DIFFERENTIAL TOPOLOGY

I. Topological spaces - a reminder

I.1. Basic definitions

Def.:
- A **topology** T on a set X is a collection of subsets of X s.t.
  (i) $\emptyset, X \in T$
  (ii) $U, V \in T \Rightarrow U \cap V \in T$
  (iii) $U_\lambda \in T \Rightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in T$
- (X, T) is called **topological space**, its elements ($U \in T$) are called **open** sets, their complements ($X \setminus U$, $U \in T$) **closed** sets.
- The **closure** $\bar{A}$ of $A \subseteq X$ is the smallest closed subset of X that contains A (i.e., the intersection of all closed sets containing A)
- The **interior** $\operatorname{int} A$ of $A \subseteq X$ is the largest open subset of X contained in A (i.e., the union of all open subsets in A)
- The **boundary** ∂A of $A \subseteq X$ is $\partial A := \bar{A} \setminus \operatorname{int}(A)$.
- $A \subseteq X$ is **dense** in X if $\bar{A} = X$.
- $U \subseteq X$ is a **neighborhood** of $x \in X$ if $\exists V \in T : x \in V \subseteq U$.
- $B \subseteq T$ is a **basis** of T if $\forall U \in T \ \exists \Lambda \subseteq B : U = \bigcup_{V \in \Lambda} V$

Def.: (Properties of top. spaces) A top. space (X, T) is called ...
- **connected** if $X = X_1 \cup X_2$, $\emptyset \neq X_i \in T$ implies $X_1 \cap X_2 \neq \emptyset$
- **Hausdorff** (a.k.a. "T2") if for all distinct $x, y \in X$ there are disjoint neighborhoods.
- **second countable** if there exists a countable basis of T.

remark: • (X,T) is connected iff $\emptyset$ and X are the only "clopen" (= closed & open) subsets of X

examples: • Metric topology of a metric space (X,d). With $B_r(x) := \{ y \in X \mid d(x,y) < r \}$ define $T := \{ U \subseteq X \mid \forall x \in U \, \exists r > 0 : B_r(x) \subseteq U \}$

Then • (X,T) is a Hausdorff space

• $\{ B_r(x) \}_{r \in \mathbb{R}_+, x \in X}$ is a basis of T

• (X,T) is second countable $\overset{ZFC}{\text{iff}}$ X has a countable dense subset (i.e., it is "separable")

• Trivial topology of a set X is $T = \{ \emptyset, X \}$ (This is not Hausdorff if $|X| \geq 2$)

• Discrete topology $T := \{ U \subseteq X \}$

(• Zariski topology defines its closed sets to be solutions of algebraic equations. It is not Hausdorff.)

I.2. Constructing new spaces from old ones

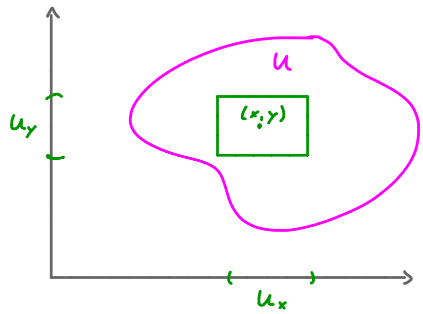
Def.: The **subspace topology** (a.k.a. "relative topology") of a subset $A \subseteq X$ of a top. space (X,T) is defined by $T|_A := \{ V \subseteq A \mid \exists U \in T : U \cap A = V \}$
(Its elements are sometimes called "relatively open")

examples: • The "n-sphere" $S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \}$ inherits a topology from Euclidean space $\mathbb{R}^{n+1}$.

• The metric topology of $(\mathbb{R}^n, \|\cdot\|_2)$ coincides with the subspace topology of $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$ with metric topology w.r.t. $\|\cdot\|_2$.

<u>Def.:</u> • The **product topology** of two top. spaces (X,T) and (Y,R) is defined as

$$\{ U \subseteq X \times Y \mid \forall (x,y) \in U \; \exists U_x \in T, U_y \in R : x \in U_x \wedge y \in U_y \wedge U_x \times U_y \subseteq U \}$$



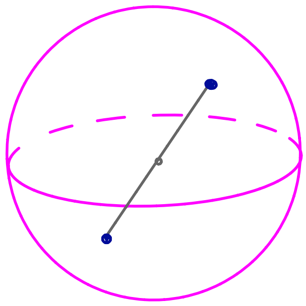
<u>remark:</u> a basis can be obtained in the form $V \times W$ by running over all elements V of a basis of X and all W of a basis of Y.

• The **quotient topology** of a quotient $X/\sim$ of (X,T) is defined as $Q := \{ V \subseteq X/\sim \mid q^{-1}(V) \in T \}$ where $q: X \to X/\sim$ is the "quotient map", i.e., $q : x \mapsto [x]$.

<u>warning:</u> Quotients can ruin Hausdorff property!

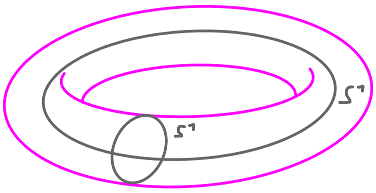
<u>examples:</u> • <u>Projective spaces</u> for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

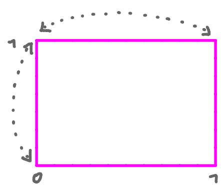
$\mathbb{K}P^n := (\mathbb{K}^{n+1} \setminus \{0\}) / \sim$ where $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{K} : x = \lambda y$



Alternatively, e.g. for $\mathbb{R}P^n := \{ \{x,-x\} \mid x \in S^n \}$

• <u>Torus</u> T^2 can be regarded as product space $S^1 \times S^1$,
as a quotient of $[0,1] \times [0,1]$ by identifying (= "gluing together") parallel edges,
or as a subspace of $\mathbb{R}^3$. Product, quotient & subspace topologies coincide.





The following is useful to verify the Hausdorff-property of a quotient space:

<u>Lemma:</u> Let (X,T) be a topological space and $q: X \to X/\sim$ an open map defining a quotient space (via $q(x) = q(y) \Leftrightarrow x \sim y$). If $\Gamma := \{(x,y) \mid x \sim y\}$ is closed w.r.t. the product topology in $X \times X$, then $X/\sim$ is Hausdorff.

<u>proof:</u> Assume $\neg(x \sim y)$.
Since $X \times X \setminus \Gamma$ is an open neighborhood of (x,y) it contains an open neighborhood of the form $V_x \times V_y$. Then $q(V_x)$ and $q(V_y)$ are open neighborhoods of $q(x)$ and $q(y)$, and $q(V_x) \cap q(V_y) = \emptyset$ since otherwise if $z \in q(V_x) \cap q(V_y)$, then there would be $V_x \ni x_z \sim y_z \in V_y$, which is excluded since $V_x \times V_y$ are in the complement of Γ. $\square$

<u>remark:</u> A related result is that (X,T) is Hausdorff iff the diagonal $\Delta := \{(x,y) \in X \times X \mid x = y\}$ is closed in $X \times X$ w.r.t. product topology.

I.3. Compactness, convergence & continuity

Def.: Let (X, T) be a top. space.

- A subset $A \subseteq X$ is called **compact** if any open cover $\left(\{U_\lambda\}_{\lambda \in \Lambda} \subseteq T \text{ with } \bigcup_{\lambda \in \Lambda} U_\lambda \supseteq A \right)$ has a finite subcover $\left(\text{i.e. } \bigcup_{i=1}^{n} U_{\lambda_i} \supseteq A \right)$.

remark: Closed subsets of compact sets are compact.

Def.: A sequence $(x_n)_{n \in \mathbb{N}}$ is said to **converge** to $x \in X$ if
$\forall U \in T: \left(x \in U \Rightarrow \exists m \in \mathbb{N} \ \forall n \geq m : x_n \in U \right)$

remarks:

- In a metric space the closure $\bar{A}$ of A is the set of limits of all sequences in A.
- In Hausdorff spaces limits are unique.

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{sequential compactness} \\ (= \text{sequences have conv. subseq.s})\end{array}} \arrow[r, leftrightarrow, "\text{metric space}"] & \boxed{\text{compactness}} \arrow[dl, "\text{Hausdorff}"'] \arrow[dr, "\text{metric space}"] & \\
& \boxed{\text{closedness}} & \boxed{\text{boundedness}}
\end{tikzcd}$$

Def.: A map $f: X \to Y$ between two top. spaces (X,T), (Y,R) is called

- **continuous** if $U \in R \Rightarrow f^{-1}(U) \in T$
- **open** if $V \in T \Rightarrow f(V) \in R$
- **homeomorphism** if it is bijective, continuous and has cont. inverse.

$C(X,Y)$:= set of all cont. maps $f: X \to Y$

X and Y are called **homeomorphic** if there exists a homeomorphism between them.

remarks:

- If $f \in C(X,Y)$ then $x_n \to x$ implies $f(x_n) \to f(x)$. In metric spaces this is equivalent to continuity
- $f: [0, 2\pi) \to S^1$, $\varphi \mapsto (\cos(\varphi), \sin(\varphi))$ is an example of a continuous bijection that is not a homeomorphism.
- continuous maps preserve compactness & connectedness

II. Topological manifolds

Def.: A second countable Hausdorff space (M,T) is a **topological manifold** of **dimension** $m \in \mathbb{N}_0$ if it is locally homeomorphic to $\mathbb{R}^m$. That is, for any $x \in M$ there is an open neighborhood $U \subseteq M$ and a homeomorphism $\varphi: U \to V \subseteq \mathbb{R}^m$.

- The pair (U, φ) is called a **chart**.
- A collection $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ of charts is called an **atlas** for M if $\bigcup_{\lambda \in \Lambda} U_\lambda \supseteq M$
- $\varphi_1, \dots, \varphi_m$ are called **coordinates** and φ^{-1} a **parametrization**.

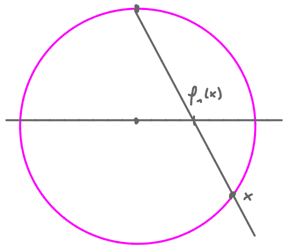
examples:

- <u>spheres</u> $\quad S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\|_2 = 1 \}$ is n-dim. top. manifold.

  Two charts are given by the 'stereographic projections'

  $$\varphi_1(x) : S^n \setminus (0, \dots, 0, 1) \to \mathbb{R}^n$$
  $$\varphi_1(x) := \frac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$$

  $$\varphi_2(x) : S^n \setminus (0, \dots, 0, -1) \to \mathbb{R}^n$$
  $$\varphi_2(x) := \frac{1}{1 + x_{n+1}} (x_1, \dots, x_n)$$

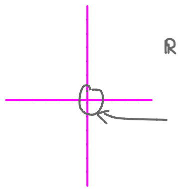


- <u>graphs</u> If $U \subseteq \mathbb{R}^m$ is open and $f \in C(U, \mathbb{R}^n)$, the graph $M := \{ (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mid y = f(x) \}$ is an m-dim. top. manifold (with subspace top. in $\mathbb{R}^m \times \mathbb{R}^n$) and $\varphi(x, y) = x$ gives a chart.

- <u>open subsets</u> of a top. manifold are again top. manifolds of the same dimension.

  E.g. $\quad GL(n, \mathbb{R}) := \{ A \in \mathbb{R}^{n \times n} \mid \det(A) \neq 0 \}$ is an open subset of $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$ and thus a top. manifold of dim. n^2.

- The following is not a top. manifold: $\quad \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \subseteq \mathbb{R}^2$



not locally homeomorphic to any $\mathbb{R}^m$.

<u>remark:</u> the dimension of a top. manifold is well-defined due to the following:

<u>Thm.:</u> (Invariance of dimension)

Two non-empty open subsets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ with $n \neq m$ cannot be homeomorphic.

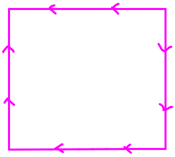
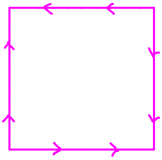
<u>proof:</u> Hard. We will prove the 'smooth' case later ... $\square$

Thm.: (Embedding into Euclidean space)
Every topological manifold can be embedded into $\mathbb{R}^N$ for some $N \in \mathbb{N}$.
That is, there exists a homeomorphism $\varphi: M \to \varphi(M) \subseteq \mathbb{R}^N$.

proof: → e.g. see book by Munkres for compact manifolds. □

remark: $N = 2m+1$ is possible. We will discuss this later in the smooth case ...
Simple examples where $N < 2m$ (with $m = 2$) is not possible, are

"Klein bottle" & "projective plane"

The proof of the embedding theorem is based on the following Lemma, which in turn exploits the property of the top. space to be second countable:

Lemma: (Existence of a partition of unity)
Let $(U_i)_{i \in \Lambda}$ be an open cover of a top. manifold M. Then there exists a 'partition of unity' subordinate to $(U_i)_{i \in \Lambda}$. That is,

- $\varphi_i \in C(M, \mathbb{R}_{\geq 0})$
- $\overline{\operatorname{supp}(\varphi_i)} \subseteq U_i$
- $\forall x \in M: \left|\{ i \in \Lambda \mid x \in \overline{\operatorname{supp}(\varphi_i)} \}\right| < \infty$
- $\sum_{i \in \Lambda} \varphi_i(x) = 1 \quad \forall x \in M.$

III. Reminder of differential calculus

Def.: Let X and Y be Euclidean spaces and $f: U \to Y$ a map from an open subset $U \subseteq X$.

- A linear map $df_x : X \to Y$ is called the **derivative** of f at $x \in U$ if

$$\lim_{\Delta \to 0} \frac{\| f(x+\Delta) - f(x) - df_x(\Delta) \|}{\| \Delta \|} = 0$$

- We write $f \in C^k(U, Y)$ for $k \in \mathbb{N}$ if all partial derivatives $\partial_{i_1} \cdots \partial_{i_k} f$ exist and are continuous.

- $C^\infty(U, Y) := \bigcap_{k \in \mathbb{N}} C^k(U, Y)$

- If $f \in C^k$ for $k \geq 1$, the matrix with entries $(df_x)_{i,j} = \partial_j f_i(x)$ is called **Jacobi-matrix**.

remarks:

- The graph of $a \mapsto (a, df_x(a) + f(x))$ is tangential to the graph of f at x.

- Chain rule: for
$$\begin{tikzcd} X \supseteq U \arrow[r, "f"] \arrow[dr, "g \circ f"'] & V \subseteq Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$
we get
$$\begin{tikzcd} X \arrow[r, "df_x"] \arrow[dr, "{d(g\circ f)_x = dg_y \circ df_x}"'] & Y \arrow[d, "dg_y"] \\ & Z \end{tikzcd}$$
where $y := f(x)$

Def.: For $k \in \mathbb{N} \cup \{\infty\}$ a bijective map $f \in C^k(U, V)$ between open sets of Euclidean spaces is called **C^k-diffeomorphism** if $f^{-1}: V \to U$ is differentiable.

Excursion: Classification of low-dim. manifolds up to homeomorphisms

d = 0: the only connected manifold is a point

d = 1: there are two connected manifolds $(0,1)$ and S^1

d = 2: Every connected compact 2-dim. topological manifold is homeomorphic to one of the following:

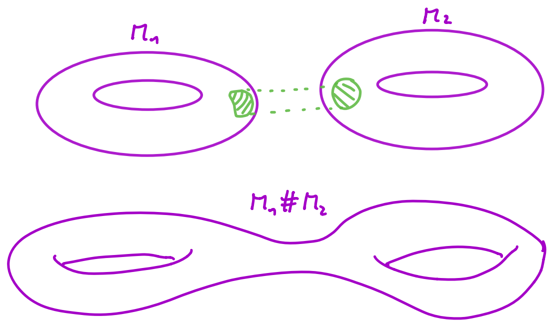
(i) S^2 (sphere)

(ii) $T^2 \# \dots \# T^2$ (orientable surface of "genus" g = nr. of T^2's)

(iii) $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$ (non-orientable surfaces of "non-orientable genus" g = nr of $\mathbb{R}P^2$'s)

remarks:

- $\#$ refers to the "connected sum":

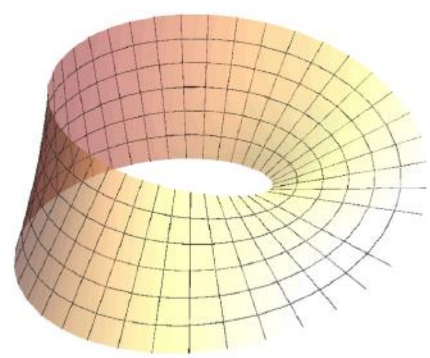


- $S^2 \# S^2 = S^2$
- $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 = \mathbb{R}P^2 \# T^2$
- $\mathbb{R}P^2 \# \mathbb{R}P^2 =$ "klein bottele"
- $\mathbb{R}P^2$ is in this context called "cross-cap"

d = 3: is the content of Thurston's geometrization conjecture proven by Perelman in 2003.

<u>Def.:</u>
- A topological manifold with boundary M is a second countable Hausdorffspace that is locally homeomorphic to $(\mathbb{R}^n)_+ := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$.
- The dimension of M is then n.
- The manifold boundary $\partial M \subset M$ is the set of points that are mapped to $x_n = 0$ by some chart, and the manifold interior $\operatorname{int} M$ the set of points that are mapped to $x_n > 0$ by some chart.

<u>Example:</u> (Möbius strip) $X/\sim$ where $X = \mathbb{R} \times [0,1]$ with $(x,y) \sim (x+1, 1-y)$



<u>Prop.:</u> If M is a topological manifold with boundary, then $\operatorname{int}(M)$ and ∂M are top. manifolds (without boundary) and
(i) $\dim(\operatorname{int}(M)) = \dim(M)$
(ii) $\partial M \neq \emptyset \Rightarrow \dim(\partial M) = \dim(M) - 1$.

<u>proof:</u> $\longrightarrow$ exercise.

Prop.: If $f: U \to V$ is a C^k-Diffeomorphism, then

(i) $U \subseteq X \wedge V \subseteq Y \Rightarrow \dim(X) = \dim(Y)$

(ii) $\forall x \in U: \det(df_x) \neq 0$

(iii) $\forall x \in U: d(f^{-1})_y = (df_x)^{-1}$ with $y = f(x)$

(iv) $f^{-1} \in C^k(V, U)$

proof: Chain rule applied to $f \circ f^{-1} = \mathbb{1}_Y$ and $f^{-1} \circ f = \mathbb{1}_X$ gives

$$df_x \, d(f^{-1})_y = \mathbb{1}_Y \quad \text{and} \quad d(f^{-1})_y \, df_x = \mathbb{1}_X .$$

So df_x is a vector-space isomorphism $\longrightarrow$ (i), (ii), (iii).

For (iv) assume $f^{-1} \in C^l$ with $l < k$. From (iii) we get

$$d(f^{-1})_y = \left(df_{f^{-1}(y)}\right)^{-1}.$$

$$\begin{tikzcd} Y \arrow[r, "d(f^{-1})"] \arrow[d, "C^l \ni f^{-1}"'] & \mathcal{B}(Y,X) \\ X \arrow[r, "df \in C^{k-1}"] & \mathcal{B}(Y,X) \arrow[u, "(\cdot)^{-1}"'] \end{tikzcd}$$

Hence $d(f^{-1}) \in C^l$

$\Rightarrow f^{-1} \in C^{l+1}$.

$\square$

Thm.: (Inverse function theorem)

Let X, Y be Euclidean spaces, $U \subseteq X$ open and $f \in C^k(U, Y)$ with $k \in \mathbb{N} \cup \{\infty\}$. If $\det(df_x) \neq 0$, there is an open neighborhood $V \ni x$ s.t. $f|_V : V \to f(V)$ is a C^k-diffeomorphism.

Corollary: (Constant rank thm.) Let $f \in C^k(U \subseteq \mathbb{R}^m, \mathbb{R}^n)$ be s.t. $\forall z \in U$: $\operatorname{rank}(df_z) = r$. Then $\forall x \in U$ there are open neighborhoods $V \ni z$, $W \ni f(z)$ and C^k-diffeomorphisms

$$\varphi : V \longrightarrow \varphi(V) \subseteq \mathbb{R}^m$$

$$\psi : W \longrightarrow \psi(W) \subseteq \mathbb{R}^n$$

such that $\psi \circ f \circ \varphi^{-1}(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0)$ for all $x \in \varphi(V)$.

proof: Set $a := (x_1, \dots, x_r)$, $b := (x_{r+1}, \dots, x_m)$ and $g = (f_1, \dots, f_r)$, $h := (f_{r+1}, \dots, f_n)$. That is, $f(a,b) = (g(a,b), h(a,b))$.

Assume w.l.o.g. $\det\left(\frac{\partial g}{\partial a}\Big|_z\right) \neq 0$.

For $\varphi(a,b) := (g(a,b), b)$ we get $d\varphi_z = \left.\begin{pmatrix} \frac{\partial g}{\partial a} & \frac{\partial g}{\partial b} \\ 0 & \mathbb{1} \end{pmatrix}\right|_z$ and $\det(d\varphi_z) \neq 0$.

By the inverse function theorem φ is a local C^k-diffeomorphism.

Define $\varphi^{-1}(a,b) =: (A(a,b), B(a,b))$

Then $(a,b) = \varphi(\varphi^{-1}(a,b)) = (g(A,B), B)$

so that $g(A,B) = a \wedge B = b$.

Therefore
$$f \circ \varphi^{-1}(a,b) = f(A,B) = (g(A,B), h(A,B))$$
$$= (a, h(A(a,b), b) =: H)$$

and $d(f \circ \varphi^{-1}) = \begin{pmatrix} \mathbb{1}_r & 0 \\ \frac{\partial H}{\partial a} & \frac{\partial H}{\partial b} \end{pmatrix}$ must be of rank r.

This requires $\frac{\partial H}{\partial b} = 0$, which implies $f \circ \varphi^{-1}(a,b) = (a, H(a))$.

Now define $\psi : (u,v) \longmapsto (u, v - H(u))$.

Since $d\psi = \begin{pmatrix} \mathbb{1} & 0 \\ -\frac{\partial H}{\partial u} & \mathbb{1} \end{pmatrix}$ is invertible

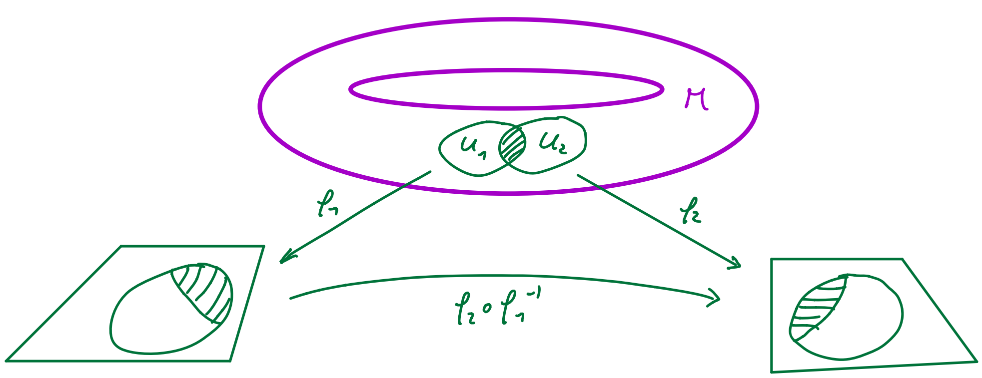
we can apply the inverse function theorem.

$\longrightarrow$ ψ is a local C^k-diffeomorphism and composing the maps yields $\psi \circ f \circ \varphi^{-1}(a,b) = (a, 0)$. $\square$

IV. DIFFERENTIABLE MANIFOLDS

Def.: Let M be an m-dim. top. manifold and $k \in \mathbb{N} \cup \{\infty\}$.

- Two charts (U_1, φ_1), (U_2, φ_2) are <u>C^k-compatible</u> if $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \subseteq \mathbb{R}^m \longrightarrow \varphi_2(U_1 \cap U_2) \subseteq \mathbb{R}^m$ is a C^k-diffeomorphism.
- A <u>C^k-atlas</u> is an atlas with pairwise C^k-comp. charts.
- A <u>C^k-structure</u> for M is a maximal C^k-atlas, i.e., one that is not contained in any strictly larger C^k-atlas.
- If $\mathcal{A}$ is a C^k-structure for M, then $(M, \mathcal{A})$ is called <u>C^k-manifold</u> of dimension m.
- <u>smooth</u> is synonym for C^∞.



Lemma: For $k \geq 1$, every C^k-atlas has a unique extension to a C^k-structure.

proof: Let $\mathcal{A}$ be a C^k-atlas for M and $\bar{\mathcal{A}}$ the set of all charts that are C^k-compatible with $\mathcal{A}$. A C^k-structure $\mathcal{S}$ satisfies $\mathcal{A} \subseteq \mathcal{S} \subseteq \bar{\mathcal{A}}$. We show that $\bar{\mathcal{A}}$ is a C^k-atlas and thus $\bar{\mathcal{A}} = \mathcal{S}$. For $(U_i, \varphi_i) \in \bar{\mathcal{A}}$ we have to show that $\varphi_2 \circ \varphi_1^{-1}$ is C^k-diff. on $\varphi_1(U_1 \cap U_2)$.

For any $y = \varphi_1(x) \in \varphi_1(U_1 \cap U_2)$ there is a chart $(W, \phi) \in \mathcal{A}$ that is C^k-compatible with $\bar{\mathcal{A}}$ and s.t. $x \in W$. Hence, $\varphi_2 \circ \varphi_1^{-1} = (\varphi_2 \circ \phi^{-1}) \circ (\phi \circ \varphi_1^{-1})$ is a composition of two local C^k-diff.s and therefore itself a local C^k-diff. $\square$

remark: This implies that two C^k-atlases determine the same C^k-structure iff their union is a C^k atlas.

examples:

- $\mathbb{R}^n$ becomes n-dim. C^∞-manifold with the single chart $(\mathbb{R}^n, id)$. The resulting smooth structure is called the "standard smooth structure". This can be applied to any open subset such as $GL(n, \mathbb{R})$.
- Spheres as C^∞-manifolds:

  For the two charts given by

  $\varphi_1(x) := \frac{1}{1 - x_{n+1}} (x_1, \dots, x_n)$ on $S^n \setminus (0, \dots 0, 1) =: U_1$

  $\varphi_2(x) := \frac{1}{1 + x_{n+1}}$ — " — on $S^n \setminus (0, \dots 0, -1) =: U_2$

  one obtains $\varphi_2 \circ \varphi_1^{-1}(z) = \frac{z}{\|z\|^2}$, which is a C^∞-diff. on $\varphi_1(U_1 \cap U_1) = \mathbb{R}^n \setminus \{0\}$.

- Projective spaces as C^∞-manifolds $\longrightarrow$ exercise
- C^k-product manifolds: If M_1, M_2 are C^k-manifolds, then $M_1 \times M_2$ becomes a C^k-manifold of $\dim(M_1 \times M_2) = \dim(M_1) + \dim(M_2)$ with charts of the form $(U \times V, \varphi \times \psi)$.

<u>Thm.:</u> [Whitney] If $k \geq 1$, every C^k-structure contains a C^∞-structure.

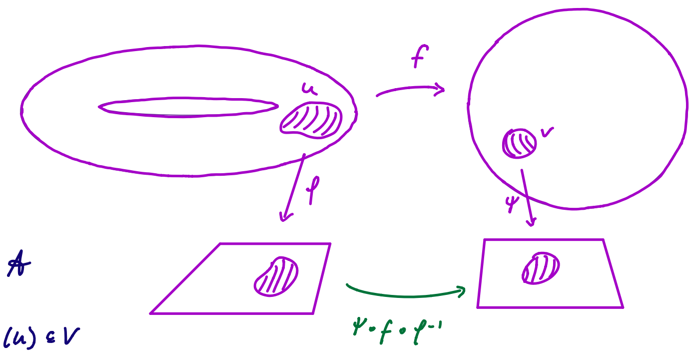
<u>proof:</u> see Whitney or Hirsch. $\square$

<u>remarks:</u>
- Motivated by this, we will only consider C^∞ (or C^0).
- There are top. manifolds that do not admit a smooth structure (e.g. the 4-dim. "E8-manifold" discovered by Freedman)
- From a given smooth structure $\{(U_\lambda, \varphi_\lambda)\}$ we can obtain another one $\{(U_\lambda, \varphi_\lambda \circ \psi)\}$ by acting with a homeomorphism ψ. Such smooth structures are called <u>equivalent</u>.

  For $\mathbb{R}^n$ with $n \neq 4$ all smooth structures are equivalent.

  For $\mathbb{R}^4$, however, there is an uncountable infinity of inequivalent smooth structures (work by Freedman & Donaldson). There are e.g. "exotic $\mathbb{R}^4$'s" with compact sets that cannot be surrounded by any smoothly embedded S^3.
- If M is an m-dim. top. manifold, then there is (up to equivalence) a unique smooth structure for $m < 4$ and finitely many (or none) for $m > 4$.
- Many Fields medalists worked on problems related to the above: Thurston, Milnor, Smale, Donaldson, Freedman, not to mention Perelman.

V. Smooth maps



Def.: Let $(M, \mathcal{A})$, $(N, \mathcal{B})$ be smooth manifolds.

- A map $f: M \to N$ is called **smooth** if for all $(U, \varphi) \in \mathcal{A}$ and all $(V, \psi) \in \mathcal{B}$ with $f(U) \subseteq V$ the map $\psi \circ f \circ \varphi^{-1}: \varphi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^n$ is smooth.
- A map $g: X \subseteq M \longrightarrow N$ is called smooth if $\forall x \in X$ there is an open neighborhood $U \subseteq M$ and a smooth map $f: U \to N$ s.t. $f = g$ on $U \cap X$.
- $g: X \subseteq M \longrightarrow Y \subseteq N$ is a **diffeomorphism** if it is bijective, smooth and has a smooth inverse. X and Y are then called **diffeomorphic**.
- $C^\infty(X, Y) :=$ all smooth maps from X to Y.

Lemma: The composition of smooth maps is smooth.

proof: $\to$ exercise $\square$

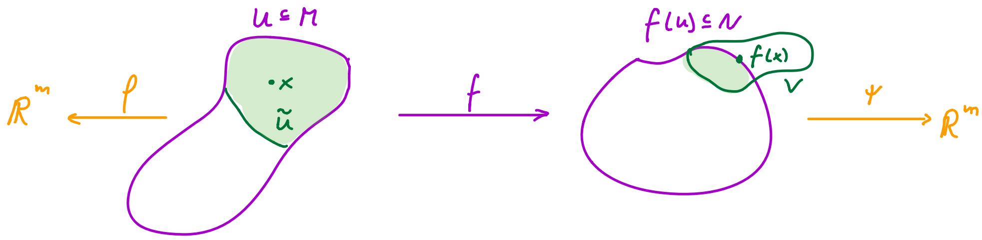
Examples:
- $\mathbb{C}P^1$ and S^2 are, with the smooth structures considered before, diffeomorphic. A diffeomorphism is $f: S^2 \to \mathbb{C}P^1 = \mathbb{C}^2/\sim$

$$f(x_0, x_1, x_2) := \begin{cases} \left[1, \dfrac{x_1 + i x_2}{1 + x_0}\right], & \text{for } x_0 \neq -1 \\ \left[\dfrac{x_1 - i x_2}{1 - x_0}, 1\right], & \text{for } x_0 \neq 1 \end{cases}.$$

Here $[z_1, z_2]$ denotes the equivalence class in $\mathbb{C}^2$ formed by $\sim$.
f is well-defined since $\left(\dfrac{x_1 + i x_2}{1 + x_0}\right)\left(\dfrac{x_1 - i x_2}{1 - x_0}\right) = 1$ on S^2.

- Similarly, $\mathbb{R}P^1$ and S^1 are diffeomorphic.

Lemma: (Smooth invariance of domain) Let M, N be smooth manifolds of equal dimension and $f: U \to f(U) \subseteq N$ a diffeomorphism from an open subset $U \subseteq M$. Then $f(U)$ is open in N.



proof: Since $f^{-1}: f(U) \longrightarrow U$ is smooth, there is, for each $x \in U$ an open neighborhood V of $f(x)$ in N to which $f^{-1}|_{V \cap f(U)}$ can be extended to a smooth map $\hat{f}: V \longrightarrow M$ s.t. $\hat{f}(y) = f^{-1}(y)$ $\forall y \in V \cap f(U)$. Due to continuity of f, $f^{-1}(V) \cap U =: \tilde{U}$ is an open neighborhood of x and $\hat{f} \circ f|_{\tilde{U}} = f^{-1} \circ f|_{\tilde{U}} = id_{\tilde{U}}$.
Using charts to pull this into Euclidean space we get:
$\varphi \hat{f} \psi^{-1} \circ \psi f \varphi^{-1} = id$ on the open set $\varphi(\tilde{U}) \subseteq \mathbb{R}^m$. Taking the derivative using the chain rule this implies that $d(\psi f \varphi^{-1})_z$ is a vector space isomorphism for all $z \in \varphi(\tilde{U})$. By the inverse func. thm. there is an open neighborhood around any such $z \in \varphi(\tilde{U})$ that is diffeomorphically mapped to an open image under $\psi f \varphi^{-1}$. Since ψ is a homeomorphism, $f(\tilde{U})$ and thus also $f(U)$ is open in N. $\square$

remarks: ◦ This is a smooth version of the general "invariance of domain thm." that states that if $U \subseteq \mathbb{R}^n$ is open and $f: U \to \mathbb{R}^n$ is a continuous injection, then $f(U)$ is open and $f: U \to f(U)$ is a homeomorphism. We will come back to this and proof this later...

- The extension of the notion "diffeomorphism" to subsets of smooth manifolds, allows (by applying it to $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$) to extend "smooth manifold" straight forwardly to "smooth manifold with boundary". Then

$$\partial M := \{ x \in M \mid \exists (U, \varphi) \in \mathcal{A} : x \in U \wedge \varphi(x)_n = 0 \} \quad \text{and}$$

$$\mathrm{Int}(M) := \{ x \in M \mid \exists (U, \varphi) \in \mathcal{A} : x \in U \wedge \varphi(x)_n > 0 \}$$

are smooth manifolds of dim. $\dim(M) - 1$ and $\dim(M)$, respectively.

The smooth invariance of domain (applied to $N = M = \mathbb{R}^n$) implies:

Corollary: If $U \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$ is open and diffeomorphic to $V \subseteq \mathbb{R}^n_+$, then $V \cap \partial \mathbb{R}^n_+ = \emptyset$.

Corollary: If M is a smooth manifold with boundary, then $M \setminus \partial M = \mathrm{Int}(M)$.

proof: Assume this is not the case, i.e., for some $x \in M$ there are charts $(\varphi_i, U_i) \in \mathcal{A}$ s.t. $\varphi_1(x) \in \partial \mathbb{R}^n_+$ and $\varphi_2(x) \notin \partial \mathbb{R}^n_+$ with $U_1 \cap U_2 \ni x$.

Since φ_2 is a homeomorphism, there is an open neighborhood $\tilde{U}_2$ of x s.t. $\varphi_2(\tilde{U}_2) \cap \partial \mathbb{R}^n_+ = \emptyset$. Define $W := U_1 \cap U_2 \cap \tilde{U}_2 \ni x$.

Since M is smooth manifold with boundary, we have

$$f := \varphi_1 \circ \varphi_2^{-1} : \varphi_2(W) \subseteq \mathbb{R}^n_+ \longrightarrow \varphi_1(W) \subseteq \mathbb{R}^n_+ \text{ is a diffeomorphism.}$$

However, $f(\varphi_2(W)) \cap \partial \mathbb{R}^n_+ \neq \emptyset$ together with $\varphi_2(W) \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$ contradicts the previous corollary. □

(With the inv. of domain thm. the same applies to top. manifolds with boundary.)

Corollary: If $f : M \to N$ is a diffeomorphism between two smooth manifolds with boundary, then $f(\mathrm{Int}(M)) = \mathrm{Int}(N)$ and $f(\partial M) = \partial N$.

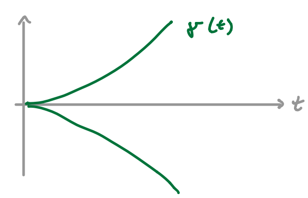
proof: Assume $f(\mathrm{Int}(M)) \cap \partial N \neq \emptyset$. Then there would be an open $U \subseteq \mathbb{R}^n_+ \setminus \partial \mathbb{R}^n_+$ that would be diffeomorphically mapped onto $V := \psi \circ f \circ \varphi^{-1}(U)$ s.t. $V \cap \partial \mathbb{R}^n_+ \neq \emptyset$. □

Def.: Let $f:(M,\mathcal{A})\longrightarrow(N,\mathcal{B})$ be a smooth map between smooth manifolds.

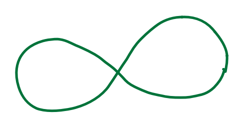
- The rank of f at $x\in M$ is the rank of $d(\psi f \varphi^{-1})_{\varphi(x)}$, where $(U,\varphi)\in\mathcal{A}$, $(V,\psi)\in\mathcal{B}$ with $x\in U$, $f(x)\in V$.
- f is an immersion if $\operatorname{rank} f = \dim(M)$ everywhere.
- f is an embedding if it is an immersion and $f: M\to f(M)$ is a homeomorphism.
- f is a submersion if $\operatorname{rank} f = \dim(N)$ everywhere.
- $y\in N$ is called a regular value of f if $\operatorname{rank} f = \dim(N)$ $\forall x\in f^{-1}(y)$. Otherwise y is called critical value of f.

remarks:
- $y\notin f(M) \Rightarrow y$ is a regular value of f.
- immersions / submersions are locally injective / surjective.
- strictly speaking it should be "smooth embedding".

examples:
- curves $\gamma:(-1,1)\to\mathbb{R}^n$ are immersions iff $\forall x\in(-1,1): \gamma'(x)\neq 0$.
  So $\gamma(t) := \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ is not an immersion ('sharp edge' at $t=0$).



- a lemniscate (figure 8) $\gamma: S^1\to\mathbb{R}^2$
  $\gamma(\cos t, \sin t) := (\sin t, \sin 2t)$
  is a non-injective immersion.



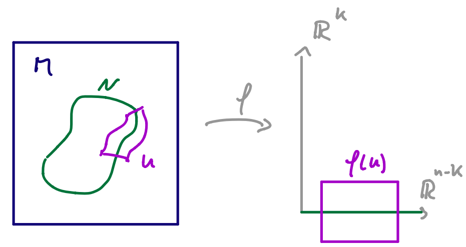
- an injective immersion, which is not an embedding: e.g. an injective curve with contact point, $\gamma:(-1,1)\to\mathbb{R}^2$ with $\lim_{t\to 1}\gamma(t)=\gamma(0)$.
  For $U:=(-\varepsilon,\varepsilon)$ the image $\gamma(U)$ is not open and thus γ not a homeomorphism.

- the map $f:\mathbb{R}^{n+1}\setminus\{0\}\longrightarrow S^n$, $f(x):=\frac{x}{\|x\|}$ is a submersion.

- inclusion maps like
  - $S^n \longrightarrow \mathbb{R}^{n+1}$, $x \mapsto x$
  - $\mathbb{R}^m \longrightarrow \mathbb{R}^{m+k}$, $x \mapsto (x_1, \dots, x_m, 0, \dots, 0)$

  are embeddings

VI. Smooth submanifolds



Def.: Let $(M, \mathcal{A})$ be a smooth manifold of dimension n. $N \subseteq M$ is a **smooth submanifold** of $(M, \mathcal{A})$ with **codimension** $k \in \{0, \dots, n\}$ if $\forall x \in N$ $\exists (U, \varphi) \in \mathcal{A}$ with $x \in U$ s.t. $U \cap N = \varphi^{-1}(\iota(\mathbb{R}^{n-k}))$ where $\iota : \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^n$, $z \mapsto (z_1, \dots, z_{n-k}, 0, \dots, 0)$.

remarks:
- a simple but important class is $M = \mathbb{R}^n$ with $\mathcal{A} = (M, \mathrm{id})$
- any linear subspace is smooth submanifold of $\mathbb{R}^n$
- the name "smooth submanifold" is justified due to the following:

Corollary: Let $(M, \mathcal{A})$ be a smooth manifold and N a smooth submanifold thereof. Then with the subspace topology N becomes a smooth manifold when equipped with a maximal atlas $\tilde{\mathcal{A}}$ containing all charts $(V, \psi) := (U \cap N, \pi \circ \varphi|_{U \cap N})$ with $(U, \varphi) \in \mathcal{A}$, $\pi : \mathbb{R}^n \to \mathbb{R}^{n-k}$, $z \mapsto (z_1, \dots, z_{n-k})$ for which $\varphi(U \cap N) = \varphi(U) \cap \iota(\mathbb{R}^{n-k})$.

proof: With the subspace topology N becomes a top. manifold. It remains to check compatibility. However, $\psi_2 \circ \psi_1^{-1} = \pi \circ \varphi_2 \circ \varphi_1^{-1} \circ \iota$ is smooth if $\varphi_2 \circ \varphi_1^{-1}$ is. $\square$

As smooth manifold the submanifold will always be understood as the pair $(N, \tilde{\mathcal{A}})$. Clearly, $\dim(N) = \dim(M) - k$.

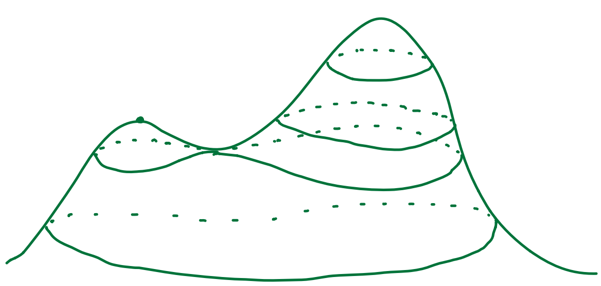
Thm.: Let $f: M \to N$ be a smooth map of constant rank r between smooth manifolds. For every $y \in f(M)$ the preimage $f^{-1}(y) \subseteq M$ is a smooth submanifold of M of codimension r.

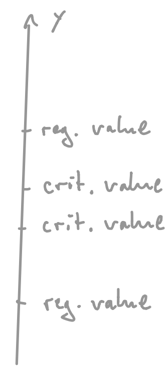
proof: By the constant rank thm. for every $x \in f^{-1}(y)$ there are charts $(U, \varphi), (V, \psi)$ with $x \in U$, $y \in V$ s.t. $\psi \circ f \circ \varphi^{-1}(z_1, \dots, z_m) = (z_1, \dots, z_r, 0 \dots 0)$ $\forall z \in \varphi(U)$ and $\psi(y) = 0$. For $z \in \varphi(U)$ we have therefore $f \circ \varphi^{-1}(z) = y \iff z = (0, \dots 0, z_{r+1}, \dots, z_m)$. Hence, $\varphi^{-1}(\iota(\mathbb{R}^{n-r})) = f^{-1}(y) \cap U$. $\square$

example: $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $x \mapsto \|x\|^2$. Then $df_x = 2x$ has rank 1 $\forall x \in \mathbb{R}^n \setminus \{0\}$, so $f^{-1}(1) = S^{n-1}$ is smooth submanifold of codimension 1.

Corollary: Let $f: M \to N$ be smooth. If $y \in N$ is a regular value, then $f^{-1}(y)$ is a smooth submanifold of M with codimension $\dim(N)$.

proof: We use that the rank is lower semicontinuous, i.e., if $(\operatorname{rank} f)(x) = r$, then there is an open neighborhood U s.t. $(\operatorname{rank} f)(x') \geq r$ $\forall x' \in U$.
This implies that there is an open set $V \subseteq M$ with $V \supseteq f^{-1}(y)$ s.t. $f: V \to M$ has constant rank $\dim(N)$. So $f^{-1}(y)$ is smooth submanifold of V and thus of M. $\square$





Example: Consider $\Lambda = \Lambda^T \in GL(n,\mathbb{R})$. Then the group $G_\Lambda := \{ A \in \mathbb{R}^{n\times n} \mid A\Lambda A^T = \Lambda \}$ is a smooth submanifold of $\mathbb{R}^{n\times n}$ of dim. $\frac{1}{2} n(n-1)$.
(Note that this includes e.g. the orthogonal and the Lorentz group.)

proof: Define $f : GL(n) \longrightarrow \mathbb{R}^{n\times n}_{sym} := \{ A \in \mathbb{R}^{n\times n} \mid A^T = A \} \simeq \mathbb{R}^{\frac{n}{2}(n+1)}$ via $f(A) := A\Lambda A^T$. Then $df_A(B) = B\Lambda A^T + A\Lambda B^T$ is, for any $A \in GL(n,\mathbb{R})$ a surjective map into $\mathbb{R}^{n\times n}_{sym}$, since for any $C = C^T$ we can choose $B := \frac{1}{2} C \Lambda^{-1} A^{-T}$ and obtain $df_A(B) = C$. Hence Λ is regular value of f. □

Thm.: Let N be a smooth manifold and $Y \subseteq N$. Then
Y is a smooth submanifold of N
$\Leftrightarrow$ Y is the image of an embedding $f : M \to N$ of a smooth manifold M.

proof: "$\Rightarrow$" Let $j : Y \to j(Y) \subseteq N$, $j(p) = p$ $\forall p \in Y$ be the inclusion map.
We want to show that j is an embedding.
Since a submanifold has the subspace topology, j is a homeomorphism onto its range. It is also an immersion since
$$\varphi \circ j \circ \varphi\big|_{U\cap Y}^{-1} \iota(z_1, \dots, z_{n-k}) = (z_1, \dots, z_{n-k}, 0 \dots 0)$$
(where $k := \operatorname{codim}(Y)$, $n := \dim(N)$), so its rank is $\dim(Y)$ everywhere.

$$\begin{tikzcd}
U \cap Y \subseteq N \arrow[r, "j"] \arrow[d, "\pi \circ \varphi|_{U\cap Y}"] & U \subseteq N \arrow[d, "\varphi"] \\
\mathbb{R}^{n-k} & \mathbb{R}^n
\end{tikzcd}$$

"$\Leftarrow$" Let $f : M \to Y = f(M) \subseteq N$ be an embedding, i.e., $f : M \to Y$ is a homeomorphism and $\operatorname{rank}(f) = \dim(M)$ everywhere. Then $\forall x \in M$ there are charts (U, φ) and (V, ψ) s.t. $x \in V$, $y = f(x) \in U$ and

$\varphi \circ f \circ \psi^{-1}(z_1, \dots, z_m) = (z_1, \dots, z_m, 0 \dots 0)$. We need to show that $U \cap \underset{f(M)}{\overset{\shortparallel}{Y}} = \varphi^{-1} \circ \iota(\mathbb{R}^m)$.

$$\begin{tikzcd} V \subseteq M \arrow[r, "f"] \arrow[d, "\psi"'] & U \subseteq N \arrow[d, "\varphi"] \\ \mathbb{R}^m & \mathbb{R}^m \times \mathbb{R}^{n-m} \end{tikzcd}$$

If $U \cap Y = f(V)$, then this is true.

Clearly, $U \cap Y \supseteq f(V)$, but similar to the "contact point" example of an injective immersion, $U \cap f(M)$ might be larger than $f(V)$. Since f is a homeomorphism onto its range, $f(V)$ is open in the subspace topology, i.e., there is an open $\tilde{U} \subseteq N$ s.t. $\tilde{U} \cap Y = f(V)$. So if U is too big, we take $\tilde{U}$ instead. $\square$

__Corollary:__ If N is a smooth submanifold of M and $f: M \to \tilde{M}$ smooth, then $f|_N : N \to \tilde{M}$ is smooth.

__proof:__ Let $j: N \to M$ be the inclusion map, then $f|_N = f \circ j$ is a composition of smooth maps. $\square$

__remark:__ similarly, one can show that if $f: N \to M$ (all smooth) and $f(N) \subseteq Y$ where Y is a smooth submanifold of M, then the induced map $f: N \to Y$ is smooth as well.

VII. The tangent bundle

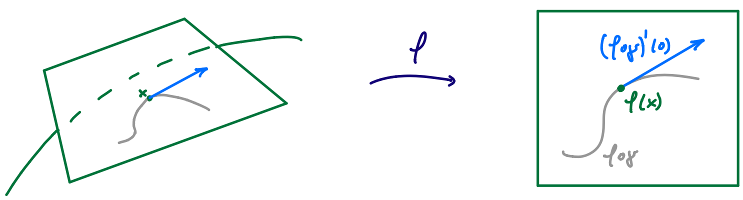
Motivating example: A smooth curve $\gamma \in C^\infty((-1,1), S^n)$ through $x := \gamma(0)$ satisfies $\|\gamma(t)\|^2 = 1$ so that $0 = \frac{d}{dt}\|\gamma(t)\|^2\Big|_{t=0} = 2\langle \gamma'(0), x\rangle$, i.e. $\gamma'(0) \perp x$.

Conversely, every $v \perp x$ is such a tangent vector for the curve

$\gamma(t) = \cos(t\|v\|)\, x + \sin(t\|v\|)\frac{v}{\|v\|}$.

The "tangent space" at x thus corresponds to the linear space $\{v \mid \langle v, x\rangle = 0\}$.

Note that it has the same dim. as the manifold.



Def.: • Let $(M, \mathcal{A})$ be a smooth manifold. A tangent vector to M at $x \in M$ is an equivalence class of curves $\gamma \in C^\infty((-1,1), M)$ through $x = \gamma(0)$ under the relation $\gamma_1 \sim \gamma_2 \iff \exists (U,\varphi) \in \mathcal{A}: \gamma_1(0) = \gamma_2(0) = x \in U$

$\wedge\ (\varphi\circ\gamma_1)'(0) = (\varphi\circ\gamma_2)'(0)$.

• The tangent space of M at x is the set of tangent vectors

$T_xM := \{[\gamma] \mid \gamma(0) = x\}$

• The principal part of the tangent vector $[\gamma] \in T_xM$ w.r.t. the chart (φ, U) is defined as $(\varphi\circ\gamma)'(0)$.

remarks: • The relation $\sim$ is indep. of the chart, since

$$(\varphi\circ\gamma)'(0) = (\varphi\circ\psi^{-1}\circ\psi\circ\gamma)'(0) \underset{\substack{\uparrow\\ \text{chain rule}}}{=} \underbrace{d(\varphi\circ\psi^{-1})_{\psi(x)}}_{\text{indep. of } \gamma}(\psi\circ\gamma)'(0)$$

That is, the principal part transform with the Jacobian of the transition map.

- The map $\Xi_{\varphi,x} : \mathbb{R}^m \to T_x M$ with $m := \dim(M)$,
$\Xi_{\varphi,x} : \xi \mapsto [\gamma(t) := \varphi^{-1}(\varphi(x) + t\xi)]$ is a bijection such that $\Xi_{\varphi,x}(\xi)$ defines an element of $T_x M$ with principle part ξ. In fact:

<u>Corollary:</u> ($T_x M$ as m-dim. vector space)

Let $(M, \mathcal{A})$ be a smooth manifold of dim. m and $(U, \varphi) \in \mathcal{A}$ s.t. $x \in U$. $T_x M$ becomes a vector space and $\Xi_{\varphi,x} : \mathbb{R}^m \to T_x M$ a vec. space isomorphism if we define for $\lambda \in \mathbb{R}$, $\eta, \xi \in \mathbb{R}^m$:

$$\lambda [\varphi^{-1}(\varphi(x) + t\xi)] + [\varphi^{-1}(\varphi(x) + t\eta)] := [\varphi^{-1}(\varphi(x) + t(\lambda\xi + \eta))] .$$

The vector space structure of $T_x M$ is indep. of the chosen chart.

<u>proof:</u> Chart-independence follows from the fact that the principal part of a tangent vector transforms (under a change of charts) according to the vector space isomorphism $d(\varphi \circ \psi^{-1})_{\psi(x)}$. □

<u>remark:</u> If $M \subseteq \mathbb{R}^n$ is a smooth submanifold embedded in some $\mathbb{R}^n$, we can define a geometrically more intuitive

$$\vec{T}_x M := \{ v \in \mathbb{R}^n \mid \gamma \in C^\infty((-1,1), M) \wedge \gamma(0) = x \wedge v = \gamma'(0) \}$$

Then $T_x M \ni [\gamma] \mapsto \gamma'(0) \in \vec{T}_x M$ turns out to be a vector space isomorphism ($\to$ exercise), so that the two spaces can be identified in many contexts.
However, $x \neq x' \Rightarrow T_x M \cap T_{x'} M = \emptyset$ whereas $\vec{T}_x M$ and $\vec{T}_{x'} M$ are not necessarily disjoint. For instance, $\vec{T}_x M = \vec{T}_{x'} M$ for $x = -x' \in S^n =: M$.

<u>Def.:</u> As a set, the <u>tangent bundle</u> TM of a smooth manifold M is defined as the disjoint union of tangent spaces: $TM := \bigsqcup_{x \in M} T_x M := \{ (x, [\gamma]) \mid x \in M, [\gamma] \in T_x M \}$

Thm.: (TM as 2m-dim. smooth manifold)

Let $(M, \mathcal{A})$ be an m-dim. smooth manifold. For every chart $(U, \varphi) \in \mathcal{A}$ define a chart (V, ϕ) for TM via $V := \{(x, [\gamma]) \in TM \mid x \in U\} = TU$ and

$\phi : V \to \phi(V) \subseteq \varphi(U) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}$, $\phi : (x, [\gamma]) \mapsto (\varphi(x), (\varphi \circ \gamma)'(0))$.

Choosing the weakest topology on TM that makes the ϕ's homeomorphisms, the resulting atlas makes TM a smooth $2m$-dim. manifold.

proof: We omit the topological part of proving that TM becomes a 2nd countable Hausdorff space and focus on C^∞-compatibility of pairs of charts $(V, \phi), (\tilde{V}, \tilde{\phi})$.

So consider $\tilde{\phi} \circ \phi^{-1} : \phi(\tilde{V} \cap V) \to \tilde{\phi}(\tilde{V} \cap V)$ and corresponding charts $(U, \varphi), (\tilde{U}, \tilde{\varphi}) \in \mathcal{A}$ with $x = \gamma(0) \in U \cap \tilde{U}$. Then

$$
\begin{aligned}
\tilde{\phi} \circ \phi^{-1}(\varphi(x), (\varphi \circ \gamma)'(0)) &= (\tilde{\varphi}(x), (\tilde{\varphi} \circ \gamma)'(0)) \\
&= (\tilde{\varphi} \circ \varphi^{-1} \circ \varphi(x), (\tilde{\varphi} \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0)) \\
&= (\tilde{\varphi} \circ \varphi^{-1}(\varphi(x)), d(\tilde{\varphi} \circ \varphi^{-1})_{\varphi(x)} (\varphi \circ \gamma)'(0))
\end{aligned}
$$

is indeed smooth, since $\tilde{\varphi} \circ \varphi^{-1}$ and $d(\tilde{\varphi} \circ \varphi^{-1})_{\varphi(x)}$ both are. $\square$

remarks:
- For $M = S^1$ the tangent bundle is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.
- However, $TS^2 \not\cong S^2 \times \mathbb{R}^2$ which is related to the hairy ball theorem. In fact, S^1, S^3 and S^7 are the only spheres with "trivial" tangent bundle. This is linked to the fact that $\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ can be equipped with a certain multiplicative structure leading to complex numbers, quaternions and octonions.
- For every $n \in \mathbb{N}$ there is a diffeomorphism $f : (TS^n) \times \mathbb{R} \to S^n \times \mathbb{R}^{n+1}$

Def.: Let $f: M \to N$ be smooth map between smooth manifolds. The differential $df_x : T_x M \to T_{f(x)} N$ of f at $x \in M$ is defined as $df_x : [\gamma] \to [f \circ \gamma]$, and the differential $df: TM \to TN$ as $df : (x, [\gamma]) \to (f(x), [f\circ\gamma])$.

remarks:
- the differential is also called tangent map or push-forward and is sometimes written Tf, f_* or f' instead of df.
- df is a smooth map.
- the principal part of $[f\circ\gamma]$ is obtained from the one of $[\gamma]$ via a linear transformation:

$$(\varphi \circ f \circ \gamma)'(0) = d(\varphi \circ f \circ \psi^{-1})_{\psi(x)} (\psi \circ \gamma)'(0) \qquad (*)$$

Since $[\gamma] \longleftrightarrow (\psi\circ\gamma)'(0)$ is a vector space isomorphism, we get:

Corollary: (i) $\forall x \in M \quad df_x : T_x M \longrightarrow T_{f(x)} N$ is linear

(ii) The chain rule $d(f\circ g)_x = df_{g(x)} \circ dg_x$ holds

(iii) $(\operatorname{rank} f)(x) = \operatorname{rank} df_x$

Thm.: Let $f: M \to N$ be smooth, $y \in N$ a regular value and $x \in f^{-1}(y) =: Z$. Then $\ker df_x = T_x Z$.

proof: If $[\gamma] \in T_x Z$, then $df_x([\gamma]) = [f\circ\gamma] = 0$ since $(f \circ \gamma)(t) = y$.
So $T_x Z \subseteq \ker df_x$. Equality holds since both are vector spaces and $\dim T_x Z = \dim Z = \dim M - \dim N$ is equal to

$$\dim \ker df_x = \dim M - \operatorname{rank} df_x \underset{\substack{\uparrow \\ f(x) \text{ is regular value}}}{=} \dim M - \dim N,$$

$\square$

remark on manifolds with boundary:

If M is a smooth submanifold of $\mathbb{R}^N$ with boundary $\partial M \neq \emptyset$, one may define a tangent vector for $x \in \partial M$ again as equivalence class of curves, where now two types of curves are considered:

(i) curves $\gamma \in C([0,1), M) \cap C^1([0,1), \mathbb{R}^N)$ starting at $x = \gamma(0)$ and

(ii) curves $\gamma \in C((-1,0], M) \cap C^1((-1,0], \mathbb{R}^N)$ ending at $x = \gamma(0)$.

The equivalence relation is again $\gamma \sim \tilde{\gamma} \Leftrightarrow \lim_{t\to 0} (f\circ\gamma)'(t) = \lim_{t\to 0} (f\circ\tilde{\gamma})'(0)$ in any chart.

In this way, T_xM is again a vector space of dim. $\dim(M)$.
Moreover, $T_x\partial M$ becomes a subspace of T_xM of codimension 1.

VIII. Sard's theorem

Def.: $X \subseteq \mathbb{R}^n$ is a set of (Lebesgue) measure zero if for any $\varepsilon > 0$ we can cover X by a set of cubes (or balls) so that their total volume is at most ε.

Lemma: Countable unions of sets of measure zero have measure zero.

proof: Let each $X_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}$ be of measure zero. Pick $\varepsilon > 0$ and a sequence of cubes $Q_i^j \subseteq \mathbb{R}^n$ s.t. $\bigcup_j Q_i^j \supseteq X_i$ with $\sum_j \mathrm{vol}(Q_i^j) < 2^{-i}\varepsilon$.
Then $\bigcup_{i,j} Q_i^j \supseteq \bigcup_i X_i$ and $\sum_{ij} \mathrm{vol}(Q_i^j) < \varepsilon \sum_i 2^{-i} = \varepsilon$. $\square$

Lemma: If $X \subseteq U \subseteq \mathbb{R}^n$ has measure zero and $f \in C^1(U, \mathbb{R}^n)$, then $f(X)$ has measure zero.

proof: Let $U \subseteq \bigcup_{i\in\mathbb{N}} B_i$ be s.t. $\forall i \,\exists k_i \in \mathbb{R}$ s.t. f is k_i-Lipschitz on B_i. If $Q_i \subseteq B_i$ is a cube of edge-length λ, then $f(Q_i)$ has edge-length at most $k_i\sqrt{n}\,\lambda$. Thus $\mathrm{vol}(f(X\cap B_i)) = 0$ so that $\mathrm{vol}(f(X)) \leq \sum_i \mathrm{vol}(\mu(f(X\cap B_i))) = 0$ $\square$

Def.: If $(M, \mathcal{A})$ is a smooth manifold of $\dim M \geq 1$, then $X \subseteq M$ is said to have **measure zero** if $\forall (U, \varphi) \in \mathcal{A}$: $\varphi(X \cap U)$ has measure zero in $\mathbb{R}^{\dim M}$.

remark: It suffices to check this for any atlas $\tilde{\mathcal{A}} \subseteq \mathcal{A}$, which can always be chosen s.t. it has only a countable number of charts. In fact, due to the 2^{nd}-countability requirement, every top. manifold is a "Lindelöf space", i.e. every open cover (like an atlas) contains a countable subcover.

Thm.: (Sard's theorem) If $f: M \to N$ is smooth, then the set of critical values of f in N has measure zero.

remark: note that the set of critical points in M, however, need not have measure zero. If for instance $f(x) = y$ is constant, then any $x \in M$ is a critical point.

proof: (for simplicity we assume $\dim M = \dim N = m$. For the general proof see, e.g. [Hirsch]) It suffices to consider a smooth map $f: [0,1]^m =: Q \to \mathbb{R}^m$. Since $f \in C^1$ and Q is compact, we have Lipschitz-continuity:

$\forall x, x' \in Q$: $\| f(x) - f(x') \| \leq L \| x - x' \|$ for some $L \in [0, \infty)$.

Let $c \in M$ be a critical point. Then $df_c(Q)$ is contained in a proper subspace of $\mathbb{R}^m$. Hence, there is a hyperplane $H \subseteq \mathbb{R}^m$ with

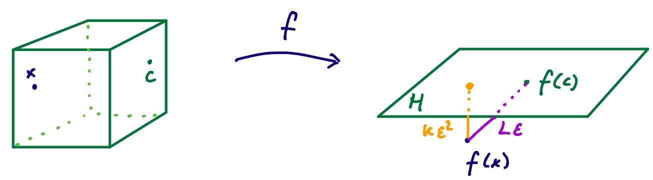
$H \supseteq \{ y \in \mathbb{R}^m \mid y \in df_c(Q) + f(c) \}$.

By Taylor's thm. with remainder there is a $K \in [0, \infty)$ s.t. $\forall x \in Q$:

$$\inf_{y \in H} \| f(x) - y \| \leq \| f(x) - (f(c) + df_c(x-c)) \| \leq K \| x - c \|^2 .$$

Thus $\|x-c\| \leq \varepsilon \Rightarrow \|f(x)-y\| \leq K\varepsilon^2$ for some $y \in H$

$\wedge \ \|f(x)-f(c)\| \leq L\varepsilon$



The image of a cube that contains x & c and has edge length $\varepsilon\sqrt{m}$ is therefore contained in a cuboid of volume $(2K\varepsilon^2)(2L\varepsilon)^{m-1} = O(\varepsilon^{m+1})$

Now consider $Q = \bigcup_{i=1}^{\Delta^m} Q_i$ subdivided into Δ^m cubes of edge length $\frac{1}{\Delta}$.

Let I be s.t. $i \in I \Leftrightarrow Q_i$ contains a critical point. Then

$$\mathrm{vol}\left[f\left(\bigcup_{i\in I} Q_i\right)\right] \leq \sum_{i\in I} \mathrm{vol}[f(Q_i)] \underset{\substack{\uparrow \\ \varepsilon\sqrt{m} \leq \frac{1}{\Delta}}}{\leq} |I|\, O(\Delta^{-m-1}) \underset{\substack{\uparrow \\ |I| \leq \Delta^m}}{\leq} O(\Delta^{-1}) \qquad \square$$

We can extend Sard's thm. to the case where M is allowed to be a smooth manifold with boundary:

<u>Thm.:</u> (Sard's thm. for manifolds with boundary)

Let $F: M \to N$ be a smooth map from a smooth manifold with boundary ∂M to a smooth manifold N. The subset of N containing points that are either critical values of F or of $f := F|_{\partial M}$ has measure zero.

<u>proof:</u> If $x \in \partial M$, then $T_x\partial M$ is a subspace of T_xM and df_x is the restriction of dF_x to that subspace. Hence, rank $df_x = \dim(N)$ implies rank $dF_x = \dim N$.

So every critical value of F is either critical for f or for $\tilde{F} := F|_{\mathrm{int}(M)}$. The claim then follows from Sard's thm. applied to f and $\tilde{F}$. $\square$

<u>Corollary:</u> Let M be a smooth manifold with (possibly empty) boundary, N a smooth manifold and $f: M \to N$ smooth. Then f has a regular value.

<u>proof:</u> If $\dim N > 0$, this follows from Sard's Theorem.

If $\dim N = 0$, i.e. N is discrete, then it is trivially true since $d(f \circ f \circ \varphi)_{\varphi(x)} : \mathbb{R}^{\dim(M)} \to \{0\}$ is always surjective. $\square$

IX. Morse Functions

Def.: Let $f: M \to \mathbb{R}$ be smooth.

- A critical point $x_0 \in M$ of f is called <u>nondegenerate</u> if there is a chart (U, φ) around x_0 s.t. the Hessian $H_g(\varphi(x_0))$ of $g := f \circ \varphi^{-1}$ is nonsingular.
- The number of negative eigenvalues of $H_g(\varphi(x_0))$ is called the <u>index</u> of the nondegenerate critical point.
- f is called a <u>Morse function</u> if all its critical points are nondegenerate.

remark: The definition is indep. of the chart. If $(\tilde{\varphi}, \tilde{U})$ is another chart, then the Hessians are related via $\tilde{H} = J^T H J$ where J is the Jacobi matrix of $d\psi_{x_0}$ with $\psi := \varphi \circ \tilde{\varphi}^{-1}$. The claim then follows from Sylvester's law of inertia together with the fact that ψ is a diffeomorphism.

There are plenty of Morse functions:

Thm.: Let $U \subseteq \mathbb{R}^n$ be open and $f: U \to \mathbb{R}$ smooth. For $a \in \mathbb{R}^n$ define $f_a(x) := f(x) + \langle a, x \rangle$. Then $\{ a \in \mathbb{R}^n \mid f_a \text{ is not Morse} \}$ has measure zero in $\mathbb{R}^n$.

proof: Define $g: U \to \mathbb{R}^n$ as $g(x) := \nabla f(x)$. Then $\nabla f_a(x) = g(x) + a$. Hence
$\nabla f_a(x) = 0 \iff g(x) = -a$
and $H_f(x) = dg_x$. So if $-a$ is regular value of g, then all critical points of f_a are nondegenerate. Now apply Sard. $\square$

Lemma: Around any invertible matrix $H \in \mathbb{R}^{n\times n}_{sym}$ there exists a neighborhood $\mathcal{U} \subseteq \mathbb{R}^{n\times n}_{sym}$ and a smooth map $\Theta : \mathcal{U} \to GL(n,\mathbb{R})$ s.t.

$$\Theta(X)\, X\, \Theta(X)^T = X \quad \forall X \in \mathcal{U}.$$

That is, there is a neighborhood of congruent matrices and the congruence transformation is smooth.

Thm.: (Morse's Lemma v.1) Let $f \in C^{k+2}(U,\mathbb{R})$ be defined on an open set $U \subseteq \mathbb{R}^m$ and $x_0 \in U$ a non-degenerate critical point of f. $k \geq 1$. There exist open sets $V \ni x_0$ and W in $\mathbb{R}^n$ and a C^k-diffeomorphism $\vartheta : V \to W$ s.t.

$$f(x) = f(x_0) + \tfrac{1}{2} \langle H_f(x_0)\vartheta(x), \vartheta(x) \rangle .$$

proof idea: W.l.o.g. $U \subseteq \mathbb{R}^m$ convex, $U \ni x_0 = 0$ and $f(x_0) = 0$.

For $x \in U \setminus \{0\}$ define $g(t) := f(tx)$. Then

$$f(x) = g(1) \overset{\text{Cauchy's version of Taylor's thm.}}{=} g(0) + g'(0) + \int_0^1 (1-t)\, g''(t)\, dt$$

$$= f(x_0) + \tfrac{1}{2} \langle A(x)\, x, x \rangle$$

$$\text{where} \quad A(x) := 2\int_0^1 (1-t)\, H_f(tx)\, dt$$

since $g'(t) = \langle \nabla f(tx), x \rangle$, $\nabla f(x_0) = 0$

and $g''(t) = \langle H_f(tx)\, x, x \rangle$

$A \in C^k(U, \mathbb{R}^{m\times m}_{sym})$ and $A(0) = H_f(x_0)$

If $V \subseteq U$ is sufficiently small, the Lemma provides a map $\vartheta \in C^k(V, GL(n,\mathbb{R}))$ s.t. $\vartheta(x)\, H_f(x_0)\, \vartheta(x) = \langle A(x)\, x, x \rangle$

Employing the inverse function thm. we can make (ϑ, V) a C^k-chart. $\square$

Cor.: (Morse's Lemma v.2)

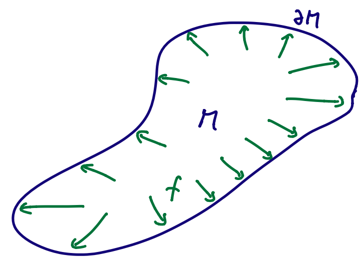
Let $x_0 \in M$ be a nondegenerate critical point of index i of $f \in C^{k+1}(M, \mathbb{R})$, $k \geq 1$. Then there is a C^k-chart (φ, U) around x_0 s.t. $f \circ \varphi^{-1}(x_1, \dots, x_m) = f(x_0) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{m} x_j^2$.

Cor.: If $f: M \to \mathbb{R}$ is a Morse function, then its critical points are isolated. If M is, in addition, compact, then there are finitely many critical points.

remarks: Morse functions contain a lot of information about the topology of the manifold. For instance:

- If $f: M \to \mathbb{R}$ is a Morse function with two critical points on a n-dim compact manifold M, then M is homeomorphic to S^n. (Milnor found 'exotic spheres' that are homeomorphic but not diffeomorphic to S^7)
- Smale used Morse theory to prove the 'h-cobordism thm.' and the Poincaré conjecture in dim ≥ 5.
- Morse's thm. states that for $f: M \to \mathbb{R}$ Morse, the 'Euler characteristic' $\chi(M)$ can be computed via $\chi(M) = \sum_i (-1)^i c_i(f)$, where c_i is the number critical points of f of index i.

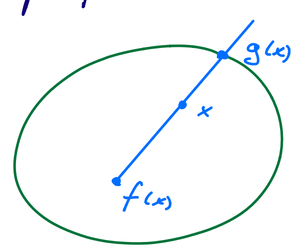
X BROUWER'S FIXED POINT THEOREM



Thm.: (No retraction thm.)
Let M be a compact smooth manifold with boundary $\partial M \neq \emptyset$. There is no smooth map $f: M \to \partial M$ s.t. $f|_{\partial M} = id$.

proof: Suppose such a map existed.
By Sard's thm. f must have a regular value $y \in \partial M$.
Then $f^{-1}(\{y\}) =: N$ is a smooth 1-dim. manifold with boundary $\partial N \ni y$. ∂N cannot contain any other point, since $\partial N \subseteq \partial M$ and for any $x \in \partial M$: $f(x) = x$.
As a closed subset of a compact space N is compact.
Any compact smooth 1-manifold, however, has an even number of boundary points. □

Thm.: (Brouwer's fixed point thm. - smooth version)
Consider $D^n := \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq 1\}$ as a smooth submanifold with boundary of $\mathbb{R}^n$. Every smooth map $f: D^n \to D^n$ has a fixed point, i.e., $\exists \hat{x} \in D^n : f(\hat{x}) = \hat{x}$.



proof: Suppose f has no fixed point. Then define $g: D^n \to \partial D^n = S^{n-1}$ s.t. $g(x) = x + t(x - f(x))$ for some $t \geq 0$. Since g is smooth, this contradicts the no-retraction thm. □

Thm.: (Brouwer's fixed point thm. - general version)

Let B be homeomorphic to D^n. If $f: B \to B$ is continuous, then it has a fixed point.

proof: It suffices to consider $B = D^n$ since if $\varphi: B \to D^n$ is a homeomorphism, then $\tilde{f} := \varphi \circ f \circ \varphi^{-1}: D^n \to D^n$ is continuous and has a fixed point iff f has one.

Since D^n is compact, we can exploit the Stone-Weierstrass thm. and approximate f by a polynomial & thus smooth map $p: D^n \to \mathbb{R}^n$ s.t. $\| f(x) - p(x) \| < \varepsilon \;\; \forall x \in D^n$.

$F(x) := (1+\varepsilon)^{-1} p(x)$ is then smooth and s.t. $F: D^n \to D^n$ since $\| F(x) \| = (1+\varepsilon)^{-1} \| p(x) \| \leq (1+\varepsilon)^{-1} \left(\| f(x) \| + \| p(x) - f(x) \| \right) \leq 1$.

Moreover, $\| F(x) - f(x) \| = (1+\varepsilon)^{-1} \| p(x) - (1+\varepsilon) f(x) \| < 2\varepsilon$.

If f had no fixed point, then $\mu := \inf\limits_{x \in D^n} \| f(x) - x \| > 0$.

With $\varepsilon := \frac{\mu}{2}$ we get $\forall x \in D^n$:

$$\begin{aligned} \| F(x) - x \| &\geq \| f(x) - x \| - \| f(x) - F(x) \| \\ &> \quad \mu \quad - \quad 2\varepsilon \quad = 0 \end{aligned}$$

So the smooth map $F: D^n \to D^n$ would have no fixed point. $\square$

Thm.: (Brouwer's invariance of domain)

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a continuous injection, then it is open.

proof (sketch):

It suffices to show that for any continuous injection $f: D^n \to \mathbb{R}^n$, $f(0) \in \operatorname{Int} f(D^n)$.

First note that $f: D^n \to f(D^n)$ is closed since a closed set $A \subseteq D^n$ is compact, mapped to a compact set $f(A)$, which is closed since it is a compact subset of a Hausdorff space.

So $f^{-1}: f(D^n) \to D^n$ is continuous.

Then it also has a cont. extension $G: \mathbb{R}^n \to \mathbb{R}^n$ (Tietze extension thm.) for which $G(f(0)) = 0$.

Assume $f(0) \in \partial D^n$, i.e. the zero of G lies on the boundary of $f(D^n)$.

Then, we can construct a perturbation $\tilde{G} \in C(f(D^n), \mathbb{R}^n)$ with $\|\tilde{G} - G\|_\infty \leq 1$ that has no zero in $f(D^n)$.

However, by Brouwer's fixed point thm., $D^n \ni x \mapsto x - \tilde{G}(f(x)) = (G - \tilde{G})(f(x))$ must have a fixed point $z \in D^n$ for which then $\tilde{G}(f(z)) = 0$. $\square$

Cor.: (invariance of dimension) There is a cont. injection $f: \mathbb{R}^n \to \mathbb{R}^m$ iff $n \leq m$.

proof: Let $n < m$, $\iota: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto (x, 0)$. Suppose $f: \mathbb{R}^m \to \mathbb{R}^n$ is a cont. injection. Then $\iota \circ f: \mathbb{R}^m \to \mathbb{R}^m$ would be a cont. injection that, however, cannot be open. $\square$

XI. EMBEDDINGS

Recall: A smooth map $f: M \to N$ between smooth manifolds is an immersion if $\forall x \in M$: $\operatorname{rank}(df_x) = \dim M$ and it is a smooth embedding if it is an immersion & $f: M \to f(M)$ a homeomorphism. Equivalently, it is a smooth embedding if $f: M \to f(M)$ is a diffeomorphism.

Lemma: If M is compact, any injective immersion is a smooth embedding.

proof: It remains to show that $f^{-1}: f(M) \to M$ is continuous, which is equivalent to $f: M \to f(M)$ being closed.
Let $A \subseteq M$ be closed. Since M is compact, A is compact, as well and so is $f(A)$ by continuity of f. Being a compact subset of a Hausdorff space, $f(A)$ is closed. $\square$

Thm.: Let M be a smooth m-manifold, $f: M \to \mathbb{R}^N$ an injective immersion and $P_v: \mathbb{R}^N \to \mathcal{H} \simeq \mathbb{R}^{N-1}$ the orthogonal projection onto the $N-1$-dim. subspace orthogonal to $v \in S^{N-1}$, i.e., $\mathcal{H} = P_v \mathbb{R}^N = \{x \in \mathbb{R}^N \mid x \perp v\}$.
If $N > 2m+1$, then for all $v \in S^{N-1}$ except a set of measure zero in S^{N-1} $P_v \circ f: M \to \mathcal{H} \simeq \mathbb{R}^{N-1}$ is an injective immersion.

proof: We first prove injectivity. To this end, define a smooth map $g: M \times M \setminus \Delta_M \to S^{N-1}$ by $g(x,y) := \dfrac{f(x)-f(y)}{\|f(x)-f(y)\|}$. Since $\Delta_M := \{(x,y) \in M \times M \mid x = y\}$ is closed, $M \times M \setminus \Delta_M$ is a smooth $2m$-manifold.
The image of g has measure zero in S^{N-1} if $2m < N-1$, as assumed.
However, if $P_v \circ f$ is not injective, then there are $x \neq y$ s.t. $f(x) - f(y) \propto v$, so necessarily $v \in g(M \times M \setminus \Delta_M)$.

To prove the immersion property, we argue similarly: define $h: TM \setminus (M \times \{0\}) \to S^{N-1}$ as $h(x,\tau) := \frac{df_x \tau}{\| df_x \tau \|}$ where $x \in M$ and $\tau \in T_x M \setminus \{0\}$. Again the image of h has measure zero if $N-1 > 2m$. For $P_v \circ f$ to be an immersion, we have to have $\operatorname{rank}(d(P_v f)_x) = m$ for all $x \in M$. However, $d(P_v f)_x = P_v \, df_x$ has rank m for all $x \in M$ unless v is in the image of h. $\square$

Corollary: (Whitney's embedding thm. - easy version)
If M is a compact manifold of dim. m, then it can be embedded in $\mathbb{R}^{2m+1}$.

proof: By a smooth version of the embedding thm. there is a smooth embedding into some $\mathbb{R}^N$. Due to compactness injective immersions are embeddings. Iterating the previous thm. we can reduce the dimension down to $2m+1$. $\square$

remark: We can learn two more things from the proof:

- embeddings are not exceptional and, in fact, dense in the set of all smooth maps into $\mathbb{R}^{2m+1}$ (see [Hirsch] for a proof).
- if we are only interested in an immersion we can reduce the dimension further by one via restricting to unit vectors in the tangent space.

Thm.: (Whitney's embedding & immersion theorem - strong version) Any smooth m-manifold can be smoothly embedded in $\mathbb{R}^{2m}$ and immersed in $\mathbb{R}^{2m-1}$ if $m > 1$.

remarks:

- this is the best possible affine bound since $\mathbb{RP}^m$ cannot be embedded in $\mathbb{R}^{2m-1}$ if m is a power of two.
- for immersions a tight bound is known (proven by Cohen in '85): any compact smooth m-manifold can be immersed in $\mathbb{R}^{2m-\alpha(m)}$, where $\alpha(m)$ is the number of ones in the binary expansion of m.

XII. HOMOTOPY & MOD2-DEGREE

"Homotopy" formalizes continuous/smooth deformations of maps.

Def.: Let $f, g : M \to N$ be smooth maps between smooth manifolds.

- f & g are called smoothly homotopic if there is a smooth map $F : M \times [0,1] \to N$ s.t. $\forall x \in M$: $F(x,0) = f(x)$, $F(x,1) = g(x)$. We write $F_t(x) := F(x,t)$.
- F is then called smooth homotopy between f & g.
- The equivalence class $[f]$ of maps smoothly homotopic to f is its smooth homotopy class.

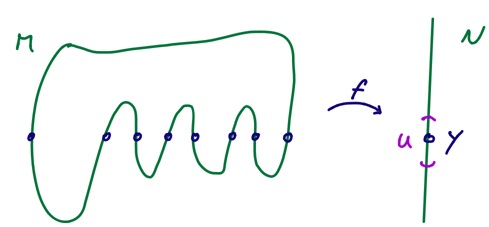
Lemma: ("stack of records")

Let M be compact, $f : M \to N$ smooth with $\dim(M) = \dim(N)$ and $y \in N$ a regular value. (i.e., $f(x) = y \Rightarrow df_x$ surjective) Then:

(i) $f^{-1}(y) = \{x_1, \dots, x_k\}$ is finite

(ii) There is an open neighborhood $U \ni y$ s.t. $f^{-1}(U) = V_1 \sqcup \dots \sqcup V_k$ where each V_j is an open neighborhood of x_j and $f : V_j \to U$ is a diffeomorphism.

In particular, $\tilde{y} \mapsto |f^{-1}(\tilde{y})|$ is constant on U and the set of regular values is open.



Our aim is to show that $|f^{-1}(y)| \bmod 2$ is independent of the regular value and constant on the smooth homotopy class of f.

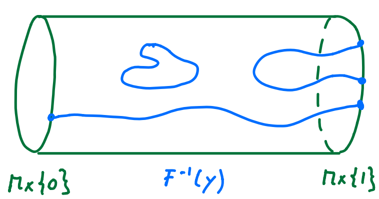
<u>Lemma:</u> M compact smooth, N smooth (possibly with boundary), $\dim(M) = \dim(N)$. If $f, g: M \to N$ are smoothly homotopic & $y \in M$ is a regular value of both f and g, then

$$|f^{-1}(y)| \bmod 2 = |g^{-1}(y)| \bmod 2 .$$

<u>proof:</u> Let $F: M \times [0,1] \to N$ be a smooth homotopy s.t. $F_0 = f$, $F_1 = g$. Assume y is a regular value for F. Then $F^{-1}(y)$ is a compact 1-dim. smooth manifold with boundary

$$\begin{aligned} \partial(F^{-1}(y)) &= F^{-1}(y) \cap \partial(M \times [0,1]) \\ &= F^{-1}(y) \cap (M \times \{0\} \cup M \times \{1\}) \\ &= f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\} \end{aligned}$$

Since $|\partial(F^{-1}(y))|$ must be even, the same has to be true for $|f^{-1}(y)| + |g^{-1}(y)|$. So $|f^{-1}(y)| \bmod 2 = |g^{-1}(y)| \bmod 2$.



If y is not a regular value of F, take U_f, U_g open neighborhoods of y s.t. $|f^{-1}(y)|$ and $|g^{-1}(y)|$ are constant on U_f & U_g.

By Sard's thm. there is a regular value $\tilde{y} \in U_f \cap U_g$ of F for which

$$\begin{aligned} \text{even} = |\partial(F^{-1}(\tilde{y}))| &= |f^{-1}(\tilde{y})| + |g^{-1}(\tilde{y})| \\ &= |f^{-1}(y)| + |g^{-1}(y)| \end{aligned}$$ $\square$

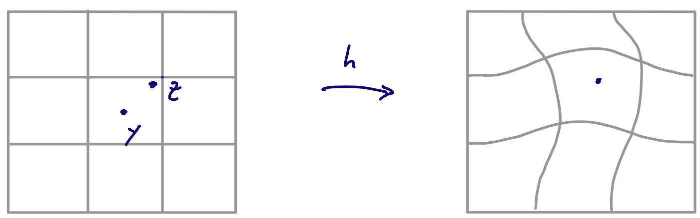
<u>Def.:</u> Two diffeomorphisms $f, g: M \to N$ are called "<u>smoothly isotopic</u>" if there is a smooth homotopy $F: M \times [0,1] \to N$ s.t. F_t is a diffeomorphism for all $t \in [0,1]$.

Lemma: Let y, z be two points in a connected smooth manifold N. There is a diffeomorphism $h: N \to N$ s.t.

(i) $h(y) = z$ and (ii) h is smoothly isotopic to the identity.

Moreover, the proof ensures that $\overline{\{x \in N \mid h_t(x) \neq x\}}$ is compact $\forall t \in [0,1]$, where h_t is the isotopy with $h_0 = h$, $h_1 = id$.

proof: The set of z's for which this is true for a given y forms an equivalence class. We will prove that this class is an open set. Since N is then a disjoint union of open sets, being connected implies that there is only one class, which then consists of the entire manifold.



$h(y) = z$

Using charts we can assume that $N = \mathbb{R}^n$, $y = 0$ and $z = (z_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Choose $f_n \in C^\infty(\mathbb{R}^n, \mathbb{R})$ s.t. $f_n(0) = 1$ and $\|x\| \geq \varepsilon \Rightarrow f_n(x) = 0$.

Define $h_t : \mathbb{R} \times \mathbb{R}^{n-1} \ni (a, b) \mapsto (a + t\, f_{n-1}(b)\, f_1(a)\, z_1,\ b)$.

Then (i) $h_1(0) = z$

(ii) $h_0 = id$

(iii) $h_t(x) = x$ if $\|x\| > \sqrt{2}\,\varepsilon$

It remains to show that h_t is a diffeomorphism for suff. small $\|z\|$.

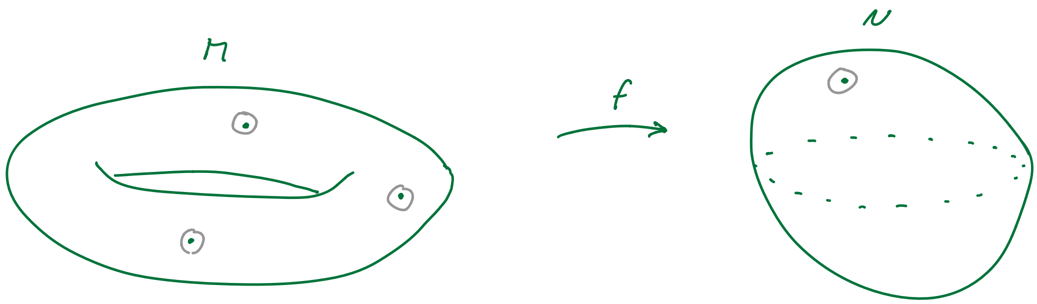
We first prove that it is bijective. Consider $g : a \mapsto f_1(a) t f_{n-1}(b) z_1$.

Then $g'(a) = 1 + f_1'(a) t f_{n-1}(b) z_1 > 0$ $\forall t \in [0,1]$ and small enough $\|z\|$.

Hence h_t is bijective. Moreover,

$$d(h_t)_{(a,b)} = \left(\begin{array}{c|c} 1 + f_1'(a)\, t\, f_{n-1}(b)\, z_1 & * * * \\ \hline 0 & \mathbb{1} \end{array}\right)$$

So by the inverse function thm. h_t is a diffeomorphism for suff. small $\|z\|$. $\square$



<u>Thm.:</u> Let M, N be smooth manifolds of equal dimension, M compact and N connected (and possibly with boundary). For a smooth map $f: M \to N$ with regular value y the "<u>mod-2-degree</u>" of f $\deg_2(f) := |f^{-1}(y)| \bmod 2$ is independent of the choice of the regular value y and depends only on the smooth homotopy class of f.

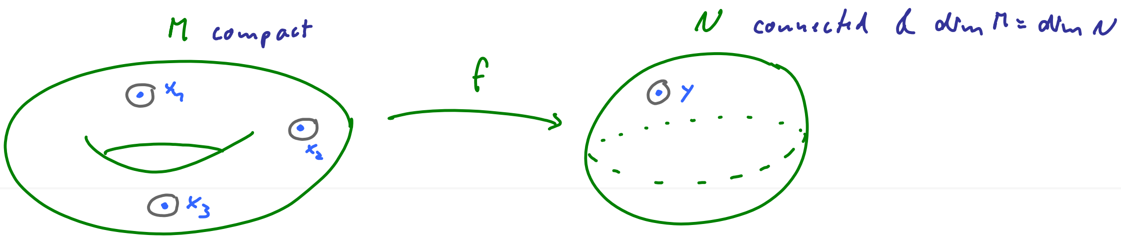
<u>proof:</u> Let y & z be two regular values and h_t an isotopy as in the previous Lemma so that $h_1(y) = z$.
Then z is a regular value of $h_1 \circ f$ and by using smooth homotopy:

$$|f^{-1}(y)| \bmod 2 \underset{y = h_1^{-1}(z)}{=} |(h_1 \circ f)^{-1}(z)| \bmod 2 \underset{h_1 \text{ smoothly homotopic to } h_0 = id}{=} |f^{-1}(z)| \bmod 2$$

If f & g are smoothly homotopic, then $\deg_2(f) = \deg_2(g)$ if there is a common regular value, which always exists by Sard's thm. $\square$

<u>remarks</u>:

- $\deg_2(f)$ is only defined if $\dim(M) = \dim(N)$, N is connected & M is compact.
- If $\deg_2(f) = 0$ we say 'f has even degree' (and 'odd degree' if $\deg_2(f) = 1$)
- beyond mod 2 a similar notion of degree ('Brouwer-degree') can be defined if M & N are 'orientable'.



Thm.: Let M, N be smooth manifolds of the same dimension, M compact and N connected (and possibly with boundary). Then for a smooth map $f: M \to N$ with regular value y the "mod 2 degree" of f $\deg_2(f) := |f^{-1}(y)| \bmod 2$ is independent of the choice of the regular value y and depends only on the smooth homotopy class of f.

proof: Let y & z be two regular values and h an isotopy as in the previous Lemma so that $h_1(y) = z$.

Then z is a regular value of $h_1 \circ f$ and by using homotopy:

$$|f^{-1}(y)| \bmod 2 \underset{\substack{\uparrow \\ y = h_1^{-1}(z)}}{=} |(h_1 \circ f)^{-1}(z)| \bmod 2 \underset{\substack{\uparrow \\ h_1 \text{ smoothly homotopic to } h_0 = id}}{=} |f^{-1}(z)| \bmod 2$$

If f & g are smoothly homotopic, then $\deg_2(f) = \deg_2(g)$ if there is a common regular value, which always exists by Sard's thm.

$\square$

remarks:
- $\deg_2(f)$ is only defined if $\dim(M) = \dim(N)$, N is connected & M is compact.
- If $\deg_2(f) = 0$ we say that f has "even degree" (and "odd degree" if $\deg_2(f) = 1$).
- beyond mod 2 the notion of a "degree" is only defined if M & N are "oriented".

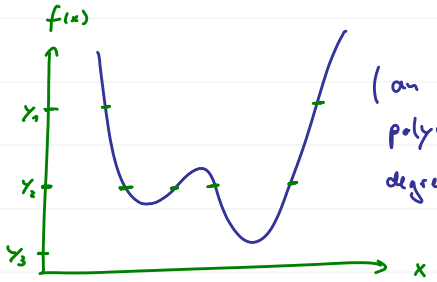
Examples:
- $id: M \to M$ has odd degree (as well as any other bijective map)
- A const. map $f: M \to M$ with $f: x \mapsto c$ has even degree, since any $y \neq c$ is a regular value & $|f^{-1}(y)| = 0$.

- More generally, if $f: M \to N$ is not surjective, f has even degree.

- As a consequence, f has even degree if N is not compact (since $f(M)$ is) or N has boundary (since then f is homotopic to a map that is not surjective - by 'contracting' $f(M)$ & moving it away from the boundary)

- Note that this generalizes the notion of the degree of a polynomial:

$|f^{-1}(y_i)| \bmod 2 = 0$



(an example of a polynomial with even degree)

XI. Winding number mod 2

Def.: Let N be a compact smooth manifold with $\dim N = n$ and let $f: N \to \mathbb{R}^{n+1}$ be a smooth map and $y \in \mathbb{R}^{n+1} \setminus f(N)$.
The "mod 2 - winding number" of f around y is defined as

$$W_2(f,y) := \deg_2(v)$$

where $v: N \to S^n$ is defined as $v(x) := \dfrac{f(x)-y}{\| f(x)-y \|}$.

remarks:
- Note that $|v^{-1}(z)|$ is the number of times $f(x)-y$ points in the same direction as z if we vary x over all of N. So $W_2(f,y)$ is the mod 2 of this number.

- If $N = S^1$ it is easy to see that $W_2(f,y)$ is indeed the mod 2 of the familiar winding number of the closed curve $f(N)$ around y.

<u>Thm.:</u> Let M be a compact n-dim. smooth manifold with boundary $\partial M \neq \emptyset$, $f: \partial M \to \mathbb{R}^n$ a smooth map and $F: M \to \mathbb{R}^n$ smooth s.t. $F|_{\partial M} = f$.
If $y \notin f(\partial M)$ is a regular value of F, then $F^{-1}(y)$ is a finite set &
$W_2(f, y) = |F^{-1}(y)| \bmod 2$.

<u>proof:</u> Suppose first that $y \notin F(M)$.

Then $V: M \to S^{n-1}$, $V(x) := \dfrac{F(x) - y}{\|F(x) - y\|}$ is well defined and $v = V|_{\partial M}$.

By Sard's thm. there exists a common regular value $z \in S^{n-1}$ of v & V.

$V^{-1}(z)$ is then a compact 1-dim. smooth manifold so that

$$\partial(V^{-1}(z)) = \partial M \cap V^{-1}(z) = v^{-1}(z) \text{ contains an even number of points.}$$

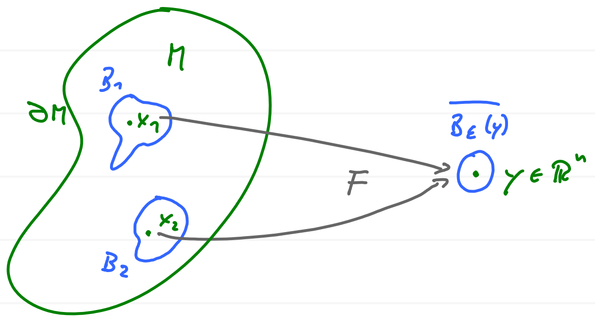
So indeed $W_2(f, y) \underset{\text{Def.}}{=} \deg_2(v) = 0 \underset{y \notin F(M)}{=} |F^{-1}(y)| \bmod 2$

Now consider the complementary case $F^{-1}(y) \neq \emptyset$

By assumption $y \notin f(\partial M) = F(\partial M)$, so that $\operatorname{Int} M \supseteq F^{-1}(y) = \{x_1, \dots, x_k\}$

According to the stack-of-records thm. there are disjoint open neighborhoods $U_j \ni x_j$ and $\tilde{U} \ni y$ s.t. $F|_{U_j}: U_j \to \tilde{U}$ are diffeomorphisms.

Take a closed ball $\overline{B_\varepsilon(y)} \subseteq \tilde{U}$ with radius $\varepsilon > 0$ around y and define $B_j \subseteq U_j$ to be the closed preimages of $\overline{B_\varepsilon(y)}$ under $F|_{U_j}$.



Define $\tilde{M} := M \setminus \left(\bigcup_{j=1}^{k} \operatorname{Int} B_j \right)$, $\tilde{F} := F|_{\tilde{M}}$, $\tilde{V} := V|_{\tilde{M}}$, $\tilde{v} := \tilde{V}|_{\partial \tilde{M}}$.

Then $y \notin \tilde{F}(\tilde{M})$, so we are back at the first case and know that $\tilde{v}$ has even degree.

Moreover, $\tilde{v}^{-1}(z) = v^{-1}(z) \sqcup v_1^{-1}(z) \sqcup \dots \sqcup v_k^{-1}(z)$ where

$$v_j : \partial B_j \to S^{n-1}, \quad v_j(x) := \frac{F(x) - y}{\| F(x) - y \|} = \frac{F(x) - y}{\varepsilon}$$

So $0 = \deg_2(\tilde{v}) = \deg_2(v) + \sum_{j=1}^{k} \deg_2(v_j) \mod 2$ and therefore

$$\deg_2(v) = \sum_{j=1}^{k} \deg_2(v_j) \mod 2$$

By the choice of B_j we have that v_j is bijective and therefore $\deg_2(v_j) = 1$.

Hence, $\deg_2(v) = k \mod 2 = |F^{-1}(y)| \mod 2$

Suppose $M \subseteq \mathbb{R}^n$ is a compact smooth submanifold and ∂M a connected "hypersurface" (i.e. of co-dimension 1) with inclusion map $f : \partial M \to \mathbb{R}^n$.
Then for $x \notin \partial M$ the value of $w_2(f, x)$ separates $\mathbb{R}^n$ in "inside" $(w_2(f,x) = 1)$ & "outside" $(w_2(f,x) = 0)$. More generally:

Thm.: (Jordan-Brouwer Separation thm.)

Let X be a compact, connected smooth submanifold of $\mathbb{R}^n$ with co-dim. 1. Then $\mathbb{R}^n \setminus X = A_0 \cup A_1$ where the A_i's are disjoint connected smooth submanifolds of $\mathbb{R}^n$, A_1 is bounded and A_0 and A_1 have point set boundaries $\partial A_0 = \partial A_1 = X$.

proof: $\to$ see [Guillemin Pollack] for the idea.

XII. The Borsuk-Ulam theorem

Def.: We call a map $f: S^n \to \mathbb{R}^m$ "odd" if $\forall x \in S^n$: $f(-x) = -f(x)$.

Examples: the antipodal map $x \mapsto -x$, $x \mapsto \sin x$ and polynomials with only odd degree terms

Thm. (Borsuk-Ulam I) Let $f: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ be an odd smooth map. Then $W_2(f, 0) = 1$.

That is, an odd map must wind around the origin an odd number of times.

here is an alternative formulation:

Thm. (Borsuk-Ulam II) Let $\phi: S^n \to S^n$ be an odd smooth map. Then $\deg_2 \phi = 1$.

proof that (BU I $\Leftrightarrow$ BU II) $\forall n \in \mathbb{N}$:

BU I $\Rightarrow$ BU II: We may consider $\phi: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$. Then by BU I

$$1 = W_2(\phi, 0) = \deg_2\left(x \mapsto \frac{\phi(x)}{\|\phi(x)\|}\right) = \deg_2(\phi)$$

BU II $\Rightarrow$ BU I: Setting $\phi(x) = \frac{f(x)}{\|f(x)\|}$ we get

$$W_2(f, 0) = \deg_2\left(x \mapsto \frac{f(x)}{\|f(x)\|}\right) = \deg_2(\phi) \qquad \square$$

proof: of BU I by induction. Assume it is true for $n-1$, $n \geq 2$.

Consider S^{n-1} to be the equator of S^n, i.e., embedded by the inclusion map $\iota: (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. So

$$S^{n-1} \simeq \iota(S^{n-1}) = \{(x, 0) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| = 1\} \subseteq S^n.$$

Define $g: S^{n-1} \to \mathbb{R}^{n+1}$ as $g(x) := f(\iota(x))$, i.e., $g = f|_{\iota(S^{n-1})}$ restricted to the equator. By Sard's thm. $\hat{g} := \frac{g}{\|g\|} : S^{n-1} \to S^n$ and $\hat{f} := \frac{f}{\|f\|} : S^n \to S^n$ have a common regular value, say $y \in S^n$. By symmetry also $-y$ is a common regular value.

Since for $\hat{g}$ the image space has larger dimension than the preimage (and therefore $d\hat{g}_x$ can never be surjective), y being a regular value means that it is not in the image of $\hat{g}$. Consequently, $g(S^{n-1})$ does not intersect the ray $\mathbb{R}\cdot y$.

For $\hat{f}$, on the other hand, y being a regular value implies that

$$|\hat{f}^{-1}(y)| \bmod 2 = \deg_2(\hat{f}) = W_2(f, 0).$$

Using symmetry we get $|\hat{f}^{-1}(y)| = \frac{1}{2}|f^{-1}(\mathbb{R}y)|$. Since f does not map points on the equator into $\mathbb{R}y$, it suffices to restrict to the upper hemisphere $S^n_+ := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \wedge x_{n+1} \geq 0\}$ and to consider $f_+ := f|_{S^n_+}$. Then $\frac{1}{2}|f^{-1}(\mathbb{R}y)| = |f_+^{-1}(\mathbb{R}y)|$ and thus

$$W_2(f, 0) = f_+^{-1}(\mathbb{R}y) \bmod 2 \qquad (1)$$

Now S^n_+ is a manifold with boundary $\partial S^n_+ = \iota(S^{n-1})$ on which we want to apply the induction hypothesis. To this end let $V \subseteq \mathbb{R}^{n+1}$ be the orthogonal complement of $\mathbb{R}y$ and $\pi: \mathbb{R}^{n+1} \to V$ the corresponding orthogonal projection. Then with $h := \pi \circ f_+ : S^n_+ \to V \cong \mathbb{R}^n$ $h|_{\iota(S^{n-1})}$ is odd (since f is odd & π is linear) and 0 is not in its image since $g(S^{n-1}) \cap \{y\mathbb{R}\} = \emptyset$. So by the induction hypothesis we have

$$W_2(h|_{\iota(S^{n-1})}, 0) = 1 \qquad (2)$$

Since $\pm y$ are regular values of $\hat{f}$ it follows (after a little computation starting from $h(x)=0 \Leftrightarrow \hat{f}(x) \in \{\pm y\}$) that 0 is a regular value of h.

We can thus exploit $\underbrace{\text{the main thm. of the previous lecture}}$ & get:

$$
\begin{aligned}
W_2\left(h|_{\iota(S^{n-1})}, 0\right) &= |h^{-1}(0)| \bmod 2 \\
&\overset{h=\pi\circ f_+}{=} |f_+^{-1}(y\mathbb{R})| \bmod 2 \\
&\overset{(1)}{=} W_2(f, 0)
\end{aligned}
$$

Together with (2) this proves the induction step.

It remains to prove the statement for $n=1$, which can be done by going to the complex plane & will be skipped here. $\square$

Recall: Borsuk-Ulam: If $f: S^n \to \mathbb{R}^{n+1}\setminus\{0\}$ is odd & smooth, then $W_2(f, 0) = |f_+^{-1}(y\mathbb{R})| \bmod 2 = 1$ where f_+ is f restricted to the upper hemisphere and y any reg. value of $x \mapsto f(x)/\|f(x)\|$.

Cor. I: If a smooth map $f: S^n \to \mathbb{R}^{n+1}\setminus\{0\}$ is odd (i.e., $f(-x) = -f(x)\ \forall x$), then $f(S^n)$ intersects every line through the origin at least once.

proof: If $f(S^n) \cap \{y\mathbb{R}\} = \emptyset$, then with the notation from the foregoing proof, $|f_+^{-1}(y\mathbb{R})| \bmod 2 = 0$. As y would be a reg. value (since it is not in the image), this would contradict B.U. ☐

Cor. II: Let $f_1, \dots, f_n : S^n \to \mathbb{R}$ be odd smooth functions. Then $\exists x \in S^n$: $f_1(x) = \dots = f_n(x) = 0$.

proof: Suppose this is not the case. Then $f: S^n \to \mathbb{R}^{n+1}\setminus\{0\}$ $f(x) := (f_1(x), \dots, f_n(x), 0)$ would not intersect the line $(0, \dots, 0, \mathbb{R})$ contradicting Cor. I. ☐

Cor. III: Let $h: S^n \to \mathbb{R}^n$ be smooth. Then $\exists x \in S^n$: $h(x) = h(-x)$.

proof: Set $f_k(x) := h_k(x) - h_k(-x)$ and insert it into the previous corollary. ☐

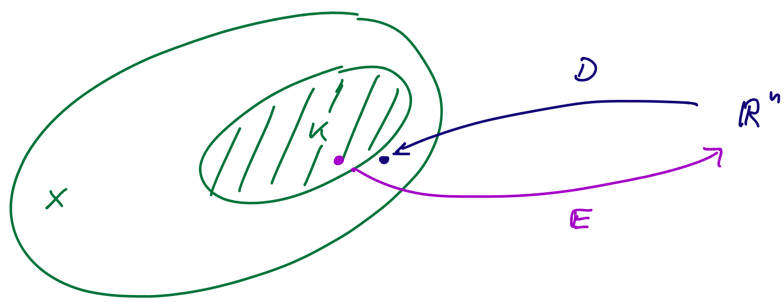
remark: By using smooth approximations one can lift this result so that 'smooth' can be replaced by 'continuous'.

Thm.: Let $B \subseteq \mathbb{R}^{n+1}$ be the unit ball of some norm in $\mathbb{R}^{n+1}$ and $f \in C(\partial B, \mathbb{R}^n)$. Then there is an $x \in \partial B$ s.t. $f(x) = f(-x)$.

proof (idea): $g: \partial B \to S^n$, $x \mapsto x/\|x\|_2$ is a homeomorphism that maps antipodal pairs to antipodal pairs. The claim then follows from the previous corollary in the continuous setting applied to $f \circ g^{-1}$. ☐

An application in approximation theory:

Q: How good can approximations by n-parameter families be?



Thm.: Let X be a normed space, $K \subseteq X$ and $E \in C(K, \mathbb{R}^n)$, $D : \mathbb{R}^n \to K$. Then $\sup_{f \in K} \| f - D \circ E(f) \| \geq \underbrace{\sup_V \sup \{ \lambda \mid \lambda B_V \subseteq K \}}_{=: b_n \text{ "Bernstein width" of } K}$, where sup is over all subspaces $V \subseteq X$ with $\dim(V) = n+1$ and $B_V := \{ x \in V \mid \|x\| \leq 1 \}$.

proof: Suppose $\lambda > 0$, $\lambda B_V \subseteq K$. $\tilde{E} := E|_{\partial \lambda B_V}$ is a cont. map from the unit sphere w.r.t. some norm to $\mathbb{R}^n$. By Borsuk-Ulam there is an $f \in \partial \lambda B_V$ with $E(f) = E(-f)$. Then

$2f = (f - D \circ E(f)) - (-f - D \circ E(-f))$. So

$2\lambda = \| 2f \| \leq \| f - D \circ E(f) \| + \| -f - D \circ E(-f) \|$.

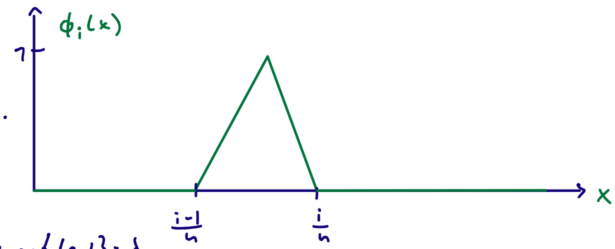
Hence, f or $-f$ is approximated with error $\geq \lambda$. $\square$

Lemma: For $X = L^\infty([0,1])$, $K := \{ f \in \underbrace{W^{1,\infty}}_{\text{Sobolev-space of Lipschitz functions}}([0,1]) \mid \| f \|_{W^{1,\infty}} := \max\{ \|f\|_\infty, \underset{\text{"weak derivative"}}{\| f' \|_\infty} \} \leq 1 \}$ we have $b_n \geq \frac{1}{2(n+1)}$.

proof: Let $\phi_i \in W^{1,\infty}([0,1])$ be a 'saw-tooth function' so that

$\phi_i(x) = 0 \quad \forall x \notin \left(\frac{i-1}{m}, \frac{i}{m}\right)$ for $i \in \{1, \dots, m\}$ and

$\|\phi_i\| = 1, \quad \|\phi_i'\| = 2m.$



$V := \operatorname{span}\{\phi_i\}_{i=1}^{m}$. So $n+1 = m$.

$f \in \partial \lambda B_V$ means

$f = \sum_{i=1}^{m} c_i \phi_i$ with $\|f\|_\infty = \max_i \{|c_i|\} = \lambda$.

On the other hand $\|f\|_{W^{1,\infty}} = \max\{\|f\|_\infty, \|f'\|_\infty\} = 2\|f\|_\infty m = 2\lambda m$

So $\lambda B_V \subseteq K \iff 2\lambda m \leq 1 \iff \lambda \leq \frac{1}{2(n+1)}$.

Taking the sup of those λ's leads to $b_n \geq \frac{1}{2(n+1)}$. □

- So, loosely speaking, approximating L-Lipschitz function in this way up to an error ε requires at least $\sim \frac{2L}{\varepsilon}$ real parameters.
- Better approximations are possible only if continuity of E is dropped.

Thm: (Parametrized Sard's theorem)

Let M, P, N be smooth manifolds of dimensions m, r, n, respectively with $m \geq n$. Assume y is regular value of a smooth map $F: M \times P \to N$. Then for allmost any $p \in P$ the map $F_p: M \to N$, $F_p(x) := F(x, p)$ has y as regular value, too.

proof: For simplicity assume $M = \mathbb{R}^m$, $P = \mathbb{R}^r$, $N = \mathbb{R}^n$. The general case follows by using charts.

Define the proj. $\tilde{\pi}: M \times P \to P$, $(x,p) \mapsto p$, the embedding $\iota: F^{-1}(\{y\}) \to M \times P: (x,p) \mapsto (x,p)$ and $\pi := \tilde{\pi} \circ \iota : F^{-1}(\{y\}) \to P$.

Assume $p \in P$ is regular value of π and x s.t. $F(x,p) = y$.

$\longrightarrow$ $d\pi_{(x,p)} : TF^{-1}(y)_{(x,p)} \to \mathbb{R}^r$ is surjective, that is

$\forall q \in \mathbb{R}^r \ \exists [\gamma] \in TF^{-1}(y)_{(x,p)}$ s.t. $d\pi_{(x,p)}[\gamma] = q$.

As $d\pi_{(x,p)} = d\tilde{\pi}_{(x,p)} \circ d\iota_{(x,p)}$, $d\tilde{\pi}_{(x,p)} = \pi$ and

$d\iota_{(x,p)} : TF^{-1}(y)_{(x,p)} \longrightarrow \vec{T}F^{-1}(y)_{(x,p)}$ represents the tangent space

in terms of the geometric tangen space, we have

$d\iota_{(x,p)} : [\gamma] \mapsto (x(q), q)$.

Moreover, since $[\gamma] \in TF^{-1}(y)_{(x,p)} \Leftrightarrow dF_{(x,p)}[\gamma] = 0$ this means that

$\forall q \in \mathbb{R}^r \ \exists x \in \mathbb{R}^m$ s.t. $dF_{(x,p)}(x, q) = 0$.

Let $\overset{}{\left(\overset{m}{A} \ \ \overset{r}{B} \right)} n$ be the Jacobi-matrix representing $dF_{(x,p)}$ then this

means that $\forall q \in \mathbb{R}^r \ \exists x \in \mathbb{R}^m : Ax + Bq = 0$

So $\operatorname{ran} A \supseteq \operatorname{ran} B$ and thus $\operatorname{ran}\left(dF_{(x,p)}\right) = \operatorname{ran} A + \operatorname{ran} B = \operatorname{ran} A$.

As (x,p) is a regular point we have $\overset{\shortparallel}{\mathbb{R}^n}$ and thus $\operatorname{ran} A = \mathbb{R}^n$.

But A is just the Jacobi-matrix of F_p at $x \in F_p^{-1}(y)$. $\square$

<u>Corollary:</u> If $M = N \times R$ where R is a 1-dim smooth manifold, then $F_p : N \times R \to N$ and each component of $F_p^{-1}(y)$ is a smooth curve for almost every $p \in P$.

The idea is know to use $t \in R$ as a 'homotopy parameter' that interpolates between an equation $F_p(x, 0) = y$ for which the solution x is known and an equation $F_p(x, 1) = y$ for which the solution is sought. The above results then can guarantee that for almost any $p \in P$ there is a smooth path (without bifurcations or crossings) that connects the two. Ideally a 'path following algorithm' then finds the sought solution.

Thm.: (Ham-sandwich thm.)

Let $\mu_1, \dots, \mu_n$ be finite Borel measures on $\mathbb{R}^n$ that all assign zero measure to hyperplanes. Then there is a halfspace $\hat{H} \subseteq \mathbb{R}^n$ s.t. $\forall k: \ \mu_k(\hat{H}) = \frac{1}{2}\mu_k(\mathbb{R}^n)$.

proof: For $v \in S^n$ define $H(v) := \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i v_i \leq v_{n+1} \}$. In this way, any halfspace corresponds to one $v \in S^n$.

Then $f: S^n \to \mathbb{R}^n$, $f(v) := (\mu_1(H(v)), \dots, \mu_n(H(v)))$ is continuous so that Borsuk-Ulam implies the existence of $x \in S^n$ s.t. $f(x) = f(-x)$.

However, $H(v) = \overline{\mathbb{R}^n \setminus H(-v)}$. $\square$

ORIENTATION

Def.:
- An ==orientation== of a real vector space is an equivalence class of ordered bases under the relation $b' \sim b \iff \det(B) > 0$ where $b = (b_1, \dots, b_n)$ and $b' = (b'_1, \dots, b'_n)$ are ordered bases of $\mathbb{R}^n$ and $B: \mathbb{R}^n \to \mathbb{R}^n$ the automorphism that maps $b'_j = B b_j$.
- An automorphism on a real vector space is called ==orientation preserving== if it has positive determinant.
- The ==standard orientation== of $\mathbb{R}^n$ contains $b = \mathbb{1} \in \mathbb{R}^{n \times n}$ and is often labelled "+1".
- It is convenient to let $\mathbb{R}^0$ also carry two possible orientations "± 1".
- A smooth manifold $(M, \mathcal{A})$ is ==orientable== if there is an atlas $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ s.t. $(U_i, \varphi_i) \in \tilde{\mathcal{A}} \wedge x \in U_i \cap U_j \Rightarrow \det\left(d(\varphi_i \varphi_j^{-1})_{\varphi_j(x)}\right) > 0$.

Lemma: A smooth manifold $(M, \mathcal{A})$ is orientable iff there exists a choice of orientations of all tangent spaces T_xM and an atlas $\hat{A} \subseteq \mathcal{A}$ s.t. for all $(U, \varphi) \in \hat{A}$ ∧ $x \in U$ the map $d\varphi_x : T_xM \to T_{\varphi(x)}\mathbb{R}^m = \mathbb{R}^m$ maps the chosen orientation of T_xM to the standard orientation of $\mathbb{R}^m$.

remark: Here $\mathbb{R}^m$ and $T_{\varphi(x)}\mathbb{R}^m$ are identified by representing $[\gamma] \in T_{\varphi(x)}\mathbb{R}^m$ by its principle part w.r.t. the chart $(\mathbb{R}^m, id)$.

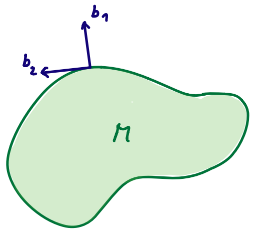
proof: "⇐" Set $\tilde{A} := \hat{A}$. Then $d(\varphi_i \varphi_j^{-1})_{\varphi_j(x)} = d(\varphi_i)_x\, d(\varphi_{j\,x})^{-1}$ is orientation preserving.

"⇒" If we set $\hat{A} := \tilde{A}$, then $(d\varphi_j)_x^{-1}$ maps the standard orientation of $\mathbb{R}^m$ in the desired way to an orientation of T_xM. □

remarks:
- Note that $\hat{A}$ then already determines the orientations on T_xM.
- $\hat{A}$ is called oriented atlas. A maximal oriented atlas is called an orientation of $(M, \mathcal{A})$.
- If M is connected and orientable, then there exist two orientations.

examples:
- The Klein-bottle, Möbius strip and the real projective plane are not orientable.
- If M is orientable and $U \subseteq M$ open, then U is orientable as a submanifold of M.
- If M, N are orientable, then so is $M \times N$.
- If M is a smooth oriented manifold with boundary, then its orientation induces an orientation on the boundary ∂M:

- $\underline{\dim M > 1:}$ $\forall x \in \partial M$ choose a basis $(b_1, ..., b_m)$ of T_xM s.t.



(i) it represents the orientation of T_xM,

(ii) $\forall i > 1$: $b_i \in T_x\partial M \subseteq T_xM$,

(iii) b_1 points outwards of M.

$(b_2, ..., b_m)$ then define an orientation of $T_x\partial M$ s.t. ∂M becomes an oriented manifold.

- $\underline{\dim M = 1:}$ $x \in \partial M$ is assigned orientation $+1$ or -1 depending on whether b_1 points outward or inward at x.

<u>examples:</u>
- S^n is orientable as the boundary of the disc D^{n+1}.
- If M is smooth oriented manifold without boundary, then

$$\partial(M \times [0,1]) = \underbrace{(-M)}_{M \text{ with negative orientation}} \times \{0\} \cup M \times \{1\}$$

<u>Def.:</u> Let $f: M \to N$ be a smooth map between smooth oriented manifolds of the same dimension. If M is compact, N connected and $y \in N$ a regular value of f, then we define

$$\boxed{\deg(f,y) := \sum_{x \in f^{-1}(y)} \operatorname{sgn}(df_x) \in \mathbb{Z}}$$

where $\operatorname{sgn}(df_x) = \pm 1$ and $+1$ iff the image of the orientation of T_xM under df_x coincides with the one of $T_{f(x)}N$.

<u>Thm.:</u> (Brouwer degree thm.)

The <u>Brouwer degree</u> $\deg(f) := \deg(f,y)$ does not depend on the choice of the regular value y and only depends on the homotopy class of f.

<u>remark:</u> note that $\deg(f) \bmod 2 = \deg_2(f)$.

The proof of the thm. is similar to that for $\deg_2(f)$.

The main ingredients are:

<u>Lemma 1:</u> If $M = \partial X$ where X is an oriented smooth compact manifold with boundary M, which inherits the orientation of X and $f: M \to N$ extends to a smooth map $F: X \to N$, then $\deg(f,y) = 0$ for any regular value y of f.

<u>proof:</u> (sketch) W.l.o.g. we can assume that y is also a regular value of F. If not, we exploit that $\deg(f,y)$ is locally constant, so we may take a regular value from the neighborhood.

Then $F^{-1}(y)$ is an oriented compact smooth 1-dim. manifold with boundary $\partial F^{-1}(y) = \partial X \cap F^{-1}(y) = f^{-1}(y)$.

Let $I \subseteq F^{-1}(y)$ be a connected component with boundary points a, b, then $\operatorname{sgn}(df_a) + \operatorname{sgn}(df_b) = 0$. □

<u>Lemma 2:</u> If f and g are smoothly homotopic and y is a common regular value, then $\deg(f,y) = \deg(g,y)$.

<u>proof</u> (sketch): Let $[0,1] \times M$ be the oriented product with boundary $\{0\} \times (-M) \cup \{1\} \times M$ and $F: [0,1] \times M \to N$ be the smooth homotopy between f and g. Then

$$\underset{\text{Lemma 1}}{\overset{\uparrow}{0 =}} \deg\Big(F\big|_{\underbrace{\partial[0,1]\times M}_{\{0\}\times(-M)\cup\{1\}\times M}}, y\Big) = \deg(g,y) - \deg(f,y).$$ □

The rest of the proof of the Brouwer degree thm. is analogous to the mod-2 case: if y & z are both reg. values for $f: M \to N$, choose a diffeomorphism $h: N \to N$ that is isotopic to the identity and s.t. $f(y) = z$ ... □

Thm.: (Multiplicativity of the Brouwer degree)

Let M, N, P be smooth orientable manifolds of the same dimension, M, N compact and N, P connected. For smooth maps $f: M \to N$, $g: N \to P$:
$$\deg(g \circ f) = \deg(g) \cdot \deg(f) \quad .$$

proof: If $p \in P$ is a reg. value of $g \circ f$, then any $y \in g^{-1}(p)$ is a reg. value of f since for any $x \in f^{-1}(y)$ the map $d(g \circ f)_x \underset{\uparrow}{=} dg_y \circ df_x$ has to be an isomorphism. So df_x has to be chain rule an isomorphism, too. Then

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(df_x) \quad \text{for any } y \in g^{-1}(p).$$

Hence,
$$\deg(g \circ f) = \sum_{x \in (g \circ f)^{-1}(p)} \operatorname{sgn}(d(gf)_x) = \sum_{y \in g^{-1}(p)} \sum_{x \in f^{-1}(y)} \operatorname{sgn}(dg_y) \operatorname{sgn}(df_x)$$
$$= \sum_{y \in g^{-1}(p)} \operatorname{sgn}(dg_y) \sum_{x \in f^{-1}(y)} \operatorname{sgn}(df_x)$$
$$= \deg(g) \cdot \deg(f). \qquad \square$$

example: The reflection $r_k: S^n \to S^n$, $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{k-1}, -x_k, x_{k+1}, \dots)$ has $\deg(r_k) = -1$. The "antipodal map" $\Theta: S^n \to S^n$, $x \mapsto -x$ can be written as $\Theta = r_1 \circ \dots \circ r_{n+1}$ so that $\deg(\Theta) = (-1)^{n+1}$.

Corollary: If n is even, then $\Theta: S^n \to S^n$, $x \mapsto -x$ is not smoothly homotopic to the identity.

remark: Hopf's thm. states that $f, g: M \to S^n$ (for M compact, orientable with $\dim(M) = n$) are homotopic iff $\deg(f) = \deg(g)$.
Similarly, if M is non-orientable, then homotopy-classes are completely characterized by $\deg_2$.

Lemma: Let $f, g: S^n \to S^n$ be smooth and s.t. $\forall x \in S^n: f(x) \neq g(x)$.

Then g is smoothly homotopic to $\Theta \circ f$.

proof: Consider $H: [0,1] \times S^n \to S^n$

$$H: (t, x) \mapsto \frac{t\,\Theta \circ f(x) + (1-t)\,g(x)}{\| \quad - \text{"} - \quad \|}.$$

This is a homotopy if the denominator does not vanish. Since $f(x) \neq g(x)$ we have that the origin is not on the line connecting $\Theta \circ f(x)$ with $g(x)$. Hence, $0 \neq t\,\Theta \circ f(x) + (1-t)\,g(x)$. □

Thm.: Let $f: S^n \to S^n$ be smooth and n even. Then f has a fixed point or sends a point to its antipode. (Hence $f(x)$ and x cannot be linearly indep. for all $x \in S^n$)

proof: If f has no fixed point, then by the previous Lemma it is smoothly homotopic to Θ.

If $f(x) \neq -x \ \forall x \in S^n$, then by the same reasoning it is smoothly homotopic to the identity.

However, if n is even, then $\deg(\Theta) \neq \deg(\mathrm{id})$. □

Corollary: ("hairy ball thm.", "hedgehog thm.")

S^n admits a non-vanishing smooth tangent vector field iff n is odd.

proof: Let n be even and $v: S^n \to \mathbb{R}^{n+1}$ s.t. $\forall x \in S^n: \langle v(x), x \rangle = 0$. If v would be non-vanishing, then $f: S^n \to S^n$, $f(x) := \frac{v(x)}{\|v(x)\|}$ would be s.t. $x \perp f(x) \ \forall x \in S^n$ contradicting the previous thm.

If n is odd, choose $v(x_1, \dots, x_{2k}) := (x_2, -x_1, x_4, -x_3, \dots)$. □

Degree theory in Euclidean space

Def.: Let $U \subseteq \mathbb{R}^n$ be open and bounded, $f: \bar{U} \to \mathbb{R}^n$ smooth and $y \in \mathbb{R}^n \setminus f(\partial U)$ a regular value of $f|_U$. Then we define the Euclidean degree

$$\deg(f, U, y) := \sum_{x \in f^{-1}(y)} \operatorname{sgn}(df_x) \in \mathbb{Z}$$

and $\deg(f, U, y) = 0$ if $f^{-1}(y) = \emptyset$.

remarks:
- $\partial U := \bar{U} \setminus U$ is the topological boundary.
- Note that this degree is well-defined: $f^{-1}(y)$ is discrete & in U and thus in the compact set $\bar{U}$ since $f^{-1}(y) \cap \partial U = \emptyset$. $f^{-1}(y)$ is also compact so that $|f^{-1}(y)| < \infty$.
- $y \mapsto \deg(f, U, y)$ is constant for all regular values in one connected component Ω of $\mathbb{R}^n \setminus f(\partial U)$. This defines $\deg(f, U, x)$ also for all singular values $x \in \Omega$.
- By approximating a continuous map by a smooth map, the definition extends to all $f \in C(\bar{U}, \mathbb{R}^n)$ and all $y \in \mathbb{R}^n \setminus f(\partial U)$.

Thm.: (Homotopy invariance) Let $U \subseteq \mathbb{R}^n$ be open and bounded, $H \in C([0,1] \times \bar{U}, \mathbb{R}^n)$ and $\gamma \in C([0,1], \mathbb{R}^n)$ a path s.t. $\gamma(t) \notin H_t(\partial U)$ $\forall t \in [0,1]$. Then $\deg(H_t, U, \gamma(t))$ does not depend on t.

Proof is very similar to previous ones. See e.g. Deimling: "Non-linear functional analysis"

Corollary: (Boundary theorem) Let $f, g \in C(\overline{U}, \mathbb{R}^n)$ be such that $f|_{\partial U} = g|_{\partial U}$. Then for any $y \notin f(\partial U) = g(\partial U)$:

$$\deg(f, U, y) = \deg(g, U, y)$$

proof: Use homotopy invariance with $H_t(x) = t f(x) + (1-t) g(x)$ and $y(t) = y$. $\square$

Consequently, the degree of f only depends on $f|_{\partial U}$.

Applications:
- One type of application is proving the existence of solutions for sets of non-linear equations: If $\deg(f, U, y) = d$, then $f(x) = y$ has at least $|d|$ solutions. Computing d can then for instance be achieved using a homotopy to a simpler set of equations.
- Another type of application exploits non-existence of homotopies such as in Brouwer's fixed point theorem or the following strengthening:

Thm.: (Rothe's fixed point theorem) Let $f \in C(\overline{B}, \mathbb{R}^n)$ on the unit ball $B := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ be s.t. $f(\partial B) \subseteq \overline{B}$, then $\exists x_0 \in \overline{B}: f(x_0) = x_0$.

proof: If there were no such x_0, then $g(x) := x - f(x)$ had no zero in $\overline{B}$. So $\deg(g, B, 0) = 0$. $H_t(x) := t\, g(x) + (1-t)\, x = x - t f(x)$, $t \in [0,1]$ is then a homotopy with $0 \notin H_t(\partial B)$. Thus $\deg(g, B, 0) = \deg(id, B, 0) = 1$, a contradiction. $\square$