde{Z}_{\lambda^*}(T)\mathcal{I}_{g_{\lambda^*}}\left(y^*;B_1,B_2\right)\left(L_{\tau_n}+N_{\tau_n}\right)\Big],
\end{aligned}$$

This proves (A.31) and hence, the second inequality in (A.26).

Part 2: We continue by proving the last inequality in (A.26). Let $V_{\lambda^*}^{v_0,\pi}$ be the wealth process of π in $\mathcal{M}_{\lambda^*}$ and set $\tilde{V}_{\lambda^*}^{v_0,\pi}(t) := P_0^{\lambda^*}(t)^{-1}\cdot V_{\lambda^*}^{v_0,\pi}(t)$. Then, through an application of Itô's product rule and a change of the diffusion from $dW(t)$ to $dW_{\lambda^*}(t)=dW(t)+\gamma_{\lambda^*}(t)dt$ we see:

$$\begin{aligned}
d(\tilde{V}_{\lambda^*}^{v_0,\pi}(t)) &= d(P_0^{\lambda^*}(t)^{-1}\cdot V_{\lambda^*}^{v_0,\pi}(t)) = P_0^{\lambda^*}(t)^{-1}dV_{\lambda^*}^{v_0,\pi}(t)+V_{\lambda^*}^{v_0,\pi}(t)d(P_0^{\lambda^*}(t)^{-1})\\
&\overset{(3.5)}{=}\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\Big(\big[r(t)+(\mu(t)-r(t)\mathbb{1})'\pi(t)\big]dt+\pi(t)'\Sigma(t)dW(t)\Big)\\
&\quad+\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\big[\delta_K(\lambda^*(t))+\pi(t)'\lambda^*(t)\big]dt-\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\left[r(t)+\delta_K(\lambda^*(t))\right]dt\\
&=\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\pi(t)'\Sigma(t)dW_{\lambda^*}(t)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&d\big(\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\cdot(L_t+N_t)\big)\\
&=\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\big(dL_t+dN_t\big)+(L_t+N_t)d\tilde{V}_{\lambda^*}^{v_0,\pi}(t)+d\langle\tilde{V}_{\lambda^*}^{v_0,\pi},N\rangle_t\\
&=\tilde{V}_{\lambda^*}^{v_0,\pi}(t)(dL_t+(\Sigma(t)^{-1}\nu(t))'dW_{\lambda^*}(t))\\
&\qquad+\tilde{V}_{\lambda^*}^{v_0,\pi}(t)(L_t+N_t)\pi(t)'\Sigma(t)dW_{\lambda^*}(t)+\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\pi(t)'\nu(t)dt\\
&=\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\Big(\Big[(\Sigma(t)^{-1}\nu(t))'+(L_t+N_t)\pi(t)'\Sigma(t)\Big]dW_{\lambda^*}(t)+\pi(t)'\nu(t)dt+dL_t\Big)
\end{aligned}\tag{A.36}$$

Define the probability measure $\tilde{Q}_{\lambda^*}^{(n)}$ via its Radon-Nikoydm derivative:

$$\left.\frac{d\tilde{Q}_{\lambda^*}^{(n)}}{dQ}\right|_{\mathcal{F}_T} := Z_{\lambda^*}(\tau_n)$$

Then, according to Girsanov's theorem, the process

$$W_{\lambda^*}^{(n)}(t) := W(t)+\int_0^t\gamma_{\lambda^*}(s)\mathbb{1}_{\{s\le\tau_n\}}ds$$

is a Wiener process with respect to $\tilde{Q}_{\lambda^*}^{(n)}$. From now on, let $\mathbb{E}_{\tilde{Q}_{\lambda^*}^{(n)}}[X]$ denote the expectation of a random variable X w.r.t $\tilde{Q}_{\lambda^*}^{(n)}$. Then, by the choice of τ_n,

$$\begin{aligned}
&\mathbb{E}_{\tilde{Q}_{\lambda^*}^{(n)}}\left[\int_0^{\tau_n}\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\big[(\Sigma(t)^{-1}\nu(t))'+(L_t+N_t)\pi(t)'\Sigma(t)\big]dW_{\lambda^*}(t)\right]\\
&=\mathbb{E}_{\tilde{Q}_{\lambda^*}^{(n)}}\left[\int_0^{T}\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\big[(\Sigma(t)^{-1}\nu(t))'+(L_t+N_t)\pi(t)'\Sigma(t)\big]\mathbb{1}_{\{t\le\tau_n\}}dW_{\lambda^*}(t)\right]\\
&=\mathbb{E}_{\tilde{Q}_{\lambda^*}^{(n)}}\left[\int_0^{T}\tilde{V}_{\lambda^*}^{v_0,\pi}(t)\big[(\Sigma(t)^{-1}\nu(t))'+(L_t+N_t)\pi(t)'\Sigma(t)\big]\mathbb{1}_{\{t\le\tau_n\}}dW_{\lambda^*}^{(n)}(t)\right]\\
&=0.
\end{aligned}\tag{A.37}$$

Furthermore, integrating over (A.36) and taking expectations with respect to $\tilde{Q}^{(n)}_{\lambda^*}$ results in

$$
\begin{aligned}
&\mathbb{E}[\tilde{Z}_{\lambda^*}(\tau_n)V^{v_0,\pi}_{\lambda^*}(\tau_n)\big(L_{\tau_n}+N_{\tau_n}\big)] = \mathbb{E}[Z_{\lambda^*}(\tau_n)\tilde{V}^{v_0,\pi}_{\lambda^*}(\tau_n)\big(L_{\tau_n}+N_{\tau_n}\big)]\\
=\ &\mathbb{E}_{\tilde{Q}^{(n)}_{\lambda^*}}[\tilde{V}^{v_0,\pi}_{\lambda^*}(\tau_n)\big(L_{\tau_n}+N_{\tau_n}\big)]\\
\overset{(A.36)}{=}\ &\mathbb{E}_{\tilde{Q}^{(n)}_{\lambda^*}}\Big[\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\nu(t)dt+dL_t]\Big]\\
&+\mathbb{E}_{\tilde{Q}^{(n)}_{\lambda^*}}\Big[\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[(\Sigma(t)^{-1}\nu(t))'+(L_t+N_t)\pi(t)'\Sigma(t)]dW_{\lambda^*}(t)\Big]\\
\overset{(A.37)}{=}\ &\mathbb{E}_{\tilde{Q}^{(n)}_{\lambda^*}}\Big[\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\nu(t)dt+dL_t]\Big]\\
=\ &\mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\nu(t)dt+dL_t]\Big].
\end{aligned}
\tag{A.38}
$$

Furthermore, since $\big(\tilde{Z}_{\lambda^*}(t)\cdot V^{v_0,\pi}_{\lambda^*}(t)\big)_{0\le t\le T}$ is a Q-supermartingale, the tower property of conditional expectations ensures

$$
\begin{aligned}
\mathbb{E}\big[V^{v_0,\pi}_{\lambda^*}(T)\tilde{Z}_{\lambda^*}(T)(L_{\tau_n}+N_{\tau_n})\big] &= \mathbb{E}\big[\mathbb{E}[V^{v_0,\pi}_{\lambda^*}(T)\tilde{Z}_{\lambda^*}(T)(L_{\tau_n}+N_{\tau_n})|\mathcal{F}_{\tau_n}]\big]\\
&= \mathbb{E}\big[(L_{\tau_n}+N_{\tau_n})\mathbb{E}[V^{v_0,\pi}_{\lambda^*}(T)\tilde{Z}_{\lambda^*}(T)|\mathcal{F}_{\tau_n}]\big]\\
&\le E\big[V^{v_0,\pi}_{\lambda^*}(\tau_n)\tilde{Z}_{\lambda^*}(\tau_n)(L_{\tau_n}+N_{\tau_n})\big].
\end{aligned}
$$

This, in combination with (A.38), proves the last inequality in (A.26).

Part 3: We can finally wrap up this proof. First, let $\nu\in\mathcal{D}$. Then, by the inequalities in (A.26)

$$
\begin{aligned}
0 &\le \mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\nu(t)+\delta_K(\nu(t))]dt\Big]\\
&= \mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{T}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\nu(t)+\delta_K(\nu(t))]\mathbb{1}_{\{t\le\tau_n\}}dt\Big]\quad\forall n\in\mathbb{N}.
\end{aligned}
$$

Assume now that the set

$$
A := \Big\{[\pi(t)'\nu(t)+\delta_K(\nu(t))]\mathbb{1}_{\{t\le\tau_n\}}<0\Big\}\subset\Omega\times[0,T]
$$

has positive product measure. Setting $\tilde{\nu}:=\nu\mathbb{1}_A\in\mathcal{D}$, we see that

$$
\begin{aligned}
&\mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)[\pi(t)'\tilde{\nu}(t)+\delta_K(\tilde{\nu}(t))]dt\Big]\\
&= \mathbb{E}\Big[\underbrace{Z_{\lambda^*}(\tau_n)}_{>0}\int_0^{\tau_n}\underbrace{\tilde{V}^{v_0,\pi}_{\lambda^*}(t)}_{>0}\underbrace{[\pi(t)'\nu(t)+\delta_K(\nu(t))]\mathbb{1}_A}_{<0}dt\Big]<0.
\end{aligned}
$$

But as $\tilde{\nu}\in\mathcal{D}$, this contradicts (A.26). Hence,

$$
\begin{gathered}
[\pi(t)'\nu(t)+\delta_K(\nu(t))]\mathbb{1}_{\{t\le\tau_n\}}\ge 0\quad\mathcal{L}[0,T]\otimes Q-a.e.\\
\downarrow n\to\infty\\
\pi(t)'\nu(t)+\delta_K(\nu(t))\ge 0\quad\mathcal{L}[0,T]\otimes Q-a.e.
\end{gathered}
$$

as $\tau_n\to T\ Q-a.s.$ for $n\to\infty$. Since the choice of $\nu\in\mathcal{D}$ was arbitrary, this allows us to conclude (A.24) by virtue of Lemma 2.1.17, (iii). In particular, the choice of $\nu=\lambda^*$ yields

$$
\pi(t)'\lambda^*(t)+\delta_K(\lambda^*(t))\ge 0\quad\mathcal{L}[0,T]\otimes Q-a.e.. \tag{A.39}
$$

On the other hand, by setting $\nu := -\lambda^*$, we obtain from (A.26)

$$0 \leq -\mathbb{E}\Big[Z_{\lambda^*}(\tau_n)\int_0^{\tau_n}\tilde{V}^{v_0,\pi}_{\lambda^*}(t)\underbrace{[\pi(t)'\lambda^*(t)+\delta_K(\lambda^*(t))]}_{\geq\ 0\ \text{by}\ (A.39)}dt\Big]$$

and this implies (A.25). □

Proof of Theorem 3.4.6. This is a direct consequence of the remarks above the statement of the corollary. □

Proof of Lemma 3.5.2. Due to G being convex and strictly decreasing, $\text{RRA}(t,z;y) \geq 0$. As $0 \in K$, we have $\delta_K(x) = -\inf_{v\in K}(v'x) \overset{v=0}{\geq} 0$. For given $(t,z;y)$, we aim to find a minimizer $x = \lambda^*(t,z;y) \in \mathbb{R}^d$ of the function

$$f(x) := \text{RRA}(t,z;y)\|\gamma + \Sigma^{-1}x\|^2 + \delta_K(x).$$

If $\text{RRA}(t,z;y) = 0$, then $x = \lambda^*(t,z;y) = 0 \in X_K$ minimizes f globally. Otherwise, f is strictly convex in x and $f(x) \to \infty$ as $\|x\| \to \infty$. Hence, f has bounded level sets and is continuous on X_K (cf. Assumption 3.5.1). Thus, f attains its global minimum at some $x = \lambda^*(t,z;y) \in X_K$. Hence, there exists a global minimizer $\lambda^*(t,z;y) \in X_K$ and it remains to show that this minimizer is uniformly bounded in $(t,z;y)$.
Since Σ^{-1} is non-singular, the matrix $C = \left(\Sigma^{-1}\right)'\Sigma^{-1}$ is symmetric and positive definite. By the symmetry of C, the spectral theorem implies that C has d orthonormal eigenvectors and d real eigenvalues. Furthermore, as C is strictly positive definite, all of its eigenvalues are strictly positive. Thus, if $c_- > 0$ is the square root of the smallest eigenvalue of C, we obtain

$$\|\Sigma^{-1}x\|^2 = x'Cx \geq (c_-)^2\|x\|^2 \text{ for all } x \in \mathbb{R}^d.$$

Further, for a given minimizer $\lambda^*(t,z;y) \in X_K$, define

$$\nu(t,z;y) := \lambda^*(t,z;y)\cdot \mathbb{1}_{\{\|\lambda^*(t,z;y)\|\leq \frac{2}{c_-}\|\gamma\|\}}$$

Then, $\nu \in X_K$ and ν coincides with λ^* whenever $\|\lambda^*(t,z;y)\| \leq \frac{2}{c_-}\|\gamma\|$. Otherwise let $\|\lambda^*(t,z;y)\| > \frac{2}{c_-}\|\gamma\|$. Then, $\nu(t,z;y) = 0$ and an application of the reverse Cauchy-Schwarz inequality

$$\|x_1+x_2\|^2 = \|x_1\|^2 + 2x_1'x_2 + \|x_2\|^2 \geq \|x_1\|^2 - 2\|x_1\|\|x_2\| + \|x_2\|^2 \quad \forall x_1,x_2 \in \mathbb{R}^d, \qquad (A.40)$$

yield

$$\begin{aligned}
&\frac{1}{2}\text{RRA}(t,z;y)\|\gamma+\Sigma^{-1}\lambda^*(t,z;y)\|^2 + \delta_K(\lambda^*(t,z;y))\\
&\overset{(A.40)}{\geq} \frac{1}{2}\text{RRA}(t,z;y)\big(\|\gamma\|^2 - 2\|\gamma\|\|\Sigma^{-1}\lambda^*(t,z;y)\| + \|\Sigma^{-1}\lambda^*(t,z;y)\|^2\big) + \delta_K(\lambda^*(t,z;y))\\
&= \frac{1}{2}\text{RRA}(t,z;y)\big(\|\gamma\|^2 + \|\Sigma^{-1}\lambda^*(t,z;y)\|(-2\|\gamma\| + \underbrace{\|\Sigma^{-1}\lambda^*(t,z;y)\|)}_{\geq c_-\|\lambda^*(t,z;y)\|\geq 2\|\gamma\|}\ \big) + \underbrace{\delta_K(\lambda^*(t,z;y))}_{\geq 0}\\
&\geq \frac{1}{2}\text{RRA}(t,z;y)\|\gamma\|^2\\
&\overset{\nu(t,z;y)=0}{=} \frac{1}{2}\text{RRA}(t,z;y)\|\gamma+\Sigma^{-1}\nu(t,z;y)\|^2 + \delta_K(\nu(t,z;y)).
\end{aligned}$$

Hence, $\nu(t,z;y)$ is also a minimizer of (3.11) and $\|\nu(t,z;y)\| \leq \frac{2}{c_-}\|\gamma\|$, for all $(t,z;y)$, which concludes the proof. □

Proof of Theorem 3.5.3. The uniform boundedness of $\lambda^*(t,z;y) \in X_K$ and the continuity of δ_K on X_K guarantee that $\|\lambda^*(t)\|$ and $|\delta_K(\lambda^*(t))|$ are $Q-a.s.$ bounded on $[0,T]$. Due to measurable selection theorems (for example Corollary 3.48 in [77]) we may w.l.o.g. assume the mapping $(t,z) \to \lambda^*(t,z;y)$ to be Borel-measurable. Hence, the corresponding stochastic process λ^* is progressively measurable and thus an element of $\mathcal{D}$.
For convenience, we define the characteristic operator $\mathcal{H}^\lambda G$ of G with respect to $\lambda \in \mathcal{D}$ as

$$\mathcal{H}^\lambda G(t,z;y) = G_t(t,z;y) - \big(r + \delta_K(\lambda(t))\big) z G_z(t,z;y) + \frac{1}{2}\|\gamma + \Sigma^{-1}\lambda(t)\|^2 z^2 G_{zz}(t,z;y).$$

Note $\mathcal{H}^\lambda G(t,z;y) \geq 0$ for any $\lambda \in \mathcal{D}$ and $\mathcal{H}^\lambda G(t,z;y) = 0$ if $\lambda = \lambda^*$.

Let now $\lambda \in \mathcal{D}$ with $\|\lambda(t) - \lambda^*(t)\| \leq 1$ $\mathcal{L}[0,T] \otimes \mathbb{Q}$-a.s., be arbitrary but fixed. Due to Lemma 3.5.2, we can assume that $C > 0$ from the polynomial growth condition was chosen large enough such that

$$\max\big(\|\gamma_\lambda(t)\|^2, \delta_K(\lambda^*(t)), \delta_K(\lambda(t))\big) \leq C \quad \mathcal{L}[0,T] \otimes \mathbb{Q}\text{-a.s.}.$$

Finally, for any $p > 0$, we define the stopping times

$$\begin{aligned}\bar{\tau}_p &= \inf\left\{S \in [t,T] \;\Big|\; \int_t^S \big(G_z(s,\tilde{Z}_\lambda(s);y)\tilde{Z}_\lambda(s)\|\gamma_\lambda(s)\|\big)^2 ds \geq p\right\}\\ \tau_p &= \min\big(\bar{\tau}_p, T\big)\end{aligned}$$

This choice of τ_p ensures that

$$\begin{aligned}&\mathbb{E}\Bigg[\underbrace{\int_t^{\tau_p} \big(G_z(s,\tilde{Z}_\lambda(s);y)\tilde{Z}_\lambda(s)\|\gamma_\lambda(s)\|\big)^2 ds}_{\leq p}\Big|\mathcal{F}_t\Bigg] \leq p\\ \Rightarrow\quad &\mathbb{E}\left[\int_t^{\tau_p} G_z(s,\tilde{Z}_\lambda(s);y)\tilde{Z}_\lambda(s)\gamma_\lambda(s)' dW(s)\Big|\mathcal{F}_t\right] = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}[G(\tau_p,\tilde{Z}_\lambda(\tau_p);y)|\mathcal{F}_t]\\ &\overset{\text{Itô}}{=} G(t,\tilde{Z}_\lambda(t);y) + \mathbb{E}\left[\int_t^{\tau_p} \underbrace{\mathcal{H}^\lambda G(s,\tilde{Z}_\lambda(s);y)}_{\geq 0} ds\Big|\mathcal{F}_t\right]\\ &\qquad + \underbrace{\mathbb{E}\left[\int_t^{\tau_p} G_z(s,\tilde{Z}_\lambda(s);y)\tilde{Z}_\lambda(s)\gamma_\lambda(s)' dW(s)\Big|\mathcal{F}_t\right]}_{=0}\\ &\geq G(t,\tilde{Z}_\lambda(t);y)\end{aligned} \tag{A.41}$$

Clearly, $\tau_p \to T$, for $p \to \infty$ and hence $G(\tau_p,\tilde{Z}_\lambda(\tau_p);y) \to G(T,\tilde{Z}_\lambda(T);y) = U^*_{g_\lambda}(y;B_1,B_2)$ for $p \to \infty$.

Inequality (A.41) holds with equality for $\lambda = \lambda^*$. Next, we show that we can interchange limit and expectation. This will be achieved using Doob's martingale inequality and the dominated convergence theorem. The polynomial growth condition gives us a dominating random variable for $G(\tau_p,\tilde{Z}_\lambda(\tau_p);y)$:

$$|G(\tau_p,\tilde{Z}_\lambda(\tau_p);y)| \leq C\Big|(\tilde{Z}_\lambda(\tau_p))^{-\alpha} + (\tilde{Z}_\lambda(\tau_p))^{\alpha}\Big|$$

$$\leq C\left(\sup_{t\leq s\leq T}(\tilde{Z}_\lambda(s))^{-\alpha}+\sup_{t\leq s\leq T}(\tilde{Z}_\lambda(s))^{\alpha}\right).$$

We now show that the right-hand side has finite expectation, due to the uniform boundedness of λ, γ_λ and $\delta_K(\lambda)$:

$$\begin{aligned}\tilde{Z}_\lambda(s)^{-\alpha} &= \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\alpha\int_t^s r+\delta_K(\lambda(u))+\frac{1}{2}\|\gamma_\lambda(u)\|^2du+\alpha\int_t^s\gamma_\lambda(u)'dW(u)\right)\\ &= \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\alpha\int_t^s r+\underbrace{\delta_K(\lambda(u))}_{\leq C}+\frac{1}{2}(1+\alpha)\underbrace{\|\gamma_\lambda(u)\|^2}_{\leq C}du\right)\\ &\quad\underbrace{\cdot\exp\left(-\frac{\alpha^2}{2}\int_t^s\|\gamma_\lambda(u)\|^2du+\alpha\int_t^s\gamma_\lambda(u)'dW(u)\right)}_{=:\,M_-(s)}\\ &\leq \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\left[\alpha(r+C)+\frac{1}{2}(\alpha+\alpha^2)C\right](s-t)\right)M_-(s)\end{aligned}$$

and M_- is a martingale by Novikov's condition (again due to the uniform boundedness of λ). Therefore, through an application of the general inequality $x\leq 1+x^2$ for all $x\in\mathbb{R}$ in (i) and an application of Doob's martingale inequality in (ii) we obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq s\leq T}(\tilde{Z}_\lambda(s))^{-\alpha}\Big|\mathcal{F}_t\right]\\ &\leq \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\left[\alpha|r+C|+\frac{1}{2}(\alpha+\alpha^2)C\right](T-t)\right)\mathbb{E}\left[\sup_{t\leq s\leq T}M_-(s)\Big|\mathcal{F}_t\right]\\ &\overset{(i)}{\leq} \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\left[\alpha|r+C|+\frac{1}{2}(\alpha+\alpha^2)C\right](T-t)\right)\mathbb{E}\left[1+\sup_{t\leq s\leq T}\big(M_-(s)\big)^2\Big|\mathcal{F}_t\right]\\ &\overset{(ii)}{\leq} \tilde{Z}_\lambda(t)^{-\alpha}\exp\left(\left[\alpha|r+C|+\frac{1}{2}(\alpha+\alpha^2)C\right](T-t)\right)\Big(1+4\underbrace{\mathbb{E}\left[M_-(T)^2\big|\mathcal{F}_t\right]}_{<\infty}\Big)<\infty,\end{aligned}$$

where the finiteness of the last expectation follows again using Novikov's condition and the uniform boundedness of λ. Similarly,

$$\begin{aligned}\tilde{Z}_\lambda(t,s)^{\alpha} &= \tilde{Z}_\lambda(t)^{\alpha}\exp\left(-\alpha\int_t^s r+\underbrace{\delta_K(\lambda(u))}_{\geq 0}+\underbrace{\frac{1}{2}\|\gamma_\lambda(u)\|^2}_{\geq 0}du-\alpha\int_t^s\gamma_\lambda(u)'dW(u)\right)\\ &\leq \tilde{Z}_\lambda(t)^{\alpha}\exp\left(-\alpha\int_t^s r-\frac{1}{2}\alpha\underbrace{\|\gamma_\lambda(u)\|^2}_{\leq C}du\right)\\ &\quad\underbrace{\cdot\exp\left(-\frac{\alpha^2}{2}\int_t^s\|\gamma_\lambda(u)\|^2du-\alpha\int_t^s\gamma_\lambda(u)'dW(u)\right)}_{=:\,M_+(s)}\\ &\leq \tilde{Z}_\lambda(t)^{\alpha}\exp\left(\left[-\alpha r+\frac{1}{2}\alpha^2C\right](s-t)\right)M_+(s)\end{aligned}$$

and M_+ is a martingale by Novikov's condition (again due to the uniform boundedness of λ). Therefore, through another application of the inequality $x\leq 1+x^2$ for all $x\in\mathbb{R}$ in (iii) and an application of Doob's martingale inequality in (iv) we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le s\le T}(\tilde{Z}_\lambda(s))^\alpha\Big|\mathcal{F}_t\right]\\
&\le \tilde{Z}_\lambda(t)^\alpha \exp\left(\left[\alpha|r|+\frac{1}{2}\alpha^2 C\right](T-t)\right)\mathbb{E}\left[\sup_{t\le s\le T}M_+(s)\Big|\mathcal{F}_t\right]\\
&\overset{(iii)}{\le} \tilde{Z}_\lambda(t)^\alpha \exp\left(\left[\alpha|r|+\frac{1}{2}\alpha^2 C\right](T-t)\right)\mathbb{E}\left[1+\sup_{t\le s\le T}(M_+(s))^2\Big|\mathcal{F}_t\right]\\
&\overset{(iv)}{\le} \tilde{Z}_\lambda(t)^\alpha \exp\left(\left[\alpha|r|+\frac{1}{2}\alpha^2 C\right](T-t)\right)\Big(1+4\underbrace{\mathbb{E}\left[M_+(T)^2\big|\mathcal{F}_t\right]}_{<\infty}\Big)<\infty.
\end{aligned}$$

The finiteness of the last expectation can again easily be verified using Novikov's condition and the uniform boundedness of λ. This allows us to apply the dominated convergence theorem

$$\begin{aligned}
G(t,\tilde{Z}_\lambda(t);y) &= \mathbb{E}\left[G(t,\tilde{Z}_\lambda(t);y)\big|\mathcal{F}_t\right]\\
&\overset{(A.41)}{\le} \lim_{p\to\infty}\mathbb{E}\left[G(\tau_p,\tilde{Z}_\lambda(\tau_p);y)\big|\mathcal{F}_t\right]\\
&\underset{\text{convergence}}{\overset{\text{dominated}}{=}} \mathbb{E}\left[\lim_{p\to\infty}G(\tau_p,\tilde{Z}_\lambda(\tau_p);y)\big|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[G(T,\tilde{Z}_\lambda(T);y)\big|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[U^*_{g_\lambda}(y;B_1,B_2)\big|\mathcal{F}_t\right],
\end{aligned}$$

with equality if $\lambda=\lambda^*$. Conditioning on $\tilde{Z}_\lambda(t)=z$ concludes the proof. □

Proof of Lemma 3.5.4. Follows immediately from Chapter 5, Theorem 6.1 in [22]. □

Proof of Lemma 3.5.5. Note that for any given constant $\beta>0$, we have

$$\beta u^2 \ge |u| \quad \forall u\in\mathbb{R} \text{ with } |u|\ge\frac{1}{\beta}$$

In particular, we then have

$$\beta u^2+\frac{1}{\beta}\ge|u| \quad \forall u\in\mathbb{R}. \tag{A.42}$$

Furthermore, for $u\in\mathbb{R}$, define $f(u):=U^*_{\hat{g}}\left(y;B_1,B_2,\exp\left(\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}u\right)\right)$. Then f satisfies the prerequisites of Lemma 3.5.4, since for any $u\in\mathbb{R}$ and any $\beta>0$

$$\begin{aligned}
|f(u)| &= \left|U^*_{\hat{g}}\left(y;B_1,B_2,\exp\left(\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}u\right)\right)\right|\\
&\overset{(3.17)}{\le} C\left(e^{-\alpha\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}u}+e^{\alpha\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}u}\right)
\end{aligned}$$

$$\leq \ 2Ce^{\alpha\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}|u|}$$
$$\overset{(A.42)}{\leq} \ 2Ce^{\alpha\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}(\beta u^2+\frac{1}{\beta})}.$$

Choosing β small enough such that $\alpha\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}\beta < \frac{1}{4T}$ guarantees that F, as in (3.16), defines a continuously differentiable function in $\tau \in (0,T]$ and twice continuously differentiable function in $u \in \mathbb{R}$, which satisfies the heat equation $F_\tau(\tau,u) = F_{uu}(\tau,u)$ for all $(\tau,u) \in (0,\infty)\times\mathbb{R}$ with initial condition $F(0,u) = f(u)$. We substitute

$$t := T-\tau \quad \text{and} \quad u := \frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\left(\log(z) - \left(r + \frac{1}{2}\|\gamma_{\lambda^*}\|^2\right)(T-t)\right)$$

and define a function $G : [0,T]\times(0,\infty) \to \mathbb{R}$ as

$$\begin{aligned}
G(t,z;y) :=& F(\tau,u) = F\left(T-t, \frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\left(\log(z) - (r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)\right)\right)\\
=& \frac{1}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}} e^{-\frac{x^2}{4(T-t)}} f\left(\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\left(\log(z) - (r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)\right) - x\right)dx\\
=& \frac{1}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}} e^{-\frac{x^2}{4(T-t)}} U^*_{\hat{g}}\left(y; B_1, B_2, ze^{-(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)-\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}\right)dx.
\end{aligned}$$

The function

$$z \to U^*_{\hat{g}}(y; B_1, B_2, z) = \sup_{B_1\leq x\leq B_2}\Big(U(x) - y_0 z x - y_1 g(x)\Big)$$

is convex and strictly decreasing in z for any $y = (y_0, y_1) \in [0,\infty)^2$. Therefore, by the linearity of the integral, G is convex and strictly decreasing in z. Furthermore, using

$$\frac{\partial\tau}{\partial t} = -1, \quad \frac{\partial u}{\partial t} = \frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}r + \frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}, \quad \frac{\partial u}{\partial z} = \frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\frac{1}{z} \quad \text{and} \quad \frac{\partial^2 u}{\partial^2 z} = -\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\frac{1}{z^2}.$$

we have

$$\begin{aligned}
G_t(t,z;y) &= \frac{\partial}{\partial t}G(t,z;y) = \frac{\partial\tau}{\partial t}F_\tau(\tau,u) + \frac{\partial u}{\partial t}F_u(\tau,u)\\
&= -F_\tau(\tau,u) + \left(\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}r + \frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}\right)F_u(\tau,u),\\
-rzG_z(t,z;y) &= -rz\frac{\partial}{\partial z}G(t,z;y) = -rz\frac{\partial u}{\partial z}F_u(\tau,u) = -r\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}F_u(\tau,u)\\
\frac{z^2}{2}\|\gamma_{\lambda^*}\|^2 G_{zz}(t,z;y) &= \frac{z^2}{2}\|\gamma_{\lambda^*}\|^2\frac{\partial}{\partial z}G_z(t,z;y) = \frac{z^2}{\sqrt{2}}\|\gamma_{\lambda^*}\|\frac{\partial}{\partial z}\left(\frac{1}{z}F_u(\tau,u)\right)\\
&= -\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}F_u(\tau,u) + z\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}\frac{\partial u}{\partial z}F_{uu}(\tau,u)\\
&= -\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}F_u(\tau,u) + F_{uu}(\tau,u)
\end{aligned}$$

and therefore

$$G_t(t,z;y) - rzG_z(t,z;y) + \frac{z^2}{2}\|\gamma_{\lambda^*}\|^2 G_{zz}(t,z;y) = -F_\tau(\tau,u) + F_{uu}(\tau,u) = 0,$$

for all $(t,z)\in[0,T)\times(0,\infty)$. The substitution of $\tau=0\Leftrightarrow t=T$ and the initial condition for F gives the terminal condition for G:

$$\begin{aligned}G(T,z;y) &\overset{t=T\Leftrightarrow\tau=0}{=} F\left(0,\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\log(z)\right)\\ &= U^*_{\hat{g}}\left(y;B_1,B_2,\exp\left(\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}\frac{\sqrt{2}}{\|\gamma_{\lambda^*}\|}\log(z)\right)\right)\\ &= U^*_{\hat{g}}(y;B_1,B_2,z).\end{aligned}$$

In total, G satisfies the PDE

$$\begin{aligned}0 &= G_t(t,z;y)-rzG_z(t,z;y)+\frac{z^2}{2}\|\gamma_{\lambda^*}\|^2G_{zz}(t,z;y)\\ &\overset{\text{Def.}\lambda^*}{=} G_t(t,z;y)-rzG_z(t,z;y)+\frac{z^2}{2}\inf_{x\in X_K}\left(\|\gamma+\Sigma^{-1}x\|^2\right)G_{zz}(t,z;y)\\ &\overset{G_{zz}\geq 0}{=} G_t(t,z;y)-rzG_z(t,z;y)+\frac{z^2}{2}\inf_{x\in X_K}\left(\|\gamma+\Sigma^{-1}x\|^2G_{zz}(t,z;y)\right)\end{aligned}$$

with terminal condition $G(T,z;y)=U^*_{\hat{g}}(y;B_1,B_2)$. Furthermore, we can bound the absolute value of G, since for $t=T$ the polynomial growth condition on $U^*_{\hat{g}}(y;B_1,B_2,z)$ holds and for $t<T$

$$\begin{aligned}|G(t,z;y)| &\leq \frac{1}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}}e^{-\frac{x^2}{4(T-t)}}\left|U^*_{\hat{g}}\left(y;B_1,B_2,ze^{-(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)-\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}\right)\right|dx\\ &\overset{(3.17)}{\leq} \frac{C}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}}e^{-\frac{x^2}{4(T-t)}}\left(ze^{-(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)-\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}\right)^{-\alpha}dx\\ &\quad+\frac{C}{\sqrt{4\pi(T-t)}}\int_{\mathbb{R}}e^{-\frac{x^2}{4(T-t)}}\left(ze^{-(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)-\frac{\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}\right)^{\alpha}dx\\ &= \frac{C}{\sqrt{4\pi(T-t)}}e^{\alpha(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)}\left(\int_{\mathbb{R}}e^{-\frac{x^2}{4(T-t)}+\frac{\alpha\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}dx\right)z^{-\alpha}\\ &\quad+\frac{C}{\sqrt{4\pi(T-t)}}e^{-\alpha(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)}\left(\int_{\mathbb{R}}e^{-\frac{x^2}{4(T-t)}-\frac{\alpha\|\gamma_{\lambda^*}\|}{\sqrt{2}}x}dx\right)z^{\alpha}\\ &\overset{(i)}{=} Ce^{\alpha(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{u^2}{2}+\alpha\|\gamma_{\lambda^*}\|\sqrt{T-t}u}du\right)}_{I_1}z^{-\alpha}\\ &\quad+Ce^{-\alpha(r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2)(T-t)}\underbrace{\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{u^2}{2}-\alpha\|\gamma_{\lambda^*}\|\sqrt{T-t}u}du\right)}_{I_2}z^{\alpha}\\ &\overset{(ii)}{\leq} \underbrace{Ce^{\alpha|r+\frac{1}{2}\|\gamma_{\lambda^*}\|^2|T+\frac{1}{2}\alpha^2\|\gamma_{\lambda^*}\|^2T}}_{=:\tilde{C}}\left(z^{-\alpha}+z^{\alpha}\right)\end{aligned}$$

where we substituted $u=\frac{x}{\sqrt{2(T-t)}}$ in (i) and used that $I_1=I_2=\exp(\frac{1}{2}\alpha^2\|\gamma_{\lambda^*}\|^2(T-t))$ in (ii), since the integrals I_1 and I_2 are the moment generating function of a standard normal random variable evaluated at $\pm\alpha\|\gamma_{\lambda^*}\|\sqrt{T-t}$. □

Proof of Lemma 3.5.6. We begin by deriving the concave conjugate of a power utility function on $(0, \infty)$. Thus, for $b < 1$, $b \neq 0$ and $y > 0$, we aim to compute

$$\sup_{x \geq 0} \Big(\frac{1}{b} x^b - yx \Big).$$

The corresponding first order optimality condition for the optimizer x^* yields

$$0 \stackrel{!}{=} \big(x^*\big)^{b-1} - y \;\Leftrightarrow\; x^* = y^{\frac{1}{b-1}}.$$

As $\frac{1}{b}x^b - yx$ is strictly concave in b, the first order optimality condition is necessary and sufficient. Hence,

$$\sup_{x \geq 0} \Big(\frac{1}{b} x^b - yx \Big) = \frac{1}{b} y^{\frac{b}{b-1}} - y y^{\frac{1}{b-1}} = \frac{1-b}{b} y^{\frac{b}{b-1}}. \tag{A.43}$$

We continue by showing that (i) and (ii) imply that the growth condition (3.17) is satisfied.

(i) If $\max\left(U(x), -g(x)\right) \leq C_+ \left(\frac{1}{b_+} x^{b_+} + 1 \right)$, then we have for any $(y_0, y_1)' \in [0, \infty)^2$

$$\begin{aligned}
U^*_{\hat{g}}(y; B_1, B_2, z) &= \sup_{B_1 \leq x \leq B_2} \Big(U(x) - y_0 x z - y_1 g(x) \Big) \\
&\leq \sup_{B_1 \leq x \leq B_2} \left(C_+(1+y_1) \left(\frac{1}{b_+} x^{b_+} + 1 \right) - y_0 x z \right) \\
&\leq \sup_{x \geq 0} \left(C_+(1+y_1) \left(\frac{1}{b_+} x^{b_+} + 1 \right) - y_0 x z \right) \\
&= C_+(1+y_1) \left(1 + \sup_{x \geq 0} \left(\frac{1}{b_+} x^{b_+} - \frac{y_0 z}{(1+y_1)C_+} x \right) \right) \\
&\stackrel{(A.43)}{=} C_+(1+y_1) \left(1 + \frac{1-b_+}{b_+} \left(\frac{y_0 z}{(1+y_1)C_+} \right)^{\frac{b_+}{b_+-1}} \right) \\
&= C_+(1+y_1) \left(1 + \frac{1-b_+}{b_+} \left(\frac{y_0}{(1+y_1)C_+} \right)^{\frac{b_+}{b_+-1}} z^{\frac{b_+}{b_+-1}} \right).
\end{aligned}$$

Using analogous arguments, we can show that $C_- \left(\frac{1}{b_-} x^{b_-} - 1 \right) \leq \min\left(U(x), -g(x)\right)$ implies

$$C_-(1+y_1) \left(-1 + \frac{1-b_-}{b_-} \left(\frac{y_0}{(1+y_1)C_-} \right)^{\frac{b_-}{b_--1}} z^{\frac{b_-}{b_--1}} \right) \leq U^*_{\hat{g}}(y; B_1, B_2, z).$$

Thus, the growth condition (3.17) is satisfied with $\alpha = \max\left(\left| \frac{b_-}{b_--1} \right|, \left| \frac{b_+}{b_+-1} \right| \right)$ and a suitably large constant $C > 0$.

(ii) If $C = |U(B_2)| + y_0(B_1 + B_2) + y_1 |g(B_2)| < \infty$, then condition (3.17) is satisfied with $\alpha = 1$ because for all $z > 0$

$$U^*_{\hat{g}}(y; B_1, B_2, z) = \sup_{B_1 \leq x \leq B_2} \big(U(x) - y_0 z x - y_1 g(x) \big) \stackrel{x = B_2}{\geq} U(B_2) - y_0 z B_2 - y_1 g(B_2)$$

and $\quad U^*_{\hat{g}}(y; B_1, B_2, z) = \sup_{B_1 \leq x \leq B_2} \big(\underbrace{U(x)}_{\leq U(B_2)} - y_0 z x \underbrace{- y_1 g(x)}_{\leq y_1 g(B_2)} \big)$

$$
\begin{aligned}
&\leq U(B_2) - y_1 g(B_2) + \sup_{B_1 \leq x \leq B_2} \big(- y_0 z x \big) = U(B_2) - y_0 z B_1 - y_1 g(B_2) \\
\Rightarrow |U^*_{\hat{g}}(y; B_1, B_2, z)| &\leq |U(B_2)| + y_0(B_1 + B_2) z + y_1 |g(B_2)| \\
&\leq (|U(B_2)| + y_1|g(B_2)|) \frac{1}{z} + (|U(B_2)| + y_0(B_1 + B_2) + y_1 |g(B_2)|) \, z \\
&\leq C(z^{-\alpha} + z^{\alpha}).
\end{aligned}
$$

□

Proof of Corollary 3.5.7. All prerequisites of Lemma 3.5.5 are satisfied and thus G (as defined in Lemma 3.5.5) satisfies the HJB equation (3.10), is convex, strictly decreasing in z and satisfies a polynomial growth condition. According to Theorem 3.5.3, as the minimizing $\lambda^*(t, z; y) = \lambda^*$ is independent of $y \in [0, \infty)^2$, this implies for all $y \in [0, \infty)^2$:

$$
\begin{aligned}
G(0, z; y) &= \mathbb{E}[U^*_{g_{\lambda^*}}(y; B_1, B_2)] \\
&\leq \mathbb{E}[U^*_{g_{\lambda}}(y; B_1, B_2)], \quad \forall \lambda \in \mathcal{D} \text{ with } \|\lambda^* - \lambda(t)\|^2 \leq 1 \;\; \mathcal{L}[0,T] \otimes Q\text{-a.s..}
\end{aligned}
$$

In particular, this holds for the choice of $y = y^*$, where y^* is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$. Finally, as $\lambda^* \in \mathcal{D}'$, Condition $(\tilde{D})$ is satisfied locally and the statement of this Corollary now follows due to the equivalence of Condition $(\tilde{B})$ and Condition $(\tilde{D})$ by virtue of Corollary 3.4.6 and Theorem 3.4.5. □

Proof of Example 3.5.10. Following Corollary 3.5.7 and Remark 3.5.8, we still need to

- verify that $U^*_{\hat{g}}$ satisfies a growth condition (3.17) in z,
- show that $\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2) = \mathrm{Cap}\big(\mathcal{I}(y^*_0), B_1, B_2\big)$,
- verify that $\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2) \in L^2_Q$,
- verify that $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2))\right]' \Delta y$ is usc at $y = y^*$ for all $\Delta \in \mathbb{R} \times [0, \infty)$.

As U is a power utility function and $g \equiv 0$, condition (i) of Lemma 3.5.6 is satisfied. Furthermore, $U(x) - y_0 \tilde{Z}_{\lambda^*}(T)$ is strictly concave in $x \geq 0$ and is maximized by $\mathcal{I}(y_0)$ over $x \in \mathbb{R}$. Hence, its maximum over $x \in [B_1, B_2]$ is attained by

$$
\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = B_1 + (\mathcal{I}(y_0) - B_1)^+ - (\mathcal{I}(y_0) - B_2)^+ = \mathrm{Cap}(\mathcal{I}(y_0), B_1, B_2).
$$

By Assumption 3.5.1, the market coefficients are deterministic and so is λ^*. Hence, $\tilde{Z}_{\lambda^*}(T)$ is log-normally distributed and all of its moments are finite. In particular, we have for all $y_0 \in (0, \infty)$

$$
\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)^2\right] \leq 2B_1^2 + 2\mathbb{E}\left[\mathcal{I}(y_0)\right] = 2B_1^2 + 2\mathbb{E}\left[\left(y_0 \tilde{Z}_{\lambda^*}(T)\right)^{\frac{2}{b-1}}\right] < \infty.
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) \tilde{Z}_{\lambda^*}(T)\right] &\leq B_1 \mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)\right] + \mathbb{E}\left[\mathcal{I}(y_0) \tilde{Z}_{\lambda^*}(T)\right] \\
&\leq B_1 \mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)\right] + y_0^{\frac{1}{b-1}} \mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)^{\frac{b}{b-1}}\right] < \infty. \qquad \text{(A.44)}
\end{aligned}
$$

Lastly, $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)$ is a concatenation of continuous functions and thereby continuous in $y_0 \in (0, \infty)$. Therefore, the monotonicity of $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)$ in y_0, equation (A.44) and the dominated convergence theorem imply the continuity of

$$y \to \mathbb{E}\left[g_{\lambda^*}\left(\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)\tilde{Z}_{\lambda^*}(T)\right)\right] = \begin{pmatrix} \mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2)\tilde{Z}_{\lambda^*}(T)\right] \\ 0 \end{pmatrix}.$$

in $y \in [0, \infty)^2$. This is also true for $y = (y_0^*, 0)'$ and thus $\lambda^* \in \mathcal{D}'$. The optimality of $\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)$ for $(\mathbf{P})$ follows from Corollary 3.5.7. □

Proof of Example 3.5.11. Following Corollary 3.5.7 and Remark 3.5.8, we still need to

- verify that $U_{\hat{g}}^*$ satisfies growth condition (3.17) in z,
- show that $\mathcal{I}_{g_{\lambda^*}}(y^*)$ is given as in the statement of the example,
- verify that $\mathcal{I}_{g_{\lambda^*}}(y^*) \in L_Q^2$,
- verify that $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right]' \Delta y$ is usc at $y = y^*$ for all $\Delta \in \mathbb{R} \times [0, \infty)$.

As U is a power utility function and $-\epsilon \leq g(x) \leq 1 - \epsilon$ for all $x \geq 0$, $U_{\hat{g}}$ satisfies the growth condition (3.17) by Lemma 3.5.6, (i). Further, we have for any $y = (y_0, y_1)' \in [0, \infty)^2$:

$$\begin{aligned}
U_{g_{\lambda^*}}^*(y) &= \sup_{x \geq 0}\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1 g(x)\right) \\
&= \sup_{x \geq 0}\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1\mathbb{1}_{\{x < B_{VaR}\}}\right) + y_1\epsilon \\
&= \max\left(\sup_{B_{VaR} \leq x}\underbrace{\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x\right)}_{=:f_1(x)}, \sup_{0 \leq x \leq B_{VaR}}\underbrace{\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1\right)}_{=:f_2(x)}\right) + y_1\epsilon \\
&= \max\left(\sup_{B_{VaR} \leq x} f_1(x), \sup_{0 \leq x \leq B_{VaR}} f_2(x)\right) + y_1\epsilon
\end{aligned}$$

Since $f_1(B_{VaR}) \geq f_2(B_2)$, the last inequality holds despite the discontinuity of $U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1\mathbb{1}_{\{x < B_{VaR}\}}$ at $x = B_{VaR}$. The functions f_1 and f_2 are both strictly concave in x and thus their maximizer over bounded intervals is obtained by capping their global maximizer at the boundaries. Hence, $x = \max(\mathcal{I}(y_0), B_{VaR})$ maximizes $f_1(x)$ over $x \in [B_{VaR}, \infty)$ and $x = \min\left(\mathcal{I}\left(y_0\right), B_{VaR}\right)$ maximizes f_2 over $x \in [0, B_{VaR}]$. It remains to determine which of these two local maximizers yields the global maximizer $\mathcal{I}_{g_{\lambda^*}}(y)$. We make a distinction between two cases:

- $B_{VaR} < \mathcal{I}(y_0)$: This implies

  $$\max(\mathcal{I}(y_0), B_{VaR}) = \mathcal{I}(y_0), \qquad \min\left(\mathcal{I}\left(y_0\right), B_{VaR}\right) = B_{VaR}.$$

  Hence,

  $$\sup_{0 \leq x \leq B_{VaR}} f_2(x) = f_2(B_{VaR}) \leq f_1(B_{VaR}) \leq \sup_{B_{VaR} \leq x} f_1(x) = f_1(\mathcal{I}(y_0))$$

  and therefore $\mathcal{I}_{g_{\lambda^*}}(y) = \mathcal{I}(y_0)$.

- $\mathcal{I}(y_0) \leq B_{VaR}$: This implies

$$\max(\mathcal{I}(y_0), B_{VaR}) = B_{VaR}, \qquad \min\left(\mathcal{I}\left(y_0\right), B_{VaR}\right) = \mathcal{I}(y_0).$$

 We then have

$$\begin{aligned} & \sup_{0\leq x\leq B_{VaR}} f_2(x) \geq \sup_{B_{VaR}\leq x} f_1(x) \quad \Leftrightarrow \quad f_2(\mathcal{I}(y_0)) \geq f_1(B_{VaR}) \\ \Leftrightarrow \quad & U(\mathcal{I}(y_0)) - \underbrace{y_0\tilde{Z}_{\lambda^*}(T)}_{=\mathcal{I}(y_0)^{b-1}}\mathcal{I}(y_0) - y_1 \geq U(B_{VaR}) - y_0\tilde{Z}_{\lambda^*}(T)B_{VaR} \\ \Leftrightarrow \quad & \frac{1-b}{b}\mathcal{I}(y_0)^b - y_1 \geq \frac{1}{b}B_{VaR}^b - y_0\tilde{Z}_{\lambda^*}(T)B_{VaR} \\ \Leftrightarrow \quad & \frac{1-b}{b}\mathcal{I}(y_0)^b + y_0\tilde{Z}_{\lambda^*}(T)B_{VaR} \geq \frac{1}{b}B_{VaR}^b + y_1. \end{aligned}$$

 Thus, if additionally

 - $\frac{1-b}{b}\mathcal{I}(y_0)^b + y_0\tilde{Z}_{\lambda^*}(T)B_{VaR} \geq \frac{1}{b}B_{VaR}^b + y_1$, then $\mathcal{I}_{g_{\lambda^*}}(y) = \mathcal{I}(y_0)$.
 - $\frac{1-b}{b}\mathcal{I}(y_0)^b + y_0\tilde{Z}_{\lambda^*}(T)B_{VaR} < \frac{1}{b}B_{VaR}^b + y_1$, then $\mathcal{I}_{g_{\lambda^*}}(y) = B_{VaR}$.

In total, the maximizing argument $\mathcal{I}_{g_{\lambda^*}}(y^*)$, where y^* is optimal for $(\mathbf{D}_{\lambda^*}^{\mathbf{V}})$, is as specified in the statement of this Example. By Assumption 3.5.1, the market coefficients are deterministic and so is λ^*. Hence, $\tilde{Z}_{\lambda^*}(T)$ is log-normally distributed and all of its moments are finite. In particular, we have for arbitrary $y = (y_0, y_1)' \in (0,\infty) \times [0,\infty)$

$$\begin{aligned} \mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y)^2\right] &\leq \mathbb{E}\left[\left(B_{VaR} + \mathcal{I}(y_0)\right)^2\right] \\ &\leq 2B_{VaR}^2 + 2\mathbb{E}\left[\left(\mathcal{I}(y_0)\right)^2\right] \\ &= 2B_{VaR}^2 + 2y_0^{\frac{2}{b-1}}\mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)^{\frac{2}{b-1}}\right] < \infty, \end{aligned}$$

i.e., $\mathcal{I}_{g_{\lambda^*}}(y) \in L_Q^2$. Similarly, we have for arbitrary $y = (y_0, y_1)' \in (0,\infty) \times [0,\infty)$

$$\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)\right] \leq B_{VaR}\mathbb{E}[\tilde{Z}_{\lambda^*}(T)] + y_0^{\frac{1}{b-1}}\mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)^{\frac{b}{b-1}}\right] < \infty.$$

For a fixed value of $\tilde{Z}_{\lambda^*}(T)$, the functions $y \to \mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)$ and $y \to g(\mathcal{I}_{g_{\lambda^*}}(y))$ are continuous at $y = y^*$ if

$$\frac{1-b}{b}\left(y_0^*\tilde{Z}_{\lambda^*}(T)\right)^{\frac{b}{b-1}} + y_0^*\tilde{Z}_{\lambda^*}(T)B_{VaR} \neq \frac{1}{b}B_{VaR}^b + y_1^*. \tag{A.45}$$

The left side of (A.45) is strictly convex in $\tilde{Z}_{\lambda^*}(T)$, while the right side is a deterministic constant. Thus, there are at most 2 values of $\tilde{Z}_{\lambda^*}(T) \in (0,\infty)$ such that we have equality in (A.45). However, $\tilde{Z}_{\lambda^*}(T)$ has a continuous distribution on $(0,\infty)$ and thus

$$\begin{aligned} & Q\left(g_{\lambda^*}\left(\mathcal{I}_{g_{\lambda^*}}(y)\right) \text{ is continuous at } y = y^*\right) \\ & \quad \geq Q\left(\frac{1-b}{b}\left(y_0^*\tilde{Z}_{\lambda^*}(T)\right)^{\frac{b}{b-1}} + y_0^*\tilde{Z}_{\lambda^*}(T)B_{VaR} \neq \frac{1}{b}B_{VaR}^b + y_1^*\right) = 1, \end{aligned}$$

i.e., $y \to g_{\lambda^*}\left(\mathcal{I}_{g_{\lambda^*}}(y)\right)$ is continuous at $y = y^*$ with probability 1. Further, for any sequence of $(y_n)_{n\in\mathbb{N}}$ taking values in $(0,\infty) \times [0,\infty)$ and converging to $y^* \in (0,\infty) \times [0,\infty)$, we know that

$$\hat{y}_0 := \inf_{n\in\mathbb{N}}\left((y_n)_0\right) > 0$$

$$\left|\mathcal{I}_{g_{\lambda^*}}(y_n)\tilde{Z}_{\lambda^*}(T)\right| \leq \left(B_{VaR} + \mathcal{I}\left((y_n)_0\right)\right)\tilde{Z}_{\lambda^*}(T) \leq \left(B_{VaR} + \mathcal{I}\left(\hat{y}_0\right)\right)\tilde{Z}_{\lambda^*}(T)$$
$$\left|g\left(\mathcal{I}_{g_{\lambda^*}}(y_n)\right)\right| \leq 1.$$

Thus, the dominated convergence theorem implies

$$\lim_{n\to\infty} \mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y_n))\right] = \lim_{n\to\infty} \begin{pmatrix} \mathbb{E}\left[-\mathcal{I}_{g_{\lambda^*}}(y_n)\tilde{Z}_{\lambda^*}(T)\right] \\ \mathbb{E}\left[-g\left(\mathcal{I}_{g_{\lambda^*}}(y_n)\right)\right] \end{pmatrix}$$
$$= \begin{pmatrix} \mathbb{E}\left[-\mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)\right] \\ \mathbb{E}\left[-g\left(\mathcal{I}_{g_{\lambda^*}}(y)\right)\right] \end{pmatrix} = \mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right].$$

Hence, $y \to \mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right]$ is continuous at $y = y^*$ and therefore $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right]'\Delta y$ is usc at $y = y^*$ for all $\Delta \in \mathbb{R} \times [0,\infty)$ and thus $\lambda^* \in \mathcal{D}'$. In total, the optimality of $\mathcal{I}_{g_{\lambda^*}}(y^*)$ for $(\mathbf{P})$ now follows from Corollary 3.5.7. □

Proof of Example 3.5.12. Following the same arguments as in the proof of Example 3.5.12, we have for any $y = (y_0, y_1) \in [0,\infty)^2$

$$\begin{aligned} U^*_{g_{\lambda^*}}(y; B_1, B_2) &= \sup_{B_1 \leq x \leq B_2} \left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1 g(x)\right) \\ &= \sup_{B_1 \leq x \leq B_2} \left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1 \mathbb{1}_{\{x < B_{VaR}\}}\right) + y_1\epsilon \\ &= \max\left(\sup_{B_{VaR} \leq x \leq B_2} \underbrace{\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x\right)}_{=f_1(x)}, \sup_{B_1 \leq x \leq B_{VaR}} \underbrace{\left(U(x) - y_0\tilde{Z}_{\lambda^*}(T)x - y_1\right)}_{=f_2(x)}\right) + y_1\epsilon \\ &= \max\left(\sup_{B_{VaR} \leq x \leq B_2} f_1(x), \sup_{B_1 \leq x \leq B_{VaR}} f_2(x)\right) + y_1\epsilon. \end{aligned}$$

Since f_1 and f_2 are strictly concave in $x \in (0,\infty)$, their constrained maximizers over intervals can be obtained by capping their global maximizers at the boundaries of the interval, i.e.,

$$\sup_{B_{VaR} \leq x \leq B_2} f_1(x) = f_1\left(\mathrm{Cap}(\mathcal{I}(y_0), B_{VaR}, B_2)\right)$$
$$\sup_{B_1 \leq x \leq B_{VaR}} f_2(x) = f_2\left(\mathrm{Cap}(\mathcal{I}(y_0), B_1, B_{VaR})\right).$$

We make a distinction between three cases:

- $B_2 < \mathcal{I}_{g_{\lambda^*}}(y)$: As $B_{VaR} < B_2$, this implies $\mathcal{I}_{g_{\lambda^*}}(y) = \mathcal{I}(y_0)$ and

  $$\mathrm{Cap}(\mathcal{I}(y_0), B_{VaR}, B_2) = B_2, \qquad \mathrm{Cap}(\mathcal{I}(y_0), B_1, B_{VaR}) = B_{VaR}.$$

  Hence,

  $$\sup_{B_1 \leq x \leq B_{VaR}} f_2(x) = f_2(B_{VaR}) \leq f_1(B_{VaR}) \leq \sup_{B_{VaR} \leq x \leq B_2} f_1(x) = f_1(B_2)$$

  and therefore $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = B_2$.

- $B_1 \leq \mathcal{I}_{g_{\lambda^*}}(y) \leq B_2$: This implies that

  $$U(\mathcal{I}_{g_{\lambda^*}}(y)) - y_0\tilde{Z}_{\lambda^*}(T)\mathcal{I}_{g_{\lambda^*}}(y) - y_1 g(\mathcal{I}_{g_{\lambda^*}}(y)) = U^*_{g_{\lambda^*}}(y)$$

$$
\begin{aligned}
&= \sup_{0\le x} \Big(U(x) - y_0 \tilde Z_{\lambda^*}(T) x - y_1 g(x) \Big)\\
&\ge \sup_{B_1 \le x \le B_2} \Big(U(x) - y_0 \tilde Z_{\lambda^*}(T) x - y_1 g(x) \Big)\\
&\ge U(\mathcal{I}_{g_{\lambda^*}}(y)) - y_0 \tilde Z_{\lambda^*}(T) \mathcal{I}_{g_{\lambda^*}}(y) - y_1 g(\mathcal{I}_{g_{\lambda^*}}(y))
\end{aligned}
$$

and therefore $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = \mathcal{I}_{g_{\lambda^*}}(y)$.

- $\mathcal{I}_{g_{\lambda^*}}(y) < B_1$: As $B_1 < B_{VaR}$, this implies $\mathcal{I}_{g_{\lambda^*}}(y) = \mathcal{I}(y_0)$ and

$$
\mathrm{Cap}(\mathcal{I}(y_0), B_{VaR}, B_2) = B_{VaR}, \qquad\qquad \mathrm{Cap}(\mathcal{I}(y_0), B_1, B_{VaR}) = B_1.
$$

  We then have

$$
\begin{aligned}
&\sup_{B_1 \le x \le B_{VaR}} f_2(x) \ge \sup_{B_{VaR} \le x \le B_2} f_1(x) \quad \Leftrightarrow \quad f_2(B_1) \ge f_1(B_{VaR})\\
\Leftrightarrow \quad & U(B_1) - y_0 \tilde Z_{\lambda^*}(T) B_1 - y_1 \ge U(B_{VaR}) - y_0 \tilde Z_{\lambda^*}(T) B_{VaR}\\
\Leftrightarrow \quad & \tilde Z_{\lambda^*}(T) \ge \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1}{y_0 (B_{VaR} - B_1)}.
\end{aligned}
$$

  Thus, if additionally

  - $\tilde Z_{\lambda^*}(T) < \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1}{y_0 (B_{VaR} - B_1)}$, then $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = B_{VaR}$.
  - $\tilde Z_{\lambda^*}(T) \ge \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1}{y_0 (B_{VaR} - B_1)}$, then $\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = B_1$.

In total, the maximizing argument $\mathcal{I}_{g_{\lambda^*}}(y^*)$ for the optimal y^* is as specified in the statement of this Example. All integrability conditions for $\mathcal{I}_{g_{\lambda^*}}(y^*; B_1, B_2)$ are satisfied by analogous arguments as in Example 3.5.11. For a fixed value of $\tilde Z_{\lambda^*}(T)$, the functions $y \to \mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) \tilde Z_{\lambda^*}(T)$ and $y \to g(\mathcal{I}_{g_{\lambda^*}}(y; B_1, B2))$ are continuous at $y = y^*$ if

$$
\frac{1-b}{b} \Big(y_0^* \tilde Z_{\lambda^*}(T) \Big)^{\frac{b}{b-1}} + y_0^* \tilde Z_{\lambda^*}(T) B_{VaR} \ne \frac{1}{b} B_{VaR}^b + y_1^*. \tag{A.46}
$$

and

$$
\tilde Z_{\lambda^*}(T) \ne \frac{\frac{1}{b}(B_{VaR}^b - B_1^b) + y_1^*}{y_0^* (B_{VaR} - B_1)}. \tag{A.47}
$$

However, as argued in the proof of Example 3.5.11, there are at most 2 values of $\tilde Z_{\lambda^*}(T)$ such that equality holds in (A.46) and there is at most one value of $\tilde Z_{\lambda^*}(T)$ such that equality holds in (A.47). Since $\tilde Z_{\lambda^*}(T)$ has a continuous distribution on $(0, \infty)$ this again implies

$$
Q\left(g_{\lambda^*}\left(\mathcal{I}_{g_{\lambda^*}}(y)\right) \text{ is continuous at } y = y^* \right) \ge Q\left(\text{(A.46) and (A.47 hold.)} \right) = 1.
$$

The rest of this example can be proved using analogous steps as in the proof of Example 3.5.13. □

Proof of Example 3.5.13. Following Corollary 3.5.7 and Remark 3.5.8, we still need to

- verify that $U_{\hat g}^*$ satisfies growth condition (3.17) in z,
- show that $\mathcal{I}_{g_{\lambda^*}}(y^*)$ is given as in the statement of the example,
- verify that $\mathcal{I}_{g_{\lambda^*}}(y^*) \in L_Q^2$,

- verify that $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right]' \Delta y$ is usc at $y = y^*$ for all $\Delta \in \mathbb{R} \times [0, \infty)$.

As U is a power utility and $-\epsilon \leq g(x) \leq B_{ES} - \epsilon$ for all $x \geq 0$, $U^{\hat{g}}$ satisfies the growth condition (3.17) by Lemma 3.5.6, (i). Further, we have for any $(y_0, y_1)' \in [0, \infty)^2$:

$$
\begin{aligned}
U^*_{g_{\lambda^*}}(y_0, y_1) &= \sup_{x \geq 0} \left(U(x) - y_0 \tilde{Z}_{\lambda^*}(T) x - y_1 g(x) \right) \\
&= \sup_{x \geq 0} \left(U(x) - y_0 \tilde{Z}_{\lambda^*}(T) x - y_1 \left(B_{ES} - x \right) \mathbb{1}_{\{x < B_{ES}\}} \right) + y_1 \epsilon \\
&= \max \left(\sup_{B_{ES} \leq x} \underbrace{\left(U(x) - y_0 \tilde{Z}_{\lambda^*}(T) x \right)}_{=: f_1(x)}, \sup_{0 \leq x \leq B_{ES}} \underbrace{\left(U(x) - (y_0 z - y_1) x - y_1 B_{ES} \right)}_{=: f_2(x)} \right) + y_1 \epsilon \\
&= \max \left(\sup_{B_{ES} \leq x} f_1(x), \sup_{0 \leq x \leq B_{ES}} f_2(x) \right) + y_1 \epsilon
\end{aligned}
$$

f_1 and f_2 are both strictly concave in x and thus their maximizer over bounded intervals is obtained by capping their global maximizer at the boundaries. Hence, $x = \max(\mathcal{I}(y_0), B_{ES})$ maximizes $f_1(x)$ over $x \in [B_{ES}, \infty)$ and $x = \min\left(\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right), B_{ES}\right)$ maximizes f_2 over $x \in [0, B_{ES}]$. It remains to determine which of these two local maximizers yields the global maximizer $\mathcal{I}_{g_{\lambda^*}}(y)$. We make a distinction between three cases

- $B_{ES} < \mathcal{I}(y_0)$: As $\mathcal{I}(y_0)$ is non-increasing in y_0, this implies $\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right) \geq \mathcal{I}(y_0) > B_{ES}$ and thus

$$
\max(\mathcal{I}(y_0), B_{ES}) = \mathcal{I}(y_0), \qquad \min\left(\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right), B_{ES}\right) = B_{ES}.
$$

  Hence,

$$
\begin{aligned}
\sup_{0 \leq x \leq B_2} f_2(x) &= f_2(B_{ES}) \\
&= U(B_{ES}) - \left(y_0 \tilde{Z}_{\lambda^*}(T) - y_1 \right) B_{ES} - y_1 B_{ES} \\
&= \underbrace{U(B_{ES}) - y_0 \tilde{Z}_{\lambda^*}(T) B_{ES}}_{= f_1(B_{ES})} \\
&< \sup_{B_{ES} \leq x} f_1(x) \\
&= f_1(\mathcal{I}(y_0))
\end{aligned}
$$

  and therefore $\mathcal{I}_{g_{\lambda^*}}(y) = \mathcal{I}(y_0)$.

- $\mathcal{I}(y_0) \leq B_{ES} \leq \mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right)$: This implies

$$
\max(\mathcal{I}(y_0), B_{ES}) = B_{ES}, \qquad \min\left(\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right), B_{ES}\right) = B_{ES}
$$

  and therefore $\mathcal{I}_{g_{\lambda^*}} = B_{ES}$.

- $\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right) < B_{ES}$: As $\mathcal{I}(y_0)$ is non-increasing in y_0, this implies $\mathcal{I}(y_0) < \mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right) < B_{ES}$ and thus and thus

$$
\max(\mathcal{I}(y_0), B_{ES}) = B_{ES}, \quad \min\left(\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right), B_{ES}\right) = \mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right).
$$

Hence,

$$\begin{aligned}
\sup_{B_{ES}\leq x} f_1(x) &= f_1(B_{ES}) \\
&= U(B_2) - y_0\tilde{Z}_{\lambda^*}(T)B_{ES} \\
&= \underbrace{U(B_2) - \left(y_0\tilde{Z}_{\lambda^*}(T) - y_1\right) B_{ES} - y_1 B_{ES}}_{=f_2(B_{ES})} \\
&\leq \sup_{0\leq x\leq B_{ES}} f_2(x) \\
&= f_2\left(\mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right)\right)
\end{aligned}$$

and therefore $\mathcal{I}_{g_{\lambda^*}} = \mathcal{I}\left(y_0 - \frac{y_1}{\tilde{Z}_{\lambda^*}(T)}\right)$.

In total, the maximizing argument $\mathcal{I}_{g_{\lambda^*}}(y^*)$, where y^* is optimal for $(\mathbf{D}^{\mathbf{V}}_{\lambda^*})$, is as specified in the statement of this Example. By Assumption 3.5.1, the market coefficients are deterministic and so is λ^*. Hence, $\tilde{Z}_{\lambda^*}(T)$ is log-normally distributed and all of its moments are finite. In particular, we have for arbitrary $y = (y_0, y_1)' \in (0,\infty)\times[0,\infty)$

$$\begin{aligned}
\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y)^2\right] &\leq \mathbb{E}\left[\left(B_{ES} + \mathcal{I}(y_0)\right)^2\right] \\
&\leq 2B_{ES}^2 + 2\mathbb{E}\left[\left(\mathcal{I}(y_0)\right)^2\right] \\
&= 2B_{ES}^2 + 2y_0^{\frac{2}{b-1}}\mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)^{\frac{2}{b-1}}\right] < \infty,
\end{aligned}$$

i.e., $\mathcal{I}_{g_{\lambda^*}}(y) \in L^2_Q$. Similarly, we have for arbitrary $y = (y_0, y_1)' \in (0,\infty)\times[0,\infty)$

$$\mathbb{E}\left[\mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)\right] \leq B_{ES}\mathbb{E}[\tilde{Z}_{\lambda^*}(T)] + y_0^{\frac{1}{b-1}}\mathbb{E}\left[\tilde{Z}_{\lambda^*}(T)^{\frac{b}{b-1}}\right] < \infty.$$

For a fixed value of $\tilde{Z}_{\lambda^*}(T)$, the functions $y \to \mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)$ and $y \to g(\mathcal{I}_{g_{\lambda^*}}(y))$ are continuous in $y \in (0,\infty)\times[0,\infty)$. Further, for any sequence of $(y_n)_{n\in\mathbb{N}}$ taking values in $(0,\infty)\times[0,\infty)$ and converging to the optimal $y^* \in (0,\infty)\times[0,\infty)$, we know that

$$\begin{aligned}
&\hat{y}_0 := \inf_{n\in\mathbb{N}}\left((y_n)_0\right) > 0 \\
&\left|\mathcal{I}_{g_{\lambda^*}}(y_n)\tilde{Z}_{\lambda^*}(T)\right| \leq \left(B_{ES} + \mathcal{I}\left((y_n)_0\right)\right)\tilde{Z}_{\lambda^*}(T) \leq \left(B_{ES} + \mathcal{I}\left(\hat{y}_0\right)\right)\tilde{Z}_{\lambda^*}(T) \\
&\left|g\left(\mathcal{I}_{g_{\lambda^*}}(y_n)\right)\right| \leq B_{ES}.
\end{aligned}$$

Thus, the dominated convergence theorem implies

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y_n))\right] &= \lim_{n\to\infty}\begin{pmatrix}\mathbb{E}\left[-\mathcal{I}_{g_{\lambda^*}}(y_n)\tilde{Z}_{\lambda^*}(T)\right] \\ \mathbb{E}\left[-g\left(\mathcal{I}_{g_{\lambda^*}}(y_n)\right)\right]\end{pmatrix} \\
&= \begin{pmatrix}\mathbb{E}\left[-\mathcal{I}_{g_{\lambda^*}}(y)\tilde{Z}_{\lambda^*}(T)\right] \\ \mathbb{E}\left[-g\left(\mathcal{I}_{g_{\lambda^*}}(y)\right)\right]\end{pmatrix} = \mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right].
\end{aligned}$$

Hence, $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y))\right]$ is continuous in $y = y^*$ and therefore $\mathbb{E}\left[g_{\lambda^*}(\mathcal{I}_{g_{\lambda^*}}(y)\right]'\Delta y$ is usc in y for all $\Delta \in \mathbb{R}\times[0,\infty)$ and thus, $\lambda^* \in \mathcal{D}'$. In total, the optimality of $\mathcal{I}_{g_{\lambda^*}}(y^*)$ for $(\mathbf{P})$ follows from Corollary 3.5.7. □

Proof of Example 3.5.14. We may proceed analogously to Example 3.5.14. Let $y \in [0,\infty)^2$. As g is convex, the maximization objective

$$U(x) - y_0 \tilde{Z}_{\lambda^*}(T)x - y_1 g(x)$$

remains strictly concave in $x \geq 0$ and is maximized globally by $x = \mathcal{I}_{g_{\lambda^*}}(y)$ over $x \in \mathbb{R}$. Hence, its maximum over $[B_1, B_2]$ is attained by

$$\mathcal{I}_{g_{\lambda^*}}(y; B_1, B_2) = \mathrm{Cap}(\mathcal{I}(y), B_1, B_2).$$

Since 'capping' is a continuous operation which only further bounds its argument, the rest of this example can be proved using analogous steps as in the proof of Example 3.5.13. □

A.3 Proofs Chapter 4

Proof of Lemma 4.3.1. From Corollary 2.1.32 it is known that

$$\begin{aligned}
0 &= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\
&\quad + v \sup_{\pi \in K} \left\{ (\mu - r\mathbb{1})'\pi G_v + \left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \right\} \\
&= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\
&\quad + v \inf_{\lambda \in \mathbb{R}^d} \sup_{\pi \in \mathbb{R}^d} \left\{ \delta_K(\lambda) + \lambda'\pi + (\mu - r\mathbb{1})'\pi G_v + \left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \right\} \\
&= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\
&\quad + v \inf_{\lambda \in \mathbb{R}^d} \sup_{\pi \in \mathbb{R}^d} \left\{ \delta_K(\underbrace{G_v}_{>0} \cdot \lambda) + \underbrace{G_v}_{>0} \cdot \lambda'\pi + (\mu - r\mathbb{1})'\pi G_v + \left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \right\}
\end{aligned}$$

By assumption, v and $G_v(t,v,z)$ are positive[1]. Using that δ_K is a support function and thus positive homogenous of order 1, yields

$$\begin{aligned}
0 &= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\
&\quad + v \inf_{\lambda \in \mathbb{R}^d} \sup_{\pi \in \mathbb{R}^d} \left\{ \left[\delta_K(\lambda) + (\mu + \lambda - r\mathbb{1})'\pi\right] G_v + \left(\Sigma^z \rho \Sigma' \pi\right)' \nabla_z (G_v) + \frac{1}{2} v \|\Sigma'\pi\|^2 G_{vv} \right\}.
\end{aligned} \tag{A.48}$$

Further, for every fixed $\lambda \in \mathbb{R}^d$, the first-order optimality condition for the maximization over $\pi \in \mathbb{R}^d$ yields the candidate optimizer π_λ through

$$\begin{aligned}
0 &\overset{!}{=} [\mu + \lambda - r\mathbb{1}] G_v + \left(\Sigma^z \rho \Sigma'\right)' \nabla_z (G_v) + v\Sigma\Sigma'\pi_\lambda G_{vv} \\
\Leftrightarrow \quad \pi_\lambda &= -\frac{1}{vG_{vv}} \left(\Sigma\Sigma'\right)^{-1} \left([\mu + \lambda - r\mathbb{1}] G_v + \left(\Sigma^z \rho \Sigma'\right)' \nabla_z (G_v) \right).
\end{aligned}$$

Note that

$$[\mu + \lambda - r\mathbb{1}]' \pi_\lambda G_v + \left(\Sigma^z \rho \Sigma' \pi_\lambda\right)' \nabla_z (G_v) = \pi_\lambda' \left[(\mu + \lambda - r\mathbb{1}) G_v + \left(\Sigma^z \rho \Sigma'\right)' \nabla_z (G_v) \right]$$

[1]Note that this assumption is valid due to the exponential structure of the wealth process $V^{v_0,\pi}$ and the strict monotonicity of the utility function U.

$$
\begin{aligned}
&= \pi_\lambda' \left[-vG_{vv}\Sigma\Sigma'\pi_\lambda \right] \\
&= -v\|\Sigma'\pi_\lambda\|^2 G_{vv}.
\end{aligned}
$$

Since we are maximizing a quadratic function with respect to π, the first-order optimality condition is both necessary and sufficient. Hence, plugging π_λ into (A.48) finally yields

$$
\begin{aligned}
0 &= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z(\Sigma^z)'\nabla_z^2 G\right] + v\inf_{\lambda\in\mathbb{R}^d}\left\{\delta_K(\lambda) - \frac{1}{2}v\|\Sigma'\pi_\lambda\|^2 G_{vv}\right\} \\
&= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z(\Sigma^z)'\nabla_z^2 G\right] \\
&\quad + v\inf_{\lambda\in\mathbb{R}^d}\left\{\delta_K(\lambda)G_v - \frac{1}{2}\frac{1}{vG_{vv}}\|\Sigma^{-1}\left[\mu+\lambda-r\mathbb{1}\right]G_v + (\Sigma^z\rho)'\nabla_z(G_v)\|^2\right\}.
\end{aligned}
$$

□

Proof of Lemma 4.3.3. For any $\pi \in \Lambda$ ($\underline{\pi} \in \underline{\Lambda}$) and $\lambda \in \mathcal{D}$ ($\underline{\lambda} \in \underline{\mathcal{D}}$) define the operator

$$
\begin{aligned}
\mathcal{H}_\lambda^\pi[G](t,v,z) &= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\mathrm{Trace}\left[\Sigma^z(\Sigma^z)'\nabla_z^2 G\right] \\
&\quad + v\Big\{\left[\delta_K(\lambda) + (\mu+\lambda-r\mathbb{1})'\pi\right]G_v + \left(\Sigma^z\rho\Sigma'\pi\right)'\nabla_z(G_v) + \frac{1}{2}v\|\Sigma'\pi\|^2 G_{vv}\Big\},
\end{aligned}
$$

Let $(\underline{\lambda}^*, \pi) \in \underline{\mathcal{D}} \times \Lambda$ satisfy Condition (UI_λ). Then, by Itô's lemma and the boundedness of the integrands, due to the definition of $\tau_{n,t}^{\underline{\lambda}^*}$, we have

$$
\begin{aligned}
&\mathbb{E}\Big[G\left(\tau_{n,t}^{\underline{\lambda}^*}, V_{\underline{\lambda}^*}^{v_0,\pi}(\tau_{n,t}^{\underline{\lambda}^*}), z(\tau_{n,t}^{\underline{\lambda}^*})\right)\ \Big|\ \mathcal{F}_t\Big] \\
&\quad = G\left(t, V_{\underline{\lambda}^*}^{v_0,\pi}(t), z(t)\right) + \mathbb{E}\Big[\int_t^{\tau_{n,t}^{\underline{\lambda}^*}} \underbrace{\mathcal{H}_{\underline{\lambda}^*}^\pi[G]\left(s, V_{\underline{\lambda}^*}^{v_0,\pi}(s), z(s)\right)}_{\le \mathcal{H}_{\underline{\lambda}^*}^{\pi^*}[G]\left(s, V_{\underline{\lambda}^*}^{v_0,\pi^*}(s), z(s)\right)=0} ds\ \Big|\ \mathcal{F}_t\Big] \\
&\qquad + \underbrace{\mathbb{E}\Big[\int_t^{\tau_{n,t}^{\underline{\lambda}^*}} V_{\underline{\lambda}^*}^{v_0,\pi}(s) G_v\left(s, V_{\underline{\lambda}^*}^{v_0,\pi}(s), z(s)\right)\cdot \pi(s)'\Sigma(s,z(s))dW(s)\ \Big|\ \mathcal{F}_t\Big]}_{=0} \\
&\qquad + \underbrace{\mathbb{E}\Big[\int_t^{\tau_{n,t}^{\underline{\lambda}^*}} \nabla_z(G)\left(s, V_{\underline{\lambda}^*}^{v_0,\pi}(s), z(s)\right)'\Sigma^z(s,z(s))dW^z(s)\ \Big|\ \mathcal{F}_t\Big]}_{=0} \\
&\quad \le G(t, V_{\underline{\lambda}^*}^{v_0,\pi}(t), z(t))
\end{aligned}
$$

Taking conditional expectations and the limit $n \to \infty$ on both sides as well as recalling Remark 4.3.2 yields

$$
\begin{aligned}
\mathbb{E}\Big[U(V_{\underline{\lambda}^*}^{v_0,\pi}(T))\ \Big|\ \mathcal{F}_t\Big] &= \mathbb{E}\Big[G(T, V_{\underline{\lambda}^*}^{v_0,\pi}(T), z(T))\ \Big|\ \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\lim_{n\to\infty} G(\tau_{n,t}^{\underline{\lambda}^*}, V_{\underline{\lambda}^*}^{v_0,\pi}(\tau_{n,t}^{\underline{\lambda}^*}), z(\tau_{n,t}^{\underline{\lambda}^*}))\ \Big|\ \mathcal{F}_t\Big] \\
&\overset{(\mathrm{UI}_\lambda)}{=} \lim_{n\to\infty}\mathbb{E}\Big[G(\tau_{n,t}^{\underline{\lambda}^*}, V_{\underline{\lambda}^*}^{v_0,\pi}(\tau_{n,t}^{\underline{\lambda}^*}), z(\tau_{n,t}^{\underline{\lambda}^*}))\ \Big|\ \mathcal{F}_t\Big] \\
&\le \lim_{n\to\infty} G\big(t, V_{\underline{\lambda}^*}^{v_0,\pi}(t), z(t)\big) \\
&= G\big(t, V_{\underline{\lambda}^*}^{v_0,\pi}(t), z(t)\big).
\end{aligned}
$$

Conditioning on $V_{\lambda^*}^{v_0,\pi}(t) = v$, $z(t) = z$ leads to (4.10).
Letting $(\lambda, \underline{\pi}^*) \in \mathcal{D} \times \underline{\Lambda}$ satisfy Condition (UI$_\lambda$) and following the analogous steps as before, we can prove equation (4.11).
As all previous inequalities become equalities if we instead consider $(\underline{\lambda}^*, \underline{\pi}^*) \in \underline{\mathcal{D}} \times \underline{\Lambda}$ satisfying Condition (UI$_\lambda$), equation (4.12) follows immediately. □

Proof of Theorem 4.3.5. We first determine the derivatives of $G = G(t, v, z)$ in terms of $A = A(T-t)$ and $B = B(T-t)$ as

$$\begin{aligned} G_t &= -\left(A_\tau + B_\tau' z\right) G, \quad G_v = \frac{b}{v} G, \quad G_{vv} = \frac{b(b-1)}{v^2} G \\ \nabla_z G &= GB, \quad \nabla_z^2 G = GBB', \quad \nabla_z \left(G_v\right) = \frac{b}{v} GB. \end{aligned} \tag{A.49}$$

Plugging the derivatives into the dual HJBI PDE (4.7), factoring $\frac{bG}{2v(1-b)} > 0$ out of the minimization, dividing by $G \neq 0$ and plugging in the optimizer $\hat{\lambda}^* = \hat{\lambda}^*(t, z, B)$ yields

$$\begin{aligned}
(*) :&= G_t + vrG_v + (\mu^z)'(\nabla_z G) + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' \nabla_z^2 G\right] \\
&\quad + v \inf_{\lambda \in \mathbb{R}^d} \left\{ \delta_K(\lambda) G_v - \frac{1}{2}\frac{1}{vG_{vv}} \|\Sigma^{-1}[\mu + \lambda - r\mathbb{1}] G_v + (\Sigma^z \rho)' \nabla_z (G_v)\|^2 \right\} \\
&= -\left(A_\tau - B_\tau' z\right) G + brG + (\mu^z)' BG + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] G \\
&\quad + v \inf_{\lambda \in \mathbb{R}^d} \left\{ \delta_K(\lambda) \frac{b}{v} G - \frac{1}{2}\frac{v}{b(b-1)G} \left\| \Sigma^{-1}[\mu + \lambda - r\mathbb{1}] \frac{b}{v} G + (\Sigma^z \rho)' \frac{b}{v} BG \right\|^2 \right\} \\
&= -\left(A_\tau - B_\tau' z\right) G + brG + (\mu^z)' BG + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] G \\
&\quad + \frac{1}{2}\frac{bG}{1-b} \inf_{\lambda \in \mathbb{R}^d} \left\{ 2(1-b)\delta_K(\lambda) + \left\|\Sigma^{-1}[\mu + \lambda - r\mathbb{1}] + (\Sigma^z \rho)' B\right\|^2 \right\} \\
&= -A_\tau - B_\tau' z + br + (\mu^z)' B + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] \\
&\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda \in \mathbb{R}^d} \left\{ 2(1-b)\delta_K(\lambda) + \left\|\Sigma^{-1}[\mu + \lambda - r\mathbb{1}] + (\Sigma^z \rho)' B\right\|^2 \right\} \\
&= -A_\tau - B_\tau' z + br + (\mu^z)' B + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] \\
&\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda \in \mathbb{R}^d} \left\{ 2(1-b)\delta_K(\lambda) + 2\lambda' \left(\Sigma\Sigma'\right)^{-1} \left(\mu - r\mathbb{1} + \left(\Sigma^z \rho \Sigma'\right)' B\right) + \left\|\Sigma^{-1}\lambda\right\|^2 \right\} \\
&\quad + \frac{1}{2}\frac{b}{1-b} \left\|\Sigma^{-1}[\mu - r\mathbb{1}] + (\Sigma^z \rho)' B\right\|^2 \\
&= -A_\tau - B_\tau' z + br + (\mu^z)' B + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] \\
&\quad + \frac{1}{2}\frac{b}{1-b} \left(2(1-b)\delta_K(\hat{\lambda}^*) + 2(\hat{\lambda}^*)' \left(\Sigma\Sigma'\right)^{-1} \left(\mu - r\mathbb{1} + \left(\Sigma^z \rho \Sigma'\right)' B\right) + \left\|\Sigma^{-1}\hat{\lambda}^*\right\|^2 \right) \\
&\quad + \frac{1}{2}\frac{b}{1-b} \left\|\Sigma^{-1}[\mu - r\mathbb{1}] + (\Sigma^z \rho)' B\right\|^2 \\
&= -A_\tau - B_\tau' z + b\left(r + \delta_K(\hat{\lambda}^*)\right) + (\mu^z)' B + \frac{1}{2}\text{Trace}\left[\Sigma^z (\Sigma^z)' BB'\right] \\
&\quad + \frac{1}{2}\frac{b}{1-b} \left(\left\|\Sigma^{-1}(\mu + \hat{\lambda}^* - r\mathbb{1})\right\|^2 + 2B'\Sigma^z \rho \Sigma^{-1} \left(\mu + \hat{\lambda}^* - r\mathbb{1}\right) + B'\Sigma^z \rho (\Sigma^z \rho)' B \right)
\end{aligned}$$

Condition (EAS) allows us to replace the market coefficients by affine functions in z. Thus, we obtain

$$\begin{aligned}(*) &= -A_\tau - B_\tau' z + b\left(p_0 + p_1' z\right) + \left(k_0 + k_1 z\right)' B + \frac{1}{2}\text{Trace}\left[\left(h_0 + h_1[z]\right) BB'\right] \\ &\quad + \frac{1}{2}\frac{b}{1-b}\Big(q_0 + q_1' z + 2B'\left(g_0 + g_1 z\right) + B'\left(l_0 + h_0 + l_1[z] + h_1[z]\right) B\Big) \\ &= -A_\tau + bp_0 + k_0' B + \frac{1}{2}\text{Trace}\left[h_0 BB'\right] + \frac{1}{2}\frac{b}{1-b}\Big(q_0 + 2g_0' B + B'\left(l_0 + h_0\right) B\Big) \\ &\quad - B_\tau' z + bp_1' z + B' k_1 z + \frac{1}{2}\text{Trace}\left[h_1[z] BB'\right] + \frac{1}{2}\frac{b}{1-b}\Big(q_1' z + 2B' g_1 z + B'\left(l_1[z] + h_1[z]\right) B\Big)\end{aligned}$$

Making use of the commutative property of the trace of a matrix as well as the matrix representation of $h_1[\cdot]$ and $l_1[\cdot]$, we have

$$\begin{aligned}\text{Trace}\left[h_0 BB'\right] &= \text{Trace}\big[\underbrace{B' h_0 B}_{\in\mathbb{R}}\big] = B' h_0 B \\ \text{Trace}\left[h_1[z] BB'\right] &= \text{Trace}\big[\underbrace{B' h_1[z] B}_{\in\mathbb{R}}\big] = B' h_1[z] B = \left(B' h_1[\cdot] B\right)' z \\ B'\left(l_1[z] + h_1[z]\right) B &= \left(B'\left(l_1[\cdot] + h_1[\cdot]\right) B\right)' z.\end{aligned}$$

Thus, as A and B are solutions to the ODEs (4.15) and (4.16)

$$\begin{aligned}(*) &= -A_\tau + bp_0 + k_0' B + \frac{1}{2} B' h_0 B + \frac{1}{2}\frac{b}{1-b}\Big(q_0 + 2g_0' B + B'\left(l_0 + h_0\right) B\Big) \\ &\quad + \left(-B_\tau + bp_1 + k_1' B + \frac{1}{2} B' h_1[\cdot] B + \frac{1}{2}\frac{b}{1-b}\Big(q_1 + 2g_1' B + B'\left(l_1[\cdot] + h_1[\cdot]\right) B\Big)\right)' z \\ &= 0.\end{aligned}$$

Hence, G is a solution to the dual HJBI PDE (4.7) and thereby, according to Lemma 4.3.1, also a solution to the primal HJB PDE (4.5). □

Proof of Theorem 4.3.7. We first derive the explicit expression (4.17) for $\underline{\pi}^*$ in terms of $\underline{\lambda}^*$ and B. Following the arguments in the proof of Lemma 4.3.1, we realize that the candidate optimal portfolio $\underline{\pi}^*$ is given as the maximizing argument π_{λ^*}, i.e.,

$$\underline{\pi}^* = \pi_{\lambda^*} = -\frac{1}{vG_{vv}}\left(\Sigma\Sigma'\right)^{-1}\left[G_v(\mu + \underline{\lambda}^* - r\mathbb{1}) + \Sigma\rho'\left(\Sigma^z\right)' \nabla_z\left(G_v\right)\right]. \tag{A.50}$$

On the other hand, since Condition (EAS) is satisfied, the exponentially affine structure of G from (4.13) implies that the derivatives of G are given by (A.49). Plugging these derivatives into (A.50) yields the candidate optimal portfolio

$$\begin{aligned}\underline{\pi}^* &= -\left(\frac{b(b-1)}{v} G\right)^{-1}\left(\Sigma\Sigma'\right)^{-1}\left[\frac{b}{v} G(\mu + \underline{\lambda}^* - r\mathbb{1}) + \Sigma\rho'\left(\Sigma^z\right)'\frac{b}{v} GB\right] \\ &= \frac{1}{1-b}\left(\Sigma\Sigma'\right)^{-1}\left[\mu + \underline{\lambda}^* - r\mathbb{1} + \left(\Sigma^z\rho\Sigma'\right)' B\right].\end{aligned}$$

We continue by proving (4.18). As G is a solution to the (dual) HJB PDE (4.7) and $\underline{\pi}^*$, $\underline{\lambda}^*$ attain the optimum in (4.7), we know for every $t \in [0, T]$

$$dG(t, V^{v_0,\underline{\pi}^*}(t), z(t)) = V^{v_0,\underline{\pi}^*}(t) G_v\left(t, V^{v_0,\underline{\pi}^*}(t), z(t)\right)\left(\underline{\pi}^*(t, V^{v_0,\underline{\pi}^*}(t), z(t))\right)' \Sigma dW(t)$$

$$+\left(\nabla_z\left(G\right)\left(t,V^{v_0,\underline{\pi}^*}(t),z(t)\right)\right)'\Sigma^z dW^z(t). \tag{A.51}$$

In particular, noting that $V^{v_0,\underline{\pi}^*}=V_0^{v_0,\underline{\pi}^*}$ and considering the stopping times $\tau_{n,t}^0$ for $n\in\mathbb{N}$ from Condition (UI_0) yields

$$\begin{aligned}
\mathbb{E}\Big[U(V^{v_0,\underline{\pi}^*}(T))\;\Big|\;\mathcal{F}_t\Big] &= \mathbb{E}\Big[G\left(T,V^{v_0,\underline{\pi}^*}(T),z(T)\right)\;\Big|\mathcal{F}_t\Big]\\
&= \mathbb{E}\Big[\lim_{n\to\infty}G\left(\tau_{n,t}^0,V^{v_0,\underline{\pi}^*}(\tau_{n,t}^0),z(\tau_{n,t}^0)\right)\;\Big|\mathcal{F}_t\Big]\\
&\overset{(UI_0)}{=}\lim_{n\to\infty}\mathbb{E}\Big[G\left(\tau_{n,t}^0,V^{v_0,\underline{\pi}^*}(\tau_{n,t}^0),z(\tau_{n,t}^0)\right)\;\Big|\mathcal{F}_t\Big]\\
&=\lim_{n\to\infty}\Big(G\left(t,V^{v_0,\underline{\pi}^*}(t),z(t)\right)\\
&\quad+\underbrace{\mathbb{E}\Big[\int_t^{\tau_{n,t}^0}V^{v_0,\underline{\pi}^*}(u)G_v\left(u,V^{v_0,\underline{\pi}^*}(u),z(u)\right)\left(\underline{\pi}^*(u,V^{v_0,\underline{\pi}^*}(u),z(u))\right)'\Sigma(u,z(u))dW(u)\Big|\mathcal{F}_t\Big]}_{=0,\text{ by choice of }\tau_{n,t}^0}\\
&\quad+\underbrace{\mathbb{E}\Big[\int_t^{\tau_{n,t}^0}\left(\nabla_z\left(G\right)\left(u,V^{v_0,\underline{\pi}^*}(u),z(u)\right)\right)'\Sigma^z(u,z(u))dW^z(u)\Big|\mathcal{F}_t\Big]}_{=0,\text{ by choice of }\tau_{n,t}^0}\Big)\\
&=G\left(t,V^{v_0,\underline{\pi}^*}(t),z(t)\right).
\end{aligned}$$

Conditioning on $V^{v_0,\underline{\pi}^*}(t)=v$, $z(t)=z$ leads to equation (4.18).

It remains to show that $\underline{\pi}^*$ dominates all other $\pi\in\Lambda_K$, i.e., we need to verify inequality (4.19). The proof idea is similar to that of Theorem 4.3 in [13], but adapted for our constrained setting. Let $t\in[0,T]$ and $\pi\in\Lambda_K(t)$ be arbitrary but fixed. Define the process $L=(L(u))_{u\in[t,T]}$ as

$$L(u)=b\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}G(u,V^{v_0,\underline{\pi}^*}(u),z(u)).$$

We proceed by first deriving the SDE of L and then showing that L is a supermartingale. By Itô's product rule we have with $\pi^*(u)=\underline{\pi}^*(u,V^{v_0,\underline{\pi}^*}(u),z(u))$

$$\begin{aligned}
d\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}\right) &= \frac{1}{V^{v_0,\underline{\pi}^*}(u)}dV^{v_0,\pi}(u)+V^{v_0,\pi}(u)d\left(\frac{1}{V^{v_0,\underline{\pi}^*}(u)}\right)+d\langle V^{v_0,\pi}(u),\frac{1}{V^{v_0,\underline{\pi}^*}(u)}\rangle_u\\
&=\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}\right)\Bigg(\left[r+(\mu-r\mathbb{1})'\pi(u)\right]du+\pi(u)'\Sigma dW(u)\\
&\qquad-\left[r+(\mu-r\mathbb{1})'\pi^*(u)-\|\Sigma'\pi^*(u)\|^2\right]du-\pi^*(u)'\Sigma dW(u)-\pi^*(u)'\Sigma\Sigma'\pi(u)du\Bigg)\\
&=\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}\right)\Bigg(\Big[(\mu-r\mathbb{1})'(\pi(u)-\pi^*(u))\underbrace{-\pi^*(u)'\Sigma\Sigma'\pi(u)+\|\Sigma'\pi^*(u)\|^2}_{=-\pi^*(u)'\Sigma\Sigma'(\pi(u)-\pi^*(u))}\Big]du\\
&\qquad+(\pi(u)-\pi^*(u))'\Sigma dW(u)\Bigg)\\
&=\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}\right)(\pi(u)-\pi^*(u))'\Bigg(\left[\mu-r\mathbb{1}-\Sigma\Sigma'\pi^*(u)\right]du+\Sigma dW(u)\Bigg).
\end{aligned}$$

Moreover, due to (A.49) and (A.51),

$$dG(u, V^{v_0,\underline{\pi}^*}(u), z(u)) \overset{(A.51)}{=} G(u, V^{v_0,\underline{\pi}^*}(u), z(u))\Big(b\pi^*(u)'\Sigma dW(u) + B(T-u)'\Sigma^z dW^z(u)\Big)$$

and

$$\begin{aligned} d\Big\langle \frac{V^{v_0,\pi}}{V^{v_0,\underline{\pi}^*}}, G(\cdot, V^{v_0,\underline{\pi}^*}, z) \Big\rangle_u \\ = \frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)} G(u, V^{v_0,\underline{\pi}^*}(u), z(u))(\pi(u) - \pi^*(u))'\Sigma \left(b\Sigma'\pi^*(u) + (\Sigma^z\rho)' B(T-u)\right) du. \end{aligned}$$

In total, this yields with $\lambda^*(u) = \underline{\lambda}^*(u, V^{v_0,\underline{\pi}^*}(u), z(u))$

$$\begin{aligned} dL(u) &= bd\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)} G(u, V^{v_0,\underline{\pi}^*}(u), z(u))\right) \\ &= b\left[\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)} dG(u, V^{v_0,\underline{\pi}^*}(u), z(u)) G(u, V^{v_0,\underline{\pi}^*}(u), z(u)) d\left(\frac{V^{v_0,\pi}(u)}{V^{v_0,\underline{\pi}^*}(u)}\right)\right] \\ &\qquad + bd\Big\langle \frac{V^{v_0,\pi}}{V^{v_0,\underline{\pi}^*}}, G(\cdot, V^{v_0,\underline{\pi}^*}, z)\Big\rangle_u \\ &= L(u)\Big[(\pi(u) - \pi^*(u))'\left[\mu - r\mathbb{1} - \Sigma\Sigma'\pi^*(u)\right] du + \Sigma dW(u) + b\pi^*(u)'\Sigma dW(u) \\ &\qquad + B(T-u)'\Sigma^z dW^z(u) + (\pi(u) - \pi^*(u))'\Sigma\left(b\Sigma'\pi^*(u) + (\Sigma^z\rho)' B(T-u)\right) du\Big] \\ &= L(u)(\pi(u) - \pi^*(u))' \underbrace{\Big[\mu - r\mathbb{1} + \left(\Sigma^z\rho\Sigma'\right)' B(T-u) - (1-b)\Sigma\Sigma'\pi^*(u)\Big]}_{\overset{(4.17)}{=} -\lambda^*(u)} du \\ &\quad + L(u)\Big((\pi(u) - (1-b)\pi^*(u))'\Sigma dW(u) + B(T-u)'\Sigma^z dW^z(u)\Big). \\ &= L(u)\Big((\pi^*(u) - \pi(u))'\lambda^*(u) du + (\pi(u) - (1-b)\pi^*(u))'\Sigma dW(u) + B(T-u)'\Sigma^z dW^z(u)\Big) \end{aligned}$$

However, Corollary 2.1.32 (resp. Theorem 2.1.31) implies

$$\begin{aligned} &\delta_K(\underline{\lambda}^*(u,v,z)) + \underline{\pi}^*(u,v,z)'\underline{\lambda}^*(u,v,z) = 0 \quad \forall (u,v,z) \in [0,T] \times (0,\infty) \times \mathbb{R}^m \\ \Rightarrow\ &\delta_K(\lambda^*(u)) + \lambda^*(u)'\pi^*(u) = 0 \quad \mathcal{L}[t,T] \otimes Q - \text{a.e..} \end{aligned}$$

Moreover, as $\pi \in \Lambda_K(t)$, $\pi(u) \in K$ holds $\mathcal{L}[t,T] \otimes Q -$ a.e. and thus

$$\delta_K(\lambda^*(u)) + \lambda^*(u)'\pi(u) = -\underbrace{\inf_{x \in K}\left(\lambda^*(u)'x\right)}_{\leq \lambda^*(u)'\pi(u)} + \lambda^*(u)'\pi(u) \geq 0 \quad \mathcal{L}[t,T] \otimes Q - \text{a.e..}$$

We finally obtain

$$dL(u) = L(u)\Big(\underbrace{-\left(\delta_K(\lambda^*(u)) + \lambda^*(u)'\pi(u)\right)}_{=(*)} du + (\pi(u) - (1-b)\pi^*(u))'\Sigma dW(u) + B(T-u)'\Sigma^z dW^z(u)\Big)$$

and for any $s \in [t,T]$, $L(s)$ can therefore be expressed as

$$L(s) = L(t)\exp\left(-\int_t^s \delta_K(\lambda^*(u)) + \lambda^*(u)'\pi(u) du\right) M(s),$$

for a supermartingale $M = (M(u))_{u \in [t,T]}$ which satisfies the SDE

$$dM(u) = M(u)\left((\pi(u) - (1-b)\pi^*(u))'\Sigma dW(u) + B(T-u)'\Sigma^z dW^z(u)\right), \quad M(t) = 1.$$

Hence, as $(*) \le 0$, L is a supermartingale, too.

To conclude the proof, recall that U is concave and therefore $U(y) \le U(x) + U'(x)(y-x)$ for all $x, y \in (0,\infty)$. This leads to

$$\begin{aligned}
&\mathbb{E}\left[U(V^{v_0,\pi}(T)) \mid \mathcal{F}_t\right] \le \mathbb{E}\left[U(V^{v_0,\underline{\pi}^*}(T)) \mid \mathcal{F}_t\right] + \mathbb{E}\left[U'(V^{v_0,\underline{\pi}^*}(T))\left(V^{v_0,\pi}(T) - V^{v_0,\underline{\pi}^*}(T)\right) \mid \mathcal{F}_t\right]\\
&\quad = \mathbb{E}\left[U(V^{v_0,\underline{\pi}^*}(T)) \mid \mathcal{F}_t\right] + \underbrace{\mathbb{E}\left[L(T) \mid \mathcal{F}_t\right]}_{\le L(t)} - \underbrace{\mathbb{E}\left[bU(V^{v_0,\underline{\pi}^*}(T)) \mid \mathcal{F}_t\right]}_{=bG(t,V^{v_0,\underline{\pi}^*}(t),z(t))}\\
&\quad \le \mathbb{E}\left[U(V^{v_0,\underline{\pi}^*}(T)) \mid \mathcal{F}_t\right] + L(t) - bG(t, V^{v_0,\underline{\pi}^*}(t), z(t))\\
&\quad = \mathbb{E}\left[U(V^{v_0,\underline{\pi}^*}(T)) \mid \mathcal{F}_t\right] + b\left(G(t, V^{v_0,\pi}(t), z(t))\frac{V^{v_0,\pi}(t)}{V^{v_0,\underline{\pi}^*}(t)} - G(t, V^{v_0,\underline{\pi}^*}(t), z(t))\right).
\end{aligned}$$

Finally, conditioning on $V^{v_0,\pi}(t) = V^{v_0,\underline{\pi}^*}(t) = v$, $z(t) = z$ yields (4.19). □

Proof of Lemma 4.4.2. In $\mathcal{M}_{BS}$, the optimizer $\hat{\lambda}^*$ from (4.14) is given as

$$\begin{aligned}
\hat{\lambda}^*(t,z,B) &= \underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\left\{2(1-b)\delta_K(\lambda) + \Big\|\underbrace{\Sigma(t,z)^{-1}}_{\Sigma(t)^{-1}}\big(\underbrace{\mu(t,z) - r(t,z)\mathbb{1}}_{=\mu(t)-r(t)\mathbb{1}} + \lambda\big) + \underbrace{(\Sigma^z(t,z)\rho(t,z))'}_{=0}B\Big\|^2\right\}\\
&= \underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\left\{2(1-b)\delta_K(\lambda) + \left\|\Sigma(t)^{-1}\left(\mu(t) - r(t)\mathbb{1} + \lambda\right)\right\|^2\right\}\\
&= \lambda^*(t),
\end{aligned}$$

i.e., $\hat{\lambda}^*(t,v,z) = \lambda^*(t)$ is a deterministic function independent of z and B. Moreover, by defining

$$\begin{aligned}
p_0(t,B) &= r(t) + \delta_K\left(\lambda^*(t)\right)\\
q_0(t,B) &= \left\|\Sigma(t)^{-1}\left(\mu(t) - r(t)\mathbb{1} + \lambda^*(t)\right)\right\|^2
\end{aligned}$$

and setting the remaining coefficients k_0, k_1, h_0, h_1, l_0, l_1, p_1, q_1, g_0 and g_1 to zero, Condition (EAS) is satisfied. The corresponding ODEs (4.15) and (4.16) simplify to

$$\begin{aligned}
A_\tau(\tau) &= b\left(r(T-\tau) + \delta_K\left(\lambda^*(T-\tau)\right)\right) + \frac{1}{2}\frac{b}{1-b}\left\|\Sigma(t)^{-1}\left(\mu(T-\tau) - r(T-\tau)\mathbb{1} + \lambda^*(T-\tau)\right)\right\|^2\\
&= br(T-\tau) + \frac{1}{2}\frac{b}{1-b}\inf_{\lambda\in\mathbb{R}^d}\left\{2(1-b)\delta_K(\lambda) + \left\|\Sigma(T-\tau)^{-1}\left(\mu(T-\tau) - r(T-\tau)\mathbb{1} + \lambda\right)\right\|^2\right\}\\
B_\tau(\tau) &= 0.
\end{aligned}$$

Hence, $B \equiv 0$ and A can be obtained through simple integration. In particular, the candidate optimal portfolio (4.17) is given through

$$\underline{\pi}^*(t,v,z) = \frac{1}{1-b}\left(\Sigma(t)\Sigma(t)'\right)^{-1}\left(\mu(t) - r(t)\mathbb{1} + \lambda^*(t)\right).$$

□

Proof of Corollary 4.4.3. We verify Condition (UI_λ) by showing the L^q boundedness of

$$G\left(\tau^0_{n,t}, V_0^{v_0,\underline{\pi}^*}(\tau^0_{n,t}), z(\tau^0_{n,t})\right)$$

in $n \in \mathbb{N}$ for arbitrary $q > 1$.
The market coefficients $\Sigma(t)$, $\mu(t)$ and $r(t)$ are continuous and therefore uniformly bounded in $t \in [0,T]$. In particular, this has the consequence that $\lambda^*(t)$ is uniformly bounded in $t \in [0,T]$, since the quadratic term in (4.14) dominates for large $\|\lambda\|$. Hence, $A(T-t)$ and the candidate optimal portfolio $\pi^*(t) := \underline{\pi}^*(t,v,z)$ are uniformly bounded in $t \in [0,T]$, too. For arbitrary $q > 1$, we can thus find a constant $C_q > 0$ such that for all $t \in [0,T]$

$$\begin{aligned}
\left|G(t, V^{v_0,\underline{\pi}^*}(t), z(t))\right|^q &\overset{\text{T. 4.3.5}}{=} \left|\frac{1}{b}\exp\left(b\ln\left(V^{v_0,\underline{\pi}^*}(t)\right) + A(T-t) + B(T-t)'z(t)\right)\right|^q \\
&= \frac{1}{|b|}\exp\left(bq\int_0^t r(s) + (\mu(s) - r(s)\mathbb{1})'\pi^*(s) - \frac{1}{2}\|\Sigma(s)'\pi^*(s)\|^2 ds + bq\int_0^t \pi^*(s)'\Sigma(s)dW(s) + qA(T-t)\right) \\
&= \frac{1}{|b|}\exp\left(bq\int_0^t r(s) + (\mu(s) - r(s)\mathbb{1})'\pi^*(s) - \frac{1-bq}{2}\|\Sigma(s)'\pi^*(s)\|^2 ds + qA(T-t)\right. \\
&\qquad\qquad \left. - \frac{1}{2}\int_0^t b^2q^2\|\Sigma(s)'\pi^*(s)\|^2 ds + bq\int_0^t \pi^*(s)'\Sigma(s)dW(s)\right) \\
&\le C_q \underbrace{\exp\left(-\frac{1}{2}\int_0^t b^2q^2\|\Sigma(s)'\pi^*(s)\|^2 ds + bq\int_0^t \pi^*(s)'\Sigma(s)dW(s)\right)}_{=:M_t} \\
&= C_q M_t. \qquad\qquad (A.52)
\end{aligned}$$

The process $M = (M_t)_{t\in[0,T]}$ is a non-negative local martingale and thus a supermartingale. Doob's optional sampling theorem ('O.S.') implies

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|G\left(\tau^0_{n,t}, V_0^{v_0,\underline{\pi}^*}(\tau^0_{n,t}), z(\tau^0_{n,t})\right)\right|^q\right] \overset{(A.52)}{\le} C_q \sup_{n\in\mathbb{N}} \mathbb{E}\left[M_{\tau^0_{n,t}}\right] \overset{O.S.}{\le} C_q M_0 = C_q < \infty.$$

Hence,

$$\left(G\left(\tau^0_{n,t}, V_0^{v_0,\underline{\pi}^*}(\tau^0_{n,t}), z(\tau^0_{n,t})\right)\right)_{n\in\mathbb{N}}$$

is bounded in L^q for any $q > 1$ and $t \in [0,T]$ and is thus uniformly integrable for any $t \in [0,T]$ (see Theorem 4.5.9 in [8] with $G(t) = t^q$). Hence, Condition (UI_λ) is satisfied and $\underline{\pi}^*$ is optimal for (**P**) by virtue of Theorem 4.3.7. □

Proof of Lemma 4.4.6. We may rewrite any $\lambda \in \mathbb{R}^d$ as

$$\lambda = \begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_m\end{pmatrix}, \quad \text{with} \quad \lambda_i \in \mathbb{R}^{d_i}, \quad i = 1, ..., m.$$

Hence, we obtain for every $\lambda \in \mathbb{R}^d$

$$\delta_K(\lambda) = -\inf_{x\in K}(x'\lambda) = -\inf_{\substack{x_i\in K_i \\ i=1,\dots,m}}\left(\sum_{i=1}^m x_i'\lambda_i\right) = -\sum_{i=1}^m \inf_{x_i\in K_i}(x_i'\lambda_i) = \sum_{i=1}^m \delta_{K_i}(\lambda_i).$$

Note that we may restrict the minimization (4.14) to $z \in (0,\infty)^m$ because Feller's condition (4.21) ensures that the m-dimensional CIR process has positive components $\mathcal{L}[0,T] \otimes Q$-a.e.. Hence, for any $(t,z,B) \in [0,T] \times (0,\infty)^m \times \mathbb{R}^m$ we have

$$\left[\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\lambda\right]'[\mu(t,z) - r(t,z)\mathbb{1}] = \begin{pmatrix} \frac{1}{z_1}\left(\Sigma_1\Sigma_1'\right)^{-1}\lambda_1 \\ \vdots \\ \frac{1}{z_m}\left(\Sigma_m\Sigma_m'\right)^{-1}\lambda_m \end{pmatrix}' \begin{pmatrix} \eta_1 z_1 \\ \vdots \\ \eta_m z_m \end{pmatrix}$$
$$= \sum_{i=1}^m \lambda_i \left(\Sigma_i\Sigma_i'\right)^{-1}\eta_i,$$

$$\left[\Sigma(t,z)^{-1}\lambda\right]'\rho'\Sigma^z(t,z)B = \begin{pmatrix} \frac{1}{\sqrt{z_1}}\Sigma_1^{-1}\lambda_1 \\ \vdots \\ \frac{1}{\sqrt{z_m}}\Sigma_m^{-1}\lambda_m \end{pmatrix}' \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_m \end{pmatrix} \begin{pmatrix} \sigma_1\sqrt{z_1}B_1 \\ \vdots \\ \sigma_m\sqrt{z_m}B_m \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{\sqrt{z_1}}\Sigma_1^{-1}\lambda_1 \\ \vdots \\ \frac{1}{\sqrt{z_m}}\Sigma_m^{-1}\lambda_m \end{pmatrix}' \begin{pmatrix} \sigma_1\sqrt{z_1}B_1\rho_1 \\ \vdots \\ \sigma_m\sqrt{z_1}B_m\rho_m \end{pmatrix}$$
$$= \sum_{i=1}^m \sigma_i B_i \left(\Sigma_i^{-1}\lambda_i\right)'\rho_i,$$

and

$$\lambda'\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\lambda = \sum_{i=1}^m \left\|\Sigma_i^{-1}\lambda_i\right\|^2 z_i.$$

Hence, the minimizer of (4.14) can be rewritten as

$$\begin{aligned}
\hat{\lambda}^*(t,z,B) &= \underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\Big\{2(1-b)\delta_K(\lambda) + 2\lambda'\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}[\mu(t,z) - r(t,z)\mathbb{1}] \\
&\qquad + 2\lambda'\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}(\Sigma^z(t,z)\rho(t,z)\Sigma(t,z)')'B + \left\|\Sigma(t,z)^{-1}\lambda\right\|^2\Big\} \\
&= \underset{\substack{\lambda=(\lambda_1,\ldots,\lambda_m)' \\ \lambda_i\in\mathbb{R}^{d_i}}}{\operatorname{argmin}}\left\{\sum_{i=1}^m 2(1-b)\delta_{K_i}(\lambda_i) + 2\left(\Sigma_i^{-1}\lambda_i\right)'\left(\Sigma_i^{-1}\eta_i + \sigma_i B_i\rho_i\right) + \left\|\Sigma_i^{-1}\lambda_i\right\|^2 z_i\right\} \\
&= \underset{\substack{\lambda=(\lambda_1,\ldots,\lambda_m)' \\ \lambda_i\in\mathbb{R}^{d_i}}}{\operatorname{argmin}}\left\{\sum_{i=1}^m z_i\left[2(1-b)\delta_{K_i}\left(\frac{\lambda_i}{z_i}\right) + 2\left(\Sigma_i^{-1}\frac{\lambda_i}{z_i}\right)'\left(\Sigma_i^{-1}\eta_i + \sigma_i B_i\rho_i\right) + \left\|\Sigma_i^{-1}\frac{\lambda_i}{z_i}\right\|^2\right]\right\} \\
&= \underset{\substack{\lambda=(\lambda_1,\ldots,\lambda_m)' \\ \lambda_i\in\mathbb{R}^{d_i}}}{\operatorname{argmin}}\left\{\sum_{i=1}^m z_i\left[2(1-b)\delta_{K_i}\left(\frac{\lambda_i}{z_i}\right) + \left\|\Sigma_i^{-1}\left(\eta_i + \frac{\lambda_i}{z_i}\right) + \sigma_i B_i\rho_i\right\|^2\right]\right\}
\end{aligned} \tag{A.53}$$

Using the change of control $\hat{\lambda}_i = \frac{\lambda_i}{z_i}$, we see that $\hat{\lambda}^*(t,z,B) = \lambda^*(t,z,B)$ from the statement of the Lemma. Letting e_i denote the i-th unit vector in $\mathbb{R}^m$, we can express the market coefficients in Condition (EAS) as

$$\mu^z(t,z) = \kappa\odot(\theta - z) = \underbrace{\kappa\odot\theta}_{=:k_0(t)} + \underbrace{\begin{pmatrix} -\kappa_1 & & 0 \\ & \ddots & \\ 0 & & -\kappa_m \end{pmatrix}}_{=:k_1(t)} z$$

$$\Sigma^z(t,z)\Sigma^z(t,z) = \begin{pmatrix} \sigma_1^2 z_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 z_m \end{pmatrix} = \underbrace{\begin{pmatrix} z'(\sigma_1^2 e_1) & & 0 \\ & \ddots & \\ 0 & & z'(\sigma_m^2 e_m) \end{pmatrix}}_{=:h_1(t)[z]}$$

$$\begin{aligned} &\Sigma^z(t,z)\rho(t,z)\left(\Sigma^z(t,z)\rho(t,z)\right)' - \Sigma^z(t,z)\Sigma^z(t,z)' \\ &= \begin{pmatrix} \sigma_1\sqrt{z_1}\rho_1' & & 0 \\ & \ddots & \\ 0 & & \sigma_m\sqrt{z_m}\rho_m' \end{pmatrix} \begin{pmatrix} \sigma_1\sqrt{z_1}\rho_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m\sqrt{z_m}\rho_m \end{pmatrix} - \begin{pmatrix} \sigma_1^2 z_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 z_m \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1^2 z_1\left(\|\rho_1\|^2 - 1\right) & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 z_m\left(\|\rho_m\|^2 - 1\right) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} z'\left(\sigma_1^2\left(\|\rho_1\|^2 - 1\right)e_1\right) & & 0 \\ & \ddots & \\ 0 & & z'\left(\sigma_m^2\left(\|\rho_m\|^2 - 1\right)e_m\right) \end{pmatrix}}_{=:l_1(t)[z]} \end{aligned}$$

$$r(t,z) + \delta_K(\hat{\lambda}^*(t,z,B)) = \underbrace{r}_{=:p_0(t,B)} + \sum_{i=1}^m \underbrace{\delta_{K_i}\left(\lambda_i^*(B_i)\right)}_{=:(p_1(t,B))_i} z_i$$

$$\left\|\Sigma^{-1}(t,z)\left(\mu(t,z) + \hat{\lambda}^*(t,z,B) - r(t,z)\mathbb{1}\right)\right\|^2 = \sum_{i=1}^m \underbrace{\left\|\Sigma_i^{-1}\left(\eta_i + \lambda_i^*(B_i)\right)\right\|^2}_{=:(q_1(t,B))_i} z_i$$

$$\begin{aligned} &\Sigma^z(t,z)\rho(t,z)\Sigma^{-1}(t,z)\left(\mu(t,z) + \hat{\lambda}^*(t,z,B) - r(t,z)\mathbb{1}\right) \\ &= \begin{pmatrix} \sigma_1\sqrt{z_1} & & 0 \\ & \ddots & \\ 0 & & \sigma_m\sqrt{z_m} \end{pmatrix} \begin{pmatrix} \rho_1' & & 0 \\ & \ddots & \\ 0 & & \rho_m' \end{pmatrix} \begin{pmatrix} \left(\Sigma_1\sqrt{z_1}\right)^{-1} & & 0 \\ & \ddots & \\ 0 & & \left(\Sigma_m\sqrt{z_m}\right)^{-1} \end{pmatrix} \begin{pmatrix} \left(\eta_1 + \lambda_1^*(B_1)\right) z_1 \\ \vdots \\ \left(\eta_m + \lambda_m^*(B_m)\right) z_m \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \sigma_1\rho_1'\Sigma_1^{-1}\left(\eta_1 + \lambda_1^*(B_1)\right) & & 0 \\ & \ddots & \\ 0 & & \sigma_m\rho_m'\Sigma_m^{-1}\left(\eta_m + \lambda_m^*(B_m)\right) \end{pmatrix}}_{=:g_1(t,B)} z. \end{aligned}$$

By setting the remaining coefficients h_0, l_0, q_0, and g_0 as zero, Condition (EAS) is satisfied. Moreover, the ODEs (4.15) and (4.16) simplify to

$$\begin{aligned} A_\tau(\tau) &= br + (\kappa \odot \theta)' B(\tau) \\ (B_\tau(\tau))_i &= b\left(p_1(T-\tau, B(\tau))\right)_i + \left(k_1(T-\tau)B(\tau)\right)_i + \frac{1}{2}\left(B(\tau)' h_1[\cdot] B(\tau)\right)_i \\ &\quad + \frac{1}{2}\frac{b}{1-b}\left[q_1(T-\tau, B(\tau)) + 2g_1(T-\tau, B(\tau))B(\tau) + B(\tau)'\left(l_1[\cdot] + h_1[\cdot]\right)B(\tau)\right]_i \\ &= b\delta_{K_i}\left(\lambda_i^*(B_i(\tau))\right) - \kappa_i B_i(\tau) + \frac{1}{2}\sigma_i^2\left(B_i(\tau)\right)^2 \\ &\quad + \frac{1}{2}\frac{b}{1-b}\left[\left\|\Sigma_i^{-1}\left(\eta_i + \lambda_i^*(B_i(\tau))\right)\right\|^2 + 2\sigma_i B_i(\tau)\rho_i'\Sigma_i^{-1}\left(\eta_i + \lambda_i^*(B_i(\tau))\right) + \sigma_i^2\|\rho_i\|^2 B_i(\tau)^2\right] \\ &= -\kappa_i B_i(\tau) + \frac{1}{2}\sigma_i^2\left(B_i(\tau)\right)^2 \\ &\quad + \frac{1}{2}\frac{b}{1-b}\left[2(1-b)\delta_{K_i}\left(\lambda_i^*(B_i(\tau))\right) + \left\|\Sigma_i^{-1}\left(\eta_i + \lambda_i^*(B_i(\tau))\right) + \sigma_i B_i(\tau)\rho_i\right\|^2\right] \end{aligned}$$

$$
\begin{aligned}
&= -\kappa_i B_i(\tau) + \frac{1}{2}\sigma_i^2 \left(B_i(\tau)\right)^2 \\
&\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda_i \in \mathbb{R}^{d_i}} \left\{ 2(1-b)\delta_{K_i}(\lambda_i) + \left\| \Sigma_i^{-1}\left(\eta_i + \lambda_i\right) + \sigma_i B_i(\tau)\rho_i \right\|^2 \right\}.
\end{aligned}
$$

Hence, according to Theorem 4.3.5,

$$
G(t, v, z) = \frac{1}{b} v^b \exp(A(T-t) + B(T-t)'z)
$$

is a solution to the dual HJBI PDE (4.7) and the corresponding candidate optimal portfolio (4.17) is given as in the statement of the lemma.
Furthermore,

$$
\begin{aligned}
&\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1} \left(\Sigma^z(t,z)\rho(t,z)\Sigma(t,z)'\right)' B(T-t) \\
&= \left(\Sigma(t,z)'\right)^{-1} \rho(t,z)'\Sigma^z(t,z)'B(T-t) \\
&= \begin{pmatrix} \left(\Sigma_1'\sqrt{z_1}\right)^{-1} & & 0 \\ & \ddots & \\ 0 & & \left(\Sigma_m'\sqrt{z_m}\right)^{-1} \end{pmatrix} \begin{pmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_m \end{pmatrix} \begin{pmatrix} \sigma_1\sqrt{z_1}B_1(T-t) \\ \vdots \\ \sigma_m\sqrt{z_m}B_m(T-t) \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1 B_1(T-t)\left(\Sigma_1^{-1}\right)'\rho_1 \\ \vdots \\ \sigma_m B_m(T-t)\left(\Sigma_m^{-1}\right)'\rho_m \end{pmatrix}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\underline{\pi}^*(t,v,z) &= \frac{1}{1-b}\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\left[\mu(t,z) + \underline{\lambda}^*(t,v,z) - r(t,z)\mathbb{1}\right] \\
&\quad + \frac{1}{1-b}\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\left(\Sigma^z(t,z)\rho(t,z)\Sigma(t,z)'\right)' B(T-t) \\
&= \frac{1}{1-b}\begin{pmatrix} (\Sigma_1\Sigma_1'z_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & (\Sigma_m\Sigma_m'z_m)^{-1} \end{pmatrix} \begin{pmatrix} \left(\eta_1 + \lambda_1^*(B_1(T-t))\right)z_1 \\ \vdots \\ \left(\eta_m + \lambda_m^*(B_m(T-t))\right)z_m \end{pmatrix} \\
&\quad + \frac{1}{1-b}\begin{pmatrix} \sigma_1 B_1(T-t)\left(\Sigma_1^{-1}\right)'\rho_1 \\ \vdots \\ \sigma_m B_m(T-t)\left(\Sigma_m^{-1}\right)'\rho_m \end{pmatrix} \\
&= \frac{1}{1-b}\begin{pmatrix} (\Sigma_1\Sigma_1')^{-1}\Big(\eta_1 + \lambda_1^*(B_1(T-t)) + \sigma_1 B_1(T-t)\Sigma_1\rho_1\Big) \\ \vdots \\ (\Sigma_m\Sigma_m')^{-1}\Big(\eta_m + \lambda_m^*(B_m(T-t)) + \sigma_m B_m(T-t)\Sigma_m\rho_m\Big) \end{pmatrix}.
\end{aligned}
$$

□

Proof of Lemma 4.4.8. In $\mathcal{M}_{OU}$, the minimizer $\hat{\lambda}^*$ from (4.14) is given as

$$
\hat{\lambda}^*(t,z,B) = \underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}}\left\{ 2(1-b)\delta_K(\lambda) + \Big\| \underbrace{\Sigma(t,z)^{-1}\big(\mu(t,z) - r(t,z)\mathbb{1} + \lambda\big)}_{\eta + \left(b(t;\hat{T})'\sigma\right)^{-1}\lambda} + \underbrace{\left(\Sigma^z(t,z)\rho(t,z)\right)'}_{=\sigma'} B \Big\|^2 \right\}
$$

$$= \underset{\lambda\in\mathbb{R}^d}{\operatorname{argmin}} \left\{ 2(1-b)\delta_K(\lambda) + \left\| \eta + \sigma' B + \left(b(t;\hat{T})'\sigma \right)^{-1} \lambda \right\|^2 \right\}$$
$$= \lambda^*(t, B),$$

i.e., $\hat{\lambda}^*(t, v, z) = \lambda^*(t, B)$ is a deterministic function independent of z, but dependent on B. Considering the market coefficients in Condition (EAS), we get

$$\mu^z(t,z) = \kappa \odot (\theta - z) = \underbrace{\kappa \odot \theta}_{=:k_0(t)} + \underbrace{\begin{pmatrix} -\kappa_1 & & 0 \\ & \ddots & \\ 0 & & -\kappa_m \end{pmatrix}}_{=:k_1(t)} z$$

$$\Sigma^z(t,z)\Sigma^z(t,z) = \underbrace{\sigma\sigma'}_{=:h_0(t)}$$

$$\Sigma^z(t,z) \underbrace{\rho(t,z)}_{=I_m} \left(\Sigma^z(t,z) \underbrace{\rho(t,z)}_{=I_m} \right)' - \Sigma^z(t,z)\Sigma^z(t,z)' = 0$$

$$r(t,z) + \delta_K(\hat{\lambda}^*(t,z,B)) = \underbrace{w_0 + \delta_K(\lambda^*(t,B))}_{=:p_0(t,B)} + \underbrace{w_1'}_{=:p_1(t,B)'} z$$

$$\left\| \Sigma^{-1}(t,z) \left(\mu(t,z) + \hat{\lambda}^*(t,z,B) - r(t,z)\mathbb{1} \right) \right\|^2 = \underbrace{\| \eta + \left(b(t;\hat{T})'\sigma \right)^{-1} \lambda^*(t,B) \|^2}_{=:q_0(t,B)}$$

$$\underbrace{\Sigma^z(t,z)}_{=\sigma} \underbrace{\rho(t,z)}_{=I_m} \underbrace{\Sigma^{-1}(t,z) \left(\mu(t,z) + \hat{\lambda}^*(t,z,B) - r(t,z)\mathbb{1} \right)}_{=\eta + \left(b(t;\hat{T})'\sigma \right)^{-1} \lambda^*(t,B)} = \underbrace{\sigma\eta + \left(b(t;\hat{T})' \right)^{-1} \lambda^*(t,B)}_{=:g_0(t,B)}.$$

By setting the remaining coefficients h_1, l_0, l_1, q_1 and g_1 as zero, Condition (EAS) is satisfied. Moreover, the ODEs (4.15) and (4.16) simplify to

$$\begin{aligned}
A_\tau(\tau) &= b\left(w_0 + \delta_K(\lambda^*(T-\tau, B(\tau)))\right) + (\kappa \odot \theta)' B(\tau) + \frac{1}{2}\|\sigma' B(\tau)\|^2 \\
&\quad + \frac{1}{2}\frac{b}{1-b} \Bigg(\left\| \eta + \left(b(T-\tau;\hat{T})'\sigma \right)^{-1} \lambda^*(T-\tau, B(\tau)) \right\|^2 \\
&\qquad + 2\left(\sigma\eta + \left(b(t;\hat{T})' \right)^{-1} \lambda^*(t, B(\tau)) \right)' B(\tau) + \left\| \sigma' B(\tau) \right\|^2 \Bigg) \\
&= bw_0 + (\kappa \odot \theta)' B(\tau) + \frac{1}{2}\|\sigma' B(\tau)\|^2 \\
&\quad + \frac{1}{2}\frac{b}{1-b} \Bigg(2(1-b)\delta_K(\lambda^*(T-\tau, B(\tau))) \\
&\qquad + \left\| \eta + \sigma' B(\tau) + \left(b(T-\tau;\hat{T})' \right)^{-1} \lambda^*(T-\tau, B(\tau)) \right\|^2 \Bigg) \\
&= bw_0 + (\kappa \odot \theta)' B(\tau) + \frac{1}{2}\|\sigma' B(\tau)\|^2 \\
&\quad + \frac{1}{2}\frac{b}{1-b} \inf_{\lambda\in\mathbb{R}^d} \left(2(1-b)\delta_K(\lambda) + \left\| \eta + \sigma' B(\tau) + \left(b(T-\tau;\hat{T})' \right)^{-1} \lambda \right\|^2 \right)
\end{aligned}$$

and

$$B_\tau(\tau) = bw_1 - \kappa \odot B(\tau).$$

Hence, according to Theorem 4.3.5,

$$G(t,v,z) = \frac{1}{b} v^b \exp(A(T-t) + B(T-t)'z)$$

is a solution to the dual HJBI PDE (4.7). The corresponding candidate optimal portfolio (4.17) is given through

$$\begin{aligned}\underline{\pi}^*(t,v,z) &= \frac{1}{1-b}\left(\Sigma(t,z)\Sigma(t,z)'\right)^{-1}\Big[\mu(t,z) + \underline{\lambda}^*(t,v,z) - r(t,z)\mathbb{1} \\ &\qquad + \left(\Sigma^z(t,z)\rho(t,z)\Sigma(t,z)'\right)' B(T-t)\Big] \\ &= \frac{1}{1-b}\left(\sigma' b(t;\hat{T})\right)^{-1}\left(\eta + \left(b(t;\hat{T})'\sigma\right)^{-1}\lambda^*(t, B(T-t)) + \sigma' B(T-t)\right).\end{aligned}$$

□

Proof of Corollary 4.4.10. We again verify Condition (UI$_\lambda$) by showing the L^q boundedness of $G\left(\tau^0_{n,t}, V_0^{v_0,\underline{\pi}^*}(\tau^0_{n,t}), z(\tau^0_{n,t})\right)$ in $n \in \mathbb{N}$ for arbitrary $q > 1$.
As per Remark 4.4.9, there exists a closed-form expression for B which is continuously differentiable. Moreover, the matrix $b(t;\hat{T})$ is continuously differentiable in t and non-singular for all $t \in [0,T]$ and therefore uniformly bounded in $t \in [0,T]$. Following the same arguments as in the proof of Corollary 4.4.3, this has the consequence that the minimizer $\lambda^*(t, B(T-t))$, $A(T-t)$ and the candidate optimal portfolio $\pi^*(t) := \underline{\pi}^*(t,v,z)$ are uniformly bounded in $t \in [0,T]$.

For arbitrary $q > 1$, we can thus find a constant $C_q > 0$ such that for all $t \in [0,T]$

$$\begin{aligned}\left|G(t, V^{v_0,\underline{\pi}^*}(t), z(t))\right|^q &= \frac{1}{|b|}\exp\Bigg(bq\int_0^t w_0 + w_1' z(s) + \eta'\sigma' b(s;\hat{T})\pi^*(s) - \frac{1}{2}\|\sigma' b(s;\hat{T})\pi^*(s)\|^2 ds \\ &\qquad + bq\int_0^t \pi^*(s)' b(s;\hat{T})'\sigma dW(s) + qA(T-t) + qB(T-t)'z(t)\Bigg) \\ &\le C_q \exp\Bigg(\underbrace{bq\int_0^t w_1' z(s) ds + bq\int_0^t \pi^*(s)' b(s;\hat{T})'\sigma dW(s) + qB(T-t)'z(t)}_{=:X_t}\Bigg) \\ &= C_q \exp(X_t).\end{aligned} \tag{A.54}$$

Since, B is continuously differentiable, we can use Itô's product rule to rewrite

$$\begin{aligned}B(T-t)'z(t) &= B(T)'z_0 + \int_0^t B(T-s)'dz(s) + \int_0^t z(s)'d\big(B(T-s)\big) + \underbrace{\langle z, B(T-\cdot)\rangle_t}_{=0} \\ &= B(T)'z_0 + \int_0^t B(T-s)'\kappa \odot [\theta - z(s)] - z(s)'B_\tau(T-s)ds + \int_0^t B(T-s)'\sigma dW^z(s)\end{aligned} \tag{A.55}$$

Due to $\rho(t,z) = I_m$, we know that $W^z(t) = W(t)$ holds $\mathcal{L}[0,T] \otimes Q$-a.e.. Hence, using (A.55) and disregarding terms of finite variation, the quadratic variation of X can be computed as

$$\begin{aligned}\langle X\rangle_t &= \left\langle bq\int_0^t w_1' z(s)ds + bq\int_0^\cdot \pi^*(s)'b(s;\hat{T})'\sigma \underbrace{dW(s)}_{=dW^z(s)} + qB(T-\cdot)'z(\cdot)\right\rangle_t \\ &= \left\langle bq\int_0^\cdot \pi^*(s)'b(s;\hat{T})'\sigma + \frac{1}{b}B(T-s)'\sigma dW^z(s)\right\rangle_t \\ &= b^2q^2\int_0^t \left\|\sigma'\left(b(s;\hat{T})\pi^*(s) + \frac{1}{b}B(T-s)\right)\right\|^2 ds.\end{aligned}$$

Since all involved functions are bounded and deterministic, $\langle X\rangle_t \le \langle X\rangle_T < \infty$ yields a deterministic upper bound on $\langle X\rangle_t$ for all $t \in [0,T]$. Therefore, we can continue equation (A.54) to obtain for all $t \in [0,T]$

$$\left|G(t, V^{v_0,\underline{\pi}^*}(t), z(t))\right|^q \overset{(A.54)}{\le} C_q \exp(X_t) \le \underbrace{C_q \exp\left(\frac{1}{2}\langle X\rangle_T\right)}_{=:\tilde{C}_q} \underbrace{\exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right)}_{=:M_t} = \tilde{C}_q M_t \tag{A.56}$$

The process $M = (M_t)_{t\in[0,T]}$ is a non-negative local martingale and thus a supermartingale. Doob's optional sampling theorem ('O.S.') implies

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left|G\left(\tau_{n,t}^0, V_0^{v_0,\underline{\pi}^*}(\tau_{n,t}^0), z(\tau_{n,t}^0)\right)\right|^q\right] \overset{(A.56)}{\le} \tilde{C}_q \sup_{n\in\mathbb{N}} \mathbb{E}\left[M_{\tau_{n,t}^0}\right] \overset{O.S.}{\le} \tilde{C}_q M_0 = \tilde{C}_q < \infty.$$

Hence,

$$\left(G\left(\tau_{n,t}^0, V_0^{v_0,\underline{\pi}^*}(\tau_{n,t}^0), z(\tau_{n,t}^0)\right)\right)_{n\in\mathbb{N}}$$

is bounded in L^q for any $q > 1$ and $t \in [0,T]$ and is thus uniformly integrable for any $t \in [0,T]$ (see Theorem 4.5.9 in [8] with $G(t) = t^q$). Hence, Condition (UI$_\lambda$) is satisfied and $\underline{\pi}^*$ is optimal for (**P**) by virtue of Theorem 4.3.7. □

A.4 Proofs Chapter 5

Proof of Lemma 5.2.2. Follows immediately from Lemma 4.4.6 with $d = m = 1$ and $\Sigma_1 = 1$. □

Proof of Lemma 5.2.3. Noting that

$$G_v(t,v,z) = \frac{b}{v}G(t,v,z) \quad \text{and} \quad G_z(t,v,z) = B(T-t)G(t,v,z),$$

this result follows immediately from Theorem 4.3.7 and Lemma 4.4.6 with $d = m = 1$ and $\Sigma_1 = 1$. □

Proof of Lemma 5.2.4. The main objective of this proof is to determine the minimizing argument $\lambda^*(B)$ for the minimization problem

$$\inf_{\lambda\in\mathbb{R}} \underbrace{\left(2(1-b)\delta_K(\lambda) + (\eta+\lambda+\sigma\rho B)^2\right)}_{=:D(\lambda,B)} = \inf_{\lambda\in\mathbb{R}} D(\lambda,B)$$

for any fixed value $B \in \mathbb{R}$. The remaining statement follows immediately after substituting the obtained minimizer $\lambda^*(B) = \lambda^*(B(\tau))$ in (5.5).

Given any constraints $K = [\alpha,\beta]$ with $-\infty \le \alpha \le \beta \le \infty$ (see (5.2)), we have for any $\lambda \in \mathbb{R}$

$$\delta_K(\lambda) = -\inf_{\alpha\le x\le\beta}(x\lambda) = -\alpha\lambda \mathbb{1}_{\{\lambda>0\}} - \beta\lambda\mathbb{1}_{\{\lambda<0\}}.$$

Thus, taking the derivative of $D(\cdot,B)$ on $(-\infty,0)$ and $(0,\infty)$ yields

$$\frac{\partial}{\partial\lambda}D(\lambda,B) = \begin{cases} -2(1-b)\alpha + 2(\eta+\lambda+\sigma\rho B), & \lambda>0 \\ -2(1-b)\beta + 2(\eta+\lambda+\sigma\rho B), & \lambda<0. \end{cases}$$

Then, by the first-order optimality condition and the fact that $D(\cdot,B)$ is quadratic on $[0,\infty)$,

$$\lambda^-(B) = \big((1-b)\alpha - (\eta+\sigma\rho B)\big)\mathbb{1}_{\{(1-b)\alpha-(\eta+\sigma\rho B)>0\}} = \big((1-b)\alpha - (\eta+\sigma\rho B)\big)\mathbb{1}_{\{\rho B<B_-\}}$$

minimises $D(\cdot,B)$ on $[0,\infty)$, and by the same argument

$$\lambda^+(B) = \big((1-b)\beta - (\eta+\sigma\rho B)\big)\mathbb{1}_{\{(1-b)\beta-(\eta+\sigma\rho B)<0\}} = \big((1-b)\beta - (\eta+\sigma\rho B)\big)\mathbb{1}_{\{\rho B>B_+\}}$$

minimises $D(\cdot,B)$ on $(-\infty,0]$. Therefore, noting that by construction

$$\begin{aligned} \max\big(D(\lambda^-(B),B),\ D(\lambda^+(B),B)\big) &= \max\Big(\underbrace{\min_{\lambda\in(-\infty,0]} D(\lambda,B)}_{\le D(0,B)},\ \underbrace{\min_{\lambda\in[0,\infty)} D(\lambda,B)}_{\le D(0,B)}\Big) \\ &\le D(0,B) = (\eta+\sigma\rho B)^2 \end{aligned} \tag{A.57}$$

and $B_- \le B_+$, we finally derive

$$\begin{aligned} &\inf\nolimits_{\lambda\in\mathbb{R}} D(\lambda,B) \\ &\quad= \min\big(D(\lambda^-(B),B),\ D(\lambda^+(B),B)\big) \\ &\quad= \min\Big(\big[2(1-b)(-\alpha)\big((1-b)\alpha-\eta-\sigma\rho B\big) + \big((1-b)\alpha\big)^2\big]\mathbb{1}_{\{\rho B<B_-\}} + \big[\eta+\sigma\rho B\big]^2\mathbb{1}_{\{B_-\le\rho B\}}, \\ &\qquad\qquad \big[2(1-b)(-\beta)\big((1-b)\beta-\eta-\sigma\rho B\big) + \big((1-b)\beta\big)^2\big]\mathbb{1}_{\{B_+<\rho B\}} + \big[\eta+\sigma\rho B\big]^2\mathbb{1}_{\{\rho B\le B_+\}}\Big) \\ &\quad\overset{(A.57)}{=} \big[2(1-b)(-\alpha)\big((1-b)\alpha-\eta-\sigma\rho B\big) + \big((1-b)\alpha\big)^2\big]\mathbb{1}_{\{B_-\le\rho B\}} + \big[\eta+\sigma\rho B\big]^2\mathbb{1}_{\{B_-\le\rho B\le B_+\}} \\ &\qquad\quad + \big[2(1-b)(-\beta)\big((1-b)\beta-\eta-\sigma\rho B\big) + \big((1-b)\beta\big)^2\big]\mathbb{1}_{\{B_+<\rho B\}} \\ &\quad= D(\lambda^-(B),B)\mathbb{1}_{\{B_-\le\rho B\}} + D(0,B)\mathbb{1}_{\{B_-\le\rho B\le B_+\}} + D(\lambda^+(B),B)\mathbb{1}_{\{B_+<\rho B\}}. \end{aligned}$$

Thus, the minimizer $\lambda^*(B)$ is given as

$$\begin{aligned} \lambda^*(B) &= \lambda^-(B)\mathbb{1}_{\{\rho B<B_-\}} + \lambda^+(B)\mathbb{1}_{\{B_+<\rho B\}} \\ &= \big((1-b)\alpha - (\eta+\sigma\rho B)\big)\mathbb{1}_{\{\rho B<B_-\}} + \big((1-b)\beta - (\eta+\sigma\rho B)\big)\lambda^+(B)\mathbb{1}_{\{B_+<\rho B\}}. \end{aligned}$$

Substituting $\lambda^*(B(\tau))$ in (5.5) and factoring $B(\tau)$ and $(B(\tau))^2$ concludes the proof. □

Proof of Theorem 5.2.8. B is specifically constructed in such a way that it is continuous and satisfies the corresponding Riccati ODE of (5.9), whenever $\rho B(\tau)$ is within each of the zones Z_-, Z_0, or Z_+. Moreover, as the right hand side of the ODE (5.5) is continuous, so are the derivatives of the constructed B. Thus, B is a solution to (5.9) and thereby (5.5). □

Proof of Lemma 5.2.10. By rewriting π^* in terms of λ^* and B, we see that

$$\frac{b\rho}{\sigma}\pi^*(t) + B(T-t) = \begin{cases} \frac{b\rho}{\sigma}\alpha + B(T-t), & \text{if } \rho B(T-t) < B_- \\ \frac{b}{1-b}\frac{\rho}{\sigma}\eta + \left(1 + \frac{b}{1-b}\rho^2\right) B(T-t), & \text{if } B_- \leq \rho B(T-t) \leq B_+ \\ \frac{b\rho}{\sigma}\beta + B(T-t), & \text{if } B_+ < \rho B(T-t) \end{cases}$$

is non-decreasing in $B(T-t)$. Hence, as B is itself monotonous (see Remark 5.2.5), we have

$$\sup_{t\in[0,T]} \left(\frac{b\rho}{\sigma}\pi^*(t) + B(T-t)\right) = \max\left(\frac{b\rho}{\sigma}\pi^*(T) + B(0), \frac{b\rho}{\sigma}\pi^*(0) + B(T)\right). \tag{A.58}$$

Hence, when first assuming that the maximum in (A.58) is attained at $t = T$, we obtain

$$\frac{b\rho}{\sigma}\underbrace{\pi^*(T)}_{\in[\alpha,\beta]} + \underbrace{B(0)}_{=0} \leq \max\left(\frac{b\rho}{\sigma}\alpha, \frac{b\rho}{\sigma}\beta\right) \overset{(5.11)}{\leq} \frac{\kappa}{\sigma^2}.$$

On the other hand, if we assume that the maximum in (A.58) is attained at $t = 0$, we need to compare the different terminal values $B(T)$ based on the coefficients of the corresponding Riccati ODE at T. Due to the monotonicity of B and the assumption that the maximum in (A.58) is attained at $t = 0$, it suffices if we can find a bound for $T \to \infty$.

- Case $\rho B(T) \in Z_+$:
  Then, $\pi^*(0) = \beta$ and $B(T)$ is given as the solution (2.23) to the Riccati ODE (2.22) with coefficients r_0^+, r_1^+, and r_2^+. In particular, we have

$$\begin{aligned} \frac{b\rho}{\sigma}\underbrace{\pi^*(0)}_{=\beta} + B(T) &\leq \frac{b\rho}{\sigma}\beta + \lim_{T\to\infty} B(T) \\ &= \frac{b\rho}{\sigma}\beta + \lim_{T\to\infty}\left(\frac{2r_2^+ r_3^+ B_0 + \left(e^{r_3^+ T} - 1\right)(r_1^+ + r_3^+)\left(r_1^+ + r_2^+ B_0 - r_3^+\right)}{2r_2^+ r_3^+ - r_2^+\left(e^{r_3^+ T} - 1\right)\left(r_1^+ + r_2^+ B_0 - r_3^+\right)}\right) \\ &= \frac{b\rho}{\sigma}\beta - \frac{r_1^+ + r_3^+}{r_2^+} = \frac{b\rho}{\sigma}\beta - \frac{b\sigma\rho\beta - \kappa}{\sigma^2} - \frac{r_3^+}{\sigma^2} = \frac{\kappa}{\sigma^2} - \underbrace{\frac{r_3^+}{\sigma^2}}_{\geq 0} \leq \frac{\kappa}{\sigma^2}. \end{aligned}$$

- Case $\rho B(T) \in Z_-$:
  Analogous arguments as in the case $\rho B(T) \in Z_+$ yield that

$$\frac{b\rho}{\sigma}\pi^*(0) + B(T) \leq \frac{\kappa}{\sigma^2}.$$

- Case $\rho B(T) \in Z_0$: Then, $\pi^*(0) = \frac{1}{1-b}(\eta + \sigma\rho B(T))$ and $B(T)$ is given as the solution (2.23) to Riccati ODE (2.22) with coefficients r_0, r_1, and r_2. In particular, we have

$$\frac{b\rho}{\sigma}\pi^*(0) + B(T) = \frac{\rho}{\sigma}\frac{b}{1-b}(\eta + \sigma\rho B(T)) + B(T) = \frac{\rho}{\sigma}\frac{b}{1-b}\eta + \left(1 + \frac{b}{1-b}\rho^2\right) B(T)$$

$$\leq \frac{\rho}{\sigma}\frac{b}{1-b}\eta + \left(1+\frac{b}{1-b}\rho^2\right)\underbrace{\lim_{T\to\infty} B(T)}_{=-\frac{r_1+r_3}{r_2}} = \frac{\rho}{\sigma}\frac{b}{1-b}\eta - \left(1+\frac{b}{1-b}\rho^2\right)\frac{r_1+r_3}{r_2}$$

$$= \frac{\rho}{\sigma}\frac{b}{1-b}\eta - \left(1+\frac{b}{1-b}\rho^2\right)\frac{r_1+r_3}{\sigma^2\left(1+\frac{b}{1-b}\rho^2\right)} = \frac{\rho}{\sigma}\frac{b}{1-b}\eta - \frac{r_1+r_3}{\sigma^2}$$

$$= \frac{\rho}{\sigma}\frac{b}{1-b}\eta - \frac{1}{\sigma^2}\left(\frac{b}{1-b}\eta\sigma\rho - \kappa\right) - \frac{r_3}{\sigma^2} = \frac{\kappa}{\sigma^2} - \underbrace{\frac{r_3}{\sigma^2}}_{\geq 0} \leq \frac{\kappa}{\sigma^2}.$$

□

Proof of Lemma 5.2.11. Note that we may express ODE (5.5) in terms of λ^* and π^* as

$$B'(\tau) = -\kappa B(\tau) + \frac{1}{2}\sigma^2 B(\tau)^2 + \frac{1}{2}\frac{b}{1-b}\left(2(1-b)\delta_K\left(\lambda^*\left(B(\tau)\right)\right) + (1-b)^2\left(\pi^*(T-\tau)\right)^2\right).$$

We distinguish between three cases, depending on whether the allocation constraint is active or not.

- Case $\pi^*(t) \in (\alpha, \beta)$:
  Then, $\lambda^*\left(B(T-t)\right) = (\lambda^*)'\left(B(T-t)\right) = \delta_K\left(\lambda^*\left(B(T-t)\right)\right) = 0$. Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-t)\right) + \sigma\rho B(T-t)\right)^2 - \frac{1}{2}b^2\rho^2\left(\pi^*(t)\right)^2 \\
&\quad + b\frac{\rho\kappa}{\sigma}\pi^*(t) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-t)\right) + \sigma\rho\right]B'(T-t) \\
=&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\sigma\rho B(T-t)\right)^2 - \frac{1}{2}b^2\rho^2\left(\pi^*(t)\right)^2 \\
&\quad + b\frac{\rho\kappa}{\sigma}\pi^*(t) + \frac{b}{1-b}\rho^2\left[-\kappa B(T-t) + \frac{1}{2}\sigma^2\left(B(T-t)\right)^2 + \frac{1}{2}b(1-b)\left(\pi^*(t)\right)^2\right] \\
=&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\sigma\rho B(T-t)\right)^2 + \frac{b}{1-b}\frac{\rho\kappa}{\sigma}\left(\eta + \sigma\rho B(T-t)\right) \\
&\quad + \rho^2\frac{b}{1-b}\left[-\kappa B(T-t) + \frac{1}{2}\sigma^2\left(B(T-t)\right)^2\right] \\
=&\frac{b}{1-b}\eta\left(\frac{\eta}{2} + \frac{\rho\kappa}{\sigma}\right)
\end{aligned}$$

- Case $\pi^*(t) = \alpha$:
  Then, $\lambda^*\left(B(T-t)\right) = (1-b)\alpha - \eta - \sigma\rho B(T-t)$ and

$$\underbrace{\left[(\lambda^*)'\left(B(T-t)\right) + \sigma\rho\right]}_{=0} B'(T-t) = 0.$$

  Hence,

$$\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\overbrace{\left(\lambda^*\left(B(T-t)\right) + \sigma\rho B(T-t)\right)^2}^{=((1-b)\alpha-\eta)^2} - \frac{1}{2}b^2\rho^2\underbrace{\left(\pi^*(t)\right)^2}_{=\alpha^2}$$

$$
\begin{aligned}
&+ b\frac{\rho\kappa}{\sigma}\underbrace{\pi^*(t)}_{=\alpha} + \frac{b}{1-b}\frac{\rho}{\sigma}\underbrace{\left[(\lambda^*)'\left(B(T-t)\right)+\sigma\rho\right]B'(T-t)}_{=0}\\
=&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left((1-b)\alpha-\eta\right)^2 - \frac{1}{2}b^2\rho^2\alpha^2 + b\frac{\rho\kappa}{\sigma}\alpha\\
=&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left((1-b)^2\alpha^2-2(1-b)\alpha\eta+\eta^2\right) - \frac{1}{2}b^2\rho^2\alpha^2 + b\frac{\rho\kappa}{\sigma}\alpha\\
=&-\frac{1}{2}\frac{b}{1-b}\left((1-b)^2\alpha^2-2(1-b)\alpha\eta\right) - \frac{1}{2}b^2\rho^2\alpha^2 + b\frac{\rho\kappa}{\sigma}\alpha\\
=&-\frac{1}{2}b(1-b)\alpha^2 + b\alpha\eta - \frac{1}{2}b^2\rho^2\alpha^2 + b\frac{\rho\kappa}{\sigma}\alpha\\
=&b\alpha\left[\eta - \frac{1}{2}\alpha + \frac{\rho\kappa}{\sigma} + \frac{1}{2}\alpha b(1-\rho^2)\right].
\end{aligned}
$$

- Case $\pi^*(t) = \beta$:
  Following the same arguments as in the case $\pi^*(t) = \alpha$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-t)\right)+\sigma\rho B(T-t)\right)^2 - \frac{1}{2}b^2\rho^2\left(\pi^*(t)\right)^2\\
&\quad + b\frac{\rho\kappa}{\sigma}\pi^*(t) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-t)\right)+\sigma\rho\right]B'(T-t)\\
=&b\beta\left[\eta - \frac{1}{2}\beta + \frac{\rho\kappa}{\sigma} + \frac{1}{2}\beta b(1-\rho^2)\right].
\end{aligned}
$$

Combining all three cases with Assumption 5.2.7 finally yields the claim. □

Proof of Theorem 5.2.12. Let B be as in Theorem 5.2.8,

$$A(\tau) = rb\tau + \kappa\theta\int_0^\tau B(s)ds.$$

and define $G : [0,T] \times (0,\infty)^2 \to \mathbb{R}$ as $G(t,v,z) = \frac{1}{b}v^b\exp(A(T-t)+B(T-t)z)$. Then, A and B are solutions to the ODEs (5.4) and (5.5) and thus continuous and bounded on $[0,T]$. Moreover, according to Lemma 5.2.2, G is a solution to the HJB PDE (5.3). Hence, as per Lemma 5.2.3, it only remains to verify that the sequence

$$\left(G(\tau^0_{n,t}, V^{v_0,\pi^*}(\tau^0_{n,t}), z(\tau^0_{n,t}))\right)_{n\in\mathbb{N}}$$

is uniformly integrable for all $t \in [0,T]$. Let $\mathcal{T}$ denote the set of all $\mathbb{F}$ stopping times taking values in $[0,T]$. We verify the stronger statement that

$$\left(G(\tau, V^{v_0,\pi^*}(\tau), z(\tau))\right)_{\tau\in\mathcal{T}}$$

is bounded in L^q for some $q > 1$ (see e.g. Theorem 4.6.2 in [27]). The argument is an adaptation of the proof of Theorem 5.3 in [56]. Consider arbitrary $t \in [0,T]$ and $q = 1+\epsilon$ with $\epsilon > 0$ and define

$$D(t) = bq\sqrt{1-\rho^2}\int_0^t \pi^*(s)\sqrt{z(s)}d\hat{W}(s).$$

Then, by defining the deterministic, continuous function

$$det_1(t) = q\left(-\ln(|b|) + b\ln(v_0) + brt + A(T-t)\right),$$

we obtain

$$
\begin{aligned}
&q \ln\left(\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|\right)\\
&\quad = q\left(-\ln(|b|) + b\ln\left(V^{v_0,\pi^*}(t)\right) + A(T-t) + B(T-t)z(t)\right)\\
&\quad = q\Bigg(-\ln(|b|) + b\ln(v_0) + b\int_0^t \left(r + \eta\pi^*(s)z(s) - \frac{1}{2}\left(\pi^*(s)\right)^2 z(s)\right)ds\\
&\qquad\quad + b\int_0^t \pi^*(s)\sqrt{z(s)}\underbrace{dW(s)}_{=\rho dW^z(s)+\sqrt{1-\rho^2}d\hat{W}(s)} + A(T-t) + B(T-t)z(t)\Bigg)\\
&\quad = det_1(t) + D(t) - \frac{1}{2}\langle D\rangle_t + \frac{1}{2}q^2b^2(1-\rho^2)\int_0^t \left(\pi^*(s)\right)^2 z(s)ds + qB(T-t)z(t)\\
&\qquad + qb\rho\int_0^t \pi^*(s)\sqrt{z(s)}dW^z(s) + qb\int_0^t \left(\pi^*(s)\eta - \frac{1}{2}\left(\pi^*(s)\right)^2\right)z(s)ds\\
&\quad = det_1(t) + D(t) - \frac{1}{2}\langle D\rangle_t + qB(T-t)z(t) + qb\rho\int_0^t \pi^*(s)\sqrt{z(s)}dW^z(s)\\
&\qquad + q\int_0^t z(s)\left\{b\left(\pi^*(s)\eta - \frac{1}{2}\left(\pi^*(s)\right)^2\right) + \frac{1}{2}qb^2\left(1-\rho^2\right)\left(\pi^*(s)\right)^2\right\}ds\\
&\quad \overset{q=1+\epsilon}{=} det_1(t) + D(t) - \frac{1}{2}\langle D\rangle_t + qB(T-t)z(t) + qb\rho\int_0^t \pi^*(s)\sqrt{z(s)}dW^z(s)\\
&\qquad + q\int_0^t z(s)\Bigg\{\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}b(1-b)\left(\pi^*(s) - \frac{\eta}{1-b}\right)^2 \qquad\qquad (A.59)\\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2\Bigg\}ds\\
&\quad \overset{(5.7)}{=} det_1(t) + D(t) - \frac{1}{2}\langle D\rangle_t + qB(T-t)z(t) + qb\rho\int_0^t \pi^*(s)\sqrt{z(s)}dW^z(s)\\
&\qquad + q\int_0^t z(s)\Bigg\{\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2\\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2\Bigg\}ds \qquad\qquad (A.60)
\end{aligned}
$$

B is continuously differentiable and monotonous, since it is the solution to an autonomous ODE (see Remark 5.2.5). Hence, π^* is monotonous and differentiable and has finite variation. Therefore, Itô's product rule yields

$$
\begin{aligned}
d\left(\pi^*(t)z(t)\right) &= \pi^*(t)dz(t) + z(t)d\pi^*(t)\\
&= \pi^*(t)dz(t) - \frac{z(t)}{1-b}\left[(\lambda^*)'\left(B(T-t)\right) + \sigma\rho\right]B'(T-t)dt\\
&= \pi^*(t)\kappa\left(\theta - z(t)\right)dt + \pi^*(t)\sigma\sqrt{z(t)}dW^z(t)\\
&\quad - \frac{z(t)}{1-b}\left[(\lambda^*)'\left(B(T-t)\right) + \sigma\rho\right]B'(T-t)dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_0^t \pi^*(s)\sqrt{z(s)}dW^z(s) &= \frac{1}{\sigma}\left(\pi^*(t)z(t) - \pi^*(0)z(0)\right) - \frac{\kappa}{\sigma}\int_0^t \pi^*(s)\left(\theta - z(s)\right)ds\\
&\quad + \frac{1}{\sigma}\frac{1}{1-b}\int_0^t \left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)z(s)ds \qquad (A.61)
\end{aligned}
$$

Substituting (A.61) in (A.60), while defining the deterministic and continuous function

$$det_2(t) = det_1(t) - \frac{qb\rho}{\sigma}\left[\pi^*(0)z(0) + \kappa\int_0^t \pi^*(s)\theta ds\right],$$

yields

$$\begin{aligned}
&q\ln\left(\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|\right) \\
&\quad = det_2(t) + D(t) - \frac{1}{2}\langle D\rangle_t + qz(t)\left[\frac{b\rho}{\sigma}\pi^*(t) + B(T-t)\right] \\
&\qquad + q\int_0^t z(s)\Bigg\{\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 \\
&\qquad\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds
\end{aligned} \tag{A.62}$$

Combining Lemma 5.2.10 with (A.62) yields

$$\begin{aligned}
&q\ln\left(\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|\right) \\
&\quad \le det_2(t) + D(t) - \frac{1}{2}\langle D\rangle_t + q\frac{\kappa}{\sigma^2}z(t) \\
&\qquad + q\int_0^t z(s)\Bigg\{\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 \\
&\qquad\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds \\
&\quad = det_2(t) + D(t) - \frac{1}{2}\langle D\rangle_t + q\frac{\kappa}{\sigma^2}\left(z(0) + \int_0^t \kappa(\theta - z(s))ds + \sigma\int_0^t \sqrt{z}(s)dW^z(s)\right) \\
&\qquad + q\int_0^t z(s)\Bigg\{\frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 \\
&\qquad\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds \\
&\quad = \underbrace{det_2(t) + q\frac{\kappa}{\sigma^2}\left(z(0) + \int_0^t \kappa\theta ds\right)}_{=:det_3(t)} + D(t) - \frac{1}{2}\langle D\rangle_t + \underbrace{\frac{q\kappa}{\sigma}\int_0^t \sqrt{z(s)}dW^z(s)}_{=:M(t)} \\
&\qquad + q\int_0^t z(s)\Bigg\{-\frac{\kappa^2}{\sigma^2} + \frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 \\
&\qquad\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds \\
&\quad = det_3(t) + D(t) - \frac{1}{2}\langle D\rangle_t + M(t) - \frac{1}{2}\langle M\rangle_t
\end{aligned}$$

$$
\begin{aligned}
&+ q\int_0^t z(s)\Bigg\{ -\frac{\kappa^2}{\sigma^2} + \frac{1}{2}\underbrace{q}_{=1+\epsilon}\frac{\kappa^2}{\sigma^2} + \frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 \\
&\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds \\
&= det_3(t) + D(t) - \frac{1}{2}\langle D\rangle_t + M(t) - \frac{1}{2}\langle M\rangle_t \qquad \text{(A.63)} \\
&+ q\int_0^t z(s)\Bigg\{ -\frac{\kappa^2}{2\sigma^2} + \frac{1}{2}\frac{b}{1-b}\eta^2 - \frac{1}{2}\frac{b}{1-b}\left(\lambda^*\left(B(T-s)\right) + \sigma\rho B(T-s)\right)^2 \\
&\qquad - \frac{1}{2}b^2\rho^2\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon b^2(1-\rho^2)\left(\pi^*(s)\right)^2 + \frac{1}{2}\epsilon\frac{\kappa^2}{\sigma^2} \\
&\qquad + b\frac{\rho\kappa}{\sigma}\pi^*(s) + \frac{b}{1-b}\frac{\rho}{\sigma}\left[(\lambda^*)'\left(B(T-s)\right) + \sigma\rho\right]B'(T-s)\Bigg\}ds \\
&= det_3(t) + D(t) - \frac{1}{2}\langle D\rangle_t + M(t) - \frac{1}{2}\langle M\rangle_t + q\int_0^t z(s)\Big\{\,(*)\,\Big\}ds \qquad \text{(A.64)}
\end{aligned}
$$

According to Lemma 5.2.11, the expression $(*)$ in (A.64) is negative for all $s \in [0,T]$ if $\epsilon = 0$. Hence, due to the continuity of $(*)$ in ϵ and the boundedness of all deterministic functions in $(*)$ w.r.t. t, there exists $\epsilon > 0$ such that $(*) < 0$ for all $s \in [0,T]$. For such a choice of ϵ we obtain

$$
\begin{aligned}
q\ln\left(\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|\right) &\le det_3(t) + D(t) - \frac{1}{2}\langle D\rangle_t + M(t) - \frac{1}{2}\langle M\rangle_t \\
&\le \sup_{s\in[0,T]}\left(det_3(s)\right) + D(t) - \frac{1}{2}\langle D\rangle_t + M(t) - \frac{1}{2}\langle M\rangle_t
\end{aligned}
$$

$$
\Leftrightarrow\quad \left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|^q \le \exp\left(\sup_{s\in[0,T]}\left(det_3(s)\right)\right)\cdot\underbrace{\exp\left(D(t) - \frac{1}{2}\langle D\rangle_t\right)\cdot\exp\left(M(t) - \frac{1}{2}\langle M\rangle_t\right)}_{=:(**)}.
$$

Since D and M are local martingales with independent diffusions, $(**)$ is a supermartingale. Hence, Doob's optional sampling theorem finally yields

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[\left|G(\tau, V^{v_0,\pi^*}(\tau), z(\tau))\right|^q\right] \\
&\qquad\le \exp\left(\sup_{s\in[0,T]}\left(det_3(s)\right)\right)\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[\exp\left(D(t) - \frac{1}{2}\langle D\rangle_t\right)\cdot\exp\left(M(t) - \frac{1}{2}\langle M\rangle_t\right)\right] \\
&\qquad\le \exp\left(\sup_{s\in[0,T]}\left(det_3(s)\right)\right)\mathbb{E}\left[\exp\left(D(0) - \frac{1}{2}\langle D\rangle_0\right)\cdot\exp\left(M(0) - \frac{1}{2}\langle M\rangle_0\right)\right] \\
&\qquad= \exp\left(\sup_{s\in[0,T]}\left(det_3(s)\right)\right) < \infty.
\end{aligned}
$$

Thus, $\left(G(\tau, V^{v_0,\pi^*}(\tau), z(\tau))\right)_{\tau\in\mathcal{T}}$ is bounded in L^q for some $q > 1$ and therefore uniformly integrable. Hence, π^* is optimal for $(\mathbf{P})$ as per Lemma 5.2.3. □

Proof of Corollary 5.2.13. Noting that $\delta_K(\lambda) = \infty$ for all $x \neq 0$ if $K = \mathbb{R}$, the minimum in (5.6) is attained by $\lambda^*(B) = 0$ for all $B \in \mathbb{R}$ and the ODE (5.5) simplifies to

$$B_u'(\tau) = -r_0 + r_1 B_u(\tau) + \frac{1}{2} r_2 \big(B_u(\tau)\big)^2$$

with coefficients

$$r_0 = -\frac{b}{2(1-b)}\eta^2, \quad r_1 = \frac{b}{1-b}\eta\sigma\rho - \kappa, \quad r_2 = \sigma^2\left(1 + \frac{b}{1-b}\rho^2\right).$$

The expression for π_u can be directly read off from Lemma 5.2.3. $\square$

Proof of Lemma 5.2.15. Let $\rho = 0$. Then, according to Remark 5.2.6 and Corollary 5.2.13, both $\hat{\pi}^*(t) = \pi_M$ and $\pi_u(t) = \pi_M$ are constant in time $t \in [0, T]$. Thus,

$$\pi^*(t) = \mathrm{Cap}(\hat{\pi}(t), \alpha, \beta) = \mathrm{Cap}(\pi_M, \alpha, \beta) = \mathrm{Cap}(\pi_u(t), \alpha, \beta) \quad \forall t \in [0, T].$$

Consider now the case $\rho \neq 0$ and $\pi_M \in K = [\alpha, \beta]$. Then, $\alpha(1-b) \leq \eta \leq \beta(1-b)$ and thus $B_- \leq 0 \leq B_+$. Following Lemma 5.2.4 and Remark 5.2.6, $0 \in Z_0$ and the Riccati ODE (5.5) has coefficients r_0, r_1, and r_2 at $\tau = 0$. Hence, $B(\tau) = B_u(\tau)$ for all $\tau \in [0, T]$ such that $\rho B(\tau) \in Z_0$. Moreover, following Remark 5.2.5, $B_u(\tau)$ and $B(\tau)$ (respectively $\pi_u(t)$ and $\hat{\pi}^*(t)$) are solutions to autonomous ODEs and thus monotone functions in τ (respectively t). Noting that $\pi_u(t) \in K = [\alpha, \beta] \Leftrightarrow \rho B_u(T-t) \in Z_0$, we know if $\pi_u(t) \in K$, then

$$\pi_u(t) = \frac{1}{1-b}\left(\eta + \sigma\rho B_u(T-t)\right) = \frac{1}{1-b}\left(\eta + \sigma\rho B(T-t)\right) = \hat{\pi}(t).$$

Due to the monotonicity of π_u and $\hat{\pi}^*$, we have $\pi_u(t) \notin (\alpha, \beta)$ if and only if $\hat{\pi}(t) \notin (\alpha, \beta)$. Thus, we have in total

$$\pi^*(t) = \mathrm{Cap}\left(\hat{\pi}^*(t), \alpha, \beta\right) = \mathrm{Cap}\left(\pi_u(t), \alpha, \beta\right) \quad \forall t \in [0, T].$$

In particular, in both cases $\rho = 0$ and $\pi_M \in K$ we have $\mathcal{P}_K^H = \mathcal{P}_K^{BS}$. $\square$

Proof of Lemma 5.2.16. The implication (ii) $\Rightarrow$ (i) is trivial, because if (ii) holds, then $\hat{\pi}^*(t) \neq \pi_u(t)$ and either $\hat{\pi}^*(t) \in (\alpha, \beta)$ or $\pi_u(t) \in (\alpha, \beta)$. Hence,

$$\pi^*(t) = \mathrm{Cap}\left(\hat{\pi}^*(t), \alpha, \beta\right) \neq \mathrm{Cap}\left(\pi_u(t), \alpha, \beta\right),$$

and therefore $\mathcal{P}_K^H\left(\pi_u, \alpha, \beta\right) \neq \mathcal{P}_K^{BS}\left(\pi_u, \alpha, \beta\right)$.
Assume now that (i) holds. According to Lemma 5.2.15, this implies $\rho \neq 0$ and $\pi_M \notin K = [\alpha, \beta]$. Moreover, as B and B_u are continuously differentiable,

$$\hat{\pi}^*(t) = \frac{1}{1-b}\left(\eta + \sigma\rho B(T-t)\right) \quad \text{and} \quad \pi_u(t) = \frac{1}{1-b}\left(\eta + \sigma\rho B_u(T-t)\right)$$

are continuously differentiable functions with $\hat{\pi}^*(T) = \pi_u(T) = \pi_M \notin K$. Thus, (i) can only be satisfied if there exists $0 < t < T$ such that

$$\{\hat{\pi}^*(t), \pi_u(t)\} \cap K \neq \emptyset, \tag{A.65}$$

as otherwise $\pi^*(t) = \mathrm{Cap}(\hat{\pi}^*(t), \alpha, \beta) = \mathrm{Cap}(\pi_M, \alpha, \beta) = \mathrm{Cap}(\pi_u(t), \alpha, \beta)$ for all $t \in [0, T]$. Thus, we define the latest such time point as

$$\hat{t} = \sup\left\{0 \leq t \leq T \;\middle|\; \{\hat{\pi}^*(t), \pi_u(t)\} \cap K \neq \emptyset\right\}.$$

Since there exists a positive $t > 0$ satisfying (A.65), we know that $\hat{t} > 0$. Moreover, as $\pi_M \notin K$ and K is closed, we also know that $\hat{t} < T$. If $\hat{\pi}^*(\hat{t}) \neq \pi_u(\hat{t})$, then only one of these portfolio processes takes a value in K and (ii) holds for $\hat{t}$.
It thus remains to verify that $\hat{\pi}^*(\hat{t}) \neq \pi_u(\hat{t})$ to conclude the proof. This will be proven by contradiction. Assume that $\hat{\pi}^*(\hat{t}) = \pi_u(\hat{t}) \in K$ which is equivalent to $\rho B(T-\hat{t}) = \rho B_u(T-\hat{t}) \in Z_0$. Further,

$$T - \hat{t} = \inf\left\{0 \leq \tau \leq T \;\middle|\; \{\rho B(\tau), \rho B_u(\tau)\} \cap Z_0 \neq \emptyset\right\}.$$

Moreover, as $\pi_M \notin K \overset{\rho\neq 0}{\Leftrightarrow} \rho B(0) = \rho B_u(0) = 0 \notin Z_0$, B and B_u are non-constant and thus strictly monotone functions. However, this implies that at $\tau = T - \hat{t}$, $\rho B(\tau)$ and $\rho B_u(\tau)$ have the same value at the boundary of Z_0 and satisfy the same ODE while taking values in Z_0. Hence, $B(\tau) = B_u(\tau)$ for all $\tau \in [0, T]$ such that $B(\tau) \in Z_0$. Noting that

$$\rho B(\tau) \in Z_0 \Leftrightarrow \hat{\pi}^*(\tau) \in K \quad \text{and} \quad \rho B_u(\tau) \in Z_0 \Leftrightarrow \pi_u(\tau) \in K,$$

finally yields for all $t \in [0, T]$:

$$\begin{aligned}\pi^*(t) &= \mathrm{Cap}\,(\hat{\pi}^*(t), \alpha, \beta) = \mathrm{Cap}\Big(\frac{1}{1-b}(\eta + \sigma \underbrace{\rho B(T-t)}_{=\rho B_u(T-t) \text{ while } \hat{\pi}^*(t)\in K}), \alpha, \beta\Big) \\ &= \mathrm{Cap}\left(\frac{1}{1-b}(\eta + \sigma\rho B_u(T-t)), \alpha, \beta\right) = \mathrm{Cap}\,(\pi_u(t), \alpha, \beta)\,,\end{aligned}$$

which is a contradiction to (i). Thus, $\hat{\pi}^*(\hat{t}) \neq \pi_u(\hat{t})$. □

Proof of Corollary 5.2.17.
Proof of (i): Let $\pi_M < \alpha$ and $0 < \alpha = 2\pi_M < \alpha\frac{2}{1-b}\eta$. Then, $\eta > 0$, $\pi_M > 0$ and $\eta = (1-b)\pi_M < (1-b)\alpha \Leftrightarrow B_- > 0$. Following Lemma 5.2.4 and Remark 5.2.6, $0 \in Z_-$ and the Riccati ODE (5.5) has coefficients r_0^-, r_1^-, and r_2^- at $\tau = 0$. However, due to the initial condition $B(0) = 0$, this implies

$$B'(0) = -r_0^- + r_1^- B(0) + \frac{1}{2}r_2^- B(0)^2 = -r_0^- = -\frac{1}{2}b\alpha\,((1-b)\alpha - 2\eta) = 0,$$

i.e., according to Remark 5.2.5, B is constant with $B(\tau) = 0$ for all $\tau \in [0, T]$. In particular, for all $t \in [0, T]$ we have $\hat{\pi}^*(t) = \pi_M$ and

$$\pi^*(t) = \mathrm{Cap}(\hat{\pi}^*(t), \alpha, \beta) = \mathrm{Cap}(\pi_M, \alpha, \beta) = \alpha.$$

On the other hand, the optimal unconstrained portfolio π_u does not violate the constraint at time t^*, i.e.,

$$\mathrm{Cap}(\pi_u(t^*), \alpha, \beta) = \pi_u(t^*) \in (\alpha, \beta),$$

and thus $\mathcal{P}_K^H \neq \mathcal{P}_K^{BS}$.
Proof of (ii): According to Corollary 5.2.13,

$$\frac{\partial}{\partial t}\left(\pi_u(t)\right) = \frac{1}{1-b}\frac{\partial}{\partial t}\left(\eta + \sigma\rho B_u(T-t)\right)$$

Since B_u is the solution to the autonomous ODE (5.13), B_u is a monotone function (cf. Remark 5.2.5). Therefore,

$$\mathrm{sign}\left(\frac{\partial}{\partial t}\left(\pi_u(t)\right)\right) = \mathrm{sign}\left(\frac{-\sigma\rho}{1-b}B_u'(T-t)\right) = \mathrm{sign}\left(\frac{-\sigma\rho}{1-b}B_u'(0)\right) \overset{(5.13)}{=} \mathrm{sign}\left(\frac{\sigma\rho}{1-b}r_0\right)$$

$$\stackrel{(5.9)}{=} \text{sign}\left(\frac{\sigma\rho}{1-b}\left(-\frac{b}{2(1-b)}\eta^2\right)\right)$$

As $\frac{\sigma\eta^2}{2(1-b)^2}$ is positive, this implies sign $\left(\frac{\partial}{\partial t}\left(\pi_u(t)\right)\right) = -\text{sign}\left(b\rho\right)$. As $\beta > 0$ and $\pi_M > \beta \Leftrightarrow 0 = \rho B(0) > B_+$, we follow the same line of argument for $\hat{\pi}^*$ to obtain

$$\begin{aligned}\text{sign}\left(\frac{\partial}{\partial t}\left(\hat{\pi}^*(t)\right)\right) &= \text{sign}\left(\frac{-\sigma\rho}{1-b}B'(0)\right) = \text{sign}\left(\frac{\sigma\rho}{1-b}r_0^+\right)\\ &= \text{sign}\left(\frac{\sigma\rho}{1-b}\frac{1}{2}b\beta\left([1-b]\beta-2\eta\right)\right) = \text{sign}\left(\frac{\sigma\rho b\beta}{2}\left(\beta-2\pi_M\right)\right)\end{aligned}$$

Disregarding the positive factor $\frac{\sigma\beta}{2}$ finally yields

$$\text{sign}\left(\frac{\partial}{\partial t}\left(\hat{\pi}^*(t)\right)\right) = \text{sign}\big(\rho b\underbrace{(\beta-2\pi_M)}_{<0}\big) = -\text{sign}\left(b\rho\right) = \text{sign}\left(\frac{\partial}{\partial t}\left(\pi_u(t)\right)\right).$$

If now $\rho < 0$ and $b < 0$, then both portfolio allocation $\pi^*(t) = \text{Cap}\left(\hat{\pi}^*(t), \alpha, \beta\right)$ and $\text{Cap}\left(\pi_u(t), \alpha, \beta\right)$ are non-increasing in time. In particular, as both allocations are equal to $\beta < \pi_M$ at $t = T$, they must be equal (to β) throughout the entire investment horizon. Hence, the projections $\mathcal{P}_K^H$ and $\mathcal{P}_K^{BS}$ coincide under these assumptions. □

Proof of Theorem 5.3.1. We transform the portfolio optimization problem (**P**) into an equivalent optimization problem by applying the change of control $\pi_A(t) := A'\pi(t)$ for any given $\pi \in \Lambda^{PCSV}$. Expressing the SDE of the wealth process $V^{v_0,\pi}$ in terms of π_A yields

$$\begin{aligned} dV^{v_0,\pi}(t) &= V^{v_0,\pi}(t)V^{v_0,\pi}(t)\left[\left(r + \underbrace{\eta' A}_{=:\eta_A'}\text{diag}(z(t))\underbrace{A'\pi(t)}_{=\pi_A(t)}\right)dt + \underbrace{\pi(t)'A}_{=\pi_A(t)'}\text{diag}(\sqrt{z(t)})dW(t)\right]\\ &= V^{v_0,\pi}(t)\left[\left(r+\eta_A'\text{diag}(z(t))\pi_A(t)\right)dt + \pi_A(t)'\text{diag}(\sqrt{z(t)})dW(t)\right] \end{aligned} \tag{A.66}$$

and $\pi(t) \in K_{PCSV} \Leftrightarrow \pi_A(t) \in \times_{i=1}^d [0, \sqrt{\beta_i}]$. In particular, we may equivalently rewrite the portfolio optimization problem in terms of π_A as

$$(\mathbf{P_A})\begin{cases}\Phi(v_0) = \sup_{\pi\in\Lambda_A}\mathbb{E}\big[U(V^{v_0,\pi}(T))\big]\\ \Lambda_A = \left\{\pi_A \in \Lambda_{PCSV} \;\middle|\; \pi_A(t) \in \times_{i=1}^d[0,\sqrt{\beta_i}]\ \mathcal{L}[0,T]\otimes Q - \text{a.e.}\right\}\end{cases}.$$

We proceed by solving ($\mathbf{P_A}$) and then inverting the change of control to obtain a solution for the original optimization problem (**P**). As $K_A = \times_{i=1}^d[0,\sqrt{\beta_i}]$ is a d-dimensional interval, ($\mathbf{P_A}$) fits into the setting of the financial market $\mathcal{M}_{CIR}$ in Definition 4.4.5 with $m = d$ and $\Sigma_i = 1$ for $i = 1, .., d$. In particular, according to Lemma 4.4.6, if $A : [0,T] \to \mathbb{R}$, $B : [0,T] \to \mathbb{R}^d$ with $A(0) = B_i(0) = 0$ for $i = 1, ..., d$ satisfy

$$\begin{aligned} A'(\tau) &= br + \sum_{i=1}^d \kappa_i\theta_i\\ B_i'(\tau) &= -\kappa_i B_i(\tau) + \frac{1}{2}\sigma_i\left(B_i(\tau)\right)^2\\ &\quad + \frac{1}{2}\frac{b}{1-b}\inf_{\lambda\in\mathbb{R}}\left\{2(1-b)\delta_{[0,\sqrt{\beta_i}]}(\lambda) + \left((\eta_A)_i + \lambda_i + \sigma_i\rho_i B_i(\tau)\right)^2\right\}, \end{aligned} \tag{A.67}$$

then the function $G(t,v,z) := \frac{1}{b}v^b \exp\left(A(T-t) + B(T-t)'z\right)$ is a solution to the HJB PDE associated with $(\mathbf{P_A})$. However, for each $i = 1, ..., d$, equation (A.67) contains the same ODE and optimization that were considered in Lemma 5.2.4 and Section 5.2. Since Assumption 5.2.7 is satisfied, the solution B_i to (A.67) is given by Theorem 5.2.8. Finally, the candidate optimal portfolio π_A^* for $(\mathbf{P})$ is given by

$$(\pi_A^*)_i(t) = \text{Cap}\left(\frac{1}{b}\left((\eta_A)_i + \sigma_i\rho_i B_i(T-t)\right), 0, \sqrt{\beta_i}\right).$$

We still need to formally verify the optimality of π_A^* for $(\mathbf{P_A})$. For this purpose, we define the sequence of stopping times $\tau_{n,t}$ as $\tau_{n,t} = \min(T, \hat{\tau}_{n,t})$, with

$$\hat{\tau}_{n,t} = \inf\Big\{t \le u \le T \;\Big|\; \int_t^u \left\| b\left(\sqrt{z(s)} \odot \pi_A(s)\right) G(s, V^{v_0,\pi^*}(s), z(s)) \right\|^2 ds \ge n, \\ \int_t^u \left\| \left(\sigma \odot \sqrt{z(s)} \odot B(T-s)\right) G(s, V^{v_0,\pi^*}(s), z(s)) \right\|^2 ds \ge n \Big\}.$$

According to Theorem 3.12 in [33], the optimality of π_A^* is verified if

$$\left(G\left(\tau_{n,t}, V^{v_0,\pi^*}(\tau_{n,t}), z(\tau_{n,t})\right)\right)_{n\in\mathbb{N}}$$

is uniformly integrable for every $t \in [0,T]$. However, as Assumptions 5.2.7 and 5.2.9 are satisfied for every $i = 1, ..., d$, following the same steps as in the proof of Theorem 5.2.12, we can show that there exists a constant $q > 1$ and a bounded, deterministic function $det(t)$ such that the local martingales

$$D_i(t) = bq\sqrt{1-\rho_i^2}\int_0^t (\pi_A)_i^*(s)\sqrt{z_i(s)}d\hat{W}_i(s)$$
$$M_i(t) = \frac{q\kappa_i}{\sigma_i}\int_0^t \sqrt{z_i(s)}dW_i^z(s)$$

can be used to bound $\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|^q$ for every $t \in [0,T]$ through

$$\left|G\left(t, V^{v_0,\pi^*}(t), z(t)\right)\right|^q \le \exp\left(\sup_{s\in[0,T]} det(s)\right)\underbrace{\exp\left(\sum_{i=1}^d D_i(t) - \frac{1}{2}\langle D_i\rangle_t + M_i(t) - \frac{1}{2}\langle M_i\rangle_t\right)}_{(*)}.$$

The diffusions of the local martingales D_i, M_i are independent and therefore $(*)$ is a supermartingale. Since the above bound holds pointwise for every $t \in [0,T]$, Doob's optional sampling theorem yields the L^q-boundedness (and thereby uniform integrability) of

$$\left(G(\tau_{n,t}^0, V^{v_0,\pi^*}(\tau_{n,t}^0), z(\tau_{n,t}^0))\right)_{n\in\mathbb{N}}.$$

Hence, π_A^* is optimal for $(\mathbf{P_A})$ and thus $\pi^*(t) = A\pi_A^*(t)$ is optimal for $(\mathbf{P})$ in $\mathcal{M}_{PCSV}$. □

Proof of Theorem 5.3.2. The proof follows similar arguments as the proof of Corollary 5.4 in [56]. For given $\pi \in \Lambda^{\gamma,\Sigma}$, let $\hat{\pi}_{BS}$ and $\hat{\pi}_H$ be defined through

$$\hat{\pi}_{BS}(t) := \Sigma(z(t))\pi(t) \qquad \text{or} \qquad \hat{\pi}_H(t) := \frac{\Sigma(z(t))}{\sqrt{z(t)}}\pi(t). \tag{A.68}$$

Proof of (i): By construction, $\pi \in \Lambda_{K(\cdot)}$ if and only if $\hat{\pi}_{BS}(t) \in \hat{K} := [\alpha, \beta]$ $\mathcal{L}[0,T] \otimes Q$-a.e., i.e., the constraint on $\hat{\pi}_{BS}$ is constant. Under the present assumptions, the wealth process satisfies

$$\begin{aligned} dV^{v_0,\pi}(t) &= V^{v_0,\pi}(t)\left([r + \eta\Sigma(z(t))\pi(t)]dt + \pi(t)\Sigma(z(t))dW(t)\right) \\ &= V^{v_0,\pi}(t)\left([r + \eta\hat{\pi}_{BS}(t)]dt + \hat{\pi}_{BS}(t)dW(t)\right). \end{aligned}$$

Hence, maximizing $\mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right]$ over $\hat{\pi}_{BS}$ subject to the constraint $\hat{\pi}_{BS}(t) \in \hat{K} = [\alpha, \beta]$ $\mathcal{L}[0,T] \otimes Q$-a.e. is equivalent to the constrained portfolio optimization problem in a Black-Scholes market $\mathcal{M}_{BS}$ with market price of risk η and volatility 1. Therefore, as discussed in Section 5.2.3, the constant-mix strategy

$$\hat{\pi}^*_{BS} = \text{Cap}\left(\pi_M, \alpha, \beta\right) \Leftrightarrow \pi^*(t) = \Sigma(z(t))^{-1}\text{Cap}\left(\pi_M, \alpha, \beta\right)$$

maximises the expected utility over all admissible $\hat{\pi}_{BS}$. Inverting the change of control through multiplication by $\Sigma(z(t))^{-1}$ yields the claim.

Proof of (ii): By construction, $\pi \in \Lambda_{K(\cdot)}$ if and only if $\hat{\pi}_H(t) \in \hat{K} := [\alpha, \beta]$ $\mathcal{L}[0,T] \otimes Q$-a.e., i.e., the constraint on $\hat{\pi}_H$ is constant. Under the present assumptions, the wealth process satisfies

$$\begin{aligned} dV^{v_0,\pi}(t) &= V^{v_0,\pi}(t)\left([r + \eta\sqrt{z(t)}\Sigma(z(t))\pi(t)]dt + \pi(t)\Sigma(z(t))dW(t)\right) \\ &= V^{v_0,\pi}(t)\left([r + \eta z(t)\hat{\pi}_H(t)]dt + \hat{\pi}_H(t)\sqrt{z(t)}dW(t)\right). \end{aligned}$$

Hence, maximizing $\mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right)\right]$ over $\hat{\pi}_H$ subject to the constraint $\hat{\pi}_H(t) \in \hat{K} = [\alpha, \beta]$ $\mathcal{L}[0,T] \otimes Q$-a.e. is equivalent to the constrained portfolio optimization problem in the Heston market $\mathcal{M}_H$, as considered in Section 5.2. As all requirements of Theorem 5.2.12 are satisfied, π^* (as defined in Theorem 5.2.12) maximises the expected utility over all admissible $\hat{\pi}_H$. Inverting the change of control through multiplication by $\sqrt{z(t)}\Sigma(z(t))^{-1}$ yields the claim. □

Proof of Lemma 5.4.1. We first verify that $G(t,v,z) = \frac{1}{b}v^b \exp(A_\pi(T-t) + B_\pi(T-t)z)$ is a solution to the Feynman-Kac PDE (5.20). The partial derivatives of G can be computed as

$$\begin{aligned} G_t(t,v,z) &= -\left(A'_\pi(T-t) + B'_\pi(T-t)z\right)G(t,v,z), \quad G_v(t,v,z) = \frac{b}{v}G(t,v,z), \\ G_{vv}(t,v,z) &= \frac{b(b-1)}{v^2}G(t,v,z), \quad G_z(t,v,z) = B_\pi(T-t)G(t,v,z), \\ G_{zz}(t,v,z) &= \left(B_\pi(T-t)\right)^2 G(t,v,z), \quad \text{and} \quad G_{zv}(t,v,z) = \frac{bB_\pi(T-t)}{v}G(t,v,z). \end{aligned}$$

Substituting these derivatives in (5.20), while omitting the arguments (t,v,z), yields

$$\begin{aligned} 0 &= G_t + (r + \eta\pi(t)z)\,vG_v + \kappa\,(\theta - z)\,G_z + \sigma\rho\pi(t)zvG_{zv} + \frac{1}{2}v^2\pi(t)^2 zG_{vv} + \frac{1}{2}\sigma^2 zG_{zz} \\ &= G\Big[-\left(A'_\pi + B'_\pi z\right) + (r + \eta\pi(t)z)\,b + \kappa\,(\theta - z)\,B_\pi \\ &\qquad + \sigma\rho\pi(t)bzB_\pi + \frac{1}{2}\pi(t)^2 zb(b-1) + \frac{1}{2}\sigma^2 zB_\pi^2\Big] \\ \overset{G\neq 0}{\Leftrightarrow} \quad 0 &= -\left(A'_\pi + B'_\pi z\right) + (r + \eta\pi(t)z)\,b + \kappa\,(\theta - z)\,B_\pi \\ &\qquad + \sigma\rho\pi(t)bzB_\pi + \frac{1}{2}\pi(t)^2 zb(b-1) + \frac{1}{2}\sigma^2 zB_\pi^2 \\ &= \underbrace{-A'_\pi + rb + \kappa\theta B_\pi}_{=0 \text{ by } (5.18)} + z\underbrace{\left[-B'_\pi + b\eta\pi(t) - \frac{1}{2}\pi(t)^2 b(1-b) + (\sigma\rho b\pi(t) - \kappa)\,B_\pi + \frac{\sigma^2}{2}B_\pi^2\right]}_{=0 \text{ by } (5.19)}. \end{aligned}$$

Moreover, G satisfies the boundary condition $G(T,v,z) = U(v)$ because $A_\pi(0) = B_\pi(0) = 0$. Hence, G is a solution to the Feynman-Kac PDE (5.20). By assumption, such a solution is unique, and therefore

$$\frac{1}{b}v^b \exp\left(A_\pi(T-t) + B_\pi(T-t)z\right) = G(t,v,z) = J^\pi(t,v,z) \quad \forall (t,v,z) \in [0,T]\times(0,\infty)\times(0,\infty).$$

□

Proof of Corollary 5.4.2. $L^\pi(t,z)$ is defined as the solution to the equation

$$\begin{aligned}\Phi(t, v(1-L^\pi(t,z)), z) &= \mathbb{E}\left[U\left(V^{v_0,\pi^*}(T)\right) \mid V^{v_0,\pi}(t) = v(1-L^\pi(t,z)),\ z(t)=z\right] \\ &= \mathbb{E}\left[U\left(V^{v_0,\pi}(T)\right) \mid V^{v_0,\pi}(t) = v,\ z(t)=z\right] \\ &= J^\pi(t,v,z)\end{aligned}$$

From Lemma 5.2.3 we know that

$$\Phi(t, v(1-L^\pi(t,z)), z) = \frac{1}{b}\left(v(1-L^\pi(t,z))\right)^b \exp(A(T-t) + B(T-t)z).$$

Similarly, we have by Lemma 5.4.1

$$J^\pi(t,v,z) = \frac{1}{b}v^b \exp\left(A_\pi(T-t) + B_\pi(T-t)z\right).$$

Thus,

$$\begin{aligned}&\Phi(t, v(1-L^\pi(t,z)), z) = J^\pi(t,v,z) \\ \Leftrightarrow\quad & \frac{1}{b}\left(v(1-L^\pi(t,z))\right)^b \exp(A(T-t)+B(T-t)z) = \frac{1}{b}v^b \exp\left(A_\pi(T-t)+B_\pi(T-t)z\right) \\ \Leftrightarrow\quad & \left(\frac{v(1-L^\pi(t,z))}{v}\right)^b = \exp\left(A_\pi(T-t) - A(T-t) + \left[B_\pi(T-t) - B(T-t)\right]z\right) \\ \Leftrightarrow\quad & L^\pi(t,z) = 1 - \exp\left(\frac{1}{b}\left(A_\pi(T-t) - A(T-t) + \left[B_\pi(T-t) - B(T-t)\right]z\right)\right).\end{aligned}$$

□

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