Trap models with vanishing drift: Scaling limits and ageing regimes

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Abstract. We discuss the long term behaviour of trap models on the integers with asymptotically vanishing drift, providing scaling limit theorems and ageing results. Depending on the tail behaviour of the traps and the strength of the drift, we identify three different regimes, one of which features a previously unobserved limit process.

1. Introduction

Trap models are a particularly simple class of stochastic processes in random environment, which have recently attracted a lot of attention. To describe the setup of most trap models, suppose a graph with finite degree is given. To each of the vertices v of the graph we associate an independent random variable τ_v chosen according to a suitable class of heavy-tailed distributions. Given this random environment, the trap model is a continuous-time nearest neighbour random walk on the graph such that the exponential holding time at a vertex v has a mean proportional to τ_v. Therefore vertices v with large values τ_v act as *traps* in which the random walk spends a larger amount of time than in vertices with small values

of v. Different trap models arise by varying the underlying graph and the drift of the random walk.

The main purpose of trap models is to serve as a phenomenological model describing how a physical system out of equilibrium moves in an energy landscape. Here vertices with large trap values represent energetically favourable states in which the system tends to remain for longer. Most results on trap models are about the phenomenon of *ageing*, which means that in such a system the time spans during which the system does not change its state are increasing as the system gets older. Trap models offer a simple explanation for ageing: Roughly speaking, the older the system, the more space it has explored, and therefore the deeper the trap it is stuck in. Let us also mention here some interesting papers exhibiting the ageing phenomenon in some other models, for example for spherical spin glasses (Ben Arous et al., 2001), the random energy model with Glauber dynamics (Ben Arous et al., 2003) and the parabolic Anderson model with heavy tailed potential (Mörters et al., 2009).

Trap models were introduced into the physics literature by Bouchaud (1992) and interest in the mathematical community was created through the pioneering work of Fontes et al. (2002) and Ben Arous et al. (2006). An excellent survey over the mathematical literature on trap models is provided in the lecture notes of Ben Arous and Černý (2006).

Understanding the ageing phenomenon is closely linked to scaling limit theorems for the trap models. For driftless trap models on the lattice $\mathbb{Z}^d$ it was shown that, on suitable path spaces,

- if $d = 1$ the rescaled trap model converges to a singular diffusion without drift, which is often called the *Fontes-Isopi-Newman diffusion*, see Fontes et al. (2002);
- if $d \geq 2$ the rescaled trap model converges to the *fractional-kinetics process*, which is a self-similar non-Markovian process, obtained as the time change of a d-dimensional Brownian motion by the inverse of an independent stable subordinator, see Ben Arous and Černý (2007).

More recently, Barlow and Černý (2010) identified the fractional-kinetics process as the scaling limit for a class of random walks with unbounded conductances and for the so-called non-symmetric trap models on $\mathbb{Z}^d$, $d \geq 3$, which have a drift depending locally on the trap environment.

In the present paper we focus on trap models on $\mathbb{Z}$ with a drift, which does not depend on the trap environment, addressing a question posed in Ben Arous and Černý (2006). In our first main result, Theorem 2.1, we look at the scaling limits of trap models with an asymptotically vanishing drift, and identify three regimes:

- In a regime where the drift vanishes *slowly*, the rescaled trap model converges to the inverse of a stable subordinator. Zindy (2009), using a different method of proof, identified the same process as the scaling limit for trap models with constant drift.
- In a regime where the drift vanishes *quickly* the rescaled trap model converges to a Fontes-Isopi-Newman diffusion, the same process as in the driftless case.

- In a critical *intermediate* regime the rescaled trap model converges to a singular diffusion with drift, which we call the Fontes-Isopi-Newman diffusion with drift. This process has not been identified as limit process in any other case before.

Our second main result, stated as Theorems 2.2 and 2.4, refer to the ageing behaviour in trap models on $\mathbb{Z}$ with vanishing drift. To this end we study the asymptotics of the depth of the trap in which the particle is at any given time, or, in other words, the environment from the point of view of the particle. This allows us to identify an *ageing exponent* $0 < \gamma \leq 1$ such that the probability

$$\mathbb{P}\big\{X_t = X_{t+s} \text{ for all } 0 \leq s \leq t^\gamma\big\} \qquad \text{as } t \uparrow \infty,$$

(averaged over the trap environment) converges to a value strictly between zero and one. Again there is a qualitatively different behaviour between the case of slowly vanishing drift on the one hand, and rapidly vanishing and critical drift on the other. Only in the case of constant drift do we have an ageing exponent $\gamma = 1$, all regimes with vanishing drift (as $t \uparrow \infty$) lead to sublinear ageing, i.e. exponents $\gamma < 1$. This is in marked contrast to the behaviour of the two-point function $\mathbb{P}\{X_t = X_{t+t^\gamma}\}$ for which we expect a nontrivial limit when $\gamma = 1$ in all cases, a fact which is rigorously established in the driftless case in Fontes et al. (2002) and in the case of fixed drift in Zindy (2009).

In the following section we give the precise formulation of our main results. We then proceed to prove our scaling limit theorems in the three regimes in Sections 3, 4 and 5, and the two regimes of ageing results in Section 6 and 7.

2. Statement of the main results

Fix $0 < \alpha < 1$ and let $(\tau_z : z \in \mathbb{Z})$ be an independent family of random variables with

$$\lim_{x \uparrow \infty} x^\alpha P\{\tau_z > x\} = 1 \,. \tag{2.1}$$

Given this *trap environment* and jump probabilities $p, q \in [0, 1]$, $q = 1 - p$, we define the Markov chain on $\mathbb{Z}$ with transition rates

$$q_{i,i+1} = p\,\tau_i^{-1}, \qquad q_{i,i-1} = q\,\tau_i^{-1} \,.$$

This is called the *(symmetric) trap model with drift*. We are mostly concerned with limit theorems for these processes in the case of vanishing drift. We therefore suppose that $\mu \geq 0$, $\beta \geq 0$ and $X^{(N)} = (X_t^{(N)} : t \geq 0)$ is defined by $X_t^{(N)} = X_{Nt}$, where $X = (X_t : t \geq 0)$ is a trap model with jump probabilities

$$p^{(N)} = \frac{1}{2}\Big(1 + \frac{\mu}{N^\beta}\Big), \quad q^{(N)} = 1 - p^{(N)} = \frac{1}{2}\Big(1 - \frac{\mu}{N^\beta}\Big).$$

(We take $\mu \leq 1$ if $\beta = 0$.). We define the following limiting processes:

- *Inverse stable subordinator.*
  For $0 < \alpha < 1$ the stable subordinator is the increasing Lévy process $(\mathsf{Sub}_t : t \geq 0)$ with

  $$\mathbb{E}\big[e^{-\lambda \mathsf{Sub}_t}\big] = \exp\big\{-t\,\Gamma(1-\alpha)\,\lambda^\alpha\big\} \,.$$

  Its right-continuous inverse $(\mathsf{Sub}_s^{-1} : s \geq 0)$ defined by

  $$\mathsf{Sub}_s^{-1} = \inf\{t > 0 : \mathsf{Sub}_t > s\}$$

is the *inverse stable subordinator* with index α.

- *Fontes Isopi Newman diffusion with drift μ.*

  Suppose $(B(t)\colon t \geq 0)$ is a Brownian motion with drift μ and $(\ell(t,x)\colon t \geq 0, x \in \mathbb{R})$ its local times. Let ρ be an independent *stable measure* with index $0 < \alpha < 1$, defined as the random measure whose cumulative distribution function is a two-sided stable subordinator with the same index. Define an increasing function

  $$\phi(t) = \int \ell(t,x)\,\rho(dx),$$

  and its inverse

  $$\psi(s) = \inf\{t > 0\colon \phi(t) > s\}.$$

  Then $(\mathsf{Fin}_s^\mu\colon s \geq 0)$ given by

  $$\mathsf{Fin}_s^\mu = B(\psi(s))$$

  is a *Fontes Isopi Newman diffusion* with drift μ.

We always denote by "$\Longrightarrow$" convergence in distribution, averaging over the trap environment.

Theorem 2.1 (Scaling limits)**.** *We have the following limit laws, where "$\Longrightarrow$" denotes convergence in distribution on the Skorokhod space $D[0,1]$ of right-continuous functions with left-hand limits.*

(a) *If $0 \leq \beta < \frac{\alpha}{\alpha+1}$ and $\mu > 0$ then*

$$\frac{X^{(N)}}{N^{\alpha(1-\beta)}} \Longrightarrow \frac{\mu^\alpha}{\Gamma(1+\alpha)}\,\mathsf{Sub}^{-1}.$$

(b) *If $\beta = \frac{\alpha}{\alpha+1}$ and $\mu > 0$ then*

$$\frac{X^{(N)}}{N^\beta} \Longrightarrow \mathsf{Fin}^\mu.$$

(c) *If $\beta > \frac{\alpha}{\alpha+1}$ or $\mu = 0$ then*

$$\frac{X^{(N)}}{N^{\frac{\alpha}{\alpha+1}}} \Longrightarrow \mathsf{Fin}^0.$$

The scaling limit in regime (c) has been identified by Fontes et al. (2002) in the case of the trap model without drift ($\mu = 0$); the inverse stable subordinator has been observed, using methods different from ours, as a scaling limit in trap models with constant drift ($\beta = 0$) by Zindy (2009). Monthus (2004) has some interesting results for the asymptotics $\alpha \downarrow 0$. The diffusion with drift, which we observe in the critical regime, represents a previously unobserved scaling behaviour.

Theorem 2.2 (Ageing in the presence of slowly vanishing drift)**.** *If $0 \leq \beta < \frac{\alpha}{\alpha+1}$ and $\mu > 0$, then there exist nonnegative nondegenerate random variables ξ_t such that*

$$\frac{\tau_{X_t^{(N)}}}{N^{1-\beta}} \Longrightarrow \xi_t.$$

Define a function $c(t) \in (0,1)$ by $c(t) = E[\exp\{-\frac{1}{\xi_t}\}]$. Then we have

$$\lim_{N\uparrow\infty} \mathbb{P}\big\{X_t^{(N)} = X_{t+s}^{(N)} \quad \text{for all} \quad 0 \leq s \leq N^{-\beta}\big\} = c(t).$$

Remark 2.3. Note that the ageing exponent defined in the introduction equals $\gamma = 1 - \beta$ in this case. The limit variable ξ_t describes the traps from the point of view of the particle. We define two independent series of nonnegative i.i.d. random variables $U_1, U_2, \ldots$ and $S_1, S_2, \ldots$ such that

- S_i is the product of two independent random variables, a Pareto variable with index α and an exponential variable with mean $1/\mu$;
- U_i is a random variable with (for some constant c depending only on α)

$$P\{U_i > x\} \sim c \frac{\mu^\alpha}{x^\alpha} \qquad \text{as } x \uparrow \infty,$$

and the law of ξ_t can be described as

$$P\{\xi_t > v\} = \sum_{j=1}^{\infty} P\Big\{ \sum_{i=1}^{j-1}(U_i + S_i) + U_j < \frac{t}{v} \leq \sum_{i=1}^{j}(U_i + S_i) \Big\}, \quad \text{for } v > 0.$$

Here, loosely speaking, the variables S_i represent periods in which the walker is in deep traps, while the U_i represent the travel times between these traps.

We also have results in the rapidly vanishing and critical drift regimes.

Theorem 2.4 (Ageing in the presence of rapidly vanishing drift)**.** *If $\beta \geq \frac{\alpha}{\alpha+1}$, then there exist nonnegative nondegenerate random variables ζ_t such that*

$$\frac{\tau_{X_t^{(N)}}}{N^{\frac{1}{\alpha+1}}} \Longrightarrow \zeta_t \,.$$

Define a function $k(t) \in (0, 1)$ by $k(t) = E[\exp\{-\frac{1}{\zeta_t}\}]$. Then we have

$$\lim_{N \uparrow \infty} \mathbb{P}\{X_t^{(N)} = X_{t+s}^{(N)} \quad \text{for all } 0 \leq s \leq N^{-\frac{\alpha}{\alpha+1}}\} = k(t) \,.$$

Remark 2.5. In this regime the ageing exponents equals $\gamma = \frac{1}{\alpha+1}$. We observe a *joint* convergence of the rescaled process and the rescaled trap environment interpreted as a random measure,

$$\Big(N^{-\frac{\alpha}{\alpha+1}} X^{(N)}, N^{-\frac{1}{\alpha+1}} \sum_{z \in \mathbb{Z}} \tau_z \, \delta_{N^{-\frac{\alpha}{\alpha+1}} z} \Big) \Longrightarrow (\mathsf{Fin}^\theta, \rho) \,,$$

where $\theta = \mu$ if $\beta = \frac{\alpha}{\alpha+1}$, and $\theta = 0$ otherwise. The limiting random variable ζ_t is then given as

$$\zeta_t = \rho(\mathsf{Fin}_t^\theta) \,.$$

Remark 2.6. In both of our ageing results, unless $\beta = 0$, the time typically spent by the process in the current state is a sublinear function of time. This kind of phenomenon is sometimes called *sub-ageing* and is exhibited by an ageing exponent $\gamma < 1$.

3. Proof of Theorem 2.1(a)

The basic idea of the proof is to show that the process $X^{(N)}$ is mostly increasing and therefore essentially invertible. The convergence of one-dimensional marginals will then be proved for the inverse process using Laplace transforms. This will be extended to finite-dimensional marginals using asymptotic independence properties, and finally to Skorokhod space by verification of a continuity criterion.

The first lemma ensures that $X^{(N)}$ is mostly increasing. We denote by T_x the first time where the process X hits level $x > 0$. We can write

$$T_x = \sum_{n=0}^{\infty} \eta_n \, \tau_{\mathcal{S}_n} 1\big\{ \max_{j \leq n} \mathcal{S}_j < x \big\},$$

where $(\mathcal{S}_n \colon n \geq 0)$ is the random walk embedded in the process X and $(\eta_n \colon n \geq 0)$ an independent family of independent standard exponential random variables.

Lemma 3.1. *There exists a constant C such that, for x large enough,*

$$\mathbb{P}\Big\{ \sup_{v \leq t} \big(\sup_{u \leq v} X_u^{(N)} - X_v^{(N)} \big) \geq x \Big\} \leq C \, t^\alpha \, N^\alpha \, \exp\big\{ - \mu \frac{x}{N^\beta} \big\},$$

for all $t > 0$.

Proof: It is clear that

$$\Big\{ \sup_{v \leq t} \big(\sup_{u \leq v} X_u^{(N)} - X_v^{(N)} \big) \geq x \Big\}$$

$$\subseteq \bigcup_{j=0}^{\infty} \Big\{ T_j < Nt; \ X_v \leq j - x \text{ for some } v > T_j \Big\}.$$

Furthermore,

$$\Big\{ T_j < Nt; \ X_v \leq j - x \text{ for some } v > T_j \Big\} \subseteq \Big\{ T_j < Nt; \ \min_{k \geq 1} \mathcal{S}_k^{(j)} \leq -x \Big\},$$

where $(\mathcal{S}_k^{(j)} \colon k = 1, 2, \ldots)$ is the random walk embedded in $(X_v - j \colon v \geq T_j)$, which is independent of $(X_v \colon v \leq T_j)$. Consequently,

$$\mathbb{P}\Big\{ \sup_{v \leq t} \big(\sup_{u \leq v} X_u^{(N)} - X_v^{(N)} \big) \geq x \Big\} \leq \mathbb{P}\Big\{ \min_{k \geq 1} \mathcal{S}_k \leq -x \Big\} \sum_{j=0}^{\infty} \mathbb{P}\big\{ T_j < Nt \big\}.$$

For the first term,

$$\mathbb{P}\Big\{ \min_{k \geq 1} \mathcal{S}_k \leq -x \Big\} = \Big(\frac{q^{(N)}}{p^{(N)}} \Big)^x = \Big(\frac{1 - \mu N^{-\beta}}{1 + \mu N^{-\beta}} \Big)^x \leq \exp\big\{ - \mu \frac{x}{N^\beta} \big\}. \tag{3.1}$$

We next note that

$$\sum_{j=0}^{\infty} \mathbb{P}\{ T_j < Nt \} \leq \sum_{j=0}^{\infty} \mathbb{P}\Big\{ \sum_{k=0}^{j-1} \tau_k \eta_k < Nt \Big\}.$$

Noting that the tail of $\tau_0 \eta_0$ is regularly varying with index α and using the renewal theorem for this class of random variables, see e.g. Erickson (1970), we see that the sum on the right is bounded by $C(Nt)^\alpha$. This completes the proof of the lemma. □

A direct consequence of Lemma 3.1 is the following limit in probability.

Lemma 3.2. *If $0 \leq \beta < \frac{\alpha}{\alpha+1}$ then*

$$\mathbb{P}\Big\{ \Big| N^{-\alpha(1-\beta)} \big(\sup_{v \leq t} X_v^{(N)} - X_t^{(N)} \big) \Big| \geq \varepsilon \Big\} \to 0.$$

The lemma implies that

$$\mathbb{P}\left\{\frac{X_t^{(N)}}{N^{\alpha(1-\beta)}} \geq a\right\} \sim \mathbb{P}\left\{T_{aN^{\alpha(1-\beta)}} \leq Nt\right\}. \tag{3.2}$$

For integers $z < x$, we denote

$$\ell^{(x)}(z) = \sum_{n=0}^{\infty} 1\{\mathcal{S}_n = z, \max_{j \leq n} \mathcal{S}_j < x\} \text{ and } \ell^{(\infty)}(z) = \sum_{n=0}^{\infty} 1\{\mathcal{S}_n = z\}. \tag{3.3}$$

Rearranging the family $(\eta_n \colon n \geq 0)$, according to the position of the random walks, as $(\eta_n(z) \colon n = 1, \ldots, \ell^{(x)}(z); z < x)$ we obtain

$$T_x = \sum_{z<x} \tau_z \sum_{n=1}^{\ell^{(x)}(z)} \eta_n(z).$$

In the following lemmas the expectations $\mathbb{E}$ are with respect to the full probability space, while the expectation E refers to the traps $(\tau_z \colon z < x)$, and the expectation E to the exponentials $(\eta_n \colon n \geq 0)$.

Lemma 3.3. *For any $\delta \in (0, \alpha)$, any $y = y(N)$ and*

$$x = x(N) \leq \min\{N^{\alpha-\delta}, y(N)\}$$

we have, for all $\lambda > 0$,

$$\mathbb{E}\left[\exp\left\{-\sum_{z<x} \tau_z \frac{\lambda}{N} \sum_{n=1}^{\ell^{(y)}(z)} \eta_n(z)\right\}\right]$$

$$= \mathbb{E}\exp\left\{-\Gamma(1-\alpha)\frac{\lambda^\alpha}{N^\alpha}(1+o(1))\sum_{z<x}\left(\ell^{(y)}(z)\right)^\alpha\right\} + o(1) \text{ as } N \uparrow \infty.$$

Proof: Taking first expectation with respect to all $\eta_i(z)$, we have

$$\mathsf{E}\left[\exp\left\{-\sum_{z<x} \tau_z \frac{\lambda}{N} \sum_{n=1}^{\ell^{(y)}(z)} \eta_n(z)\right\}\right] = \left[\prod_{z<x}\left(1 + \frac{\lambda\tau_z}{N}\right)^{-\ell^{(y)}(z)}\right]$$

$$= \exp\left\{-\sum_{z<x} \ell^{(y)}(z) \log\left(1 + \frac{\lambda\tau_z}{N}\right)\right\}.$$

It follows from (3.1) that $\mathbb{P}\{\ell^{(y)}(-N^{\beta+\varepsilon}) > 0\} = o(1)$ for any $\varepsilon > 0$. Furthermore, we can choose $\varepsilon > 0$ such that

$$P\left\{\max_{z \in [-N^{\beta+\varepsilon}, x]} \tau_z > N^{1-\varepsilon}\right\} \leq (x + N^{\beta+\varepsilon}) P\{\tau_0 > N^{1-\varepsilon}\} = o(1).$$

Therefore,

$$\mathbb{E}\left[\exp\left\{-\sum_{z<x} \tau_z \frac{\lambda}{N} \sum_{n=1}^{\ell^{(y)}(z)} \eta_n(z)\right\}\right]$$

$$= \mathbb{E}\left[\exp\left\{-\sum_{z<x} \ell^{(y)}(z) \log\left(1 + \frac{\lambda\tau_z}{N}\right)\right\}; \mathcal{A}\right] + o(1),$$

where

$$\mathcal{A} = \left\{\max_{z \in [-N^{\beta+\varepsilon}, x]} \tau_z \leq N^{1-\varepsilon}\right\} \cap \left\{\ell^{(y)}(-N^{\beta+\varepsilon}) = 0\right\}.$$

Next we note that $\log\left(1 + \frac{\lambda\tau_z}{N}\right) = \frac{\lambda\tau_z}{N}(1 + O(N^{-\varepsilon}))$ on the event $\{\tau_z \leq N^{1-\varepsilon}\}$. Hence

$$\mathbb{E}\left[\exp\left\{-\sum_{z<x}\tau_z\frac{\lambda}{N}\sum_{n=1}^{\ell^{(y)}(z)}\eta_n(z)\right\}\right]$$

$$= \mathbb{E}\left[\exp\left\{-\frac{\lambda}{N}(1 + O(N^{-\varepsilon}))\sum_{z<x}\tau_z\ell^{(y)}(z)\right\}; \mathcal{A}\right] + o(1).$$

By the Markov property, for $k \geq 1$,

$$\mathbb{P}\{\ell^{(\infty)}(0) = k\} = (2p^{(N)} - 1)(2 - 2p^{(N)})^{k-1} = \frac{\mu}{N^\beta}\left(1 - \frac{\mu}{N^\beta}\right)^{k-1}. \tag{3.4}$$

Using this one can easily obtain the bound

$$\mathbb{P}\{\ell^{(\infty)}(z) > N^{1-\varepsilon}\} \leq \exp\{-\mu N^{1-\beta-\varepsilon}\}.$$

This implies that

$$\mathbb{P}\left\{\max_{z\in[-N^{\beta+\varepsilon},x]}\ell^{(y)}(z) > N^{1-\varepsilon}\right\} \leq (x + N^{\beta+\varepsilon})\mathbb{P}\{\ell^{(\infty)}(z) > N^{1-\varepsilon}\} = o(1).$$

Recall from (2.1) in conjunction with XIII (5.22) of Feller (1971) that

$$Ee^{-\lambda\tau_z} = e^{-\Gamma(1-\alpha)\lambda^\alpha}(1 + o(\lambda^\alpha)) = \exp\{-\Gamma(1 - \alpha)\lambda^\alpha(1 + o(1))\}, \quad \lambda \downarrow 0.$$

Hence, on the event $\{\ell^{(y)}(z) \leq N^{1-\varepsilon}\}$,

$$E\exp\left\{-\frac{\lambda}{N}\tau_z\ell^{(y)}(z)\right\} = \exp\left\{-\Gamma(1 - \alpha)\frac{\lambda^\alpha\left(\ell^{(y)}(z)\right)^\alpha}{N^\alpha}(1 + o(1))\right\}.$$

Consequently,

$$\mathbb{E}\left[\exp\left\{-\sum_{z<x}\tau_z\frac{\lambda}{N}\sum_{n=1}^{\ell^{(y)}(z)}\eta_n(z)\right\}; \mathcal{B}\right]$$

$$= \exp\left\{-\Gamma(1 - \alpha)\frac{\lambda^\alpha}{N^\alpha}(1 + o(1))\sum_{z<x}\left(\ell^{(y)}(z)\right)^\alpha\right\} + o(1),$$

where $\mathcal{B} = \mathcal{A} \cap \{\max_{z\in[-N^{\beta+\varepsilon},x]}\ell^{(y)}(z) \leq N^{1-\varepsilon}\}$. This completes the proof. $\quad\square$

Lemma 3.4. *Let* $0 \leq \beta < \frac{\alpha}{\alpha+1}$ *and suppose* $\psi\colon [0,\infty) \to [0,\infty)$ *is increasing and satisfies*

$$\int_0^\infty \psi^4(y)e^{-\mu y}\,dy < \infty.$$

Then we have the following limit in distribution (averaged over the trap environment):

$$\lim_{\substack{N\to\infty \\ x/N^\beta\to\infty}}\frac{1}{x}\sum_{z=-\infty}^{x-1}\psi\left(\frac{\ell^{(x)}(z)}{N^\beta}\right) = \begin{cases} \int_0^\infty \psi(y)\mu e^{-\mu y}\,dy, & \text{if } \beta > 0, \\ \sum_{k=1}^\infty \mu(1 - \mu)^{k-1}\psi(k), & \text{if } \beta = 0. \end{cases}$$

Proof: We give the proof for the case $\beta > 0$ only, as the case $\beta = 0$ differs only in one minor point. For every $z < 0$ we have the inequalities

$$\mathbb{E}\psi\left(\frac{\ell^{(x)}(z)}{N^\beta}\right) \leq \mathbb{E}\psi\left(\frac{\ell^{(\infty)}(z)}{N^\beta}\right) \leq \mathbb{P}\{\min_{k\geq 1}\mathcal{S}_k \leq z\}\mathbb{E}\psi\left(\frac{\ell^{(\infty)}(0)}{N^\beta}\right).$$

Applying (3.1), we obtain, for $z > 0$,

$$\mathbb{E}\psi\left(\frac{\ell^{(x)}(z)}{N^\beta}\right) \leq \exp\left\{\mu\frac{z}{N^\beta}\right\}\mathbb{E}\psi\left(\frac{\ell^{(\infty)}(0)}{N^\beta}\right).$$

Therefore,

$$\mathbb{E} \sum_{z=-\infty}^{-1} \psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big) \leq CN^\beta \mathbb{E}\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big). \tag{3.5}$$

From (3.4) we conclude, using here that $\beta > 0$, that

$$\lim_{N \to \infty} \mathbb{E}\psi^r\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big) = \int_0^\infty \psi^r(x)\mu e^{-\mu x}\, dx, \tag{3.6}$$

for $0 < r \leq 4$. Furthermore, by (3.4), the random variable $\ell^{(\infty)}(0)$ has a geometric distribution and

$$\sup_{N \geq 1} \mathbb{E}\psi^r\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big) < \infty. \tag{3.7}$$

Combining (3.5) and (3.7), we conclude that (with a constant C not depending on x)

$$\mathbb{E} \sum_{z=-\infty}^{-1} \psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big) \leq CN^\beta. \tag{3.8}$$

It follows from this bound and the condition $xN^{-\beta} \to \infty$ that

$$\frac{1}{x} \sum_{z=-\infty}^{-1} \psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big) \Longrightarrow 0. \tag{3.9}$$

For every $z > 0$ we define

$$\sigma_z := \min\{k \geq 1 : \mathcal{S}_k = z\} \text{ and } A_z := \{\mathcal{S}_k > 0 \text{ for all } k > \sigma_z\}. \tag{3.10}$$

Then

$$\mathbb{E}\big[\psi\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\,\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big]$$
$$= \mathbb{E}\big[\psi\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\,\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big); A_z\big] + \mathbb{E}\big[\psi\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\,\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big); A_z^c\big]$$
$$=: E_1 + E_2.$$

Using the Cauchy-Schwarz inequality, we obtain

$$E_2 \leq \mathbb{P}^{1/2}(A_z^c)\mathbb{E}^{1/2}\big[\psi^2\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\psi^2\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big]$$
$$\leq \mathbb{P}^{1/2}(A_z^c)\mathbb{E}^{1/4}\big[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big]\,\mathbb{E}^{1/4}\big[\psi^4\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big]$$
$$\leq \mathbb{P}^{1/2}(A_z^c)\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big].$$

Noting that $\mathbb{P}(A_z^c) = \mathbb{P}\{\min_{k \geq 1} \mathcal{S}_k \leq -z\}$ and applying (3.1), we get

$$E_2 \leq \exp\big\{-\mu\tfrac{z}{2N^\beta}\big\}\mathbb{E}^{1/2}[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)]. \tag{3.11}$$

Since $\ell^{(x)}(0) = \ell^{(z)}(0)$ on the event A_z, using the Markov property,

$$E_1 = \mathbb{E}\big[\psi\big(\tfrac{\ell^{(z)}(0)}{N^\beta}\big)\,\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big); A_z\big]$$
$$= \mathbb{E}\big[\psi\big(\tfrac{\ell^{(z)}(0)}{N^\beta}\big)\big]\,\mathbb{E}\big[\psi\big(\tfrac{\ell^{(x-z)}(0)}{N^\beta}\big); \min_{k \geq 1} \mathcal{S}_k > -z\big]$$
$$\leq \mathbb{E}\big[\psi\big(\tfrac{\ell^{(z)}(0)}{N^\beta}\big)\big]\mathbb{E}\big[\psi\big(\tfrac{\ell^{(x-z)}(0)}{N^\beta}\big)\big]$$
$$\leq \mathbb{E}\big[\psi\big(\tfrac{\ell^{(z)}(0)}{N^\beta}\big)\big]\mathbb{E}\big[\psi\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big]. \tag{3.12}$$

Combining (3.11) and (3.12) gives

$$\mathbb{C}ov\big(\psi\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big), \psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big) \leq \exp\big\{-\mu\tfrac{z}{2N^\beta}\big\}\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big].$$

Therefore,

$$\mathbb{V}ar\Big[\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\Big] = \sum_{z=0}^{x-1}\mathbb{V}ar\big[\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big]+$$

$$2\sum_{z=0}^{x-1}\sum_{y=z+1}^{x-1}\mathbb{C}ov\big(\psi\big(\tfrac{\ell^{(x)}(y)}{N^\beta}\big),\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\big)$$

$$\leq x\mathbb{E}\big[\psi^2\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]+$$

$$2\sum_{z=0}^{x-1}\sum_{y=z+1}^{x-1}\exp\big\{-\mu\tfrac{y-z}{2N^\beta}\big\}\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big]$$

$$\leq x\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]+$$

$$x\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(x)}(0)}{N^\beta}\big)\big]\sum_{z=0}^{\infty}\exp\big\{-\mu\tfrac{z}{2N^\beta}\big\}$$

$$\leq CxN^\beta\mathbb{E}^{1/2}\big[\psi^4\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big].$$

From this bound and Chebyshev's inequality we get

$$\mathbb{P}\Big\{\Big|\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)-\mathbb{E}\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\Big|>\varepsilon x\Big\}\leq\frac{CxN^\beta\mathbb{E}^{1/2}[\psi^4\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)]}{\varepsilon^2 x^2}.$$

Applying (3.6) and (3.7), we have

$$\mathbb{P}\Big\{\Big|\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)-\mathbb{E}\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\Big|>\varepsilon x\Big\}\leq\frac{CN^\beta}{\varepsilon^2 x}. \tag{3.13}$$

We now estimate the expectation $\mathbb{E}\sum_{z=0}^{x-1}\psi(\ell^{(x)}(z)/N^\beta)$. On the one hand,

$$\mathbb{E}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)\leq\mathbb{E}\psi\big(\tfrac{\ell^{(\infty)}(z)}{N^\beta}\big)=\mathbb{E}\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big).$$

On the other hand,

$$\mathbb{E}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)=\mathbb{E}\psi\big(\tfrac{\ell^{(x-z)}(0)}{N^\beta}\big)\geq\mathbb{E}\big[\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big);A_{x-z}\big]$$

$$=\mathbb{E}\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)-\mathbb{E}\big[\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big);A_{x-z}^c\big]$$

$$\geq\mathbb{E}\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)-\mathbb{P}^{1/2}(A_{x-z}^c)\mathbb{E}^{1/2}\big[\psi^2\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big].$$

Consequently,

$$\Big|\mathbb{E}\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)-x\mathbb{E}\big[\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]\Big|\leq\mathbb{E}^{1/2}\big[\psi^2\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]\sum_{z=0}^{\infty}\mathbb{P}^{1/2}\big(A_z^c\big).$$

Applying (3.1) and (3.7), we conclude that

$$\limsup_{\substack{N\to\infty\\x/N^\beta\to\infty}}\mathbb{E}^{1/2}\big[\psi^2\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]\times\frac{1}{x}\sum_{z=0}^{\infty}\mathbb{P}^{1/2}(A_z^c)$$

$$\leq\sup_{N\geq 1}\mathbb{E}^{1/2}\big[\psi^2\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big]\times\limsup_{x/N^\beta\to\infty}\frac{1}{x}\sum_{z=0}^{\infty}\exp\{-\mu\tfrac{z}{2N^\beta}\}=0.$$

Using also (3.6), we infer that

$$\lim_{\substack{N\to\infty \\ x/N^\beta\to\infty}} \frac{1}{x}\mathbb{E}\sum_{z=0}^{x-1}\psi\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big) = \lim_{N\to\infty}\mathbb{E}\big[\psi\big(\tfrac{\ell^{(\infty)}(0)}{N^\beta}\big)\big] = \int_0^\infty \psi(y)\mu e^{-\mu y}\,dy. \qquad (3.14)$$

Combining (3.9), (3.13) and (3.14) finishes the proof of the lemma. □

We now start with the proof of Theorem 2.1 (a), first considering one-dimensional marginals. Using Lemma 3.3 for $x = y = aN^{\alpha(1-\beta)}$ we have

$$\mathbb{E}\exp\big\{-\tfrac{\lambda}{N}T_{aN^{\alpha(1-\beta)}}\big\} = \mathbb{E}\exp\big\{-\Gamma(1-\alpha)\tfrac{\lambda^\alpha}{N^{\alpha(1-\beta)}}(1+o(1))\sum_{z<x}\big(\tfrac{\ell^{(x)}(z)}{N^\beta}\big)^\alpha\big\}.$$

We therefore conclude from Lemma 3.4, taking $\psi(x) = x^\alpha$,

$$\lim_{N\uparrow\infty}\mathbb{E}\exp\big\{-\tfrac{\lambda}{N}T_{aN^{\alpha(1-\beta)}}\big\} = \exp\big\{-a\,\lambda^\alpha\,\Gamma(1-\alpha)\int_0^\infty u^\alpha\,\mu e^{-\mu u}\,du\big\}.$$

Finally, note that

$$\int_0^\infty u^\alpha\,\mu e^{-\mu u}\,du = \Gamma(1+\alpha)\mu^{-\alpha}\,.$$

Hence,

$$\lim_{N\uparrow\infty}\mathbb{E}\exp\big\{-\tfrac{\lambda}{N}T_{a\mu^\alpha N^{\alpha(1-\beta)}/\Gamma(1+\alpha)}\big\} = \exp\big\{-a\,\lambda^\alpha\,\Gamma(1-\alpha)\big\} \qquad (3.15)$$

as required, in the light of (3.2), to complete the convergence of one-dimensional marginals.

Next, we show convergence of finite-dimensional distributions. It is easy to see that, for all $0 \le x < y$,

$$T_y - T_x = \sum_{z=x}^{y-1}\tau_z\sum_{j=1}^{\ell^{(y)}(z)}\eta_j(z) + \sum_{z<x}\tau_z\sum_{j=\ell^{(x)}(z)+1}^{\ell^{(y)}(z)}\eta_j(z).$$

Using Lemma 3.3 and inequality (3.8) we see that

$$\mathbb{E}\exp\big\{-\frac{\lambda}{N}\sum_{z<x}\tau_z\sum_{j=\ell^{(x)}(z)+1}^{\ell^{(y)}(z)}\eta_j(z)\big\} = \mathbb{E}\exp\big\{-\frac{\lambda}{N}\sum_{z<0}\tau_z\sum_{j=1}^{\ell^{(y-x)}(z)}\eta_j(z)\big\}$$

$$= \mathbb{E}\exp\big\{-\frac{\lambda^\alpha\Gamma(1-\alpha)(1+o(1))}{N^{\alpha(1-\beta)}}$$

$$\sum_{z<0}\big(\frac{\ell^{(y-x)}(z)}{N^\beta}\big)^\alpha\big\}$$

$$\longrightarrow 1, \qquad \text{as } N\uparrow\infty,$$

for every $\lambda > 0$ (recall $\beta < \alpha/(\alpha+1)$). Therefore,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{z<x}\tau_z\sum_{j=\ell^{(x)}(z)+1}^{\ell^{(y)}(z)}\eta_j(z) = 0 \quad \text{in probability.}$$

Since T_x and $\sum_{z=x}^{y-1}\tau_z\sum_{j=1}^{\ell^{(y)}(z)}\eta_j(z)$ are independent, we therefore conclude that T_x/N and $(T_y - T_x)/N$ are asymptotically independent.

Now fix $0 \leq t_1 < \cdots < t_k$ and $0 < a_1 < \cdots < a_k$. By the argument above one can easily show that $T_{a_1}/N, (T_{a_2} - T_{a_1})/N, \ldots, (T_{a_k} - T_{a_{k-1}})/N$ are asymptotically independent. Noting also that, for any $0 < a < b$,

$$\frac{1}{N}(T_{bN^{\alpha(1-\beta)}} - T_{aN^{\alpha(1-\beta)}}) \to \mathsf{Sub}_b - \mathsf{Sub}_a$$

we obtain the convergence of the finite-dimensional distributions,

$$\left(\frac{1}{N}T_{a_1 N^{\alpha(1-\beta)}}, \ldots, \frac{1}{N}T_{a_k N^{\alpha(1-\beta)}}\right) \longrightarrow (\mathsf{Sub}_{a_1}, \ldots, \mathsf{Sub}_{a_k}).$$

Finally, to prove the tightness of $X^{(N)}$ in the Skorokhod space $D[0,1]$, we first note that the convergence of one-dimensional distributions and Lemma 3.1 imply that, for any $\delta > 0$ and $\varepsilon > 0$, for some constant $c = c(\alpha, \mu)$,

$$\lim_{N \to \infty} \mathbb{P}\left\{\sup_{t \leq \delta}|X_t^{(N)} - X_0^{(N)}| \geq \varepsilon\right\} = \mathbb{P}\left\{\mathsf{Sub}_\varepsilon \leq c\delta\right\}.$$

Then, since the increments of $X^{(N)}$ are homogeneous in time,

$$\limsup_{N \to \infty} \mathbb{P}\left\{\max_{0 \leq i \leq \delta^{-1}} \sup_{t \leq \delta}|X_{i\delta+t}^{(N)} - X_{i\delta}^{(N)}| \geq \varepsilon\right\} \leq (1 + \delta^{-1})\,\mathbb{P}\left\{\mathsf{Sub}_\varepsilon \leq c\delta\right\}.$$

Let

$$\omega'(f, \delta) = \inf_{\substack{0 = t_0 < t_1 < \cdots < t_v = 1 \\ t_i - t_{i-1} > \delta}} \sup_{\substack{1 \leq i \leq v \\ t_{i-1} \leq s, t < t_i}} |f(s) - f(t)|$$

denote the standard continuity modulus in the Skorokhod space $D[0,1]$. It is easy to see that

$$\omega'(f, \delta) \leq 2 \max_{0 \leq i \leq \delta^{-1}} \sup_{t \leq \delta}|f(i\delta + t) - f(i\delta)|\,.$$

Therefore,

$$\limsup_{N \to \infty} \mathbb{P}\left\{\omega'(X^{(N)}, \delta) \geq \varepsilon\right\} \leq (1 + \delta^{-1})\mathbb{P}\left\{\mathsf{Sub}_{\varepsilon/2} \leq c\delta\right\}.$$

Noting that all negative moments of Sub are finite, we conclude that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \mathbb{P}\left\{\omega'(X^{(N)}, \delta) \geq \varepsilon\right\} = 0.$$

This, according to Theorem 13.2 in Billingsley (1999), ensures the convergence in the path space $D[0,1]$.

4. Proof of Theorem 2.1(b)

The idea is to represent the processes $X^{(N)}$ as time and scale change of a Brownian motion $(B(t)\colon t \geq 0)$ with drift μ. A result of Stone (1963) allows us to infer convergence of $X^{(N)}$ from convergence of the parameters in this representation.

We now recall the results of Stone (1963) showing how to represent random walks as time and scale-changed Brownian motions. Let

$$\nu = \sum_{i \in \mathbb{Z}} w_i \delta_{y_i}$$

be an atomic measure called the *speed measure* with atoms $\{y_i \colon i \in \mathbb{Z}\}$ indexed in increasing order. Let S be a strictly increasing function on this set of atoms, which

is called the *scale function*. Let $(\ell(t,x)\colon x \in \mathbb{R}, t \geq 0)$ be the local time field of the Brownian motion. Define

$$\phi[\nu, S](t) := \phi(t) := \int \ell(t, S(x))\, \nu(dx)$$

and

$$\psi[\nu, S](t) := \psi(t) := \inf\{s > 0\colon \phi(s) > t\}.$$

We define the process $(Y[\nu, S](t)\colon t \geq 0)$ by

$$Y[\nu, S](t) := Y(t) := S^{-1}\big(B(\psi(t))\big).$$

Lemma 4.1. *Define* $u\colon \mathbb{R} \to \mathbb{R}$ *as*

$$u(x) = \begin{cases} \frac{1 - e^{-2\mu x}}{2\mu} & \text{if } \mu > 0, \\ x & \text{if } \mu = 0. \end{cases}$$

Then $(Y(t)\colon t \geq 0)$ *is a nearest-neighbour random walk on* $\{y_i\colon i \in \mathbb{Z}\}$. *The waiting time in the state* y_i *is exponentially distributed with mean*

$$2w_i \frac{(u(S(y_{i+1})) - u(S(y_i)))(u(S(y_i)) - u(S(y_{i-1})))}{u(S(y_{i+1})) - u(S(y_{i-1}))}$$

and after leaving state y_i *the process jumps to state* y_{i-1} *and* y_{i+1} *with respective probabilities*

$$\frac{u(S(y_{i+1})) - u(S(y_i))}{u(S(y_{i+1})) - u(S(y_{i-1}))} \quad \text{and} \quad \frac{u(S(y_i)) - u(S(y_{i-1}))}{u(S(y_{i+1})) - u(S(y_{i-1}))}.$$

Proof: For the case of a driftless Brownian motion this construction is carried out in Section 3 of Stone (1963), see also Proposition 3.6 in Ben Arous and Černý (2006). In order to extend this to the case of a Brownian motion with drift, one has to compute the exit probabilities, see Formula 3.0.4, Page 309 in Borodin and Salminen (1996), and the expected local time at the origin, see Formula 3.3.1 on Page 310 in Borodin and Salminen (1996), of a Brownian motion with drift μ, which is started at the origin and killed upon leaving the interval $(-a, b)$, for $a, b > 0$. □

Let

$$h^{(N)} = \frac{1}{2\mu} \log\left(\frac{1 + \mu N^{-\beta}}{1 - \mu N^{-\beta}}\right),$$

and define the speed measure

$$\nu^{(N)} = \frac{1}{N^{\frac{1}{\alpha+1}}} \sum_{i \in \mathbb{Z}} \tau_i\, \delta_{ih^{(N)}}$$

and the identity as scale function. By Lemma 4.1 the corresponding process $(Y^{(N)}(t)\colon t \geq 0)$ is a nearest-neighbour random walk on $h^{(N)}\mathbb{Z}$ which moves to the left with probability

$$\frac{1 - e^{-2\mu h^{(N)}}}{e^{2\mu h^{(N)}} - e^{-2\mu h^{(N)}}} = q^{(N)}.$$

Furthermore, the waiting time in $ih^{(N)}$ is exponentially distributed with mean

$$2\tau_i \frac{1}{N^{\frac{1}{\alpha+1}}} \frac{1}{2\mu} \frac{(e^{2\mu h^{(N)}} - 1)(1 - e^{-2\mu h^{(N)}})}{e^{2\mu h^{(N)}} - e^{-2\mu h^{(N)}}} = \tau_i N^{-1}$$

(recalling $\beta = \alpha/(\alpha+1)$). Hence, we have shown that the distributions of $h^{(N)} X_t^{(N)}$ and $Y^{(N)}(t)$ are equal. Noting that

$$h^{(N)} \sim N^{-\beta} \quad \text{as } N \to \infty,$$

one can easily verify that $\nu^{(N)} \to \rho$ vaguely in distribution. At this point, one can not yet apply Stone's Theorem since it refers to deterministic speed measures. However, the above convergence can be made almost sure on a suitably defined probability space, see Section 3.2.3 in Ben Arous and Černý (2006). By Theorem 1 in Stone (1963), we then obtain

$$N^{-\beta} X^{(N)} \sim h^{(N)} X^{(N)} \stackrel{d}{=} Y[\nu^{(N)}, \mathrm{id}] \implies X,$$

where X is a diffusion with speed measure ρ. This is a Fontes-Isopi-Newman diffusion with drift μ, completing the proof of the part (b).

5. Proof of Theorem 2.1(c)

Here we represent $X^{(N)}$ as a time-scale change of the driftless Brownian motion. The proof repeats mainly the corresponding proof in Fontes et al. (2002) and Ben Arous and Černý (2006). We explain the needed changes only. Define

$$S^{(N)}(x) = N^{-\frac{\alpha}{\alpha+1}} \frac{1 - e^{-\mu^{(N)} N^{\frac{\alpha}{\alpha+1}} x}}{\mu^{(N)}},$$

where

$$\mu^{(N)} = \log\left(\frac{1 + \mu N^{-\beta}}{1 - \mu N^{-\beta}}\right).$$

Furthermore, define the speed measure

$$\nu^{(N)} = \frac{c^{(N)}}{N^{\frac{1}{\alpha+1}}} \sum_{i \in \mathbb{Z}} \tau_i \, \delta_{y_i},$$

where $y_i = N^{-\frac{\alpha}{\alpha+1}} i$ and

$$c^{(N)} = \frac{\mu^{(N)}}{2\mu N^{-\beta}} \to 1.$$

By Lemma 4.1 the process $Y[\nu^{(N)}, S^{(N)}]$ is a random walk on $N^{-\frac{\alpha}{\alpha+1}}\mathbb{Z}$ with transition probabilities

$$\frac{S^{(N)}(y_{i+1}) - S^{(N)}(y_i)}{S^{(N)}(y_{i+1}) - S^{(N)}(y_{i-1})} = q^{(N)} \quad \text{and} \quad \frac{S^{(N)}(y_i) - S^{(N)}(y_{i-1})}{S^{(N)}(y_{i+1}) - S^{(N)}(y_{i-1})} = p^{(N)}$$

and the waiting time at y_i is exponentially distributed with mean

$$2\frac{\tau_i c^{(N)}}{N^{\frac{1}{\alpha+1}}} \frac{(S^{(N)}(y_{i+1}) - S^{(N)}(y_i))(S^{(N)}(y_i) - S^{(N)}(y_{i-1}))}{S^{(N)}(y_{i+1}) - S^{(N)}(y_{i-1})} = \frac{\tau_i}{N}.$$

Consequently, $N^{-\frac{\alpha}{\alpha+1}} X^{(N)} \stackrel{d}{=} Y[\nu^{(N)}, S^{(N)}]$. Since $\beta > \frac{\alpha}{\alpha+1}$ and $\mu^{(N)} \sim N^{-\beta}$, we have $S^{(N)}(x) \to x$ uniformly on compact subsets of $\mathbb{R}$. Moreover, $\nu^{(N)} \to \rho$ vaguely in distribution, so that the result follows from Theorem 1 in Stone (1963).

6. Proof of Theorem 2.2

The main task here is to study limits of

$$\mathbb{P}\big\{\tau_{X_t^{(N)}} > vN^{1-\beta}\big\} \qquad \text{for } v > 0.$$

Our strategy is to look at the deep traps $z \in \mathbb{Z}$ defined by $\tau_z > vN^{1-\beta}$ and at the 'travelling intervals' defined as the times which the process X spends travelling from one deep trap to the next one to its right. We show that during the intermediate intervals, which separate the travelling intervals, the process spends most of its time in a deep trap. The length of travelling intervals and intermediate intervals are both of order N and we determine the asymptotic distribution of their lengths, which enables us to find the limit above. The further statements in Theorem 2.2 follow easily from this.

Define the sequence of deep traps (with $x_0 = 0$), as

$$x_j = \min\{z > x_{j-1} : \tau_z > vN^{1-\beta}\} \text{ for } j \geq 1.$$

The following lemma reveals the typical distance of two successive traps.

Lemma 6.1. *For any $u > 0$ and $j \geq 0$, we have*

$$\lim_{N \to \infty} P\big\{x_{j+1} - x_j \geq uN^{\alpha(1-\beta)}\big\} = e^{-u/v^\alpha}.$$

Proof: Using the tail behaviour of the random variables τ_z given by (2.1), we have for any $r > 0$,

$$P\{x_{j+1} - x_j \geq r\} = \left(1 - \left(\frac{1+o(1)}{vN^{1-\beta}}\right)^\alpha\right)^r,$$

from which the result follows by Euler's formula. $\qquad \square$

Next, we investigate the time spent in a deep trap before the next deep trap is hit for the first time.

Lemma 6.2. *For all $j \geq 0$,*

$$\frac{\ell^{(x_{j+1})}(x_j)}{N^\beta} \Longrightarrow \xi,$$

where ξ is exponentially distributed with mean $1/\mu$ if $\beta > 0$, and geometrically distributed with mean $1/\mu$ if $\beta = 0$.

Proof: Recall (3.10) and let

$$A_{x,y} = \big\{\mathcal{S}_k > x \text{ for all } k \geq \sigma_y\big\} \text{ for } x < y.$$

Keeping $x_1, x_2, \dots$ fixed by conditioning on the trap environment, we have

$$\frac{\ell^{(\infty)}(x_j)}{N^\beta} \geq \frac{\ell^{(x_{j+1})}(x_j)}{N^\beta} \geq \frac{\ell^{(\infty)}(x_j)}{N^\beta} \mathbf{1}_{A_{x_j,x_{j+1}}}.$$

Observe that $\ell^{(\infty)}(x_j)$ is geometrically distributed with success parameter μ/N^β. Therefore

$$\frac{\ell^{(\infty)}(x_j)}{N^\beta} \Longrightarrow \xi,$$

where ξ is exponentially distributed with parameter μ. It therefore suffices to show that

$$\frac{\ell^{(\infty)}(x_j)}{N^\beta} \mathbf{1}_{A_{x_j,x_{j+1}}^c}$$

converges weakly to zero. As the first factor converges, it further suffices to show that $\mathbb{P}(A^c_{x_j,x_{j+1}})$ converges to zero. By (3.1) we have

$$\mathbb{P}\big(A^c_{x_j,x_{j+1}}\big) \le \mathbb{E}\exp\Big\{ -\tfrac{\mu(x_{j+1}-x_j)}{N^\beta} \Big\}.$$

Averaging over the trap environment and using Lemma 6.1 together with the fact that $\alpha(1-\beta) > \beta$, we see that the right hand side converges to zero. □

Lemma 6.3. *For τ as in* (2.1), *we have as $y \uparrow \infty$ and $\lambda/y \downarrow 0$,*

$$E\big[e^{-\frac{\lambda}{y}\tau} \mid \tau \le y\big] = 1 + y^{-\alpha}\Big(1 - \Gamma(1-\alpha)\lambda^\alpha - \int_1^\infty e^{-\lambda z}\tfrac{\alpha}{z^{\alpha+1}}\,dz\Big) + o\big((\lambda+1)^\alpha y^{-\alpha}\big).$$

Proof: We have $P\{\tau \le y\} = 1 - y^{-\alpha} + o(y^{-\alpha})$, and hence

$$\frac{1}{P\{\tau \le y\}} = 1 + y^{-\alpha} + o(y^{-\alpha}).$$

Moreover, using integration by parts,

$$E\big[e^{-\frac{\lambda}{y}\tau}; \tau > y\big] = \int_y^\infty e^{-\frac{\lambda}{y}x}\,P\{\tau \in dx\}$$

$$= e^{-\lambda}P\{\tau > y\} - \tfrac{\lambda}{y}\int_y^\infty P\{\tau > x\}e^{-\frac{\lambda}{y}x}\,dx$$

$$= e^{-\lambda}P\{\tau > y\} - \lambda\int_1^\infty P\{\tau > yz\}e^{-\lambda z}\,dz.$$

As, for all $z > 1$,

$$\frac{P\{\tau > yz\}}{P\{\tau > y\}} \longrightarrow z^{-\alpha},$$

we obtain, by dominated convergence,

$$\frac{\int_1^\infty P\{\tau > yz\}e^{-\lambda z}\,dz}{P\{\tau > y\}} \longrightarrow \int_1^\infty e^{-\lambda z}z^{-\alpha}\,dz.$$

Therefore

$$E\big[e^{-\frac{\lambda}{y}\tau}; \tau > y\big] = P\{\tau > y\}\Big[e^{-\lambda} - \lambda\int_1^\infty e^{-\lambda z}z^{-\alpha}\,dz\Big](1+o(1))$$

$$= y^{-\alpha}(1+o(1))\int_1^\infty e^{-\lambda z}\tfrac{\alpha}{z^{\alpha+1}}\,dz.$$

Recalling that $Ee^{-\frac{\lambda}{y}\tau} = 1 - \Gamma(1-\alpha)(\tfrac{\lambda}{y})^\alpha + o(\lambda^\alpha y^{-\alpha})$ and summarising,

$$E\big[e^{-\frac{\lambda}{y}\tau} \mid \tau \le y\big]$$

$$= \frac{1}{P\{\tau \le y\}}\Big(Ee^{-\frac{\lambda}{y}\tau} - E\big[e^{-\frac{\lambda}{y}\tau}; \tau > y\big]\Big)$$

$$= 1 + y^{-\alpha}\Big(1 - \Gamma(1-\alpha)\lambda^\alpha - \int_1^\infty e^{-\lambda z}\tfrac{\alpha}{z^{\alpha+1}}\,dz\Big) + o\big((\lambda+1)^\alpha y^{-\alpha}\big).$$

This completes the proof.   □

We now define quantities, which will be shown to converge in distribution to the families $U_1, U_2, \dots$ and $S_1, S_2, \dots$ described in Remark 2.3. Fix $j \ge 1$ and let $\mathcal{S}^{(j)} = (\mathcal{S}_n^{(j)}: n = 0, \dots, \zeta^{(j)})$ be the embedded random walk started from the first

hitting of x_{j-1} and stopped upon hitting x_j, such that $\mathcal{S}_0^{(j)} = x_{j-1}$ and $\mathcal{S}_{\zeta^{(j)}}^{(j)} = x_j$. Let $\ell_j(x)$ be the local time in x of the embedded random walk and let

$$0 = n_1 < n_2 < \cdots < n_m < \zeta^{(j)}$$

with $m = \ell_j(x_{j-1})$ be the complete list of visits to x_{j-1} by $\mathcal{S}^{(j)}$. Define

$$U_j^{(N)} = \sum_{i=n_m+1}^{\zeta^{(j)}-1} \tau_{\mathcal{S}_i^{(j)}} \eta_i\big(\mathcal{S}_i^{(j)}\big)$$

and

$$S_{j-1}^{(N)} = \tau_{x_{j-1}} \sum_{i=1}^{\ell_j(x_{j-1})} \eta_{n_i}(x_{j-1}),$$

where $(\eta_i(x) : i \in \mathbb{N}, x \in \mathbb{Z})$ is a family of independent standard exponential variables, independent of everything else. Observe that, roughly speaking, $U_j^{(N)}/N$ is the time the process $X^{(N)}$ requires to travel from x_{j-1} to x_j and $S_{j-1}^{(N)}/N$ the time spent in the trap x_{j-1} before the first hit of x_j.

It is important to note that $U_j^{(N)}$ is independent of $U_1^{(N)}, \ldots, U_{j-1}^{(N)}$ and of $S_1^{(N)}, \ldots, S_j^{(N)}$, and also $S_j^{(N)}$ is independent of $S_1^{(N)}, \ldots, S_{j-1}^{(N)}$.

For $1 \le l \le \ell_j(x_{j-1}) - 1$ we define

$$Q_{j,l}^{(N)} = \sum_{n=n_l+1}^{n_{l+1}-1} \tau_{\mathcal{S}_n^{(j)}} \eta_n\big(\mathcal{S}_n^{(j)}\big),$$

such that $Q_{j,l}^{(N)}/N$ is the time spent by $X^{(N)}$ in the l-th excursion from x_{j-1} to x_{j-1} before reaching x_j. Further define the set

$$\mathcal{R}_j^{(N)} = \bigcup_{i=1}^{\ell_j(x_{j-1})-1} \left(\tau_{x_{j-1}} \sum_{l=1}^{i} \eta_{n_l}(x_{j-1}) + \sum_{l=1}^{i-1} Q_{j,l}^{(N)}, \tau_{x_{j-1}} \sum_{l=1}^{i} \eta_{n_l}(x_{j-1}) + \sum_{l=1}^{i} Q_{j,l}^{(N)} \right).$$

The set $\frac{1}{N}\mathcal{R}_j^{(N)}$ is the (random) set of times which $X^{(N)}$ spends in excursions from x_{j-1} (either to the left or to the right) which return to x_{j-1}. We first show that the time spent in these excursions is negligible.

Lemma 6.4. *Let* $R_j^{(N)} = |\mathcal{R}_j^{(N)}|$. *Then*

(a) $\dfrac{R_j^{(N)}}{N} \Rightarrow 0$ *as* $N \uparrow \infty$;

(b) *for every* $t > 0$, *we have* $\lim_{N\uparrow\infty} \mathbb{P}\big\{Nt \in \mathcal{R}_j^{(N)}\big\} = 0$.

Proof: If (b) holds, then $\mathbb{E}R_j^{(N)} = \int_0^N \mathbb{P}\{t \in \mathcal{R}_j^{(N)}\}\,dt = o(N)$, hence (a) is an immediate consequence of (b).

It remains to show (b). By Lemma 6.1 the distance between x_{j-2} and x_{j-1} is of order $N^{\alpha(1-\beta)}$. Then, using (3.1), we conclude that the probability that X hits x_{j-2} after hitting x_{j-1} converges to zero as $N \to \infty$. Consequently,

$$\mathbb{P}\big\{t \in \mathcal{R}_j^{(N)}\big\} = \mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) + o(1),$$

where $\mathcal{A}_j = \{X_t^{(N)} > x_{j-2} \text{ for all } t > U_{x_{j-1}}\}$. It follows from the definition of $Q_{j,l}^{(N)}$, that, conditioned on $(\tau_z \colon z \in \mathbb{Z}^d)$,

$$\mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) =$$

$$\mathbb{E} \sum_{k=0}^{\ell_j(x_{j-1})-1} \mathsf{P}\Big(\Big\{\sum_{i=1}^{k} U_i + \tau_{x_{j-1}}\eta_{k+1}(x_{j-1}) < Nt < \sum_{i=1}^{k+1} D_i\Big\} \cap \mathcal{A}_j\Big),$$

where

$$D_i = \tau_{x_{j-1}}\eta_i(x_{j-1}) + Q_{j,i}^{(N)}.$$

Let $\mathcal{B}_{j,i}^{(N)}$ denote the event that the corresponding excursion does not hit neither x_{j-2} nor x_j. Furthermore, denote

$$\widetilde{Q}_{j,i}^{(N)} = Q_{j,i}^{(N)} 1_{\mathcal{B}_{j,i}^{(N)}} \quad \text{and} \quad V_i = \tau_{x_{j-1}}\eta_i(x_{j-1}) + \widetilde{Q}_{j,i}^{(N)}.$$

Then (dropping the subindex $k-1$ when it is convenient)

$$\mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) \leq \sum_{k=0}^{\infty} \mathbb{P}\Big\{\sum_{i=1}^{k} V_i + \tau_{x_{j-1}}\eta_{k+1}(x_{j-1}) < Nt < \sum_{i=1}^{k+1} V_i\Big\}$$

$$= \sum_{k=0}^{\infty} \mathbb{P}\Big\{\sum_{i=1}^{k} V_i \in \big(Nt - \tau_{x_{j-1}}\eta_{k+1}(x_{j-1}) - \widetilde{Q}_{j,k+1}^{(N)}, \ Nt - \tau_{x_{j-1}}\eta_{k+1}(x_{j-1})\big)\Big\}$$

$$= \mathbb{E}\Big[H\big(Nt - \tau_{x_{j-1}}\eta(x_{j-1})\big) - H\big(Nt - \tau_{x_{j-1}}\eta(x_{j-1}) - \widetilde{Q}_j^{(N)}\big)\Big]$$

$$+ \mathbb{P}\Big\{Nt - \tau_{x_{j-1}}\eta(x_{j-1}) - \widetilde{Q}_j^{(N)} < 0 < Nt - \tau_{x_{j-1}}\eta(x_{j-1})\Big\},$$

where $H(x)$ denotes the renewal function corresponding to the sequence $(V_i \colon i = 0, 1, \dots)$. This renewal function satisfies the inequality

$$H(x+y) - H(x) \leq \min\Big\{1, \frac{y}{\tau_{x_{j-1}}}\Big\}\Big(1 + \frac{y}{\tau_{x_{j-1}}}\Big) \leq 2\frac{y}{\tau_{x_{j-1}}}, \quad x, y > 0. \quad (6.1)$$

(We postpone the derivation of this inequality to the end of the proof.) Using this bound we get

$$\mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) \leq 2\mathbb{E}\big[\tfrac{\widetilde{Q}_j^{(N)}}{\tau_{x_{j-1}}}\big] + \mathbb{P}\Big\{\eta(x_{j-1}) \in \big(\tfrac{z-\widetilde{Q}_j^{(N)}}{\tau_{x_{j-1}}}, \tfrac{z}{\tau_{x_{j-1}}}\big)\Big\}.$$

Noting that

$$\mathbb{P}\Big\{\eta(x_{j-1}) \in \big(\tfrac{z-\widetilde{Q}_j^{(N)}}{\tau_{x_{j-1}}}, \tfrac{z}{\tau_{x_{j-1}}}\big)\Big\} \leq \mathbb{E}\big[\tfrac{\widetilde{Q}_j^{(N)}}{\tau_{x_{j-1}}}\big],$$

we have

$$\mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) \leq 3\mathbb{E}\big[\tfrac{\widetilde{Q}_j^{(N)}}{\tau_{x_{j-1}}}\big].$$

Recalling also that $\tau_{x_{j-1}} \geq vN^{1-\beta}$, we arrive finally at the bound

$$\mathbb{P}\big(\{t \in \mathcal{R}_j^{(N)}\} \cap \mathcal{A}_j\big) \leq \frac{3}{vN^{1-\beta}} \mathbb{E}\widetilde{Q}_j^{(N)}.$$

Going back to the unconditioned probability, we have, uniformly in t,

$$\mathbb{P}\{t \in \mathcal{R}_j^{(N)}\} \leq \frac{3}{vN^{1-\beta}} \mathbb{E}\widetilde{Q}_j^{(N)} + o(1).$$

It follows from (2.1) that $E[\tau_z \,|\, \tau_z \le vN^{1-\beta}] \le CN^{(1-\alpha)(1-\beta)}$. Then,

$$\mathbb{E}\widetilde{Q}_j^{(N)} \le CN^{(1-\alpha)(1-\beta)}\mathbb{E}[L-1; L < \infty],$$

where L is the length of an excursion of the embedded random walk.

As $\mathbb{P}\{L = k\} = 0$, if k is odd, and, with $p = p^{(N)}$, (see e.g. III.9 in Feller, 1968)

$$\mathbb{P}\{L = k\} = \frac{2}{j}\binom{2j-2}{j-1}p^j(1-p)^j,$$

if $k = 2j$ is even, we obtain

$$\sum_{k=1}^{\infty}(k-1)\,\mathbb{P}\{L = k\} = 2\sum_{j=1}^{\infty}\frac{2j-1}{j}\binom{2j-2}{j-1}p^j(1-p)^j$$

$$\le \sum_{j=0}^{\infty}\binom{2j}{j}\left(p(1-p)\right)^j = \mathbb{E}[\ell^{(\infty)}(0)] = \frac{1}{2p-1}.$$

Note that $2p - 1 = \frac{\mu}{N^\beta}$. Thus, $\mathbb{E}[L-1; L < \infty] \le CN^\beta$. Hence, we have

$$\mathbb{E}\widetilde{Q}_j^{(N)} \le CN^{\beta+(1-\alpha)(1-\beta)} = o(N^{1-\beta}).$$

Therefore, it remains to prove (6.1).

Let θ_x denote the first time when the random walk $\sum_{i=1}^n V_i$ leaves the interval $(-\infty, x)$, that is,

$$\theta_x := \min\{n \ge 1 : \sum_{i=1}^n V_i \ge x\}.$$

Then, for every $y > 0$,

$$\mathbb{P}\left\{\sum_{i=1}^{\theta_x} V_i \le x + y\right\} = \int_0^x \mathbb{P}\left\{\sum_{i=1}^{\theta_x - 1} V_i \in du\right\}\mathbb{P}\{V_1 \in (x-u, x+y-u)\} \le y \sup_{z>0} f(z),$$

where f is the density of V_1. This function is the convolution of densities of $\tau_{x_{j-1}}\eta$ and $Q_j^{(N)}$. Then $\sup_{z>0} f(z)$ does not exceed the maximal value of the density of $\tau_{x_{j-1}}\eta$, which is equal to $1/\tau_{x_{j-1}}$. Consequently,

$$\mathbb{P}\left\{\sum_{i=1}^{\theta_x} V_i \le x + y\right\} \le \min\left\{1, \frac{y}{\tau_{x_{j-1}}}\right\}. \tag{6.2}$$

Using the Markov property, we obtain

$$H(x+y) - H(x) = \mathbb{E}\left[1\left\{\sum_{i=1}^{\theta_x} V_i \le x + y\right\}\right.$$

$$\left.\left(1 + \sum_{k=1}^{\infty}1\left\{\sum_{i=\theta_x+1}^{\theta_x+k} V_i \le x + y - \sum_{i=1}^{\theta_x} V_i\right\}\right)\right]$$

$$= \mathbb{E}\left[1\left\{\sum_{i=1}^{\theta_x} V_i \le x + y\right\}H\left(x + y - \sum_{i=1}^{\theta_x} V_i\right)\right]$$

$$\le H(y)\mathbb{P}\left\{\sum_{i=1}^{\theta_x} V_i \le x + y\right\}.$$

We now note that from the definition of V_1 follows the inequality

$$H(y) \leq 1 + \sum_{k=1}^{\infty} \mathbb{P}\Big\{ \sum_{i=1}^{k} \eta_i \leq y\tau_{x_{j-1}} \Big\} \leq 1 + \frac{y}{\tau_{x_{j-1}}}.$$

Combining this with (6.2), we arrive at (6.1). This completes the proof of the lemma.   □

Lemma 6.5. *For $N \uparrow \infty$,*

$$\frac{S_j^{(N)}}{vN} \Longrightarrow S_j,$$

where S_j is the product of two independent random variables, a Pareto variable with index α, and an exponential variable with mean $1/\mu$.

Proof: First note that, for $y \geq v$ and $x \in \mathbb{Z}$,

$$\lim_{N \uparrow \infty} P\big\{ \tau_x > yN^{1-\beta} \,\big|\, \tau_x > vN^{1-\beta} \big\} = \Big(\frac{v}{y} \Big)^{\alpha}.$$

Therefore we have, for all $j \in \mathbb{N}$,

$$\lim_{N \uparrow \infty} P\big\{ \tau_{x_j} > yN^{1-\beta} \big\} = \Big(\frac{v}{y} \Big)^{\alpha} \mathbf{1}\{y \geq v\}. \tag{6.3}$$

We write

$$\frac{S_j^{(N)}}{vN} = \Big(\frac{\tau_{x_j}}{vN^{1-\beta}} \Big) \Big(\frac{\ell_{j+1}(x_j)}{N^{\beta}} \Big) \Big(\frac{1}{\ell_{j+1}(x_j)} \sum_{i=1}^{\ell_{j+1}(x_j)} \eta_i(x_j) \Big)$$

and observe convergence of all three factors on the right hand side.

Indeed, the first factor converges in distribution to a Pareto law, by (6.3), and the second factor to an exponential law with mean $1/\mu$, by Lemma 6.2. Moreover, the second factor is independent of the first. To understand the third factor, recall from the discussion of the second factor that $\ell_{j+1}(x_j)$ converges to infinity in probability. Thus, by the weak law of large numbers, the third factor converges to one in probability. Hence, the product $S_j^{(N)}/vN$ converges to the product of an independent Pareto and exponential law.   □

Lemma 6.6. *For $N \uparrow \infty$,*

$$\frac{U_j^{(N)}}{vN} \Longrightarrow U_j,$$

where U_j is a random variable with

$$\mathbb{P}\{U_j > x\} \sim c\,\frac{\mu^{\alpha}}{x^{\alpha}} \qquad \text{as } x \uparrow \infty,$$

for some $c > 0$ depending only on α.

Proof: Recall that

$$U_j^{(N)} = \sum_{i=n_k+1}^{\zeta^{(j)}-1} \tau_{\mathcal{S}_i^{(j)}}\, \eta_i\big(\mathcal{S}_i^{(j)}\big).$$

Conditional on x_{j-1}, x_j the random variables τ_x, $x = x_{j-1}+1, \ldots, x_j - 1$, are still independent with τ_x conditioned to satisfy $\tau_x \leq vN^{1-\beta}$.

We first consider the case $\beta > 0$. Writing Laplace transforms

$$\mathbb{E} \exp\{ - \lambda \tfrac{U_j^{(N)}}{vN} \}$$

$$\sim \mathbb{E}E\Big[\prod_{x=x_{j-1}+1}^{x_j-1} \mathsf{E} \exp\{ - \tfrac{\lambda}{vN} \tau_x \sum_{i=1}^{\ell_j(x)} \eta_i(x) \} \Big]$$

$$= \mathbb{E}\Big[\prod_{x=x_{j-1}+1}^{x_j-1} E\big[\big(1 + \tfrac{\lambda \tau_x}{vN}\big)^{-\ell_j(x)} \mid \tau_x \le vN^{1-\beta} \big] \Big]$$

$$= \mathbb{E} \prod_{x=x_{j-1}+1}^{x_j-1} E\Big[\exp\{ - \tfrac{\lambda}{vN} \tau_x \ell_j(x) \big(1 + O(N^{-\beta})\big) \} \,\Big|\, \tau_x \le vN^{1-\beta} \Big],$$

where in the last step we have used that, on the event $\{\tau_x \le vN^{1-\beta}\}$,

$$\big(1 + \tfrac{\lambda \tau_x}{vN}\big)^{-\ell_j(x)} = \exp\{ - \ell_j(x) \log\big(1 + \tfrac{\lambda \tau_x}{vN}\big) \}$$

$$= \exp\{ - \tfrac{\lambda}{vN} \tau_x \ell_j(x) \big(1 + O(N^{-\beta})\big) \}.$$

By Lemma 6.3, we have

$$E\big[\exp\{ - \tfrac{\lambda}{vN} \tau_x \ell_j(x) \big(1 + O(N^{-\beta})\big) \} \mid \tau_x \le vN^{1-\beta} \big]$$

$$= 1 + \frac{1 + o(1)}{N^{\alpha(1-\beta)} v^\alpha} \psi_\lambda\big(\tfrac{\ell_j(x)}{N^\beta}\big),$$

for

$$\psi_\lambda(y) = (1 - \Gamma(1-\alpha))\lambda^\alpha y^\alpha - \int_1^\infty e^{-\lambda yz} \tfrac{\alpha}{z^{\alpha+1}} \, dz.$$

By Lemma 3.4 and (3.9) we have

$$\frac{1}{x_j - x_{j-1}} \sum_{x=x_{j-1}}^{x_j-1} \psi_\lambda\big(\tfrac{\ell_j(x)}{N^\beta}\big) \Rightarrow \int \psi_\lambda(y) \, \mu e^{-\mu y} \, dy.$$

Altogether,

$$\lim_{N\uparrow\infty} \mathbb{E} \exp\{ - \lambda \tfrac{U_j^{(N)}}{vN} \}$$

$$= \lim_{N\uparrow\infty} \mathbb{E} \exp\Big\{ \tfrac{x_j - x_{j-1}}{v^\alpha N^{\alpha(1-\beta)}} \int_0^\infty \psi_\lambda(y) \, \mu e^{-\mu y} \, dy \, (1 + o(1)) \Big\}$$

$$= \int_0^\infty e^{-u} \exp\Big\{ u \int_0^\infty \psi_\lambda(y) \, \mu e^{-\mu y} \, dy \Big\} du$$

$$= \Big(\int_0^\infty \Gamma(1-\alpha)\lambda^\alpha y^\alpha \, \mu e^{-\mu y} \, dy + \int_0^\infty \int_1^\infty e^{-\lambda yz} \tfrac{\alpha}{z^{\alpha+1}} \, dz \, \mu e^{-\mu y} \, dy \Big)^{-1}$$

$$= \Big(\tfrac{\alpha\pi}{\sin\alpha\pi} \big(\tfrac{\lambda}{\mu}\big)^\alpha + \alpha \int_1^\infty \tfrac{dz}{z^{\alpha+1}(1 + z\frac{\lambda}{\mu})} \Big)^{-1}.$$

The limit is continuous at $\lambda = 0$ and hence, by Bochner's theorem, see e.g. Theorem 5.22 in Kallenberg (2002), it is the Laplace transform of some random variable U_j. Theorem 4 in XIII.5 of Feller (1971) implies the statement about the tail behaviour.

Assume now that $\beta = 0$. Set $\theta_j(x) = \sum_{i=1}^{\ell_j(x)} \eta_i(x)$. Then

$$\mathbb{E} \exp\{ - \lambda \tfrac{U_j^{(N)}}{vN} \}$$

$$\sim \mathbb{E}\mathsf{E}E\Big[\prod_{x=x_{j-1}+1}^{x_j-1} \exp\big\{ - \tfrac{\lambda}{vN}\, \tau_x\, \theta_j(x) \big\} \Big]$$

$$= \mathbb{E}\mathsf{E}\Big[\prod_{x=x_{j-1}+1}^{x_j-1} E\big[\exp\big\{ - \tfrac{\lambda}{vN}\, \tau_x\, \theta_j(x) \big\} \mid \tau_x \le vN \big] \Big].$$

Using Lemma 6.3 once again, we get

$$E\big[\exp\big\{ - \tfrac{\lambda}{vN}\, \tau_x\, \theta_j(x) \big\} \mid \tau_x \le vN \big] = 1 + \frac{1+o(1)}{N^\alpha v^\alpha}\, \psi_\lambda\big(\theta_j(x)\big).$$

Repeating the arguments of Lemma 3.4, one can easily see that

$$\tfrac{1}{x_j - x_{j-1}} \sum_{x=x_{j-1}}^{x_j-1} \psi_\lambda\big(\theta_j(x)\big) \Rightarrow \mathbb{E}\psi_\lambda(\theta),$$

where $\theta = \sum_{i=1}^{\ell^{(\infty)}(0)} \eta_i(0)$. Since $\ell^{(\infty)}(0)$ is geometrically distributed, θ is exponentially distributed with mean $1/\mu$. Thus,

$$\mathbb{E}\psi_\lambda(\theta) = \int \psi_\lambda(y)\, \mu e^{-\mu y}\, dy.$$

This means that the remaining part of the proof coincides with that for the case $\beta > 0$. □

Recall that $R_j^{(N)} = |\mathcal{R}_j^{(N)}|$. Then

$$\frac{1}{N}\Big(\sum_{i=1}^{j-1}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)}) + U_j^{(N)} \Big)$$

is the total time $X^{(N)}$ takes to hit x_j. For the lower bound in Theorem 2.2 we use Lemmas 6.4, 6.5, and 6.6, and get, for any $M > 0$,

$$\mathbb{P}\Big\{ \tau_{X_t^{(N)}/N^{1-\beta}} > v \Big\}$$

$$\ge \sum_{j=1}^{M} \Big(\mathbb{P}\Big\{ \sum_{i=1}^{j-1}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)}) + U_j^{(N)}$$

$$< Nt \le \sum_{i=1}^{j}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)}) \Big\} - \mathbb{P}\{ Nt \in \mathcal{R}_j^{(N)} \} \Big)$$

$$\longrightarrow \sum_{j=1}^{M} \mathbb{P}\Big\{ \sum_{i=1}^{j-1}(U_i + S_i) + U_j < \frac{t}{v} \le \sum_{i=1}^{j}(U_i + S_i) \Big\}, \quad \text{as } N \uparrow \infty,$$

and we get the required lower bound by letting $M \uparrow \infty$.

For the upper bound, we have, for any $M > 0$,

$$\mathbb{P}\left\{\tau_{X_t^{(N)}}/N^{1-\beta} > v\right\}$$

$$\leq \sum_{j=1}^{M} \mathbb{P}\left\{\sum_{i=1}^{j-1}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)}) + U_j^{(N)} < Nt \leq \sum_{i=1}^{j}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)})\right\}$$

$$+ \mathbb{P}\left\{\sum_{i=1}^{M}(U_i^{(N)} + S_i^{(N)} + R_i^{(N)}) + U_{M+1}^{(N)} < Nt\right\}$$

$$\longrightarrow \sum_{j=1}^{M} \mathbb{P}\left\{\sum_{i=1}^{j-1}(U_i + S_i) + U_j < \frac{t}{v} \leq \sum_{i=1}^{j}(U_i + S_i)\right\}$$

$$+ \mathbb{P}\left\{\sum_{i=1}^{M}(U_i + S_i) + U_{M+1} < \frac{t}{v}\right\}, \qquad \text{as } N \uparrow \infty,$$

and, as $M \uparrow \infty$ the additional term on the right converges to zero, because $U_i + S_i$ are independent, nonnegative random variables. This completes the proof of the first statement in Theorem 2.2.

For the second statement we evaluate the probability of the process $X^{(N)}$ staying put conditional on the environment as

$$\mathbb{P}\left\{X_{t+s}^{(N)} = X_t^{(N)} \text{ for all } 0 \leq s \leq N^{-\beta} \mid (\tau_z, z \in \mathbb{Z}^d), X_t^{(N)}\right\} = \exp\left\{-\frac{N^{1-\beta}}{\tau_{X_t^{(N)}}}\right\}.$$

As the right hand side is a continuous and bounded function of $\tau_{X_t^{(N)}}/N^{1-\beta}$, we obtain from the first statement that

$$\lim_{N\to\infty} \mathbb{E}\exp\left\{-\frac{N^{1-\beta}}{\tau_{X_t^{(N)}}}\right\} = \mathbb{E}\exp\{-1/\xi_t\},$$

which is the second statement of Theorem 2.2.

7. Proof of Theorem 2.4

We follow the framework of Fontes et al. (2002) and start with a discussion of the notion of convergence of atomic measures in the *point process sense*, which is crucial for this argument. Let

$$\nu^{(N)} = \sum_{i\in\mathbb{Z}} w_i^{(N)} \delta_{y_i^{(N)}}, \qquad \nu = \sum_{i\in\mathbb{Z}} w_i \delta_{y_i}$$

be atomic measures. If, for every open set $G \subset \mathbb{R} \times (0, \infty)$ whose closure in $\mathbb{R} \times (0, \infty)$ is compact with $\rho(\partial G) = 0$, we have, for all sufficiently large N,

$$\#\left\{(y_i^{(N)}, w_i^{(N)}) \in G\right\} = \#\left\{(y_i, w_i) \in G\right\}$$

we say that $\nu^{(n)} \to \nu$ in the point process sense.

Lemma 7.1. *Suppose that $\nu^{(N)} \to \nu$ in the point process sense and the scale functions $S^{(N)}$ converge uniformly on compact intervals to the identity then, for any $t > 0$,*

$$\nu^{(N)}(\{Y[\nu^{(N)}, S^{(N)}](t)\}) \Longrightarrow \nu(\{Y[\nu, \mathrm{id}](t)\}) \quad \text{for } N \to \infty.$$

Proof: By Theorem 2.1 in Fontes et al. (2002), the law of $Y[\nu^{(N)}, S^{(N)}](t)$ converges to the law of $Y[\nu, \mathrm{id}](t)$ weakly as well as in the point process sense. Given an open set $G \subset \mathbb{R} \times (0, \infty)$ as above, let $x_1, \ldots, x_l$ be the positions of the atoms in G. Then, by Condition 1 in Fontes et al. (2002), there exists N_0 such that for all $N \geq N_0$ the values $x_1^{(N)}, \ldots, x_l^{(N)}$ are the positions of the atoms of $\nu^{(N)}$ in G, and

$$\lim_{N \to \infty} x_i^{(N)} = x_i \text{ and } \lim_{N \to \infty} \nu^{(N)}(\{x_i^{(N)}\}) = \nu(\{x_i\}) \quad \text{for all } i \in \{1, \ldots, l\}.$$

Using the convergence of the distributions we further have

$$\lim_{N \to \infty} \mathbb{P}\{Y[\nu^{(N)}, S^{(N)}](t) = x_i^{(N)}\} = \mathbb{P}\{Y[\nu, \mathrm{id}](t) = x_i\} \quad \text{for all } i \in \{1, \ldots, l\}.$$

Observe now that, because $Y[\nu^{(N)}, S^{(N)}](t)$ converges in law, the sequence is uniformly tight, more precisely for each $\varepsilon > 0$ there exists an open ball $B \subset \mathbb{R}$ with

$$\sup_{N \geq 1} \mathbb{P}\{Y[\nu^{(N)}, S^{(N)}](t) \notin B\} < \varepsilon.$$

Now given $0 < u < v$ we let $G = B \times (u, v)$ and assume that u, v are not weights of atoms of ν. With the notation from above we have

$$\lim_{N \to \infty} \mathbb{P}\{\nu^{(N)}(\{Y[\nu^{(N)}, S^{(N)}](t)\}) \in (u, v), Y[\nu^{(N)}, S^{(N)}](t) \in B\}$$
$$= \lim_{N \to \infty} \sum_{i=1}^{l} \mathbb{P}\{Y[\nu^{(N)}, S^{(N)}](t) = x_i^{(N)}\} = \sum_{i=1}^{l} \mathbb{P}\{Y[\nu, \mathrm{id}](t) = x_i\}$$
$$= \mathbb{P}\{\nu(\{Y[\nu, \mathrm{id}](t)\}) \in (u, v), Y[\nu, \mathrm{id}](t) \in B\},$$

which completes the proof as $\varepsilon > 0$ was arbitrary. $\qquad\square$

By a classical stable limit theorem, see Proposition 3.1 in Fontes et al. (2002), there exists a coupling of the measures $\nu^{(N)}$ in the proof of Theorem 2.1 (b) and (c) such that, almost surely, $\nu^{(N)}$ converges to ρ in the point process sense. Obviously, $S^{(N)}$ converges to the identity uniformly on compact sets, and hence Lemma 7.1 shows that

$$\lim_{N \to \infty} \frac{\tau_{X_t^{(N)}}}{N^{\frac{1}{\alpha+1}}} = \lim_{N \to \infty} \nu^{(N)}(\{Y[\nu^{(N)}, S^{(N)}](t)\}) = \rho(Y[\rho, \mathrm{id}])(t) = \rho(\mathsf{Fin}_t^\theta) \quad \text{in law.}$$

The ageing result follows by the same argument as in Theorem 2.2.

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