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Geometry of rational double points and del Pezzo surfaces

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Title: Geometry of rational double points and del Pezzo surfaces

Abstract: The topic of this thesis is the interplay between rational double point singularities and del Pezzo surfaces. Chapter I determines which rational double points occur on del Pezzo surfaces, extending work of Du Val to positive characteristic. Chapter II classifies weak del Pezzo surfaces with global vector fields. The corresponding problem for RDP del Pezzo surfaces in odd characteristic is solved in Chapter III. As an application of the techniques developed in this thesis, rational (quasi-)elliptic surfaces with global vector fields are treated in Chapter IV.

Key words: rational double points, del Pezzo surfaces, vector fields, positive characteristic

Titel: Geometrie rationaler Doppelpunkte und del Pezzo Flächen

Kurzfassung: Das Thema dieser Arbeit ist das Zusammenspiel zwischen rationalen Doppelpunkt-Singularitäten und del Pezzo Flächen. Kapitel I bestimmt, welche rationalen Doppelpunkte auf del Pezzo Flächen auftreten, und verallgemeinert damit Arbeiten von Du Val auf den Fall positiver Charakteristik. Kapitel II klassifiziert schwache del Pezzo Flächen mit globalen Vektorfeldern. Das entsprechende Problem für RDP del Pezzo Flächen in ungerader Charakteristik wird in Kapitel III gelöst. Als eine Anwendung der in dieser Arbeit entwickelten Techniken werden in Kapitel IV rationale (quasi-)elliptische Flächen mit globalen Vektorfeldern behandelt.

Stichworte: rationale Doppelpunkte, del Pezzo Flächen, Vektorfelder, positive Charakteristik

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Introduction

Throughout this thesis, unless explicitly mentioned otherwise, we work over an algebraically closed field k of characteristic $\mathrm{char}(k) = p \geq 0$.

0.1. Rational double points. A *rational double point* is a rational normal surface singularity of multiplicity two [Art66]. Despite the modern mathematical language used in the above definition, the starting point of the investigation of these singularities may be traced back as far as 400-300 BC, when the Greek mathematician Theaetetus and the Pythagoreans ([Euc56, Book XIII], [Wat72]) first discovered and mathematically described the regular solids in three-dimensional space: tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron. These regular polyhedra are called *Platonic solids* in honor of the Greek philosopher Plato and the historically first attempt of a characterization of matter in terms of mathematical models in his famous Timaios dialogue [Pla25, Timaeus, 53e]. It is believed that Euclid's books [Euc56] were written in order to provide a proof of Theaethetus' classification of the regular solids – considered the most important result in ancient mathematics –, a hypothesis that is further corroborated by the fact that Euclid devoted the final volume XIII of his "Elements" to this topic. Around two thousand years later in 1596, Kepler's model of the solar system [Kep96] inscribed the five Platonic solids between two successive orbits of the six planets – only six of them were known at that time – circling around the sun (see also [Slo83]). Although we now know the flaws of this model, it should not be underestimated that, after Copernicus, this constituted the first physical theory of heliocentrism.

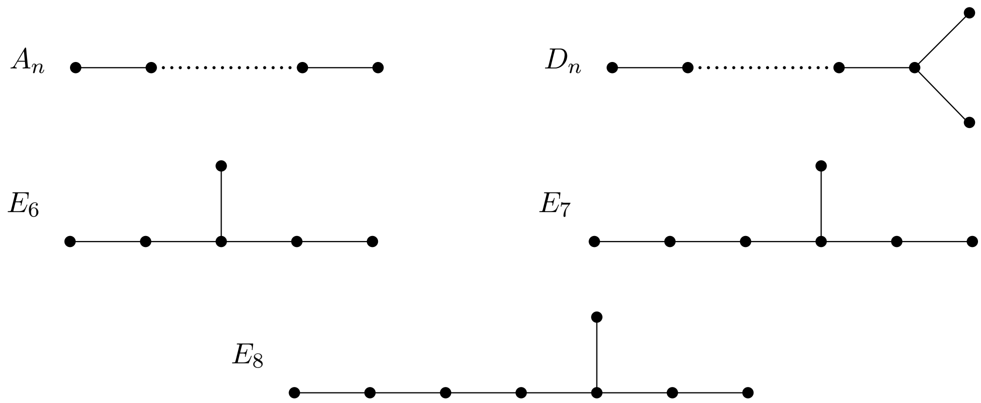
In the nineteenth century, the symmetry groups of the Platonic solids reappeared in Klein's classification of finite subgroups G of $\mathrm{SL}_2(\mathbb{C})$: Up to conjugation, these are the cyclic, the binary dihedral, as well as the binary tetrahedral, binary octahedral and binary icosahedral groups, and correspond to the symmetry groups of plane regular n-gons, pyramids over them, and the Platonic solids [Kle93], [Slo83]. In this context, in Klein's lectures on the icosahedron [Kle93] in 1884, the equations of complex rational double points first occurred as the relations between the generators of the invariant ring $\mathbb{C}[x, y]^G$ under actions by the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Half a century later, these quotient singularities appeared again in Du Val's classification of surface singularities that "do not affect the conditions of adjunction" [DV34]. In the modern language of the minimal model program, Du Val's characterization can be rephrased as saying that rational double points are precisely the *canonical singularities* of normal surfaces.

The first complete proof of the existence of resolutions of surface singularities goes back to Walker [Wal35] – after preliminary work by Del Pezzo [DP92], Segre [Seg97], Levi

[Lev97], Severi [Sev14], Chisini [Chi21], Albanese [Alb24] – and was extended to fields of characteristic 0 by Zariski [Zar39], [Zar42] and to positive characteristic by Abhyankar [Abh56]. It follows from Castelnuovo's contraction theorem [CE01] (see also [Har77, Chapter V, Theorem 5.7]) that there is a unique minimal resolution for surface singularities.

In the case of rational double points, the exceptional locus of the minimal resolution consists of (-2)-curves (that is, smooth rational curves of self-intersection (-2)) whose dual graph is a Dynkin diagram of type A_n, D_n, E_6, E_7, or E_8 as depicted below [Dyn47]. (In these resolution graphs each vertex corresponds to an irreducible component E_i of the exceptional locus and two such vertices E_i, E_j are joined by $E_i.E_j$ many lines.) Conversely, every surface singularity such that the exceptional divisor of the minimal resolution consists of (-2)-curves forming an ADE-diagram is a rational double point (see [Dur79] or [Sta18b] for various equivalent definitions). Moreover, the analytic isomorphism class of a rational double point is uniquely determined by its resolution graph (this property is called *tautness*). In other words, over the complex numbers, rational double points are classified by Dynkin diagrams of types A_n, D_n, E_6, E_7, and E_8.



Dynkin diagrams of types $A_n (n \geq 1), D_n (n \geq 4), E_6, E_7$, and E_8

This suggests that there is a relation between rational double points and other mathematical objects that are classified by Dynkin diagrams, such as simple algebraic groups and their Lie algebras ([Che51], [Che55], [Bor56], [Sem58]; see also [Hum75]). Indeed, as Grothendieck had conjectured, the following connections between rational double points and simple Lie algebras (analogously, simple algebraic groups) of ADE-type were established: For a Lie algebra $\mathfrak{g}$ of some Dynkin type, its variety $N(\mathfrak{g})$ of nilpotent elements intersected with a so-called transverse slice S to a subregular orbit is again a surface with an isolated singularity that is a rational double point of the same Dynkin type as $\mathfrak{g}$ ([Bri71]; see also [Esn76], [Ste65]). Moreover, the restriction of the adjoint quotient $\mathfrak{g} \to \mathfrak{t}/W$ to the transverse slice realizes the miniversal deformation of the rational double point in $S \cap N(\mathfrak{g})$, and, by work of Grothendieck and Springer [Spr69], this description of the miniversal deformation can be used to construct simultaneous resolutions of families of rational double points after a finite base change with the Weyl group W as covering group ([Bri71], [Slo80]; see also [Art74], [SB21]).

In positive characteristic, the dual resolution graph of a rational double point is still a Dynkin diagram of ADE-type, but it turns out that there can be several non-isomorphic rational double points admitting the same resolution graph in characteristic $p \in \{2, 3, 5\}$, that is, rational double points in these characteristics are not necessarily *taut*. Nevertheless, there are only finitely many formal isomorphism classes of rational double points with the same resolution graph and these isomorphism classes, together with their defining equations, were classified by Lipman [Lip69] in the case of E_8-singularities and by Artin [Art77] in the other cases. Artin shows that rational double points with the same resolution graph are distinguishable in terms of their deformation theory ([Art76, Chapter 1, §4]) and assigned a coindex to the formal isomorphism classes which we call *Artin coindex*. The connection between rational double points, simple algebraic groups and simultaneous resolutions persists in positive characteristic if p is not too small, as explored by Slodowy [Slo80], and extended by Shepherd-Barron [SB01] if p is *very good* (depending on the type of algebraic group resp. rational double point).

0.2. Del Pezzo surfaces. If $X \subseteq \mathbb{P}^n$ is a projective variety of degree d and dimension m such that X is non-degenerate, that is, such that X is not contained in a hyperplane, then a simple projection argument shows that $d \geq 1 + n - m$. If equality holds, then X is called a *variety of minimal degree*. The classification of surfaces of minimal degree goes back to del Pezzo [dP85] and was later generalized to higher dimensions by Bertini [Ber07] (see also [EH87]). The result is that the varieties of minimal degree are exactly quadric hypersurfaces, the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, cones over the Veronese surface, or rational normal scrolls. The classification is the same in arbitrary characteristic (see for example [Dol12, Section 8.1.1], [GH94, Chapter 4, 3., p.522ff], or [Har95, Theorem 19.9.]).

Restricting ourselves to dimension 2, it is a natural question to ask what the non-degenerate projective surfaces of *almost minimal degree* $d = n$ are. This question was studied by del Pezzo in [dP87]. Among these surfaces are cones over curves of degree d in $\mathbb{P}^{d-1}$ and projections of surfaces of minimal degree d in $\mathbb{P}^{d+1}$ [Dol12, Section 8.1.1]. For surfaces of almost minimal degree that are not in these two classes, del Pezzo showed that $3 \leq d \leq 9$ and that all such surfaces are normal, have at worst rational double points as singularities and – in modern terminology – have very ample anti-canonical sheaf ω_X^{-1} (see [Dol12, Proposition 8.1.8, Theorem 8.1.11]). Weakening the condition "very ample" to "ample" leads to the following more general modern definition of del Pezzo surfaces as (canonical) Fano varieties of dimension 2 and of weak del Pezzo surfaces as their minimal resolutions.

DEFINITION. Let X and $\widetilde{X}$ be projective surfaces.

- X is a *del Pezzo surface* if it is smooth and ω_X^{-1} is ample.
- X is an *RDP del Pezzo surface* if all of its singularities are rational double points (RDPs) and ω_X^{-1} is ample.
- $\widetilde{X}$ is a *weak del Pezzo surface* if it is the minimal resolution of a (RDP) del Pezzo surface.

In all the above cases, the number $\deg(X) := K_X^2$ (resp. $\deg(\widetilde{X}) := K_{\widetilde{X}}^2$) is called the *degree of* X (resp. $\widetilde{X}$).

The RDP del Pezzo surfaces that were classically studied by del Pezzo are those with $\deg(X) \geq 3$ and they are also exactly those where ω_X^{-1} is very ample. In degree $\deg(X) = 2$, the surface X is a double cover of $\mathbb{P}^2$ branched over a quartic curve (if $p \neq 2$) and in degree $\deg(X) = 1$, the surface X is a double cover of a quadratic cone in $\mathbb{P}^3$ branched over a sextic curve (if $p \neq 2$). It is well-known (see [Dol12, Section 8.1.3]) that weak del Pezzo surfaces are exactly the smooth projective surfaces $\widetilde{X}$ with $\omega_{\widetilde{X}}^{-1}$ big and nef, and that all of them are either $\mathbb{P}^1 \times \mathbb{P}^1$, the second Hirzebruch surface $\mathbb{F}_2$, or can be obtained by blowing up at most 8 points in $\mathbb{P}^2$ in almost general position.

0.3. Which rational double points occur on del Pezzo surfaces? Given that the only singularities that can occur on del Pezzo surfaces are rational double points, it is a natural question to ask exactly which rational double points (and configurations of them) actually do occur on RDP del Pezzo surfaces. Over the complex numbers, the answer to this question is classical: By work of Timms [Tim28], Schläfli [Sch63], and Du Val [DV34], a configuration of rational double points Γ occurs on an RDP del Pezzo surface X if and only if the corresponding (-2)-curve configuration Γ' occurs on a weak del Pezzo surface. The latter happens if and only if the lattice Γ' embeds into the E_8-lattice and $\Gamma' \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$.

In Chapter I, we extend these classical results to all positive characteristics. As explained in the above introduction to rational double points, the classification is more subtle in small characteristics, since rational double points may no longer be taut if $\mathrm{char}(k) = p \in \{2, 3, 5\}$. So, a complete answer to the question "Which rational double points occur on del Pezzo surfaces?" must be more involved than over the complex numbers, since determining the possible RDP configurations Γ on RDP del Pezzo surfaces is a priori not equivalent to only determining the possible configurations Γ' of (-2)-curves on weak del Pezzo surfaces. Nevertheless, it is a consequence of our classification (Theorem 1.2 in Chapter I) that, in characteristic different from 2, the two classifications coincide, that is, whenever a configuration of (-2)-curves occurs on a weak del Pezzo surface, then the corresponding configurations of rational double points with all possible Artin coincides occur on some RDP del Pezzo surface. If $p = 2$, this correspondence fails, but we are still able to achieve a complete classification of all configurations of rational double points that occur. Moreover, as a byproduct of this classification, we obtain simplified equations for all RDP del Pezzo surfaces of degree 1 containing non-taut rational double points.

For further details and the complete classification, we refer the reader to Chapter I of this thesis and Theorem 1.2 therein.

0.4. Simultaneous resolutions of rational double points via del Pezzo surfaces. In the above, we studied rational double points on del Pezzo surfaces because they are the only singularities that can possibly occur on these surfaces (at least with our definition, inspired by del Pezzo's classification of varieties of almost minimal degree). However, del Pezzo surfaces can also be used to study the geometry of rational double points: Indeed, even before simultaneous resolutions of rational double points were studied using linear algebraic groups as explained above, there have been explicit constructions of such simultaneous resolutions using (weak) del Pezzo surfaces by Brieskorn [Bri66], [Bri68], and Tjurina

[Tju70]. In the following, we recall Tjurina's description of a simultaneous resolution of a versal deformation of the E_8-singularity in characteristic 0 [Tju70]:

Consider a cuspidal cubic $C \subseteq \mathbb{P}^2$, let $P \in C$ be an inflection point, and let $U \subseteq C$ be a sufficiently small neighborhood of P. Then, the family $\widetilde{\gamma} : \widetilde{\mathcal{X}} \to U^8$ obtained by blowing up (the strict transforms of) $\mathbb{P}^2_{U^8}$ in the eight sections determined by the inclusion $U \subseteq \mathbb{P}^2$ is a family of weak del Pezzo surfaces. The fiber of $\widetilde{\gamma}$ over $(P, \ldots, P)$ is a weak del Pezzo surface with an E_8-configuration of (-2)-curves and Tjurina shows, by taking the relative anti-canonical model, that $\widetilde{\mathcal{X}}$ can be blown down to a family of RDP del Pezzo surfaces $\gamma : \mathcal{X} \to U^8$ that induces a deformation of the E_8-singularity in the fiber of γ over $(P, \ldots, P)$ whose base is a finite cover of a versal deformation space of the singularity. In other words, by this construction, any deformation of E_8 admits a simultaneous resolution after a finite base change. In particular, up to finite base change, every deformation of the E_8-singularity is induced by a deformation of an RDP del Pezzo surface X with a singularity of type E_8.

We note that Tjurina's construction of a simultaneous resolution for the E_8-singularity cannot work in arbitrary characteristic: For example, if $\mathrm{char}(k) = p = 2$, only three of the five types of E_8-singularities occur on del Pezzo surfaces (see Chapter I), and among the three E_8^r-singularities that do occur, two have miniversal deformation spaces of dimension strictly larger than 8 (namely E_8^0 and E_8^3). Therefore, the space U^8 from Tjurina's construction cannot provide a finite cover of the miniversal deformation space of the singularity. However, one may wonder whether the additional deformations of the singularity are still induced by a deformation of a (RDP) del Pezzo surface, possibly via a more involved construction than Tjurina's.

This begs for the more general question about the relationship between deformations of RDP del Pezzo surfaces, their minimal resolutions weak del Pezzo surfaces, and the rational double points that they contain. As the deformation spaces of rational double points are well-known in all characteristics ([Art77], [Art76], [Slo80]; see also [Sta18b]), the first step towards understanding this relation is to determine the deformation spaces of RDP and weak del Pezzo surfaces.

Recall that the locally trivial deformations of a projective variety X are governed by the cohomology groups $H^0(X, T_X)$, $H^1(X, T_X)$, and $H^2(X, T_X)$, where T_X denotes the tangent sheaf of X. By work of Matsumura and Oort [MO68], the automorphism functor Aut_X of X over k is representable by a group scheme locally of finite type over k whose tangent space at the identity can be identified canonically with the space of global vector fields $H^0(X, T_X)$. Note that this tangent space only depends on the identity component Aut_X^0 of the representing group scheme Aut_X. By [Ser06, Proposition 1.2.9], the space $H^1(X, T_X)$ is the tangent space to the functor of locally trivial deformations of X, and, by [Ser06, Proposition 2.4.6], the space $H^2(X, T_X)$ is an obstruction space for this functor.

Thus, the first step towards understanding deformations of weak and RDP del Pezzo surfaces is to determine the cohomology groups of the tangent sheaf.

Note that in the case of a weak del Pezzo surface $\widetilde{X}$ we have by Serre-duality and [DI87, Corollaire 2.8]

$$H^2(\widetilde{X}, T_{\widetilde{X}}) \cong H^0(\widetilde{X}, \Omega_{\widetilde{X}} \otimes \omega_{\widetilde{X}})^\vee = 0.$$

By the Hirzebruch–Riemann–Roch-Theorem and formulae for Chern- and Todd-classes (see [Har77, Appendix A, 4]), one can check that

$$
\begin{aligned}
\chi(T_{\widetilde{X}}) &= h^0(\widetilde{X}, T_{\widetilde{X}}) - h^1(\widetilde{X}, T_{\widetilde{X}}) \\
&= \deg\left(\mathrm{ch}(T_{\widetilde{X}}).\mathrm{td}(T_{\widetilde{X}})\right)_2 = \ldots = 2K_{\widetilde{X}}^2 - 10.
\end{aligned}
\tag{0.1}
$$

Therefore, since the miniversal deformation space of $\widetilde{X}$ is formally smooth, it is determined by its dimension $h^1(\widetilde{X}, T_{\widetilde{X}})$ and by Equation 0.1 it is thus sufficient to compute $h^0(\widetilde{X}, T_{\widetilde{X}})$.

0.5. Weak del Pezzo surfaces with global vector fields. In Chapter II of this thesis, we obtain the dimension of $H^0(\widetilde{X}, T_{\widetilde{X}})$ as a byproduct of the determination of the identity component $\mathrm{Aut}^0_{\widetilde{X}}$ of the automorphism scheme for every weak del Pezzo surface $\widetilde{X}$ in arbitrary degree and arbitrary characteristic.

The structure of $\mathrm{Aut}^0_{\widetilde{X}}$ is well-known if $\widetilde{X}$ is not a blow-up of $\mathbb{P}^2$, that is, if $\widetilde{X} \in \{\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2\}$. The key tool for the classification of the remaining weak del Pezzo surfaces with global vector fields is Blanchard's Lemma (see [BSU13, Proposition 4.2.1] and [Bla56, §I.1]), which implies that connected group scheme actions descend along birational morphisms between normal varieties, allowing us to describe $\mathrm{Aut}^0_{\widetilde{X}}$ as an iterated stabilizer of the points blown up in a realization of $\widetilde{X}$ as a blow-up of $\mathbb{P}^2$.

Using this approach, we give a complete description of the configuration of negative curves – which, by adjunction, are either (-1)- or (-2)-curves – on weak del Pezzo surfaces with global vector fields and calculate the identity component $\mathrm{Aut}^0_{\widetilde{X}}$ of their automorphism schemes. It turns out that there are 53 distinct families of such surfaces if $p \neq 2, 3$, while there are 61 such families if $p = 3$, and 75 such families if $p = 2$. Each of these families has at most one moduli. Note that this also yields a classification of weak (and RDP) del Pezzo surfaces with infinite automorphism group, independently obtained over the complex numbers by Cheltsov and Prokhorov [CP21]. However, in positive characteristic – and, by our classification, a posteriori only in characteristic 2 and 3 – there can exist weak del Pezzo surfaces with global vector fields and finite automorphism group, since $\mathrm{Aut}^0_{\widetilde{X}}$ can be non-reduced.

Further details and the complete classification can be found in Chapter II and the Main Theorem therein.

0.6. Deformation spaces of weak del Pezzo surfaces. As we explained above, the knowledge of $h^0(\widetilde{X}, T_{\widetilde{X}})$ for any weak del Pezzo surface $\widetilde{X}$, as obtained in Chapter II, allows us to calculate $h^1(\widetilde{X}, T_{\widetilde{X}})$ for all such surfaces: Since $H^2(\widetilde{X}, T_{\widetilde{X}}) = 0$, the deformation space of $\widetilde{X}$ is formally smooth and hence uniquely determined by its dimension $h^1(\widetilde{X}, T_{\widetilde{X}})$. By the above Equation 0.1 and the classification of Chapter II, we can determine its dimension for all weak del Pezzo surfaces in all degrees and characteristics. For the convenience of the reader, we collected the results on dimensions of deformation spaces of weak del Pezzo surfaces in Table 0 in the Appendix 0 of this thesis.

The next step in understanding the relation between deformations of weak del Pezzo surfaces, RDP del Pezzo surfaces, and RDPs is determining the deformations spaces of RDP del Pezzo surfaces. There, the situation is more delicate, for a number of reasons:

Firstly, not every deformation of an RDP del Pezzo surface X is locally trivial and, in fact, the most interesting deformations here are those that induce non-trivial deformations of the rational double points and these deformations are never locally trivial. Secondly, in positive characteristic, the vanishing of $H^2(X, T_X)$ does not follow from a general vanishing result and, indeed, examples of RDP del Pezzo surfaces X with $H^2(X, T_X) \neq 0$ in characteristic $p = 2, 3$ have recently been exhibited by Kawakami and Nagaoka [KN22, Remark 1.8]. Finally, an RDP del Pezzo surface X can have more global vector fields than its minimal resolution, since there is no a priori reason for an action of a non-reduced group scheme to lift to the resolution.

Nevertheless, even though obtaining a full description of the deformation space of an RDP del Pezzo surface X is more difficult than in the case of weak del Pezzo surfaces, the first question that needs to be answered is how big $H^0(X, T_X)$ can be.

0.7. RDP del Pezzo surfaces with global vector fields in odd characteristic. As in the case of weak del Pezzo surfaces, we obtain the dimension of $H^0(X, T_X)$ as a byproduct of the determination of Aut_X^0 for all RDP del Pezzo surfaces X in all degrees and odd characteristics.

Note that Blanchard's Lemma applied to the minimal resolution $\pi : \widetilde{X} \to X$ yields a closed immersion $\mathrm{Aut}_{\widetilde{X}}^0 \hookrightarrow \mathrm{Aut}_X^0$. We prove that this closed immersion is an isomorphism in characteristic $p \notin \{2, 3, 5, 7\}$, hence the classification of RDP del Pezzo surfaces with global vector fields is equivalent to the classification of weak del Pezzo surfaces with global vector fields in these characteristics. If $p \in \{3, 5, 7\}$, we classify the cases where this closed immersion is not an isomorphism. There are two such surfaces in characteristic 7, nine such surfaces in characteristic 5, and 56 families of RDP del Pezzo surfaces X where Aut_X^0 is strictly larger than $\mathrm{Aut}_{\widetilde{X}}^0$ in characteristic 3. Moreover, we give explicit equations for all such RDP del Pezzo surfaces in all possible degrees. As an application, we construct regular non-smooth RDP del Pezzo surfaces over imperfect fields of characteristic 7, thereby showing that the known bound $p \leq 7$ [BT20, Proposition 5.2] for the characteristics, where such a surface can exist, is sharp.

In the local setting, the problem that group scheme actions on X do not necessarily lift to $\widetilde{X}$ is mirrored by the fact that not every vector field on a rational double point singularity lifts to its minimal resolution [Wah75, (5.18.1) Remarks]. Rational double points where all vector fields lift are called *equivariant* and were classified by Hirokado [Hir19]. The idea of our proof is to show first that $\mathrm{Aut}_{\widetilde{X}}^0 \hookrightarrow \mathrm{Aut}_X^0$ is an isomorphism if X contains only equivariant RDPs, and to give a criterion for liftability of group scheme actions for the simplest example of a non-equivariant RDP in characteristic p given by the A_{p-1}-singularity. This reduces the problem of determining Aut_X^0 for general RDP del Pezzo surfaces to RDP del Pezzo surfaces containing non-equivariant RDPs, a list of which can now be compiled easily from the results of Chapter I and [Hir19]. For these *non-equivariant RDP del Pezzo surfaces* we set up an inductive argument, where we first classify the RDP del Pezzo surfaces of highest possible degree with $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and a given configuration of non-equivariant RDPs, and then use Blanchard's Lemma to obtain a classification in smaller degrees.

For further details and the complete classification of RDP del Pezzo surfaces with global vector fields in characteristic different from 2, we refer the reader to Chapter III and Theorem 1.1 and Theorem 1.2 therein.

0.8. On rational (quasi-)elliptic surfaces with global vector fields. One of the key tools we used in our classification of rational double points on RDP del Pezzo surfaces in Chapter I was the close connection between weak (resp. RDP) del Pezzo surfaces and Jacobian rational (quasi-)elliptic surfaces (resp. their Weierstraß models). More precisely, the blow-up of the unique base point of the anti-canonical system on a weak del Pezzo surface of degree 1 yields a Jacobian rational (quasi-)elliptic surface and the contraction of any section on a Jacobian rational (quasi-)elliptic surface recovers the weak del Pezzo surface. By Blanchard's Lemma and the fact that $\mathrm{Aut}^0_{\widetilde{X}}$ fixes the base point, this yields an identification of the automorphism schemes of the surfaces that correspond to each other under this bijection.

This begs for the question whether one can achieve a similar classification for arbitrary rational (quasi-)elliptic surfaces that do not necessarily admit a section. In Chapter IV, we recall the classical description of these surfaces as the resolution of base points of a Halphen pencil [Hal82] of curves of degree $3m$ in $\mathbb{P}^2$. In particular, this description shows that non-Jacobian rational (quasi-)elliptic surfaces with a multiple fiber of multiplicity m are also blow-ups of weak del Pezzo surfaces of degree 1, but, instead of the base point of $|-K_{\widetilde{X}}|$, one blows up a point $\widetilde{P}$ of exact order m on the identity component of an anti-canonical curve. Thus, even though there is no longer a correspondence between such rational (quasi-)elliptic surfaces $\widetilde{Z}$ and weak del Pezzo surfaces $\widetilde{X}$ of degree 1, we can still use Blanchard's Lemma to calculate $\mathrm{Aut}^0_{\widetilde{Z}}$ as the stabilizer of $\widetilde{P}$ with respect to the action of $\mathrm{Aut}^0_{\widetilde{X}}$. In particular, if $\widetilde{Z}$ has global vector fields, then so does $\widetilde{X}$ and, going through the list of such $\widetilde{X}$ in Chapter II, one can classify all $\widetilde{Z}$ with global vector fields.

This approach works in arbitrary characteristic, but since the final chapter of this thesis is more of an outlook – because it is only tangentially related to del Pezzo surfaces and rational double points – we restrict ourselves to the four families of weak del Pezzo surfaces of degree 1 in the list of Chapter II that occur also in larger characteristics. This will allow us to keep Chapter IV relatively short while still illustrating that the techniques developed in the previous chapters also apply to more general classes of surfaces. At the same time, we establish a complete classification of all rational (quasi-)elliptic surfaces – both Jacobian and non-Jacobian – with global vector fields in characteristics $p \neq 2, 3$.

For further details, the complete classification in characteristic different from 2 and 3, as well as discussions of the geometry of some explicit examples in characteristic 2 and 3, we refer the reader to Chapter IV and Theorem 1.1 therein.

Conventions, official announcements, and acknowledgements. Concerning the numbering of definitions, propositions, theorems, tables, figures etc. we apply the following conventions: The numbering will be by chapter and hence it will be reset at the start of every new chapter. Whenever, in one of the four main chapters of this thesis, we refer to a result with a number only and do not mention the chapter it is contained in, the result will be found in the same chapter as the reference. Whenever we wish to refer to a result of another chapter, the chapter this result can be found in will be mentioned explicitly, together with its label in that respective chapter.

In the digital version of this thesis, every reference will have an underlying link attached to it, which leads the reader to the corresponding part of this thesis. In the Appendix, all classification tables are collected chapter-wise for the sake of user-friendliness and accessibility of the results of this thesis. There, each caption encompasses a link leading to the first occurrence of the respective result in the text.

In case a reader is mainly interested in one of the four chapters of this composition, sections called "Motivation and summary" and recaps of the necessary background material are included in each chapter such that, despite the chapters building upon one another, each of them will nevertheless be also readable on its own.

Parts of this thesis have been made accessible to the academic community in the form of preprints on the ArXiv, and have been published resp. accepted resp. submitted for publication in peer-reviewed journals:

- Chapter I follows the article [Sta21], which has been published in *Épijournal de Géométrie Algébrique*. Moreover, it has officially been acknowledged for my master's thesis in the TopMath program at TUM. [1]
- Chapter II follows the article [MS20], which is joint work with Gebhard Martin and has been accepted for publication in *Geometry and Topology*. [2]
- Chapter III follows the article [MS22], which is joint work with Gebhard Martin and has been submitted to a peer-reviewed journal. [2]
- Chapter IV has not yet been made available to the academic community.

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[2] Bei [MS20] und [MS22] war Claudia Stadlmayr federführend im Sinne von II.2.b) Punkt 4 von *Klassische Promotion. Fakultätsinterne Standards [...] für die Prüfungsphase (Stand 11.7.2018)*.

CHAPTER I

Which rational double points occur on del Pezzo surfaces?

Up to minor modifications, this chapter is taken from the article "Which rational double points occur on del Pezzo surfaces?" of the author. The article is published in *Épijournal de Géométrie Algébrique* and can also be found on the ArXiv (see [Sta21]).

1. Motivation and summary

Throughout this chapter, we work over an algebraically closed field k of characteristic $\mathrm{char}(k) = p \geq 0$. Quite classically, del Pezzo surfaces admitting at worst rational double point singularities (also called *RDP del Pezzo surfaces*, *Gorenstein log del Pezzo surfaces* or *Du Val del Pezzo surfaces*) first appeared as non-degenerate surfaces of degree d in $\mathbb{P}^d$ which are not cones or projections of surfaces of minimal degree. A first natural question to ask is the following:

QUESTION 1.1. Which rational double points occur on RDP del Pezzo surfaces?

For $1 \leq d \leq 9$, we define the lattice E_{9-d} as the orthogonal complement of the vector $(-3, 1, \ldots, 1)$ in the unimodular lattice $\mathrm{I}^{1,9-d}$ of signature $(1, 9-d)$ defined by the matrix $\mathrm{diag}(1, -1, \ldots, -1)$. Over the complex numbers, the answer to Question 1.1 is old:

- If $d \geq 3$, a configuration Γ of rational double points occurs on an RDP del Pezzo surface X_d of degree d if and only if the lattice Γ' spanned by the irreducible components of the exceptional divisor of the minimal resolution $\widetilde{X}$ of X embeds into the lattice E_{9-d}. If $d \geq 5$, this is elementary to check, while if $d = 4$ it was proven by Timms [Tim28] in 1928, and if $d = 3$ it was proven by Schläfli [Sch63] in 1863.
- If $d = 2$, then X_2 is a double cover of $\mathbb{P}^2$ branched over a quartic curve. Simple singularities of plane quartics, and thus RDPs on X_2, were classified by Du Val [DV34] in 1934. It turns out that Γ occurs on some X_2 if and only if Γ' embeds into E_7 and Γ is not of type $7A_1$.
- If $d = 1$, then X_1 is a double cover of a quadric cone in $\mathbb{P}^3$ branched over a sextic curve and the possible singularities of the branch locus have also been classified by Du Val [DV34]. It turns out that Γ occurs on some X_1 if and only if Γ' embeds into E_8 and $\Gamma \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$.

Besides these very classical sources, we refer the reader to [Ura83] for a more modern treatment of the cases where $d \in \{1, 2\}$. From the above discussion, we see that if we do not care about the degree of X, but only about whether Γ occurs on *some* RDP del Pezzo surface, then the lattice E_8 – being the biggest possible orthogonal complement of $-K_X$ –

plays a central rôle and we note that the classification of root sublattices of E_8 is also quite classical and goes back to Dynkin [Dyn52].

Thus, we are mainly interested in Question 1.1 in the case $p > 0$, even though our methods also recover Du Val's results over the complex numbers. While an answer to Question 1.1 may be known to the experts if $p \neq 2, 3, 5$, it becomes particularly subtle in small characteristics, where a rational double point is not necessarily *taut*, that is, it is not necessarily uniquely determined by its dual resolution graph. Such non-taut rational double points can only occur in characteristic $2, 3$, and 5, and have first been studied and classified by Artin [Art77]: There are only finitely many rational double points with the same resolution graph and they are distinguished by a coindex that we call *Artin coindex* (see Table 7 for a summary of all non-taut rational double points).

Very recently, there has been substantial progress on RDP del Pezzo surfaces in positive characteristic:

- In characteristic at least 5, RDP del Pezzo surfaces of Picard rank 1 have been classified by Lacini in [Lac20], generalizing work of Ye [Ye02] and Furushima [Fur86] (see also [MZ88] and [MZ93]).
- In characteristic 2 and 3, Kawakami and Nagaoka [KN20] classify RDP del Pezzo surfaces of Picard rank 1 and determine some, but not all, of the Artin coindices of the rational double points that occur. In [KN22], they also investigate in detail some interesting pathological examples in characteristic 2 that will also appear as exceptional cases in the present chapter (see Proposition 3.2).

In this chapter, instead of studying RDP del Pezzo surfaces of small Picard rank, which play a prominent rôle in the minimal model program (see e.g. [MZ93, Lemma 2]), we want to approach the problem from a more classical angle and try to find a satisfying positive characteristic analogue of Du Val's work relating Question 1.1 to the lattice E_8.

Before stating our main result, let us fix some terminology. We say that a sum $\Gamma' = \sum_i \Gamma_{i,n_i}$ of root lattices (that is, $\Gamma_i \in \{A, D, E\}$ and n_i is the number of simple roots) occurs on a weak del Pezzo surface $\widetilde{X}$ if it is isomorphic to the lattice spanned by all (-2)-curves on $\widetilde{X}$.

Then, Theorem 1.2 gives a complete answer to Question 1.1 in arbitrary degree and Picard rank over an algebraically closed field of arbitrary characteristic $\mathrm{char}(k) = p \geq 0$.

THEOREM 1.2. *Let $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ be an RDP configuration with Artin coindices k_i and let $\Gamma' = \sum_i \Gamma_{i,n_i}$ be the lattice spanned by the irreducible components of the exceptional divisor of its minimal resolution.*

(1) If $p \neq 2$, then the following are equivalent:
- *Γ occurs on an RDP del Pezzo surface.*
- *Γ' occurs on a weak del Pezzo surface.*
- *Γ' embeds into E_8 and $\Gamma' \notin \{D_4 + 4A_1, 8A_1, 7A_1\}$.*

(2) If $p = 2$, then the following are equivalent:
- *Γ' occurs on a weak del Pezzo surface.*
- *Γ' embeds into E_8 and $\Gamma' \notin \{2A_3 + 2A_1, A_3 + 4A_1, 6A_1\}$.*

If all $\Gamma_{i,n_i}^{k_i}$ are taut, these statements are also equivalent to the following:
- Γ *occurs on an RDP del Pezzo surface.*

(3) *If $p = 2$ and some $\Gamma_{i,n_i}^{k_i}$ is non-taut, then the following are equivalent:*
- Γ *occurs on an RDP del Pezzo surface.*
- Γ *occurs in Table 4, 5, or 6.*

(4) *Moreover, in the Tables 2, 3, 4, 5, and 6, we give equations for all RDP del Pezzo surfaces of degree 1 containing a non-taut rational double point.*

For the convenience of the reader, we list all possible Γ' embedding into E_8 (see [Dyn52, Table 11]) and state whether the respective Γ' occurs on a weak del Pezzo surface or not. For the possible Artin coindices of the corresponding RDP configurations in the non-taut case if $p = 2$, we refer the reader to Tables 4, 5, and 6 (see also Remark 4.6).

REMARK 1.3. We will see in Proposition 3.2, that if $p \neq 2$, all rational double points that occur on some RDP del Pezzo surface also occur on an RDP del Pezzo surface of degree 1. This fails for precisely four RDP configurations in characteristic 2, namely for $\Gamma \in \{E_7^0, D_6^0 + A_1, D_4^0 + 3A_1, 7A_1\}$, each of which occurs on a unique RDP del Pezzo surface of degree 2. These four surfaces coincide with the surfaces described in [KN22, Theorem 1.4(2)]. Note, however, that the (-2)-curve configurations of types E_7 and $D_6 + A_1$ do occur on weak del Pezzo surfaces of degree 1 in characteristic 2, whereas the configurations of (-2)-curves $D_4 + 3A_1$ and $7A_1$ do not.

REMARK 1.4. One way of classifying RDP configurations on RDP del Pezzo surfaces over the complex numbers is to reduce to the case of RDP del Pezzo surfaces of Picard rank 1 or 2 as described in [MZ93, Lemma 2, Lemma 4]. However, as it is unclear whether this reduction also works in positive characteristic (in particular if $p = 2, 3, 5$), and since the classification of RDP del Pezzo surfaces of Picard rank 2 in positive characteristic is not available yet, we will pursue a different approach in this chapter.

The structure of this chapter and thus also the structure of the proof of Theorem 1.2 is as follows: After recalling the classification of non-taut rational double points in Section 2, we show in Section 3.1 that an RDP configuration occurs on an RDP del Pezzo surface if and only if it occurs on an RDP del Pezzo surface of degree 1, with the exception of the four configurations mentioned in Remark 1.3. Then, in Section 3.2, we recall the well-known connection between RDP del Pezzo surfaces of degree 1 and Weierstraß models of rational (quasi-)elliptic surfaces. In Section 3.3, we explain how this connection, and the theory of Mordell–Weil groups, can be exploited to classify all configurations of (-2)-curves that can occur on weak del Pezzo surfaces. This reduces Question 1.1 to non-taut RDPs and thus to characteristics $2, 3$, and 5. Finally, the bulk of the chapter is devoted to the classification of RDP del Pezzo surfaces of degree 1 with at least one non-taut rational double point in characteristic $2, 3$, and 5. This is achieved by using the classification of singular fibers of rational (quasi-)elliptic surfaces due to Ito [Ito92], [Ito94], and Jarvis–Lang–Rimmasch–Rogers–Summers–Petrosyan [JLR+05], [Lan00] to derive simple equations for these RDP del Pezzo surfaces that allow us to explicitly determine the Artin coindices of the rational double points that occur.

$\Gamma' \hookrightarrow E_8$	occurs if		$\Gamma' \hookrightarrow E_8$	occurs if		$\Gamma' \hookrightarrow E_8$	occurs if	
	$p \neq 2$	$p = 2$		$p \neq 2$	$p = 2$		$p \neq 2$	$p = 2$
A_1	✓	✓	$A_3 + 3A_1$	✓	✓	$A_5 + A_2$	✓	✓
$2A_1$	✓	✓	$3A_2$	✓	✓	$D_5 + A_2$	✓	✓
A_2	✓	✓	$A_3 + A_2 + A_1$	✓	✓	$A_6 + A_1$	✓	✓
$3A_1$	✓	✓	$A_4 + 2A_1$	✓	✓	$D_6 + A_1$	✓	✓
$A_2 + A_1$	✓	✓	$D_4 + 2A_1$	✓	✓	$E_6 + A_1$	✓	✓
A_3	✓	✓	$2A_3$	✓	✓	A_7	✓	✓
$4A_1$	✓	✓	$A_4 + A_2$	✓	✓	D_7	✓	✓
$A_2 + 2A_1$	✓	✓	$D_4 + A_2$	✓	✓	E_7	✓	✓
$2A_2$	✓	✓	$A_5 + A_1$	✓	✓	$8A_1$	×	✓
$A_3 + A_1$	✓	✓	$D_5 + A_1$	✓	✓	$D_4 + 4A_1$	×	✓
A_4	✓	✓	A_6	✓	✓	$4A_2$	✓	✓
D_4	✓	✓	D_6	✓	✓	$2A_3 + 2A_1$	✓	×
$5A_1$	✓	✓	E_6	✓	✓	$A_5 + A_2 + A_1$	✓	✓
$A_2 + 3A_1$	✓	✓	$7A_1$	×	✓	$D_6 + 2A_1$	✓	✓
$2A_2 + A_1$	✓	✓	$A_3 + 4A_1$	✓	×	$2A_4$	✓	✓
$A_3 + 2A_1$	✓	✓	$3A_2 + A_1$	✓	✓	$2D_4$	✓	✓
$A_3 + A_2$	✓	✓	$A_3 + A_2 + 2A_1$	✓	✓	$D_5 + A_3$	✓	✓
$A_4 + A_1$	✓	✓	$D_4 + 3A_1$	✓	✓	$E_6 + A_2$	✓	✓
$D_4 + A_1$	✓	✓	$2A_3 + A_1$	✓	✓	$A_7 + A_1$	✓	✓
A_5	✓	✓	$A_4 + A_2 + A_1$	✓	✓	$E_7 + A_1$	✓	✓
D_5	✓	✓	$A_5 + 2A_1$	✓	✓	A_8	✓	✓
$6A_1$	✓	×	$D_5 + 2A_1$	✓	✓	D_8	✓	✓
$A_2 + 4A_1$	✓	✓	$A_4 + A_3$	✓	✓	E_8	✓	✓
$2A_2 + 2A_1$	✓	✓	$D_4 + A_3$	✓	✓			

Table 1. $\Gamma' \subseteq E_8$ occurring on weak del Pezzo surfaces

NOTATION. In Tables 2, 3, 4, 5, and 6 we list all possible RDP configurations containing a non-taut rational double point in Column 1. In Column 2, we give simplified Weierstraß equations for all RDP del Pezzo surfaces of degree 1 containing the respective configuration. If an extra condition on parameters in such an equation leads to extra RDPs, the condition is written under the respective equation separated by a dashed line. The equation above a dashed line is assumed to satisfy none of the conditions listed below it. In Columns 3 and 4 we give the discriminant Δ and the j-invariant j (see Subsection 4.1 for explicit formulae) of the corresponding rational (quasi-)elliptic surface, whose type in the notation of Lang/Ito/Miranda–Persson is given in Column 5 and we note in Column 6 whether the fibration is elliptic or quasi-elliptic.

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$	$\Delta =$	$j =$	Miranda's & Persson's type
$\mathbf{E_8}$				
E_8^0	$y^2 = x^3 + t^5 s$	$-2t^{10}s^2$	0	X_{22}
E_8^1	$y^2 = x^3 + t^4 x + t^5 s$	$t^{10}(t^2 - 2s^2)$	$\frac{3t^{12}}{\Delta}$	X_{211}

Table 2. E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 5$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{E_6}$				
E_6^0	$y^2 = x^3 + t^4x + t^4s^2$	$-t^{12}$	0	6C	ell
E_6^0	$y^2 = x^3 + t^3sx + a_{6,5}t^5s + t^4s^2$	$-t^9s^3$	0	6A	ell
$+ \quad A_1$	if $a_{6,5} = 0$			6A 5.	ell
$E_6^0 \quad + \quad A_2$	$y^2 = x^3 + t^4s^2$	0		3.3(2)	q-ell
E_6^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s + t^4s^2$	$-t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	6B	ell
$+ \quad A_1$	if $a_{6,5} = 0$			6B 2.	ell
E_6^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3$	$-t^9s(a_{6,5}t^2 + a_{6,4}ts + s^2)$	$\frac{t^{12}}{\Delta}$	6A	ell
$+ \quad A_1$	if $a_{6,4} \neq 0$ and $(a_{6,5} = 0$ or $a_{6,5} = a_{6,4}^2)$			6A 2.	ell
$+ \quad A_2$	if $a_{6,5} = 0$ and $a_{6,4} = 0$			6A 1.	ell
	$\mathbf{E_7}$				
E_7^0	$y^2 = x^3 + t^3sx + t^5s$	$-t^9s^3$	0	7	ell
$E_7^0 \quad + \quad A_1$	$y^2 = x^3 + t^3sx$	$-t^9s^3$	0	7 5.	ell
E_7^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s - t^4s^2 + t^3s^3$	$-t^9s(a_{6,5}t^2 - ts + s^2)$	$\frac{t^{12}}{\Delta}$	7	ell
$+ \quad A_1$	if $a_{6,5} \in \{0,1\}$			7 2.	ell
	$\mathbf{E_8}$				
E_8^0	$y^2 = x^3 + t^5s$	0		3.3(1)	q-ell
E_8^1	$y^2 = x^3 + t^4x + t^5s$	$-t^{12}$	0	8B	ell
E_8^2	$y^2 = x^3 + t^2x^2 + t^5s$	$-t^{11}s$	$-\frac{t}{s}$	8A 1.	ell

Table 3. E_6-, E_7- and E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 3$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{D_4}$				
D_4^0	$y^2 + t^3y = x^3 + (a_{4,2}s^2 + a_{4,3}ts + a_{4,4}t^2)t^2x + s^3t^3$	t^{12}	0	12B	ell
D_4^0	$y^2 + t^2sy = x^3 + a_{2,1}tsx^2 + a_{4,3}t^3sx + a_{6,6}t^6 + t^3s^3$	t^8s^4	0	12A	ell
$+ \quad A_1$	if $a_{6,6} = 0$ and $a_{4,3} \neq 0$			12A 10A	ell
$+ \quad A_2$	if $a_{6,6} = a_{4,3} = 0$			12A 11	ell
$D_4^0 \quad + \quad 3A_1$	occurs only in degree 2 (see Proposition 3.2 (C.))				
$D_4^0 \quad + \quad 4A_1$	$y^2 = x^3 + (t^3s + a_{4,2}t^2s^2 + ts^3)x$ with $a_{4,2} \neq 0$	0		5.2.(f)	q-ell
$D_4^0 \quad + \quad D_4^0$	$y^2 = x^3 + a_{4,2}t^2s^2x + t^3s^3$	0		5.2.(d)	q-ell
D_4^1	$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3$	$t^9s(a_{6,5}t^2 + a_{6,4}ts + s^2)$	$\frac{t^{12}}{\Delta}$	4B.	ell
$+ \quad A_1$	if $(a_{6,5} = 0$ and $a_{6,4} \neq 0)$ or $(a_{6,5} \neq 0$ and $a_{6,4} = 0)$			4B. 2.	ell
$+ \quad A_2$	if $a_{6,5} = a_{6,4} = 0$			4B. 3.	ell
D_4^1	$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + a_{6,3}t^3s^3 + t^2s^4$ with $a_{2,1} + a_{6,3} \neq 0$	$t^8s(a_{6,5}t^3 + a_{6,4}t^2s + a_{6,3}ts^2 + s^3)$	$\frac{t^{12}}{\Delta}$	4A.	ell
$+ \quad A_1$	if $(a_{6,5} = 0$ and $a_{6,3} \neq 0)$ or $a_{6,5} = a_{6,4}a_{6,3} \neq 0$			4A. 2.	ell
$+ \quad 2A_1$	if $a_{6,5} = a_{6,3} = 0$ and $a_{6,4} \neq 0$			4A. 4.	ell
$+ \quad A_2$	if $(a_{6,5} = a_{6,4} = 0$ and $a_{6,3} \neq 0)$ or $(a_{6,3}^2 = a_{6,4}$ and $a_{6,3}^3 = a_{6,5} \neq 0)$			4A. 3.	ell
$+ \quad A_3$	if $a_{6,5} = a_{6,4} = a_{6,3} = 0$			4A. 5.	ell
	$\mathbf{D_5}$				
D_5^0	$y^2 + t^2sy = x^3 + (a_{2,2}t^2 + ts)x^2 + a_{6,5}t^5s$	t^8s^4	0	13A	ell
$+ \quad A_1$	if $a_{6,5} = 0$ and $a_{2,2} \neq 0$			13A 10A	ell
$+ \quad A_2$	if $a_{6,5} = a_{2,2} = 0$			13A 11	ell
D_5^1	$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + a_{6,3}t^3s^3 + t^2s^4$ with $a_{2,1} = a_{6,3}$	$t^8s(a_{6,5}t^3 + a_{6,4}t^2s + a_{6,3}ts^2 + s^3)$	$\frac{t^{12}}{\Delta}$	5A.	ell
$+ \quad A_1$	if $(a_{6,5} = 0$ and $a_{6,3} \neq 0)$ or $a_{6,5} = a_{6,4}a_{6,3} \neq 0$			5A. 2.	ell
$+ \quad 2A_1$	if $a_{6,5} = a_{6,3} = 0$ and $a_{6,4} \neq 0$			5A. 4.	ell
$+ \quad A_2$	if $(a_{6,5} = a_{6,4} = 0$ and $a_{6,3} \neq 0)$ or $(a_{6,3}^2 = a_{6,4}$ and $a_{6,3}^3 = a_{6,5} \neq 0)$			5A. 3.	ell
$+ \quad A_3$	if $a_{6,5} = a_{6,4} = a_{6,3} = 0$			5A. 5.	ell

Table 4. D_4- and D_5-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 2$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{D_6}$				
$D_6^0 + A_1$	occurs only in degree 2 (see Proposition 3.2 (B.))				
$D_6^0 + 2A_1$	$y^2 = x^3 + (t^3s + t^2s^2)x$	0		5.2.(e)	q-ell
D_6^1	$y^2 + t^3y = x^3 + (a_{2,2}t^2 + a_{2,1}ts)x^2 + t^3sx$	t^{12}	0	13B	ell
D_6^2	$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + t^4s^2$ with $a_{2,1} \neq 0$	$t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	5B.	ell
$+ A_1$	if $a_{6,5} = 0$			5B. 2.	ell
	$\mathbf{D_7}$				
D_7^1	$y^2 + t^3y = x^3 + tsx^2$	t^{12}	0	13C	ell
D_7^2	$y^2 + txy = x^3 + a_{2,1}tsx^2 + t^5s$ with $a_{2,1} \neq 0$	$t^{11}s$	$\frac{t}{s}$	5C.	ell
	$\mathbf{D_8}$				
D_8^0	$y^2 = x^3 + t^2s^2x + t^5s$	0		5.2.(b)	q-ell
D_8^3	$y^2 + txy = x^3 + tsx^2 + a_{6,6}t^6$ with $a_{6,6} \neq 0$	$a_{6,6}t^{12}$	$\frac{1}{a_{6,6}}$	5D.	ell

Table 5. D_6-, D_7- and D_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 2$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
$\mathbf{E_6}$					
E_6^0	$y^2 + t^2sy = x^3 + a_{2,2}t^2x^2 + a_{6,5}t^5s$	t^8s^4	0	14	ell
$+ \quad A_1$	if $a_{6,5} = 0$ and $a_{2,2} \neq 0$			14 10A	ell
$+ \quad A_2$	if $a_{6,5} = a_{2,2} = 0$			14 11	ell
E_6^1	$y^2 + txy = x^3 + tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3 + t^2s^4$	$t^8s(a_{6,5}t^3 + a_{6,4}t^2s + ts^2 + s^3)$	$\frac{t^{12}}{\Delta}$	6.	ell
$+ \quad A_1$	if ($a_{6,5} = 0$ and $a_{6,4} \neq 0$) or $a_{6,5} = a_{6,4} \notin \{0,1\}$			6. 2.	ell
$+ \quad A_2$	$a_{6,5} = a_{6,4} \in \{0,1\}$			6. 3.	ell
$\mathbf{E_7}$					
E_7^0	occurs only in degree 2 (see Proposition 3.2 (A.))				
$E_7^0 \; + \; A_1$	$y^2 = x^3 + t^3sx$	0		5.2.(c)	q-ell
E_7^2	$y^2 + t^3y = x^3 + t^3sx$	t^{12}	0	15	ell
E_7^3	$y^2 + txy = x^3 + a_{6,5}t^5s + t^4s^2$	$t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	7.	ell
$+ \quad A_1$	if $a_{6,5} = 0$			7. 2.	ell
$\mathbf{E_8}$					
E_8^0	$y^2 = x^3 + t^5s$	0		5.2.(a)	q-ell
E_8^3	$y^2 + t^3y = x^3 + t^5s$	t^{12}	0	16	ell
E_8^4	$y^2 + txy = x^3 + t^5s$	$t^{11}s$	$\frac{t}{s}$	8.	ell

Table 6. E_6-, E_7- and E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 2$

2. Non-taut rational double points in positive characteristic

Recall that *rational double point* (RDP) is one of the names for a canonical surface singularity. The $n \geq 1$ irreducible components of the exceptional divisor of its minimal resolution span a negative definite root lattice Γ_n of type $\Gamma \in \{A, D, E\}$. If $p \neq 2, 3, 5$, every rational double point is *taut*, that is, its formal isomorphism class is uniquely determined by Γ_n. It turns out that this fails for certain rational double points in small characteristics. Nevertheless, Artin [Art77] was able to give a classification of formal isomorphism classes of these non-taut rational double points and, in particular, he proved that there are only finitely many isomorphism classes $\Gamma_n^1, \ldots, \Gamma_n^{k_n}$ for each fixed Γ_n. The number k in Γ_n^k is called *Artin coindex* of the rational double point (e.g., if $p = 5$, there are two distinct rational double points E_8^0 and E_8^1 both of which have resolution graph of type E_8). We call a formal sum $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ of such RDPs $\Gamma_{i,n_i}^{k_i}$ an *RDP configuration*.

In the following Table 7 (where $n \geq 2$ and $1 \leq r \leq n-1$), we listed Artin's equations for the non-taut rational double points together with the dimension m of their miniversal deformation spaces. Here, we observe that the Artin coindices for a given Dynkin type can be distinguished by m:

OBSERVATION 2.1. *The completions of two rational double points are isomorphic if and only if they have the same resolution graph and the dimensions m of their miniversal deformation spaces coincide.*

The following well-known Proposition 2.2 provides a way to calculate m for hypersurface singularities, and thus in particular for the rational double points.

PROPOSITION 2.2. *[Art76, Chapter 1, §4] Let the local ring $R = k[x, y, z]_{(x,y,z)}/(f)$ be a normal surface singularity given by one equation $f \in k[x, y, z]$. Then, the tangent space T_f of the deformation functor $\mathrm{Def}_{\mathrm{Spec}\, R}$ of $\mathrm{Spec}\, R$ is given by*

$$T_f \coloneqq \mathrm{Def}_{\mathrm{Spec}\, R}(k[\epsilon]/(\epsilon^2)) \cong k[x, y, z]_{(x,y,z)}/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}).$$

Characteristic 5

Type	Equation	m
$\mathbf{E_8}$		
E_8^0	$z^2+x^3+y^5$	10
E_8^1	$z^2+x^3+y^5+xy^4$	8

Characteristic 3

Type	Equation	m
$\mathbf{E_6}$		
E_6^0	$z^2+x^3+y^4$	9
E_6^1	$z^2+x^3+y^4+x^2y^2$	7
$\mathbf{E_7}$		
E_7^0	$z^2+x^3+xy^3$	9
E_7^1	$z^2+x^3+xy^3+x^2y^2$	7
$\mathbf{E_8}$		
E_8^0	$z^2+x^3+y^5$	12
E_8^1	$z^2+x^3+y^5+x^2y^3$	10
E_8^2	$z^2+x^3+y^5+x^2y^2$	8

Characteristic 2

Type	Equation	m
$\mathbf{D_n}$		
D_{2n}^0	$z^2+x^2y+xy^n$	$4n$
D_{2n}^r	$z^2+x^2y+xy^n+xy^{n-r}z$	$4n-2r$
D_{2n+1}^0	$z^2+x^2y+y^nz$	$4n$
D_{2n+1}^r	$z^2+x^2y+y^nz+xy^{n-r}z$	$4n-2r$
$\mathbf{E_6}$		
E_6^0	$z^2+x^3+y^2z$	8
E_6^1	$z^2+x^3+y^2z+xyz$	6
$\mathbf{E_7}$		
E_7^0	$z^2+x^3+xy^3$	14
E_7^1	$z^2+x^3+xy^3+x^2yz$	12
E_7^2	$z^2+x^3+xy^3+y^3z$	10
E_7^3	$z^2+x^3+xy^3+xyz$	8
$\mathbf{E_8}$		
E_8^0	$z^2+x^3+y^5$	16
E_8^1	$z^2+x^3+y^5+xy^3z$	14
E_8^2	$z^2+x^3+y^5+xy^2z$	12
E_8^3	$z^2+x^3+y^5+y^3z$	10
E_8^4	$z^2+x^3+y^5+xyz$	8

Table 7. Types of non-taut rational double points in $\mathrm{char}(k)=2,3,5$

3. From del Pezzo surfaces to rational (quasi-)elliptic surfaces

In this section, we reduce Question 1.1 to the corresponding question for Weierstraß models of rational (quasi-)elliptic surfaces. On the way, we recall the necessary background on del Pezzo surfaces and rational (quasi-)elliptic surfaces as well as their connection.

3.1. From del Pezzo surfaces to del Pezzo surfaces of degree 1**.** Recall the following related notions of *del Pezzo surfaces*.

DEFINITION 3.1. Let X and $\widetilde{X}$ be projective surfaces.

- X is a *del Pezzo surface* if it is smooth and $-K_X$ is ample.
- $\widetilde{X}$ is a *weak del Pezzo surface* if it is smooth and $-K_{\widetilde{X}}$ is big and nef. The lattice $\sum_i \Gamma_{i,n_i} \subseteq \mathrm{Pic}(\widetilde{X})$ spanned by all the (-2)-curves on $\widetilde{X}$ is called *configuration of* (-2)*-curves on* $\widetilde{X}$.
- X is an *RDP del Pezzo surface* if all its singularities are rational double points and $-K_X$ is ample. The formal sum $\sum_i \Gamma_{i,n_i}^{k_i}$ of the formal isomorphism classes of all the rational double points on X is called *RDP configuration of* X.

In all the above cases, the number $\deg(X) = K_X^2$ (resp. $\deg(\widetilde{X}) = K_{\widetilde{X}}^2$) is called the *degree of* X (resp. $\widetilde{X}$).

Note that weak del Pezzo surfaces are precisely the minimal resolutions of RDP del Pezzo surfaces and every RDP del Pezzo surface X is obtained by contracting all the (-2)-curves on a weak del Pezzo surface $\widetilde{X}$. If the RDP configuration of X is $\sum_i \Gamma_{i,n_i}^{k_i}$, then the configuration of (-2)-curves on $\widetilde{X}$ is $\sum_i \Gamma_{i,n_i}$.

The following observation tells us that, in order to understand rational double points on RDP del Pezzo surfaces, it suffices to understand them on RDP del Pezzo surfaces of degree 1 with precisely four exceptions in characteristic 2:

PROPOSITION 3.2. *Let* $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ *be an RDP configuration. If* Γ *occurs on an RDP del Pezzo surface, but not on an RDP del Pezzo surface of degree* 1*, then* $p = 2$ *and* Γ *is one of the following:*

(A.) $\Gamma = E_7^0$
(B.) $\Gamma = D_6^0 + A_1$
(C.) $\Gamma = D_4^0 + 3A_1$
(D.) $\Gamma = 7A_1$

Moreover, there is a unique RDP del Pezzo surface (necessarily of degree 2*) realizing each of these exceptional cases.*

PROOF. If X is an RDP del Pezzo surface of degree $d \geq 2$ with RDP configuration Γ, we want to construct an RDP del Pezzo surface X_1 of degree 1 with the same configuration Γ by finding a point $p \in \widetilde{X}$, where $\widetilde{X}$ is the minimal resolution of X, such that $\mathrm{Bl}_p(\widetilde{X})$ is again a weak del Pezzo surface and such that $\mathrm{Bl}_p(\widetilde{X})$ and $\widetilde{X}$ have the same configuration of (-2)-curves. Contracting the (-2)-curves on $\mathrm{Bl}_p(\widetilde{X})$ yields an RDP del Pezzo surface of degree $(d-1)$ with RDP configuration Γ, so the claim will follow by induction.

We fix a realization of $\widetilde{X}$ as a blow-up of $\mathbb{P}^2$ in (possibly infinitely near) points $p_1, \ldots, p_{9-d}$. Using the precise description of (-2)-curves on $\widetilde{X}$ (see e.g. Lemma 2.8 (i) in Chapter II), we see that the blow-up of $\widetilde{X}$ in a point $p \in \widetilde{X}$ will be a weak del Pezzo surface with the same configuration of (-2)-curves as $\widetilde{X}$ if and only if the following two conditions are satisfied:

(1) p does not lie on a (-1)- or (-2)-curve on $\widetilde{X}$, and
(2) if $d = 2$, then p is not the singular point of the strict transform $\widetilde{C}$ of an irreducible singular cubic $C \subseteq \mathbb{P}^2$ through $p_1, \ldots, p_7$.

Since there are only finitely many negative curves on $\widetilde{X}$, we can always find a p that satisfies Condition (1). Thus, we may assume that $d = 2$. To deal with Condition (2), note that every $\widetilde{C}$ as in Condition (2) is a member of the two-dimensional linear system $|-K_{\widetilde{X}}|$ and consider the variety

$$\mathfrak{I} = \{(\widetilde{C}, p) \mid \widetilde{C} \in |-K_{\widetilde{X}}| \text{ is integral and singular, and } p \text{ is its singular point}\}.$$

Denoting the sublocus of singular curves in $|-K_{\widetilde{X}}|$ as $|-K_{\widetilde{X}}|_{\text{sing}}$, we have a correspondence

$$\begin{tikzcd} \mathfrak{I} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \widetilde{X} \\ |-K_{\widetilde{X}}|_{\text{sing}} \arrow[r, hook] & |-K_{\widetilde{X}}|, \end{tikzcd}$$

where pr_1 is quasi-finite. If the image of pr_2 is not dense in $\widetilde{X}$, then we can find a p satisfying Conditions (1) and (2). Hence, we have to show that if pr_2 is dominant, then Γ is one of the four Exceptions (A.), (B.), (C.), or (D.)

If pr_2 is dominant, then

$$2 = \dim |-K_{\widetilde{X}}| \geq \dim |-K_{\widetilde{X}}|_{\text{sing}} = \dim \mathfrak{I} \geq \dim(\widetilde{X}) = 2,$$

and since $|-K_{\widetilde{X}}|_{\text{sing}}$ is closed in $|-K_{\widetilde{X}}| \cong \mathbb{P}^2$, we have $|-K_{\widetilde{X}}|_{\text{sing}} = |-K_{\widetilde{X}}|$, that is, every anti-canonical curve on $\widetilde{X}$ is singular. By [KN22, Theorem 1.4(2)] the (-2)-curve configuration Γ' associated to Γ is one of the following:

(A'.) $\Gamma' = E_7$
(B'.) $\Gamma' = D_6 + A_1$
(C'.) $\Gamma' = D_4 + 3A_1$
(D'.) $\Gamma' = 7A_1$

So, we still have to determine the Artin coindices in Cases (A'.), (B'.) and (C'.). By [KN22, Lemma 4.5(5)–(7)] and [Ito94, Theorem 5.2.], the non-taut rational double points in these cases are isomorphic to RDPs that occur in partial resolutions of the affine surfaces in $\mathbb{A}^3$ given by

(A".) $y^2 = x^3 + t^5$,
(B".) $y^2 = x^3 + t^3x$, and
(C".) $y^2 = x^3 + (t^3 + t)x$, respectively.

We see from Table 7 that the first two cases are E_8^0 and E_7^0, respectively. In Case (C".) one can apply Proposition 2.2 to see that the singularity is of type D_6^0. By [Sta18b, Theorem 2.70., Table 10], the Artin coindex of a rational double point appearing in a partial resolution of a rational double point with coindex 0 is itself 0. Hence, the Artin coindices in the exceptional Cases (A'.), (B'.), and (C'.) are all 0.

The existence and uniqueness of the four exceptional cases was proved in [KN22, Theorem 1.4, Table 1]. □

Thus, Proposition 3.2 reduces the initial Question 1.1 to the following one:

QUESTION 3.3. Which rational double points occur on RDP del Pezzo surfaces of degree 1?

3.2. From del Pezzo surfaces of degree 1 to rational (quasi-)elliptic surfaces. To answer Question 3.3, we will exploit the connection between weak (resp. RDP) del Pezzo surfaces of degree 1 and (Jacobian) rational (quasi-)elliptic surfaces (resp. Weierstraß models of those). For this, let us first recall their definition (see for example [CD89, Chapter V]) or [SS10]).

DEFINITION 3.4. Let Y and $\widetilde{Y}$ be projective surfaces.

- $\widetilde{Y}$ is a *rational (quasi-)elliptic surface* if it is smooth, rational, and admits a morphism $f : \widetilde{Y} \to \mathbb{P}^1$ such that the following conditions hold:
  - f is surjective with $f_*\mathcal{O}_{\widetilde{Y}} = \mathcal{O}_{\mathbb{P}^1}$,
  - the generic fiber of f is a regular curve of arithmetic genus 1,
  - there are no (-1)-curves in fibers of f, and
  - f admits a section $\sigma_0 : \mathbb{P}^1 \to \widetilde{Y}$.

  Moreover, the group $\mathrm{MW}(f)$ of sections of f is called *Mordell–Weil group* of $f : \widetilde{Y} \to \mathbb{P}^1$.
- The *Weierstraß model Y of $\widetilde{Y}$* is the surface obtained from $\widetilde{Y}$ by contracting all components of fibers of f that do not meet $\sigma_0(\mathbb{P}^1)$.

REMARK 3.5. In the literature one usually finds the definition of a (quasi-)elliptic surface as a pair of a surface and a (quasi-)elliptic fibration. Since $\widetilde{Y}$ is rational, the canonical bundle formula shows that it admits a unique (quasi-)elliptic fibration induced by $|-K_{\widetilde{Y}}|$, so we do not need to specify the fibration. Similarly, while a priori the Weierstraß model Y seems to depend on the chosen section σ_0 of f, any two such sections are interchanged by an automorphism of $\widetilde{Y}$, so the associated Weierstraß models are isomorphic, and thus we will not keep track of the section.

Note that, because all components of reducible fibers of f are (-2)-curves, the Weierstraß model Y of $\widetilde{Y}$ has only rational double points as singularities. So, analogously to Section 3.1, we define the *configuration of (-2)-curves on $\widetilde{Y}$* and the *RDP configuration of Y*. Note, however, that the configuration of (-2)-curves on $\widetilde{Y}$ is not a sum of root lattices in general.

We assume that the reader is familiar with the Kodaira–Néron classification of singular fibers of (quasi-)elliptic surfaces [Kod60], [Kod63], [Ń64] as described for example in

[Sil94, Table 4.1, p.365]. In the following Table 8, we summarize which Kodaira–Néron type of a fiber in $\widetilde{Y}$ leads to which rational double point on its image in Y. Here, we denote a smooth point by A_0, and we have $n \geq 1$ for type I_n and $n \geq 0$ for type I_n^*.

Kodaira–Néron type	I_0	I_n	II	III	IV	I_n^*	IV^*	III^*	II^*
Rational double point	A_0	A_{n-1}	A_0	A_1	A_2	D_{4+n}	E_6	E_7	E_8

Table 8. Kodaira–Néron types and corresponding rational double points

By [Dol12, Section 8.3.2] an RDP del Pezzo surface X of degree 1 is isomorphic to a sextic hypersurface $V(f_6) \subseteq \mathbb{P}(1,1,2,3)$ and, conversely, a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ with at worst rational double point singularities defines an RDP del Pezzo surface of degree 1. Such sextics are of the form

$$\text{(W)} \qquad y^2 \;+\; a_1xy \;+\; a_3y \;=\; x^3 \;+\; a_2x^2 \;+\; a_4x \;+\; a_6,$$

where the $a_i \in k[t,s]$ are homogeneous of degree i and t,s,x,y are of degrees $1,1,2,3$, respectively.

Projecting $\mathbb{P}(1,1,2,3)$ onto t and s yields a rational map $\mathbb{P}(1,1,2,3) \dashrightarrow \mathbb{P}^1$, which, when restricted to the RDP del Pezzo $X = V(f_6)$ is given by the linear system $|-K_X|$ and has precisely one base point (at $s = t = 0, y^2 = x^3$). Blowing up the base point yields the Weierstraß model $Y \to \mathbb{P}^1$ of a rational (quasi-)elliptic surface $f : \widetilde{Y} \to \mathbb{P}^1$, where the zero section σ_0 on Y resp. $\widetilde{Y}$ is the exceptional (-1)-curve of this blow-up. Conversely, for a rational (quasi-)elliptic surface $f : \widetilde{Y} \to \mathbb{P}^1$ with chosen section σ_0, contracting all components of fibers not meeting σ_0 yields its Weierstraß model Y, and contracting also σ_0, we obtain an RDP del Pezzo surface X of degree 1. In turn, X is the anti-canonical model of a weak del Pezzo surface $\widetilde{X}$ of degree 1, which, when blown-up in the base point of its anti-canonical linear system $|-K_{\widetilde{X}}|$ gives back $\widetilde{Y}$. This connection is summarized in the following commutative diagram:

$$\text{(3.1)} \qquad \begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & Y \arrow[r] \arrow[d] & \mathbb{P}^1 \\ \widetilde{X} \arrow[r] & X \arrow[ur, dashed] & \end{tikzcd}$$

In particular, since the morphism $Y \to X$ is the blow-up of a smooth point, this diagram shows the following:

OBSERVATION 3.6. *A configuration of rational double points occurs on an RDP del Pezzo surface X of degree 1 if and only if it occurs on the Weierstraß model of a rational (quasi-)elliptic surface.*

Thus, Question 3.3 is equivalent to the following one:

QUESTION 3.7. Which rational double points occur on Weierstraß models of rational (quasi-)elliptic surfaces?

3.3. Configurations of (-2)-curves on weak del Pezzo surfaces: Reduction to non-taut RDPs. Let $\Gamma = \sum_i \Gamma_{i,n_i}$ be a configuration of (-2)-curves. Consider the following three conditions, where $p = \mathrm{char}(k)$ and $\ell \neq p$ is a prime and the respective torsion parts are denoted by $[\ldots]$.

(E8) There is an embedding $\iota : \Gamma \hookrightarrow E_8$.

(E8+T[ℓ]) (E8) and $(E_8/\iota(\Gamma))[\ell] \subseteq (\mathbb{Z}/\ell\mathbb{Z})^2$.

(E8+T[p])
(E8), and if $p > 0$, $(E_8/\iota(\Gamma))[p] \subseteq \mathbb{Z}/p\mathbb{Z}$ or $(E_8/\iota(\Gamma)) \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some $n \geq 0$.

The reason why we consider the above three conditions is the following lemma.

LEMMA 3.8. *Let $\Gamma = \sum_i \Gamma_{i,n_i}$ be a configuration of (-2)-curves. If Γ occurs on a weak del Pezzo surface of degree 1, then it satisfies Condition* (E8+T[p]) *and* (E8+T[ℓ]) *for all $\ell \neq p = \mathrm{char}(k)$.*

PROOF. Let $\widetilde{X}$ be a weak del Pezzo surface of degree 1 realizing Γ. Let $f : \widetilde{Y} \to \mathbb{P}^1$ be the associated rational (quasi-)elliptic surface as in Diagram (3.1) with Mordell–Weil group $\mathrm{MW}(f)$.

By [OS91, Theorem 3.1], there is an embedding $\iota : \Gamma \hookrightarrow E_8$ such that $\mathrm{rk}(\mathrm{MW}(f)) = 8 - \mathrm{rk}(\Gamma) = \mathrm{rk}(E_8/\iota(\Gamma))$ and $\mathrm{MW}(f)_{\mathrm{tors}} = (E_8/\iota(\Gamma))_{\mathrm{tors}}$. For all $n \geq 0$, we have $(E_8/\iota(\Gamma))[n] = (\mathrm{MW}(f))[n] \subseteq X_{\bar\eta}^{sm}[n]$, where $X_{\bar\eta}^{sm}$ is the smooth locus of the geometric generic fiber of f. If f is elliptic, then $X_{\bar\eta}^{sm}[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^2$ and $X_{\bar\eta}^{sm}[p] \subseteq \mathbb{Z}/p\mathbb{Z}$, so the Conditions (E8+T[p]), and (E8+T[ℓ]) are satisfied for all $\ell \neq p = \mathrm{char}(k)$. If f is quasi-elliptic, then $\mathrm{MW}(f)$ is a finitely generated subgroup of $X_{\bar\eta}^{sm} \cong \mathbb{G}_a$, so $\mathrm{MW}(f) \cong (\mathbb{Z}/p\mathbb{Z})^n$ for some $n \geq 0$. In particular, $E_8/\iota(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^n$, so again both conditions are satisfied. $\square$

The root sublattices of E_8 have been classified by Dynkin [Dyn52, §5., Table 11, p.385]. We can easily check which of them satisfy the conditions above.

LEMMA 3.9. *Let $\Gamma = \sum_i \Gamma_{i,n_i} \subseteq E_8$ be a configuration of (-2)-curves. Then, the following hold:*

(1) Γ satisfies (E8+T[q]) *for $q \neq 2$,*
(2) If $p \neq 2$, then (E8+T[ℓ = 2]) *is satisfied if and only if $\Gamma \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$.*
(3) If $p = 2$, then (E8+T[p=2]) *is satisfied if and only if $\Gamma \not\in \{D_4 + 3A_1, 2A_3 + 2A_1, A_3 + 4A_1, 7A_1, 6A_1\}$.*

PROOF. The groups $(E/\Gamma)_{\mathrm{tors}}$ have been calculated by Oguiso and Shioda in [OS91, Corollary 2.1] for all Γ except $D_4 + 4A_1, 8A_1$, and $7A_1$. Using their results, we leave it to the reader to check that for $\Gamma \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$ the Conditions (E8+T[ℓ]) are satisfied for all ℓ, and Γ does not satisfy Condition (E8+T[p]) if and only if $p = 2$ and $\Gamma \in \{D_4 + 3A_1, A_3 + 4A_1, 2A_3 + 2A_1, 6A_1\}$. We will now treat the three remaining cases.

- If $\Gamma = D_4 + 4A_1$, then $E_8/\Gamma = (\mathbb{Z}/2\mathbb{Z})^3$ by [Ito94, Table 1]. So, Γ satisfies (E8+T[p]), and it satisfies (E8+T[ℓ]) if and only if $\ell \neq 2$.

- If $\Gamma = 8A_1$, then $E_8/\Gamma = (\mathbb{Z}/2\mathbb{Z})^4$ by [Ito94, Table 1]. So, Γ satisfies (E8+T[p]), and it satisfies (E8+T[ℓ]) if and only if $\ell \neq 2$.
- Finally, if $\Gamma = 7A_1$, then there is a unique embedding of $7A_1$ into E_8 by [Dyn52, §5., Table 11, p.385]. So, this embedding coincides with $7A_1 \hookrightarrow 8A_1 \hookrightarrow E_8$ and thus $E_8/7A_1$ has rank 1 and $E_8/7A_1[2]$ contains $(\mathbb{Z}/2\mathbb{Z})^3$. Hence, Γ never satisfies (E8+T[2]), but it satisfies all (E8 + T[q]) with $q \neq 2$.

$\square$

PROPOSITION 3.10. *Let $\Gamma := \sum_i \Gamma_{i,n_i}$ be a configuration of (-2)-curves. Then, the following are equivalent:*

(1) Γ occurs on a weak del Pezzo surface of degree 1.
(2) Γ satisfies (E8+T[2]).

PROOF. The implication (1) to (2) follows immediately from Lemma 3.8.

For the converse, we have to show that every configuration of (-2)-curves that satisfies (E8+T[2]) occurs on a weak del Pezzo surface of degree 1, or, equivalently, as the configuration of (-2)-curves that do not meet a fixed chosen section σ_0 on a rational (quasi-)elliptic surface. We remind the reader that we summarized the relation between these curve configurations and the corresponding Kodaira–Néron types in Table 8.

For $p \neq 2, 3$, this is precisely the content of [SS19, Theorem 8.9] (see also [OS91, Remark 2.7]). For $p = 3$, it follows from the classification of singular fibers of elliptic surfaces in characteristic 3 [JLR+05], that every Γ that satisfies (E8+T[2]) occurs on an elliptic surface, except $\Gamma = 4A_2$. By [Ito92, Theorem 3.3], this Γ is realized on a quasi-elliptic surface. Similarly, if $p = 2$, one can use [Lan00] and [Ito94] to check that every Γ that satisfies (E8+T[2]) occurs on some (quasi-)elliptic surface. $\square$

Combining the results of this section, we can give a proof of Theorem 1.2 for configurations of taut rational double points. In particular, this proves Theorem 1.2 in characteristic different from $2, 3$, and 5:

PROOF OF THEOREM 1.2 FOR TAUT RDPS. Let $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ be an RDP configuration and assume that all the $\Gamma_{i,n_i}^{k_i}$ are taut, so Γ is uniquely determined by its associated configuration of (-2)-curves $\Gamma' := \sum_i \Gamma_{i,n_i}$. Further assume that $\Gamma \neq 7A_1$ if $p = 2$. Then, we have the following equivalences:

Γ occurs on an RDP del Pezzo surface

$\overset{\text{Prop. } 3.2}{\Longleftrightarrow}$ Γ occurs on an RDP del Pezzo surface of degree 1

$\overset{\text{all } \Gamma_{i,n_i}^{k_i} \text{ taut}}{\Longleftrightarrow}$ Γ' occurs on a weak del Pezzo surface of degree 1

$\overset{\text{Prop. } 3.10}{\Longleftrightarrow}$ Γ' satisfies (E8+T[2])

$\overset{\text{Lem. } 3.9}{\Longleftrightarrow}$ Γ' satisfies (E8) and

$$\Gamma' \notin \begin{cases} \{D_4 + 4A_1, 8A_1, 7A_1\} & \text{if } p \neq 2, \\ \{2A_3 + 2A_1, A_3 + 4A_1, 6A_1\} & \text{if } p = 2. \end{cases}$$

Note that for the last equivalence we did not have to consider D_4+3A_1 since D_4 is not taut if $p=2$.

Since we already know by Proposition 3.2 that $\Gamma = 7A_1$ occurs on an RDP del Pezzo surface of degree 2, this proves Theorem 1.2 in the case where all the $\Gamma_{i,n_i}^{k_i}$ are taut. $\square$

Thus, we have reduced Question 3.3 to the following:

QUESTION 3.11. Which non-taut RDPs occur on Weierstraß models of rational (quasi-) elliptic surfaces?

The remainder of this chapter will be devoted to finding an answer to this question.

4. Classification of non-taut RDP del Pezzo surfaces of degree 1

In this section, we will explain how to derive simple equations for non-taut RDP del Pezzo surfaces of degree 1 from the classification of singular fibers of rational (quasi-) elliptic surfaces given in [Mir90], [JLR+05], [Lan00], [Ito92], and [Ito94]. In the (quasi-) elliptic case, the equations found by Ito are already simplified, which is why we will focus on the elliptic case in Subsections 4.1, 4.2 and 4.3. Using these simplified equations, it is straightforward to determine the Artin coindices of the rational double points that occur by applying Proposition 2.2. We will exemplify this final step in Example 4.3 and Example 4.5 in characteristic 2 and 3, respectively.

4.1. Tate's algorithm for determining the type of a singular fiber in an elliptic pencil. Given a Weierstraß equation for the Weierstraß model of a rational elliptic surface, it is fairly standard to determine the Kodaira–Néron types (see Table 8) of its singular fibers by carrying out Tate's algorithm [Tat75], [Del75]. For the sake of self-containedness, we quickly recall Tate's algorithm: Let

$$\text{(W)} \qquad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

be a Weierstraß equation of an elliptic curve over the function field $k(C)$ of a smooth curve C. Choose a closed point $c \in C$ and write $\widehat{\mathcal{O}}_{C,c} \cong k[[t]]$, assume $a_i \in k[[t]]$ and write $\nu(\ ) = \mathrm{ord}_t$ for the valuation on $k[[t]]$. Then, Tate defines the following quantities:

$$b_2 := a_1^2 + 4a_2, \qquad b_4 := a_1a_3 + 2a_4, \qquad b_6 := a_3^2 + 4a_6,$$

$$b_8 := a_1^2a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2,$$

$$c_4 := b_2^2 - 24b_4, \qquad c_6 := -b_2^3 + 36b_2b_4 - 216b_6,$$

$$\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \neq 0, \qquad j = \frac{c_4^3}{\Delta}.$$

Then, excluding the subalgorithm for determining $v > 0$ in Step 7 if $p = 2$ (see [Tat75, p.50-51] for details in this case), Tate's algorithm is as follows, where F denotes the fiber over $t = 0$.

ALGORITHM.

Step 1: If $t\nmid\Delta$, then F is of type I_0. Else ...
Step 2: Change coordinates such that $t|a_3, t|a_4$ and $t|a_6$.
If $t\nmid b_2$, then F is of type I_v for $v=\nu(\Delta)$. Else ...
Step 3: If $t^2\nmid a_6$, then F is of type II. Else ...
Step 4: If $t^3\nmid b_8$, then F is of type III. Else ...
Step 5: If $t^3\nmid b_6$, then F is of type IV. Else ...
Step 6: Change coordinates such that $t|a_1, t|a_2, t^2|a_3, t^2|a_4$ and $t^3|a_6$.
Consider the polynomial $P(T)=T^3+a_2\frac{T^2}{t}+a_4\frac{T}{t^2}+a_6\frac{1}{t^3}$.
If P has three distinct roots, then F is of type I_0^*. Else ...
Step 7: If P has one single or one double root, then F is of type I_v^* for some $v>0$ (if $p\neq 2$, then $v=\nu(\Delta)-6$). Else ...
Step 8: Change variables such that the triple root is 0, and $t^2|a_2, t^3|a_4$ and $t^4|a_6$.
Consider the polynomial $Q(Y)=Y^2+\frac{a_3}{t^2}Y-a_6\frac{1}{t^4}$.
If Q has distinct roots, then F is of type IV^*. Else ...
Step 9: Change variables such that the double root is 0 and $t^3|a_3$ and $t^5|a_6$.
If $t^4\nmid a_4$, then F is of type III^*. Else ...
Step 10: If $t^6\nmid a_6$, then F is of type II^*. Else ...
Step 11: Divide each a_i by t^i and repeat from Step 1.

4.2. Simplified Weierstraß equations. Depending on the characteristic, the Weierstraß equation (W) (if $p=2$, under the additional assumption that it contains a non-taut rational double point) for an RDP del Pezzo surface X of degree 1 with associated rational elliptic surface $f:\widetilde{Y}\to\mathbb{P}^1$ can be simplified to an equation of the following form:

(W0)		$y^2 = x^3 + a_4x + a_6$	if $p\neq 2,3$,
(W3)		$y^2 = x^3 + a_2x^2 + a_4x + a_6$	if $p=3$,
(W2)	or	$y^2 + a_1xy = x^3 + a_2x^2 + a_4x + a_6$	if $p=2$.
(W2')		$y^2 + a_3y = x^3 + a_2x^2 + a_4x + a_6$	

This is well-known if $p\neq 2$. If $p=2$, to see that we can simplify (W) to an equation of the form (W2) or (W2'), we first observe that by Table 8 the non-taut rational double points (see Table 7) correspond to certain additive fibers of f. Hence, Tate's algorithm shows that, if X contains a non-taut rational double point, then in (W) we may assume $t\mid a_1$ and $t\mid a_3$. Thus, if $a_1=0$, we get (W2') and if $a_1\neq 0$, we can assume that $\frac{a_3}{a_1}$ is a polynomial and $x\mapsto x+\frac{a_3}{a_1}$ transforms (W) to an equation of the form (W2).

4.3. Automorphisms of $\mathbb{P}(1,1,2,3)$ preserving simplified Weierstraß equations. In order to find simple equations for degree 1 RDP del Pezzo surfaces with non-taut rational double points, let us have a look at which automorphisms of $\mathbb{P}(1,1,2,3)$ send an equation of the form (W0), (W3), (W2) or (W2') to an equation of the same form.

First, observe that substitutions in t and s only, always preserve these types of equations. Thus, let us focus on those automorphisms of $\mathbb{P}(1,1,2,3)$ fixing t and s, that is, those inducing the trivial automorphism on $\mathbb{P}^1$.

4.3.1. *Automorphisms of* $\mathbb{P}(1,1,2,3)$ *over* $\mathbb{P}^1$ *if* $p \neq 2, 3$. A general substitution fixing t and s and sending a Weierstraß equation of the form (W0) to one of the same form is given by

$$x \mapsto \lambda^2 x, \quad y \mapsto \lambda^3 y \qquad \text{with } \lambda \in k^*.$$

This sends (W0) to $y^2 = x^3 + \frac{1}{\lambda^4} a_4 x + \frac{1}{\lambda^6} a_6$.

4.3.2. *Automorphisms of* $\mathbb{P}(1,1,2,3)$ *over* $\mathbb{P}^1$ *if* $p = 3$. A general substitution fixing t and s and sending a Weierstraß equation of the form (W3) to one of the same form is given by

$$\begin{array}{rcl} x & \mapsto & \lambda^2 x + f \\ y & \mapsto & \lambda^3 y \end{array}$$

with $\lambda \in k^*$ and $f \in k[t, s]$ homogeneous of degree 2. This sends (W3) to

$$y^2 = x^3 + \frac{1}{\lambda^2} a_2 x^2 + \frac{1}{\lambda^4}(a_4 + 2a_2 f)x + \frac{1}{\lambda^6}(a_6 + a_4 f + a_2 f^2 + f^3).$$

4.3.3. *Automorphisms of* $\mathbb{P}(1,1,2,3)$ *over* $\mathbb{P}^1$ *if* $p = 2$.

(W2) A general substitution fixing t and s and sending a Weierstraß equation of the form (W2) to one of the same form is given by

$$\begin{array}{rcl} x & \mapsto & \lambda^2 x \\ y & \mapsto & \lambda^3 y + fx + g \end{array}$$

with $\lambda \in k^*$ and $f, g \in k[t, s]$ homogeneous of degree 1 and 3, respectively. This sends (W2) to

$$y^2 + \frac{1}{\lambda} a_1 xy = x^3 + \frac{1}{\lambda^6}(\lambda^4 a_2 + \lambda^2 a_1 f + f^2)x^2 + \frac{1}{\lambda^4}(a_4 + a_1 g)x + \frac{1}{\lambda^6}(a_6 + g^2).$$

(W2') A general substitution fixing t and s and sending a Weierstraß equation of the form (W2') to one of the same form is given by

$$\begin{array}{rcl} x & \mapsto & \lambda^2 x + f \\ y & \mapsto & \lambda^3 y + gx + h \end{array}$$

with $\lambda \in k^*$ and $f, g, h \in k[t, s]$ homogeneous of degree 2, 1 and 3, respectively. This sends (W2') to

$$\begin{array}{rcl} y^2 + \dfrac{1}{\lambda^3} a_3 y & = & x^3 + \dfrac{1}{\lambda^6}(\lambda^4 a_2 + g^2 + \lambda^4 f)x^2 + \dfrac{1}{\lambda^6}(\lambda^2 a_4 + a_3 g + \lambda^2 f^2)x \\ & & + \dfrac{1}{\lambda^6}(a_6 + a_4 f + a_3 h + a_2 f^2 + f^3 + h^2). \end{array}$$

4.4. Proof of Theorem 1.2 in Characteristic 5. Assume $p = 5$. Let $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ be an RDP configuration containing a non-taut RDP and let $\Gamma' = \sum_i \Gamma_{i,n_i}$ be the associated configuration of (-2)-curves. By Table 7, we have $\Gamma_{i,n_i} = E_8$ for some i and by Proposition 3.2 and Lemma 3.8, Γ can only occur on an RDP del Pezzo surface if Γ' embeds into E_8. Thus, to prove Theorem 1.2 in characteristic 5, it suffices to consider $\Gamma \in \{E_8^0, E_8^1\}$. Note that Γ' embeds into E_8 and $\Gamma' = E_8 \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$. On the other hand, the following proposition shows that both of these rational double points occur, so Theorem 1.2 holds in characteristic 5.

PROPOSITION 4.1. *Each of the rational double points E_8^0 and E_8^1 occurs on an RDP del Pezzo surface X of degree 1. Moreover, every RDP del Pezzo surface of degree 1 containing a non-taut rational double point is given by an equation as in Table 2.*

PROOF. By Table 8, the rational elliptic surface associated to an RDP del Pezzo surface of degree 1 with a singularity of type E_8 admits a fiber of type II^*. By [Lan94, Theorem 4.1.] and [MP86, Theorem 4.1., Tables 5.1 and 5.2] there are precisely two such elliptic surfaces and their Weierstraß equations in $\mathbb{P}(1,1,2,3)$ are

$$y^2 = x^3 + t^5 s \tag{4.1}$$

and

$$y^2 = x^3 + t^4 x + t^5 s. \tag{4.2}$$

Considering the affine chart $s = 1$ and comparing with Table 7, we see that Equation (4.1) defines a singularity of type E_8^0 and Equation (4.2) defines a singularity of type E_8^1. $\square$

4.5. Proof of Theorem 1.2 in Characteristic 3. Assume $p = 3$. Let $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ be an RDP configuration containing a non-taut RDP and let $\Gamma' = \sum_i \Gamma_{i,n_i}$ be the associated configuration of (-2)-curves. By Table 7, we have $\Gamma_{i,n_i} \in \{E_6, E_7, E_8\}$ for some i and by Proposition 3.2 and Lemma 3.8, Γ can only occur on an RDP del Pezzo surface if Γ' embeds into E_8. Thus, by Dynkin's classification [Dyn52, Table 11], to prove Theorem 1.2 in characteristic 3, it suffices to consider

$$\Gamma \in \{E_8^0, E_8^1, E_8^2, E_7^0{+}A_1, E_7^0, E_7^1{+}A_1, E_7^1, E_6^0{+}A_2, E_6^0{+}A_1, E_6^0, E_6^1{+}A_2, E_6^1{+}A_1, E_6^1\}. \tag{4.3}$$

Note that for all of the Γ above, we have $\Gamma' \not\in \{D_4 + 4A_1, 8A_1, 7A_1\}$. On the other hand, the following proposition shows that all these possiblities occur on some RDP del Pezzo surface, so Theorem 1.2 holds in characteristic 3.

PROPOSITION 4.2. *Each RDP configuration Γ in the List (4.3) occurs on an RDP del Pezzo surface of degree 1. Moreover, every RDP del Pezzo surface of degree 1 containing a non-taut rational double point admits an equation as in Table 3.*

PROOF. By Table 8, we have to study those RDP del Pezzo surfaces X whose associated rational (quasi-) elliptic surface $\widetilde{Y}$ has a singular fiber of type IV^*, III^*, or II^*. In the elliptic case and in the notation of [JLR[+]05], these correspond to the Types $6A$, $6B$, $6C$, 7, $8A$, and $8B$. In the quasi-elliptic case and in the notation of [Ito92, Theorem 3.3], these correspond to Cases (1) and (2).

All the Weierstraß equations for $X \subseteq \mathbb{P}(1,1,2,3)$ given in [JLR⁺05] and [Ito92] are already of the form (W3). To simplify them and determine the rational double points that occur, we will proceed along the following steps:

(1.) Carry out a substitution in t and s only.
(2.) Apply an automorphism of $\mathbb{P}(1,1,2,3)$ over $\mathbb{P}^1$ preserving the form (W3) as in Subsection 4.3.2.
(3.) Check for additional rational double points (e.g. using Tate's algorithm (see Subsection 4.1) to determine the other reducible fibers of the underlying rational (quasi-)elliptic surface).
[(4.) Determine the Artin coindices as described in Section 2, e.g. via Proposition 2.2. This will be left to the reader, but we will show how it works in Example 4.3.]

Lang's Type 6A (IV*). X is given by $y^2 = x^3 + c_0 t^2 x^2 + c_1 t^3 x + c_2 t^4$ with $t \nmid c_1, t \nmid c_2$ and $c_i \in k[t,s]$ homogeneous of degree i. From now on, let us distinguish the cases $c_0 = 0$ and $c_0 \neq 0$.

- $c_0 = 0$:
  (1.) Since $t \nmid c_1$, we can apply an automorphism of $\mathbb{P}^1$ to assume $c_1 = s$. Then, scaling $s \mapsto \lambda^3 s, t \mapsto \lambda^{-1} t$ for an appropriate λ, we can write $c_2 = s^2 + c_{2,1} ts + c_{2,2} t^2$.
  (2.) $x \mapsto x - \sqrt[3]{c_{2,2}} t^2, y \mapsto y$ yields the equation $y^2 = x^3 + t^3 sx + a_{6,5} t^5 s + t^4 s^2$.
  (3.) We have $\Delta = -t^9 s^3$ and by Tate's algorithm the fiber at $s = 0$ is reducible if and only if $a_{6,5} = 0$ in which case it has two components; so the RDP configuration on X is $E_6 + A_1$ in this case.
- $c_0 \neq 0$:
  (1.) Rescaling t and s, we can assume $c_0 = 1$. Then, we have $\Delta = -t^9(c_2 t - c_1^2 t + c_1^3)$. Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of $-t^9 s^3$ in Δ is 1.
  (2.) $x \mapsto x + c_1 t, y \mapsto y$ yields the equation $y^2 = x^3 + t^2 x^2 + a_{6,5} t^5 s + a_{6,4} t^4 s^2 + t^3 s^3$.
  (3.) We have $\Delta = -t^9 s(a_{6,5} t^2 + a_{6,4} ts + s^2)$ and we see by Tate's algorithm that the RPD configuration on X is $E_6 + A_2$ if $a_{6,5} = a_{6,4} = 0$, that it is $E_6 + A_1$ if $a_{6,4} \neq 0$ and ($a_{6,5} = 0$ or $a_{6,5} = a_{6,4}^2$), and E_6 otherwise.

Lang's Type 6B (IV*). X is given by $y^2 = x^3 + c_0 t^2 x^2 + d_0 t^4 x + c_2 t^4$ with $t \nmid c_0, t \nmid c_2$ and $c_i, d_i \in k[t,s]$ homogeneous of degree i.

(1.) Rescaling t and s, we can assume $c_0 = 1$. Then, we have $\Delta = -t^{10}(c_2 - d_0^2 t^2 + d_0^3 t^2)$. Since $t \nmid c_2$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of $-t^{10} s^2$ in Δ is 1.
(2.) $x \mapsto x + d_0 t^2, y \mapsto y$ yields the equation $y^2 = x^3 + t^2 x^2 + a_{6,5} t^5 s + t^4 s^2$.
(3.) We have $\Delta = -t^{10} s(a_{6,5} t + s)$ and we see by Tate's algorithm that the RDP configuration on X is $E_6 + A_1$ if $a_{6,5} = 0$, and E_6 otherwise.

Lang's Type 6C (IV^*). X is given by $y^2 = x^3 + d_0t^4x + c_2t^4$ with $t \nmid d_0, t \nmid c_2$ and $c_i, d_i \in k[t,s]$ homogeneous of degree i.

(1.) Using an automorphism of $\mathbb{P}^1$ we can assume that $d_0 = 1$ and $c_2 = s^2 + c_{2,2}t^2$.
(2.) $x \mapsto x+\lambda t^2, y \mapsto y$ with $\lambda^3+\lambda+c_{2,2} = 0$ yields the equation $y^2 = x^3+t^4x+t^4$.
(3.) Since $\Delta = -t^{12}$, X has no other singularities apart from E_6.

Lang's Type 7 (III^*). X is given by $y^2 = x^3 + c_0t^2x^2 + c_1t^3x + d_1t^5$ with $t \nmid c_1$ and $c_i, d_i \in k[t,s]$ homogeneous of degree i. From now on, let us distinguish the cases $c_0 = 0$ and $c_0 \neq 0$.

- $c_0 = 0$:
  (1.) Since $t \nmid c_1$, we can apply an automorphism of $\mathbb{P}^1$ to assume $c_1 = s$. Write $d_1 = d_{1,0}s + d_{1,1}t$. Then, scaling $s \mapsto \lambda^3 s, t \mapsto \lambda^{-1}t$ for an appropriate λ, we can assume $d_{1,0}^3 - d_{1,1} \in \{0,1\}$.
  (2.) $x \mapsto x - \sqrt[3]{d_{1,1}}t^2, y \mapsto y$ yields the equation $y^2 = x^3 + t^3sx + (d_{1,0} - \sqrt[3]{d_{1,1}})t^5s$.
  (3.) We have $\Delta = -t^9s^3$ and by Tate's algorithm the fiber at $s = 0$ is reducible if and only if $d_{1,0}^3 - d_{1,1} = 0$ in which case it has two components; so the RDP configuration on X is $E_7 + A_1$ in this case, and E_7 if $d_{1,0}^3 - d_{1,1} = 1$.
- $c_0 \neq 0$:
  (1.) Rescaling t and s, we can assume $c_0 = 1$. Then, we have $\Delta = -t^9(d_1t^2 - c_1^2t + c_1^3)$. Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of $-t^9s^3$ in Δ is 1.
  (2.) $x \mapsto x+c_1t, y \mapsto y$ yields the equation $y^2 = x^3+t^2x^2+a_{6,5}t^5s-t^4s^2+t^3s^3$.
  (3.) We have $\Delta = -t^9s(a_{6,5}t^2 - ts + s^2)$ and we see by Tate's algorithm that the RPD configuration on X is $E_7 + A_1$ if $a_{6,5} \in \{0,1\}$, and only E_7 otherwise.

Lang's Types 8A and 8B (II^*). These equations have been simplified by Lang in [Lan94] and they are as described in the second column of Table 3.

Quasi-elliptic surfaces: Ito's Types 3.3(1) (II^*) *and 3.3(2)* (IV^*). These equations have been simplified by Ito in [Ito92, Theorem 3.3] and they are as described in the second column of Table 3. □

EXAMPLE 4.3. Consider Lang's Type 7 with $c_0 \neq 0$ simplified as in the proof of Proposition 4.2 and localized at the E_7-singularity at $t = x = y = 0$:

$$f = y^2 - (x^3 + t^2x^2 + a_{6,5}t^5 - t^4 + t^3) = 0.$$

Applying Proposition 2.2, we have

$$\begin{aligned} T_f &= k[t,x,y]_{(t,x,y)}/(f, tx^2 + a_{6,5}t^4 - t^3, xt^2, -y) \\ &\cong k[t,x]_{(t,x)}/(x^3 + a_{6,5}t^5 - t^4 + t^3, tx^2 + a_{6,5}t^4 - t^3, xt^2) \\ &\cong k[t,x]_{(t,x)}/(x^3 + t^3, tx^2 - t^3, xt^2) =: R, \end{aligned}$$

where for the second isomorphism we have used that $0 = t^2x^2 + a_{6,5}t^5 + t^4 = t^4(1 + a_{6,5}t)$, hence $t^4 = 0$, as $1 + a_{6,5}t$ is a unit. Now, it is easy to check that R is generated as a k-vector space by $1, x, x^2, x^3, t, tx, tx^2, tx^3, t^2$, hence $\dim T_f = 7$. Therefore, by Table 7, the Artin coindex of this E_7-singularity is 1.

4.6. Proof of Theorem 1.2 in Characteristic 2. Assume $p = 2$. Let $\Gamma = \sum_i \Gamma_{i,n_i}^{k_i}$ be an RDP configuration containing a non-taut RDP and let $\Gamma' = \sum_i \Gamma_{i,n_i}$ be the associated configuration of (-2)-curves. By Table 7, we have $\Gamma_{i,n_i} \in \{D_n, E_6, E_7, E_8\}$ for some i.

First, observe that if $\Gamma \in \{E_7^0, D_6^0 + A_1, D_4^0 + 3A_1\}$, then Γ occurs on a weak del Pezzo surface of degree 2 by Proposition 3.2. Moreover, its associated Γ' embeds into E_8 and $\Gamma' \notin \{2A_3 + 2A_1, A_3 + 4A_1, 6A_1\}$, so Theorem 1.2 holds for these three exceptional cases.

Next, if $\Gamma \notin \{E_7^0, D_6^0 + A_1, D_4^0 + 3A_1\}$, then, by Proposition 3.2, Γ occurs on an RDP del Pezzo surface if and only if it occurs on an RDP del Pezzo surface of degree 1. Hence, by Lemma 3.8, Theorem 1.2 holds for all Γ such that Γ' does not embed into E_8. Thus, we may assume that Γ' embeds into E_8 and we note that $\Gamma' \notin \{2A_3 + 2A_1, A_3 + 4A_1, 6A_1\}$, since Γ contains a non-taut summand. On the other hand, it will follow from Proposition 4.4 that every such Γ' occurs on some weak del Pezzo surface of degree 1. Thus, the following Proposition 4.4 finishes the proof of Theorem 1.2.

PROPOSITION 4.4. *An RDP configuration Γ containing a non-taut rational double point occurs on an RDP del Pezzo surface if and only if it occurs in Table 4, 5, or 6. Moreover, every RDP del Pezzo surface of degree 1 containing a non-taut rational double point is given by an equation in one of these tables.*

PROOF. This time, not all of the Weierstraß equations for $X \subseteq \mathbb{P}(1,1,2,3)$ in the classification of rational (quasi-)elliptic surfaces in [Lan00], [Lan94] and [Ito94] are of the form (W2) or (W2'). Thus, to simplify them, we have to add a 0th Step, before we can go on with our procedure as follows:

(0.) Transform the Weierstraß equation into the form (W2) or (W2').
(1.) Carry out a substitution in t and s only.
(2.) Apply an automorphism of $\mathbb{P}(1,1,2,3)$ over $\mathbb{P}^1$ preserving the form (W2), or (W2'), respectively, as in Subsection 4.3.3.
(3.) Check for additional rational double points (e.g. using Tate's algorithm (see Subsection 4.1) to determine the other reducible fibers of the underlying rational (quasi-)elliptic surface).
[(4.) Determine the Artin coindices as described in Section 2, e.g. via Proposition 2.2. This will be left to the reader, but we will show how it works in Example 4.5.]

Lang's Type 4A (I_0^*). X is given by $y^2 + txy + c_1t^2y = x^3 + d_1tx^2 + e_1t^3x + c_3t^3$ with $t \nmid c_1, t \nmid c_3$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_1t$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_2t^2x + c_3t^3$ with $t \nmid e_2$ and $t \nmid (e_2d_1 + c_3)$.
(1.) For this new equation, we have $\Delta = t^8(c_3t + e_2^2)$. Since $t \nmid e_2$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of t^8s^4 in Δ is 1.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_2t$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation

$$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + a_{6,3}t^3s^3 + t^2s^4 \text{ with } a_{2,1} + a_{6,3} \neq 0.$$

(3.) We have $\Delta = t^8s(a_{6,5}t^3 + a_{6,4}t^2s + a_{6,3}ts^2 + s^3)$. By Tate's algorithm we see that the RDP configuration on X is $D_4 + A_3$ if $a_{6,5} = a_{6,4} = a_{6,3} = 0$, that it is $D_4 + A_2$ if ($a_{6,5} = a_{6,4} = 0$ and $a_{6,3} \neq 0$) or ($a_{6,3}^2 = a_{6,4}$ and $a_{6,3}^3 = a_{6,5} \neq 0$), that it is $D_4 + 2A_1$ if $a_{6,5} = a_{6,3} = 0$ and $a_{6,4} \neq 0$, that it is $D_4 + A_1$ if ($a_{6,5} = 0$ and $a_{6,3} \neq 0$) or $a_{6,5} = a_{6,4}a_{6,3} \neq 0$, and that it is only D_4 otherwise.

Lang's Type 4B (I_0^*). X is given by $y^2 + txy + c_0t^3y = x^3 + d_1tx^2 + e_1t^3x + c_3t^3$ with $t \nmid c_3$ and $c_i, d_i, e_i \in k[t, s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_0t^2$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_1t^3x + c_3t^3$ with $t \nmid c_3$.

(1.) For this new equation, we have $\Delta = t^9(c_3 + e_1^2t)$. Since $t \nmid c_3$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of t^9s^3 in Δ is 1.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_1t^2$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation $y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3$.

(3.) We have $\Delta = t^9s(a_{6,5}t^2 + a_{6,4}ts + s^2)$. By Tate's algorithm we see that the RDP configuration on X is $D_4 + A_2$ if $a_{6,5} = a_{6,4} = 0$, that it is $D_4 + A_1$ if ($a_{6,5} = 0$ and $a_{6,4} \neq 0$) or ($a_{6,5} \neq 0$ and $a_{6,4} = 0$), and that it is only D_4 otherwise.

Lang's Type 5A (I_1^*). X is given by $y^2 + txy + c_1t^2y = x^3 + d_1tx^2 + e_1t^3x + c_2t^4$ with $t \nmid c_1, t \nmid d_1$ and $c_i, d_i, e_i \in k[t, s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_1t$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_2t^2x + c_3t^3$ with $t \nmid e_2$ and $t \mid (e_2d_1 + c_3)$.

(1.) For this new equation, we have $\Delta = t^8(c_3t + e_2^2)$. Since $t \nmid e_2$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of t^8s^4 in Δ is 1.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_2t$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation

$$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + a_{6,3}t^3s^3 + t^2s^4 \text{ with } a_{2,1} + a_{6,3} = 0.$$

(3.) We have $\Delta = t^8s(a_{6,5}t^3 + a_{6,4}t^2s + a_{6,3}ts^2 + s^3)$. By Tate's algorithm we see that the RDP configuration on X is $D_5 + A_3$ if $a_{6,5} = a_{6,4} = a_{6,3} = 0$, that it is $D_5 + A_2$ if ($a_{6,5} = a_{6,4} = 0$ and $a_{6,3} \neq 0$) or ($a_{6,3}^2 = a_{6,4}$ and $a_{6,3}^3 = a_{6,5} \neq 0$), that it is $D_5 + 2A_1$ if $a_{6,5} = a_{6,3} = 0$ and $a_{6,4} \neq 0$, that it is $D_5 + A_1$ if ($a_{6,5} = 0$ and $a_{6,3} \neq 0$) or $a_{6,5} = a_{6,4}a_{6,3} \neq 0$, and that it is only D_5 otherwise.

Lang's Type 5B (I_2^*). First, we note that there seems to be a typo in [Lan00, Case 5B., p.5825] in the sense that the fiber type should be (I_2^*) instead of (I_1^*). X is given by $y^2+txy+c_0t^3y = x^3 + d_1tx^2 + e_1t^3x + c_1t^5$ with $t \nmid d_1, t \nmid e_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_0t^2$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_1t^3x + c_1t^5$ with $t \nmid d_1$ and $t \nmid e_1$.

(1.) For this new equation, we have $\Delta = t^{10}(e_1^2 + c_1t)$. Since $t \nmid e_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of $t^{10}s^2$ in Δ is 1.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$, where $d_{1,0} \neq 0$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_1t^2$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation $y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + t^4s^2$ with $a_{2,1} \neq 0$.

(3.) We have $\Delta = t^{10}s(a_{6,5}t + s)$. By Tate's algorithm we see that the RDP configuration on X is $D_6 + A_1$ if $a_{6,5} = 0$, and D_6 otherwise.

Lang's Type 5C (I_3^*). X is given by $y^2 + txy + c_0t^3y = x^3 + d_1tx^2 + e_0t^4x + d_0t^6$ with $t \nmid c_0, t \nmid d_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_0t^2$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_0t^4x + c_1t^5$ with $t \nmid d_1$ and $t \nmid c_1$.

(1.) For this new equation, we have $\Delta = t^{11}(c_1 + e_0^2t)$. Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$, i.e., $(c_1 + e_0^2t) = s$.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$, where $d_{1,0} \neq 0$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_0t^3$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation $y^2 + txy = x^3 + a_{2,1}tsx^2 + t^5s$ with $a_{2,1} \neq 0$.

(3.) We have $\Delta = t^{11}s$ and the RDP configuration on X is D_7.

Lang's Type 5D (I_4^*). This equation has been simplified by Lang in [Lan94] and it is as described in the second column of Table 5.

Lang's Type 6 (IV^*). X is given by $y^2 + txy + c_1t^2y = x^3 + d_0t^2x^2 + e_1t^3x + c_2t^4$ with $t \nmid c_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_1t$ to obtain the new equation $y^2 + txy = x^3 + d_1tx^2 + e_2t^2x + c_3t^3$ with $t \nmid e_2, t \mid (d_1^2 + e_2)$ and $t \mid (d_1e_2 + c_3)$.

(1.) For this new equation, we have $\Delta = t^8(c_3t + e_2^2)$. Since $t \nmid e_2$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of t^8s^4 in Δ is 1.

(2.) We write $d_1 = d_{1,0}s + d_{1,1}t$, where $d_{1,0} \neq 0$. Then, $x \mapsto x, y \mapsto y + \lambda tx + e_2t$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_{1,1}$. This yields the equation

$$y^2 + txy = x^3 + tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3 + t^2s^4.$$

(3.) We have $\Delta = t^8 s(a_{6,5}t^3 + a_{6,4}t^2 s + ts^2 + s^3)$. By Tate's algorithm we see that the RDP configuration on X is $E_6 + A_2$ if $a_{6,5} = a_{6,4} \in \{0, 1\}$, that it is $E_6 + A_1$ if ($a_{6,5} = 0$ and $a_{6,4} \neq 0$) or $a_{6,5} = a_{6,4} \notin \{0, 1\}$, and that it is only E_6 otherwise.

Lang's Type 7 (III*). X is given by $y^2 + txy + c_0 t^3 y = x^3 + d_0 t^2 x^2 + e_1 t^3 x + f_1 t^5$ with $t \nmid e_1$ and $c_i, d_i, e_i, f_i \in k[t, s]$ homogeneous of degree i.

(0.) We want to transform the Weierstraß equation into one of the form (W2). For this, send $x \mapsto c_0 t^2$ to obtain the new equation $y^2 + txy = x^3 + d_0 t^2 x^2 + e_1 t^3 x + f_1 t^5$ with $t \nmid e_1$.

(1.) For this new equation, we have $\Delta = t^{10}(e_1^2 + f_1 t)$. Since $t \nmid e_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $s \mid \Delta$ and the coefficient of $t^{10}s^2$ in Δ is 1.

(2.) Then, $x \mapsto x, y \mapsto y + \lambda tx + e_1 t^2$, where λ is chosen in such a way that $\lambda^2 + \lambda = d_0$. This yields the equation $y^2 + txy = x^3 + a_{6,5}t^5 s + t^4 s^2$.

(3.) We have $\Delta = t^{10}s(a_{6,5}t + s)$. By Tate's algorithm we see that the RDP configuration on X is $E_7 + A_1$ if $a_{6,5} = 0$, and E_7 otherwise.

Lang's Type 8 (II*). This equation has been simplified by Lang in [Lan94] and it is as described in the second column of Table 5.

Lang's Type 12A (I_0^*). X is given by $y^2 + c_1 t^2 y = x^3 + d_1 tx^2 + e_1 t^3 x + c_3 t^3$ with $t \nmid c_1, t \nmid c_3$ and $c_i, d_i, e_i \in k[t, s]$ homogeneous of degree i.

(0.) The Weierstraß equation is already of the form (W2').

(1.) We have $\Delta = t^8 c_1^4$. Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $c_1 = s$. We write $c_3 = c_{3,0}s^3 + c_{3,1}ts^2 + c_{3,2}t^2 s + c_{3,3}t^3$. Then, since $t \nmid c_3$, scaling $s \to \lambda^2 s, t \mapsto \lambda^{-1}t$ for an appropriate λ yields $c_{3,0} = 1$.

(2.) Let us write $d_1 = d_{1,0}s + d_{1,1}t$ and $e_1 = e_{1,0}s + e_{1,1}t$. Then, we carry out the substitution

$$x \mapsto x + \sqrt{e_{1,1}}t^2, \quad y \mapsto y + \sqrt{d_{1,1} + \sqrt{e_{1,1}}}tx + \lambda t^2 s + (c_{3,2} + e_{1,0}\sqrt{e_{1,1}} + d_{1,0}e_{1,1})t^3,$$

where $\lambda^2 + \lambda = c_{3,1}$. This yields $y^2 + t^2 sy = x^3 + a_{2,1}tsx^2 + a_{4,3}t^3 sx + a_{6,6}t^6 + t^3 s^3$.

(3.) We have $\Delta = t^8 s^4$. By Tate's algorithm we see that the RDP configuration on X is $D_4 + A_2$ if $a_{6,6} = a_{4,3} = 0$, that it is $D_4 + A_1$ if $a_{6,6} = 0$ and $a_{4,3} \neq 0$, and that it is D_4 otherwise.

Lang's Type 12B (I_0^*). X is given by $y^2 + c_0 t^3 y = x^3 + d_1 tx^2 + c_1 t^3 x + c_3 t^3$ with $t \nmid c_0, t \nmid c_3$ and $c_i, d_i \in k[t, s]$ homogeneous of degree i. For the simplification of this equation, we will not follow the procedure described in the beginning of the proof, but perform the substitutions in a different order.

(a.) First, applying $x \mapsto x + d_1 t$ and then, scaling $x \mapsto \lambda^3 x, y \mapsto \lambda^2 y$ for an appropriate λ yields the new equation $y^2 + t^3 y = x^3 + c_2 t^2 x + c_3 t^3$ with $t \nmid c_3$.

(b.) Since $t \nmid c_3$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ to obtain a c_3 of the form $c_3 = s^3 + a^2 s^2 t + ast^2 + c_{3,3}t^3$ for some $a \in k$.

(c.) Finally, $y \mapsto y + ast^2 + \mu t^3$ with $\mu^2 + \mu = c_{3,3}$ yields

$$y^2 + t^3 y = x^3 + (a_{4,2}s^2 + a_{4,3}ts + a_{4,4}t^2)t^2 x + s^3 t^3.$$

(3.) Since $\Delta = t^{12}$, there are no other reducible fibers and the RDP configuration on X is D_4.

Lang's Type 13A (I_1^*). X is given by $y^2 + c_1 t^2 y = x^3 + d_1 t x^2 + e_1 t^3 x + d_2 t^4$ with $t \nmid c_1, t \nmid d_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(1.) We have $\Delta = t^8 c_1^4$. Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $c_1 = s$. Then, since $t \nmid d_1$, we can scale s and t such that $d_1 = s + d_{1,1}t$.

(2.) Let us write $e_1 = e_{1,0}s + e_{1,1}t$ and $d_2 = d_{2,0}s^2 + d_{2,1}ts + d_{2,2}t^2$. The substitution $x \mapsto x + \sqrt{e_{1,1}}t^2, y \mapsto y + e_{1,0}tx + \lambda t^2 s + \sqrt{d_{1,1}e_{1,1} + d_{2,2}}t^3$ with $\lambda^2 + \lambda = d_{2,0}$ yields the new equation $y^2 + t^2 sy = x^3 + (a_{2,2}t^2 + ts)x^2 + a_{6,5}t^5 s$.

(3.) We have $\Delta = t^8 s^4$. By Tate's algorithm we see that the RDP configuration on X is $D_5 + A_2$ if $a_{6,5} = a_{2,2} = 0$, that it is $D_5 + A_1$ if $a_{6,5} = 0$ and $a_{2,2} \neq 0$, and that it is D_5 otherwise.

Lang's Type 13B (I_2^*). X is given by $y^2 + c_0 t^3 y = x^3 + d_1 t x^2 + e_1 t^3 x + f_1 t^5$ with $t \nmid c_0, t \nmid e_1$ and $c_i, d_i, e_i, f_i \in k[t,s]$ homogeneous of degree i.

(1.) Since $t \nmid e_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $e_1 = s$. Then, since $t \nmid c_0$, we can scale s and t such that $c_0 = 1$.

(2.) Let $d_1 = d_{1,0}s + d_{1,1}t$ and $f_1 = f_{1,0}s + f_{1,1}t$. The substitution $x \mapsto x + \lambda t^2, y \mapsto y + \lambda^2 tx + \mu t^3$, where λ and μ are chosen such that $d_{1,0}\lambda^2 + \lambda = f_{1,0}$ and $\mu^2 + \mu = \lambda^3 + d_{1,1}\lambda^2 + f_{1,1}$, yields the new equation $y^2 + t^3 y = x^3 + (a_{2,2}t^2 + a_{2,1}ts)x^2 + t^3 sx$.

(3.) Since $\Delta = t^{12}$, the RDP configuration on X is D_6.

Lang's Type 13C (I_3^*). X is given by $y^2 + c_0 t^3 y = x^3 + d_1 t x^2 + d_0 t^4 x + e_0 t^6$ with $t \nmid c_0, t \nmid d_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(1.) Since $t \nmid d_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $d_1 = s$. Then, since $t \nmid c_0$, we can scale s and t such that $c_0 = 1$. Further, we apply the $\mathbb{P}^1$-automorphism $s \mapsto s + d_0^2 t$ to obtain $y^2 + t^3 y = x^3 + tsx^2 + d_0^2 t^2 x^2 + d_0 t^4 x + e_0 t^6$.

(2.) Then the substitution $y \mapsto y + d_0 tx + \lambda t^3$ with $\lambda^2 + \lambda = e_0$ yields $y^2 + t^3 y = x^3 + tsx^2$.

(3.) Since $\Delta = t^{12}$, the RDP configuration on X is D_7.

Lang's Type 14 (IV^*). X is given by $y^2 + c_1 t^2 y = x^3 + d_0 t^2 x^2 + e_1 t^3 x + d_2 t^4$ with $t \nmid c_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(1.) Since $t \nmid c_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $c_1 = s$.

(2.) Let us write $e_1 = e_{1,0}s + e_{1,1}t$ and $d_2 = d_{2,0}s^2 + d_{2,1}ts + d_{2,2}t^2$. The substitution $x \mapsto x + \sqrt{e_{1,1}}t^2,\ y \mapsto y + e_{1,0}tx + \lambda t^2 s + \sqrt{d_0 e_{1,1} + d_{2,2}}t^3$ with $\lambda^2 + \lambda = d_{2,0}$ yields the new equation $y^2 + t^2 sy = x^3 + a_{2,2}t^2x^2 + a_{6,5}t^5 s$.

(3.) We have $\Delta = t^8 s^4$. By Tate's algorithm we see that the RDP configuration on X is $E_6 + A_2$ if $a_{6,5} = a_{2,2} = 0$, that it is $E_6 + A_1$ if $a_{6,5} = 0$ and $a_{2,2} \neq 0$, and that it is E_6 otherwise.

Lang's Type 15 (III*). X is given by $y^2 + c_0 t^3 y = x^3 + d_0 t^2 x^2 + e_1 t^3 x + d_1 t^5$ with $t \nmid c_0, t \nmid e_1$ and $c_i, d_i, e_i \in k[t,s]$ homogeneous of degree i.

(1.) Since $t \nmid e_1$, we can apply a substitution of the form $s \mapsto \mu s + \lambda t$ for appropriate $\mu, \lambda \in k$ such that $e_1 = s$. Then, since $t \nmid c_0$, we can scale s and t such that $c_0 = 1$. Let us write $d_1 = d_{1,0}s + d_{1,1}t$ and apply the $\mathbb{P}^1$-automorphism $s \mapsto s + (d_{1,0}^2 + \sqrt{d_0 + d_{1,0}})t$ to obtain $y^2 + t^3 y = x^3 + d_0 t^2 x^2 + t^3 sx + (d_{1,0}^2 + \sqrt{d_0 + d_{1,0}})t^4 x + d_{1,0}t^5 s + d_{1,0}(d_{1,0}^2 + \sqrt{d_0 + d_{1,0}})t^6 + d_{1,1}t^6$.

(2.) Finally, the substitution $x \mapsto x + d_{1,0}t^2, y \mapsto y + \sqrt{d_0 + d_{1,0}}tx + \lambda t^3$ with $\lambda^2 + \lambda = d_{1,0}^3 + d_0 d_{1,0}^2 + d_{1,1}$ yields the simplified equation $y^2 + t^3 y = x^3 + t^3 sx$.

(3.) Since $\Delta = t^{12}$, the RDP configuration on X is E_7.

Lang's Type 16 (II*). This equation has been simplified by Lang in [Lan94] and it is as described in the second column of Table 5.

Quasi-elliptic surfaces:
Ito's Types 5.2.(a) (II*), *5.2.(b)* (I$_4^*$), *5.2.(c)* (III*), *5.2.(d)* (2I$_0^*$), *5.2.(e)* (I$_2^*$) *and 5.2.(f)* (I$_0^*$). These equations have been simplified by Ito in [Ito94, Theorem 5.2.] and they are as described in the second column of Table 3, where we only simplified the equation for 5.2.(e) in order to put the D_6-singularity at $(t,x,y) = (0,0,0)$. □

EXAMPLE 4.5. Consider Lang's Type 5B. simplified as in the proof of Proposition 4.4 and localized at the D_6-singularity at $t = x = y = 0$:

$$f = y^2 + txy + (x^3 + a_{2,1}tx^2 + a_{6,5}t^5 + t^4) = 0.$$

Applying Proposition 2.2, we have

$$\begin{aligned} T_f &= k[t,x,y]_{(t,x,y)}/(f, a_{2,1}x^2 + a_{6,5}t^4 + xy, x^2 + ty, tx) \\ &\cong k[t,x,y]_{(t,x,y)}/(y^2 + a_{6,5}t^5 + t^4, a_{2,1}x^2 + a_{6,5}t^4 + xy, x^2 + ty, tx) \\ &\cong k[t,x,y]_{(t,x,y)}/(y^2 + t^4, a_{2,1}x^2 + a_{6,5}t^4 + xy, x^2 + ty, tx) =: R, \end{aligned}$$

where for the first isomorphism we have used that $x^3 = xty = 0$ and for the second isomorphism we have used that $a_{6,5}t^5 = a_{2,1}x^2 t + xyt = 0$. Now, it is easy to check that R is generated as a k-vector space by $1, x, y, t, x^2, y^2, t^2, t^3$, hence $\dim T_f = 8$. Therefore, by Table 7, the Artin coindex of this D_6-singularity is 2.

REMARK 4.6. Using our results, it is straightforward to list all RDP configurations Γ such that the associated configuration of (-2)-curves Γ' occurs on a weak del Pezzo surface, but Γ itself does not occur on any RDP del Pezzo surface. By Theorem 1.2, this phenomenon happens only in characteristic 2 and there precisely if

$$\begin{aligned}\Gamma \in \{ & E_8^1, E_8^2, E_7^1 + A_1, E_7^1, E_7^2 + A_1, D_8^1, D_8^2, D_7^0, D_6^0, D_6^1 + 2A_1, D_6^1 + A_1, D_6^2 + 2A_1, \\ & D_5^0 + A_3, D_5^0 + 2A_1, D_4^0 + D_4^1, D_4^0 + A_3, D_4^1 + D_4^1, D_4^1 + 4A_1, D_4^1 + 3A_1\}.\end{aligned}$$

It would be interesting to find an abstract reason for the non-existence of those Artin coindices on RDP del Pezzo surfaces.

CHAPTER II

Weak del Pezzo surfaces with global vector fields

Up to minor modifications, this chapter is taken from the article "Weak del Pezzo surfaces with global vector fields", which is joint work of the author with G. Martin. The article has been accepted for publication in *Geometry and Topology* and can be found on the ArXiv (see [MS20]).

1. Motivation and summary

Recall that a weak del Pezzo surface over an algebraically closed field k is a smooth projective surface $\widetilde{X}$ with anti-canonical divisor class $-K_{\widetilde{X}}$ big and nef, or, equivalently, $\widetilde{X}$ is $\mathbb{P}^1 \times \mathbb{P}^1$, the second Hirzebruch surface $\mathbb{F}_2$, or the blow-up of at most 8 points in $\mathbb{P}^2$ in almost general position. More classically, weak del Pezzo surfaces appear as the minimal resolutions of surfaces of degree d in $\mathbb{P}^d$ which are neither cones nor projections of surfaces of minimal degree d in $\mathbb{P}^{d+1}$ [Dol12, Definition 8.1.5].

By a result of Matsumura and Oort [MO68], the automorphism functor $\mathrm{Aut}_{\widetilde{X}}$ of a proper variety $\widetilde{X}$ over k is representable by a group scheme locally of finite type over k. Since $\mathrm{Aut}_{\widetilde{X}}$ is well-known for surfaces of minimal degree (that is for quadric surfaces, the Veronese surface, and rational normal scrolls [Dol12, Corollary 8.1.2]), weak del Pezzo surfaces form the first class of smooth projective surfaces for which the study of $\mathrm{Aut}_{\widetilde{X}}$ is interesting. In this chapter, we are concerned with the identity component $\mathrm{Aut}^0_{\widetilde{X}}$ of $\mathrm{Aut}_{\widetilde{X}}$, which can be non-reduced in positive characteristic.

While this non-reducedness phenomenon does not occur for smooth projective curves, we will see that it appears for one of the first non-trivial classes of smooth projective surfaces, namely for weak del Pezzo surfaces (see also [Neu79]), at least in characteristic 2 and 3. This means that for a weak del Pezzo surface $\widetilde{X}$ in characteristic 2 and 3 we may have $h^0(\widetilde{X}, T_{\widetilde{X}}) > \dim \mathrm{Aut}^0_{\widetilde{X}}$, that is, $\widetilde{X}$ may have more global vector fields than expected.

More classically, automorphisms of (weak) del Pezzo surfaces are being studied in the context of the plane Cremona group, i.e. the group of birational automorphisms of $\mathbb{P}^2$. The main reason for this is that automorphisms of (weak) del Pezzo surfaces yield birational automorphisms of $\mathbb{P}^2$ that do not necessarily extend to biregular automorphisms. For the action of $\mathrm{Aut}^0_{\widetilde{X}}$ on a weak del Pezzo surface $\widetilde{X}$, the situation is very different, since this action always descends to an action on the whole minimal model of $\widetilde{X}$ by Blanchard's Lemma 2.10.

This special feature of the connected automorphism scheme $\mathrm{Aut}^0_{\widetilde{X}}$ will enable us to calculate it explicitly for all weak del Pezzo surfaces that are blow-ups of $\mathbb{P}^2$ in terms of stabilizers as a subgroup scheme of PGL_3. Using this, we will classify all weak del Pezzo surfaces $\widetilde{X}$ with non-trivial $\mathrm{Aut}^0_{\widetilde{X}}$ and determine their configurations of (-2)- and (-1)-curves, as well as their number of moduli, which is the content of the following Main Theorem:

MAIN THEOREM. *Let $\widetilde{X}$ be a weak del Pezzo surface over an algebraically closed field. If $h^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0$, then $\widetilde{X}$ is one of the surfaces in Table 1, 2, 3, 4, 5, or Table 6. All cases exist and have an irreducible moduli space of the stated dimension.*

In Tables 2, 3, 4, 5, 6, the figure describing the configuration of (-2)-curves and (-1)-curves ("lines") on these surfaces is given in column 2. In these figures, a "thick" curve denotes a (-2)-curve, while a "thin" curve denotes a (-1)-curve. The intersection multiplicity of two such curves is no more than 3 at every point; intersection multiplicities 1 and 2 will be clear from the picture, whereas we write a small 3 next to the point of intersection if the intersection multiplicity is 3. Recall that the dual graph of all (-2)-curves on a weak del Pezzo surface is a union of Dynkin diagrams of types A_n, D_n and E_n. This graph can be read off from the corresponding figure, but for ease of reference we give its Dynkin type in column 3. For the same reason, in column 4, we list the number of (-1)-curves on these surfaces. In column 5, we describe a general S-valued point of $\mathrm{Aut}^0_{\widetilde{X}}$, where S is a k-scheme. In particular, the dimension of $H^0(\widetilde{X}, T_{\widetilde{X}}) = \mathrm{Aut}^0_{\widetilde{X}}(k[\epsilon]/(\epsilon^2))$ can be read off from this description and is listed in column 6 for the convenience of the reader. Comparing this with the dimension of $\mathrm{Aut}^0_{\widetilde{X}}$, it can be checked whether $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth or not. This is done in column 7. If there is more than one weak del Pezzo surface with the configuration of curves and with the automorphism scheme as in the previous columns, we give the dimension of a modular family of such surfaces in column 8. If, instead, there is a unique surface of this type, we write "$\{\mathrm{pt}\}$" in column 8 in order to emphasize that the surface is unique. Finally, in column 9, we give the characteristic(s) in which the respective surface(s) exist(s).

In particular, our classification also gives a complete list of weak del Pezzo surfaces with non-reduced automorphism schemes. In the following corollary, we list the characteristics p and degrees d for which every weak del Pezzo surface of degree d in characteristic p has reduced automorphism scheme.

COROLLARY 1.1. *Let k be an algebraically closed field of characteristic $p \geq 0$. Then, every weak del Pezzo surface $\widetilde{X}$ of degree d over k has reduced automorphism scheme if and only if one of the following three conditions holds:*

(1) $p \neq 2, 3$,
(2) $p = 3$ *and* $d \geq 4$,
(3) $p = 2$ *and* $d \geq 5$.

Moreover, if $\mathrm{Aut}_{\widetilde{X}}$ is non-reduced, then the number of (-2)-curves on $\widetilde{X}$ is at least $(7-d)$.

In particular, the above corollary recovers the result that the automorphism scheme of every del Pezzo surface (where $-K_{\widetilde{X}}$ is ample) is smooth, which is in fact easier to prove and has already been observed by Dolgachev and Duncan (see [DD19, Theorem 2.4.]).

REMARK 1.2. Independently, shortly after the upload of the article [MS20] to the ArXiv, and using a completely different approach, Cheltsov and Prokhorov [CP21] classified all RDP del Pezzo surfaces X over an algebraically closed field k of characteristic 0 such that $\mathrm{Aut}_X(k)$ is infinite. Now, $\mathrm{Aut}_X(k)$ is infinite if and only if $\mathrm{Aut}_X^0(k)$ is infinite if and only if $\mathrm{Aut}_{\widetilde{X}}^0(k)$ is infinite, where $\widetilde{X}$ is the weak del Pezzo surface that is the minimal resolution of X. Since $\mathrm{Aut}_{\widetilde{X}}^0$ is always smooth in characteristic 0 by Cartier's Theorem (see e.g. [Per76, Corollaire 4.2.8]), $\mathrm{Aut}_{\widetilde{X}}^0(k)$ is infinite if and only if $\widetilde{X}$ admits global vector fields. So, the classification in [CP21] is equivalent to the characteristic 0 part of our Main Theorem.

The structure of this chapter is as follows: Section 2 is devoted to setting up the framework for the study of the geometry of (most) weak del Pezzo surfaces as blow-ups of $\mathbb{P}^2$ including the notion of their "height" in Subsection 2.1. The necessary background on automorphism schemes as well as the key ingredient Blanchard's Lemma are treated in Subsection 2.2. This enables us to set up an inductive strategy for the classification in Section 3. Finally, this strategy is carried out in Section 4, where we go through all possible heights of weak del Pezzo surfaces, realizing each such surface as a blow-up of a surface of height one less, starting from the height 0 surface $\mathbb{P}^2$.

Case	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}_{\widetilde{X}}^0$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}_{\widetilde{X}}^0$ smooth?	Moduli	$\mathrm{char}(k)$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\emptyset$	0	$\mathrm{PGL}_2 \times \mathrm{PGL}_2$	6	✓	$\{\mathrm{pt}\}$	any
$\mathbb{F}_2$	A_1	0	$(\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$ $= (\mathbb{G}_a^3 \rtimes \mathrm{GL}_2)/\mu_2$	7	✓	$\{\mathrm{pt}\}$	any

Table 1. Weak del Pezzo surfaces of degree 8 that are not blow-ups of $\mathbb{P}^2$

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
degree 9								
$9A$		$\emptyset$	0	PGL_3	8	✓	{pt}	any
degree 8								
$8A$	Fig. 5	$\emptyset$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix}$	6	✓	{pt}	any
degree 7								
$7A$	Fig. 4	$\emptyset$	3	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	{pt}	any
$7B$	Fig. 26	A_1	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	{pt}	any
degree 6								
$6A$	Fig. 3	$\emptyset$	6	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
$6B$	Fig. 24	A_1	4	$\begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix}$	3	✓	{pt}	any
$6C$	Fig. 2	A_1	3	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix}$	3	✓	{pt}	any
$6D$	Fig. 25	$2A_1$	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	{pt}	any
$6E$	Fig. 51	A_2	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix}$	4	✓	{pt}	any
$6F$	Fig. 52	$A_2 + A_1$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	{pt}	any
degree 5								
$5A$	Fig. 1	A_1	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$5B$	Fig. 22	$2A_1$	5	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
$5C$	Fig. 18	A_2	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
$5D$	Fig. 23	$A_2 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix}$	3	✓	{pt}	any
$5E$	Fig. 50	A_3	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix}$	3	✓	{pt}	any
$5F$	Fig. 60	A_4	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix}$	4	✓	{pt}	any

Table 2. Weak del Pezzo surfaces with global vector fields of degree ≥ 5 that are blow-ups of $\mathbb{P}^2$

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$4A$	Fig. 13	$2A_1$	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$4B$	Fig. 14	$3A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$4C$	Fig. 15	$A_2 + A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$4D$	Fig. 17	A_3	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$4E$	Fig. 42	A_3	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2$
$4F$	Fig. 21	$4A_1$	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\text{pt}\}$	any
$4G$	Fig. 20	$A_2 + 2A_1$	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\text{pt}\}$	any
$4H$	Fig. 43	$A_3 + A_1$	3	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}$	2	✓	$\{\text{pt}\}$	any
$4I$	Fig. 49	A_4	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	$\{\text{pt}\}$	any
$4J$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix}$	2	✓	$\{\text{pt}\}$	$\neq 2$
$4K$	Fig. 48	$A_3 + 2A_1$	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix}$	3	✓	$\{\text{pt}\}$	any
$4L$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix}$	3	✓	$\{\text{pt}\}$	$\neq 2$
$4M$	Fig. 42	A_3	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	2	×	$\{\text{pt}\}$	$= 2$
$4N$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	2	✓	$\{\text{pt}\}$	$= 2$
$4O$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix}$	3	✓	$\{\text{pt}\}$	$= 2$
$4P$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	3	✓	$\{\text{pt}\}$	$= 2$
$4Q$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix}$	4	✓	$\{\text{pt}\}$	$= 2$

Table 3. Weak del Pezzo surfaces of degree 4 with global vector fields

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$3A$	Fig. 10	$2A_2$	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$3B$	Fig. 16	D_4	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$3C$	Fig. 11	$2A_2 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$3D$	Fig. 12	$A_3 + 2A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$3E$	Fig. 41	$A_4 + A_1$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$3F$	Fig. 46	A_5	3	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$\neq 3$
$3G$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$\neq 2$
$3H$	Fig. 19	$3A_2$	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$3I$	Fig. 47	$A_5 + A_1$	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$3J$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$\neq 2, 3$
$3K$	Fig. 46	A_5	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, e^3 = 1$	2	×	$\{\mathrm{pt}\}$	$= 3$
$3L$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$= 3$
$3M$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	$= 3$
$3N$	Fig. 33	A_4	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	$\{\mathrm{pt}\}$	$= 2$
$3O$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$= 2$
$3P$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$= 2$
$3Q$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & b & c \\ & 1 & b^2+b \\ & & 1 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$= 2$
$3R$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & b & c \\ & e & b^2 e \\ & & e^3 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	$= 2$

Table 4. Weak del Pezzo surfaces of degree 3 with global vector fields

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$2A$	Fig. 7	$2A_3$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$2B$	Fig. 39	$D_5 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$2C$	Fig. 64	E_6	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	{pt}	$\neq 2$
$2D$	Fig. 8	$2A_3 + A_1$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$2E$	Fig. 9	$D_4 + 3A_1$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$2F$	Fig. 40	$A_5 + A_2$	3	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$2G$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	{pt}	$\neq 2$
$2H$	Fig. 56	A_7	2	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$\neq 2$
$2I$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix}$	1	✓	{pt}	$\neq 2, 3$
$2J$	Fig. 45	A_6	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	×	{pt}	$= 3$
$2K$	Fig. 54	D_6	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	×	{pt}	$= 3$
$2L$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$= 3$
$2M$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix}$	2	✓	{pt}	$= 3$
$2N$	Fig. 30	A_5	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	1 dim	$= 2$
$2O$	Fig. 38	D_5	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	{pt}	$= 2$
$2P$	Fig. 32	$A_5 + A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	{pt}	$= 2$
$2Q$	Fig. 31	$A_5 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	{pt}	$= 2$
$2R$	Fig. 54	D_6	3	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	1 dim	$= 2$
$2S$	Fig. 64	E_6	4	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, f^2 = 0$	2	×	{pt}	$= 2$
$2T$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$= 2$
$2U$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	{pt}	$= 2$
$2V$	Fig. 56	A_7	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, e^4 = 1$	2	×	{pt}	$= 2$
$2W$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$= 2$
$2X$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & b & c \\ & 1 & b^2 \\ & & 1 \end{pmatrix}$	2	✓	{pt}	$= 2$
$2Y$	Fig. 67	E_7	1	$\begin{pmatrix} 1 & b & c \\ & e & b^2 e \\ & & e^3 \end{pmatrix}$	3	✓	{pt}	$= 2$

Table 5. Weak del Pezzo surfaces of degree 2 with global vector fields

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$1A$	Fig. 6	$2D_4$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$1B$	Fig. 37	$E_6 + A_2$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$1C$	Fig. 63	$E_7 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2$
$1D$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2, 3$
$1E$	Fig. 53	D_7	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^3 = 1$	1	$\times$	$\{\text{pt}\}$	$= 3$
$1F$	Fig. 62	E_7	5	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	$\times$	$\{\text{pt}\}$	$= 3$
$1G$	Fig. 44	A_8	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	$\times$	$\{\text{pt}\}$	$= 3$
$1H$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	$\{\text{pt}\}$	$= 3$
$1I$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix}$	2	✓	$\{\text{pt}\}$	$= 3$
$1J$	Fig. 35	E_6	13	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	$\times$	1 dim	$= 2$
$1K$	Fig. 34	$E_6 + A_1$	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	$\times$	$\{\text{pt}\}$	$= 2$
$1L$	Fig. 27	A_7	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	$\times$	1 dim	$= 2$
$1M$	Fig. 61	E_7	5	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}, f^2 = 0$	1	$\times$	$\{\text{pt}\}$	$= 2$
$1N$	Fig. 29	$D_6 + 2A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	$\times$	$\{\text{pt}\}$	$= 2$
$1O$	Fig. 28	$A_7 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	$\times$	$\{\text{pt}\}$	$= 2$
$1P$	Fig. 63	$E_7 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, f^2 = 0$	2	$\times$	$\{\text{pt}\}$	$= 2$
$1Q$	Fig. 55	D_8	2	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	1 dim	$= 2$
$1R$	Fig. 55	D_8	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, e^4 = 1$	2	$\times$	$\{\text{pt}\}$	$= 2$
$1S$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$	1	✓	$\{\text{pt}\}$	$= 2$
$1T$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix}, b^4 = 0$	3	$\times$	$\{\text{pt}\}$	$= 2$

Table 6. Weak del Pezzo surfaces of degree 1 with global vector fields

2. Generalities

This section provides the necessary background on the two main topics of this chapter: weak del Pezzo surfaces and automorphism schemes. Throughout, we will be working over an algebraically closed field k.

2.1. Geometry of weak del Pezzo surfaces and their "height". We recall that every weak del Pezzo surface $\widetilde{X}$ (except $\widetilde{X} = \mathbb{P}^1 \times \mathbb{P}^1$ and the second Hirzebruch surface $\widetilde{X} = \mathbb{F}_2$) is a successive blow-up of $\mathbb{P}^2$ satisfying certain properties (see Lemma 2.5 and Lemma 2.7), and we define the notion of "height", which is a measure for the complexity of $\widetilde{X}$. We describe the set of all (-2)- and (-1)-curves on $\widetilde{X}$ in terms of a realization of $\widetilde{X}$ as a blow-up of $\mathbb{P}^2$.

DEFINITION 2.1. A *weak del Pezzo surface* is a smooth projective surface $\widetilde{X}$ with nef and big anti-canonical class $-K_{\widetilde{X}}$. The number $\deg(\widetilde{X}) = K_{\widetilde{X}}^2$ is called the *degree of* $\widetilde{X}$.

Recall that every birational morphism $\pi : \widetilde{X}' \to \widetilde{X}$ of smooth projective surfaces can be factored as

$$\pi : \widetilde{X}' \xrightarrow{\varphi} \widetilde{X}'^{(n)} \xrightarrow{\pi^{(n-1)}} \widetilde{X}'^{(n-1)} \xrightarrow{\pi^{(n-2)}} \ldots \xrightarrow{\pi^{(1)}} \widetilde{X}'^{(1)} \xrightarrow{\pi^{(0)}} \widetilde{X}'^{(0)} = \widetilde{X},$$

where φ is an isomorphism and each $\pi^{(i)} : \widetilde{X}'^{(i+1)} \to \widetilde{X}'^{(i)}$ is the blow-up of a number of distinct closed points on $\widetilde{X}'^{(i)}$. The isomorphism φ can be neglected by identifying $\widetilde{X}'$ with $\widetilde{X}'^{(n)}$ via φ. Then, the above factorization becomes unique (up to unique isomorphism for every $n \geq i \geq 1$) if in each step we blow up the maximal number of distinct closed points of $\widetilde{X}'^{(i)}$. In this case, we call the above factorization of π *minimal*.

DEFINITION 2.2. Let $\widetilde{X}$ and $\widetilde{X}'$ be two smooth projective surfaces.

- For every birational morphism $\pi : \widetilde{X}' \to \widetilde{X}$, let $\pi = \pi^{(0)} \circ \ldots \circ \ldots \pi^{(n-1)}$ be its minimal factorization. The *height of* π is defined as
$$\mathrm{ht}(\pi) := n.$$
- If $\widetilde{X}'$ admits some birational morphism to $\widetilde{X}$, we define the *height of* $\widetilde{X}'$ *over* $\widetilde{X}$ as
$$\mathrm{ht}(\widetilde{X}'/\widetilde{X}) := \min_{\pi : \widetilde{X}' \to \widetilde{X}} \{\mathrm{ht}(\pi)\},$$
where the minimum is taken over all birational morphisms $\pi : \widetilde{X}' \to \widetilde{X}$.
- If $\widetilde{X}$ is a weak del Pezzo surface which is a successive blow-up of $\mathbb{P}^2$, then we define
$$\mathrm{ht}(\widetilde{X}) := \mathrm{ht}(\widetilde{X}/\mathbb{P}^2)$$
and if $\widetilde{X}$ is not a blow-up of $\mathbb{P}^2$, we set $\mathrm{ht}(\widetilde{X}) = 0$.

REMARK 2.3. The reader should compare our notion of height with the height function on the bubble space of $\widetilde{X}$ considered in [Dol12, Section 7.3.2].

NOTATION 2.4. Let $\pi : \widetilde{X} \to \mathbb{P}^2$ be a birational morphism of height n, and let $\pi = \pi^{(0)} \circ \ldots \circ \pi^{(n-1)}$ be its minimal factorization. Then, we fix the following notation:

- For each $0 \leq i < n$, we let $p_{1,i}, \ldots, p_{n_i,i} \in \widetilde{X}^{(i)}$ be the points blown up under $\pi^{(i)}$.

- The exceptional divisor $(\pi^{(i)})^{-1}(p_{j,i}) \subseteq \widetilde{X}^{(i+1)}$ over a closed point $p_{j,i} \in \widetilde{X}^{(i)}$ will be denoted by $E_{j,i}$ for $j = 1, \ldots, n_i$.
- For every $0 \leq i \leq k \leq n$, the strict transform of a curve $C \subseteq \widetilde{X}^{(i)}$ along $\pi^{(i)} \circ \ldots \circ \pi^{(k-1)}$ is denoted by $C^{(k)}$.

Using this notation, we can now state a necessary and sufficient criterion for a successive blow-up of $\mathbb{P}^2$ to be a weak del Pezzo surface.

LEMMA 2.5. *[Dém80] [Dol12, Section 8.1.3] With Notation 2.4, let $\pi : \widetilde{X} \to \mathbb{P}^2$ be a birational morphism of height n. Then, $\widetilde{X}$ is a weak del Pezzo surface if and only if the following three conditions hold.*

- *On each $E_{j,i}$ there is at most one $p_{k,i+1}$.*
- *For every line $\ell \subseteq \mathbb{P}^2$, there are at most three $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$, where i ranges over $0, \ldots, n-1$.*
- *For every irreducible conic $Q \subseteq \mathbb{P}^2$, there are at most six $p_{j,i}$ with $p_{j,i} \in Q^{(i)}$, where i ranges over $0, \ldots, n-1$.*

NOTATION 2.6. By Lemma 2.5, there is at most one $p_{k,i+1}$ on each $E_{j,i}$. Therefore, it makes sense to rename the $p_{k,i+1}$ so that $p_{k,i+1}$ lies on $E_{k,i}$. We will adopt this convention from now on.

If the above three conditions of Lemma 2.5 are satisfied, we say that the points $p_{j,i}$ are in *almost general position*. Using this terminology, there is the following well-known characterization of weak del Pezzo surfaces.

LEMMA 2.7. *[Dol12, Section 8.1.3] If $\widetilde{X}$ is a weak del Pezzo surface, then*

(i) $\widetilde{X} \cong \mathbb{P}^1 \times \mathbb{P}^1$, *or*
(ii) $\widetilde{X} \cong \mathbb{F}_2$, *the* second Hirzebruch surface, *or*
(iii) $\widetilde{X}$ *is the successive blow-up of* $\mathbb{P}^2$ *in* $n \leq 8$ *points in* almost general position.

In particular, we have $1 \leq \deg(\widetilde{X}) \leq 9$, and $\operatorname{ht}(\widetilde{X}) = 0$ if and only if $\widetilde{X} \in \{\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2\}$.

All the possible classes of (-2)- and (-1)-curves in the odd unimodular lattice $\operatorname{Pic}(\widetilde{X}) = \mathrm{I}_{1,9-\deg(\widetilde{X})}$ of signature $(1, 9-\deg(\widetilde{X}))$ are well-known and described in [Man86, Definition 23.7., Proposition 26.1.] and [Dol12, Proposition 8.2.7]. This lattice-theoretic description can be translated into geometry (see [Man86, Theorem 26.2. (ii)] for the case of del Pezzo surfaces). A straightforward adaption of Manin's approach to our situation of *weak* del Pezzo surfaces yields the following description of (-2)- and (-1)-curves on $\widetilde{X}$.

LEMMA 2.8. *Let $\widetilde{X}$ be a weak del Pezzo surface and let $\pi : \widetilde{X} = \widetilde{X}^{(n)} \to \mathbb{P}^2$ be a birational morphism of height n.*

(i) A curve on $\widetilde{X}$ is a (-2)-curve if and only if it is of one of the following four types:

- *the strict transform $E_{j,i}^{(n)}$ of an exceptional curve such that there is exactly one $p_{j,i+1}$ on $E_{j,i}$,*
- *the strict transform $\ell^{(n)}$ of a line $\ell \subseteq \mathbb{P}^2$ such that there are exactly three $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$,*

- *the strict transform $C^{(n)}$ of an irreducible conic $C \subseteq \mathbb{P}^2$ such that there are exactly six $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, or*
- *the strict transform $C^{(n)}$ of an irreducible singular cubic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that one of the $p_{j,0}$ is the singular point of C.*

(ii) A curve on $\widetilde{X}$ is a (-1)-curve if and only if it is of one of the following seven types:
- *the strict transform $E_{j,i}^{(n)}$ of an exceptional curve such that there is no $p_{k,i+1}$ on $E_{j,i}$,*
- *the strict transform $\ell^{(n)}$ of a line $\ell \subseteq \mathbb{P}^2$ such that there are exactly two $p_{j,i}$ with $p_{j,i} \in \ell^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible conic $C \subseteq \mathbb{P}^2$ such that there are exactly five $p_{j,i}$ with $p_{j,i} \in C^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible singular cubic $C \subseteq \mathbb{P}^2$ such that there are exactly seven $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that one of the $p_{j,0}$ is the singular point of C,*
- *the strict transform $C^{(n)}$ of an irreducible singular quartic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly three of the $p_{j,i}$ are double points of $C^{(i)}$,*
- *the strict transform $C^{(n)}$ of an irreducible singular quintic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly six of the $p_{j,i}$ are double points of $C^{(i)}$, or*
- *the strict transform $C^{(n)}$ of an irreducible singular sextic $C \subseteq \mathbb{P}^2$ such that there are exactly eight $p_{j,i}$ with $p_{j,i} \in C^{(i)}$, and such that exactly seven of the $p_{j,i}$ are double points of $C^{(i)}$ and exactly one of the $p_{j,0}$ is a triple point of C.*

REMARK 2.9. In particular, it can be seen that the criterion given in Lemma 2.5 simply tells us that a successive blow-up of $\mathbb{P}^2$ in at most 8 points is a weak del Pezzo surface if and only if we have never blown up a point on a (-2)-curve.

2.2. Automorphism schemes of blow-ups of smooth surfaces. By a result of Matsumura and Oort [MO68], the automorphism functor Aut_X^0 of a proper variety over k is representable and it is well-known that the tangent space of Aut_X^0 can be identified naturally with $H^0(X, T_X)$. The main tool in our study of automorphism schemes of weak del Pezzo surfaces is the following lemma of Blanchard (see [Bri17, Theorem 7.2.1]).

LEMMA 2.10. (Blanchard's Lemma) *Let $f : Y \to X$ be a morphism of proper schemes over k with $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then, f induces a homomorphism of group schemes $f_* : \mathrm{Aut}_Y^0 \to \mathrm{Aut}_X^0$. If f is birational, then f_* is a closed immersion.*

Thus, if f is birational, we can and will identify Aut_Y^0 with its image under f_* in the following. If f is the blow-up of a smooth surface X in a closed point p, it is possible to describe the image of f_* (see [Neu79, Lemma 1.1] and [Mar22, Proposition 2.7]) as an intersection of successive stabilizer group schemes (see Definition 3.5 and Notation 3.6).

LEMMA 2.11. *Let $f : Y \to X$ be the blow-up of a smooth projective surface X in n distinct points $p_1, \ldots, p_n \in X$. Then, we have $\mathrm{Aut}_Y^0 = (\bigcap_{i=1}^n (\mathrm{Stab}(p_i))^0)^0$.*

PROOF. We prove the claim by induction on n with the case $n = 0$ being trivial. For the inductive step, let Y' be the blow-up of X in $p_1, \ldots, p_{n-1}$. Then, $f' : Y \to Y'$ is the blow-up in p_n and we have $\mathrm{Aut}^0_{Y'} = (\bigcap_{i=1}^{n-1}(\mathrm{Stab}(p_i))^0)^0$ by the induction hypothesis. Note that the identity component of the stabilizer of $p_n \in Y'$, with respect to the action of $\mathrm{Aut}^0_{Y'}$, is precisely $(\bigcap_{i=1}^{n}(\mathrm{Stab}(p_i))^0)^0$. By [Mar22, Remark 2.8], the Aut^0_Y-action on Y preserves the exceptional divisor of f', hence Aut^0_Y, being connected, is contained in $(\bigcap_{i=1}^{n}(\mathrm{Stab}(p_i))^0)^0$. Conversely, by [Mar22, Proposition 2.7], the $(\bigcap_{i=1}^{n}(\mathrm{Stab}(p_i))^0)^0$-action on Y' lifts to Y and since $(\bigcap_{i=1}^{n}(\mathrm{Stab}(p_i))^0)^0$ is connected, it actually lifts to a subgroup scheme of Aut^0_Y. This finishes the proof. $\square$

Let $\pi : \widetilde{X}'^{(n)} \to \widetilde{X}$ be a birational morphism of smooth projective surfaces $\widetilde{X}$ and $\widetilde{X}'^{(n)}$. Let $E \subseteq \widetilde{X}'^{(n)}$ be a π-exceptional irreducible curve. Recall that the left-action of $\mathrm{Aut}^0_{\widetilde{X}}$ on $\mathrm{Hilb}_{\widetilde{X}}$ is given on S-valued points by

$$\begin{array}{rcl} \mathrm{Aut}^0_{\widetilde{X}}(S) \times \mathrm{Hilb}_{\widetilde{X}}(S) & \xrightarrow{\rho(S)} & \mathrm{Hilb}_{\widetilde{X}}(S) \\ (g : \widetilde{X}_S \to \widetilde{X}_S, \iota : Z \hookrightarrow \widetilde{X}_S) & \longmapsto & (Z \times_{\iota, \widetilde{X}_S, g^{-1}} \widetilde{X}_S \hookrightarrow \widetilde{X}_S), \end{array}$$

where $\widetilde{X}_S := \widetilde{X} \times S$, and this induces a natural action ρ of $\mathrm{Aut}^0_{\widetilde{X}'^{(n)}} \subseteq \mathrm{Aut}^0_{\widetilde{X}}$ on $\mathrm{Hilb}_{\widetilde{X}}$. For a *pencil* (that is, a 1-dimensional linear system) $f : \mathcal{C} \to \mathbb{P}^1 \subseteq \mathrm{Hilb}_{\widetilde{X}}$ of curves on $\widetilde{X}$ we will identify a point $p \in \mathbb{P}^1(S)$ with its fiber $\mathcal{C}_p$ under f. Let $V \subseteq \mathbb{P}^1$ be an open subset such that any two fibers $\mathcal{C}_p$ and $\mathcal{C}_q$ with $p, q \in V$ (as well as their strict transforms in all the $\widetilde{X}'^{(i)}$) have the same multiplicity at the $p_{j,i}$. Then, the rational map

$$\begin{array}{rcl} \mathbb{P}^1 \supseteq V & \longrightarrow & \mathrm{Hilb}_E \\ p & \longmapsto & \mathcal{C}_p^{(n)} \cap E, \end{array} \tag{2.1}$$

can be extended to a morphism φ from $\mathbb{P}^1$, since every irreducible component of Hilb_E is proper.

DEFINITION 2.12. Let $\pi : \widetilde{X}'^{(n)} \to \widetilde{X}$ be a birational morphism of smooth projective surfaces $\widetilde{X}$ and $\widetilde{X}'^{(n)}$. Let $E \subseteq \widetilde{X}'^{(n)}$ be a π-exceptional irreducible curve. A pencil of curves $f : \mathcal{C} \to \mathbb{P}^1$ is called *adapted* (to E and π) (or *E-adapted*), if the morphism φ of (2.1) factors through an isomorphism $\mathbb{P}^1 \xrightarrow{\cong} E \subseteq \mathrm{Hilb}_E$.

For an adapted pencil $\mathcal{C} \to \mathbb{P}^1$, we can transfer the $\mathrm{Aut}^0_{\widetilde{X}'^{(n)}}$-action on E via φ to an action on the pencil. Over V, we can describe this action explicitly on S-valued points as follows: For $\mathcal{C}_p \in V(S) \subseteq \mathbb{P}^1(S)$ with embedding $\iota : \mathcal{C}_p \to \widetilde{X}_S$, an element $g \in \mathrm{Aut}_{\widetilde{X}'^{(n)}}(S)$ sends $\mathcal{C}_p$ to the unique curve $\mathcal{C}_{g(p)} \in \mathbb{P}^1(S)$ such that $(\mathcal{C}_p \times_{\iota, \widetilde{X}_S, g^{-1}} \widetilde{X}_S)^{(n)} \cap E_S = \varphi(\mathcal{C}_{g(p)})$. The action of $\mathrm{Aut}^0_{\widetilde{X}'^{(n)}}$ transferred from E to the pencil is the unique extension of the above action from V to $\mathbb{P}^1$. In particular, orbits and stabilizers of the $\mathrm{Aut}^0_{\widetilde{X}'^{(n)}}$-action on E can be calculated on $\mathbb{P}^1$, which we are going to exploit throughout.

REMARK 2.13. In most of the cases occurring in our classification we can choose the adapted pencil $\mathcal{C} \to \mathbb{P}^1$ to be stable under the natural action of $\mathrm{Aut}^0_{\widetilde{X}'^{(n)}}$ on $\mathrm{Hilb}_{\widetilde{X}}$. In this case, we have $\mathcal{C}_{g(p)} = \mathcal{C}_p \times_{\iota, \widetilde{X}_S, g^{-1}} \widetilde{X}_S$.

EXAMPLE 2.14. $\mathrm{Aut}^0_{\widetilde{X}'(n)}$-stable adapted pencils do not always exist, even for blow-ups of $\mathbb{P}^2$:
Consider the morphism $\pi : \widetilde{X}'^{(2)} \to \mathbb{P}^2$ of height 2 given by blowing up the points $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$, $p_{3,0} = [1:1:0]$ and $p_{1,1} := \ell_y^{(1)} \cap E_{1,0}$, where $\ell_y = V(y)$. Then, $\widetilde{X}'^{(2)}$ is the surface of Case $5C$. In the classification in Section 4 (see Case $5C$), we use an $E_{1,1}$-adapted pencil which is not $\mathrm{Aut}^0_{\widetilde{X}'(2)}$-stable to show that $\mathrm{Aut}^0_{\widetilde{X}'(2)}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$ acts on $E_{1,1}$ as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$. For this morphism π, there is no $E_{1,1}$-adapted pencil which is also $\mathrm{Aut}^0_{\widetilde{X}'(2)}$-stable:

Indeed, seeking a contradiction, assume that there exists such a pencil whose fiber over $[\lambda : \mu] \in \mathbb{P}^1$ is $C_{\lambda,\mu} = V(\lambda f_1 + \mu f_2)$ with f_1, f_2 homogeneous of the same degree. By the previous paragraph, the subgroup scheme $\mathbb{G}_a \subseteq \mathrm{Aut}^0_{\widetilde{X}'(2)}$ of automorphisms with $i = 1$ acts trivially on $E_{1,1}$. By Remark 2.13, this implies that every $C_{\lambda,\mu}$ is stable under this $\mathbb{G}_a$-action. In particular, every $C_{\lambda,\mu}$ is a union of orbits of the $\mathbb{G}_a$-action on $\mathbb{P}^2$. The closures of the $\mathbb{G}_a$-orbits are the lines through $[1:0:0]$ except $V(z)$, and every point on $V(z)$. Therefore, each $C_{\lambda,\mu}$ is a union of lines through $[1:0:0]$, hence $\varphi(C_{\lambda,\mu}) = n(\ell_y^{(2)} \cap E_{1,1})$ for some $n \geq 0$ and thus the pencil is not $E_{1,1}$-adapted, contradicting our assumption.

REMARK/NOTATION 2.15. If $\widetilde{X} = \mathbb{P}^2$ and f_1, f_2 are homogeneous equations of the same degree, we say that $\lambda f_1 + \mu f_2$ is *adapted* (to π and E) if the pencil spanned by $C_1 = \mathcal{V}(f_1)$ and $C_2 = \mathcal{V}(f_2)$ is adapted to π and E and if, in addition, we identified C_1 and C_2 with $[1:0]$ and $[0:1]$ in $\mathbb{P}^1$, respectively. We will use this choice of coordinates to determine the orbits and stabilizers of the $\mathrm{Aut}^0_{\widetilde{X}'(n)}$-action on E explicitly by reducing it to a calculation on the pencil $[\lambda : \mu]$.

3. Strategy of proof

For the proof of our Main Theorem we are going to argue inductively by going through all possible weak del Pezzo surfaces with non-trivial connected automorphism scheme in the order given by their height, i.e., we start with del Pezzo surfaces of height 0, which are $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$. Then, by Lemma 2.7, to study del Pezzo surfaces of height 1 we have to study blow-ups of $\mathbb{P}^2$ in a number of distinct "honest" points. After that, for height 2, we have to consider del Pezzos that arise as blow-ups of points on exceptional divisors of blow-ups of points in $\mathbb{P}^2$ (sometimes we will also refer to such points as *infinitely near* points of the first order, as was introduced in [Dol12, Section 7.3.2]). Continuing this pattern, increasing the height by 1 means that we have to study those surfaces that arise as blow-ups of points on the "latest exceptional divisor".

In this subsection, we are going to further specify our strategy of proof and explain why the classification of weak del Pezzo surfaces with non-trivial vector fields obtained via our inductive procedure is indeed complete.

3.1. Inductive strategy.

Assume we have a complete set $\mathcal{L}_i = \{\widetilde{X}_k\}_{k \in K_i}$, for some index set K_i, of representatives of weak del Pezzo surfaces of height i that are blow-ups of $\mathbb{P}^2$ with $H^0(\widetilde{X}_k, T_{\widetilde{X}_k}) \neq 0$, where for every $\widetilde{X}_k$ we have fixed a birational morphism $\psi_k : \widetilde{X}_k \to \mathbb{P}^2$ of height i. Further assume that we have calculated $(\psi_k)_*(\mathrm{Aut}^0_{\widetilde{X}_k}) \subseteq \mathrm{PGL}_3$ (see Lemma 2.11) for every k. If $i = 0$, such a list is given by $\mathcal{L}_0 = \{\mathbb{P}^2\}$ with $\mathrm{Aut}^0_{\mathbb{P}^2} = \mathrm{PGL}_3$. Using the list $\mathcal{L}_i$, we produce a list $\mathcal{L}_{i+1}$ as follows:

PROCEDURE 3.1.

Step 1: Choose $\widetilde{X} \in \mathcal{L}_i$ with $\psi : \widetilde{X} \to \mathbb{P}^2$ and let $\psi : \widetilde{X} \overset{\psi^{(i-1)}}{\longrightarrow} \widetilde{X}^{(i-1)} \overset{\psi^{(i-2)}}{\longrightarrow} \ldots \overset{\psi^{(0)}}{\longrightarrow} \widetilde{X}^{(0)} = \mathbb{P}^2$ be the minimal factorization of ψ.

Step 2: If $i = 0$, let $E := \widetilde{X} = \mathbb{P}^2$. Otherwise, let

$$E := \left(\mathrm{Exc}(\psi^{(i-1)}) - \bigcup_{j=0}^{i-2} \mathrm{Exc}(\psi^{(j)}) \right) - D,$$

where D is the union of all (-2)-curves on $\widetilde{X}$. Note that, if $i > 0$, then E is the set of points on the "latest" exceptional divisors that do not lie on (-2)-curves. Using the description of $\mathrm{Aut}^0_{\widetilde{X}}$ as a subgroup scheme of PGL_3, we calculate the orbits and stabilizers of the action of $\mathrm{Aut}^0_{\widetilde{X}}$ on E using $E_{j,i-1}$-adapted pencils.

Step 3: Choose a set of points $\{p_{1,i}, \ldots, p_{n_i,i}\} \subseteq E$ such that $(\bigcap_{j=1}^{n_i} (\mathrm{Stab}(p_{j,i}))^0)^0$ is non-trivial and such that the blow-up $\psi' : \widetilde{X}' \to \widetilde{X}$ in these points is still a weak del Pezzo surface (see the criterion given in Lemma 2.8). In particular, since there is at most one of the $p_{j,i}$ on every exceptional curve, we may assume that $p_{j,i} \in E_{j,i-1}$. Note that we obtain isomorphic surfaces if we replace a point $p_{j,i}$ by a point in the same orbit under the action of $\bigcap_{k \neq j} \mathrm{Stab}(p_{k,i}) \subseteq \mathrm{Aut}_{\widetilde{X}}$.

Step 4: If $\widetilde{X}'$ is isomorphic to a surface already contained in $\mathcal{L}_j$ for some $j \leq i+1$, discard this case. Otherwise, add $\widetilde{X}'$ to $\mathcal{L}_{i+1}$, choose the blow-up realization $\psi \circ \psi' : \widetilde{X}' \to \mathbb{P}^2$, and calculate

$$(\psi \circ \psi')_*(\mathrm{Aut}^0_{\widetilde{X}'}) = (\psi_*)(\bigcap_{j=1}^{n_i} (\mathrm{Stab}(p_{j,i}))^0)^0 \subseteq \mathrm{PGL}_3 .$$

We do this by describing the group $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ for an arbitrary local k-algebra R (see Subsection 3.2).

Step 5: Repeat Steps 3 and 4 for all possible point combinations $\{p_{1,i}, \ldots, p_{n_i,i}\}$.

Step 6: Then, repeat Steps 1–5 for all $\widetilde{X} \in \mathcal{L}_i$.

LEMMA 3.2. *For every i, the above Procedure 3.1 yields a complete set $\mathcal{L}_{i+1} = \{\widetilde{X}_k\}_{k \in K_{i+1}}$ of representatives of isomorphism classes of weak del Pezzo surfaces of height $(i+1)$ with non-trivial global vector fields, that are blow-ups of $\mathbb{P}^2$.*

PROOF. We prove the claim by induction on the height i. The case $i = 0$ with $\mathcal{L}_0 = \{\mathbb{P}^2\}$ is clear by Lemma 2.11. Therefore, assume that the claim holds for $(i-1) \geq 0$ and that we have a list $\mathcal{L}_i$.

Let $\widetilde{X}'$ be a weak del Pezzo surface of height $(i+1)$ with $h^0(\widetilde{X}', T_{\widetilde{X}'}) \neq 0$. Choose a birational morphism $\pi : \widetilde{X}' \to \mathbb{P}^2$ with minimal factorization

$$\pi : \widetilde{X}' = \widetilde{X}'^{(i+1)} \xrightarrow{\pi^{(i)}} \widetilde{X}'^{(i)} \xrightarrow{\pi^{(i-1)}} \ldots \xrightarrow{\pi^{(0)}} \widetilde{X}'^{(0)} = \mathbb{P}^2$$

such that for every birational morphism $\pi' : \widetilde{X}' \to \mathbb{P}^2$, the number of exceptional curves for $\pi'^{(i)}$ is at least as great as the number of exceptional curves for $\pi^{(i)}$, i.e. such that the number of points blown up by the last step $\pi^{(i)}$ is minimal. By Lemma 2.10, there is an inclusion

$$(\pi^{(i)})_*(\mathrm{Aut}^0_{\widetilde{X}'}) \subseteq \mathrm{Aut}^0_{\widetilde{X}'^{(i)}} .$$

In particular, we have $h^0(\widetilde{X}'^{(i)}, T_{\widetilde{X}'^{(i)}}) \neq 0$ since $\mathrm{Aut}^0_{\widetilde{X}'} \neq \{\mathrm{id}\}$ and $(\pi^{(i)})_*$ is a closed immersion. Hence, by the induction hypothesis, there is $\widetilde{X} \in \mathcal{L}_i$ such that there exists an isomorphism $\phi : \widetilde{X}'^{(i)} \to \widetilde{X}$ and $\widetilde{X}$ comes with a birational morphism $\psi : \widetilde{X} \to \mathbb{P}^2$.

To prove the claim, it suffices to show that $\phi \circ \pi^{(i)}$ is the blow-up of $\widetilde{X}$ in a set of points $p_{1,i}, \ldots, p_{n_i,i}$ on E defined as in Procedure 3.1. Indeed, once we prove this, it will follow from Lemma 2.11 and the assumption $h^0(\widetilde{X}', T_{\widetilde{X}'}) \neq 0$ that $\mathrm{Aut}^0_{\widetilde{X}'} = (\bigcap_{j=1}^{n_i} (\mathrm{Stab}(p_{j,i}))^0)^0$ is non-trivial.

Now, note that the condition that the $p_{j,i}$ lie on E is equivalent to $\phi \circ \pi^{(i)}$ being the first step in the minimal factorization of

$$\psi' := \psi \circ \phi \circ \pi^{(i)} : \widetilde{X}' \to \widetilde{X}'^{(i)} \to \widetilde{X} \to \mathbb{P}^2.$$

Thus, we take the minimal factorization of ψ' and let $\psi'^{(i)} : \widetilde{X}' \to \widetilde{X}''$ be the first morphism in the minimal factorization of ψ'. Since $\widetilde{X}$ has height i, the morphism $\phi \circ \pi^{(i)} : \widetilde{X}' \to \widetilde{X}$ factors through $\psi'^{(i)}$, which means there is a morphism $f : \widetilde{X}'' \to \widetilde{X}$ such that $f \circ \psi'^{(i)} = \phi \circ \pi^{(i)}$. In particular, the number of points blown up under $\psi'^{(i)}$ is at most the number of points blown up under $\pi^{(i)}$. As we chose the number of points blown up under $\pi^{(i)}$ to be minimal, this shows that f is an isomorphism. In fact, since f is an isomorphism over $\mathbb{P}^2$, this isomorphism is unique and we can identify $\widetilde{X}''$ with $\widetilde{X}$. $\square$

One technical question that arises in Procedure 3.1 is how one checks in Step 4 whether $\widetilde{X}'$ is isomorphic to a surface in one of our lists $\mathcal{L}_j$ with $j \leq i+1$. Clearly, a necessary condition for this is that $\widetilde{X}'$ has the same configuration of negative curves as one of the surfaces $\widetilde{X}_k \in \mathcal{L}_j$ for some $j \leq i+1$. By Lemma 3.2, we have the following converse:

COROLLARY 3.3. *Let $\widetilde{X}'$ be a weak del Pezzo surface with non-trivial global vector fields that arises in Step 3 of Procedure 3.1. Assume that $\widetilde{X}'$ has the same configuration of negative curves as a surface in $\mathcal{L}_j$ for some $j < i+1$. Then $\widetilde{X}'$ is isomorphic to a surface already contained in $\mathcal{L}_j$.*

PROOF. If $\widetilde{X}'$ has the same configuration as a surface in $\mathcal{L}_j$, then there is a sequence of contractions of (-1)-curves on $\widetilde{X}'$ that realizes $\widetilde{X}'$ as a weak del Pezzo surface of height $j < i+1$, and then Lemma 3.2 shows that $\widetilde{X}'$ is isomorphic to a surface in $\mathcal{L}_j$. $\square$

REMARK 3.4. If, instead, $\widetilde{X}'$ has the same configuration of negative curves as a surface in $\mathcal{L}_{i+1}$, then we cannot immediately use Lemma 3.2, since the list $\mathcal{L}_{i+1}$ is not yet complete at that point. Whenever this happens in Section 4, we will describe an explicit way of blowing down $\widetilde{X}'$ to a surface with the same configuration as (hence, by Lemma 3.2, isomorphic to) some $\widetilde{X}_k \in \mathcal{L}_i$ in such a way that the image of the exceptional locus lies in the set $E \subseteq \widetilde{X}_k$. If Steps 1 to 5 of Procedure 3.1 have already been carried out for $\widetilde{X}_k \in \mathcal{L}_i$, this implies that $\widetilde{X}'$ is isomorphic to a surface already contained in $\mathcal{L}_{i+1}$.

Since we distinguish the families of weak del Pezzo surfaces with global vector fields according to their configuration of negative curves and automorphism schemes, once we know that $\widetilde{X}'$ is isomorphic to a surface in $\mathcal{L}_j$, we can determine the family to which it belongs by describing its configuration of negative curves and by computing its automorphism scheme.

3.2. On the calculation of stabilizers. Before starting our classification, let us explain how to calculate the scheme-theoretic stabilizers of the points $p_{j,i} \in E_{j,i-1}$ occurring in Step 4 of Procedure 3.1. First, recall the definition of the scheme-theoretic stabilizer.

DEFINITION 3.5. Let $\rho : G \times X \to X$ be an action of a group scheme G on a scheme X over k. Let $Z \subseteq X$ be a closed subscheme. The *stabilizer* $\mathrm{Stab}_G(Z) \subseteq G$ of Z with respect to ρ is defined as

$$\begin{array}{rcl} \mathrm{Stab}_G(Z): & (Sch/k) & \to (Sets) \\ & S & \mapsto \{g \in G(S) \mid g(Z_S) = Z_S\}, \end{array}$$

where $g \in G(S)$ is considered as an element of $\mathrm{Aut}(X_S)$ via the homomorphism $G(S) \to \mathrm{Aut}(X_S)$ induced by ρ, and $Z_S = Z \times_k S \hookrightarrow X_S$ is the closed subscheme obtained by base changing $Z \hookrightarrow X$ to S.

NOTATION 3.6. Throughout this chapter, we will omit the subscript G in the case where Z is a point p and simply write $\mathrm{Stab}(p)$ instead of $\mathrm{Stab}_G(p)$, whenever G is clear from the context.

The stabilizer $\mathrm{Stab}(p) \subseteq G$ is a closed subgroup scheme of G. As mentioned in Step 4 of Procedure 3.1, we will describe only the R-valued points of the stabilizers occurring in our classification, where R is a local k-algebra. This is sufficient, since in each case – all the conditions on the matrices in $\mathrm{PGL}_3(R)$ of Tables 2–6 being given by polynomial equations which respect the group structure on PGL_3 – there will be an obvious closed subgroup scheme G of PGL_3 that admits the same R-valued points as the given stabilizer. The group scheme G will then be equal to the stabilizer because of the following well-known lemma.

LEMMA 3.7. *Let $Z_1, Z_2 \subseteq X$ be two closed subschemes of a scheme X over a field k. If $Z_1(R) = Z_2(R) \subseteq X(R)$ for all local k-algebras R, then $Z_1 = Z_2$ as closed subschemes of X.*

The advantage of only considering R-valued points of PGL_n lies in the fact that R-valued points $\mathbb{P}^n$ are simply given by $(n+1)$-tuples of elements in R up to units in R such that at least one of the elements in the $(n+1)$-tuple is a unit. This allows us to describe the action of $\mathrm{Aut}^0_{\widetilde{X}}(R)$ on $E_{j,i-1}(R) \cong \mathbb{P}^1(R)$ explicitly using adapted pencils, so that the calculation of the scheme-theoretic stabilizer of a k-valued point $p_{j,i} \in E_{j,i-1}$ becomes straightforward (by Lemma 3.7). Thus, R will denote a local k-algebra from now on.

4. Proof of Main Theorem: Classification

In this section, we will carry out Procedure 3.1 to obtain the classification of weak del Pezzo surfaces with global regular vector fields and prove our Main Theorem.

Firstly, note that there are two weak del Pezzo surfaces which do not fit into the framework of Procedure 3.1, namely those which are not blow-ups of $\mathbb{P}^2$. By Lemma 2.7, these are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$. As is well-known, we have $\mathrm{Aut}_{\mathbb{P}^1\times\mathbb{P}^1} = \mathrm{PGL}_2 \times \mathrm{PGL}_2$. As for $\mathrm{Aut}_{\mathbb{F}_2}$, we make use of the fact that this group scheme is smooth and connected by [Mar71, Theorem 1 and Lemma 10]. An explicit description of this group scheme is given in [Mar71]. Alternatively, one can blow-down the unique (-2)-curve on $\mathbb{F}_2$ to obtain the weighted projective plane $\mathbb{P}(1,1,2)$ and use the fact that $(\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$ fixes the unique singular point on $\mathbb{P}(1,1,2)$. Hence, this action lifts to $\mathbb{F}_2$ and we get $\mathrm{Aut}_{\mathbb{F}_2} = (\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$. These results are summarized in Table 1.

For the remaining cases, we can apply Procedure 3.1 and we will subdivide the proof into subsections according to the height of our weak del Pezzo surfaces. Throughout, we write $\ell_f := \mathcal{V}(f)$ for the line given by $f = 0$ in $\mathbb{P}^2$. Recall that in the following figures a "thick" curve denotes a (-2)-curve, while a "thin" curve denotes a (-1)-curve. The intersection multiplicity of two such curves is at most 3 at every point; intersection multiplicities 1 and 2 will be clear from the picture, whereas we write a small 3 next to the point of intersection if the intersection multiplicity is 3.

4.1. Height 0.

We have $\mathcal{L}_0 = \{\widetilde{X}_{9A}\}$, where $\widetilde{X}_{9A} := \mathbb{P}^2$ with $\mathrm{Aut}_{\mathbb{P}^2} = \mathrm{PGL}_3$.

4.2. Height 1.

Case 9A. In this case, $\widetilde{X} = \mathbb{P}^2$ and $\psi = \mathrm{id}$. We have $E = \mathbb{P}^2$ and the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathrm{PGL}_3$ on E is transitive. Now, note that if $p_{1,0}, \ldots, p_{n_0,0} \in \mathbb{P}^2$ are points such that at least four of them are in general position, then

$$\mathrm{Aut}^0_{\widetilde{X}'} = (\bigcap_{j=1}^{n_0} (\mathrm{Stab}(p_{j,0}))^0) = \{*\}.$$

On the other hand, according to Lemma 2.5, to guarantee that $\widetilde{X}'$ is a weak del Pezzo surface, no more than three of the $p_{j,0}$ may be on a line. Up to isomorphism, this leaves the following five possibilities for $p_{1,0}, \ldots, p_{n_0,0}$:

(1) $n = 4$ and $p_{1,0}, p_{2,0}, p_{4,0}$ on a line ℓ, $p_{3,0} \notin \ell$: Using the action of PGL_3, we may assume that $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0], p_{3,0} = [0:0:1], p_{4,0} = [1:1:0]$ and $\ell = \ell_z$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $\ell_z^{(1)}$
- (-1)-curves: $E_{1,0}, E_{2,0}, E_{3,0}, E_{4,0}, \ell_x^{(1)}, \ell_y^{(1)}, \ell_{x-y}^{(1)}$
- with configuration as in Figure 1.

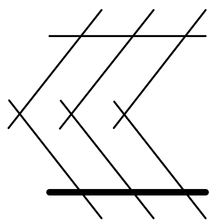


Figure 1

This is case 5A.

(2) $n = 3$, all points on a line ℓ: We may assume that $p_{1,0} = [1:0:0], p_{2,0} = [0:1:0], p_{3,0} = [1:1:0]$ and $\ell = \ell_z$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \left(\begin{smallmatrix} 1 & & c \\ & 1 & f \\ & & i \end{smallmatrix} \right) \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $\ell_z^{(1)}$
- (-1)-curves: $E_{1,0}, E_{2,0}, E_{3,0}$
- with configuration as in Figure 2.

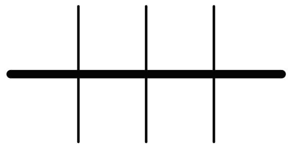


Figure 2

This is case $6C$.

(3) $n = 3$, not all points on a line: We may assume that $p_{1,0} = [1:0:0], p_{2,0} = [0:1:0], p_{3,0} = [0:0:1]$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \left(\begin{smallmatrix} 1 & & \\ & e & \\ & & i \end{smallmatrix} \right) \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: none
- (-1)-curves: $E_{1,0}, E_{2,0}, E_{3,0}, \ell_x^{(1)}, \ell_y^{(1)}, \ell_z^{(1)}$
- with configuration as in Figure 3.

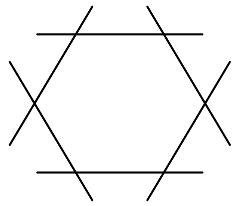


Figure 3

This is case $6A$.

(4) $n = 2$: We may assume that $p_{1,0} = [1:0:0]$, $p_{2,0} = [0:1:0]$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \left(\begin{smallmatrix} 1 & & c \\ & e & f \\ & & i \end{smallmatrix} \right) \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: none
- (-1)-curves: $E_{1,0}, E_{2,0}, \ell_z^{(1)}$
- with configuration as in Figure 4.

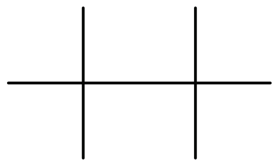


Figure 4

This is case $7A$.

(5) $n = 1$: We may assume that $p_{1,0} = [1:0:0]$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \left(\begin{smallmatrix} 1 & b & c \\ & e & f \\ & h & i \end{smallmatrix} \right) \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: none
- (-1)-curves: $E_{1,0}$
- with configuration as in Figure 5.

Figure 5

This is case $8A$.

Summarizing, we obtain $\mathcal{L}_1 = \{\widetilde{X}_{5A}, \widetilde{X}_{6C}, \widetilde{X}_{6A}, \widetilde{X}_{7A}, \widetilde{X}_{8A}\}$.

4.3. Height 2.

Case 5A. We have $E = (\bigcup_{j=1}^4 E_{j,0}) - \ell_z^{(1)}$. Recall that the R-valued points of $\mathrm{Aut}^0_{\widetilde{X}}$ are given by $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & i \end{smallmatrix} \right) \in \mathrm{PGL}_3(R) \right\}$. We calculate the action of $\mathrm{Aut}^0_{\widetilde{X}}$ on the $E_{j,0}$ using adapted pencils:
- $\lambda y + \mu z$ is $E_{1,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$
- $\lambda x + \mu z$ is $E_{2,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$
- $\lambda x + \mu y$ is $E_{3,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu]$

- $\lambda(x-y)+\mu z$ is $E_{4,0}$-adapted and $\operatorname{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda:\mu]\mapsto[\lambda:i\mu]$

In particular, there is one unique point with non-trivial stabilizer on $E\cap E_{1,0}$, $E\cap E_{2,0}$, and $E\cap E_{4,0}$, respectively. Since $p_{1,0}, p_{2,0}$ and $p_{4,0}$ can be interchanged by automorphisms of $\mathbb{P}^2$ preserving $p_{3,0}$, we have the following ten possibilities for $p_{1,1},\dots,p_{n,1}$:

(1) $p_{1,1}=E_{1,0}\cap\ell_y^{(1)}, p_{2,1}=E_{2,0}\cap\ell_x^{(1)}, p_{3,1}=E_{3,0}\cap\ell_{x+\alpha y}^{(1)}$ with $\alpha\notin\{0,-1\}$, $p_{4,1}=E_{4,0}\cap\ell_{x-y}^{(1)}$
  - $\operatorname{Aut}^0_{\widetilde{X}'}(R)=\left\{\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}\in\mathrm{PGL}_3(R)\right\}$
  - (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}, \ell_{x-y}^{(2)}$
  - (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, E_{4,1}, \ell_{x+\alpha y}^{(2)}$
  - with configuration as in Figure 6.

  This is case $1A$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{1A,\alpha}$ depending on the parameter α.

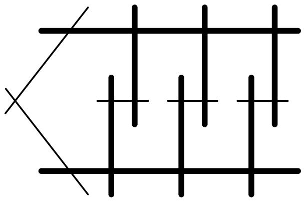


Figure 6

(2) $p_{1,1}=E_{1,0}\cap\ell_y^{(1)}, p_{2,1}=E_{2,0}\cap\ell_x^{(1)}, p_{3,1}=E_{3,0}\cap\ell_{x+\alpha y}^{(1)}$ with $\alpha\notin\{0,-1\}$
  - $\operatorname{Aut}^0_{\widetilde{X}'}(R)=\left\{\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}\in\mathrm{PGL}_3(R)\right\}$
  - (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
  - (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, E_{4,0}^{(2)}, \ell_{x-y}^{(2)}, \ell_{x+\alpha y}^{(2)}$
  - with configuration as in Figure 7.

  This is case $2A$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{2A,\alpha}$ depending on the parameter α.

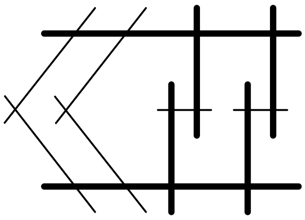


Figure 7

(3) $p_{1,1}=E_{1,0}\cap\ell_y^{(1)}, p_{2,1}=E_{2,0}\cap\ell_x^{(1)}, p_{3,1}=E_{3,0}\cap\ell_{x-y}^{(1)}$
  - $\operatorname{Aut}^0_{\widetilde{X}'}(R)=\left\{\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}\in\mathrm{PGL}_3(R)\right\}$
  - (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}, \ell_{x-y}^{(2)}$
  - (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, E_{4,0}^{(2)}$
  - with configuration as in Figure 8.

  This is case $2D$.

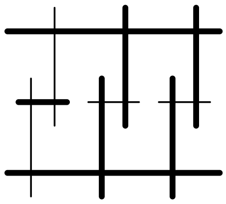


Figure 8

(4) $p_{1,1}=E_{1,0}\cap\ell_y^{(1)}, p_{2,1}=E_{2,0}\cap\ell_x^{(1)}, p_{4,1}=E_{4,0}\cap\ell_{x-y}^{(1)}$
  - $\operatorname{Aut}^0_{\widetilde{X}'}(R)=\left\{\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}\in\mathrm{PGL}_3(R)\right\}$
  - (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}, \ell_{x-y}^{(2)}$
  - (-1)-curves: $E_{1,1}, E_{2,1}, E_{4,1}, E_{3,0}^{(2)}$
  - with configuration as in Figure 9.

  This is case $2E$.

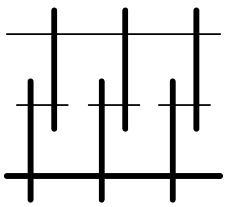


Figure 9

(5) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{3,1} = E_{3,0} \cap \ell_{x+\alpha y}^{(1)}$ with $\alpha \notin \{0, -1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{3,0}^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{3,1}, E_{2,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_{x-y}^{(2)}, \ell_{x+\alpha y}^{(2)}$
- with configuration as in Figure 10.

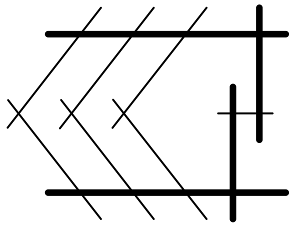


Figure 10

This is case $3A$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{3A,\alpha}$ depending on the parameter α.

(6) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{3,1} = E_{3,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{3,1}, E_{2,0}^{(2)}, E_{4,0}^{(2)}, \ell_{x-y}^{(2)}$
- with configuration as in Figure 11.

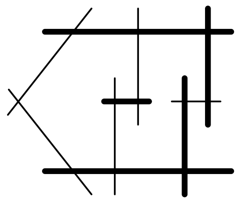


Figure 11

This is case $3C$.

(7) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,0}^{(2)}, E_{4,0}^{(2)}, \ell_{x-y}^{(2)}$
- with configuration as in Figure 12.

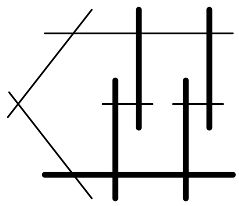


Figure 12

This is case $3D$.

(8) $p_{3,1} = E_{3,0} \cap \ell_{x+\alpha y}^{(1)}$ with $\alpha \notin \{0, -1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{3,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{3,1}, E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_{x-y}^{(2)}, \ell_{x+\alpha y}^{(2)}$
- with configuration as in Figure 13.

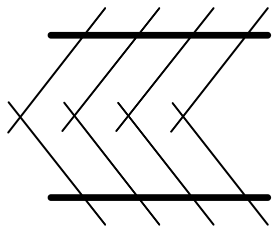


Figure 13

This is case $4A$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{4A,\alpha}$ depending on the parameter α.

(9) $p_{3,1} = E_{3,0} \cap \ell_y^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{3,0}^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{3,1}, E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_{x-y}^{(2)}$
- with configuration as in Figure 14.

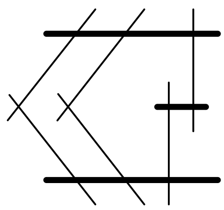


Figure 14

This is case $4B$.

(10) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, E_{4,0}^{(2)}, \ell_x^{(2)}, \ell_{x-y}^{(2)}$
- with configuration as in Figure 15.

This is case $4C$.

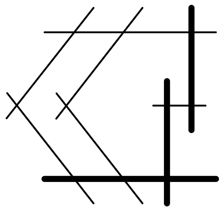


Figure 15

Case 6C. We have $E = (\bigcup_{j=1}^{3} E_{j,0}) - \ell_z^{(1)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y + \mu z$ is $E_{1,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + f\lambda]$
- $\lambda x + \mu z$ is $E_{2,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$
- $\lambda(x - y) + \mu z$ is $E_{3,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + (c - f)\lambda]$

Since $p_{1,0}, p_{2,0}$ and $p_{3,0}$ can be interchanged by automorphisms of $\mathbb{P}^2$ and the action of $\mathrm{Aut}^0_{\widetilde{X}}$ is transitive on every $E \cap E_{i,0}$, we have the following three possibilities for $p_{1,1}, \dots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}, p_{3,1} = E_{3,0} \cap \ell_{x-y}^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_{x-y}^{(2)}$
- with configuration as in Figure 16.

This is case $3B$.

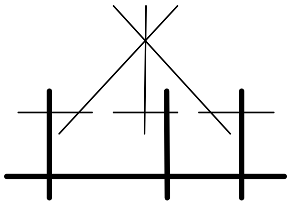


Figure 16

(2) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}$
- with configuration as in Figure 17.

This is case $4D$.

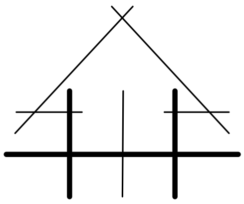


Figure 17

(3) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_y^{(2)}$
- with configuration as in Figure 18.

This is case $5C$.

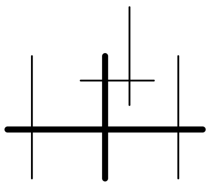


Figure 18

Case 6*A*. We have $E = \bigcup_{j=0}^{3} E_{j,0}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y + \mu z$ is $E_{1,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i\mu]$
- $\lambda x + \mu z$ is $E_{2,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$
- $\lambda x + \mu y$ is $E_{3,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e\mu]$

Since $p_{1,0}, p_{2,0}$ and $p_{3,0}$ can be permuted arbitrarily by automorphisms of $\mathbb{P}^2$, we have the following nine possibilities for $p_{1,1}, \ldots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}, p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}, p_{3,1} = E_{3,0} \cap \ell^{(1)}_{x}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(2)}_{1,0}, E^{(2)}_{2,0}, E^{(2)}_{3,0}, \ell^{(2)}_{x}, \ell^{(2)}_{z}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, \ell^{(2)}_{y}, \ell^{(2)}_{y-z}$
- with configuration as in Figure 11, that is, as in case 3*C*.

Blowing down the two right-most (-1)-curves in Figure 11, we see that $\widetilde{X}'$ arises as a blow-up of $\widetilde{X}_{5A}$ in 2 points on E and $\widetilde{X}' \cong \widetilde{X}_{3C}$ by Remark 3.4.

(2) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}, p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}, p_{3,1} = E_{3,0} \cap \ell^{(1)}_{y}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(2)}_{1,0}, E^{(2)}_{2,0}, E^{(2)}_{3,0}, \ell^{(2)}_{y}, \ell^{(2)}_{z}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}, \ell^{(2)}_{x}, \ell^{(1)}_{y-z}$
- with configuration as in Figure 12, that is, as in case 3*D*.

Blowing down the two (-1)-curves in Figure 12 that are not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ arises as a blow-up of $\widetilde{X}_{5A}$ in 2 points on E and $\widetilde{X}' \cong \widetilde{X}_{3D}$ by Remark 3.4.

(3) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{z}, p_{2,1} = E_{2,0} \cap \ell^{(1)}_{x}, p_{3,1} = E_{3,0} \cap \ell^{(1)}_{y}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(2)}_{1,0}, E^{(2)}_{2,0}, E^{(2)}_{3,0}, \ell^{(2)}_{x}, \ell^{(2)}_{y}, \ell^{(2)}_{z}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,1}$
- with configuration as in Figure 19.

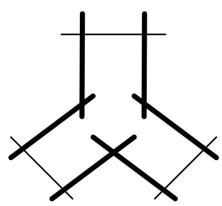


Figure 19

This is case 3*H*.

(4) $p_{1,1} = E_{1,0} \cap \ell^{(1)}_{y-z}, p_{2,1} = E_{2,0} \cap \ell^{(1)}_{z}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(2)}_{1,0}, E^{(2)}_{2,0}, \ell^{(2)}_{z}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E^{(2)}_{3,0}, \ell^{(2)}_{x}, \ell^{(2)}_{y}, \ell^{(2)}_{y-z}$
- with configuration as in Figure 15, that is, as in case 4*C*.

Blowing down the (-1)-curve in Figure 15 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ arises as a blow-up of $\widetilde{X}_{5A}$ in 1 point on E and hence $\widetilde{X}' \cong \widetilde{X}_{4C}$ by Remark 3.4.

(5) $p_{1,1} = E_{1,0} \cap \ell_{y-z}^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_x^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,0}^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}, \ell_{y-z}^{(2)}$
- with configuration as in Figure 14, that is, as in case $4B$.

Blowing down one of the (-1)-curves in Figure 15 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ arises as a blow-up of $\widetilde{X}_{5A}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{4B}$ by Remark 3.4.

(6) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_x^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,0}^{(2)}, \ell_y^{(2)}$
- with configuration as in Figure 20.

This is case $4G$.

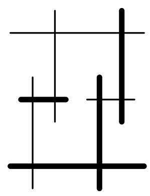


Figure 20

(7) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, E_{3,0}^{(2)}, \ell_z^{(2)}$
- with configuration as in Figure 21.

This is case $4F$.

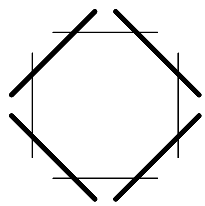


Figure 21

(8) $p_{1,1} = E_{1,0} \cap \ell_{y-z}^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}, \ell_{y-z}^{(2)}$
- with configuration as in Figure 1, that is, as in case $5A$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{5A}$.

(9) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}, E_{3,0}^{(2)}, \ell_x^{(2)}, \ell_y^{(2)}$
- with configuration as in Figure 22.

This is case $5B$.

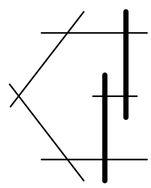


Figure 22

Case 7A. We have $E = E_{1,0} \cup E_{2,0}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y + \mu z$ is $E_{1,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i\mu + f\lambda]$
- $\lambda x + \mu z$ is $E_{2,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$

Since $p_{1,0}$ and $p_{2,0}$ can be interchanged by an automorphism of $\mathbb{P}^2$, we have the following four possibilities for $p_{1,1}, \ldots, p_{n,1}$:

(1) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, \ell_x^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- with configuration as in Figure 22, that is, as in case 5B.

Blowing down the (-1)-curve in Figure 22 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ arises as a blow-up of $\widetilde{X}_{6A}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{5B}$ by Remark 3.4.

(2) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}, p_{2,1} = E_{2,0} \cap \ell_x^{(1)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, E_{2,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,1}, \ell_x^{(2)}$
- with configuration as in Figure 23.

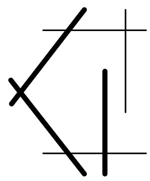


Figure 23

This is case 5D.

(3) $p_{1,1} = E_{1,0} \cap \ell_y^{(1)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}, \ell_y^{(2)}, \ell_z^{(2)}$
- with configuration as in Figure 24.

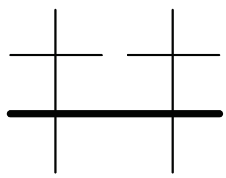


Figure 24

This is case 6B.

(4) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}, \ell_z^{(2)}$
- (-1)-curves: $E_{1,1}, E_{2,0}^{(2)}$
- with configuration as in Figure 25.

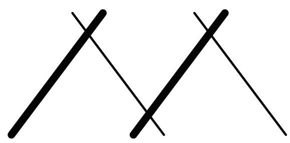


Figure 25

This is case 6D.

Case 8A. We have $E = E_{1,0}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y + \mu z$ is $E_{1,0}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda + h\mu : i\mu + f\lambda]$

Therefore, there is a unique possibility for $p_{1,1}, \ldots, p_{n,1}$ up to isomorphism:

(1) $p_{1,1} = E_{1,0} \cap \ell_z^{(1)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(2)}$
- (-1)-curves: $E_{1,1}, \ell_z^{(2)}$
- with configuration as in Figure 26.

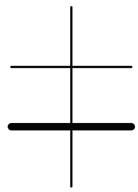


Figure 26

This is case 7B.

Summarizing, we obtain

$$\mathcal{L}_2 = \{\widetilde{X}_{1A,\alpha}, \widetilde{X}_{2A,\alpha}, \widetilde{X}_{2D}, \widetilde{X}_{2E}, \widetilde{X}_{3A,\alpha}, \widetilde{X}_{3C}, \widetilde{X}_{3D}, \widetilde{X}_{4A,\alpha}, \widetilde{X}_{4B}, \widetilde{X}_{4C}, \widetilde{X}_{3B}, \widetilde{X}_{4D}, \widetilde{X}_{5C}, \widetilde{X}_{3H}, \widetilde{X}_{4G}, \widetilde{X}_{4F}, \widetilde{X}_{5B}, \widetilde{X}_{5D}, \widetilde{X}_{6B}, \widetilde{X}_{6D}, \widetilde{X}_{7B}\}.$$

4.4. Height 3.

Case 2A. We have $E = \bigcup_{j=1}^3 E_{j,1} - (\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda y^2 + \mu(x + \alpha y)z$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$

Note that $\widetilde{X}$ has degree 2, therefore we are only allowed to blow up one more point $p_{j,2}$. Moreover, the involution $x \leftrightarrow \alpha y$ of $\mathbb{P}^2$ lifts to an involution of $\widetilde{X}$ interchanging $E_{1,1}$ and $E_{2,1}$, thus we may assume without loss of generality that $j = 1$ or $j = 3$. Finally, if $j = 3$, then the stabilizer of $p_{3,2} \in E \cap E_{3,1}$ is trivial unless $p_{3,2}$ lies on the strict transform of $\ell_{x+\alpha y}$. Moreover, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$. Hence, we have the following two possibilities:

(1) $p_{3,2} = E_{3,1} \cap \ell_{x+\alpha y}^{(2)}$ with $\alpha \notin \{0, -1\}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{3,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}, \ell_{x+\alpha y}^{(3)}$
- (-1)-curves: $E_{3,2}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, E_{4,0}^{(3)}, \ell_{x-y}^{(3)}$
- with configuration as in Figure 6, that is, as in case 1A.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1A,\alpha'}$ for some α'.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,1}^{(3)}, E_{4,0}^{(3)}, \ell_{x-y}^{(3)}, \ell_{x+\alpha y}^{(3)}$, $C_1^{(3)}, C_2^{(3)}$ with $\alpha \notin \{0, -1\}$ and $C_2 = \mathcal{V}(x^3y + xy^3 + x^2z^2 + \alpha^2y^2z^2)$
- with configuration as in Figure 27.

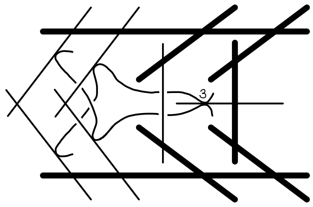


Figure 27

This is case $1L$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{1L,\alpha}$ depending on the parameter α.

<u>*Case 2D*</u>. We have $E = \bigcup_{j=1}^3 E_{j,1} - (\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_{x-y}^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda y^2 + \mu(x - y)z$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$

Note that $\widetilde{X}$ has degree 2, thus we are only allowed to blow up one more point $p_{j,2}$. Next, note that the stabilizer of every point on $E \cap E_{3,1}$ is trivial, hence we may assume $j = 1$ or $j = 2$. Similar to Case $2A$, the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of $\widetilde{X}$ interchanging $E_{1,1}$ and $E_{2,1}$, thus we may assume without loss of generality that $j = 1$. Hence, there is the following unique choice for $p_{j,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}, \ell_{x-y}^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,1}^{(3)}, E_{4,0}^{(3)}, C^{(3)}$
- with configuration as in Figure 28.

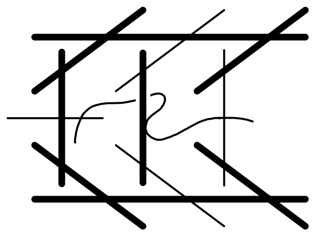


Figure 28

This is case $1O$.

Case 2E. We have $E = (E_{1,1} \cup E_{2,1} \cup E_{4,1}) - (E_{1,0}^{(2)} \cup E_{2,0}^{(2)} \cup E_{4,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_{x-y}^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda(x-y)x + \mu z^2$ is $E_{4,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

Note that $\widetilde{X}$ has degree 2, thus we are only allowed to blow up one more point $p_{j,2}$. Next, the automorphisms of $\mathbb{P}^2$ interchanging $p_{1,0}, p_{2,0}$ and $p_{4,0}$ and preserving $p_{3,0}$ lift to $\widetilde{X}$ and interchange $E_{1,1}, E_{2,1}$ and $E_{4,1}$, thus we may assume $j = 1$. Finally, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$, hence we have the following unique choice for $p_{j,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{4,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}, \ell_{x-y}^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{4,1}^{(3)}, E_{3,0}^{(3)}, C_1^{(3)}, C_2^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- with configuration as in Figure 29.

This is case $1N$.

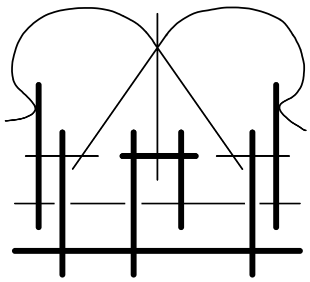


Figure 29

Case 3A. We have $E = (E_{1,1} \cup E_{3,1}) - (E_{1,0}^{(2)} \cup E_{3,0}^{(2)} \cup \ell_y^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda y^2 + \mu(x + \alpha y)z$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$

Note that there is one unique point with non-trivial stabilizer on $E \cap E_{3,1}$, while $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$. Hence, we have the following three choices up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{3,2} = E_{3,1} \cap \ell_{x+\alpha y}^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$ and $\alpha \notin \{0, -1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{3,1}^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}, \ell_{x+\alpha y}^{(3)}$
- (-1)-curves: $E_{1,2}, E_{3,2}, E_{2,0}^{(3)}, E_{4,0}^{(3)}, \ell_x^{(3)}, \ell_{x-y}^{(3)}, C_2^{(3)}, C_3^{(3)}$ with $C_2 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2), C_3 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2 + y^3)$

- with configuration as in Figure 27, that is, as in case $1L$.

Blowing down the right-most (-1)-curve in Figure 27, we see that $\widetilde{X}'$ is the blow-up of some $\widetilde{X}_{2A,\alpha}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1L,\alpha'}$ for some α' by Remark 3.4.

(2) $p_{3,2} = E_{3,1} \cap \ell^{(2)}_{x+\alpha y}$ with $\alpha \notin \{0, -1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{3,0}, E^{(3)}_{3,1}, \ell^{(3)}_{y}, \ell^{(3)}_{z}, \ell^{(3)}_{x+\alpha y}$
- (-1)-curves: $E_{3,2}, E^{(3)}_{1,1}, E^{(3)}_{2,0}, E^{(3)}_{4,0}, \ell^{(3)}_{x}, \ell^{(3)}_{x-y}$
- with configuration as in Figure 7, that is, as in case $2A$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2A,\alpha'}$ for some α'.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{3,0}, E^{(3)}_{1,1}, \ell^{(3)}_{y}, \ell^{(3)}_{z}$
- (-1)-curves: $E_{1,2}, E^{(3)}_{3,1}, E^{(3)}_{2,0}, E^{(3)}_{4,0}, \ell^{(3)}_{x}, \ell^{(3)}_{x-y}, \ell^{(3)}_{x+\alpha y}$ with $\alpha \notin \{0, -1\}$
- with configuration as in Figure 30.

This is case $2N$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{2N,\alpha}$ depending on the parameter α.

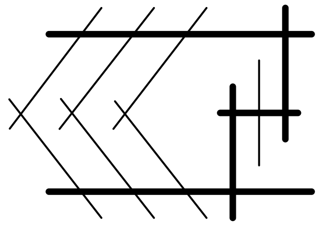


Figure 30

Case 3C. We have $E = (E_{1,1} \cup E_{3,1}) - (E^{(2)}_{1,0} \cup E^{(2)}_{3,0} \cup \ell^{(2)}_{y} \cup \ell^{(2)}_{x})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda xz + \mu y^2$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : \mu]$

Note that the stabilizer of every point in $E \cap E_{3,1}$ is trivial while $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$. Hence, we have the following unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{3,0}, E^{(3)}_{1,1}, \ell^{(3)}_x, \ell^{(3)}_y, \ell^{(3)}_z$
- (-1)-curves: $E_{1,2}, E^{(3)}_{3,1}, E^{(3)}_{2,0}, E^{(3)}_{4,0}, \ell^{(3)}_{x-y}$
- with configuration as in Figure 31.

This is case $2Q$.

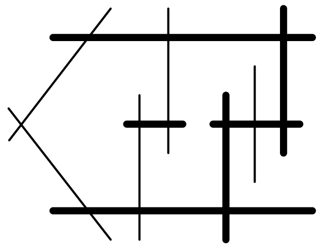


Figure 31

Case $3D$. We have $E = (E_{1,1} \cup E_{2,1}) - (E^{(2)}_{1,0} \cup E^{(2)}_{2,0} \cup \ell^{(2)}_y \cup \ell^{(2)}_x)$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of $\widetilde{X}$ interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively and with finite stabilizers on both $E \cap E_{1,1}$ and $E \cap E_{2,1}$. Hence, we have the following three possibilities for $p_{1,2}, \ldots, p_{n,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}, p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{2,0}, E^{(3)}_{1,1}, E^{(3)}_{2,1}, \ell^{(3)}_x, \ell^{(3)}_y, \ell^{(3)}_z, C^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E^{(3)}_{3,0}, E^{(3)}_{4,0}, \ell^{(3)}_{x-y}$
- with configuration as in Figure 28, that is, as in case $1O$.

Blowing down the left-most (-1)-curve in Figure 28, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{2D}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1O}$ by Remark 3.4.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2), C_2 = \mathcal{V}(xy + \alpha z^2)$, $\alpha \notin \{0,1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}, E_{4,0}^{(3)}, \ell_{x-y}^{(3)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}$ with $C_3 = \mathcal{V}(x^3y^2 + x^2y^3 + xz^4 + \alpha^2 yz^4)$
- with configuration as in Figure 27, that is, as in case $1L$.

Blowing down the right-most (-1)-curve in Figure 27, we see that $\widetilde{X}'$ is the blow-up of some $\widetilde{X}_{2A,\alpha}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1L,\alpha'}$ for some α' by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,0}^{(3)}, E_{4,0}^{(3)}, \ell_{x-y}^{(3)}, C^{(3)}$
- with configuration as in Figure 32.

This is case $2P$.

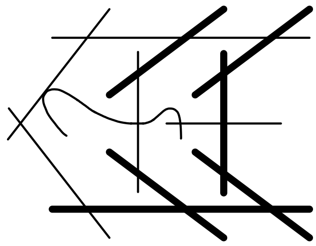


Figure 32

<u>*Case* $4A$</u>. We have $E = E_{3,1} - E_{3,0}^{(2)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y^2 + \mu(x + \alpha y)z$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$

Note that there is one unique point on $E \cap E_{3,1}$ with non-trivial stabilizer, leading to the following unique choice for $p_{3,2}$:

(1) $p_{3,2} = E_{3,1} \cap \ell_{x+\alpha y}^{(2)}$ with $\alpha \notin \{0, -1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{3,0}^{(3)}, E_{3,1}^{(3)}, \ell_z^{(3)}, \ell_{x+\alpha y}^{(3)}$
- (-1)-curves: $E_{3,2}, E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{4,0}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_{x-y}^{(3)}$
- with configuration as in Figure 10, that is, as in case $3A$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3A,\alpha'}$ for some α'.

Case 4B. We have $E = E_{3,1} - (E_{3,0}^{(2)} \cup \ell_y^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2 + \mu yz$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$

There is no point on $E \cap E_{3,1}$ with non-trivial stabilizer, so we get no new cases by further blowing up $\widetilde{X}$.

Case 4C. We have $E = E_{1,1} - (E_{1,0}^{(2)} \cup \ell_y^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

In particular, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$. We get the following unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{1,1}^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{4,0}^{(3)}, \ell_x^{(3)}, \ell_{x-y}^{(3)}$
- with configuration as in Figure 33.

This is case $3N$.

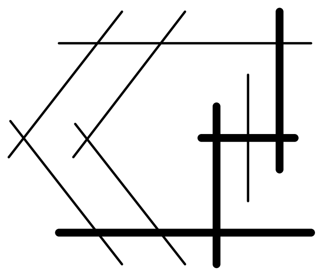


Figure 33

Case 3B. We have $E = \bigcup_{j=1}^3 E_{j,1} - \bigcup_{j=1}^3 E_{j,0}^{(2)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$
- $\lambda(x-y)x + \mu z^2$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

Note that automorphisms of $\mathbb{P}^2$ fixing $[0 : 0 : 1]$ and interchanging the $p_{j,0}$ lift to automorphisms of $\widetilde{X}$ interchanging the $E_{j,1}$. Moreover, since $\widetilde{X}$ has degree 3, we are only allowed to blow up two more points. Finally, on every $E \cap E_{j,1}$, the action of $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits and one of them is a fixed point. Hence, we get the following six possibilities for $p_{1,2}, \ldots, p_{3,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_1^{(2)}$ with
$C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}, C_1^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_{x-y}^{(3)}, C_2^{(3)}, C_3^{(3)}$
  with $C_2 = \mathcal{V}(xy + y^2 + z^2), C_3 = \mathcal{V}(xy + x^2 + z^2)$
- with configuration as in Figure 34.

This is case $1K$.

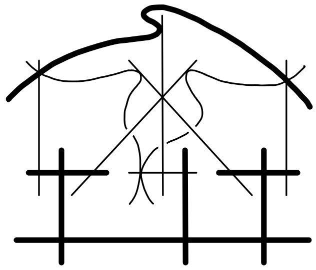


Figure 34

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with
$C_1 = \mathcal{V}(xy + z^2), C_2 = \mathcal{V}(xy + \alpha z^2), \alpha \notin \{0, 1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_{x-y}^{(3)}, C_1^{(3)}, C_2^{(3)}$,
  $C_4^{(3)}, C_5^{(3)}, C_6^{(3)}, C_7^{(3)}$ with $C_3 = \mathcal{V}(xy + y^2 + z^2)$,
  $C_4 = \mathcal{V}(xy + x^2 + \alpha z^2)$,
  $C_5 = \mathcal{V}(x^2y^2 + xy^3 + \alpha y^2 z^2 + z^4)$,
  $C_6 = \mathcal{V}(x^2y^2 + x^3y + x^2z^2 + \alpha^2 z^4)$,
  $C_7 = \mathcal{V}(x^3y^2 + x^2y^3 + xz^4 + \alpha^2 yz^4)$
- with configuration as in Figure 35.

This is case $1J$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{1J,\alpha}$ depending on the parameter α.

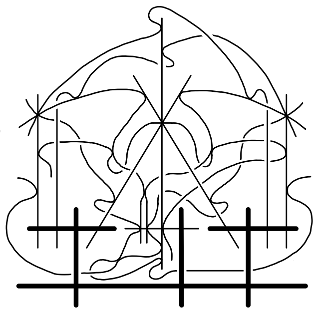


Figure 35

REMARK 4.1. Figure 35 is by far the most complicated configuration that occurs in our classification. To make Figure 35 easier to digest for the reader, we will now break our habit of describing the curve configuration only via an intuitive picture and also describe the *dual graph* of the configuration. Each white vertex in the dual graph corresponds to a (-1)-curve and each black vertex corresponds to a (-2)-curve. The number of edges between two vertices corresponding to curves C_1 and C_2 is equal to the intersection number of C_1 and C_2. With these conventions, the dual graph of Figure 35 looks as follows:

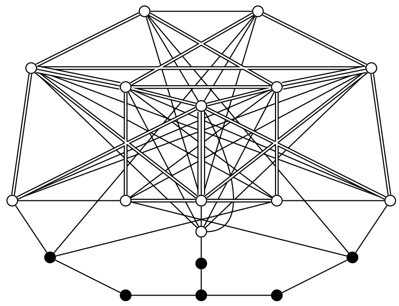


Figure 36. Dual graph of Figure 35

Note that, in general, a dual graph carries less information than the non-dual picture. In our case, we see from Figure 35 that every simply laced triangle of vertices in Figure 36 corresponds to three curves meeting in a single point, every double edge corresponds to two curves meeting in a single point with multiplicity 2, and every triple edge corresponds to two curves meeting in two distinct points with multiplicities 2 and 1, respectively. While the symmetry group of Figure 36 is the dihedral group D_{12} of order 12, the interested reader can use the additional information from Figure 35 to check that the only involution in D_{12} that can actually come from an automorphism of $\widetilde{X}$ is the unique central involution. And indeed, the pencil of cubic curves through the 8 points $p_{i,j}$ contains the curve $\mathcal{V}(z^3)$ and the smooth curve $\mathcal{V}(z^3 + z^2x + \alpha z^2y + x^2y + xy^2)$, hence it is an elliptic pencil and the inverse in the group structure on the generic fiber of the associated rational elliptic surface (classically called "Bertini involution" associated to the points $p_{i,j}$) induces the central $\mathbb{Z}/2\mathbb{Z}$-symmetry of the above graph.

(3) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,1}^{(3)}, \ell_y^{(3)}, \ell_{x-y}^{(3)}, C_1^{(3)}, C_2^{(3)}, C_3^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2), C_3^{(3)} = \mathcal{V}(x^2y^2 + xy^3 + z^4)$
- with configuration as in Figure 34, that is, as in case $1K$.

Blowing down the left-most and the right-most (-1)-curve in Figure 34, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{3B}$ in 2 points on E which do not lie on the intersection of E with the other (-1)-curves on $\widetilde{X}_{3B}$ and $\widetilde{X}' \cong \widetilde{X}_{1K}$ by Remark 3.4.

(4) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}, p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,1}^{(3)}, \ell_{x-y}^{(3)}$
- with configuration as in Figure 37.

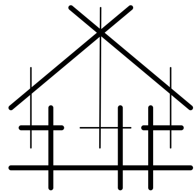


Figure 37

This is case $1B$.

(5) $p_{1,2} = E_{1,1} \cap C_1^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_{x-y}^{(3)}, C_1^{(3)}, C_2^{(3)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- with configuration as in Figure 38.

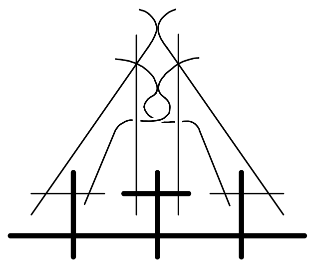


Figure 38

This is case $2O$.

(6) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,1}^{(3)}, \ell_x^{(3)}, \ell_{x-y}^{(3)}$
- with configuration as in Figure 39.

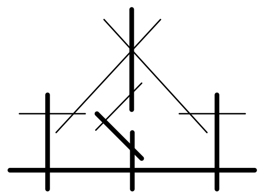


Figure 39

This is case $2B$.

<u>*Case $4D$*</u>. We have $E = \bigcup_{j=1}^2 E_{j,1} - \bigcup_{j=1}^2 E_{j,0}^{(2)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

Note that automorphisms of $\mathbb{P}^2$ fixing $[0 : 0 : 1]$ and interchanging $p_{1,0}$ and $p_{2,0}$ lift to automorphisms of $\widetilde{X}$ interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on each $E \cap E_{j,1}$, one of which is a fixed point. Hence, we get the following six possibilities for $p_{1,2}, p_{2,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}, p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}, C^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}$
- with configuration as in Figure 31, that is, as in case $2Q$.

Blowing down the right-most (-1)-curve in Figure 31, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{3C}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{2Q}$ by Remark 3.4.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2), C_2 = \mathcal{V}(xy + \alpha z^2)$, $\alpha \notin \{0, 1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, C_1^{(3)}, C_2^{(3)}$
- with configuration as in Figure 30, that is, as in case $2N$.

Blowing down the right-most (-1)-curve in Figure 30, we see that $\widetilde{X}'$ is a blow-up of some $\widetilde{X}_{3A,\alpha}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{2N,\alpha'}$ for some α' by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}, p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}, \ell_y^{(3)}, C^{(3)}$
- with configuration as in Figure 31, that is, as in case $2Q$.

Blowing down the right-most (-1)-curve in Figure 31, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{3C}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{2Q}$ by Remark 3.4.

(4) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}, p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}$
- with configuration as in Figure 40.

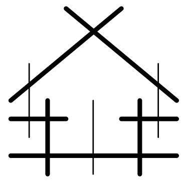


Figure 40

This is case $2F$.

(5) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,0}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, C^{(3)}$
- with configuration as in Figure 33, that is, as in case $3N$.

Blowing down the right-most (-1)-curve in Figure 33, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{4C}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{3N}$ by Remark 3.4.

(6) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,0}^{(3)}, \ell_x^{(3)}$
- with configuration as in Figure 41.

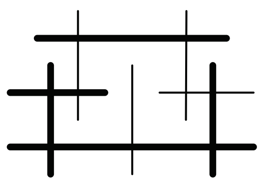


Figure 41

This is case $3E$.

Case 5C. We have $E = E_{1,1} - E_{1,0}^{(2)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^2\mu]$

Note that this is the first case in which there exists no $\mathrm{Aut}^0_{\widetilde{X}}$-stable $E_{1,1}$-adapted pencil (see Example 2.14). We remind the reader that we explained how to calculate the $\mathrm{Aut}^0_{\widetilde{X}}$-action on exceptional curves using not necessarily $\mathrm{Aut}^0_{\widetilde{X}}$-stable adapted pencils after Definition 2.12. From now on, we will no longer explicitly point out when a non-$\mathrm{Aut}^0_{\widetilde{X}}$-stable adapted pencil is used and assume that the reader is familiar with the techniques explained in Subsection 2.2.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on $E \cap E_{1,1}$, we get the following two possibilities for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\} & \text{if } p = 2 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:
- (-2)-curves: $E_{1,0}^{(3)}, E_{1,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, \ell_y^{(3)}$
- with configuration as in Figure 42.

This is case $4E$ if $p \neq 2$, and case $4M$ if $p = 2$.

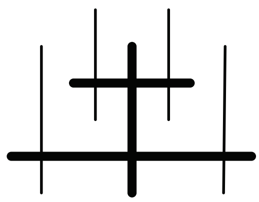


Figure 42

(2) $p_{1,2} = E_{1,1} \cap \ell_y^{(2)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{1,1}^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,0}^{(3)}, E_{3,0}^{(3)}$
- with configuration as in Figure 43.

This is case $4H$.

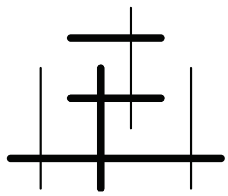


Figure 43

Case $3H$. We have $E = \bigcup_{j=1}^3 E_{j,1} - (\bigcup_{j=1}^3 E_{j,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)} \cup \ell_z^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$
- $\lambda xy + \mu z^2$ is $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$
- $\lambda yz + \mu x^2$ is $E_{3,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [ei\lambda : \mu]$

Note that all automorphisms of $\mathbb{P}^2$ inducing cyclic permutations of $p_{1,0}, p_{2,0}$, and $p_{3,0}$ lift to automorphisms of $\widetilde{X}$ and since $\widetilde{X}$ has degree 3, we can only blow up two additional points. Moreover, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on every $E \cap E_{j,1}$. Hence, we get the following two possibilities for $p_{1,2}, \ldots, p_{3,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2), C_2 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,1}^{(3)}$
- with configuration as in Figure 44.

This is case $1G$.

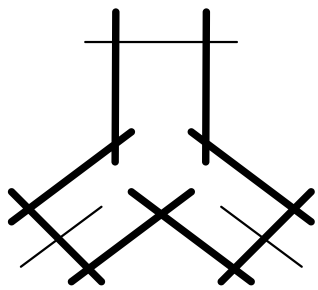


Figure 44

(2) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{3,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_y^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,1}^{(3)}$
- with configuration as in Figure 40, that is, as in case $2F$.

Blowing down the left-most and the right-most (-1)-curve in Figure 40, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{4D}$ in 2 points on E and $\widetilde{X}' \cong \widetilde{X}_{2F}$ by Remark 3.4.

<u>*Case 4G*</u>. We have $E = \bigcup_{j=1}^2 E_{j,1} - (\bigcup_{j=1}^2 E_{j,0}^{(2)} \cup \ell_z^{(2)} \cup \ell_x^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$
- $\lambda xy + \mu z^2$ is $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on every $E \cap E_{j,1}$, we get the following three possibilities for $p_{1,2}, p_{2,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2), C_2 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E_{3,0}^{(3)}, \ell_y^{(3)}$
- with configuration as in Figure 45.

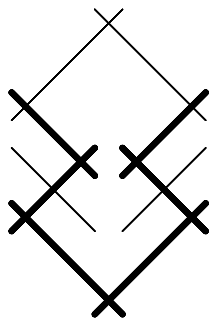


Figure 45

This is case $2J$.

(2) $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{2,2}, E_{1,1}^{(3)}, E_{3,0}^{(3)}, \ell_y^{(3)}$
- with configuration as in Figure 41, that is, as in case $3E$.

Blowing down the left-most (-1)-curve in Figure 41, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{4D}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{3E}$ by Remark 3.4.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,1}^{(3)}, E_{3,0}^{(3)}, \ell_y^{(3)}$
- with configuration as in Figure 41, that is, as in case $3E$.

Blowing down the left-most (-1)-curve in Figure 41, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{4D}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{3E}$ by Remark 3.4.

<u>*Case* $4F$</u>. We have $E = (E_{1,1} \cup E_{2,1}) - (E_{1,0}^{(2)} \cup E_{2,0}^{(2)} \cup \ell_x^{(2)} \cup \ell_y^{(2)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an involution of $\widetilde{X}$ interchanging $E_{1,1}$ and $E_{2,1}$. Moreover, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on both $E \cap E_{1,1}$ and $E \cap E_{2,1}$, but the stabilizer of every point on $E \cap E_{1,1}$ acts trivially on $E \cap E_{2,1}$. Hence, we have the following three possibilities up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}, p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{2,0}, E^{(3)}_{1,1}, E^{(3)}_{2,1}, \ell^{(3)}_x, \ell^{(3)}_y, C^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, E^{(3)}_{3,0}, \ell^{(3)}_z$
- with configuration as in Figure 8, that is, as in case $2D$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2D}$.

(2) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xy + z^2), C_2 = \mathcal{V}(xy + \alpha z^2)$, $\alpha \notin \{0, 1\}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{2,0}, E^{(3)}_{1,1}, E^{(3)}_{2,1}, \ell^{(3)}_x, \ell^{(3)}_y$
- (-1)-curves: $E_{1,2}, E_{2,2}, E^{(3)}_{3,0}, \ell^{(3)}_z, C^{(3)}_1, C^{(3)}_2$
- with configuration as in Figure 7, that is, as in case $2A$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2A,\alpha'}$ for some α'.

(3) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{2,0}, E^{(3)}_{1,1}, \ell^{(3)}_x, \ell^{(3)}_y$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,1}, E^{(3)}_{3,0}, \ell^{(3)}_z, C^{(3)}$
- with configuration as in Figure 12, that is, as in case $3D$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3D}$.

Case 5B. We have $E = E_{1,1} - (E^{(2)}_{1,0} \cup \ell^{(2)}_z)$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$, we have the following unique choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{1,1}, \ell^{(3)}_z$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,0}, E^{(3)}_{3,0}, \ell^{(3)}_x, \ell^{(3)}_y$
- with configuration as in Figure 17, that is, as in case $4D$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{4D}$.

Case 5D. We have $E = \bigcup_{j=1}^2 E_{j,1} - (\bigcup_{j=1}^2 E^{(2)}_{j,0} \cup \ell^{(2)}_z)$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$
- $\lambda xy + \mu z^2$ is $E_{2,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$

Note that $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$, and with two orbits, one of which is a fixed point, on $E \cap E_{2,1}$. Hence, we have the following five choices for $p_{1,2}, p_{2,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C_1^{(2)}, p_{2,2} = E_{2,1} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz+y^2), C_2 = \mathcal{V}(xy+z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 3$ and $p = 3$ simultaneously:
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}, \ell_x^{(3)}$
- with configuration as in Figure 46.

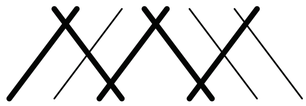


Figure 46

This is case $3F$ if $p \neq 3$, and case $3K$ if $p = 3$.

(2) $p_{1,2} = E_{1,1} \cap C^{(2)}, p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$ with $C = \mathcal{V}(xz+y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{1,1}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}, E_{2,2}$
- with configuration as in Figure 47.

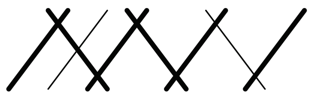


Figure 47

This is case $3I$.

(3) $p_{2,2} = E_{2,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy+z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & i^2 & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{2,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{2,2}, E_{1,1}^{(3)}, \ell_x^{(3)}$
- with configuration as in Figure 43, that is, as in case $4H$.

Blowing down the (-1)-curve in the middle of Figure 43, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{5C}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{4H}$ by Remark 3.4.

(4) $p_{2,2} = E_{2,1} \cap \ell_x^{(2)}$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{2,0}^{(3)}, E_{2,1}^{(3)}, \ell_x^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{2,2}, E_{1,1}^{(3)}$
- with configuration as in Figure 48.

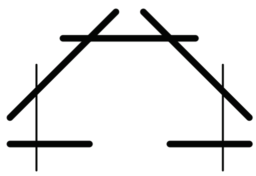


Figure 48

This is case $4K$.

(5) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{2,0}, E^{(3)}_{1,1}, \ell^{(3)}_z$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,1}, \ell^{(3)}_x$
- with configuration as in Figure 49.

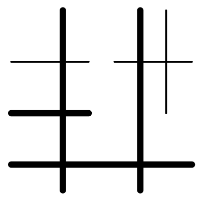


Figure 49

This is case $4I$.

Case 6B. We have $E = E_{1,1} - E^{(2)}_{1,0}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy + \mu z^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : i^2\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on $E \cap E_{1,1}$, we have the following two choices for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & i^2 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{1,1}$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,0}, \ell^{(3)}_y, \ell^{(3)}_z$
- with configuration as in Figure 18, that is, as in case $5C$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{5C}$.

(2) $p_{1,2} = E_{1,1} \cap \ell^{(2)}_y$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{1,1}, \ell^{(3)}_y$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,0}, \ell^{(3)}_z$
- with configuration as in Figure 23, that is, as in case $5D$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{5D}$.

Case 6D. We have $E = E_{1,1} - (E^{(2)}_{1,0} \cup \ell^{(2)}_z)$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,1}$, there is only one choice for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(3)}_{1,0}, E^{(3)}_{1,1}, \ell^{(3)}_z$
- (-1)-curves: $E_{1,2}, E^{(3)}_{2,0}$
- with configuration as in Figure 50.

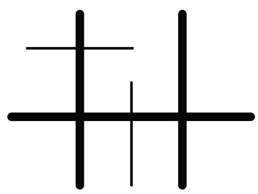


Figure 50

This is case $5E$.

Case 7B. We have $E = E_{1,1} - E^{(2)}_{1,0}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xz + \mu y^2$ is $E_{1,1}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^2\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on $E \cap E_{1,1}$, there are the following two choices for $p_{1,2}$ up to isomorphism:

(1) $p_{1,2} = E_{1,1} \cap C^{(2)}$ with $C = \mathcal{V}(xz + y^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{1,1}^{(3)}$
- (-1)-curves: $E_{1,2}, \ell_z^{(3)}$
- with configuration as in Figure 51.

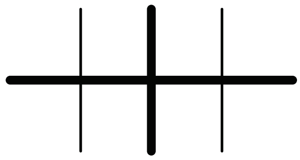


Figure 51

This is case $6E$.

(2) $p_{1,2} = E_{1,1} \cap \ell_z^{(2)}$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(3)}, E_{1,1}^{(3)}, \ell_z^{(3)}$
- (-1)-curves: $E_{1,2}$
- with configuration as in Figure 52.

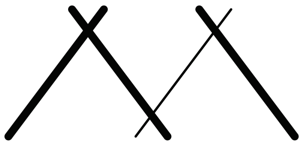


Figure 52

This is case $6F$.

Summarizing, we obtain

$$\begin{aligned} \mathcal{L}_3 \quad = \quad & \{\widetilde{X}_{1L,\alpha}, \widetilde{X}_{1O}, \widetilde{X}_{1N}, \widetilde{X}_{2N,\alpha}, \widetilde{X}_{2Q}, \widetilde{X}_{2P}, \widetilde{X}_{3N}, \widetilde{X}_{1K}, \widetilde{X}_{1J,\alpha}, \widetilde{X}_{1B}, \widetilde{X}_{2O}, \widetilde{X}_{2B}, \widetilde{X}_{2F}, \\ & \widetilde{X}_{3E}, \widetilde{X}_{4E}, \widetilde{X}_{4M}, \widetilde{X}_{4H}, \widetilde{X}_{1G}, \widetilde{X}_{2J}, \widetilde{X}_{3F}, \widetilde{X}_{3K}, \widetilde{X}_{3I}, \widetilde{X}_{4K}, \widetilde{X}_{4I}, \widetilde{X}_{5E}, \widetilde{X}_{6E}, \widetilde{X}_{6F}\}. \end{aligned}$$

4.5. Height 4.

Case $2N$. This case exists only if $p = 2$.

We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.
- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with non-trivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{3,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{3,1}^{(4)}, E_{2,0}^{(4)}, E_{4,0}^{(4)}, \ell_x^{(4)}, \ell_{x-y}^{(4)}, \ell_{x+\alpha y}^{(4)}, C_1^{(4)}, C_2^{(4)}, C_3^{(4)}, C_4^{(4)}, C_5^{(4)}, C_6^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2), C_3 = \mathcal{V}(x^2y + xz^2 + \alpha yz^2)$, $C_4 = \mathcal{V}(x^2y + xz^2 + y^3 + \alpha yz^2), C_5 = \mathcal{V}(x^2y^2 + x^2z^2 + x^3y + \alpha^2y^2z^2)$, $C_6 = \mathcal{V}(xy^3 + x^2z^2 + x^3y + \alpha^2y^2z^2), \alpha \notin \{0, -1\}$
- with configuration as in Figure 35, that is, as in case $1J$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1J,\alpha'}$ for some α'.

Case 2Q. This case exists only if $p = 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with non-trivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ $= \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{3,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}$, $\ell_x^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{3,1}^{(4)}, E_{2,0}^{(4)}, E_{4,0}^{(4)}$, $\ell_{x-y}^{(4)}, C_1^{(4)}, C_2^{(4)}, C_3^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$, $C_3 = \mathcal{V}(xz^2 + x^2y + y^3)$
- with configuration as in Figure 34, that is, as in case 1K.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1K}$.

Case 2P. This case exists only if $p = 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : \mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with non-trivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ $= \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}$, $\ell_x^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}, C_1^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, E_{3,0}^{(4)}, E_{4,0}^{(4)}$, $\ell_{x-y}^{(4)}, C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- with configuration as in Figure 29, that is, as in case 1N.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1N}$.

Case 3N. This case exists only if $p = 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with non-trivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$
$= \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,0}^{(4)}, E_{3,0}^{(4)}, E_{4,0}^{(4)}$, $\ell_x^{(4)}, \ell_{x-y}^{(4)}, C_1^{(4)}, C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- with configuration as in Figure 38, that is, as in case $2O$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2O}$.

Case $2O$. This case exists only if $p = 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu]$.

Note that there is only one point on $E \cap E_{1,2}$ with non-trivial stabilizer, hence we have the following unique choice for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$
$= \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{3,0}^{(4)}, E_{1,1}^{(4)}$, $E_{1,2}^{(4)}, \ell_z^{(4)}, C_1^{(4)}, C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, E_{3,1}^{(4)}, \ell_x^{(4)}$, $\ell_y^{(4)}, \ell_{x-y}^{(4)}$
- with configuration as in Figure 29, that is, as in case $1N$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1N}$.

Case $2B$. We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_y^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$.

Hence, we have the following unique choice for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2y + z^3)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{3,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, E_{3,1}^{(4)}, \ell_x^{(4)}, \ell_{x-y}^{(4)}$
- with configuration as in Figure 53.

This is case $1E$.

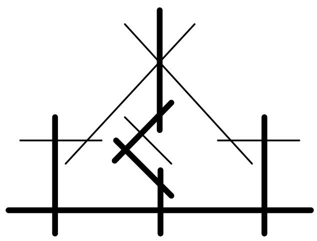


Figure 53

Case 2F. We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)} \cup \ell_x^{(3)} \cup \ell_y^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$.
- $\lambda x y^2 + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$.

Note that the involution $x \leftrightarrow y$ of $\mathbb{P}^2$ lifts to an automorphism of $\widetilde{X}$ interchanging $E_{1,2}$ and $E_{2,2}$. Moreover, since $\widetilde{X}$ has degree 2, we are only allowed to blow up one more point. Hence, we have the following unique choice for $p_{1,3}, p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2 y + z^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_x^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}, E_{3,0}^{(4)}$
- with configuration as in Figure 44, that is, as in case 1G.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1G}$.

Case 3E. We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_y^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2 y + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$.

Hence, we have the following unique choice for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2 y + z^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, E_{3,0}^{(4)}, \ell_x^{(4)}$
- with configuration as in Figure 45, that is, as in case 2J.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2J}$.

Case 4E. This case exists only if $p \neq 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2 y + x z^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - c\lambda]$

In particular, the stabilizer of every point on $E \cap E_{1,2}$ is trivial, hence this case does not lead to additional weak del Pezzo surfaces with global vector fields.

Case 4M. This case exists only if $p = 2$.
We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$.

- $\lambda(x^2y + xz^2) + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i\mu + c\lambda]$

In particular, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,2}$, so there is the following unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}$ with $C_1 = \mathcal{V}(xy + z^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^2 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,0}^{(4)}, E_{3,0}^{(4)}, \ell_y^{(4)}, C_1^{(4)}, C_2^{(4)}$ with $C_2 = \mathcal{V}(xy + y^2 + z^2)$
- with configuration as in Figure 33, that is, as in case 3*N*.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3N}$.

Case 4H. We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_y^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2y + \mu z^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : i^3\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,2}$, there is the following unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2y + z^3)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 3 \\ \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| i^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 3$ and $p = 3$ simultaneously:
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_y^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,0}^{(4)}, E_{3,0}^{(4)}$
- with configuration as in Figure 46, that is, as in case 3*F* or 3*K*.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3F}$ if $p \neq 3$, and $\widetilde{X}' \cong \widetilde{X}_{3K}$ if $p = 3$.

Case 2J. This case exists only if $p = 3$.
We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$.

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu]$.
- $\lambda(xy^2 + yz^2) + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu]$.

Note that $\widetilde{X}$ has degree 2, hence we are only allowed to blow up one more point. Moreover, there is a unique point on $E \cap E_{1,2}$ and on $E \cap E_{2,2}$ with non-trivial stabilizer. Therefore, we have the following two possibilities for $p_{1,3}$ and $p_{2,3}$:

(1) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{2,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{2,3}, E_{1,2}^{(4)}, E_{3,0}^{(4)}, \ell_y^{(4)}, C^{(4)}$
- with configuration as in Figure 53, that is, as in case $1E$.

Blowing down the (-1)-curve in Figure 53 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{2B}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1E}$ by Remark 3.4.

(2) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}, E_{3,0}^{(4)}, \ell_y^{(4)}, C^{(4)}$
- with configuration as in Figure 53, that is, as in case $1E$.

Blowing down the (-1)-curve in Figure 53 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{2B}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1E}$ by Remark 3.4.

Case 3F. This case exists only if $p \neq 3$.
We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - 2f\lambda]$
- $\lambda(xy^2 + yz^2) + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - f\lambda]$

If $p \neq 2$, then $\mathrm{Aut}^0_{\widetilde{X}}$ acts simply transitively on both $E \cap E_{1,2}$ and $E \cap E_{2,2}$, hence we cannot blow up $\widetilde{X}$ any further and still obtain a weak del Pezzo surface with global vector fields. If $p = 2$, then $\mathrm{Aut}^0_{\widetilde{X}}$ still acts transitively on $E \cap E_{2,2}$, but now it acts trivially on $E \cap E_{1,2}$. This leads to the following possibilities for $p_{1,3}$:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + \alpha y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2,3 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}, \ell_x^{(4)}$
- with configuration as in Figure 54.

This is case $2R$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{2R,\alpha}$ depending on the parameter α.

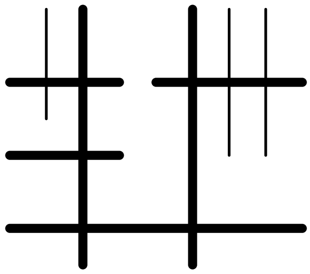


Figure 54

Case $3K$. This case exists only if $p = 3$.

We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$.

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu - 2ef\lambda]$
- $\lambda(xy^2 + yz^2) + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : \mu - ef\lambda]$

Note that $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on both $E \cap E_{1,2}$ and $E \cap E_{2,2}$. The stabilizer of every point on $E \cap E_{1,2}$ is isomorphic to μ_3 and this μ_3 has a unique fixed point on $E \cap E_{2,2}$. This leads to the following three possibilities for $p_{1,3}, p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}, p_{2,3} = E_{2,2} \cap C_2^{(2)}$ with $C_1 = \mathcal{V}(xz + y^2), C_2 = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, E_{2,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,3}, \ell_x^{(4)}, C_2^{(4)}, C_3^{(4)}$ with $C_3 = \mathcal{V}(x^2y^2 + x^3z + z^4)$
- with configuration as in Figure 53, that is, as in case $1E$.

Blowing down the (-1)-curve in Figure 53 that is not adjacent to any other (-1)-curve, we see that $\widetilde{X}'$ is a blow-up of $\widetilde{X}_{2B}$ in 1 point on E and $\widetilde{X}' \cong \widetilde{X}_{1E}$ by Remark 3.4.

(2) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy + z^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{2,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{2,3}, E_{1,2}^{(4)}, \ell_x^{(4)}, C^{(4)}$
- with configuration as in Figure 45, that is, as in case $2J$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{2J}$.

(3) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$
  $= \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}, \ell_x^{(4)}$
- with configuration as in Figure 54.

This is case $2K$.

Case 3I. We have $E = (E_{1,2} \cup E_{2,2}) - (E_{1,1}^{(3)} \cup E_{2,1}^{(3)} \cup \ell_x^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2z+xy^2)+\mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$
- $\lambda xy^2 + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^6\mu]$

Note that $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{2,2}$. If $p \neq 2$ (resp. $p = 2$), then $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively (resp. with two orbits) on $E \cap E_{1,2}$. We have the following five possibilities for $p_{1,3}, p_{2,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C_1^{(3)}, p_{2,3} = E_{2,2} \cap C_2^{(3)}$ with
$C_1 = \mathcal{V}(x^2z + xy^2 + y^3), C_2 = \mathcal{V}(xy^2 + \alpha z^3), \alpha \neq 0$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$
  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, E_{2,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,3}$
- with configuration as in Figure 55.

This is case $1Q$ and we see that we get a 1-dimensional family of such surfaces $\widetilde{X}_{1Q,\alpha}$ depending on the parameter α.

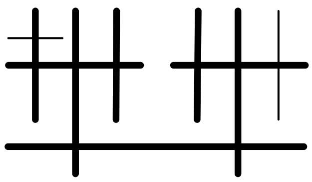


Figure 55

(2) $p_{1,3} = E_{1,2} \cap C_1^{(3)}, p_{2,3} = E_{2,2} \cap C_2^{(3)}$ with $C_1 = \mathcal{V}(xz + y^2), C_2 = \mathcal{V}(xy^2 + z^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, e^4 = 1 \right\} & \text{if } p = 2 \end{cases}$

  Hence, $\widetilde{X}'$ has global vector fields only if $p = 2$. Therefore, we assume $p = 2$ when describing the configuration of negative curves.
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, E_{2,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,3}$
- with configuration as in Figure 55.

This is case $1R$.

(3) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy^2 + z^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, e^4 = 1 \right\} & \text{if } p = 2 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{2,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{2,3}, E_{1,2}^{(4)}$
- with configuration as in Figure 56.

This is case $2H$ if $p \neq 2$, and case $2V$ if $p = 2$.

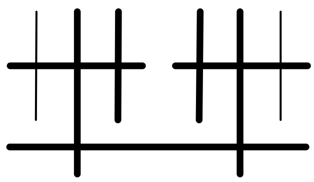


Figure 56

(4) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}$
- with configuration as in Figure 57.

This is case $2G$ if $p \neq 2$, and case $2U$ if $p = 2$.

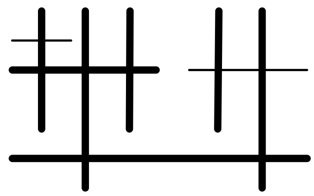


Figure 57

(5) Let $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{2,1}^{(4)}, E_{1,2}^{(4)}, \ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,2}^{(4)}$
- with configuration as in Figure 57.

This is case $2T$.

Case 4K. We have $E = E_{2,2} - (E_{2,1}^{(3)} \cup \ell_x^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda xy^2 + \mu z^3$ is $E_{2,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : i^3\mu]$

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{2,2}$, there is a unique possibility for $p_{2,3}$ up to isomorphism:

(1) $p_{2,3} = E_{2,2} \cap C^{(3)}$ with $C = \mathcal{V}(xy^2 + z^3)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ $= \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^2 = i^3 \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{2,1}^{(4)}, E_{2,2}^{(4)}$, $\ell_x^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{2,3}, E_{1,1}^{(4)}$
- with configuration as in Figure 47, that is, as in case $3I$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3I}$.

Case 4I. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$

If $p \neq 2$, then $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,2}$, while if $p = 2$, then the $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on $E \cap E_{1,2}$. Hence, if $p = 2$, there is only one possibility for $p_{1,3}$ and if $p = 2$, there are two possibilities up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, \ell_x^{(4)}$
- with configuration as in Figure 58.

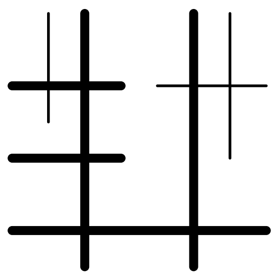


Figure 58

This is case $3G$ if $p \neq 2$, and case $3P$ if $p = 2$.

(2) Let $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{2,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,1}^{(4)}, \ell_x^{(4)}$
- with configuration as in Figure 58.

This is case $3O$.

Case 5E. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2z+xy^2)+\mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - 2ef\lambda]$

As in the previous case, if $p \neq 2$, there is only one possibility for $p_{1,3}$ up to isomorphism, and if $p = 2$, there are two possibilities up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$

We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:

- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,0}^{(4)}$
- with configuration as in Figure 59.

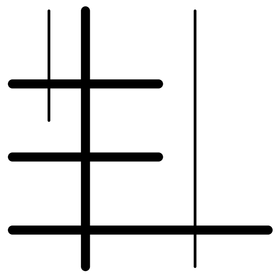


Figure 59

This is case 4J if $p \neq 2$, and case 4O if $p = 2$.

(2) Let $p = 2$ and $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + xy^2 + y^3)$.

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}, E_{2,0}^{(4)}$
- with configuration as in Figure 59.

This is case 4N.

Case 6E. We have $E = E_{1,2} - E_{1,1}^{(3)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x^2z + xy^2) + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^3\mu - be^2\lambda - 2ef\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,2}$, there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & -2fe^{-1} & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}$
- (-1)-curves: $E_{1,3}, \ell_z^{(4)}$
- with configuration as in Figure 50, that is, as in case 5E.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{5E}$.

Case 6F. We have $E = E_{1,2} - (E_{1,1}^{(3)} \cup \ell_z^{(3)})$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2z + \mu y^3$ is $E_{1,2}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [i\lambda : e^3\mu]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,2}$, there is a unique possibility for $p_{1,3}$ up to isomorphism:

(1) $p_{1,3} = E_{1,2} \cap C^{(3)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(4)}, E_{1,1}^{(4)}, E_{1,2}^{(4)}, \ell_z^{(4)}$
- (-1)-curves: $E_{1,3}$
- with configuration as in Figure 60.

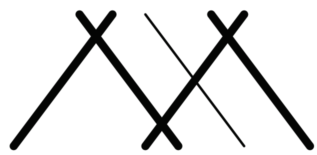


Figure 60

This is case $5F$.

Summarizing, we obtain

$$\begin{aligned} \mathcal{L}_4 \quad = \quad & \{\widetilde{X}_{1E}, \widetilde{X}_{2R,\alpha}, \widetilde{X}_{2K}, \widetilde{X}_{1Q,\alpha}, \widetilde{X}_{1R}, \widetilde{X}_{2H}, \widetilde{X}_{2V}, \widetilde{X}_{2G}, \\ & \widetilde{X}_{2U}, \widetilde{X}_{2T}, \widetilde{X}_{3G}, \widetilde{X}_{3P}, \widetilde{X}_{3O}, \widetilde{X}_{4J}, \widetilde{X}_{4O}, \widetilde{X}_{4N}, \widetilde{X}_{5F}\}. \end{aligned}$$

4.6. Height 5.

Case $2R$. This case exists only if $p = 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x + \alpha y)^2(xz + y^2 + \alpha yz) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (\alpha f + f^2)\lambda]$.

Therefore, if $\alpha \neq 0$, then the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence there is no way of further blowing up $\widetilde{X}$ and still obtaining a weak del Pezzo surface with global vector fields. If $\alpha = 0$, then there is the following unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| f^2 = 0 \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,2}^{(5)}, \ell_x^{(5)}, C_1^{(5)}, C_2^{(5)}$ with $C_2 = \mathcal{V}(x^2y^2 + x^3z + z^4)$
- with configuration as in Figure 61.

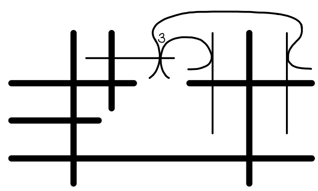


Figure 61

This is case $1M$.

Case $2K$. This case exists only if $p = 3$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$.

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e\mu]$.

Note that there is a unique point on $E \cap E_{1,3}$ with non-trivial stabilizer. This leads to the following unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(xz + y^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,2}^{(5)}, \ell_x^{(5)}, C_1^{(5)}, C_2^{(5)}$ with $C_2 = \mathcal{V}(x^2y^2 + x^3z + z^4 + 2xyz^2)$
- with configuration as in Figure 62.

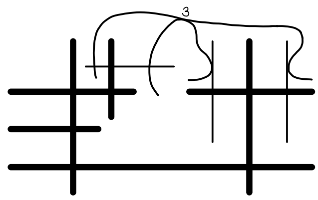


Figure 62

This is case $1F$.

Case 2H. This case exists only if $p \neq 2$.
We have $E = E_{2,3} - E_{2,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda y(xy^2 + z^3) + \mu z^4$ is $E_{2,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu - 2f\lambda]$.

In particular, since $p \neq 2$ the stabilizer of every point on $E \cap E_{2,3}$ is trivial, hence there is no way of further blowing up $\widetilde{X}$ and obtaining a weak del Pezzo surface with global vector fields.

Case 2V. This case exists only if $p = 2$.
We have $E = E_{2,3} - E_{2,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^4 = 1 \right\}$.

- $\lambda y(xy^2 + z^3) + \mu z^4$ is $E_{2,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^3\lambda : \mu]$.

This leads to the following possibilities for $p_{1,4}$:

(1) $p_{2,4} = E_{2,3} \cap C^{(4)}$ with $C = \mathcal{V}(xy^3 + yz^3 + \alpha z^4), \alpha \neq 0$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{2,2}^{(5)}, E_{2,3}^{(5)}, \ell_x^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{2,4}, E_{1,2}^{(5)}$
- with configuration as in Figure 55, that is, as in case $1Q$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1Q,\alpha'}$ for some α'.

(2) $p_{2,4} = E_{2,3} \cap C^{(4)}$ with $C = \mathcal{V}(xy^2 + z^3)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^4 = 1 \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{2,2}^{(5)}, E_{2,3}^{(5)}, \ell_x^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{2,4}, E_{1,2}^{(5)}$
- with configuration as in Figure 55, that is, as in case $1R$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1R}$.

Case 2G. This case exists only if $p \neq 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^2\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,3}$ such that the identity component of its stabilizer is non-trivial. This leads to the following unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_x^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,2}^{(5)}, C^{(5)}$
- with configuration as in Figure 63.

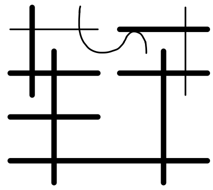


Figure 63

This is case 1C.

Case 2U. This case exists only if $p = 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz+y^2)+\mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^4\mu+f^2\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,3}$, there is a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ $= \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, f^2 = 0 \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{2,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_x^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,2}^{(5)}, C^{(5)}$
- with configuration as in Figure 63.

This is case 1P.

Case 2T. This case exists only if $p = 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x + y)(x^2z + xy^2 + y^3 + y^2z) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (f + f^2)\lambda]$.

Note that the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence we cannot blow up further and still obtain a weak del Pezzo surface with global vector fields.

Case 3G. This case exists only if $p \neq 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^2\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,3}$ for which the identity component of the stabilizer is non-trivial. This leads to the following unique possibility for $p_{1,4}$:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,1}^{(5)}, \ell_x^{(5)}, C^{(5)}$
- with configuration as in Figure 64.

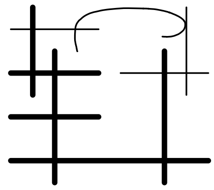


Figure 64

This is case $2C$.

Case $3P$. This case exists only if $p = 2$.

We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz+y^2)+\mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^4\mu+f^2\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,3}$, there is a unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$ $= \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \,\middle|\, f^2 = 0 \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{2,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,1}^{(5)}, \ell_x^{(5)}, C^{(5)}$
- with configuration as in Figure 64.

This is case $2S$.

Case $3O$. This case exists only if $p = 2$.

We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x+y)(x^2z+xy^2+y^3+y^2z)+\mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (f+f^2)\lambda]$.

In particular, the identity component of the stabilizer of every point on $E \cap E_{1,3}$ is trivial, hence we cannot blow up further.

Case $4J$. This case exists only if $p \neq 2$.

We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz+y^2)+\mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^2\mu + c\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,3}$, we have the following unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,0}^{(5)}, C^{(5)}$
- with configuration as in Figure 58, that is, as in case $3G$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3G}$.

Case 4O. This case exists only if $p = 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(xz + y^2) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e^2\lambda : e^4\mu + (ce^2 + f^2)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,3}$, we have the following unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & f^2e^{-2} \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,0}^{(5)}, C^{(5)}$
- with configuration as in Figure 58, that is, as in case $3P$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3P}$.

Case 4N. This case exists only if $p = 2$.
We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda(x + y)(x^2z + xy^2 + y^3 + y^2z) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (c + f + f^2)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,3}$, we have the following unique possibility for $p_{1,4}$ up to isomorphism:

(1) $p_{1,4} = E_{1,3} \cap C_1^{(4)}$ with $C_1 = \mathcal{V}(x^2z + xy^2 + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & f+f^2 \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}, E_{2,0}^{(5)}, C_2^{(5)}$ with $C_2 = \mathcal{V}(xz + yz + y^2)$
- with configuration as in Figure 58, that is, as in case $3O$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{3O}$.

Case 5F. We have $E = E_{1,3} - E_{1,2}^{(4)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x(x^2z + y^3) + \mu y^4$ is $E_{1,3}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e\mu - 2b\lambda]$.

Therefore, if $p \neq 2$, we have one unique possibility for $p_{1,4} \in E \cap E_{1,3}$, while if $p = 2$, there are two possibilities:

(1) $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2 \\ \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 2 \end{cases}$

We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2$ and $p = 2$ simultaneously:

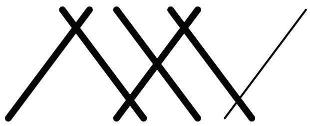


Figure 65

- (-2)-curves: $E_{1,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}$
- with configuration as in Figure 65.

This is case $4L$ if $p \neq 2$, and case $4Q$ if $p = 2$.

(2) Let $p = 2$ and $p_{1,4} = E_{1,3} \cap C^{(4)}$ with $C = \mathcal{V}(x^3z + xy^3 + y^4)$.

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(5)}, E_{1,1}^{(5)}, E_{1,2}^{(5)}, E_{1,3}^{(5)}, \ell_z^{(5)}$
- (-1)-curves: $E_{1,4}$
- with configuration as in Figure 65.

This is case $4P$.

Summarizing, we obtain

$$\mathcal{L}_5 \quad = \quad \{\widetilde{X}_{1M}, \widetilde{X}_{1F}, \widetilde{X}_{1C}, \widetilde{X}_{1P}, \widetilde{X}_{2C}, \widetilde{X}_{2S}, \widetilde{X}_{4L}, \widetilde{X}_{4Q}, \widetilde{X}_{4P}\}.$$

4.7. Height 6.

Case $2C$. This case exists only if $p \neq 2$.

We have $E = E_{1,4} - E_{1,3}^{(5)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^3(xz + y^2) + \mu y^5$ is $E_{1,4}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Note that if $p \neq 3$, then there is a unique point on $E \cap E_{1,4}$ such that the identity component of its stabilizer is non-trivial. If $p = 3$, this identity component is non-trivial for every point. In all characteristics, the action of $\mathrm{Aut}^0_{\widetilde{X}}$ on $E \cap E_{1,4}$ has two orbits. Hence, we have the following two possibilities for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C_1^{(5)}$ with $C_1 = \mathcal{V}(x^4z + x^3y^2 + y^5)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \{\mathrm{id}\} & \text{if } p \neq 2, 3 \\ \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| e^3 = 1 \right\} & \text{if } p = 3 \end{cases}$

Hence, $\widetilde{X}'$ has global vector fields only if $p = 3$. Therefore, we assume $p = 3$ when describing the configuration of negative curves.

- (-2)-curves: $E_{1,0}^{(6)}, E_{2,0}^{(6)}, E_{1,1}^{(6)}, E_{1,2}^{(6)}, E_{1,3}^{(6)}, E_{1,4}^{(6)}, \ell_z^{(6)}$
- (-1)-curves: $E_{1,5}, E_{2,1}^{(6)}, \ell_x^{(6)}, C_2^{(6)}, C_3^{(6)}$ with $C_2 = \mathcal{V}(xz + y^2)$, $C_3 = \mathcal{V}(xy^4 - xyz^3 - x^2y^2z + x^3z^2 - y^3z^2 - z^5)$
- with configuration as in Figure 62, that is, as in case $1F$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1F}$.

(2) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{2,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z, C^{(6)}$
- (-1)-curves: $E_{1,5}, E^{(6)}_{2,1}, \ell^{(6)}_x$
- with configuration as in Figure 63, that is, as in case $1C$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1C}$.

Case $2S$. This case exists only if $p = 2$.
We have $E = E_{1,4} - E^{(5)}_{1,3}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| f^2 = 0 \right\}$.

- $\lambda x^3(xz + y^2) + \mu y^5$ is $E_{1,4}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts on $E \cap E_{1,4}$ with two orbits, we have the following two possibilities for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C_1^{(5)}$ with $C_1 = \mathcal{V}(x^4z + x^3y^2 + y^5)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| f^2 = 0 \right\}$
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{2,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z$
- (-1)-curves: $E_{1,5}, E^{(6)}_{2,1}, \ell^{(6)}_x, C^{(6)}_2, C^{(6)}_3$ with $C_2 = \mathcal{V}(xz + y^2)$, $C_3 = \mathcal{V}(xy^4 + x^3z^2 + z^5)$
- with configuration as in Figure 61, that is, as in case $1M$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1M}$.

(2) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(xz + y^2)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| f^2 = 0 \right\}$
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{2,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z, C^{(6)}$
- (-1)-curves: $E_{1,5}, E^{(6)}_{2,1}, \ell^{(6)}_x$
- with configuration as in Figure 63, that is, as in case $1P$.

By Corollary 3.3, we have $\widetilde{X}' \cong \widetilde{X}_{1P}$.

Case $4L$. This case exists only if $p \neq 2$.
We have $E = E_{1,4} - E^{(5)}_{1,3}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(x^2z + y^3) + \mu y^5$ is $E_{1,4}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : e^3\mu - 3f\lambda]$.

In particular, if $p \neq 3$, then $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,4}$ and we have only one choice for $p_{1,5}$ up to isomorphism, and if $p = 3$, then $\mathrm{Aut}^0_{\widetilde{X}}$ acts with two orbits on $E \cap E_{1,4}$, hence we have two choices up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \begin{cases} \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p \neq 2,3 \\ \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\} & \text{if } p = 3 \end{cases}$

  We describe the configurations of negative curves on $\widetilde{X}'$ for $p \neq 2,3$ and $p = 3$ simultaneously:
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z$
- (-1)-curves: $E_{1,5}$
- with configuration as in Figure 66.

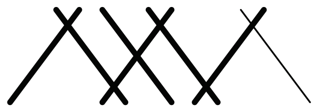


Figure 66

This is case $3J$ if $p \neq 2,3$, and case $3M$ if $p = 3$.

(2) Let $p = 3$ and $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^4z + x^2y^3 + y^5)$.

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z$
- (-1)-curves: $E_{1,5}$
- with configuration as in Figure 66.

This is case $3L$.

<u>*Case* $4Q$</u>. This case exists only if $p = 2$.

We have $E = E_{1,4} - E^{(5)}_{1,3}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(x^2z + y^3) + \mu y^5$ is $E_{1,4}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [e\lambda : e^3\mu + (b^2e + f)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,4}$, there is a unique choice for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z$
- (-1)-curves: $E_{1,5}$
- with configuration as in Figure 66.

This is case $3R$.

<u>*Case* $4P$</u>. This case exists only if $p = 2$.

We have $E = E_{1,4} - E^{(5)}_{1,3}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x(x^3z + xy^3 + y^4) + \mu y^5$ is $E_{1,4}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (b + b^2 + f)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,4}$, we have the following unique choice for $p_{1,5}$ up to isomorphism:

(1) $p_{1,5} = E_{1,4} \cap C^{(5)}$ with $C = \mathcal{V}(x^3z + xy^3 + y^4)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2+b \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(6)}_{1,0}, E^{(6)}_{1,1}, E^{(6)}_{1,2}, E^{(6)}_{1,3}, E^{(6)}_{1,4}, \ell^{(6)}_z$
   - (-1)-curves: $E_{1,5}$
   - with configuration as in Figure 66.

   This is case $3Q$.

Summarizing, we obtain

$$\mathcal{L}_6 \quad = \quad \{\widetilde{X}_{3J}, \widetilde{X}_{3M}, \widetilde{X}_{3L}, \widetilde{X}_{3R}, \widetilde{X}_{3Q}\}.$$

4.8. Height 7.

<u>*Case* $3J$</u>. This case exists only if $p \neq 2, 3$.
We have $E = E_{1,5} - E^{(6)}_{1,4}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu + 2c\lambda]$.

Since $p \neq 2$, $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,5}$, so there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(7)}_{1,0}, E^{(7)}_{1,1}, E^{(7)}_{1,2}, E^{(7)}_{1,3}, E^{(7)}_{1,4}, E^{(7)}_{1,5}, \ell^{(7)}_z$
   - (-1)-curves: $E_{1,6}$
   - with configuration as in Figure 67.

   This is case $2I$.

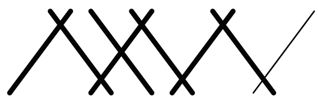


Figure 67

<u>*Case* $3M$</u>. This case exists only if $p = 3$.
We have $E = E_{1,5} - E^{(6)}_{1,4}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu + 2c\lambda]$.

As in the previous case, there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(7)}_{1,0}, E^{(7)}_{1,1}, E^{(7)}_{1,2}, E^{(7)}_{1,3}, E^{(7)}_{1,4}, E^{(7)}_{1,5}, \ell^{(7)}_z$
   - (-1)-curves: $E_{1,6}$
   - with configuration as in Figure 67.

   This is case $2M$.

Case 3L. This case exists only if $p = 3$.
We have $E = E_{1,5} - E_{1,4}^{(6)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x(x^4z + x^2y^3 + y^5) + \mu y^6$ is $E_{1,5}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + 2c\lambda]$.

As in the previous case, there is a unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^4z + x^2y^3 + y^5)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(7)}, E_{1,1}^{(7)}, E_{1,2}^{(7)}, E_{1,3}^{(7)}, E_{1,4}^{(7)}, E_{1,5}^{(7)}, \ell_z^{(7)}$
- (-1)-curves: $E_{1,6}$
- with configuration as in Figure 67.

This is case 2L.

Case 3R. This case exists only if $p = 2$.
We have $E = E_{1,5} - E_{1,4}^{(6)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^3(x^2z + y^3) + \mu y^6$ is $E_{1,5}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^3\mu]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ has two orbits on $E \cap E_{1,5}$, we have the following two choices for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^5z + x^3y^3 + y^6)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2 \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(7)}, E_{1,1}^{(7)}, E_{1,2}^{(7)}, E_{1,3}^{(7)}, E_{1,4}^{(7)}, E_{1,5}^{(7)}, \ell_z^{(7)}$
- (-1)-curves: $E_{1,6}$
- with configuration as in Figure 67.

This is case 2X.

(2) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^2z + y^3)$
- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E_{1,0}^{(7)}, E_{1,1}^{(7)}, E_{1,2}^{(7)}, E_{1,3}^{(7)}, E_{1,4}^{(7)}, E_{1,5}^{(7)}, \ell_z^{(7)}$
- (-1)-curves: $E_{1,6}$
- with configuration as in Figure 67.

This is case 2Y.

Case 3Q. This case exists only if $p = 2$.
We have $E = E_{1,5} - E_{1,4}^{(6)}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2+b \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(x^3z + xy^3 + y^4) + \mu y^6$ is $E_{1,5}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (b^2 + b)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,5}$, we have the following unique choice for $p_{1,6}$ up to isomorphism:

(1) $p_{1,6} = E_{1,5} \cap C^{(6)}$ with $C = \mathcal{V}(x^3z + xy^3 + y^4)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(7)}_{1,0}, E^{(7)}_{1,1}, E^{(7)}_{1,2}, E^{(7)}_{1,3}, E^{(7)}_{1,4}, E^{(7)}_{1,5}, \ell^{(7)}_z$
   - (-1)-curves: $E_{1,6}$
   - with configuration as in Figure 67.

   This is case $2W$.

Summarizing, we obtain

$$\mathcal{L}_7 \quad = \quad \{\widetilde{X}_{2I}, \widetilde{X}_{2M}, \widetilde{X}_{2L}, \widetilde{X}_{2X}, \widetilde{X}_{2Y}, \widetilde{X}_{2W}\}.$$

4.9. Height 8.

Case $2I$. This case exists only if $p \neq 2, 3$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^4\mu]$.

Since $p \neq 2$, there is a unique point on $E \cap E_{1,6}$ whose stabilizer has non-trivial identity component. This leads to the following unique choice for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(8)}_{1,0}, E^{(8)}_{1,1}, E^{(8)}_{1,2}, E^{(8)}_{1,3}, E^{(8)}_{1,4}, E^{(8)}_{1,5}, E^{(8)}_{1,6}, \ell^{(8)}_z$
   - (-1)-curves: $E_{1,7}$
   - with configuration as in Figure 68.

   This is case $1D$.

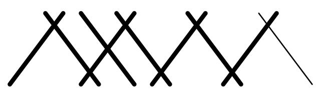


Figure 68

Case $2M$. This case exists only if $p = 3$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^4\mu]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts with two orbits on $E \cap E_{1,6}$, we have the following two choices for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^6z + x^4y^3 + y^7)$
   - $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
   - (-2)-curves: $E^{(8)}_{1,0}, E^{(8)}_{1,1}, E^{(8)}_{1,2}, E^{(8)}_{1,3}, E^{(8)}_{1,4}, E^{(8)}_{1,5}, E^{(8)}_{1,6}, \ell^{(8)}_z$
   - (-1)-curves: $E_{1,7}$
   - with configuration as in Figure 68.

   This is case $1H$.

(2) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(8)}_{1,0}, E^{(8)}_{1,1}, E^{(8)}_{1,2}, E^{(8)}_{1,3}, E^{(8)}_{1,4}, E^{(8)}_{1,5}, E^{(8)}_{1,6}, \ell^{(8)}_z$
- (-1)-curves: $E_{1,7}$
- with configuration as in Figure 68.

This is case $1I$.

Case $2L$. This case exists only if $p = 3$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^2(x^4z + x^2y^3 + y^5) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + f\lambda]$.

Hence, the stabilizer of every point on $E \cap E_{1,6}$ is trivial, therefore we cannot blow up $\widetilde{X}$ further and still obtain a weak del Pezzo surface with global vector fields.

Case $2X$. This case exists only if $p = 2$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & 1 & b^2 \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x(x^5z + x^3y^3 + y^6) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + (b + b^4)\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,6}$, there is a unique choice for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^5z + x^3y^3 + y^6)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$
- (-2)-curves: $E^{(8)}_{1,0}, E^{(8)}_{1,1}, E^{(8)}_{1,2}, E^{(8)}_{1,3}, E^{(8)}_{1,4}, E^{(8)}_{1,5}, E^{(8)}_{1,6}, \ell^{(8)}_z$
- (-1)-curves: $E_{1,7}$
- with configuration as in Figure 68.

This is case $1S$.

Case $2Y$. This case exists only if $p = 2$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^4(x^2z + y^3) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : e^4\mu + b^4\lambda]$.

Since $\mathrm{Aut}^0_{\widetilde{X}}$ acts transitively on $E \cap E_{1,6}$, there is a unique choice for $p_{1,7}$ up to isomorphism:

(1) $p_{1,7} = E_{1,6} \cap C^{(7)}$ with $C = \mathcal{V}(x^2z + y^3)$

- $\mathrm{Aut}^0_{\widetilde{X}'}(R)$
  $= \left\{ \begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix} \in \mathrm{PGL}_3(R) \middle| b^4 = 0 \right\}$
- (-2)-curves: $E^{(8)}_{1,0}, E^{(8)}_{1,1}, E^{(8)}_{1,2}, E^{(8)}_{1,3}, E^{(8)}_{1,4}, E^{(8)}_{1,5}, E^{(8)}_{1,6}, \ell^{(8)}_z$
- (-1)-curves: $E_{1,7}$
- with configuration as in Figure 68.

This is case $1T$.

<u>*Case* $2W$</u>. This case exists only if $p = 2$.
We have $E = E_{1,6} - E^{(7)}_{1,5}$ and $\mathrm{Aut}^0_{\widetilde{X}}(R) = \left\{ \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(R) \right\}$.

- $\lambda x^3(x^3z + xy^3 + y^4) + \mu y^7$ is $E_{1,6}$-adapted and $\mathrm{Aut}^0_{\widetilde{X}}(R)$ acts as $[\lambda : \mu] \mapsto [\lambda : \mu + c\lambda]$.

In particular, the identity component of the stabilizer of every point on $E \cap E_{1,6}$ is trivial, hence we cannot blow up further and still obtain a weak del Pezzo surface with global vector fields.

Summarizing, we obtain

$$\mathcal{L}_8 \quad = \quad \{\widetilde{X}_{1D}, \widetilde{X}_{1H}, \widetilde{X}_{1I}, \widetilde{X}_{1S}, \widetilde{X}_{1T}\}.$$

□

CHAPTER III

RDP del Pezzo surfaces with global vector fields in odd characteristic

Up to minor modifications, this chapter is taken from the article "RDP del Pezzo surfaces with global vector fields in odd characteristic", which is joint work of the author with G. Martin. Currently, the article is submitted to a peer-reviewed journal and can be found on the ArXiv (see [MS22]).

1. Motivation and summary

We are working over an algebraically closed field k of characteristic $p \geq 0$. Let X be a del Pezzo surface with at worst rational double points as singularities and let $\pi : \widetilde{X} \to X$ be its minimal resolution, so that, by definition, $\widetilde{X}$ is a weak del Pezzo surface. Since $-K_X$ is ample, Aut_X is an affine group scheme of finite type, hence its group of automorphisms $\mathrm{Aut}(X) := \mathrm{Aut}_X(k)$ is infinite if and only if the automorphism scheme Aut_X is positive-dimensional. Moreover, by Blanchard's Lemma [Bri17, Theorem 7.2.1], and since X is the anti-canonical model of $\widetilde{X}$, there is a closed immersion of group schemes $\pi_* : \mathrm{Aut}_{\widetilde{X}} \hookrightarrow \mathrm{Aut}_X$. We call X *equivariant*, if π_* is an isomorphism. Summarizing, for all characteristics, there is the following chain of implications:

$$|\mathrm{Aut}(X)| = |\mathrm{Aut}(\widetilde{X})| = \infty \implies H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0 \implies H^0(X, T_X) \neq 0 \tag{1.1}$$

Over the complex numbers, every RDP del Pezzo surface X is equivariant, so in particular we have $H^0(X, T_X) = H^0(\widetilde{X}, T_{\widetilde{X}})$, and by Cartier's theorem (see e.g. [Per76, Corollaire 4.2.8]) Aut_X^0 is smooth, hence it is positive-dimensional if and only if $H^0(X, T_X) \neq 0$. In other words, in characteristic 0, all implications in (1.1) are in fact equivalences.

In the previous Chapter II, we obtained the classification of weak del Pezzo surfaces with global vector fields over algebraically closed fields of arbitrary characteristic (over the complex numbers, an independent proof was given by Cheltsov and Prokhorov in [CP21]). By (1.1), this includes the classification of all RDP del Pezzo surfaces with infinite automorphism group, but we note that if $p = 2, 3$, there are RDP del Pezzo surfaces with finite automorphism group whose minimal resolution has global vector fields, so the first implication in (1.1) is not an equivalence precisely if $p = 2, 3$. In other words, we have

$$|\mathrm{Aut}(X)| = |\mathrm{Aut}(\widetilde{X})| = \infty \overset{p \neq 2,3}{\Longleftrightarrow} H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0 \implies H^0(X, T_X) \neq 0 \tag{1.2}$$

and a classification of $\widetilde{X}$ with $H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0$ in all characteristics.

Now, the missing piece is a classification of RDP del Pezzo surfaces X with $H^0(X, T_X) \neq 0$. What makes this subtle in positive characteristic is the existence of non-equivariant rational double points. Recall that the dual graph of the exceptional locus of the minimal resolution of a rational double point is a Dynkin diagram of type A_n, D_n, E_6, E_7, or E_8. In positive characteristic, the Dynkin diagram does not always determine the formal isomorphism class of the singularity, but for each fixed graph Γ_n, there are only finitely many isomorphism classes Γ_n^r of RDPs with resolution graph Γ_n. These isomorphism classes have been classified by Lipman [Lip69] and Artin [Art77]. As a first step towards the classification of RDP del Pezzo surfaces with global vector fields, we extend Hirokado's [Hir19] results on the liftability of vector fields to group scheme actions on RDP del Pezzo surfaces as follows:

THEOREM 1.1 (= Theorem 6.1). *Let X be an RDP del Pezzo surface and let $\pi : \widetilde{X} \to X$ be its minimal resolution. Assume that one of the following conditions holds:*

(1) $p \notin \{2, 3, 5, 7\}$,
(2) $p = 7$ and X does not contain an RDP of type A_6.
(3) $p = 5$ and X does not contain an RDP of type A_4 or E_8^0.
(4) $p = 3$ and X does not contain an RDP of type $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$ or E_8^1.
(5) $p = 2$ and X does not contain an RDP of type $A_1, A_3, A_5, A_7, D_n^r, E_6^0, E_7^0, E_7^1$, $E_7^2, E_7^3, E_8^0, E_8^1, E_8^2$ or E_8^3, where $n \leq 8$.

Then, $\mathrm{Aut}_X = \mathrm{Aut}_{\widetilde{X}}$*, and thus, in particular,* $H^0(\widetilde{X}, T_{\widetilde{X}}) = H^0(X, T_X)$*. Therefore,* $H^0(X, T_X) \neq 0$ *if and only if X is the anti-canonical model of one of the surfaces in the classification Tables 1, 2, 3, 4, 5 and 6 of Chapter II.*

Thus, in order to classify RDP del Pezzo surfaces X with global vector fields, we may restrict our attention to RDP del Pezzo surfaces containing a configuration Γ of RDPs excluded in Theorem 6.1. In Theorem 7.1, we give a criterion for X to be the blow-up of an RDP del Pezzo surface of higher degree with the same configuration Γ. In the language of the Minimal Model Program, this means that we give a sufficient criterion for the existence of a $K_{\widetilde{X}}$-negative extremal ray on $\widetilde{X}$ which lies in the orthogonal complement of the exceptional locus over Γ. Using Blanchard's Lemma, this allows us to set up an inductive argument for the classification of non-equivariant RDP del Pezzo surfaces X with RDP configuration Γ. This strategy will be carried out in Sections 8.1, 8.2, and 8.3, for characteristic $p = 7$, $p = 5$, and $p = 3$, respectively. The following theorem is obtained by combining Theorem 8.3, Theorem 8.6, and Theorem 8.8.

THEOREM 1.2. *Let X be an RDP del Pezzo surface and let $\pi : \widetilde{X} \to X$ be its minimal resolution. Assume that $H^0(X, T_X) \neq 0$. Then, the following hold:*

(1) If $p = 7$ and X contains an RDP of type A_6, then X is one of the 2 surfaces in Table 1.
(2) If $p = 5$ and X contains an RDP of type A_4 or E_8^0, then X is one of the 9 surfaces in Table 2.
(3) If $p = 3$ and X contains an RDP of type $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$ or E_8^1, then X is a member of one of the 56 families of surfaces in Tables 3, 4, 5, and 6.

For each of these surfaces, we have $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$ *and in particular* $\mathrm{Aut}_X \neq \mathrm{Aut}_{\widetilde{X}}$. *Moreover,* Aut_X^0 *is as described in the respective table.*

REMARK 1.3. The reason why we do not treat the case $p = 2$ is due to the sheer amount of RDP del Pezzo surfaces with global vector fields in characteristic 2. Indeed, by Theorem 6.1, it is unclear whether $\mathrm{Aut}_{\widetilde{X}} = \mathrm{Aut}_X$ as soon as X has a single node and in fact, even for the quadratic cone $\{x_0^2 - x_1x_2 = 0\} \subseteq \mathbb{P}^3$ in characteristic 2 it is not true that every vector field lifts to its minimal resolution; consider for example $x_3\partial_{x_0}$. However, in principle, our approach would also work if $p = 2$.

Comparing Tables 1, 2, 3, 4, 5, and 6 with the classification in Chapter II, we see that in characteristics $p = 3, 5$ and 7, there exists an RDP del Pezzo surface X with $H^0(X, T_X) \neq 0$ whose minimal resolution admits no non-trivial global vector fields. In other words, we have the following picture, where the implications from right to left hold *only* in the indicated characteristics:

(1.3)

$$|\mathrm{Aut}(X)| = |\mathrm{Aut}(\widetilde{X})| = \infty \quad \underset{\Longrightarrow}{\overset{p\neq 2,3}{\Longleftarrow}} \quad H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0 \quad \underset{\Longrightarrow}{\overset{p\neq 2,3,5,7}{\Longleftarrow}} \quad H^0(X, T_X) \neq 0.$$

d	singularities	equation of X	Aut_X^0
2	A_6	$w^2 = x^3y + y^3z + z^3x$	$\mu_7 : [\lambda x : \lambda^4 y : \lambda^2 z : w]$
1	$A_6 + A_1$	$y^2 = x^3 + ts^3x + t^5s$	$\mu_7 : [\lambda s : \lambda^4 t : x : y]$

Table 1. Non-equivariant RDP del Pezzo surfaces with global vector fields in characteristic 7

d	RDPs	equation(s) of X	Aut_X^0
5	A_4	$\begin{aligned} x_0x_2 - x_1^2 &= 0 \\ x_0x_3 - x_1x_4 &= 0 \\ x_2x_4 - x_1x_3 &= 0 \\ x_1x_2 + x_4^2 + x_0x_5 &= 0 \\ x_2^2 + x_3x_4 + x_1x_5 &= 0 \end{aligned}$	$\langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$ with $\alpha_5 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2\varepsilon^2 & 2\varepsilon^3 & \varepsilon & 2\varepsilon^4 \\ 0 & 0 & 1 & 2\varepsilon & 0 & -\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & \varepsilon & \varepsilon^2 & 1 & -2\varepsilon^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_4	$\begin{aligned} x_0x_1 - x_2x_3 &= 0 \\ x_0x_4 + x_1x_2 + x_3^2 &= 0 \end{aligned}$	$\langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$ with $\alpha_5 : \begin{pmatrix} 1 & -\varepsilon^3 & -2\varepsilon & 2\varepsilon^2 & 2\varepsilon^4 \\ 0 & 1 & 0 & 0 & 2\varepsilon \\ 0 & -\varepsilon^2 & 1 & -2\varepsilon & \varepsilon^3 \\ 0 & \varepsilon & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
3	A_4	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_0 = 0$	$\mu_5 : [x_0 : \lambda x_1 : \lambda^4 x_2 : \lambda^3 x_3]$
	$A_4 + A_1$	$x_0x_1x_3 + x_0x_2^2 + x_1^2x_2 = 0$	$\alpha_5 \rtimes \mathbb{G}_m$ with $\alpha_5 : \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & -2\varepsilon^3 \\ 0 & 1 & 2\varepsilon & -\varepsilon^2 \\ 0 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m : [x_0 : \lambda x_1 : \lambda^2 x_2 : \lambda^3 x_3]$
2	$A_4 + A_1$	$w^2 = x^4 + xy^2z + yz^3$	$\mu_5 : [x : \lambda y : \lambda^3 z : w]$
	$A_4 + A_2$	$w^2 = xy^3 + yz^3 + x^2z^2$	$\mu_5 : [\lambda^2 x : \lambda y : \lambda^3 z : w]$
1	$A_4 + A_2 + A_1$	$y^2 = x^3 + s^3tx + s^2t^4$	$\mu_5 : [s : \lambda t : \lambda^3 x : \lambda^2 y]$
	$2A_4$	$y^2 = x^3 + t^4x + s^5t$	$\alpha_5 \rtimes \mu_5 : [\lambda s + \varepsilon t : t : x : y]$
	E_8^0	$y^2 = x^3 + s^5t$	$\alpha_5 \rtimes \mathbb{G}_m : [\lambda s + \varepsilon t : \lambda^{-5} t : x : y]$

Table 2. Non-equivariant RDP del Pezzo surfaces with global vector fields in characteristic 5

d	RDPs	equation(s) of X	Aut^0_X
6	A_2	$\begin{array}{lcl} x_0x_5 - x_3x_4 & = & 0 \\ x_0x_6 - x_1x_4 & = & 0 \\ x_0x_6 - x_2x_3 & = & 0 \\ x_3x_6 - x_1x_5 & = & 0 \\ x_4x_6 - x_2x_5 & = & 0 \\ x_1x_6 + x_3^2 + x_3x_4 & = & 0 \\ x_2x_6 + x_3x_4 + x_4^2 & = & 0 \\ x_6^2 + x_3x_5 + x_4x_5 & = & 0 \\ x_1x_2 + x_0x_3 + x_0x_4 & = & 0 \end{array}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & -\varepsilon & 0 & 1 & 0 & 0 & 0 \\ -\varepsilon^2 & 0 & \varepsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon^2 & -\varepsilon^2 & -\varepsilon & \varepsilon & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{array}{lcl} x_0^2 - x_1x_5 & = & 0 \\ x_0x_2 - x_1x_4 & = & 0 \\ x_0x_3 - x_2x_4 & = & 0 \\ x_0x_4 - x_2x_5 & = & 0 \\ x_0x_5 - x_2x_6 & = & 0 \\ x_1x_3 - x_2^2 & = & 0 \\ x_3x_5 - x_4^2 & = & 0 \\ x_3x_6 - x_4x_5 & = & 0 \\ x_4x_6 - x_5^2 & = & 0 \end{array}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
5	A_2	$\begin{array}{lcl} x_0x_2 - x_1x_5 & = & 0 \\ x_0x_2 - x_3x_4 & = & 0 \\ x_0x_3 + x_1^2 + x_1x_4 & = & 0 \\ x_0x_5 + x_1x_4 + x_4^2 & = & 0 \\ x_3x_5 + x_1x_2 + x_2x_4 & = & 0 \end{array}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & \varepsilon & 0 & -\varepsilon^2 & -\varepsilon & -\varepsilon^2 \\ 0 & 1 & -\varepsilon^2 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 1 & 0 & 0 \\ 0 & 0 & -\varepsilon^2 & 0 & 1 & -\varepsilon \\ 0 & 0 & -\varepsilon & 0 & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{array}{lcl} x_0^2 - x_1x_4 & = & 0 \\ x_0x_2 - x_1x_3 & = & 0 \\ x_0x_3 - x_2x_4 & = & 0 \\ x_0x_4 - x_2x_5 & = & 0 \\ x_3x_5 - x_4^2 & = & 0 \end{array}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_2	$\begin{array}{lcl} x_0x_1 + x_2x_4 + x_3x_4 & = & 0 \\ x_0x_4 + x_1x_4 + x_2x_3 & = & 0 \end{array}$	$\mu_3 : [x_0 : x_1 : \lambda x_2 : \lambda x_3 : \lambda^2 x_4]$
	$A_2 + A_1$	$\begin{array}{lcl} x_0x_1 - x_2x_3 & = & 0 \\ x_1x_2 + x_2x_4 + x_3x_4 & = & 0 \end{array}$	$\alpha_3 \rtimes \mathbb{G}_m$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & 1 & \varepsilon & -\varepsilon & 0 \\ -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & 0 \\ -\varepsilon^2 & 0 & -\varepsilon & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m : [\lambda^2 x_0 : x_1 : \lambda x_2 : \lambda x_3 : x_4]$
	$A_2 + 2A_1$	$\begin{array}{lcl} x_0^2 - x_3x_4 & = & 0 \\ x_0x_3 - x_1x_2 & = & 0 \end{array}$	$\alpha_3 \rtimes \mathbb{G}_m^2$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m^2 : [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : \lambda_1\lambda_2 x_3 : (\lambda_1\lambda_2)^{-1} x_4]$

Table 3. Non-equivariant RDP del Pezzo surfaces of degree at least 4 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
3	A_2	$x_0^2x_1 + x_0x_1^2 + x_2^2x_3 + x_2x_3^2 = 0$	$\mu_3 : [x_0 : x_1 : \lambda x_2 : \lambda x_3]$
	$A_2 + 2A_1$	$x_0^2x_1 + x_0^2x_2 + x_0x_3^2 + x_1x_2x_3 = 0$	$\mu_3 : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$2A_2$	$x_0^3 + x_1x_2x_3 + x_0x_1^2 + ax_0^2x_1 = 0$ with $a^2 \neq 1$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_2 : x_1 : x_2 : a\varepsilon x_0 - \varepsilon x_1 - a\varepsilon^2 x_2 + x_3]$ $\alpha_3 : [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 - \varepsilon x_1 + x_2 - a\varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m : [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$2A_2 + A_1$	$x_0^3 + x_1x_2x_3 + x_0^2x_1 = 0$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_2 : x_1 : x_2 : \varepsilon x_0 - \varepsilon^2 x_2 + x_3]$ $\alpha_3 : [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 + x_2 - \varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m : [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$3A_2$	$x_0^3 + x_1x_2x_3 = 0$	$\alpha_3^3 \rtimes \mathbb{G}_m^2$ with $\alpha_3^3 : [x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 : x_1 : x_2 : x_3]$ $\mathbb{G}_m^2 : [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : (\lambda_1\lambda_2)^{-1} x_3]$
	A_5	$x_0^3 + x_0x_2x_3 + x_1^2x_2 + x_2^3 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mu_3 \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mu_3 : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$A_5 + A_1$	$x_0^3 + x_0x_2x_3 + x_1^2x_2 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mathbb{G}_m : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	E_6^0	$x_0^3 + x_1^2x_2 + x_2^2x_3 = 0$	$\langle G, \mathbb{G}_a^2 \rtimes \mathbb{G}_m \rangle$ with $\mathbb{G}_a : [x_0 + \varepsilon x_2 : x_1 : x_2 : -\varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a : [x_0 : x_1 + \varepsilon x_2 : x_2 : \varepsilon^3 x_1 - \varepsilon^2 x_2 + x_3]$ $\mathbb{G}_m : [x_0 : \lambda x_1 : x_2 : \lambda^{-2} x_2 : \lambda^4 x_3]$ and G non-commutative, $\lvert G \rvert = 27$, acting as $[x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_3 : x_1 + \varepsilon_1^3 x_3 : -\varepsilon_1^3 x_1 + x_2 + \varepsilon_1^6 x_3 : x_3]$ where $\varepsilon_1^9 = \varepsilon_2^3 = 0$
	E_6^1	$x_0^3 + x_1^3 + x_0x_1x_2 + x_2^2x_3 = 0$	$\langle \mu_3, \mathbb{G}_a^2 \rangle$ with $\mu_3 : [\lambda x_0 : \lambda^2 x_1 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 - \varepsilon x_2 : x_1 : x_2 : \varepsilon x_1 + \varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a : [x_0 : x_1 - \varepsilon x_2 : x_2 : \varepsilon x_0 + \varepsilon^3 x_2 + x_3]$

Table 4. Non-equivariant RDP del Pezzo surfaces of degree 3 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
2	A_2+3A_1	$w^2=z(xy(x+y)+z^3)$	$\mu_3:[x:y:\lambda z:\lambda^{-1}w]$
	A_2+A_3	$w^2=x^4+a^3x^2yz+xy^3+y^2z^2$ with $a^2\neq 1$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$A_2+A_3+A_1$	$w^2=x^2yz+xy^3+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_2+A_4	$w^2=(xz+y^2)^2+y^3z$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	$2A_2$	$w^2=x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $(b^3-a^2b^2)^2\neq a^3b^3, b\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$2A_2$	$w^2=(xz+y^2)^2+x^3z+a^6z^4$ with $a\neq 0$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	$2A_2+A_1$	$w^2=ax^2yz+xy^3+xz^3+y^2z^2$ with $a\neq 0,1$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$3A_2$	$w^2=y^4+x^2y^2+xz^3$	$\alpha_3^2\rtimes\mu_3:[x:y:\varepsilon_1x+\varepsilon_2y+\lambda z:w]$
	A_5	$w^2=x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $(b^3-a^2b^2)^2=a^3b^3, b\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2=x^4+ax^2yz+xy^3+xz^3$ with $a\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2=(xz+y^2)^2+x^3z$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	A_5	$w^2=z(z(xz+y^2)+x^3)$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	A_5+A_1	$w^2=x^2yz+xy^3+xz^3$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_1	$w^2=x^2yz+xy^3+xz^3+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_2	$w^2=x^2y^2+xz^3$	$\alpha_3^2\rtimes\mathbb{G}_m:[x:\lambda^3y:\varepsilon_1x+\varepsilon_2y+\lambda^2z:\lambda^3w]$
	E_6^0	$w^2=y^4+xz^3$	$\langle G,\mathbb{G}_m\rangle$ with $\mathbb{G}_m:[x:\lambda^3y:\lambda^4z:\lambda^6w]$ and G non-commutative, $\lvert G\rvert=27$, acting as $[x:y-\varepsilon_1^3x:\varepsilon_2x+\varepsilon_1y+z:w]$ where $\varepsilon_1^9=\varepsilon_2^3=0$
	E_6^1	$w^2=(y^3+z^3)x+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	E_7^0	$w^2=x^3y+xz^3$	$\langle\alpha_3,\mathbb{G}_a\rtimes\mathbb{G}_m\rangle$ with $\alpha_3:[x:y:z+\varepsilon y:w]$ $\mathbb{G}_a:[x:y+\varepsilon^3x:z-\varepsilon x:w]$ $\mathbb{G}_m:[x:\lambda^6y:\lambda^2z:\lambda^3w]$

Table 5. Non-equivariant RDP del Pezzo surfaces of degree 2 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
1	A_2+D_4	$y^2=x^3+stx^2+a^3s^6+s^3t^3$ with $a\neq 0$	$\mu_3:[\lambda s:\lambda^{-1}t:x:y]$
	A_2+D_4	$y^2=x^3+s^2t^2x+t^6$	$\mu_3:[\lambda s:\lambda^{-1}t:x:y]$
	$2A_2$	$y^2=x^3+stx^2+a^3s^6+b^3s^3t^3+t^6$ with $a\neq 0, b^2\neq a$	$\mu_3:[\lambda s:\lambda^{-1}t:x:y]$
	$2A_2$	$y^2=x^3+s^2t^2x+a^3s^6+t^6$ with $a\neq 0$	$\mu_3:[\lambda s:\lambda^{-1}t:x:y]$
	$3A_2$	$y^2=x^3+s^2x^2+st^3x+a^3s^3t^3+b^3t^6$ with $a\notin\{0,(b-1)^2\}, b\neq 0$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	$3A_2$	$y^2=x^3+s^2x^2+a^3s^3t^3+t^6$ with $a\neq 0$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	$3A_2+A_1$	$y^2=x^3+s^2x^2+st^3x+a^3s^3t^3$ with $a\notin\{0,1\}$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	$4A_2$	$y^2=x^3+s^4t^2+s^2t^4$	$\alpha_3^3\rtimes\mu_3:[\lambda s:\lambda t:x+\varepsilon_1s^2+\varepsilon_2st+\varepsilon_3t^2:y]$
	A_5	$y^2=x^3+stx^2+b^6s^6+b^3s^3t^3+t^6$ with $b\neq 0$	$\mu_3:[\lambda s:\lambda^{-1}t:x:y]$
	A_5+A_2	$y^2=x^3+s^2x^2+st^3x+b^3t^6$ with $b\neq 0,1$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	A_5+A_2	$y^2=x^3+s^2x^2+t^6$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	$A_5+A_2+A_1$	$y^2=x^3+s^2x^2+st^3x$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	E_6^0	$y^2=x^3+st^3x+as^3t^3+t^6$ with $a\neq 0$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	E_6^0	$y^2=x^3+s^4x+t^6$	$\mu_3:[s:\lambda t:x:y]$
	$E_6^0+A_1$	$y^2=x^3+st^3x+s^3t^3$	$\alpha_3\rtimes\mu_9:[\lambda^6s:\varepsilon s+\lambda t:x+(1-\lambda^3)s^2:y]$
	$E_6^0+A_2$	$y^2=x^3+s^4t^2$	$\langle G,\mathbb{G}_m\rangle$ with $\mathbb{G}_m:[\lambda s:\lambda^{-2}t:x:y]$ and G non-commutative, $\lvert G\rvert=81$, acting as $\mathbb{G}_m:[s:t-\varepsilon_2^3s:x+\varepsilon_1s^2+\varepsilon_2st+\varepsilon_3t^2:y]$ with $\varepsilon_1^3=\varepsilon_2^9=\varepsilon_3^3=0$
	$E_6^1+A_2$	$y^2=x^3+s^2x^2+s^3t^3$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	E_7^0	$y^2=x^3+st^3x+t^6$	$\alpha_3\rtimes\mu_3:[s:\varepsilon s+\lambda t:x:y]$
	$E_7^0+A_1$	$y^2=x^3+st^3x$	$\alpha_3\rtimes\mathbb{G}_m:[\lambda^{-3}s:\varepsilon s+\lambda t:x:y]$
	A_8	$y^2=x^3+s^2x^2+st^3x+t^6$	$\alpha_9\rtimes\mu_3:[s:\varepsilon s+\lambda t:x+\varepsilon^3s^2:y]$
	E_8^0	$y^2=x^3+s^5t$	$\langle\alpha_3^2,\mathbb{G}_a\rtimes\mathbb{G}_m\rangle$ with $\mathbb{G}_a:[s:t-a^3s:x+as^2:y]$ $\mathbb{G}_m:[\lambda s:\lambda^{-5}t:x:y]$ $\alpha_3^2:[s:t:x+\varepsilon_1st+\varepsilon_2t^2:y]$
	E_8^1	$y^2=x^3+s^4x+s^3t^3$	$\mathbb{G}_a\rtimes\mu_3$ with $\mathbb{G}_a:[s:t-(a^3+a)s:x+as^2:y]$ $\mu_3:[s:\lambda t:x:y]$

Table 6. Non-equivariant RDP del Pezzo surfaces of degree 1 with global vector fields in characteristic 3

2. An application: Regular inseparable twists of RDP del Pezzo surfaces

A *twisted form* of a k-scheme X over a field extension $L \supseteq k$ is a scheme Y over L such that $Y_{\bar{L}} \cong X_{\bar{L}}$, where $\bar{L}$ is an algebraic closure of L. If X is a proper scheme over k, then smoothness of Aut_X is intimately related with properties of twisted forms of X, as the following proposition shows. Even though this proposition should be well-known, we include the proof for the convenience of the reader.

PROPOSITION 2.1. *Let $k \subseteq L$ be a field extension. Let X be a proper scheme over k and let Y be a twisted form of X over L. Assume that Aut_X is smooth. Then, the following hold:*

(1) If L is separably closed, then $Y \cong X_L$.
(2) If $\mathcal{P}$ is a property of schemes that is stable under field extensions and local in the étale topology, then, if X satisfies $\mathcal{P}$, also Y satisfies $\mathcal{P}$.

PROOF. Let us first prove Claim (1). As explained for example in [Mil16, p.134], an isomorphism $\bar{\varphi} : Y_{\bar{L}} \cong X_{\bar{L}}$ gives rise to a Čech cocycle on the fppf site of $\mathrm{Spec}\, L$, hence to an element in $\check{H}^1_{\text{fppf}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L})$. By [Mil16, Chapter III: Theorem 4.3.(b), Corollary 4.7, Remark 4.8], the smoothness of Aut_X implies that

$$\check{H}^1_{\text{fppf}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L}) = \check{H}^1_{\text{ét}}(\mathrm{Spec}\, L, \mathrm{Aut}_{X_L})$$

and since L is separably closed, the latter is trivial. Hence, Y and X_L are already isomorphic over L.

For Claim (2), let L^{sep} be the separable closure of L in $\bar{L}$. By (1), we have $X_{L^{sep}} \cong Y_{L^{sep}}$. Since X is proper, this isomorphism is defined over a finite subextension $L \subseteq L' \subseteq L^{sep}$, so that $X_{L'} \cong Y_{L'}$. The morphism $\mathrm{Spec}\, L' \to \mathrm{Spec}\, L$ is finite and étale. Hence, by our assumptions on $\mathcal{P}$, if X satisfies $\mathcal{P}$, then $X_{L'}$ satisfies $\mathcal{P}$, and thus also Y satisfies $\mathcal{P}$. $\square$

Choosing for $\mathcal{P}$ the property that the singular locus is non-empty and specializing to the case where X is an RDP del Pezzo surface, we obtain the following:

COROLLARY 2.2. *Let X be an RDP del Pezzo surface over k. Let $k \subseteq L$ be a field extension and let Y be a twisted form of X over L. If X has at least one singular point and Y is regular, then Aut_X is non-smooth.*

At a first glance, these twists seem rather hard to get a grip on geometrically, but it turns out that they can be written down explicitly if one has explicit descriptions of X and Aut_X. For example, consider $\mu_p \subseteq \mathrm{PGL}_{n+1,k}$ embedded diagonally with weights $(0, a_1, \ldots, a_{n-1}, 1)$, that is, given on the level of scheme valued points as $\lambda \mapsto (1, \lambda^{a_1}, \ldots, \lambda^{a_{n-1}}, \lambda)$. Alternatively, the μ_p-action is given by the p-closed derivation $D = \sum_{i=1}^n a_i x_i \partial_{x_i}$ with $a_n = 1$. We can write μ_p as the kernel of the surjective homomorphism

$$\begin{array}{rccl} f: & \mathbb{G}_m^n & \to & \mathbb{G}_m^n \\ & (1, u_1, \ldots, u_n) & \mapsto & (1, u_1 u_n^{-a_1}, \ldots, u_{n-1} u_n^{-a_{n-1}}, u_n^p). \end{array}$$

By [Mil16, Proposition 4.5] and Hilbert 90, for every field extension $k \subseteq L$, this yields a short exact sequence of abelian groups

$$0 \to \mathbb{G}_m^n(L) \overset{f(L)}{\to} \mathbb{G}_m^n(L) \overset{d}{\to} \check{H}^1_{\mathrm{fppf}}(\operatorname{Spec} L, \mu_p) \to 0.$$

Here, for $g \in \mathbb{G}_m^n(L)$, the element $d(g)$ is defined by choosing $\bar{g} \in \mathbb{G}_m^n(\bar{L})$ such that $f(\bar{L})(\bar{g}) = g_{\bar{L}}$ and setting $d(g)$ to be the image of the cocycle $(\bar{g} \otimes 1)^{-1}(1 \otimes \bar{g}) \in \mu_p(\bar{L} \otimes_L \bar{L})$. If $X \subseteq \mathbb{P}^n_k$ is a subvariety stabilized by μ_p – which, on the level of derivations, means that $D(I_X) \subseteq I_X$, where I_X is the ideal of X – then $\bar{g}^{-1}(X_{\bar{L}}) \subseteq \mathbb{P}^n_{\bar{L}}$ is defined over L and the cocycle one associates to this twisted form is in the same class as $d(g)$. In other words, we can realize every twist of X corresponding to an element of $\check{H}^1_{\mathrm{fppf}}(\operatorname{Spec} L, \mu_p)$ by choosing $\bar{g} \in \mathbb{G}_m^n(\bar{L})$ such that $f(\bar{L})(\bar{g})$ is defined over L and translating X along $\bar{g}^{-1}$.

EXAMPLE 2.3. Consider the quartic curve $Q = \{x^3y + y^3z + z^3x = 0\} \subseteq \mathbb{P}^2_k$ in characteristic 7. It is stable under the μ_7-action with weights $(0, 3, 1)$. Let $L = k(t)$ and consider $\bar{g} = (1, t^{3/7}, t^{1/7}) \in \mathbb{G}_m^2(\bar{L}) \subseteq \mathrm{PGL}_3(\bar{L})$. Then,

$$\bar{g}^{-1}(Q) = \{t^{3/7}x^3y + t^{10/7}y^3z + t^{3/7}z^3x = 0\} = \{x^3y + ty^3z + z^3x = 0\}.$$

Spreading out over $\mathbb{A}^1_k - \{0\} = \operatorname{Spec} k[t, t^{-1}]$, we obtain a fibered surface S over $\mathbb{A}^1_k - \{0\}$ which is easily checked to be smooth using the Jacobian criterion. Its generic fiber is a twisted form of Q over $k(t)$ and this twisted form is regular, because it is the generic fiber of a flat morphism between smooth k-schemes. Taking the double cover of $(\mathbb{A}^1 - \{0\}) \times \mathbb{P}^2$ branched over S, we obtain a smooth fibered threefold T over $\mathbb{A}^1 - \{0\}$ whose generic fiber is the regular del Pezzo surface Y of degree 2 given by the equation

$$\{w^2 = x^3y + ty^3z + z^3x\} \subseteq \mathbb{P}_{k(t)}(1, 1, 1, 2).$$

As before, Y is regular, being the generic fiber of a flat morphism of smooth k-schemes. Observe, however, that Y is not smooth, because $Y_{\overline{k(t)}} \cong X$, where X is the del Pezzo surface of degree 2 with a singularity of type A_6 given in our Table 1.

REMARK 2.4. Example 2.3 shows that the bound $p \leq 7$ given in [BT20, Proposition 5.2] for the characteristics in which non-smooth regular RDP del Pezzo surfaces can exist is sharp. Using the approach explained in the beginning of Example 2.3, it is not hard to construct similar examples if $p = 2, 3, 5$, but since this is not the topic of this chapter, we leave these constructions to the interested reader. Finally, we note that it is no mere coincidence that the Klein quartic in characteristic 7 appears in this context and refer the reader to [Stö04] for a closer study of this curve and its regular twists.

3. Preliminaries on (RDP) del Pezzo surfaces

In this section, we recall the definition of RDP del Pezzo surfaces and weak del Pezzo surfaces, which occur as minimal resolutions of RDP del Pezzo surfaces, as well as their basic properties.

DEFINITION 3.1. Let X and $\widetilde{X}$ be projective surfaces.

- X is a *del Pezzo surface* if it is smooth and $-K_X$ is ample.
- $\widetilde{X}$ is a *weak del Pezzo surface* if it is smooth and $-K_{\widetilde{X}}$ is big and nef.
- X is an *RDP del Pezzo surface* if all its singularities are rational double points and $-K_X$ is ample.

In all the above cases, the number $\deg(X) = K_X^2$ (resp. $\deg(\widetilde{X}) = K_{\widetilde{X}}^2$) is called the *degree of* X (resp. $\widetilde{X}$).

Recall (e.g. from [Dol12, Theorem 8.1.15, Corollary 8.1.24]) that $1 \leq \deg(X) = \deg(\widetilde{X}) \leq 9$ and that every weak del Pezzo surface of degree d and different from $\mathbb{P}^1 \times \mathbb{P}^1$ and the second Hirzebruch surface $\mathbb{F}_2$ can be realized as a blow-up of $\mathbb{P}^2$ in $9 - d$ (possibly infinitely near) points in almost general position.

As mentioned in the beginning of this section, weak del Pezzo surfaces arise as the minimal resolutions of RDP del Pezzo surfaces and, conversely, every RDP del Pezzo surface X is the anti-canonical model of a weak del Pezzo surface $\widetilde{X}$. The linear systems $|-nK_{\widetilde{X}}|$ are well studied (see e.g. [BT20, Proposition 2.14, Theorem 2.15] for proofs in positive characteristic). We denote the morphism induced by a linear system $|D|$ by $\varphi_{|D|}$ and recall that a curve singularity is called *simple* if its completion is isomorphic to one of the normal forms given in [GK90, Section 1]. The following description of the geometric picture is well-known [Dol12, Theorem 8.3.2].

THEOREM 3.2. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree d.*

(1) If $d \geq 3$, then $\varphi_{|-K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xhookrightarrow{\varphi_{|-K_X|}} \mathbb{P}^d$, where $\varphi_{|-K_X|}$ is a closed immersion that realizes X as a surface of degree d.

(2) If $d = 2$, then $\varphi_{|-K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{\varphi_{|-K_X|}} \mathbb{P}^2$, where $\varphi_{|-K_X|}$ is finite flat of degree 2.
If $p \neq 2$, then $\varphi_{|-K_X|}$ is branched over a quartic curve Q with simple singularities.

(3) If $d = 1$, then the $\varphi_{|-2K_{\widetilde{X}}|}$ factors as $\widetilde{X} \xrightarrow{\pi} X \xrightarrow{\varphi_{|-2K_X|}} \mathbb{P}(1,1,2) \subseteq \mathbb{P}^3$, where $\mathbb{P}(1,1,2)$ is the quadratic cone and $\varphi_{|-2K_X|}$ is finite flat of degree 2.
If $p \neq 2$, then $\varphi_{|-2K_X|}$ is branched over a sextic curve S with simple singularities.

Next, we recall the notion of a *marking* of a weak del Pezzo surface $\widetilde{X} \notin \{\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2\}$ (see [Dol12, Definition 8.1.21]) and explain how to describe the negative curves on $\widetilde{X}$ in terms of such a marking.

- A *marking* of $\widetilde{X}$ is an isomorphism $\phi : \mathrm{I}^{1,9-d} \xrightarrow{\sim} \mathrm{Pic}(\widetilde{X})$, where $\mathrm{I}^{1,9-d}$ is the lattice of rank $10-d$ with quadratic form given by the diagonal matrix $(1,-1,\ldots,-1)$ with respect to a basis $e_0,\ldots,e_{9-d}$.
- A realization $\pi : \widetilde{X} \to \mathbb{P}^2$ of $\widetilde{X}$ as an iterated blow-up of $\mathbb{P}^2$ induces a marking ϕ with $\phi(e_0) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $\phi(e_i)$ is the class of the preimage in $\widetilde{X}$ of the i-th point blown up by π. A marking that arises in this way is called *geometric*.
- If ϕ is a geometric marking of $\widetilde{X}$, then $\phi^{-1}(K_{\widetilde{X}}) = (-3,1,\ldots,1) =: k_{9-d}$.
- The lattice E_{9-d} is defined as $\langle k_{9-d}\rangle^{\perp} \subseteq \mathrm{I}^{1,9-d}$.

- For $d = 1, 2, 3$, the lattices E_{9-d} are precisely the three exceptional irreducible root lattices.
- For $d = 4, 5, 6$, there are identifications $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \oplus A_1$.
- For $d = 7, 8$, the lattices E_{9-d} are no root lattices. Every maximal root lattice contained in E_2 is isomorphic to A_1, and E_1 does not contain any (-2)-vectors.

REMARK 3.3. In particular, the number of (-2)-curves in the minimal resolution of an RDP del Pezzo surface of degree d is bounded above by $9 - d$.

Following [Dol12, Section 8.2], we let

$$\mathrm{Exc}_{9-d} := \{v \in \mathrm{I}^{1,9-d} \mid v^2 = -1, v.k_{9-d} = -1\} \subseteq \mathrm{I}^{1,9-d}$$

be the subset of *exceptional vectors*. Let $\mathcal{R}$ be a set of linearly independent (-2)-vectors in E_{9-d}, and define the cone

$$C_{\mathcal{R}} := \{v \in \mathrm{I}^{1,9-d} \otimes \mathbb{R} \mid v.w \geq 0 \text{ for all } w \in \mathcal{R}\}.$$

For a sublattice Λ of E_{9-d}, we denote the Weyl group of Λ by $W(\Lambda)$. That is, $W(\Lambda)$ is the subgroup of the orthogonal group $\mathrm{O}(\mathrm{I}^{1,9-d})$ generated by reflections along (-2)-vectors in Λ. With this notation, $W(\Lambda)$ preserves Exc_{9-d} and, for $\Lambda = \langle \mathcal{R} \rangle$, $C_{\mathcal{R}}$ is a fundamental domain for the action of $W(\Lambda)$ on $\mathrm{I}^{1,9-d} \otimes \mathbb{R}$.

LEMMA 3.4. *With the above notation, we have the following description of certain sets of (-1)-curves on $\widetilde{X}$:*

(1) If $\mathcal{R}$ is the pre-image of the set of all *(-2)-curves on $\widetilde{X}$ under a geometric marking ϕ, then ϕ induces a bijection*

$$\{(-1)\text{-curves on } \widetilde{X}\} \longleftrightarrow C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cong \mathrm{Exc}_{9-d}/W(\Lambda).$$

(2) If $\mathcal{R}' \subseteq \mathcal{R}$ with $\Lambda' := \langle \mathcal{R}' \rangle$, then ϕ induces a bijection

$$\{(-1)\text{-curves on } \widetilde{X} \text{ disjoint from } \phi(\mathcal{R}')\} \longleftrightarrow C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d}^{W(\Lambda')}.$$

(3) If, moreover, Λ' is a sum of connected components of Λ, then ϕ induces a bijection

$$\{(-1)\text{-curves on } \widetilde{X} \text{ disjoint from } \phi(\mathcal{R}')\} \longleftrightarrow (\mathrm{Exc}_{9-d}^{W(\Lambda')})/W(\Lambda).$$

PROOF. For (1) and (2), see [Dol12, Lemma 8.2.22 and Proposition 8.2.34]. To prove (3), we note that we have an orthogonal decomposition $\Lambda = \Lambda' \oplus \Lambda''$, where $\Lambda = \langle \mathcal{R} \rangle$ and $\Lambda'' = \langle \mathcal{R} \setminus \mathcal{R}' \rangle$. Therefore, the $W(\Lambda)$-action preserves $\mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = \mathrm{Exc}_{9-d}^{W(\Lambda')}$ and, by (1), we can write

$$C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d} \cap (\Lambda')^{\perp} = (C_{\mathcal{R}} \cap \mathrm{Exc}_{9-d})^{W(\Lambda')} \cong (\mathrm{Exc}_{9-d}^{W(\Lambda')})/W(\Lambda).$$

□

4. Group scheme actions on anti-canonical models

The purpose of this section is to recall some basic facts about group scheme actions on (blow-ups of) normal projective surfaces and to describe the automorphism scheme of an RDP del Pezzo surface in terms of the anti-(bi-)canonical morphisms recalled in Theorem 3.2.

Quite generally, the key tool to control the behavior of group scheme actions under birational morphisms is Blanchard's Lemma [Bri17, Theorem 7.2.1]. Note that Blanchard's Lemma was already used as the crucial ingredient in Chapter II (see Lemma 2.10), but for the reader's convenience we will recall it here and give a generalization of Lemma 2.11 of Chapter II to the setting of normal surfaces with at worst rational double points in the following Proposition 4.2.

THEOREM 4.1. (Blanchard's Lemma) *Let $f : Y \to X$ be a morphism of proper schemes with $f_*\mathcal{O}_Y = \mathcal{O}_X$. Then, f induces a homomorphism of group schemes $f_* : \mathrm{Aut}_Y^0 \to \mathrm{Aut}_X^0$. If f is birational, then f_* is a closed immersion.*

Given an action of a group scheme G on a scheme X and a closed subscheme $Z \subseteq X$, we let $\mathrm{Stab}_G(Z) \subseteq \mathrm{Aut}_X$ be the *stabilizer subgroup scheme* of Z (recall Definition 3.5 in Chapter II). The following proposition describes the image of π'_* if π' is a blow-up of a closed point on a normal surface with at worst rational double points.

PROPOSITION 4.2. *Let X be a normal surface with at worst rational double points and let $\pi' : X' \to X$ be the blow-up of X in a closed point P. Then, $\pi'_*(\mathrm{Aut}^0_{X'}) = (\mathrm{Stab}_{\mathrm{Aut}_X}(P))^0$.*

PROOF. By [Mar22, Proposition 2.7], it suffices to find an infinitesimal rigid subscheme $E \subseteq X'$ whose schematic image is P. Thus, let E be the exceptional divisor of π', with scheme structure given by the inverse image ideal sheaf of P. In particular, E is a Cartier divisor on X'.

If P is smooth, then E is a (-1)-curve, hence E is infinitesimally rigid. If P is not smooth, then P is a rational double point. In particular, π' factors the minimal resolution $\pi : \widetilde{X} \to X$ and $\pi^*\omega_X \cong \omega_{\widetilde{X}}$. Since X' has at worst rational double points as well and $\widetilde{X} \to X'$ is its minimal resolution, we obtain $\pi'^*\omega_X \cong \omega_{X'}$ using the projection formula. Now, ω_X is trivial in a neighborhood of P, so $\omega_{X'}$ is trivial in a neighborhood of E. Thus, the normal sheaf $\mathcal{N}_{E/X'}$ of E in X' coincides with ω_E by adjunction.

On the other hand, the rational double point $P \in X$ has multiplicity 2 and embedding dimension 3, hence E is isomorphic to a (possibly non-reduced) conic in $\mathbb{P}^2$. As such, it satisfies $\omega_E \cong \mathcal{O}_{\mathbb{P}^2}(-1)|_E$. Hence,

$$h^0(E, \mathcal{N}_{E/X'}) = h^0(E, \mathcal{O}_{\mathbb{P}^2}(-1)|_E) = 0,$$

so, by [Ser06, Proposition 3.2.1.(ii)], E is infinitesimally rigid. This finishes the proof. $\square$

If X is an RDP del Pezzo surface, then we have the following description of Aut_X in terms of the anti-canonical morphisms $\varphi_{|-nK_X|}$ of X.

PROPOSITION 4.3. *Let X be an RDP del Pezzo surface of degree d. Then, $\varphi_{|-nK_X|}$ is Aut_X-equivariant for all $n \geq 0$ and the following hold.*

(1) If $d \geq 3$, then $\operatorname{Aut}_X = \operatorname{Stab}_{\mathrm{PGL}_{d+1}}(X)$.

(2) If $d = 2$ and $p \neq 2$, then there is an exact sequence of group schemes

$$1 \to \underline{\mathbb{Z}/2\mathbb{Z}} \to \operatorname{Aut}_X \to \operatorname{Stab}_{\mathrm{PGL}_3}(Q) \to 1,$$

where Q is the branch quartic of the anti-canonical morphism $X \to \mathbb{P}^2$.

(3) If $d = 1$ and $p \neq 2$, then there is an exact sequence of group schemes

$$1 \to \underline{\mathbb{Z}/2\mathbb{Z}} \to \operatorname{Aut}_X \to \operatorname{Stab}_{\operatorname{Aut}_{\mathbb{P}(1,1,2)}}(S) \to 1,$$

where S is the branch sextic of the anti-bi-canonical morphism $X \to \mathbb{P}(1,1,2) \subseteq \mathbb{P}^3$.

PROOF. By [Bri18, Remark 2.15.(iv)], the line bundles $\omega_X^{\otimes(-n)}$ admit natural Aut_X-linearizations for all $n \geq 0$ and hence the natural action of Aut_X on the space of global sections of $\omega_X^{\otimes(-n)}$ induces a homomorphism $f_n : \operatorname{Aut}_X \to \mathrm{PGL}_{N+1}$, where $N = \dim(H^0(X, \omega_X^{\otimes(-n)}))-1$, making the rational map $\varphi_{|-nK_X|} : X \dashrightarrow \mathbb{P}^N$ Aut_X-equivariant.

If $d \geq 3$, then the anti-canonical map is an embedding by Theorem 3.2, hence f_1 is a monomorphism. By [Mar22, Lemma 2.5], f_1 factors through $\operatorname{Stab}_{\mathrm{PGL}_{d+1}}(X)$. Conversely, restricting the G-action on $\mathbb{P}^d$ to X yields a left-inverse $g_1 : \operatorname{Stab}_{\mathrm{PGL}_{d+1}}(X) \to \operatorname{Aut}_X$ to f_1. Since X is not contained in a proper linear subspace of $\mathbb{P}^d$ and the fixed locus of a subgroup scheme of PGL_{d+1} is a linear subspace, g_1 has to be a monomorphism. Hence f_1 is an isomorphism.

If $d = 2$ and $p \neq 2$, then the anti-canonical map is a finite flat cover of $\mathbb{P}^2$ branched over a quartic curve $Q \subseteq \mathbb{P}^2$ by Theorem 3.2. Let K be the kernel of f_1. Restricting the action of K on X to the generic point of X yields a $k(\mathbb{P}^2)$-linear action of $K_{k(\mathbb{P}^2)}$ on the degree 2 field extension $k(X)$ of $k(\mathbb{P}^2)$. Since $p \neq 2$, the field extension $k(\mathbb{P}^2) \subseteq k(X)$ is Galois, which shows $K = \underline{\mathbb{Z}/2\mathbb{Z}}$. Since K is normal in Aut_X, the action of Aut_X on X preserves the fixed locus X^K, hence, by [Mar22, Lemma 2.5], the induced action of Aut_X on $\mathbb{P}^2$ preserves the scheme-theoretic image of X^K under $\varphi_{|-K_X|}$, which is nothing but Q. Hence, f_1 factors through $\operatorname{Stab}_{\mathrm{PGL}_3}(Q)$. In order to show faithful flatness of $f_1' : \operatorname{Aut}_X \to \operatorname{Stab}_{\mathrm{PGL}_3}(Q)$, we write X as $\{w^2 = Q(x,y,z)\} \subseteq \mathbb{P}(1,1,1,2)$. For every k-algebra R and every automorphism σ of $\mathbb{P}^2_R$ preserving $Q_R = \{Q(x,y,z) = 0\} \subseteq \mathbb{P}^2_R$, that is, mapping $Q(x,y,z)$ to $\lambda Q(x,y,z)$ for some $\lambda \in R^\times$, we can pass to the faithfully flat ring extension $R' := R[\sqrt{\lambda}]$ of R and lift σ to an automorphism of $X_{R'}$ by mapping w to $\pm\sqrt{\lambda}w$. Hence, f_1' is faithfully flat and thus the sequence in (2) is exact.

If $d = 1$ and $p \neq 2$, we can apply essentially the same argument as in the previous paragraph to the anti-bi-canonical morphism $\varphi_{|-2K_X|} : X \to \mathbb{P}^3$: Indeed, the argument for $K = \underline{\mathbb{Z}/2\mathbb{Z}}$ is exactly the same as in the previous paragraph. To prove faithful flatness of $f_2' : \operatorname{Aut}_X \to \operatorname{Stab}_{\operatorname{Aut}_{\mathbb{P}(1,1,2)}}(S)$, we can write X as a hypersurface in $\mathbb{P}(1,1,2,3)$ by identifying the quadratic cone with $\mathbb{P}(1,1,2)$ and then argue as above. $\square$

5. On equivariant resolutions

It is an immediate consequence of the uniqueness of the minimal resolution $\widetilde{X}$ of a projective surface X that the action of the automorphism *group* $\mathrm{Aut}_X(k)$ on X lifts to $\widetilde{X}$. Over the complex numbers, this implies that the action of the automorphism *scheme* Aut_X lifts to $\widetilde{X}$. In general, this is no longer true in positive characteristic. In this section, we will study this phenomenon.

DEFINITION 5.1. Let $\pi : \widetilde{X} \to X$ be a proper birational morphism of schemes. Assume that X is integral and normal.

(1) The morphism π is called T_X*-equivariant* if the natural map $\pi_* T_{\widetilde{X}} \to T_X$ is an isomorphism.
(2) Assume additionally that X is proper (in particular, this implies $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$). Then, π is called Aut_X^0*-equivariant* if the closed immersion $\pi_* : \mathrm{Aut}^0_{\widetilde{X}} \hookrightarrow \mathrm{Aut}^0_X$ induced by Blanchard's Lemma is an isomorphism.

REMARK 5.2. Note that T_X-equivariance is local on X and implies $H^0(\widetilde{X}, T_{\widetilde{X}}) \cong H^0(X, T_X)$. If π is T_X-equivariant, then $\pi_* : \mathrm{Aut}^0_{\widetilde{X}} \hookrightarrow \mathrm{Aut}^0_X$ is an isomorphism on tangent spaces.

The study of the T_X-equivariance of the minimal resolution of a rational double point has been initiated by Wahl [Wah75] and extended to all positive characteristics by Hirokado [Hir19]. There, T_X-equivariance is simply called "equivariance". For the convenience of the reader, we will recall the classification of RDPs whose minimal resolution is not T_X-equivariant (see [Hir19, Theorem 1.1]).

PROPOSITION 5.3. *Let $\pi : \widetilde{X} \to X$ be the minimal resolution of a rational double point (X, x). Then, π is not T_X equivariant if and only if (X, x) is of type*

(1) A_n with $p \mid (n+1)$,
(2) E_8^0 if $p = 5$,
(3) $E_6^0, E_6^1, E_7^0, E_8^0, E_8^1$ if $p = 3$, or
(4) $D_n^r, E_6^0, E_7^0, E_7^1, E_7^2, E_7^3, E_8^0, E_8^1, E_8^2, E_8^3$ if $p = 2$.

In the next sections, we would like to apply the notions of Aut^0_X-equivariance and T_X-equivariance to RDP del Pezzo surfaces and their partial resolutions.

DEFINITION 5.4. Let X be a proper surface. A *partial resolution* of X is a proper birational morphism $\pi' : X' \to X$ such that the minimal resolution of X factors through π'.

We thank the referee for suggesting a simplified proof of the following result.

PROPOSITION 5.5. *Let X be a normal proper surface and let $\pi' : X' \to X$ be a partial resolution of X. Assume that there exists an open subset $U \subseteq X$ such that $\pi'^{-1}(U) \to U$ is an isomorphism and all singularities in $X \setminus U$ admit a T_X-equivariant minimal resolution. Then, π' is T_X-equivariant.*

PROOF. Let $\pi : \widetilde{X} \to X$ be the minimal resolution of X. By definition of a partial resolution, π factors through π'. We get an induced sequence

$$\pi_* T_{\widetilde{X}} \xrightarrow{\sigma} \pi'_* T_{X'} \xrightarrow{\tau} T_X.$$

By assumption, τ is an isomorphism on U and $\tau \circ \sigma$ is an isomorphism in a neighborhood of every point in $X \setminus U$. Hence, τ is surjective and generically an isomorphism. Since $\pi'_* T_{X'}$ is torsion-free, this implies that τ is an isomorphism, hence π' is equivariant, as claimed. $\square$

In some situations, Aut_X^0-equivariance can be deduced immediately from the simpler notion of T_X-equivariance, which is the content of the following proposition.

PROPOSITION 5.6. *Let $\pi : \widetilde{X} \to X$ be a birational morphism of proper k-schemes. If $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth and π is T_X-equivariant, then π is Aut^0_X-equivariant.*

PROOF. If π is T_X-equivariant, then

$$\dim \mathrm{Aut}^0_{\widetilde{X}} \leq \dim \mathrm{Aut}^0_X \leq \dim_k H^0(X, T_X) = \dim_k H^0(\widetilde{X}, T_{\widetilde{X}})$$

and since $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth, all inequalities above are in fact equalities. Thus, Aut^0_X is smooth and of the same dimension as $\mathrm{Aut}^0_{\widetilde{X}}$. Hence, we must have $\mathrm{Aut}^0_{\widetilde{X}} = \mathrm{Aut}^0_X$, that is, π is Aut^0_X-equivariant. $\square$

REMARK 5.7. In particular, if, in the situation of Proposition 5.5, we assume in addition that $\mathrm{Aut}^0_{X'}$ is smooth, then π' is Aut^0_X-equivariant.

To the best of our knowledge, the question whether partial resolutions of a proper normal surface with rational double points are Aut^0_X-equivariant has not been studied. In the following, we prove Aut^0_X-equivariance for A_n-singularities with $n < p-1$ and bound the failure of Aut^0_X-equivariance for A_{p-1}-singularities. While this does not cover all rational double points and not even all A_n-singularities, it will come in handy for the calculation of the automorphism schemes of non-equivariant RDP del Pezzo surfaces in Section 8.

PROPOSITION 5.8. *Let X be a proper surface. Let $\pi' : X' \to X$ be a partial resolution of X and assume that the only singularities over which π' is not an isomorphism are A_n-singularities with $n \leq p-1$. Then,*

$$\mathrm{length}\left(\mathrm{Aut}^0_X / \mathrm{Aut}^0_{X'}\right) \leq p^m,$$

where m is the number of A_{p-1}-singularities on X over which π' is not an isomorphism. In particular, if $m = 0$, then π' is Aut_X-equivariant.

PROOF. It suffices to prove the statement if $\pi' : X' \to X$ is not an isomorphism only over a single singularity P of type A_n. By Proposition 4.2, it suffices to show that $G := \mathrm{Aut}^0_X$ fixes P if $n < p-1$ and that the stabilizer of P has index 1 or p in G if $n = p-1$. To see this, we equip the singular locus X_{sing} of X with a scheme structure using Fitting ideals (see [Sta18a, Tag 07Z6]) and we let Y be the irreducible component of X_{sing} containing P. Since the scheme structure on X_{sing} is canonical and G is connected, G preserves Y (see [Sta18a, Tag 07ZA]), so we get a homomorphism $\varphi : G \to \mathrm{Aut}_Y$. To prove the proposition, it suffices to show that the stabilizer of P in Aut_Y has index 1 or p, with the latter only occurring for $n = p-1$.

Since an A_n-singularity is given in a formal neighborhood by the equation $z^{n+1} + xy$, Y is isomorphic to

$$\begin{aligned} Y_n &:= \operatorname{Spec}\left(k[z]/(z^{n+1})\right) \quad \text{if} \quad n < p-1 \\ Y_{p-1} &:= \operatorname{Spec}\left(k[z]/(z^p)\right) \quad \text{if} \quad n = p-1. \end{aligned}$$

Now, we calculate Aut_{Y_i} by computing its R-valued points for an arbitrary local k-algebra R. An element of $\mathrm{Aut}_{Y_i}(R)$ is an R-linear automorphism φ of $R[z]/(z^{i+1})$, hence it is determined by where it sends z. Let $a_0, \ldots, a_i \in R$ such that $\varphi(z) = \sum_{j=0}^{i} a_j z^j$. Let $\mathfrak{m}$ be the maximal ideal of R, so that $(\mathfrak{m}, z)$ is the maximal ideal of $R[z]/(z^{i+1})$. Since φ is an automorphism, it maps $(\mathfrak{m}, z)$ to itself, hence $a_0 \in \mathfrak{m}$. If $a_1 \in \mathfrak{m}$, then the coefficient of z in every $\varphi(z^j)$ is in $\mathfrak{m}$, so z would not lie in the image of φ, which is absurd. Hence, $a_1 \in R \setminus \mathfrak{m} = R^\times$. Next, we know that $\varphi(z^{i+1}) = 0$. Since the degree 0 term of $\varphi(z^{i+1})$ is a_0^{i+1}, we have $a_0^{i+1} = 0$. The degree j term of $\varphi(z^{i+1})$ is of the form $\binom{i+1}{j} a_0^{i+1-j} a_1^j + a_0^{i+2-j} b_j$ for some $b_j \in R$. If $i < p-1$, then $p \nmid \binom{i+1}{j}$ for all j, hence solving the above equations inductively shows $a_0 = 0$. If $i = p-1$, then $p \mid \binom{i+1}{j}$ for all $j > 0$, so we only get $a_0^p = 0$.

Conversely, given a sequence $(a_0, \ldots, a_i)$ in R with $a_1 \in R^\times$ and $a_0 = 0$ if $i < p-1$ (resp. $a_0^p = 0$ if $i = p$), the morphism induced by $z \mapsto \sum_{j=0}^{i} a_j z^j$ is an automorphism, since it is well-defined and its inverse is given by $z \mapsto (\sum_{j=1}^{i} a_j z^{j-1})^{-1} z - a_0$.

Summarizing, we have natural identifications

$$\begin{aligned} \mathrm{Aut}_{Y_i}(R) &= \{(0, a_1, \ldots, a_i) \mid a_j \in R, a_1 \in R^\times\} \text{ for } i < p-1 \\ \mathrm{Aut}_{Y_{p-1}}(R) &= \{(a_0, a_1, \ldots, a_{p-1}) \mid a_j \in R, a_1 \in R^\times, a_0^p = 0\}. \end{aligned}$$

In both cases, the corresponding automorphism of Y_i preserves $P \times \operatorname{Spec} R \subseteq Y_i \times \operatorname{Spec} R$ if and only if $a_0 = 0$, since the ideal of $P \times \operatorname{Spec} R$ is (z). Hence, the index of the stabilizer of P in Aut_{Y_i} is 1 if $i < p-1$, and p if $i = p-1$. This finishes the proof. $\square$

REMARK 5.9. The strategy of proof of Proposition 5.8 would, in principle, also apply to other rational double points. However, there are two obstacles to overcome:

(1) The automorphism scheme of the singular locus is more complicated for more general RDPs, since the singular locus has a more complicated scheme structure in general. For example, if $p = 5$ and X admits an RDP of type E_8^0, then, in a neighorhood of this singularity, the singular locus of X looks like $\operatorname{Spec} k[[x, y]]/(x^2, y^5)$. This also makes the calculation of the stabilizer of the closed point more complicated.

(2) If $\pi : \widetilde{X} \to X$ is the minimal resolution of an RDP surface, then $\mathrm{Aut}^0_{\widetilde{X}}$ is the intersection of all stabilizers of all singularities that occur in the blow-ups making up π. For example, if $p = 3$ and X admits a single RDP of type A_4, then the argument of Proposition 5.8 shows that $\mathrm{Aut}^0_{X'} = \mathrm{Aut}^0_X$, where X' is the blow-up of the closed point of P, but X' has a singularity of type A_2, so the approach of Proposition 5.8 only shows that $\operatorname{length}\left(\mathrm{Aut}^0_X / \mathrm{Aut}^0_{\widetilde{X}}\right) \leq 3$ even though we would expect the two group schemes to be equal by Proposition 5.3.

One case where the argument of Proposition 5.8 goes through essentially unchanged is if $p = 3$ and the morphism $\pi : \widetilde{X} \to X$ is the minimal resolution of an RDP of type A_5. In this case, π factors as a composition of three blow-ups $\widetilde{X} \to X'' \to X' \to X$, where X' has an A_3-singularity and X'' has an A_1-singularity. Then, the argument of Proposition 5.8 shows that $\mathrm{Aut}^0_{\widetilde{X}} = \mathrm{Aut}^0_{X''} = \mathrm{Aut}^0_{X'}$ and $\mathrm{length}\left(\mathrm{Aut}^0_X / \mathrm{Aut}^0_{X'}\right) \leq 3$, hence $\mathrm{length}\left(\mathrm{Aut}^0_X / \mathrm{Aut}^0_{\widetilde{X}}\right) \leq 3$.

6. Automorphism schemes of equivariant RDP del Pezzo surfaces

In Chapter II, we classified all weak del Pezzo surfaces $\widetilde{X}$ with global vector fields and calculated the identity component $\mathrm{Aut}^0_{\widetilde{X}}$ of their automorphism schemes. In particular, if X is a projective surface, whose minimal resolution $\pi : \widetilde{X} \to X$ is Aut^0_X-equivariant and such that $\widetilde{X}$ is a weak del Pezzo surface, then $\mathrm{Aut}^0_X = \mathrm{Aut}^0_{\widetilde{X}}$ and thus, if Aut^0_X is non-trivial, then $\widetilde{X}$ appears in the classification tables of Chapter II.

In the following, we will observe that all RDP del Pezzo surfaces in characteristic $p \geq 11$ fall into the above category and we will give a list of possible candidates for exceptions in small characteristics.

THEOREM 6.1. *Let X be an RDP del Pezzo surface over an algebraically closed field of characteristic p and let $\pi : \widetilde{X} \to X$ be its minimal resolution. Assume that one of the following conditions holds:*

(1) $p \notin \{2, 3, 5, 7\}$.
(2) $p = 7$ and X does not contain an RDP of type A_6.
(3) $p = 5$ and X does not contain an RDP of type A_4 or E_8^0.
(4) $p = 3$ and X does not contain an RDP of type $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$ or E_8^1.
(5) $p = 2$ and X does not contain an RDP of type $A_1, A_3, A_5, A_7, D_n^r, E_6^0, E_7^0, E_7^1, E_7^2, E_7^3, E_8^0, E_8^1, E_8^2$ or E_8^3, where $n \leq 8$.

Then, $\mathrm{Aut}_X = \mathrm{Aut}_{\widetilde{X}}$, and thus, in particular, $H^0(\widetilde{X}, T_{\widetilde{X}}) = H^0(X, T_X)$. Therefore, $H^0(X, T_X) \neq 0$ if and only if X is the anti-canonical model of one of the surfaces in the classification Tables 1, 2, 3, 4, 5 and 6 of Chapter II.

PROOF. By Proposition 5.5 and Proposition 5.6, the theorem holds for those RDP del Pezzo surfaces that satisfy the following two conditions:

(a) all singularities of X admit a T_X-equivariant minimal resolution, and
(b) $\mathrm{Aut}^0_{\widetilde{X}}$ is smooth.

By Proposition 5.3 and Remark 3.3, Condition (a) holds if we exclude the types of RDPs in the statement of the theorem. In particular, note that if $p \geq 11$, we do not have to exclude any RDPs.

Once we exclude those types of RDPs, then, by Tables 1–6 in Chapter II, Condition (b) is also satisfied unless we are in one of the following three cases, where Γ is the RDP configuration on X and $d = \deg(X)$:

(i) $p = 3$, $d = 2$, $\Gamma = A_6$
(ii) $p = 3$, $d = 2$, $\Gamma = D_6$
(iii) $p = 2$, $d = 3$, $\Gamma = A_4$

In Chapter II, these exceptions correspond to cases $2J$, $2K$ and $3N$, respectively, and there is a unique weak del Pezzo surface of each of these types. In all cases, we have $h^0(\widetilde{X}, T_{\widetilde{X}}) = 1$, hence $h^0(X, T_X) = 1$ by Proposition 5.3, so the only remaining statement we have to show in these three cases is that π is Aut_X^0-equivariant. We will check this via explicit calculation:

(i) Assume $p = 3$. Consider the surface
$$X \coloneqq \{w^2 = x^2z^2 + xy^2z + y^4 + x^3y\} \subseteq \mathbb{P}(1,1,1,2)$$
and let Q be the branch quartic of the induced double cover $X \to \mathbb{P}^2$. An elementary calculation shows that X admits an A_6-singularity over $[0 : 0 : 1]$ and no other singularity. By Proposition 4.3, we have $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$. The diagonal μ_3-action with weights $(0,1,2)$ on $\mathbb{P}^2$ preserves Q, hence $\mu_3 \subseteq \mathrm{Aut}_X^0$. Since π is T_X-equivariant by Proposition 5.3, $\widetilde{X}$ must be the surface of type $2J$ in Chapter II, hence $\mathrm{Aut}_{\widetilde{X}}^0 = \mu_3$. Since $h^0(X, T_X) = 1$, we have $\mathrm{Aut}_X^0[F] = \mu_3$, where $\mathrm{Aut}_X^0[F]$ denotes the kernel of Frobenius on Aut_X^0. Hence, μ_3 is normal in $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$ and thus $\mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$ preserves the eigenspaces of the μ_3-action, hence $\mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0$ acts diagonally. With this restriction, it is easy to compute that $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_3}(Q)^0 = \mu_3$. Therefore, π is Aut_X^0-equivariant, which is what we wanted to show.

(ii) Assume $p = 3$. Consider the surface
$$X \coloneqq \{w^2 = x(x^3 + y^3 + xyz)\} \subseteq \mathbb{P}(1,1,1,2)$$
and let Q be the branch quartic of the induced double cover $X \to \mathbb{P}^2$. Note that Q is the union of a nodal cubic and one of its nodal tangents, with the node located at $[0 : 0 : 1]$, hence X has a D_6-singularity at $[0 : 0 : 1 : 0]$ and no other singularities. The diagonal μ_3-action with weights $(0,1,2)$ preserves Q, hence $h^0(X, T_X) \neq 0$, and thus $\widetilde{X}$ is the surface of type $2K$ in Chapter II. The rest of the argument is the same as in the previous Case (i), and shows that π is Aut_X^0-equivariant.

(iii) Assume $p = 2$. Consider the surface
$$X \coloneqq \{x_0x_1x_3 + x_1^2x_2 + x_0x_2^2 + x_0^2x_2\} \subseteq \mathbb{P}^3,$$
which is a cubic surface with a single singularity, which is of type A_4, at $[0 : 0 : 0 : 1]$ (see e.g. [Roc96, Case B]). It admits a diagonal μ_2-action with weights $(0,1,0,1)$, hence $h^0(X, T_X) \neq 0$ and therefore, as π is T_X-equivariant by Proposition 5.3, X is the anti-canonical model of the surface of type $3N$ in Chapter II. Straightforward calculation, again using that μ_2 is normal in $\mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$, shows that $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0 = \mu_2$, hence π is Aut_X^0-equivariant.

□

7. Finding (-1)-curves in the equivariant locus

In view of Theorem 6.1, in order to classify RDP del Pezzo surfaces with global vector fields in odd characteristic, it remains to study RDP del Pezzo surfaces X which are not T_X-equivariant. Let Γ' be the configuration of rational double points on X which are not T_X-equivariant and let $\pi : \widetilde{X} \to X$ be the minimal resolution of X.

In this section, we will describe a criterion for X to be the anti-canonical model of a blow-up of an RDP del Pezzo surface X' of higher degree containing Γ' such that $X' \dashrightarrow X$ is an isomorphism around Γ'. On the corresponding minimal resolutions, this will amount to finding (-1)-curves away from the configuration of (-2)-curves over Γ'. In other words, we are trying to find (-1)-curves on $\widetilde{X}$ that map to the T_X-equivariant locus of π. In Section 8, this criterion will allow us to give a complete classification of non-equivariant RDP del Pezzo surfaces with global vector fields by setting up an inductive argument depending on the degree of the surface.

THEOREM 7.1. *Let X_d be an RDP del Pezzo surface of degree $d \leq 8$ and let Γ' be a configuration of rational double points on X_d. Assume that its minimal resolution $\widetilde{X}_d$ is a blow-up of $\mathbb{P}^2$ and let $\Lambda' \subseteq \mathrm{Pic}(\widetilde{X}_d)$ be the sublattice generated by the components of the exceptional locus over Γ'. Then the following are equivalent:*

(1) There exists a (-1)-curve on $\widetilde{X}_d$ whose image in X_d does not pass through Γ'.
(2) X_d is the anti-canonical model of a blow-up in a smooth point P of an RDP del Pezzo surface X_{d+1} of degree $(d+1)$ containing Γ' such that $X_d \dashrightarrow X_{d+1}$ is an isomorphism around Γ'.
(3) The map $\Lambda' \hookrightarrow \langle K_{\widetilde{X}_d}\rangle^{\perp} \cong E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$.

PROOF. First, we show $(1) \Rightarrow (2)$. Let $\widetilde{C}$ be the (-1)-curve whose existence is asserted in (1). Contracting $\widetilde{C}$, we obtain a weak del Pezzo surface $\widetilde{X}_{d+1}$ of degree $(d+1)$ such that $\widetilde{X}_d$ is the blow-up of $\widetilde{X}_{d+1}$ in a smooth point $\widetilde{P}$. Let X_{d+1} be the anti-canonical model of $\widetilde{X}_{d+1}$ and let P be the image of $\widetilde{P}$ in X_{d+1}. By our choice of $\widetilde{C}$, all components of the exceptional locus over Γ' stay (-2)-curves in $\widetilde{X}_{d+1}$, hence X_{d+1} contains Γ'.

Since $\widetilde{X}_d$ is a weak del Pezzo surface, $\widetilde{P}$ cannot lie on a (-2)-curve (otherwise the strict transform of such a curve would have negative intersection with $-K_{\widetilde{X}_d}$, which is impossible as $-K_{\widetilde{X}_d}$ is nef), hence P is a smooth point on X_{d+1}. Thus, blowing up $P \in X_{d+1}$, we obtain a surface Y_d with the same singularities as X_{d+1}. In particular, Y_d has only rational double points as singularities and its minimal resolution is $\widetilde{X}_d$. Therefore, pullback of sections induces isomorphisms $H^0(Y_d, -nK_{Y_d}) \cong H^0(\widetilde{X}_d, -nK_{\widetilde{X}_d})$ for all $n \geq 0$, where the surjectivity follows from the fact that Y_d is normal. Thus, the anti-canonical model of Y_d coincides with X_d. The situation is summarized in the following Figure 1.

Note that $X_d \dashrightarrow Y_d \to X_{d+1}$ is an isomorphism in a neighborhood of Γ', since $\widetilde{C}$ is disjoint from the exceptional locus over Γ'.

Next, we show $(2) \Rightarrow (3)$. We have $\langle K_{\widetilde{X}_d}\rangle^{\perp} \cong E_{9-d}$ and $\langle K_{\widetilde{X}_{d+1}}\rangle^{\perp} \cong E_{8-d}$. Since X_{d+1} contains Γ', the embedding $\Lambda' \hookrightarrow \mathrm{Pic}(\widetilde{X}_d)$ factors through the pullback map $\mathrm{Pic}(\widetilde{X}_{d+1}) \hookrightarrow \mathrm{Pic}(\widetilde{X}_d)$, which maps $\langle K_{\widetilde{X}_{d+1}}\rangle^{\perp}$ to $\langle K_{\widetilde{X}_d}\rangle^{\perp}$. Hence (3) follows.

$$\begin{tikzcd}
\widetilde{C} \subseteq \widetilde{X}_d \arrow[r, "\text{contract } \widetilde{C}", "\text{blow-up } \widetilde{P}"'] \arrow[d] \arrow[dd, bend right=60, "\text{anti-can.}"'] & \widetilde{X}_{d+1} \ni \widetilde{P} \arrow[d, "\text{anti-can.}"] \\
C \subseteq Y_d \arrow[r, "\text{contract } C", "\text{blow-up } P"'] \arrow[d, "\text{anti-can.}"] & X_{d+1} \ni P \\
X_d &
\end{tikzcd}$$

Figure 1. Contracting a (-1)-curve disjoint from the singular locus

Finally, to show that (3) $\Rightarrow$ (1), we identify $\mathrm{Pic}(\widetilde{X}_d)$ and $\mathrm{I}^{1,9-d}$ via a geometric marking. We have to show that there is a (-1)-curve $\widetilde{C}$ on $\widetilde{X}_d$ that does not meet the set $\mathcal{R}'$ of exceptional curves over Γ'. Let Λ be the sublattice of $\mathrm{Pic}(\widetilde{X}_d)$ spanned by the classes of all (-2)-curves. Since Λ' is a sum of connected components of Λ, Lemma 3.4 shows that it suffices to prove

$$(\mathrm{Exc}_{9-d}^{W(\Lambda')})/W(\Lambda) \neq \emptyset.$$

Clearly, this is the case if and only if $\mathrm{Exc}_{9-d}^{W(\Lambda')} \neq \emptyset$. Since $\Lambda' \hookrightarrow E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$, we have

$$\mathrm{Exc}_{9-d}^{W(E_{8-d})} \subseteq \mathrm{Exc}_{9-d}^{W(\Lambda')},$$

so it suffices to show that $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$. Since the action of $W(E_{9-d})$ on $\mathrm{Pic}(\widetilde{X}_d)$ preserves Exc_{9-d}, the condition $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$ depends on the embedding $E_{8-d} \hookrightarrow E_{9-d}$ only up to conjugation by elements of $W(E_{9-d})$ and up to automorphisms of E_{8-d}.

If $d \leq 5$, then E_{8-d} is a root lattice. By [Dyn52, Table 11] and [Mar03, Exercise 4.2.1, 4.6.2], the embedding $\iota : E_{8-d} \hookrightarrow E_{9-d}$ is unique up to the action of $\mathrm{O}(E_{9-d})$. Since $\mathrm{O}(E_{9-d})$ is generated by $\{\pm\mathrm{id}\}$ and $W(E_{9-d})$ in every case (see e.g. [Mar03, Proposition 4.2.2, Theorem 4.3.3, 4.5.2, 4.5.3]), ι is unique up to the action of $W(E_{9-d})$ and up to automorphisms of E_{8-d}. Therefore, in order to show that $\mathrm{Exc}_{9-d}^{W(E_{8-d})} \neq \emptyset$, it suffices to show that there exists *some* $\widetilde{X}_d$ containing a configuration of (-2)-curves of type E_{8-d} and such that a (-1)-curve disjoint from this configuration exists. This is known and can be seen for example in Figures 23, 49, 58, 64, and 62 in Chapter II.

If $d \geq 7$, then E_{8-d} does not contain any (-2)-vectors, hence $\Lambda' = 0$ and the implication (3) $\Rightarrow$ (1) holds, since $\widetilde{X}_d$ is a blow-up of $\mathbb{P}^2$ by assumption.

Finally, if $d = 6$, then the maximal root lattice contained in $E_{8-d} = E_2$ is A_1. Thus, we may assume $\Lambda' = A_1$, for otherwise we can argue as in the previous Case $d \geq 7$. Up to the action of $\mathrm{O}(E_3) = \{\pm\mathrm{id}\} \times W(E_3)$, there are two embeddings of A_1 into $E_3 = A_2 \oplus A_1$. It is easy to check that $\iota : A_1 \hookrightarrow E_3$ factors through E_2 if and only if ι factors through the A_2-summand of E_3 and then ι is unique up to the action of $W(E_3)$ and up to automorphisms of A_1. Hence, similar to what we did in the case $d \leq 5$, it suffices to find some $\widetilde{X}_6$ containing a (-2)-curve and a disjoint (-1)-curve. Again, this is known, see Figure 24 in Chapter II. $\square$

COROLLARY 7.2. *Let X_d be an RDP del Pezzo surface of degree $d \leq 8$, let Γ' be a configuration of rational double points on X_d, and let Λ' be the root lattice associated to Γ'. Assume that Γ' occurs on an RDP del Pezzo surface of degree $(d+1)$ and satisfies one of the following conditions:*

(1) $d \neq 4, 2, 1$
(2) $d = 4$ *and* $\Lambda' \neq A_3$
(3) $d = 2$ *and* $\Lambda' \notin \{A_5 + A_1, A_5, A_3 + 2A_1, A_3 + A_1, 4A_1, 3A_1\}$
(4) $d = 1$ *and* $\Lambda' \notin \{A_7, 2A_3, A_5 + A_1, A_3 + 2A_1, 4A_1\}$

Then, X_d is the anti-canonical model of a blow-up in a smooth point P of an RDP del Pezzo surface X_{d+1} of degree $(d+1)$ containing Γ' such that $X_d \dashrightarrow X_{d+1}$ is an isomorphism around Γ'.

PROOF. The embeddings of root lattices into E_6, E_7, and E_8 have been classified by Dynkin [Dyn52, Table 11] and for the embeddings of root lattices into $E_3 = A_2 \oplus A_1, E_4 = A_4$, and $E_5 = D_5$, we refer the reader to [Mar03, Exercise 4.2.1, 4.6.2]. It follows from these classifications that if an embedding of Λ' into E_{9-d} exists, then this embedding is unique (up to the action of $\mathrm{O}(E_{9-d})$), except precisely in the cases excluded in $(2), (3)$ and (4). Hence, if one embedding $\Lambda' \hookrightarrow E_{9-d}$ factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$, then *every* embedding factors through an embedding $E_{8-d} \hookrightarrow E_{9-d}$. If Γ' occurs on some RDP del Pezzo surface of degree $(d+1)$, then an embedding of Λ' with such a factorization exists and the claim follows from Theorem 7.1. $\square$

8. Automorphism schemes of non-equivariant RDP del Pezzo surfaces

Throughout this section, X denotes an RDP del Pezzo surface and $\pi : \widetilde{X} \to X$ is its minimal resolution. In this section, we will prove Theorem 1.2, that is, we will classify all X with $H^0(X, T_X) \neq 0$ over a field of characteristic $p \in \{3, 5, 7\}$ and such that X contains one of the RDPs excluded in Theorem 6.1. We will treat the cases $p = 7$, $p = 5$, and $p = 3$, in Sections 8.1, 8.2, and 8.3, respectively. This will complete the classification of all RDP del Pezzo surfaces with global vector fields in odd characteristic.

The strategy of proof is as follows: First, for each degree $1 \leq d \leq 9$, we give the list of RDP configurations Γ that can occur on an RDP del Pezzo surface X of degree d and that contain at least one RDP whose minimal resolution is not T_X-equivariant. Then, starting with the highest possible degree and working our way down with Theorem 7.1, we classify those X containing Γ and satisfying $H^0(X, T_X) \neq 0$. In each step, we give explicit equations and calculate Aut_X^0 using Proposition 4.2, Proposition 4.3, and Proposition 5.8.

NOTATION 8.1. If $d \geq 3$, we use the notation $x_0, \ldots, x_d$ for the coordinates of $\mathbb{P}^d$. If $d = 2$, we use the notation x, y, z and w for the coordinates of $\mathbb{P}(1,1,1,2)$, where w has weight 2. Finally, if $d = 1$, we use the notation s, t, x and y for the coordinates of $\mathbb{P}(1,1,2,3)$, where x has weight 2 and y has weight 3. In Tables 1, 2, 3, 4, 5, and 6, we describe Aut_X^0 as follows:

- We only describe the R-valued points of Aut_X^0, where R is an arbitrary local k-algebra. By Lemma 3.7 in Chapter II, this suffices to describe the scheme structure of Aut_X^0 completely. We do this by either describing a general R-valued point as a matrix or by describing the image of $[x_0 : \ldots : x_n]$ (resp. $[x : y : z : w]$, resp. $[s : t : x : y]$) under a general R-valued automorphism of X.

- We often describe Aut_X^0 as the group scheme $\langle G_1, G_2 \rangle$ generated by subgroup schemes G_1 and G_2 of Aut_X^0. By this we mean that we describe Aut_X^0, using Proposition 4.3, as the smallest subgroup scheme of PGL_{d+1} (resp. $\mathrm{Aut}_{\mathbb{P}(1,1,1,2)}$ if $d = 2$, resp. $\mathrm{Aut}_{\mathbb{P}(1,1,2,3)}$ if $d = 1$) containing both G_1 and G_2.
- We use the variables λ or λ_i for R-valued points of $\mathbb{G}_m$ and μ_{p^n} (where $\lambda^{p^n} = 1$), and the variables ε or ε_i for R-valued points of $\mathbb{G}_a$ and α_{p^n} (where $\varepsilon^{p^n} = 0$).

8.1. In characteristic 7. By Theorem 6.1, we have to list all RDP configurations containing A_6 that can occur on an RDP del Pezzo surface in characteristic 7.

LEMMA 8.2. *If $p = 7$, $\deg(X) = d$, and X contains an A_6-singularity, then d and the configuration Γ of RDPs on X is one of the cases in Table 7.*

d	Γ	$\subseteq \langle k_{9-d} \rangle^{\perp}$
2	A_6	$\subseteq E_7$
1	$A_6, \quad A_6 + A_1$	$\subseteq E_8$

Table 7. Non-equivariant RDP configurations in characteristic 7

PROOF. Since A_6 has rank 6 and discriminant 7, it does not embed into E_{9-d} with $d \geq 3$. By [Dyn52, Table 11], the only root lattice containing A_6 as a direct summand and embedding into E_7 is A_6 itself and there are precisely two root lattices containing A_6 and embedding into E_8, namely A_6 and $A_6 + A_1$. $\square$

THEOREM 8.3. *Assume that $p = 7$ and X contains an RDP of type A_6. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Table 1. Moreover, Aut_X^0 is as in Table 1, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

PROOF. By Lemma 8.2, we have $\deg(X) = d \leq 2$. Assume $H^0(X, T_X) \neq 0$.

<u>If $d = 2$</u>, then, by Theorem 3.2, the anti-canonical system of X realizes X as a double cover of $\mathbb{P}^2$ branched over a quartic curve Q with a simple singularity of type A_6. Over the complex numbers, there is a unique such Q (see [BG81, Proposition 1.3.II]: Up to projective equivalence, one can choose $L = x$, $B = 2y^2$, and $\varphi = y^4 + x^3 y$ in the notation of loc. cit.) and the argument carries over without change to characteristic 7. Now, an elementary calculation shows that the Klein quartic equation

$$x^3 y + y^3 z + z^3 x = 0$$

defines such a Q with an A_6-singularity at $[1 : 2 : -3]$. Thus, X is given by the equation in Table 1. Clearly, the μ_7-action described in Table 1 preserves X. Since $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial by Chapter II, Proposition 5.8 implies that $\mathrm{Aut}_X^0 = \mu_7$.

If $d = 1$, then, by Theorem 3.2, the anti-bi-canonical system of X realizes X as a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. By Table 7, the RDP configuration on X is either A_6 or $A_6 + A_1$. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_1 in a smooth point P of the surface X_2 of Case $d = 2$. By Proposition 5.8, the morphism $Y_1 \to X$ is Aut_X^0-equivariant, since it is an isomorphism around the A_6-singularity. Hence, by Proposition 4.2, we have $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_1}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_2}^0}(P)^0 = \mathrm{Stab}_{\mu_7}(P)^0$. So, since μ_7 is simple, Y_1 is the blow-up of X_2 in a smooth fixed point P of the μ_7-action on X_2, and $\mathrm{Aut}_X^0 = \mu_7$.

Next, we prove the uniqueness of X. We may assume that X_2 is given by the equation in Table 1. The fixed points of the μ_7-action are the three points $[1:0:0:0]$, $[0:1:0:0]$ and $[0:0:1:0]$. The automorphism $x \mapsto y \mapsto z \mapsto x$ of X_2 permutes these fixed points. So, Y_1 is unique and hence so is X.

Therefore, X is the unique RDP del Pezzo surface of degree 1 with an A_6-singularity and non-zero global vector fields. Now, the equation

$$y^2 = x^3 + ts^3x + t^5s$$

defines such an RDP del Pezzo surface of degree 1 in $\mathbb{P}(1,1,2,3)$ with an A_6-singularity at $[1:-3:1:0]$ and, additionally, an A_1-singularity at $[1:0:0:0]$. Clearly, the μ_7-action described in Table 1 preserves X. Hence, this is the surface we were looking for. $\square$

REMARK 8.4. We remark that by Chapter II, in both cases of Theorem 8.3 the minimal resolution $\widetilde{X}$ of X does not admit any non-trivial global vector fields. In particular, $\pi : \widetilde{X} \to X$ is not T_X-equivariant.

8.2. In characteristic 5. By Theorem 6.1, we have to list all RDP configurations containing A_4 or E_8^0 that can occur on an RDP del Pezzo surface in characteristic 5.

LEMMA 8.5. *If $p = 5$, $\deg(X) = d$, and X contains a singularity of type A_4 or E_8^0, then d and the configuration Γ of RDPs on X is one of the cases in Table 8.*

d	Γ	$\subseteq \langle k_{9-d}\rangle^\perp$
5	A_4	$\subseteq A_4$
4	A_4	$\subseteq D_5$
3	$A_4,\quad A_4 + A_1$	$\subseteq E_6$
2	$A_4,\quad A_4 + A_1,\quad A_4 + A_2$	$\subseteq E_7$
1	$A_4,\quad A_4 + A_1,\quad A_4 + 2A_1,\quad A_4 + A_2,$ $A_4 + A_2 + A_1,\quad A_4 + A_3,\quad 2A_4,\quad E_8^0$	$\subseteq E_8$

Table 8. Non-equivariant RDP configurations in characteristic 5

PROOF. Since A_4 has rank 4 and discriminant 5, it does not embed into E_{9-d} with $d \geq 6$, and the only root lattice containing A_4 as a direct summand and embedding into E_{9-d} with $d \in \{4, 5\}$ is A_4 itself. The other cases can be found in [Dyn52, Table 11]. $\square$

THEOREM 8.6. *Assume that $p = 5$ and X contains an RDP of type A_4 or E_8^0. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Table 2. Moreover, Aut_X^0 is as in Table 2, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

PROOF. By Lemma 8.5, we have $d \leq 5$.

<u>If $d = 5$</u>, then X is a quintic surface in $\mathbb{P}^5$. By Table 8, the RDP configuration on X is A_4. By the same argument as in characteristic 0 (going through the possible configurations of four (possibly infinitely) near points in $\mathbb{P}^2$), there is a unique quintic surface in $\mathbb{P}^5$ containing an A_4-singularity. It is given by the equations in Table 2 (see [Der14, Section 3.3., p.657]) with singular point at $[0 : 0 : 0 : 0 : 0 : 1]$. The α_5-action given in Table 2 preserves X and does not preserve the singular point, hence $\alpha_5 \cap \mathrm{Aut}_{\widetilde{X}}^0 = \{\mathrm{id}\}$. By Proposition 5.8, this implies that $\mathrm{Aut}_X^0 = \langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.

<u>If $d = 4$</u>, then X is a quartic surface in $\mathbb{P}^4$. By Table 8, the RDP configuration on X is A_4. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_4 in a smooth point P of the surface X_5 of Case $d = 5$.

Such an X is in fact unique: By Chapter II, we have $\dim \mathrm{Aut}_{X_5}^0 = \dim \mathrm{Aut}_{\widetilde{X}_5}^0 = 4$, and since the orbit of P is at most 2-dimensional, the stabilizer of P is positive-dimensional, so $H^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0$. By Chapter II, there is a unique quartic del Pezzo surface with an A_4-singularity and whose minimal resolution has global vector fields, hence X is unique.

In Table 2, we give equations for such a surface (which we took from [Der14]), hence this is our X. The singular point is located at $[0 : 0 : 0 : 0 : 1]$. The α_5-action given in Table 2 preserves X and does not preserve the singular point. Again, by Proposition 5.8, this implies that $\mathrm{Aut}_X^0 = \langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.

<u>If $d = 3$</u>, then X is a cubic surface in $\mathbb{P}^3$. By Table 8, the RDP configuration on X is A_4 or $A_4 + A_1$. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_3 in a smooth point P of the surface X_4 of Case $d = 4$.

Next, we show that there are at most two non-isomorphic such X. By Chapter II, we have $\dim \mathrm{Aut}_{X_4}^0 = \dim \mathrm{Aut}_{\widetilde{X}_4}^0 = 2$. There is at most one 2-dimensional orbit on X_4, hence there is at most one X whose minimal resolution does not admit global vector fields. On the other hand, by Chapter II, there is precisely one X whose minimal resolution does have global vector fields.

In Table 2, we give two (non-isomorphic) equations for cubic surfaces with an A_4-singularity, distinguished by their RDP configuration Γ, hence these are the two possible X:

(1) If $\Gamma = A_4$, the singular point is located at $[-2 : -1 : 2 : 1]$. We describe a μ_5-action on X in Table 2. In this case, $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial by Chapter II, hence $\mathrm{Aut}_X^0 = \mu_5$ by Proposition 5.8.
(2) If $\Gamma = A_4 + A_1$, the A_4-singularity is $[0 : 0 : 0 : 1]$ while the A_1-singularity is $[1 : 0 : 0 : 0]$. We describe an $\alpha_5 \rtimes \mathbb{G}_m$-action on X in Table 2. In this case, $\mathrm{Aut}_{\widetilde{X}}^0 = \mathbb{G}_m$ by Chapter II, hence $\mathrm{Aut}_X^0 = \alpha_5 \rtimes \mathbb{G}_m$ by Proposition 5.8.

<u>If $d = 2$</u>, then X is a double cover of $\mathbb{P}^2$ branched over a quartic curve Q. By Table 8, the possible RDP configurations on X are A_4, A_4+A_1, and A_4+A_2. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_2 in a smooth point P of an RDP del Pezzo surface

X_3 of degree 3 with an A_4-singularity. Since $Y_2 \to X_3$ and $Y_2 \to X$ are isomorphisms around the A_4-singularities, Proposition 5.8 yields $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_2}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}^0}(P)^0$. Hence, for each of the surfaces X_3 in Case $d = 3$, we have to determine the points with non-trivial stabilizer.

- If the RDP configuration on X_3 is A_4, then the points with non-trivial stabilizer under the action of $\mathrm{Aut}_{X_3}^0 = \mu_5$ are $[1:0:0:0], [0:1:0:0], [0:0:1:0]$, and $[0:0:0:1]$ and these fixed points are permuted by the automorphism $x_0 \mapsto x_1 \mapsto x_2 \mapsto x_3 \mapsto x_0$. Hence, there is a unique choice for P up to isomorphism.
- If the RDP configuration on X_3 is $A_4 + A_1$, then there are four lines on X_3: The lines $\ell_1 = \{x_0 = x_1 = 0\}, \ell_2 = \{x_0 = x_2 = 0\}, \ell_3 = \{x_1 = x_2 = 0\}$ pass through the A_4-singularity at $[0:0:0:1]$ and the line $\ell_4 = \{x_2 = x_3 = 0\}$ passes through the A_1-singularity at $[1:0:0:0]$, but not through A_4. Moreover, ℓ_2 and ℓ_4 intersect in $[0:1:0:0]$. Straightforward calculation shows that the points with non-trivial stabilizer in $X_3 \setminus \{\ell_1, \ell_2, \ell_3\}$ are precisely those lying on the hyperplane $H = \{x_3 = 0\}$. The intersection $X_3 \cap H$ is the union of the conic $C = \{x_3 = x_0x_2 + x_1^2 = 0\}$ and the line ℓ_4. Hence, either $P \in C \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = C \setminus \{[1:0:0:0], [0:0:1:0]\}$ or $P \in \ell_4 \setminus (\ell_1 \cup \ell_2 \cup \ell_3) = \ell_4 \setminus \{[1:0:0:0], [0:1:0:0]\}$. The group scheme $\mathrm{Aut}_{X_3}^0$ acts transitively on both loci, hence there are only two choices for P up to isomorphism. In both cases, one checks that $\mathrm{Aut}_{Y_2}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}}(P)^0 = \mu_5$. One of the two choices of P can be reduced to a previous case as follows:
  - If $P \in C \setminus \{[1:0:0:0], [0:0:1:0]\}$, then the RDP configuration on X is $A_4 + A_1$. The strict transform C' of C on the blow-up X' of X in the A_1-singularity is a (-1)-curve that passes through the (-2)-curve over A_1 and which is disjoint from A_4. If we contract C', we obtain an RDP del Pezzo surface X_3' of degree 3 which contains an A_4-singularity as its only singularity and such that $H^0(X_3', T_{X_3'}) \neq 0$ (by Proposition 5.5 and Blanchard's Lemma). Hence X_3' is isomorphic to the cubic surface with RDP configuration A_4 in Table 8, and thus X coincides with the surface we constructed in the previous bullet point.

Summarizing, there are at most two RDP del Pezzo surfaces of degree 2 which contain an A_4-singularity and which admit a non-trivial global vector field. Moreover, $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_3}^0}(P) = \mu_5$ in both cases. In Table 2, we give two equations of such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_4 + A_1$, the A_4-singularity is $[-2:1:2:0]$, the A_1-singularity is $[0:1:0:0]$, and the corresponding $\mathrm{Aut}_X^0 = \mu_5$-action is as in Table 2
(2) If $\Gamma = A_4 + A_2$, the A_4-singularity is $[2:1:-1:0]$, the A_2-singularity is $[1:0:0:0]$, and the corresponding $\mathrm{Aut}_X^0 = \mu_5$-action is as in Table 2.

<u>If $d = 1$</u>, then X is a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. We consider separately the three cases where X contains a single A_4-singularity (and possibly equivariant RDPs of other types), two A_4-singularities, and an E_8^0-singularity, respectively:

(a) X contains a single A_4-singularity. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_1 in a smooth point P of an RDP del Pezzo surface X_2 with an A_4-singularity. By Proposition 5.8, we have $\mathrm{Aut}_X^0 = \mathrm{Aut}_{Y_1}^0 = \mathrm{Stab}_{\mathrm{Aut}_{X_2}^0}(P)^0$. Since $\mathrm{Aut}_{X_2}^0 = \mu_5$, we thus have to determine the fixed points of the μ_5-action on X_2.

- If the RDP configuration on X_2 is $A_4 + A_2$, then the fixed points of the μ_5-action are $[1:0:0:0], [0:1:0:0]$, and $[0:0:1:0]$, and we recall that $[1:0:0:0]$ is the A_2-singularity. Hence, there are two choices for P. In fact, one of them does not occur, as the following argument shows:
  - If $P = [0:0:1:0]$, let Q be the branch quartic of $X_2 \to \mathbb{P}^2$. The line $\ell = \{y = 0\}$ is tangent to Q at $[1:0:0]$ and $[0:0:1]$, hence the preimage of ℓ in X_2 consists of two smooth rational curves C, C' meeting in P and the A_2-singularity. In the minimal resolution $\widetilde{Y}_1$ of Y_1, their strict transforms $\widetilde{C}$ and $\widetilde{C}'$ are (-2)-curves, which, together with the two exceptional curves over the A_2-singularity, form an A_4-configuration of (-2)-curves. Thus, X contains two A_4-singularities, contradicting our assumption.
- If the RDP configuration on X_2 is $A_4 + A_1$, then the fixed points of the μ_5-action are $[1:0:0:1], [1:0:0:-1], [0:1:0:0]$, and $[0:0:1:0]$. The first two points are interchanged by $w \mapsto -w$ and the point $[0:1:0:0]$ is the A_1-singularity on X_2, hence we have two choices for P up to isomorphism. Let Q be the branch quartic of $X_2 \to \mathbb{P}^2$ and recall that the A_4-singularity is located at $[-2:1:2:0]$. We will now show that both choices for P lead to the same surface as the one in the previous bullet point:
  - If $P = [1:0:0:1]$, then the image $[1:0:0]$ of P in $\mathbb{P}^2$ lies on the two bitangents $\ell_1 = \{y = 0\}$ and $\ell_2 = \{z = 0\}$ of Q. Let C_i, C_i' be the two irreducible components of the preimage of ℓ_i for $i = 1, 2$ and assume that P lies on C_1 and C_2. On $\widetilde{Y}_1$, the strict transforms $\widetilde{C}_1$ and $\widetilde{C}_2$ are (-2)-curves, while the strict transforms $\widetilde{C}_1'$ and $\widetilde{C}_2'$ remain (-1)-curves. Thus, the RDP configuration on X is $A_4 + A_2 + A_1$, where the A_2 is obtained from the A_1 of Y_1 by also contracting $\widetilde{C}_2$, and the A_1 arises from the contraction of $\widetilde{C}_1$. Therefore, if we contract the image of C_1' in the surface Y_1' obtained from Y_1 by contracting C_2, we obtain an RDP del Pezzo surface of degree 2 with global vector fields and containing an RDP configuration of type $A_4 + A_2$, hence this case is reduced to the previous bullet point.
  - If $P = [0:0:1:0]$, then the image $[0:0:1]$ of P in $\mathbb{P}^2$ lies on the bitangent $\ell = \{y = 0\}$ and P is the non-transversal intersection point of the two irreducible components C and C' of the preimage of

ℓ in X_2. Thus, the strict transforms $\widetilde{C}$ and $\widetilde{C}'$ of C and C' on $\widetilde{Y}_1$ are (-2)-curves, hence the RDP configuration on X is $A_4+A_2+A_1$, where the A_2 is obtained by contracting $\widetilde{C}$ and $\widetilde{C}'$. The preimage D of the line $\{x = 0\}$ in X_2 is a cuspidal cubic with cusp at $[0 : 1 : 0 : 0]$. On $\widetilde{X}_2$, the strict transform of D is a smooth rational curve of self-intersection 0 by adjunction. Hence, the strict transform $\widetilde{D}$ of D on $\widetilde{Y}_1$ is a (-1)-curve which passes through the exceptional curve over the A_1-singularity. Contracting the image of $\widetilde{D}$ on the surface Y_1' obtained from Y_1 by contracting C and C' and resolving the A_1-singularity, we obtain an RDP del Pezzo surface of degree 2 with global vector fields containing an RDP configuration of type $A_4 + A_2$, hence this case is also reduced to the previous bullet point.

Summarizing, there is at most one RDP del Pezzo surface of degree 1 with global vector fields and containing a single A_4-singularity. In Table 2, we give an equation of such a surface, hence this is our X. The singularities of X are as follows: A_4 at $[1 : -2 : 2 : 0]$, A_2 at $[0 : 1 : 0 : 0]$, and A_1 at $[1 : 0 : 0 : 0]$. The μ_5-action we describe in Table 2 preserves the equation, hence $\mathrm{Aut}_X^0 = \mu_5$.

(b) X contains two A_4-singularities. By [Lan94, Theorem 4.1.], there is a unique RDP del Pezzo surface of degree 1 with RDP configuration $A_4 + A_4$, namely the Weierstrass model of the rational elliptic surface with singular fibers of type $\mathrm{I}_5, \mathrm{I}_5, \mathrm{II}$. Its equation is given in Table 2. The $\alpha_5 \rtimes \mu_5$-action we describe in Table 2 preserves the equation. By Chapter II, $\mathrm{Aut}_{\widetilde{X}}^0$ is trivial, hence $\mathrm{Aut}_X^0 = \alpha_5 \rtimes \mu_5$ follows from Proposition 5.8.

(c) X contains an E_8^0-singularity. By [Lan94], there are two RDP del Pezzo surfaces of degree 1 with an RDP of type E_8. The one whose equation we give in Table 2 has an E_8^0-singularity, while the other one has an E_8^1-singularity (see Table 2 in Chapter I). The $\alpha_5 \rtimes \mathbb{G}_m$-action we describe in Table 2 preserves the equation. We leave it to the reader to check that $\mathrm{Aut}_X^0 = \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S)^0 = \alpha_5 \rtimes \mathbb{G}_m$.

$\square$

8.3. In characteristic 3. By Theorem 6.1, we have to list all RDP configurations containing $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$ or E_8^1 that can occur on an RDP del Pezzo surface.

LEMMA 8.7. *If* $p = 3$, $\deg(X) = d$*, and* X *contains a singularity of type* $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$ *or* E_8^1*, then* d *and the configuration* Γ *of RDPs on* X *is one of the cases in Table 9.*

PROOF. The maximal root lattice contained in E_2 is isomorphic to A_1, hence none of the root lattices in the statement of the lemma embed into E_{9-d} with $d \geq 7$. The list for $4 \leq d \leq 6$ follows from [Mar03, Exercise 4.2.1, 4.6.2] and the one for $7 \leq d \leq 9$ follows from [Dyn52, Table 11] (note that the lattice $A_6 + A_2$ in Dynkin's table of root lattices in E_8 should be $E_6 + A_2$), where we marked those RDP configurations whose associated root lattice embeds in two non-conjugate ways into E_{9-d} by a prime $'$. $\square$

d	Γ	$\subseteq \langle k_{9-d} \rangle^{\perp}$
6	$A_2, \quad A_2 + A_1$	$\subseteq A_2 + A_1$
5	$A_2, \quad A_2 + A_1$	$\subseteq A_4$
4	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1$	$\subseteq D_5$
3	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad 2A_2 + A_1, \quad A_5,$ $3A_2, \quad A_5 + A_1, \quad E_6^0, \quad E_6^1$	$\subseteq E_6$
2	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad A_2 + 3A_1, \quad 2A_2 + A_1,$ $A_3 + A_2, \quad (A_5)', \quad 3A_2, \quad A_3 + A_2 + A_1, \quad A_4 + A_2,$ $(A_5 + A_1)', \quad E_6^0, \quad E_6^1, \quad A_5 + A_2, \quad E_7^0$	$\subseteq E_7$
1	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad A_2 + 3A_1, \quad 2A_2 + A_1,$ $A_3 + A_2, \quad A_5, \quad A_2 + 4A_1, \quad 2A_2 + 2A_1, \quad 3A_2, \quad A_3 + A_2 + A_1,$ $A_4 + A_2, \quad D_4 + A_2, \quad (A_5 + A_1)', \quad E_6^0, \quad E_6^1,$ $3A_2 + A_1, \quad A_3 + A_2 + 2A_1, \quad A_4 + A_2 + A_1, \quad A_5 + 2A_1,$ $A_5 + A_2, \quad D_5 + A_2, \quad E_6^0 + A_1, \quad E_6^1 + A_1, \quad E_7^0,$ $4A_2, \quad A_5 + A_2 + A_1, \quad E_6^0 + A_2, \quad E_6^1 + A_2, \quad A_8, \quad E_8^0, \quad E_8^1$	$\subseteq E_8$

Table 9. Non-equivariant RDP configurations in characteristic 3

THEOREM 8.8. *Assume that $p = 3$ and X contains an RDP of type $A_2, A_5, A_8, E_6^0, E_6^1, E_7^0, E_8^0$, or E_8^1. Then, $H^0(X, T_X) \neq 0$ if and only if X is given by an equation as in Tables 3, 4, 5 and 6. Moreover, Aut_X^0 is as given in these tables, so that $\mathrm{Aut}_{\widetilde{X}}^0 \subsetneq \mathrm{Aut}_X^0$ and even $h^0(X, T_X) > h^0(\widetilde{X}, T_{\widetilde{X}})$.*

PROOF. By Lemma 8.7, we have $d \leq 6$.

<u>If $d = 6$</u>, then X is a sextic surface in $\mathbb{P}^6$. The RDP configuration Γ on X is either A_2 or $A_2 + A_1$ by Table 9. In both cases, X is uniquely determined by its singularities, by the same argument as in characteristic 0, that is, by checking the possible configurations of infinitely near points in $\mathbb{P}^2$ (see e.g. [Dol12, Section 8.4.2]).

(a) If $\Gamma = A_2$ then, by [Der14, Section 3.2.], X is given by the equations in Table 3. The A_2-singularity is $[1 : 0 : 0 : 0 : 0 : 0 : 0]$ and we give an α_3-action on X which does not preserve the singularity in Table 3. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.
(b) If $\Gamma = A_2 + A_1$, then, by [KN09, Appendix, p.3], X is given by the equation in Table 3. The A_2-singularity is $[0 : 1 : 0 : 0 : 0 : 0 : 0]$ and the A_1-singularity is $[0 : 0 : 0 : 0 : 0 : 0 : 1]$. We give an α_3-action on X which does not fix the A_2-singularity in Table 3. By Proposition 5.8, this implies $\mathrm{Aut}_X^0 = \langle \alpha_3, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$.

<u>If $d = 5$</u>, then X is a quintic surface in $\mathbb{P}^5$. By Table 9, the RDP configuration Γ on X is either A_2 or $A_2 + A_1$. As in the Case $d = 6$, X is uniquely determined by Γ, as can be seen by checking the possible configurations of four infinitely near points in $\mathbb{P}^2$ (see e.g. [Dol12, Section 8.5.1]).

(a) If $\Gamma = A_2$, then, by [Der14, Section 3.3.], X is given by the equation in Table 3. The A_2-singularity is $[0 : 0 : 1 : 0 : 0 : 0]$ and we give an α_3-action on X

which does not preserve the singularity in Table 3. By Proposition 5.8, this implies $\mathrm{Aut}^0_X = \langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$.

(b) If $\Gamma = A_2 + A_1$, then by [KN09, Appendix, p.5] is given by the equation in Table 3. The A_2-singularity is $[0:1:0:0:0:0]$ and the A_1-singularity is $[0:0:0:0:0:1]$. We give an α_3-action on X which does not fix the A_2-singularity in Table 3. By Proposition 5.8, this implies $\mathrm{Aut}^0_X = \langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$.

<u>If $d = 4$</u>, then X is a quartic surface in $\mathbb{P}^4$. By Table 9, the RDP configuration on X is A_2, $A_2 + A_1$, or $A_2 + 2A_1$. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_4 in a smooth point P of an RDP del Pezzo surface X_5 of degree 5 with an A_2-singularity.

Next, we show that there are at most three non-isomorphic such X. By Chapter II, there are at most two X whose minimal resolution has global vector fields. Assume that $\mathrm{Aut}^0_{\widetilde{X}} = \{\mathrm{id}\}$. Then, by Proposition 4.2 and Proposition 5.8, the stabilizer of P is 0-dimensional, hence $\dim \mathrm{Aut}^0_{\widetilde{X}_5} = \dim \mathrm{Aut}^0_{X_5} \leq 2$. Hence, by Chapter II, X_5 is the RDP del Pezzo surface of degree 5 whose RDP configuration is A_2. For this surface, $\dim \mathrm{Aut}^0_{X_5} = 2$, hence there is at most one 2-dimensional orbit on X_5, so there is at most one X whose minimal resolution does not have global vector fields. In Table 3, we give equations for three such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_2$, the A_2-singularity is at $[1:1:1:1:1]$. We give a μ_3-action on X in Table 3, while the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by Chapter II. By Proposition 5.8, this implies $\mathrm{Aut}^0_X = \mu_3$.
(2) If $\Gamma = A_2 + A_1$, the A_2-singularity is $[1:0:0:0:0]$ while the A_1-singularity is $[0:0:0:0:1]$. We give a $\alpha_3 \rtimes \mathbb{G}_m$-action on X in Table 3 , while $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$ by Chapter II, hence $\mathrm{Aut}^0_X = \alpha_3 \rtimes \mathbb{G}_m$ by Proposition 5.8.
(3) If $\Gamma = A_2 + 2A_1$, the A_2-singularity is $[0:0:0:0:1]$ and the two A_1-singularities are $[0:1:0:0:0]$ and $[0:0:1:0:0]$. We give a $\alpha_3 \rtimes \mathbb{G}_m^2$-action on X in Table 3 , while $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m^2$ by Chapter II, hence $\mathrm{Aut}^0_X = \alpha_3 \rtimes \mathbb{G}_m^2$ by Proposition 5.8.

<u>If $d = 3$</u>, then X is a cubic surface in $\mathbb{P}^3$. By Table 9, X contains either a single A_2, two A_2s, three A_2s, one A_5, one E_6^0, or one E_6^1. In the following, we consider these six cases separately.

(a) X contains a single A_2-singularity. By Corollary 7.2, X is the anti-canonical model of a blow-up Y_3 in a smooth point P of an RDP del Pezzo surface X_4 with an A_2-singularity. More precisely, since $X \dashrightarrow X_4$ is an isomorphism around the only non-equivariant RDP on X and all other singularities of X are A_1-singularities by Table 9, Proposition 5.8 and Proposition 4.2 imply that $\mathrm{Aut}^0_X = \mathrm{Aut}^0_{Y_3} = \mathrm{Stab}_{\mathrm{Aut}^0_{X_4}}(P)^0$. In particular, P is a point with non-trivial stabilizer on one of the surfaces X_4 in Table 3. Before we go on, note that by Chapter II, the group scheme $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence the stabilizer of P is 0-dimensional.
   - Assume the RDP configuration on X_4 is $A_2 + 2A_1$. In this case, $\mathrm{Aut}^0_{X_4}$ is 2-dimensional, so there is at most one 2-dimensional orbit for this action, hence there is at most one choice for P up to isomorphism.

- Assume the RDP configuration on X_4 is $A_2 + A_1$. The surface X_4 contains the six lines $\ell_1 = \{x_0 = x_2 = x_3 = 0\}, \ell_2 = \{x_0 = x_2 = x_4 = 0\}, \ell_3 = \{x_0 = x_3 = x_1 + x_4 = 0\}, \ell_4 = \{x_1 = x_2 = x_3 = 0\}, \ell_5 = \{x_1 = x_2 = x_4 = 0\}, \ell_6 = \{x_1 = x_3 = x_4 = 0\}$. The lines ℓ_4, ℓ_5, and ℓ_6 pass through the A_2-singularity at $[1 : 0 : 0 : 0 : 0]$ and the others do not. The lines ℓ_1 and ℓ_4 pass through the A_1-singularity at $[0 : 0 : 0 : 0 : 1]$ and the others do not. Using the description of the $\mathrm{Aut}^0_{X_4}$-action in Table 3, one easily checks that the points with non-trivial and 0-dimensional stabilizer on X_4 are precisely those on ℓ_2 and those on ℓ_3, except for $[0 : 1 : 0 : 0 : 0]$, $[0 : 0 : 1 : 0 : 0], [0 : 0 : 0 : 1 : 0]$, and $[0 : 1 : 0 : 0 : -1]$. Since the automorphism $x_2 \leftrightarrow x_3, x_4 \leftrightarrow -x_4 - x_1$ interchanges the two lines ℓ_2 and ℓ_3 and $\mathbb{G}_m \subseteq \mathrm{Aut}_{X_4}$ acts transitively on the locus of points with 0-dimensional stabilizer on each of ℓ_2 and ℓ_3, there is a unique choice for P up to isomorphism. We will now reduce this case to the previous bullet point.
  - Without loss of generality, assume that $P \in \ell_2 \setminus \{[0 : 1 : 0 : 0 : 0], [0 : 0 : 0 : 1 : 0]\}$. Then, the strict transform of ℓ_2 on Y_3 is a (-2)-curve and the RDP configuration on X is $A_2 + 2A_1$. Since ℓ_3 is disjoint from ℓ_2, it remains a (-1)-curve on X, hence we can contract it to obtain a realization of X as a blow-up of an RDP del Pezzo surface X_4' with RDPs of type $A_2 + 2A_1$. Hence, this case is reduced to the previous bullet point.
- Assume the RDP configuration on X_4 is A_2. Using the description of the $\mathrm{Aut}^0_{X_4} = \mu_3$-action given in Table 3, we see that the points P with non-trivial stabilizer on X_4 are $[1 : 0 : 0 : 0 : 0], [0 : 1 : 0 : 0 : 0], [0 : 0 : 1 : 0 : 0], [0 : 0 : 0 : 1 : 0]$, and $[0 : 0 : 0 : 0 : 1]$. The surface X_4 admits the two involutions $x_0 \leftrightarrow x_1$ and $(x_0, x_1) \leftrightarrow (x_2, x_3)$. Hence, blowing up any of the first four points leads to the same surface. In fact, we already treated the resulting surface, as the following argument shows:
  - Assume without loss of generality that $P = [1 : 0 : 0 : 0 : 0]$. There are two lines on X_4 passing through P, namely $\ell_1 = \{x_1 = x_2 = x_4 = 0\}$ and $\ell_2 = \{x_1 = x_3 = x_4 = 0\}$, and both of these lines do not pass through the A_2-singularity at $[1 : 1 : 1 : 1 : 1]$. Their strict transforms on Y_3 are disjoint (-2)-curves. Thus, the RDP configuration on X is A_2+2A_1. Now, the conic $C = \{x_1 = x_2+x_3 = x_0x_4-x_3^2 = 0\}$ meets ℓ_1 and ℓ_2 transversally at P and does not pass through $[1 : 1 : 1 : 1 : 1]$, hence we can contract the image of the strict transform of C in X and obtain an RDP del Pezzo surface X_4' with RDP configuration A_2+2A_1 and with global vector fields, hence X_4' is the surface in the first bullet point.

Summarizing, there are at most two RDP del Pezzo surfaces of degree 3 with global vector fields and containing a single A_2-singularity. Moreover, $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{Aut}^0_{X_4}}(P)^0 = \mu_3$. In Table 4, we give equations for two such surfaces, distinguished by their RDP configuration Γ:

(1) If $\Gamma = A_2$, the A_2-singularity is at $[1:1:-1:-1]$. Clearly, the μ_3-action we give preserves the equation.

(2) If $\Gamma = A_2 + 2A_1$, the A_2-singularity is at $[1:-1:-1:1]$ and the two A_1-singularities are at $[0:1:0:0]$ and $[0:0:1:0]$. Again, the μ_3-action we give preserves the equation.

(b) X contains two A_2-singularities. By Table 9, the RDP configuration Γ on X is $2A_2$ or $2A_2 + A_1$. Simplifying the normal form of Roczen given in [Roc96], we obtain the equations given in Table 4. In particular, there is a 1-dimensional family of X with $\Gamma = 2A_2$ and a unique X with $\Gamma = 2A_2 + A_1$.

(1) If $\Gamma = 2A_2$, the two A_2-singularities are at $[0:0:1:0]$ and $[0:0:0:1]$. The two α_3-actions and the $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$-action we give in Table 4 preserve the equation. Each of the α_3-actions fixes one of the A_2-singularities and does not preserve the other one, hence Proposition 5.8 shows $\mathrm{Aut}^0_X = \langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$.

(2) If $\Gamma = 2A_2 + A_1$, the two A_2-singularities are at $[0:0:1:0]$ and $[0:0:0:1]$ and the A_1-singularity is at $[0:1:0:0]$. The two α_3-actions and the $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$-action we describe in Table 4 preserve the equation. By the same argument as in (1), we have $\mathrm{Aut}^0_X = \langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$.

(c) X contains three A_2-singularities. Again, we can simplify the normal form of Roczen given in [Roc96] and obtain the equation in Table 4, which admits A_2-singularities at $[0:1:0:0]$, $[0:0:1:0]$, and $[0:0:0:1]$. In Table 4, we give an action of $\alpha_3^3 \rtimes \mathbb{G}_m^2$ on X. By Chapter II, we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m^2$. Each of the the three factors of the α_3^3-action preserves two of the A_2-singularities and moves the other one, hence Proposition 5.8 yields $\mathrm{Aut}^0_X = \alpha_3^3 \rtimes \mathbb{G}_m^2$.

(d) X contains an A_5-singularity. As above, we simplify Roczen's normal form [Roc96] to the two equations in Table 4. Let Γ be the RDP configuration on X:

(1) If $\Gamma = A_5$, then the A_5-singularity is at $[0:0:0:1]$. The α_3-action we describe preserves the equation and does not fix the A_5-singularity. By Chapter II, we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mu_3$ and we describe this action in terms of the equation in Table 4. By Remark 5.9, we have $\mathrm{Aut}^0_X = \langle \alpha_3, \mathbb{G}_a \rtimes \mu_3 \rangle$.

(2) If $\Gamma = A_5 + A_1$, then the A_5-singularity is at $[0:0:0:1]$ and the A_1-singularity is at $[0:0:1:0]$. The α_3-action we describe preserves the equation. By Chapter II, we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mathbb{G}_m$ and we describe this action in terms of the equation in Table 4. By Remark 5.9, we have $\mathrm{Aut}^0_X = \langle \alpha_3, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$.

(e) X contains an E_6^0-singularity. By [Roc96, Case C3], X is given by the equation in Table 4 and the singularity is at $[0:0:0:1]$. In Table 4, we give an action of a group scheme G of order 27 such that no subgroup scheme of G lifts to $\widetilde{X}$, as well as the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a^2 \rtimes \mathbb{G}_m$ in terms of the equation. We leave it to the reader to check that these two actions generate $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$.

(f) X contains an E_6^1-singularity. By [Roc96], X is given by the equation in Table 4 and the singularity is at $[0:0:0:1]$. In Table 4, we give a μ_3-action that does not preserve the singular point, as well as the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a^2$ in terms of the equation. We leave it to the reader to check that these two actions generate $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_4}(X)^0$.

If $d = 2$, then X is a double cover of $\mathbb{P}^2$ branched over a quartic curve Q. In this case, we will take a slightly different approach which will turn out to be more economical than using Corollary 7.2. Namely, we classify all possible Q with global vector fields. If Q admits a global vector field, then it also admits an additive or multiplicative global vector field. This vector field is induced by a vector field on $\mathbb{P}^2$. Up to conjugation, there are two non-zero vector fields D on $\mathbb{P}^2$ with $D^3 = D$ and two non-zero vector fields with $D^3 = 0$. They correspond to the following four matrices in Jordan normal form in the Lie algebra of PGL_3:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Integrating the corresponding vector fields (see e.g. [Tzi17, Proposition 3.1]), we obtain the following four μ_3- and α_3-actions on $\mathbb{P}^2$:

(a) $\mu_3 : [x : y : z] \mapsto [x : \lambda y : \lambda^2 z]$
(b) $\mu_3 : [x : y : z] \mapsto [x : y : \lambda z]$
(c) $\alpha_3 : [x : y : z] \mapsto [x + \varepsilon y : y : z]$
(d) $\alpha_3 : [x : y : z] \mapsto [x + \varepsilon y - \varepsilon^2 z : y + \varepsilon z : z]$

For each of these actions, we will now classify the quartics that are invariant under it:

(a) There are three types of quartics which are invariant under this μ_3-action, namely

$$\begin{gathered} ax^4 + bx^2yz + cxy^3 + dxz^3 + ey^2z^2 \\ ax^3y + bx^2z^2 + cxy^2z + dy^4 + eyz^3 \\ ax^3z + bx^2y^2 + cxyz^2 + dy^3z + ez^4 \end{gathered}$$

These families are identified by permuting x, y and z, so it suffices to study the first one. Now, we simplify this equation and identify the singularities. We sort the classification according to the number of coefficients that are 0. Since Q is reduced, at least 2 coefficients are non-zero. Except in the case where $c = d = 0$, we can scale three of the non-zero coefficients to 1.

- If three of the coefficients are 0, the other two can be scaled to 1. The fact that Q is reduced leaves us with the following three cases, after using the symmetry $y \leftrightarrow z$:

$$x^4 + y^2z^2 \tag{8.1}$$
$$x^2yz + xy^3 \tag{8.2}$$
$$x^2yz + y^2z^2 \tag{8.3}$$

In Case (8.1), X has two A_3-singularities, one at $[0 : 1 : 0 : 0]$ and one at $[0 : 0 : 1 : 0]$. In particular, X contains only equivariant RDPs, so it does not appear in Table 5.

In Case (8.2), X has a D_6-singularity at $[0:0:1:0]$ and an A_1-singularity at $[1:0:0:0]$. As before, X contains only equivariant RDPs, so it does not appear in Table 5.
In Case (8.3), X has an A_1-singularity at $[1:0:0:0]$ and two A_3-singularities at $[0:1:0:0]$ and $[0:0:1:0]$. Again, X contains only equivariant RDPs, so it does not appear in Table 5.

- If two of the coefficients are 0, we get the following cases, again using scaling and the symmetry $y \leftrightarrow z$:

$$x^4 + ax^2yz + y^2z^2 \tag{8.4}$$
$$x^4 + x^2yz + xy^3 \tag{8.5}$$
$$x^4 + y^2z^2 + xy^3 \tag{8.6}$$
$$x^2yz + y^2z^2 + xy^3 \tag{8.7}$$
$$x^2yz + xy^3 + xz^3 \tag{8.8}$$
$$xy^3 + xz^3 + y^2z^2 \tag{8.9}$$

In Case (8.4), we must have $a^2 \neq 1$, otherwise Q is a double conic. Then, Q is the union of two conics, tangent at the points $[0:1:0]$ and $[0:0:1]$. The singularities of X over these two points are two A_3-singularities, hence X contains only equivariant RDPs and does not appear in Table 5.
In Case (8.5), X has a D_6-singularity at $[0:0:1:0]$, hence X contains only equivariant RDPs and does not appear in Table 5.
In Case (8.6), X has an A_3-singularity at $[0:0:1:0]$ and an A_2-singularity at $[1:-1:0:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.
In Case (8.7), X has an A_3-singularity at $[0:0:1:0]$, an A_2-singularity at $[1:1:1:0]$, and an A_1-singularity at $[1:0:0:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.
In Case (8.8), X has an A_5-singularity at $[0:1:-1:0]$ and an A_1-singularity at $[1:0:0:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence Proposition 5.8 implies $\mathrm{Aut}^0_X = \mu_3$.
In Case (8.9), X has an E_6^1-singularity at $[1:0:0:0]$. It is elementary to check that $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{PGL}_3}(Q) = \mu_3$ in this case. Alternatively, observe that this case does not occur in the classification of invariant quartics for the actions (b), (c), and (d) below, so $\mathrm{Aut}^0_X[F] = \mu_3$, where $\mathrm{Aut}^0_X[F]$ is the Frobenius kernel of Aut^0_X. In particular, μ_3 is normal in Aut^0_X, so the Aut^0_X-action preserves the eigenspaces of the μ_3-action, hence the induced action on $\mathbb{P}^2$ is diagonal. This simplifies the calculation of the stabilizer of Q considerably.

- If only one coefficient is 0, then we get the following cases, again using scaling and the symmetry $y \leftrightarrow z$:

$$x^4 + ax^2yz + xy^3 + xz^3 \tag{8.10}$$
$$x^4 + a^3x^2yz + xy^3 + y^2z^2 \tag{8.11}$$
$$x^4 + xy^3 + xz^3 + ay^2z^2 \tag{8.12}$$
$$ax^2yz + xy^3 + xz^3 + y^2z^2 \tag{8.13}$$

In Case (8.10), X has an A_5-singularity at $[0 : 1 : -1 : 0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, so $\mathrm{Aut}^0_X = \mu_3$ by Remark 5.9.

Consider Case (8.11). If $a^2 = 1$, then X has an A_6-singularity at $[0 : 0 : 1 : 0]$, hence X contains only equivariant RDPs and does not occur in Table 5. If $a^2 \neq 1$, then X has an A_3-singularity at $[0 : 0 : 1 : 0]$ and an A_2-singularity at $[a^2 - 1 : a^4 + a^2 + 1 : a^3 : 0]$. In this case, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by Chapter II, so $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8.

In Case (8.12), X has two A_2-singularities, one at $[1 : 0 : -1 : 0]$ and one at $[1 : -1 : 0 : 0]$. We leave it to the reader to check that $\mathrm{Aut}^0_X = \mu_3$ in this case.

In Case (8.13), X has an A_1-singularity at $[1 : 0 : 0 : 0]$, and additional singularities at $[u : au^2 : 1]$, where u is a solution of $a^3u^6 - au^3 + 1 = 0$. If $a = 1$, then $u = -1$ is unique and the resulting singularity of X is of type A_5. If $a \neq 1$, then X has two singularities of type A_2. In both cases, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by Chapter II, so $\mathrm{Aut}^0_X = \mu_3$ if $a = 1$ by Remark 5.9. If $a \neq 1$, one can check $\mathrm{Aut}^0_X = \mu_3$ directly.
- If no coefficient is 0, we get the following case

$$x^4 + xy^3 + xz^3 + ax^2yz + by^2z^2 \tag{8.14}$$

Here, X has singularities at the points $[bu : au^2 : b]$, where u is a solution of $a^3u^6 + (b^3 - a^2b^2)u^3 + b^3 = 0$. If $(b^3 - a^2b^2)^2 = a^3b^3$, then x is unique and the resulting singularity of X is of type A_5. If $(b^3 - a^2b^2)^2 \neq a^3b^3$, then X has two singularities of type A_2. In both cases, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial by Chapter II, so $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8 and Remark 5.9.

(b) There are three types of quartics which are invariant under this μ_3-action, namely

$$f_2(x, y)z^2$$
$$f_3(x, y)z + f_0z^4$$
$$f_4(x, y) + f_1(x, y)z^3$$

where the f_i are homogeneous polynomials of degree i in x and y. All quartics in the first family contain a double line, hence they lead to non-normal X. For the same reason, we have $f_3, f_4 \neq 0$ in the latter two families. In the third family, we have $f_1 \neq 0$, for otherwise Q has a quadruple point and the corresponding singularity on X is not an RDP. The GL_2-action on x, y normalizes the μ_3-action, hence acts on the space of invariant quartics. Similarly, substitutions of the form $z \mapsto z + \beta x + \gamma y$ with $\beta, \gamma \in k$ act on the space of invariant quartics. Using these substitutions, and keeping in mind that Q must be reduced, we obtain the following simplified normal forms:

$$xy(x+y)z \tag{8.15}$$
$$xy(x+y)z+z^4 \tag{8.16}$$
$$x^2yz+z^4 \tag{8.17}$$
$$y^4+xz^3 \tag{8.18}$$
$$y^4+x^2y^2+xz^3 \tag{8.19}$$
$$x^2y^2+xz^3 \tag{8.20}$$
$$x^3y+xz^3 \tag{8.21}$$

In Case (8.15), X has a D_4-singularity at $[0:0:1:0]$ and three A_1-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, and $[1:-1:0:0]$. In particular, X contains only equivariant RDPs, so it does not occur in Table 5.
In Case (8.16), X has an A_2-singularity at $[1:1:1:0]$ and three A_1-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, and $[1:-1:0:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence $\mathrm{Aut}^0_X = \mu_3$ by Proposition 5.8.
In Case (8.17), X has a D_5-singularity at $[0:1:0:0]$ and an A_1-singularity at $[1:0:0:0]$. In particular, X contains only equivariant RDPs, so it does not occur in Table 5.
In Case (8.18), X has a singularity of type E_6^0 at $[1:0:0:0]$. In Table 5, we give an action of a group scheme G of length 27 which preserves X and such that no subgroup scheme of G lifts to $\widetilde{X}$. Moreover, we give an action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$ on X. As in the corresponding case in degree 3, we leave it to the reader to check that these two actions generate Aut^0_X.
In Case (8.19), X has three A_2-singularities, at $[1:0:0:0]$, $[1:-1:1:0]$, and $[1:1:1:0]$. In Table 5, we describe an action of $\alpha_3^2 \rtimes \mu_3$ on X. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence $\mathrm{Aut}^0_X = \alpha_3^2 \rtimes \mu_3$ by Proposition 5.8.
In Case (8.20), X has an A_5-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$. In Table 5, we give an action of $\alpha_3^2 \rtimes \mathbb{G}_m$ on X. By Chapter II, we have $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_m$, so Proposition 5.8 and Remark 5.9 show that $\mathrm{Aut}^0_X = \alpha_3^2 \rtimes \mathbb{G}_m$.
In Case (8.21), X admits a singularity of type E_7^0 at $[0:1:0:0]$. In Table 5, we give an action of α_3 on X that does not fix the E_7^0-singularity as well as the action of $\mathrm{Aut}^0_{\widetilde{X}} = \mathbb{G}_a \rtimes \mathbb{G}_m$. We leave it to the reader to check that these two actions generate Aut^0_X.

(c) We can write the equation of Q as

$$f_0x^4 + x^3f_1(y,z) + x^2f_2(y,z) + xf_3(y,z) + f_4(y,z),$$

where the f_i are homogeneous of degree i in y and z. Applying the α_3-action, we obtain

$$\begin{aligned}f_0x^4 + \varepsilon f_0x^3y + x^3f_1(y,z) + (x^2 - \varepsilon xy + \varepsilon^2y^2)f_2(y,z)\\ +(x+\varepsilon y)f_3(y,z) + f_4(y,z).\end{aligned}$$

By considering the non-zero term of highest degree in x, we see that this is a multiple of the original equation if and only if it is equal to it. Comparing the coefficients of ε^2 and ε, we see that this happens if and only if $f_0 = f_2 = f_3 = 0$. Hence, Q is of the form

$$x^3f_1(y,z) + f_4(y,z).$$

But then Q is invariant under the μ_3-action $[x : y : z] \mapsto [\lambda x : y : z]$ and hence, after interchanging x and z, Q is as in Cases (8.18), (8.19), (8.20), and (8.21).

(d) We write the equation of Q as $\sum a_{ijk}x^iy^jz^k$. As in Case (c), one checks that Q is preserved by the α_3-action if and only if its equation is preserved by the α_3-action. Applying the α_3-action and comparing the coefficients of ε and ε^2, respectively, we see that Q is α_3-invariant if and only if the following two conditions are satisfied:

$$\begin{aligned}a_{400}x^3y + a_{310}x^3z - a_{220}x^2yz + a_{211}x^2z^2 - a_{220}xy^3\\ - a_{211}xy^2z - (a_{202} + a_{121})xyz^2 + a_{112}xz^3 + a_{130}y^4\\ + (a_{121} + a_{040})y^3z + a_{112}y^2z^2 + (a_{103} - a_{022})yz^3 + a_{013}z^4 &= 0\end{aligned}$$

$$\begin{aligned}- a_{400}x^3z + a_{220}x^2z^2 - a_{220}xy^2z + (a_{121} + a_{202})xz^3\\ + a_{220}y^4 + (a_{211} - a_{130})y^3z + (a_{121} + a_{202})y^2z^2\\ + (a_{022} - a_{103})z^4 &= 0\end{aligned}$$

In other words, Q is α_3-invariant if and only if it is given by an equation of the form

$$a(xz+y^2)^2 + bz^2(xz+y^2) + cx^3z + dy^3z + ez^4.$$

The substitutions of the form $x \mapsto \beta^2x + \beta\gamma y + \delta z$, $y \mapsto \beta y + \gamma z$ with $\beta \in k^\times$ and $\gamma, \delta \in k$ normalize the α_3-action, hence they preserve the space of α_3-invariant quartics. If $a \neq 0$, we can scale it to $a = 1$ and use δ to kill b. If $a = 0$, then $b \neq 0$, otherwise Q contains a triple line. Then, we can assume $b = 1$. Using the other substitutions, we arrive at one of the following five simplified normal forms:

$$(xz+y^2)^2 + z^4 \tag{8.22}$$

$$(xz+y^2)^2 + y^3z \tag{8.23}$$

$$(xz+y^2)^2 + x^3z + a^6z^4 \tag{8.24}$$

$$z^2(xz+y^2) + y^3z \tag{8.25}$$

$$z^2(xz+y^2) + x^3z \tag{8.26}$$

In Case (8.22), X has an A_7-singularity at $[1 : 0 : 0 : 0]$, hence X contains only equivariant RDPs and does not occur in Table 5.

In Case (8.23), X has an A_4-singularity at $[1:0:0:0]$ and an A_2-singularity at $[0:0:1:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence $\mathrm{Aut}^0_X = \alpha_3$ by Proposition 5.8.
In Case (8.24), X is singular precisely at $[-a^2 : \pm a : 1]$. If $a = 0$, then this singularity is of type A_5 and, by Chapter II and Remark 5.9, we have $\mathrm{Aut}^0_X = \alpha_3$. If $a \neq 0$, then both singularities are of type A_2. Direct calculation shows that $\mathrm{Aut}^0_X = \alpha_3$.
In Case (8.25), X has an RDP of type E_7^1 at $[1:0:0:0]$. In particular, X contains only equivariant RDPs and does not occur in Table 5.
Finally, in Case (8.26), has an RDP of type A_5 at $[0:1:0:0]$. By Chapter II, $\mathrm{Aut}^0_{\widetilde{X}}$ is trivial, hence, by Remark 5.9, we have $\mathrm{Aut}^0_X = \alpha_3$. Note that this surface is not isomorphic to the one in Case (8.24) (with $a = 0$), since here, Q contains a line, while in the other case, Q is irreducible.

If $d = 1$, then X is a double cover of the quadratic cone in $\mathbb{P}^3$ branched over a sextic curve S. In particular, X is given by an equation in Weierstrass form

$$y^2 = x^3 + a_2(s,t)x^2 + a_4(s,t)x + a_6(s,t),$$

where a_i is a homogeneous polynomial of degree i in s and t, and S is given by the equation

$$x^3 + a_2(s,t)x^2 + a_4(s,t)x + a_6(s,t) = 0 \tag{8.27}$$

in $\mathbb{P}(1,1,2)$. By Proposition 4.3, we have $\mathrm{Aut}^0_X = \mathrm{Stab}_{\mathrm{Aut}_{\mathbb{P}(1,1,2)}}(S)^0$. Since $\mathbb{P}(1,1,2)$ is an Aut_X-equivariant RDP del Pezzo surface of degree 8, Theorem 6.1 and Chapter II show that $\mathrm{Aut}_{\mathbb{P}(1,1,2)} = (\mathbb{G}_a^3 \rtimes \mathrm{GL}_2)/(\mathbb{Z}/2\mathbb{Z})$, which acts via substitutions of the form

$$\begin{array}{rcl} x & \mapsto & x + f_2(s,t) \\ s & \mapsto & \alpha s + \beta t \\ t & \mapsto & \gamma s + \delta t \end{array}$$

where f_2 is a homogeneous polynomial of degree 2 in s and t, and $\alpha, \beta, \gamma, \delta$ are scalars such that $\alpha\delta - \beta\gamma$ is invertible.

Now, assume Aut^0_X is non-trivial. Then, it contains $G \in \{\alpha_3, \mu_3\}$. If $G = \mu_3$, we claim that there are only three embeddings of G into $\mathrm{Aut}^0_{\mathbb{P}(1,1,2)} = \mathbb{G}_a^3 \rtimes \mathrm{GL}_2$. First, by counting the possible Jordan normal forms, observe that there are only three conjugacy classes of embeddings of G into GL_2. Then, applying [Ray66, Théorème 5.1.1 (ii) (b)], we see that every such embedding lifts uniquely, up to conjugation by $\mathbb{G}_a^3$, to an embedding of μ_3 into $\mathbb{G}_a^3 \rtimes \mathrm{GL}_2$. Hence, if $G = \mu_3$, we may assume that it acts in one of the following three ways on $\mathbb{P}(1,1,2)$:

(a) $\mu_3 : [s:t:x] \mapsto [s:\lambda t:x]$,
(b) $\mu_3 : [s:t:x] \mapsto [\lambda s:\lambda t:x]$,
(c) $\mu_3 : [s:t:x] \mapsto [\lambda s:\lambda^{-1} t:x]$.

Next, assume that $G = \alpha_3$ and the image of G in GL_2 is trivial. Then, the embedding of G into $\mathbb{G}_a^3 \rtimes \mathrm{GL}_2$ is given by a homomorphism $\alpha_3 \to \mathbb{G}_a^3$, which in turn corresponds to a choice of homogeneous polynomial f_2 of degree 2 in s and t. According to whether this polynomial has one double zero or two simple zeroes, we can conjugate the embedding of α_3 using the GL_2-action to get one of the following two α_3-actions:

(d) $\alpha_3 : [s : t : x] \mapsto [s : t : x + \varepsilon s^2]$,

(e) $\alpha_3 : [s : t : x] \mapsto [s : t : x + \varepsilon st]$.

Finally, assume that $G = \alpha_3$ and the image of G in GL_2 is non-trivial. After conjugating by elements of GL_2, we can assume that the image of G in GL_2 acts as $(s, t) \mapsto (s + \varepsilon t, t)$. An action of α_3 on $\mathbb{P}(1, 1, 2)$ that lifts this embedding of α_3 into GL_2 acts on x via

$$x \mapsto x + p(\varepsilon)s^2 + q(\varepsilon)st + r(\varepsilon)t^2$$

where p, q, r are polynomials of degree 2 in ε satisfying the following conditions:

$$\begin{aligned} p(0) = q(0) = r(0) &= 0 \\ p(\varepsilon + \varepsilon') &= p(\varepsilon) + p(\varepsilon') \\ q(\varepsilon + \varepsilon') &= q(\varepsilon) + q(\varepsilon') - p(\varepsilon)\varepsilon' \\ r(\varepsilon + \varepsilon') &= r(\varepsilon) + r(\varepsilon') + q(\varepsilon)\varepsilon' + p(\varepsilon)\varepsilon'^2 \end{aligned}$$

Solving this system of equations, we obtain $p = 0$, $q(\varepsilon) = \alpha\varepsilon$ and $r(\varepsilon) = \beta\varepsilon - \alpha\varepsilon^2$ for scalars $\alpha, \beta \in k$. Conjugating with the substitution $x \mapsto x - \alpha s^2 + \beta st$, we obtain that our α_3-action is conjugate to the following:

(f) $\alpha_3 : [s : t : x] \mapsto [s + \varepsilon t : t : x]$.

We will now classify the sextics as in Equation (8.27) which are reduced with only simple singularities and invariant under one of the above actions. In particular, note that if all the a_i are scalar multiples of the i-th power of the same linear polynomial in s and t, then S has a non-simple singularity, so it will not appear in our list. Calculating Aut_X^0 is straightforward here, using our description of $\mathrm{Aut}_{\mathbb{P}(1,1,2)}$ above, so it will be left to the reader without further mention. The results can be found in Table 6:

(a) The sextic S is invariant if and only if the t-degree of every monomial that occurs in the equation of S is divisible by 3. Note that the substitutions

$$\begin{array}{rcl} x & \mapsto & x + \alpha s^2 \\ s & \mapsto & \beta s \\ t & \mapsto & \gamma t + \delta s \end{array}$$

act on the space of sextics satisfying the condition on the t-degree, hence we can apply them to arrive at the following normal forms for S:

$$x^3 + s^4x + t^6 \tag{8.28}$$

$$x^3 + s^4x + s^3t^3 \tag{8.29}$$

$$x^3 + st^3x \tag{8.30}$$

$$x^3 + st^3x + as^3t^3 + t^6 \tag{8.31}$$

$$x^3 + st^3x + s^3t^3 \tag{8.32}$$

$$x^3 + s^2x^2 + s^3t^3 \tag{8.33}$$

$$x^3 + s^2x^2 + a^3s^3t^3 + t^6 \tag{8.34}$$

$$x^3 + s^2x^2 + st^3x + a^3s^3t^3 + b^3t^6 \tag{8.35}$$

In Case (8.28), X has an E_6^0-singularity at $[0:1:-1:0]$.
In Case (8.29), X has an E_8^1-singularity at $[0:1:0:0]$.
In Case (8.30), X has an E_7^0-singularity at $[1:0:0:0]$ and an A_1-singularity at $[0:1:0:0]$.
In Case (8.31), if $a \neq 0$, then X has an E_6^0-singularity at $[1:0:0:0]$. If $a = 0$, then X has an E_7^0-singularity at $[1:0:0:0]$.
In Case (8.32), X has an E_6^0-singularity at $[1:0:0:0]$ and an A_1-singularity at $[0:1:0:0]$.
In Case (8.33), X has an E_6^1-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$.
In Case (8.34), X is singular at $[1:0:0:0]$, $[1:-a:0:0]$, and $[0:1:-1:0]$. If $a \neq 0$, then all singular points are A_2-singularities. If $a = 0$, then the first two combine to an A_5-singularity, while the latter stays an A_2-singularity.
Consider Case (8.35). Here, X is singular at $[1:0:0:0]$ and $[u:1:u^{-1}:0]$, where u is a solution of $au^2+(b-1)u+1=0$. If $b=0$, then X has an additional A_1-singularity at $[0:1:0:0]$. If the first three singular points are distinct, which happens if and only if $a \neq 0, (b-1)^2$, then they are A_2-singularities. Now, consider the case where $a = 0$ or $a = (b-1)^2$, but not both: Then $[1:0:0:0]$ is an A_5-singularity and $[u:1:u^{-1}:0]$ an A_2-singularity if $0 = a \neq (b-1)^2$, and the other way round if $0 \neq a = (b-1)^2$. Note that the substitution $t \mapsto t+(b-1)s, x \mapsto x+(b-1)^3s^2$ maps the family with $a = (b-1)^2$ to the one with $a = 0$, which is why only the latter occurs in Table 6. Finally, if $a = 0$ and $b = 1$, then X has an A_8-singularity at $[1:0:0:0]$.

(b) The sextic S is invariant if and only if the degrees of the a_i are divisible by 3. This happens if and only if $a_2 = a_4 = 0$. Using a substitution from $\mathrm{Aut}_{\mathbb{P}(1,1,2)}$, we obtain the following normal forms:

$$x^3 + s^5t \tag{8.36}$$

$$x^3 + s^4t^2 \tag{8.37}$$

$$x^3 + s^4t^2 + s^2t^4 \tag{8.38}$$

In Case (8.36), X has an E_8^0-singularity at $[0:1:0:0]$.
In Case (8.37), X has an E_6^0-singularity at $[0:1:0:0]$ and an A_2-singularity at $[1:0:0:0]$.
In Case (8.38), X has four A_2-singularities, at $[1:0:0:0]$, $[0:1:0:0]$, $[1:-1:0:0]$, and $[1:1:0:0]$.

(c) The sextic S is invariant if and only if for every monomial in the equation of S, the difference between the s- and t-degree is divisible by 3. We may assume that not both a_2 and a_4 are zero, otherwise we get Cases (8.36), (8.37), and (8.38). Note that the substitutions

$$\begin{array}{rcl} x & \mapsto & x + \alpha st \\ s & \mapsto & \beta s \\ t & \mapsto & \gamma t \end{array}$$

normalize our μ_3-action, hence we can apply them to arrive at the following normal forms for S:

$$x^3 + s^2t^2x \tag{8.39}$$

$$x^3 + s^2t^2x + t^6 + a^3s^6 \tag{8.40}$$

$$x^3 + stx^2 + s^6 \tag{8.41}$$

$$x^3 + stx^2 + a^3s^6 + s^3t^3 \tag{8.42}$$

$$x^3 + stx^2 + a^3s^6 + b^3s^3t^3 + t^6 \tag{8.43}$$

In Case (8.39), X has two D_4-singularities, one at $[1:0:0:0]$ and one at $[0:1:0:0]$, hence X contains only equivariant RDPs and does not occur in Table 6.

Consider Case (8.40). If $a = 0$, then X has a D_4-singularity at $[1:0:0:0]$ and an A_2-singularity at $[0:1:-1:0]$. If $a \neq 0$, then X has two A_2-singularities, one at $[1:0:-a:0]$ and one at $[0:1:-1:0]$.

In Case (8.41), X has an RDP of type D_7 at $[0:1:0:0]$. In particular, X contains only equivariant RDPs and does not occur in Table 6.

Consider Case (8.42). If $a = 0$, then X has RDPs of type D_4 at $[1:0:0:0]$ and $[0:1:0:0]$, hence X contains only equivariant RDPs and does not occur in Table 6. If $a \neq 0$, then X has an RDP of type D_4 at $[0:1:0:0]$ and an A_2-singularity at $[1:-a:0:0]$.

Consider Case (8.43). If $a = 0$, we can interchange s and t to reduce to one of the previous two cases. Here, X has singularities at $[1:u:0:0]$, where u is a solution of $u^2 + bu + a = 0$. If $b^2 = a$, then the unique singularity of X is an A_5-singularity. If $b^2 \neq a$, then X has two A_2-singularities.

(d) If $a_2 \neq 0$ or $a_4 \neq 0$, then S cannot be α_3-invariant. Hence, $a_2 = a_4 = 0$. But then S is given by one of the equations in (b), so we are done.

(e) By the same argument as in Case (d), we can reduce this to Case (b).

(f) The sextic S is α_3-invariant if and only if each a_i is invariant under the α_3-action. This happens if and only if the s-degree of each monomial in the equation of S is divisible by 3. Interchanging the roles of s and t, we can therefore reduce this Case to Case (a). In fact, this explains why each of the surfaces in Case (a) admits an α_3-action of this form.

□

CHAPTER IV

On rational (quasi-)elliptic surfaces with global vector fields

1. Motivation and summary

Recall from Chapters I and II that the blow-up of a weak del Pezzo surface $\widetilde{X}$ of degree 1 in the unique base point of $|-K_{\widetilde{X}}|$ yields a Jacobian rational (quasi-)elliptic surface $\widetilde{Y}$ (see Definition 3.4 in Chapter I) and, conversely, the contraction of any section of a Jacobian rational (quasi-)elliptic surface yields a weak del Pezzo surface of degree 1 such that the image of the exceptional curve is exactly the base point of its anti-canonical linear system. Similarly, the Weierstraß model Y of $\widetilde{Y}$ is the blow-up of the anti-canonical model X of $\widetilde{X}$ in the unique base point of $|-K_X|$. As in Chapters I, II, and III, we will denote by $\widetilde{X}$ a weak del Pezzo surface (here of degree 1) and by X the corresponding RDP del Pezzo surface (of degree 1). We recall (see Diagram 3.1 in Chapter I) and summarize the connection between these four kinds of surfaces in the following commutative diagram:

(1.1)
$$\begin{tikzcd} \widetilde{Y} \arrow[r] \arrow[d] \arrow[rr, bend left, "f"] & Y \arrow[r] \arrow[d] & \mathbb{P}^1 \\ \widetilde{X} \arrow[r] & X \arrow[ur, dashed] & \end{tikzcd}$$

This close connection between *Jacobian* rational (quasi-)elliptic surfaces and del Pezzo surfaces has already been exploited at several points in this thesis:

- In Chapter I, we classified the rational double point configurations that occur on Weierstraß models Y of Jacobian rational (quasi-)elliptic surfaces, thereby classifying which rational double point configurations occur on RDP del Pezzo surfaces X.
- In Subsection 0.8 of the main introduction to this thesis, we noted that Blanchard's Lemma implies that $\mathrm{Aut}^0_{\widetilde{Y}} \cong \mathrm{Aut}^0_{\widetilde{X}}$, so Table 6 in Chapter II already gives a complete classification of Jacobian rational (quasi-)elliptic surfaces with global vector fields (see also Corollary 3.1 below).
- Similarly, for the corresponding Weierstraß models Y of Jacobian rational (quasi-) elliptic surfaces resp. the anti-canonical models X of weak del Pezzo surfaces of degree 1, we have $\mathrm{Aut}^0_Y \cong \mathrm{Aut}^0_X$, so Theorem 6.1 and Tables 1, 2, and 6 in Chapter III give a classification of Weierstraß models of Jacobian rational (quasi-)elliptic surfaces with global vector fields in characteristic different from 2.

In this chapter, we will explain how to use the results of the previous chapters to classify *non-Jacobian* rational (quasi-)elliptic surfaces $f : \widetilde{Z} \to \mathbb{P}^1$ with global vector fields, where for the notion of *non-Jacobian* such surfaces we simply drop the last assumption that f admits a section in Definition 3.4 in Chapter I.

For this, we will first recall that all such surfaces are obtained by blowing up the 9 base points of a Halphen pencil of degree $3m$ and deduce that they are blow-ups of weak del Pezzo surfaces of degree 1 in a point which is *different from the base point of the anti-canonical system* (in contrast to the *Jacobian case*), but satisfies other restrictive properties which we make precise in Section 2 below.

Then, again using Blanchard's Lemma, we can apply the approach of Chapter II and calculate stabilizers of the $\mathrm{Aut}^0_{\widetilde{X}}$-action on $\widetilde{X}$, or we calculate the stabilizers of the Aut^0_X-action on X using the Weierstraß equations of Chapter III (see Sections 3 and 4 for a combination of these two approaches).

Since the classification of rational (quasi-)elliptic surfaces with global vector fields is a byproduct of this thesis and only tangentially related to the study of the geometry of rational double points and del Pezzo surfaces, we will mainly be interested in explaining how our previous results make such a classification possible. We will illustrate the approach for rational (quasi-)elliptic surfaces that are blow-ups of the surfaces of type $1A$, $1B$, $1C$, or $1D$ in Table 6 of Chapter II (see Section 4). Note that, if $p \neq 2, 3$, then every not necessarily Jacobian rational (quasi-)elliptic surface with global vector fields is a blow-up of surfaces of these four types, so we will in fact achieve a complete classification of all rational (quasi-)elliptic surfaces with global vector fields in characteristic $p \neq 2, 3$ (see Corollary 1.2). At the same time, we will make some progress towards the full classification in the remaining characteristics $p = 2$ and $p = 3$.

THEOREM 1.1. *Let $\widetilde{Z}$ be a rational (quasi-)elliptic surface with $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$. Assume that $\widetilde{Z}$ is a blow-up of a weak del Pezzo surface of type $1A$, $1B$, $1C$, or $1D$ in Table 6 of Chapter II. Then $\widetilde{Z}$ is one of the surfaces in Table 1.*

Type	blow up $\widetilde{X}$ in	Jac. or non-Jac.	Multiple fiber	Reducible fibers	$\mathrm{Aut}^0_{\widetilde{Z}}$	$h^0(\widetilde{Z}, T_{\widetilde{Z}})$	Moduli of $\widetilde{Z}$	$\mathrm{char}(k) = p$
$1A$	base pt.	Jac.	none	$\mathrm{I}_0^* + \mathrm{I}_0^*$	$\mathbb{G}_m$	1	1 dim	any
$1A$	2-tors. pt.	non-Jac.	2II	I_4^*	μ_2	1	1 dim	$= 2$
$1B$	base pt.	Jac.	none	$\mathrm{IV}^* + \mathrm{IV}$	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	any
$1B$	3-tors. pt.	non-Jac.	3II	II^*	μ_3	1	$\{\mathrm{pt}\}$	$= 3$
$1B$	2-tors. pt.	non-Jac.	$2\mathrm{IV}^*$	$\mathrm{IV}^* + \mathrm{I}_3$	μ_2	1	$\{\mathrm{pt}\}$	$= 2$
$1C$	base pt.	Jac.	none	$\mathrm{III}^* + \mathrm{III}$	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	$\neq 2$
$1C$	3-tors. pt.	non-Jac.	$3\mathrm{III}^*$	$\mathrm{III}^* + \mathrm{I}_2$	μ_3	1	$\{\mathrm{pt}\}$	$= 3$
$1D$	base pt.	Jac.	none	II^*	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	$\neq 2, 3$
$1D$	5-tors. pt.	non-Jac.	$5\mathrm{II}^*$	II^*	μ_5	1	$\{\mathrm{pt}\}$	$= 5$

Table 1. Rational (quasi-)elliptic surfaces $\widetilde{Z}$ with global vector fields that are blow-ups of weak del Pezzo surfaces $\widetilde{X}$ of types $1A$, $1B$, $1C$ and $1D$

In Table 1, the first column denotes the type of a weak del Pezzo surface $\widetilde{X}$ of degree 1 of which $\widetilde{Z}$ is a blow-up and the second column denotes the kind of point that is blown-up. Note that the type of $\widetilde{X}$ is not uniquely determined – we will see this in Remark 3.3 –, that is, blow-ups of non-isomorphic weak del Pezzo surfaces of degree 1 can be isomorphic rational (quasi-)elliptic surfaces. Hence, the first and second column of Table 1 should be understood as instructions for the construction of these surfaces $\widetilde{Z}$ from Theorem 1.1. The third column indicates whether $\widetilde{Z}$ is Jacobian or non-Jacobian, where only in the latter case there exists a multiple fiber that is specified in the fourth column. Moreover, we determine all reducible fibers (fifth column) of the (quasi-)elliptic fibration $\widetilde{Z} \to \mathbb{P}^1$ and compute the connected component of the identity of its automorphism scheme (sixth column), both in the Jacobian case (Corollary 3.1) and in the non-Jacobian case (Subsections 4.1, 4.2, 4.3, and 4.4). In seven of the nine cases in Table 1, there is a unique $\widetilde{Z}$ with global vector fields (compare column eight), whereas the 1-dimensional moduli in the second and third row of Table 1 come from the fact that the family $1A$ of weak del Pezzo surfaces is 1-dimensional (see Table 6 in Chapter II). Comparing the third and last columns of Table 1 with Table 6 from Chapter II, we see that in characteristic $p \neq 2, 3$ there is only one non-Jacobian rational (quasi-)elliptic surface with global vector fields, and this one only occurs in characteristic 5. In particular, the results of this chapter imply the following.

COROLLARY 1.2. *Let* $\mathrm{char}(k) = p \neq 2, 3$. *Let* $\widetilde{Z}$ *be a non-Jacobian rational (quasi-) elliptic surface.*

(1) If $p \neq 5$*, then* $h^0(\widetilde{Z}, T_{\widetilde{Z}}) = 0$.
(2) If $p = 5$*, then* $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$ *if and only if* $\widetilde{Z}$ *is the surface in row ten of Table 1.*

Although the classification of rational (quasi-)elliptic surfaces with global vector fields is complete only if $p \neq 2,3$, we nevertheless determined the connected components of the automorphism schemes and (-2)-curve configurations for those surfaces that are blow-ups of types $1A$, $1B$, $1C$ and $1D$ if $p = 2,3$, in order to illustrate the interplay of the classifications of previous chapters as well as to reap the benefits of our meticulous drawings of configurations of negative curves on weak del Pezzo surfaces in Chapter II. In this sense, the parts of this chapter exceeding the $p \neq 2,3$ cases should be seen as an outlook on possible future projects and should highlight the strength of the classifications of the previous chapters of this thesis.

2. Non-Jacobian rational (quasi-)elliptic surfaces and Halphen pencils

As before, we work over an algebraically closed field k of characteristic $\mathrm{char}(k) = p \geq 0$. For the sake of completeness, we recall the definition of not necessarily Jacobian rational (quasi-)elliptic surfaces (compare with Definition 3.4 in Chapter I in the Jacobian case).

DEFINITION 2.1. A projective surface $\widetilde{Z}$ is called a *rational (quasi-)elliptic surface* if it is smooth, rational, and admits a morphism $f : \widetilde{Z} \to \mathbb{P}^1$ such that the following conditions hold:

- f is surjective and $f_*\mathcal{O}_{\widetilde{Z}} \cong \mathcal{O}_{\mathbb{P}^1}$,
- the generic fiber of f is a regular curve of arithmetic genus 1, and
- there are no (-1)-curves in fibers of f.

We call f *elliptic* if the generic fiber of f is smooth, and *quasi-elliptic* if the generic fiber is singular.

As in the Jacobian case, if f is quasi-elliptic, then the geometric generic fiber has to be a cuspidal curve and such fibrations can only occur in characteristics $p = 2$ and $p = 3$.

Now, let $f : \widetilde{Z} \to \mathbb{P}^1$ be a rational (quasi-)elliptic surface. We denote the fiber of f over $P \in \mathbb{P}^1$ by F_P. The greatest common divisor m_P of the multiplicities of the components of F_P is called the *multiplicity* of the fiber F_P and we set $\overline{F}_P := \frac{1}{m_P}F_P$. The fibers with $m_P > 1$ are called *multiple fibers*. With this notation, the canonical bundle formula (see for example [BŎ1, Theorem 7.15.]) yields

$$\omega_{\widetilde{Z}} \;\cong\; f^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\widetilde{Z}}\left(\sum_{P \in \mathbb{P}^1} (m_P - 1)\overline{F}_P\right). \tag{2.1}$$

Since $\widetilde{Z}$ is rational, for any $n > 0$ we have $h^0(\widetilde{Z}, \omega_{\widetilde{Z}}^{\otimes n}) = p_n(\widetilde{Z}) = p_n(\mathbb{P}^2) = 0$, hence, by Equation 2.1, there is at most one multiple fiber and we denote its multiplicity by m. Hence, the class of a fiber of f in the Picard group is $-K_{\widetilde{Z}}$ if there are no multiple fibers, and $-mK_{\widetilde{Z}}$ if there is a multiple fiber. Thus, by adjunction, all (-2)-curves on $\widetilde{Z}$ have to lie in fibers of f.

For $l > 0$, an *l-section* of f is an irreducible curve C on $\widetilde{Z}$ such that $C.F_P = l$, where F_P is a general fiber of f. The *index* of f is the minimal l for which there exists an l-section. By [CD89, Chapter V, Proposition 5.6.1.(vi)], the index of a rational (quasi-)elliptic surface is equal to the maximal multiplicity of its fibers. In particular, it is 1 if and only if f does not admit a multiple fiber. In this case, f admits a section and is called *Jacobian* (see Definition 3.4 in Chapter I). Moreover, by contracting a section of f, one obtains a weak del Pezzo surface $\widetilde{X}$ of degree 1. If f admits a multiple fiber, then the index of f is equal to the multiplicity m of this unique multiple fiber. Moreover, note that in this case, every (-1)-curve on $\widetilde{Z}$ is an m-section of f and intersects each fiber with multiplicity m (as can be seen by adjunction).

As recalled above and explained in Subsection 3.2 of Chapter I, every *Jacobian* rational (quasi-)elliptic surface is a blow-up of a weak del Pezzo surface of degree 1. This connection persists for *non-Jacobian* rational (quasi-)elliptic fibrations, as we are going to recall in the following (we refer the reader to [B01, Chapter 7], [CD89, Chapter V], [Dol66], [CD12, Section 2.], or [Hal82] for further details).

Let $\widetilde{X}$ be the contraction of any m-section of f. Since $K_{\widetilde{Z}}^2 = 0$ and $-K_{\widetilde{Z}}$ is nef, the anti-canonical divisor $-K_{\widetilde{X}}$ of $\widetilde{X}$ is nef with $K_{\widetilde{X}}^2 = 1$, so $\widetilde{X}$ is a weak del Pezzo surface of degree 1. Hence, we can choose a realization of $\widetilde{X}$ as a blow-up of $\mathbb{P}^2$ in 8 points and obtain a birational morphism $\pi : \widetilde{Z} \to \widetilde{X} \to \mathbb{P}^2$ which realizes $\widetilde{Z}$ as the blow-up of 9 (possibly infinitely near) points. This yields the classical description of f as the resolution of the base points of a *Halphen pencil of index* m, which is a pencil of curves of degree $3m$, all of which have multiplicity m at the 9 points prescribed by π. The image of the multiple fiber of f under π is a cubic through these 9 points, taken with multiplicity m.

If $\widetilde{X}$ is a weak del Pezzo surface of degree 1, then every curve in $|-K_{\widetilde{X}}|$ is of arithmetic genus one and marked with the base point of $|-K_{\widetilde{X}}|$. Note that this base point is a smooth point on every member of $|-K_{\widetilde{X}}|$, since $K_{\widetilde{X}}^2 = 1$. Hence, for every curve $C \in |-K_{\widetilde{X}}|$, the smooth locus of the irreducible component of C that contains this base point becomes a group scheme (with the base point as neutral element). We call this group scheme the *identity component* of C and denote it by C^0.

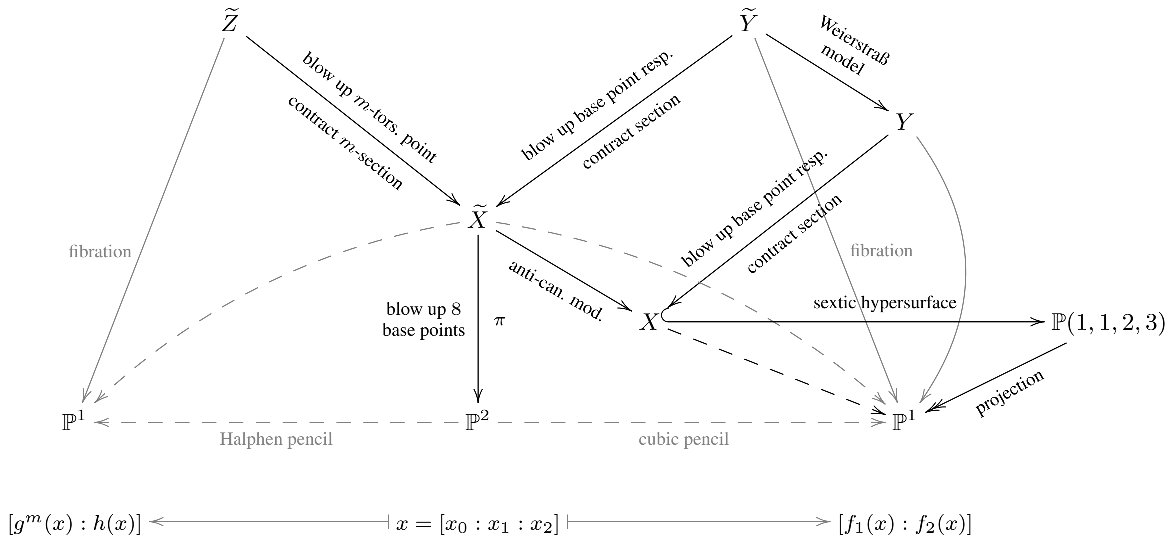
The description of the previous paragraph has the following important consequence, which is explained in [DM22, Theorem 4.2]:

PROPOSITION 2.2. *Let $f : \widetilde{Z} \to \mathbb{P}^1$ be a rational (quasi-)elliptic surface of index m. Then, $\widetilde{Z}$ is the blow-up of a weak del Pezzo surface $\widetilde{X}$ in a point $\widetilde{P}$ that satisfies the following conditions:*

(1) If $m = 1$, then $\widetilde{P}$ is the base point of $|-K_{\widetilde{X}}|$.
(2) If $m > 1$, then $\widetilde{P}$ lies on a unique curve $C \in |-K_{\widetilde{X}}|$. More precisely, $\widetilde{P}$ is a point of exact order m on C^0.

Using this, we can now extend the above Diagram 1.1 with non-Jacobian fibrations $f : \widetilde{Z} \to \mathbb{P}^1$ of index m and Halphen pencils: Let f_1, f_2 be equations of cubics in $\mathbb{P}^2$ such that 8 of their (possibly infinitely near) 9 points of intersection lie in almost general position (see Subsection 2.1 in Chapter II). Let h be the equation of a curve of degree $3m$ in $\mathbb{P}^2$ with multiplicity m at the above 8 points in almost general position in $\{f_1 = 0 = f_2\}$ and with one more point of multiplicity m at a point P_9 different from the 9 base points of the cubic pencil spanned by f_1 and f_2. Then, there exists precisely one curve in this cubic pencil passing through P_9. Let g be the cubic equation of this curve. By Proposition 2.2, the point $P_9 \in \{g = 0\}$ is a point of exact order m on $\{g = 0\}$. Moreover, the Halphen pencil corresponding to $\widetilde{Z}$ is spanned by g^m and h, and $\widetilde{Z}$ is obtained by blowing up $\mathbb{P}^2$ in the 8 common base points of the cubic pencil and the Halphen pencil, and P_9. Note that simply blowing up the 9 base points of the cubic pencil $\langle f_1, f_2 \rangle$ yields a Jacobian rational (quasi-)elliptic surface $\widetilde{Y}$. The situation is summarized in the following Diagram 2.2.

(2.2)



3. Automorphism schemes: From weak del Pezzo surfaces of degree 1 to rational (quasi-)elliptic surfaces

Now that since we have realized each rational (quasi-)elliptic surface $\widetilde{Z}$ as a blow-up of a weak del Pezzo surface $\widetilde{X}$ of degree 1, the approach of Chapter II gives us a method of calculating $\mathrm{Aut}^0_{\widetilde{Z}}$ from our knowledge of $\mathrm{Aut}^0_{\widetilde{X}}$.

First, we note the following corollary, which we have already mentioned in Subsection 0.8 of the main introduction of this thesis and which enables us to reduce the classification of Jacobian rational (quasi-)elliptic surfaces to weak del Pezzo surfaces of degree 1.

COROLLARY 3.1. *There is a bijection of isomorphism classes*

$$\left\{ \begin{array}{c} \textit{Weak del Pezzo surfaces} \\ \textit{of degree } 1 \end{array} \right\} / \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{c} \textit{Jacobian rational} \\ \textit{(quasi-)elliptic surfaces} \end{array} \right\} / \cong$$

$$\begin{array}{rcl} \widetilde{X} & \longmapsto & \begin{array}{l} \textit{Blow-up of } \widetilde{X} \\ \textit{in the base point of } |-K_{\widetilde{X}}| \end{array} \\ \textit{Contraction of a section} & \longleftarrow\!\shortmid & \widetilde{Y} \end{array}$$

which preserves the subsets of surfaces with global vector fields and identifies the identity components of the automorphism schemes of these surfaces.

PROOF. First, we observe that the map from the left-hand side to the right-hand side is well-defined, since every isomorphism between two weak del Pezzo surfaces of degree 1 identifies the unique base points of the anti-canonical system and hence lifts to the blow-up.

Similarly, since any two sections of a Jacobian rational (quasi-)elliptic surface can be mapped to each other by a suitable translation, the isomorphism class of the weak del Pezzo surface resulting from the contraction of a section does not depend on the choice of a section. Hence the map from the right-hand side to the left-hand side is well-defined.

Finally, if $\widetilde{X}$ is the contraction of a Jacobian rational (quasi-)elliptic surface $\widetilde{Y}$, then $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathrm{Aut}^0_{\widetilde{Y}}$ by Lemma 2.10 and Lemma 2.11 in Chapter II. $\square$

In the case of not necessarily Jacobian rational (quasi-)elliptic surfaces $\widetilde{Z}$, the correspondence is more subtle, since contractions of two different m-sections might lead to non-isomorphic weak del Pezzo surfaces (see Remark 3.3). Nevertheless, let us note the following corollary to Proposition 2.2, combined with Lemma 2.10 and Lemma 2.11 from Chapter II:

COROLLARY 3.2. *Every rational (quasi-)elliptic surface $\widetilde{Z}$ of index $m > 1$ with $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$ is obtained by blowing up a weak del Pezzo surface $\widetilde{X}$ of degree 1 with $h^0(\widetilde{X}, T_{\widetilde{X}}) \neq 0$ in a point $\widetilde{P} \in \widetilde{X}$ such that the following conditions hold:*

(1) $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0$ is non-trivial.

(2) $\widetilde{P}$ is a point of exact order m on the identity component C^0 of the unique curve $C \in |-K_{\widetilde{X}}|$ that contains $\widetilde{P}$.

Moreover, we have $\mathrm{Aut}^0_{\widetilde{Z}} \cong (\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0$.

Thus, in order to classify rational (quasi-)elliptic surfaces with global vector fields, we have to calculate the stabilizers of the action of $\mathrm{Aut}^0_{\widetilde{X}}$, where $\widetilde{X}$ is a weak del Pezzo surface of degree 1 with global vector fields.

REMARK 3.3. Note that the above corollary does not yield a bijection between isomorphism classes of weak del Pezzo surfaces of degree 1 together with a point to be blown up and rational (quasi-)elliptic surfaces with multiple fibers. Indeed, we we will see in the discussions of configurations of (-2)-curves for Family $1B$ and Family $1C$ that blow-ups of non-isomorphic weak del Pezzo surfaces can yield isomorphic rational (quasi-)elliptic surfaces $\widetilde{Z}$: For example, if $p = 2$, the unique $\widetilde{Z}$ with $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$ obtained from type $1B$ is a blow-up of a weak del Pezzo of type $1K$, and if $p = 3$, the unique $\widetilde{Z}$ with $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$ obtained from type $1C$ is a blow-up of a weak del Pezzo of type $1F$.

4. Four families of weak del Pezzo surfaces of degree 1 with global vector fields

Throughout this section, we use the notation summarized in Diagram 2.2.

By Corollary 3.1, the classification of Jacobian rational (quasi-)elliptic surfaces with global vector fields follows immediately from Table 6 of Chapter II. In this section, we will carry out the approach suggested by Corollary 3.2 for the non-Jacobian rational (quasi-) elliptic surfaces that are blow-ups of the surfaces of types $1A$, $1B$, $1C$ and $1D$ in Table 6 of Chapter II. In particular, the results of this section will prove Theorem 1.1.

By Corollary 3.2 and our classification in Chapter II, every rational (quasi-)elliptic surface is of this form as long as $p \neq 2, 3$. So, Corollary 1.2 will follow immediately from Theorem 1.1.

In the following Table 2, we recall the configurations of (-1)- and (-2)-curves (thin resp. thick lines) on these four families of del Pezzo surfaces (compare Table 6 of Chapter II). The Weierstraß equations for their anti-canonical models X are taken from [MP86], [Lan94], [Ito92], and [Ito94] (resp. Tables 2–6 in Chapter I), and simplified to a characteristic-free form. In each case, we know from Table 6 of Chapter II that $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathbb{G}_m$, hence $(\mathrm{Aut}^0_X)_{\mathrm{red}} \cong \mathbb{G}_m$, since smooth group scheme actions lift to $\widetilde{X}$ by the universal property of the minimal resolution and $\mathrm{Aut}^0_{\widetilde{X}} \hookrightarrow \mathrm{Aut}^0_X$ by Blanchard's Lemma. Therefore, the faithful $\mathbb{G}_m$-actions on X that we describe in Table 2 lift to $\widetilde{X}$ and must coincide with the known $\mathrm{Aut}^0_{\widetilde{X}}$-actions.

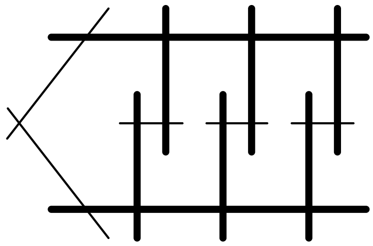
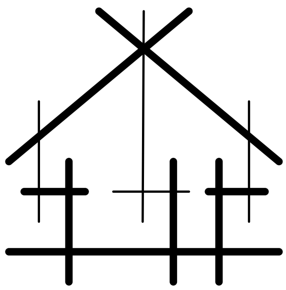
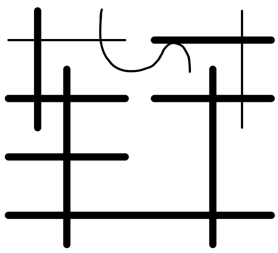
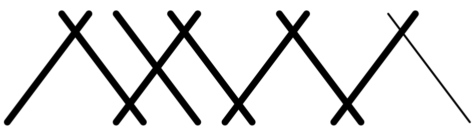
Case	(-2)-curves on $\widetilde{X}$	Configuration of negative curves on $\widetilde{X}$	Action of $\mathrm{Aut}^0_{\widetilde{X}}$ on the Weierstraß equation of X	$\mathrm{char}(k)$
$1A$	$2D_4$		$\mathbb{G}_m : [\lambda s : \lambda^{-1} t : x : y]$ acting on $y^2 = x^3 + astx^2 + s^2t^2x$ where $a \in k$ and $a^2 \neq 4$	any
$1B$	$E_6 + A_2$		$\mathbb{G}_m : [\lambda^2 s : \lambda^{-1} t : x : y]$ acting on $y^2 + st^2y = x^3$	any
$1C$	$E_7 + A_1$		$\mathbb{G}_m : [\lambda^3 s : \lambda^{-1} t : x : y]$ acting on $y^2 = x^3 + st^3x$	$\neq 2$
$1D$	E_8		$\mathbb{G}_m : [\lambda^5 s : \lambda^{-1} t : x : y]$ acting on $y^2 = x^3 + st^5$	$\neq 2, 3$

Table 2. Four families of weak del Pezzo surfaces of degree 1 with global vector fields

NOTATION 4.1. In the following, we say that $C \in |-K_{\widetilde{X}}|$ is of type Σ, where Σ is a Kodaira–Néron type, if the strict transform of C on the Jacobian (quasi-)elliptic surface $\widetilde{Y}$ associated to $\widetilde{X}$ is a fiber of type Σ. For the readers convenience, we recall these fiber types Σ resp. the extended Dynkin types corresponding to the configurations of (-2)-curves in reducible fibers of $\widetilde{Y} \to \mathbb{P}^1$, as well as the respective types of rational double points on the corresponding Weierstraß models Y resp. the RDP del Pezzo surfaces X (compare Chapter I, Subsection 3.2) in the following table (see also [CD89, Chapter V] or [LLR04], [LLR18], [Kod60], [Kod63], [N64́]):

Kodaira–Néron type	I_0	I_n	II	III	IV	I_n^*	IV^*	III^*	II^*
dual graph	$\widetilde{A}_0$	$\widetilde{A}_{n-1}$	$\widetilde{A}_0$	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{D}_{4+n}$	$\widetilde{E}_6$	$\widetilde{E}_7$	$\widetilde{E}_8$
corresponding rational double point	smooth	A_{n-1}	smooth	A_1	A_2	D_{4+n}	E_6	E_7	E_8

Table 3. Kodaira–Néron types and dual graphs

Now, we are ready to outline the structure of the following subsections in the following Strategy 4.2 and to explain how they imply Theorem 1.1.

STRATEGY 4.2. Let $\widetilde{X}$ be a weak del Pezzo surface in family $1A$, $1B$, $1C$ or $1D$. Since the identity components of the curves in $|-K_{\widetilde{X}}|$ are not contracted by the anti-canonical model $\widetilde{X} \to X$, we can use the explicit $\mathbb{G}_m$-actions in Table 2 to calculate the stabilizers of the $\mathrm{Aut}^0_{\widetilde{X}}$-action on the identity components C^0 of the curves C in $|-K_{\widetilde{X}}|$. In particular, we will obtain a classification of the points on $\widetilde{X}$ satisfying the assumptions of Corollary 3.2 in Propositions 4.5, 4.9, 4.13 and 4.17. After that, in Corollaries 4.6, 4.10, 4.14 and 4.18, we will determine which of these points lead to isomorphic non-Jacobian rational (quasi-)elliptic surfaces with global vector fields. At this point, we know the Kodaira–Néron type as well as the multiplicity m of the multiple fiber of $\widetilde{Z}$, but the configuration of all (-2)-curves on $\widetilde{Z}$ is not clear in general. Indeed, it might happen that curves with non-negative self-intersection on $\widetilde{X}$ – which are therefore not visible in the configurations of Table 2 – become (-2)-curves on $\widetilde{Z}$.

We solve this issue in Discussions 4.7, 4.11, 4.15 and 4.19: First, we study what happens to the negative curves on $\widetilde{X}$ under the blow-up $\widetilde{Z} \to \widetilde{X}$. In each case, the following Lemma 4.3 shows that there is at most one (-2)-curve on $\widetilde{Z}$ which does not come from a negative curve on $\widetilde{X}$. Using a realization of $\widetilde{Z}$ as a blow-up of a weak del Pezzo surface of another type (see Table 6 in Chapter II) by contracting a suitable (-1)-curve on $\widetilde{Z}$ as well as using the smoothness of the fixed locus of linearly reductive group scheme actions [CGP15, Proposition A.8.10(2)], we determine how this "new" (-2)-curve fits into the configuration of "known" curves that come from negative curves on $\widetilde{X}$. As a byproduct, we will sometimes obtain additional (-1)-curves on $\widetilde{Z}$ that come from non-negative curves on

$\widetilde{X}$, but we do not claim completeness of the configuration of (-1)-curves on $\widetilde{Z}$ that occur in our diagrams.

In particular, these discussions will allow us to determine the Kodaira–Néron types of the reducible fibers of $\widetilde{Z}$ in Corollaries 4.8, 4.12, 4.16, and 4.20. Note that the configuration of (-2)-curves on $\widetilde{Y}$ is usually different from the configuration of (-2)-curves on $\widetilde{Z}$ (so, contrary to what one might expect at first, $\widetilde{Y}$ is, in general, not isomorphic to the Jacobian of $\widetilde{Z}$ – compare [LLR04], [LLR18]). Taken together, these results will yield Table 1.

Recall that, by adjunction, (-2)-curves on $\widetilde{Z}$ lie in fibers of $f : \widetilde{Z} \to \mathbb{P}^1$ and that, conversely, all components of reducible fibers have to be (-2)-curves. In order to determine the configurations of (-2)-curves on $\widetilde{Z}$, we will need the following lemma.

LEMMA 4.3. *Let $f : \widetilde{Z} \to \mathbb{P}^1$ be a not necessarily Jacobian rational (quasi-)elliptic fibration. Let e_P be the number of irreducible components of a fiber F_P over $P \in \mathbb{P}^1$. Then,*

$$\sum_{P \in \mathbb{P}^1} (e_P - 1) \leq 8.$$

PROOF. By [LLR04, Theorem 6.6.], [LLR18], it suffices to prove the statement if f has a section σ, i.e., if f is Jacobian. Then, $\langle \sigma, F_P \rangle^{\perp} \subseteq \mathrm{Pic}(\widetilde{Z})$ has rank 8. For every fiber F_P, the irreducible components of F_P that do not meet σ span a negative definite lattice of rank $(e_P - 1)$ in $\langle \sigma, F_P \rangle^{\perp}$. Hence $\sum_{P \in \mathbb{P}^1} (e_P - 1) \leq \mathrm{rank}(\langle \sigma, F_P \rangle^{\perp}) = 8$. $\square$

NOTATION 4.4. The convention according to which we draw curves in the qualitative pictures in Discussions 4.7, 4.11, 4.15 and 4.19 is as follows:

As for the curve configurations in the above Table 2 and the figures in Chapter II, thick curves always depict (-2)-curves. On $\widetilde{Z}$ resp. $\widetilde{X}$ thin curves depict (-1)-curves. Whereas on each $\widetilde{X}$ resp. $\widetilde{Y}$ the picture shows the configuration of *all* negative curves resp. fibers, for the study of curves on $\widetilde{Z}$ we restrict ourselves to the description of configurations of all (-2)-curves, i.e. reducible fibers, and do not claim completeness of the configuration of (-1)-curves on $\widetilde{Z}$.

Intersection multiplicities 1 and 2 will be clear from the pictures, whereas we write a small 3 next to the point of intersection if the intersection multiplicity is 3. Base points and sections, as well as their (pre)images, will be marked in gray. The blown up points $\widetilde{P} \in \widetilde{X}$ and their (pre)images are drawn in red.

If a figure contains five configurations with no specific labels (see Figures 1, 5, 6, 10, and 11), they will be arranged according to the following diagram:

$$\begin{tikzcd} \widetilde{Z} \arrow[r] & \widetilde{X} \arrow[d] & \widetilde{Y} \arrow[l] \arrow[d] \\ & X & Y \arrow[l] \end{tikzcd}$$

4.1. Family $1A$. These surfaces occur for arbitrary $\mathrm{char}(k) = p \geq 0$.

PROPOSITION 4.5. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree 1 of type $1A$. Let $\widetilde{P} \in \widetilde{X}$ be a point which is a non-trivial torsion point of order m on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$.*

(0) If $p \neq 2$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 = \{*\}$.

(1) If $p = 2$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \neq \{*\}$ if and only if C is of type II and $\widetilde{P}$ lies on a (-1)-curve. Moreover, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \cong \mu_2$ and $m = 2$.

PROOF. As explained in Strategy 4.2 for such a $\widetilde{P} \in \widetilde{X}$, we can compute $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P})$ as $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}) \cong \mathrm{Stab}_{\mathbb{G}_m}(P)$, where P is the image of $\widetilde{P}$ in X under the minimal resolution and where X is explicitly given as a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ by

$$y^2 = x^3 + astx^2 + s^2t^2x \quad \text{with} \quad a \in k \quad \text{and} \quad a^2 \neq 4. \tag{4.1}$$

with $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathbb{G}_m$ acting as $[s : t : x : y] \mapsto [\lambda s : \lambda^{-1} t : x : y]$ (see Table 2). We note that the two D_4-singularities are at $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$. To find points $P = [s : t : x : y] \in X$ with non-trivial $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$, which are distinct from the base point of $|-K_X|$ (i.e., with s and t not both zero), we consider the following cases:

(a) If $s = 0$, we can assume $t = 1$. For the action $[0 : 1 : x : y] \mapsto [0 : 1 : \lambda^2 x : \lambda^3 y]$ to fix P, we must have either $x = y = 0$, in which case P would be a D_4-singularity, or $x, y \neq 0$ and $\lambda = 1$, in which case $\mathrm{Stab}_{\mathbb{G}_m}(P)$ is trivial.

(b) Thus, we can assume $s = 1$. Exploiting the symmetry between s and t, we can assume $t \neq 0$ by (a).

  (1) For the action $[1 : t : x : y] \mapsto [1 : \lambda^{-2} t : \lambda^{-2} x : \lambda^{-3} y]$ to fix P, we immediately see that $\lambda^2 = 1$ must hold. If furthermore y was non-zero, this would imply $\lambda = \lambda^3 = 1$. Thus, we can assume $y = 0$.
  Then, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$ is non-trivial if and only if $p = 2$ and

$$P = [1 : t : x : 0] \quad \text{with} \quad t \neq 0 \quad \text{and} \quad x^3 + atx^2 + t^2x = 0.$$

  We note that $x^3 + atx^2 + t^2x = x(x + bt)(x + (a+b)t)$ for $b \in k$ a solution of $z^2 + az + 1 = 0$, and thus

$$P \in \{[1 : t : 0 : 0], [1 : t : bt : 0], [1 : t : (a+b)t]\}.$$

  For such points P, we have $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 \cong \mu_2$. Moreover, from the location of the singular points of X, we see that $\widetilde{P}$ lies in an irreducible fiber C of $\widetilde{X} \to X \dashrightarrow \mathbb{P}^1$. Since $p = 2$, our equation for X is the Weierstraß equation (4.1) of a quasi-elliptic fibration [Ito94, Theorem 5.2.(d)], hence C is of type II, $C^0 \cong \mathbb{G}_a$ and $m = 2$ by Corollary 3.2. Finally, note that the equations $x = 0, x + bt = 0$, and $x + (a+b)t = 0$ are exactly the equations of the (-1)-curves on $\widetilde{X}$ that are not contained in members of $|-K_{\widetilde{X}}|$.

□

COROLLARY 4.6. *Let $\widetilde{Z}$ be a non-Jacobian rational (quasi-)elliptic surface that is a blow-up of a weak del Pezzo surface $\widetilde{X}$ of type $1A$ in a point $\widetilde{P}$ on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$. Assume that $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$. Then, $p = 2$, such surfaces $\widetilde{Z}$ form a 1-dimensional family, each of them has one multiple fiber $2\mathrm{II}$, and $\mathrm{Aut}^0_{\widetilde{Z}} \cong \mu_2$.*

PROOF. Everything except the number of moduli follows by combining Corollary 3.2 with the above Proposition 4.5. To see that these surfaces form a 1-dimensional family, note that weak del Pezzo surfaces of type $1A$ form a 1-dimensional family, so it suffices to show that for every fixed $\widetilde{X}$ of type $1A$, the choice of $\widetilde{P}$ is unique up to automorphisms of the surface. For this, firstly, we observe that all the curves $C \in |-K_{\widetilde{X}}|$ of type II are conjugate under $\mathrm{Aut}(\widetilde{X})$, and, secondly, that in every such fiber C, the three points $\widetilde{P}$, whose blow-up yields $\widetilde{Z}$ are permuted by an S_3-action on X. Both follow from our description of the $\mathbb{G}_m$-action on X in Table 2 and the proof of Proposition 4.5(1): First, $\mathbb{G}_m$ sends a fiber $\{[1 : t : x : y] \mid y^2 = x^3 + atx^2 + t^2x\}$ over $[1 : t], t \neq 0$, to the fiber over $[1 : \lambda^{-2}t]$, hence all such fibers are conjugate under $\mathbb{G}_m$. Second, for fixed $t \neq 0$, the involutions $x \mapsto x + bst$ (resp. $x \mapsto x + (a+b)st$) of X interchange $[1 : t : 0 : 0]$ and $[1 : t : bt : 0]$ (resp. $[1 : t : (a+b)t : 0]$). □

DISCUSSION 4.7. Note that, in the explicit description of the possibly blown up points $P \in X$ in the proof of Proposition 4.5(1) and their identification via automorphisms of $\widetilde{X}$ in Corollary 4.6, we see the structure of the Mordell–Weil group of the Jacobian rational quasi-elliptic fibration $\widetilde{Y} \to \mathbb{P}^1$ associated to $\widetilde{X}$: By [OS91] the Mordell–Weil group is $\mathrm{MW}(\widetilde{Y} \to \mathbb{P}^1) \cong (\mathbb{Z}/2\mathbb{Z})^2$, and, since $Y \to X$ is the contraction of the zero-section, the three sections different from the zero-section are visible in the equation of X; namely as $X \cap \{x = 0\}$, $X \cap \{x = bst\}$ and $X \cap \{x = (a+b)st\}$. The involutions $x \mapsto x + bst$ and $x \mapsto x + (a+b)st$ generate $\mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2) = S_3$ and permute these sections resp. the three possibilities for P on each fiber over $[1 : t], t \neq 0$. The strict transforms of these sections in $\widetilde{X}$ are the three (-1)-curves intersecting only (-2)-curves. Thus, $\widetilde{Z}$ contains a configuration of nine (-2)-curves of Kodaira–Néron type I_4^*. By Lemma 4.3, $\widetilde{Z}$ cannot contain any further (-2)-curves. This situation is summarized in Figure 1 and Corollary 4.8.

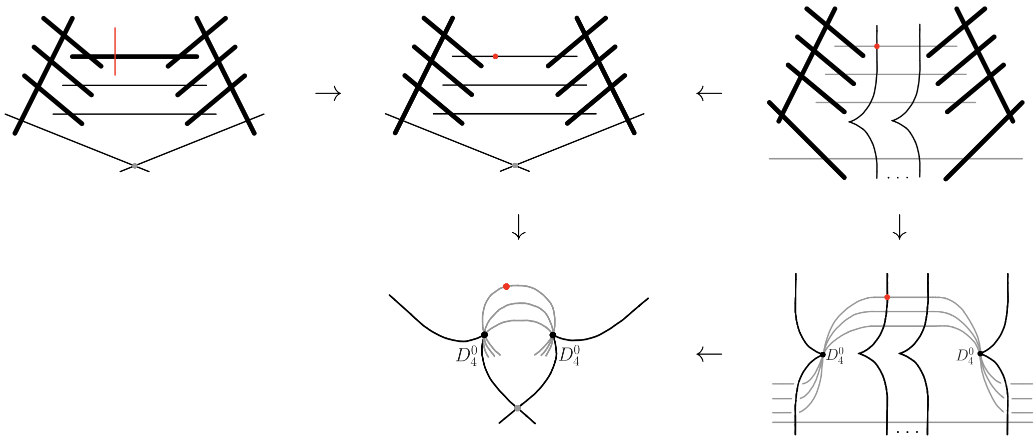


Figure 1. Non-Jacobian and Jacobian fibrations with global vector fields originating from $\widetilde{X}$ of type $1A$ ($p = 2$)

COROLLARY 4.8. *Each of the non-Jacobian rational (quasi-)elliptic surfaces $\widetilde{Z}$ of Corollary 4.6 contains nine (-2)-curves with dual graph of type $\widetilde{D}_8$ forming configuration I_4^*.*

4.2. Family $1B$.

Again, these surfaces occur for arbitrary $\mathrm{char}(k) = p \geq 0$.

PROPOSITION 4.9. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree 1 of type $1B$. Let $\widetilde{P} \in \widetilde{X}$ be a point which is a non-trivial torsion point of order m on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$.*

(0) If $p \neq 2, 3$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 = \{*\}$.

(1) If $p = 2$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \neq \{*\}$ if and only if C is of type IV^*. Moreover, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \cong \mu_2$ and $m = 2$.

(2) If $p = 3$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \neq \{*\}$ if and only if C is of type II and $\widetilde{P}$ lies on a (-1)-curve. Moreover, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \cong \mu_3$ and $m = 3$.

PROOF. As explained in Strategy 4.2 for such a $\widetilde{P} \in \widetilde{X}$, we can compute $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P})$ as $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}) \cong \mathrm{Stab}_{\mathbb{G}_m}(P)$, where P is the image of $\widetilde{P}$ in X under the minimal resolution and where X is explicitly given as a sextic hypersurface in $\mathbb{P}(1, 1, 2, 3)$ by

$$y^2 + st^2y = x^3 \tag{4.2}$$

with $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathbb{G}_m$ acting as $[s : t : x : y] \mapsto [\lambda^2 s : \lambda^{-1} t : x : y]$ (see Table 2). We note that the E_6-singularity is at $[1 : 0 : 0 : 0]$, whereas the A_2-singularity is at $[0 : 1 : 0 : 0]$. To find points $P = [s : t : x : y] \in X$ with non-trivial $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$, which are distinct from the base point of $|-K_X|$ (i.e., with s and t not both zero), we consider the following cases:

(a) If $s = 0$, we can assume $t = 1$. For the action $[0 : 1 : x : y] \mapsto [0 : 1 : \lambda^2 x : \lambda^3 y]$ to fix P, we must either have $x = y = 0$, in which case P would be the A_2-singularity, or $x, y \neq 0$ and $\lambda = 1$, in which case $\mathrm{Stab}_{\mathbb{G}_m}(P)$ is trivial.

(b) Thus, we can assume $s = 1$.

  (1) If $t = 0$, then for the action $[1 : 0 : x : y] \mapsto [1 : 0 : \lambda^{-4}x : \lambda^{-6}y]$ to fix P, we must either have $x = y = 0$, in which case P would be the E_6-singularity, or $x, y \neq 0$ and $\lambda^2 = 1$. Thus, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$ is non-trivial if and only if $p = 2$ and

$$P = [1 : 0 : x : y] \;\text{ with }\; x, y \neq 0 \;\text{ and }\; y^2 = x^3.$$

  In this case, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 \cong \mu_2$. Moreover, since P and the E_6-singularity lie on the same fiber of the projection $\mathbb{P}(1,1,2,3) \supseteq X \dashrightarrow \mathbb{P}^1$ onto s and t, $\widetilde{P}$ lies on the identity component of a curve $C \in |-K_{\widetilde{X}}|$ of type IV^*. Since P lies on the cuspidal curve $X \cap \{t = 0\}$, we have $C^0 \cong \mathbb{G}_a$ as group schemes and thus $m = 2$ by Corollary 3.2.

  (2) If $t \neq 0$, then for the action $[1 : t : x : y] \mapsto [1 : \lambda^{-3}t : \lambda^{-4}x : \lambda^{-6}y]$ to fix P, we immediately see that $\lambda^3 = 1$ must hold. If furthermore x was non-zero, this would imply $\lambda = \lambda^4 = 1$. Thus, we can assume $x = 0$.
  Then, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$ is non-trivial if and only if $p = 3$ and

$$P = [1 : t : 0 : y] \;\text{ with }\; t \neq 0 \;\text{ and }\; y \in \{0, -t^2\}.$$

  For such points, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 \cong \mu_3$. Moreover, from the location of the singular points of X, we see that $\widetilde{P}$ lies in an irreducible fiber C of $\widetilde{X} \to X \dashrightarrow \mathbb{P}^1$. Since $p = 3$, our equation (4.2) for X is the Weierstraß equation of a quasi-elliptic fibration [Ito92, Theorem 3.3(2)], hence C is of type II, $C^0 \cong \mathbb{G}_a$ and $m = 3$ by Corollary 3.2. Finally, note that the equations $y = 0$ and $y = -t^2$ are exactly the equations of the (-1)-curves on $\widetilde{X}$ that are not contained in members of $|-K_{\widetilde{X}}|$.

□

COROLLARY 4.10. *Let $\widetilde{Z}$ be a non-Jacobian rational (quasi-)elliptic surface that is a blow-up of a weak del Pezzo surface $\widetilde{X}$ of type $1B$ in a point $\widetilde{P}$ on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$. Assume that $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$. Then,*

(1) either $p = 2$, the surface $\widetilde{Z}$ is unique up to isomorphism, has one multiple fiber $2\mathrm{IV}^*$, and $\mathrm{Aut}^0_{\widetilde{Z}} \cong \mu_2$,

(2) or $p = 3$, the surface $\widetilde{Z}$ is unique up to isomorphism, has one multiple fiber $3\mathrm{II}$, and $\mathrm{Aut}^0_{\widetilde{Z}} \cong \mu_3$.

PROOF. Everything except the uniqueness follows by combining Corollary 3.2 with the above Proposition 4.9.

(1) If $p = 2$, for the uniqueness of $\widetilde{Z}$ in (1), it suffices to observe that all points $\widetilde{P} \in C^0$, where $C \in |-K_{\widetilde{X}}|$ is *the* curve of type IV^* are conjugate under

$\mathrm{Aut}(\widetilde{X})$. This follows from our description of the $\mathbb{G}_m$-action on X in Table 2 and the proof of Proposition 4.9(1): $\mathbb{G}_m$ sends a point of the form $[1:0:x:y]$, where $x, y \neq 0$ and $y^2 = x^3$, to $[1:0:\lambda^{-4}x:\lambda^{-6}y]$, so all such points are in the same $\mathbb{G}_m$-orbit.

(2) If $p = 3$, for the uniqueness of $\widetilde{Z}$ in (2), firstly, we observe that all the curves $C \in |-K_{\widetilde{X}}|$ of type II are conjugate under $\mathrm{Aut}(\widetilde{X})$, and, secondly, that in every such fiber C, the two points $\widetilde{P}$, whose blow-up yields $\widetilde{Z}$ are interchanged simultaneously by an automorphism of X. Both follow from our description of the $\mathbb{G}_m$-action on X in Table 2 and the proof of Proposition 4.9(2): First, $\mathbb{G}_m$ sends a fiber $\{[1:t:x:y] \mid y^2 + t^2 y = x^3\}$ over $[1:t], t \neq 0$, to the fiber over $[1:\lambda^{-3}t]$, hence all such fibers are conjugate under $\mathbb{G}_m$. Second, for fixed $t \neq 0$, the two points $[1:t:0:0]$ and $[1:t:0:-t^2]$ are interchanged by the involution of X given by $y \mapsto -y - st^2$.

$\square$

DISCUSSION 4.11. In both of the above cases, we again see the Mordell–Weil group of the Jacobian rational (quasi-)elliptic fibration $\widetilde{Y} \to \mathbb{P}^1$ associated to $\widetilde{X}$: By [OS91] $\mathrm{MW}(\widetilde{Y} \to \mathbb{P}^1) \cong \mathbb{Z}/3\mathbb{Z}$ and the two sections that are visible in the equation for X are given by $X \cap \{y = 0\}$ and $X \cap \{y = -st^2\}$. These sections are interchanged by the automorphism $y \mapsto -y - st^2$. By [Lan94] and [Ito92], $\widetilde{Y}$ is elliptic with singular fibers IV^* and IV if $p = 2$, and quasi-elliptic with reducible fibers IV^* and IV if $p = 3$.

To determine the number and configuration of (-2)-curves on $\widetilde{Z}$ we treat cases (1) and (2) of Corollary 4.10 separately.

(1) By the proof of Proposition 4.9(1), we know that $\widetilde{P} \in \widetilde{X}$ lies on a (-1)-curve intersecting the E_6- but not on a (-1)-curve intersecting the A_2-configuration of (-2)-curves. Hence, $\widetilde{Z}$ contains configuration IV^*.

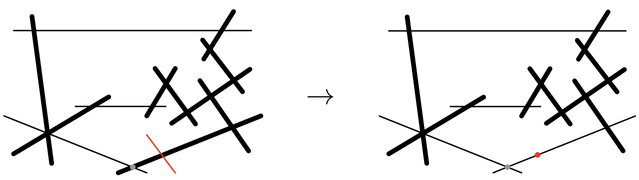


Figure 2. $\widetilde{Z} \to \widetilde{X}$ with incomplete (-2)-curve configuration on $\widetilde{Z}$

From the Kodaira–Néron classification of fiber types and Lemma 4.3 we see that the other two obvious (-2)-curves on $\widetilde{Z}$ (in the left of the picture for $\widetilde{Z}$ in Figure 2) have to constitute a dual graph $\widetilde{A}_2$ together with another (-2)-curve, that we were not yet able to see as a negative curve on $\widetilde{X}$. We will now determine the precise configuration – either IV or I_3 – of the two "known" (-2)-curves with the "new" (-2)-curve: Taking into account that, by Proposition 4.9(1), every (-1)-curve on $\widetilde{Z}$ is a 2-section, we obtain that the (-1)-curve in Figure 2 that intersects both the "known" (-2)-curves in one point cannot intersect the "new" (-2)-curve. Hence, configuration IV is not possible.

We are therefore left with the four possibilities for the intersection behavior of I_3 as in Figure 3, where the "known" (-2)-curves are still drawn in black and the "new" (-2)-curve is drawn in purple. Note that we also assigned other colors to some of the remaining negative curves in order to be able to better refer to them in the argument. We remark that, in order to not overload these drawings, we did not yet include the intersection behavior of the red exceptional curve on $\widetilde{Z}$ with other curves.

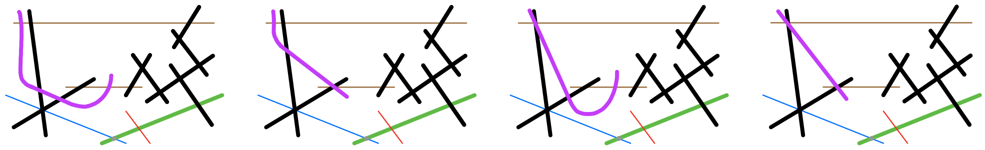


Figure 3. Four possibilities for configurations of (-2)-curves on $\widetilde{Z}$

When contracting the blue (-1)-curve in Figure 3, we obtain a realization of $\widetilde{Z}$ as blow-up of another weak del Pezzo surface with global vector fields containing an E_6-configuration of (-2)-curves and at least five (-1)-curves. So, by the classification in Chapter II, this weak del Pezzo surface is either of type $1K$ or $1J$. Since the blue (-1)-curve does not intersect the purple (-2)-curve, its image under the contraction is still a (-2)-curve that does not intersect the E_6-configuration of (-2)-curves on the contraction. So, $\widetilde{Z}$ is a blow-up of the weak del Pezzo surface $\widetilde{X}_{1K}$ of type $1K$. We can identify some of the curves in Figure 3 with curves on $\widetilde{X}_{1K}$ according to the color they are given below and learn about their intersection behavior.

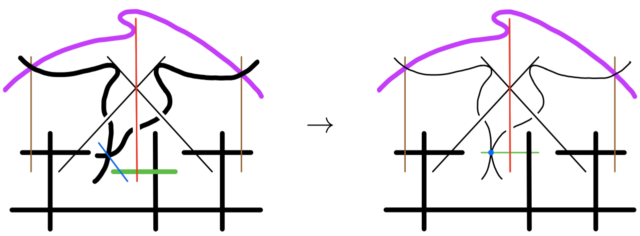


Figure 4. $\widetilde{Z}$ as a blow-up of a weak del Pezzo surface of type $1K$ $(p = 2)$

From a comparison with Figure 4 we see that the fourth configuration in Figure 3 is correct. Moreover, there are (at least) three more (-1)-curves on $\widetilde{Z}$ than visible in Figure 2, and the red exceptional curve intersects the purple (-2)-curve in one point with multiplicity 2. The results of this discussion are summarized in the following Figure 5.

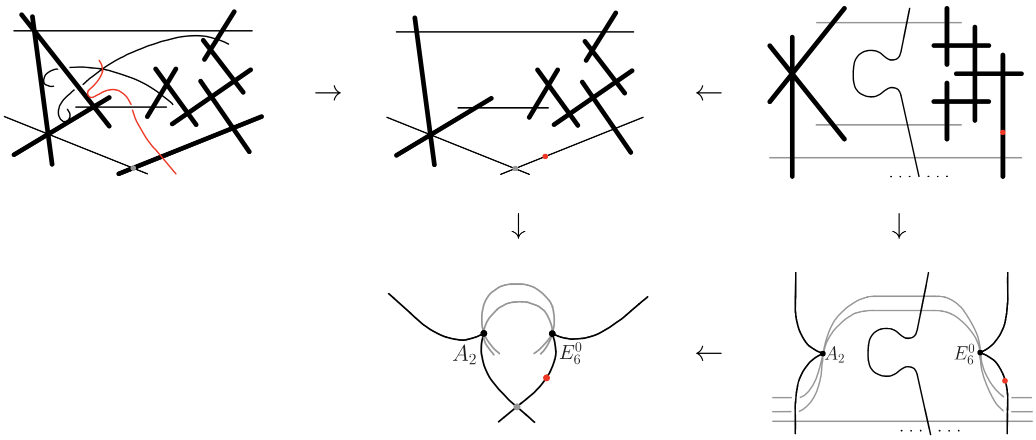


Figure 5. Non-Jacobian and Jacobian fibrations with global vector fields originating from $\widetilde{X}$ of type $1B$ ($p = 2$)

(2) By the proof of Proposition 4.9(2), $\widetilde{P}$ lies on a (-1)-curve connecting the E_6- and the A_2-configuration of (-2)-curves on $\widetilde{X}$. Thus, $\widetilde{Z}$ contains (-2)-curves forming a configuration of type II^* and, by Lemma 4.3 no further ones. As in the previous cases, the situation is illustrated below.

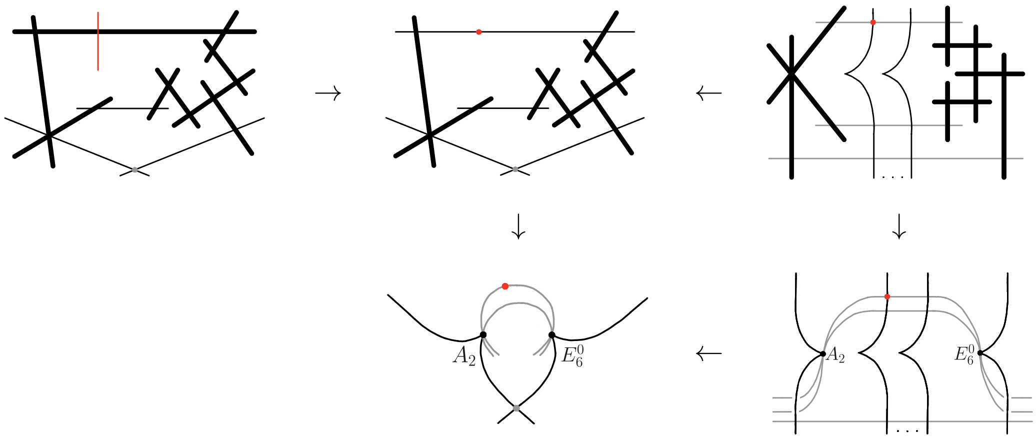


Figure 6. Non-Jacobian and Jacobian fibrations with global vector fields originating from $\widetilde{X}$ of type $1B$ ($p = 3$)

Hence, we have the following summarizing corollary.

COROLLARY 4.12. *Let $\widetilde{Z}$ be one of the non-Jacobian rational (quasi-)elliptic surfaces of Corollary 4.10. Then, the following hold.*

(1) If $p = 2$, $\widetilde{Z}$ contains ten (-2)-curves with dual graph of type $\widetilde{E}_6 + \widetilde{A}_2$ forming configurations IV^* and I_3. Moreover, IV^* is the unique multiple fiber and of multiplicity $m = 2$.

(2) If $p = 3$, $\widetilde{Z}$ contains nine (-2)-curves with dual graph of type $\widetilde{E}_8$ forming configuration II^*.

4.3. Family $1C$. This family exists only if $\mathrm{char}(k) = p \neq 2$.

PROPOSITION 4.13. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree 1 of type $1C$. Let $\widetilde{P} \in \widetilde{X}$ be a point which is a non-trivial torsion point of order m on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$.*

(0) If $p \neq 3$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 = \{*\}$.

(1) If $p = 3$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \neq \{*\}$ if and only if C is of type III^*. Moreover, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \cong \mu_3$ and $m = 3$.

PROOF. As explained in Strategy 4.2 for such a $\widetilde{P} \in \widetilde{X}$, we can compute $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P})$ as $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}) \cong \mathrm{Stab}_{\mathbb{G}_m}(P)$, where P is the image of $\widetilde{P}$ in X under the minimal resolution and where X is explicitly given as a sextic hypersurface in $\mathbb{P}(1, 1, 2, 3)$ by

$$y^2 = x^3 + st^3x \tag{4.3}$$

with $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathbb{G}_m$ acting as $[s : t : x : y] \mapsto [\lambda^3 s : \lambda^{-1} t : x : y]$ (see Table 2). We note that the E_7-singularity is at $[1 : 0 : 0 : 0]$, whereas the A_1-singularity is at $[0 : 1 : 0 : 0]$. To find points $P = [s : t : x : y] \in X$ with non-trivial $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$, which are distinct from the base point of $|-K_X|$ (i.e., with s and t not both zero), we consider the following cases:

(a) If $s = 0$, we can assume $t = 1$. For the action $[0 : 1 : x : y] \mapsto [0 : 1 : \lambda^2 x : \lambda^3 y]$ to fix P, we must either have $x = y = 0$, in which case P would be the A_1-singularity, or $x, y \neq 0$ and $\lambda = 1$, in which case $\mathrm{Stab}_{\mathbb{G}_m}(P)$ is trivial.

(b) Thus, we can assume $s = 1$. If $t \neq 0$, then for the action $[1 : t : x : y] \mapsto [1 : \lambda^{-4} t : \lambda^{-6} x : \lambda^{-9} y]$ to fix P, we see that $\lambda^4 = 1$ must hold. Since $p \neq 2$, this implies $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 = \{*\}$.

  (a) So, we can assume $t = 0$ and see that for the above action to fix P we get either $x = y = 0$, in which case P would be the E_7-singularity, or $x, y \neq 0$ and $\lambda^3 = 1$. Thus $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$ is non-trivial if and only if $p = 3$ and

$$P = [1 : 0 : x : y] \quad \text{with} \quad x, y \neq 0 \quad \text{and} \quad y^2 = x^3.$$

  In this case, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 \cong \mu_3$. Moreover, since P and the E_7-singularity lie on the same fiber of the projection $\mathbb{P}(1, 1, 2, 3) \supseteq X \dashrightarrow \mathbb{P}^1$ onto s and t, $\widetilde{P}$ lies on the identity component of a curve $C \in |-K_{\widetilde{X}}|$ of type III^*. Since

P lies on the cuspidal curve $X \cap \{t = 0\}$, we have $C^0 \cong \mathbb{G}_a$ and thus $m = 3$ by Corollary 3.2.

$\square$

COROLLARY 4.14. *Let $\widetilde{Z}$ be a non-Jacobian rational (quasi-)elliptic surface that is a blow-up of a weak del Pezzo surface of type 1C in a point $\widetilde{P}$ on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$. Assume that $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$. Then, $p = 3$, the surface $\widetilde{Z}$ is unique up to isomorphism, has one multiple fiber* 3III*, *and* $\mathrm{Aut}^0_{\widetilde{Z}} \cong \mu_3$.

PROOF. Everything except the uniqueness follows by combining Corollary 3.2 with Proposition 4.13. To show that $\widetilde{Z}$ is unique up to isomorphism, it suffices to observe that all points $\widetilde{P} \in C^0$ where $C \in |-K_{\widetilde{X}}|$ is the curve of type III* are conjugate under $\mathrm{Aut}(\widetilde{X})$. This follows from our description of the $\mathbb{G}_m$-action on the Weierstraß model in Table 2: The points on C^0 which do not lie on the zero section are of the form $[1 : 0 : x : y]$ with $x, y \neq 0, y^2 = x^3$, and $\mathbb{G}_m$ sends such a point to $[1 : 0 : \lambda^{-6}x : \lambda^{-9}y]$, so all such points are in the same $\mathbb{G}_m$-orbit. $\square$

DISCUSSION 4.15. In the configuration of curves on $\widetilde{X}$, the base point has to be the intersection of the two intersecting (-1)-curves. Blowing it up yields $\widetilde{Y}$, which is elliptic with singular fibers III* and III by [Lan94] and has Mordell–Weil group $\mathbb{Z}/2\mathbb{Z}$ by [OS91]. Thus, the (-1)-curve on $\widetilde{X}$ which does not intersect another (-1)-curve corresponds to the non-zero section on $\widetilde{Y}$.

We have seen in the proof of Proposition 4.13 that $\widetilde{P} \in \widetilde{X}$ lies on the (-1)-curve that contains the base point and intersects the E_7-configuration of (-2)-curves. Thus, $\widetilde{Z}$ contains configuration III*.

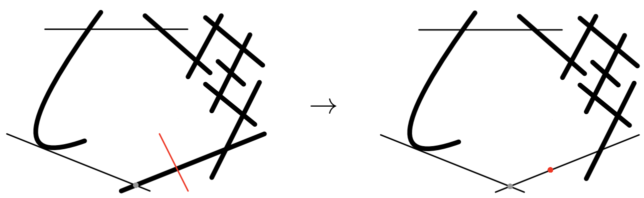


Figure 7. $\widetilde{Z} \to \widetilde{X}$ with incomplete (-2)-curve configuration on $\widetilde{Z}$

Aiming for the entire configuration of (-2)-curves on $\widetilde{Z}$, we recall from the classification of reducible fibers of $\widetilde{Z}$ and Lemma 4.3 that the single (-2)-curve on $\widetilde{Z}$ has to constitute a dual graph $\widetilde{A}_1$ together with another (-2)-curve, that we were not yet able to see as a negative curve on $\widetilde{X}$. We will now determine the precise configuration – either I_2 or III – of the "known" (-2)-curve with the "new" one on $\widetilde{Z}$: Taking into account that every (-1)-curve on $\widetilde{Z}$ is a 3-section, we have the following ten possibilities for their intersection behavior, where the first two rows in Figure 8 show the I_2-cases and the third row shows the two III-cases. Here, the "known" resp. "new" (-2)-curves are drawn in black resp.

purple. Note that we also assigned other colors to some of the remaining negative curves to be able to better refer to them in the argument. We remark that, in order to not overload these drawings, we did not yet include the intersection behavior of the red exceptional curve on $\widetilde{Z}$ with other curves.

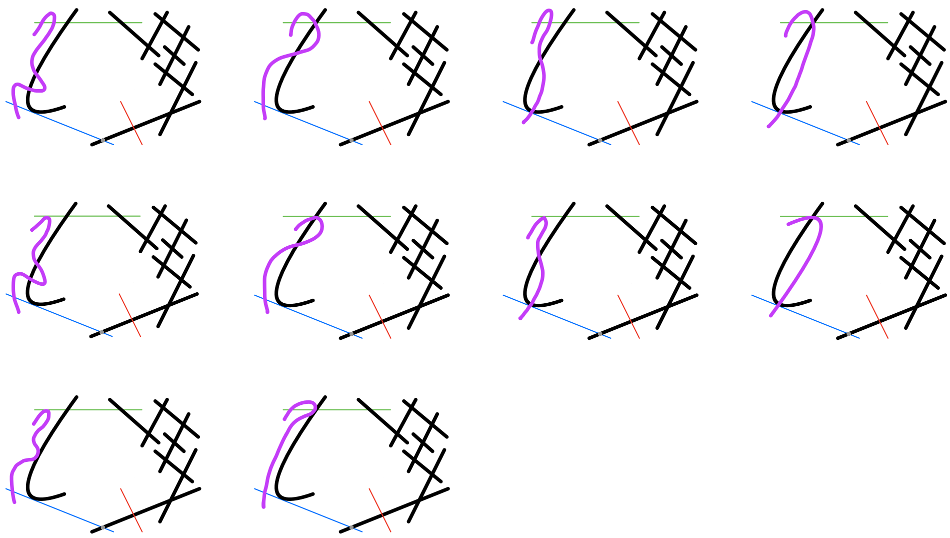


Figure 8. Ten possibilities for configurations of (-2)-curves on $\widetilde{Z}$

When contracting the blue (-1)-curve in Figure 8, this yields a realization of $\widetilde{Z}$ as a blow-up of another weak del Pezzo surface with global vector fields and an E_7-configuration of (-2)-curves and two (-1)-curves with intersection number 2. By the classification of Chapter II, this must be $\widetilde{X}_{1F}$ of type $1F$. The blown-up point $\widetilde{P}_{1F} \in \widetilde{X}_{1F}$ has to be one of the intersection points of the horizontal (-1)-curve with the curved ones in Figure 62 in Chapter II. By symmetry of the configuration on $\widetilde{X}_{1F}$, we can choose $\widetilde{P}_{1F}$ to be the blue point in Figure 9. Although, when contracting the blue (-1)-curve, the "known" (-2)-curve becomes a curve of non-negative self-intersection and hence is no longer visible in the curve configuration below, we can identify some images of the curves in Figure 8 with curves on $\widetilde{X}_{1F}$ according to the color they are given below and learn about their intersection behavior.

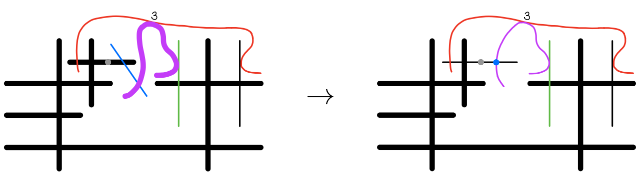


Figure 9. $\widetilde{Z}$ as a blow-up of a weak del Pezzo surface $\widetilde{X}_{1F}$ of type $1F$

Thus, in Figure 8 the purple (-2)-curve and the green (-1)-curve intersect in one point with multiplicity two, which rules out the first four possibilities in Figure 8. Moreover, we see that the red (-1)-curve meets the purple (-2)-curve in one point with multiplicity 3. To find the true configuration among the remaining six possibilities we need stronger techniques using the μ_3-action on $\widetilde{Z}$:

Since μ_3 is linearly reductive, its fixed locus on $\widetilde{Z}$ is smooth by [CGP15, Proposition A.8.10(2)]. Moreover, μ_3 preserves every negative curve and hence transverse intersections of negative curves are fixed points. This excludes the first configuration in the second row of Figure 8. Indeed, since there are at least 3 fixed points on the purple curve and on the black "known" (-2)-curve, and since μ_3 has at most 2 fixed points on $\mathbb{P}^1$ (see for example [Mar22, Lemma 2.34(i)]), both these curves have to be fixed pointwise. This contradicts the smoothness of the fixed locus $\widetilde{Z}^{\mu_3}$.

To exclude the sixth, seventh and tenth configuration of Figure 8, we refine the above argument and carry it out for the sixth configuration (the other two use the analogous argument for differently colored curves): The purple curve contains 3 fixed points, hence is fixed pointwise. Let Q be the point on the green (-1)-curve C, where the purple (-2)-curve touches C. Since μ_3 fixes their intersection $C_1 = \operatorname{Spec} k[x]/(x^2)$, the non-reduced C_1 is contained in the fixed locus C^{μ_3}, which is smooth by [CGP15, Proposition A.8.10(2)] applied to C. Thus, μ_3 has to act trivially on C. The purple curve and C being contained in $\widetilde{Z}^{\mu_3}$ yields a contradiction to smoothness of $\widetilde{Z}^{\mu_3}$ (again applying [CGP15, Proposition A.8.10(2)] to $\widetilde{Z}$).

To exclude the ninth configuration in Figure 8, we cannot immediately tell that the green, blue, purple or black curve is fixed pointwise since there are not enough transverse intersections. To overcome this, let us have a closer look at a point where two of these curves meet: μ_3 acts on the first order neighborhood $C_1 := k[x]/(x^2)$ of such a point. In the proof of Proposition 5.8 in Chapter III, we saw $\operatorname{Aut}_{C_1} \cong \mathbb{G}_m$ acting as $x \mapsto a_1 x$ if $p \neq 2$. Thus, the closed point of C_1 is fixed by $\mathbb{G}_m$ and thus also by μ_3 since $C_1^{\mu_3} \supseteq C_1^{\mathbb{G}_m}$. Therefore, the green, blue, purple and black curve are fixed pointwise, which contradicts smoothness of $\widetilde{Z}^{\mu_3}$.

So, we showed that $\widetilde{Z}$ contains configurations III^* and I_2 as depicted in the eighth configuration of Figure 8. Note that from the comparison with Figure 9 we see that there is an additional (-1)-curve on $\widetilde{Z}$ intersecting the red (-1)-curve in a point with multiplicity 2 and the "known" (-2)-curve in a point of multiplicity 3 (the latter follows from the smooth fixed loci argument that we used above). We summarize the results of the previous discussion in Figure 10 and Corollary 4.16 below.

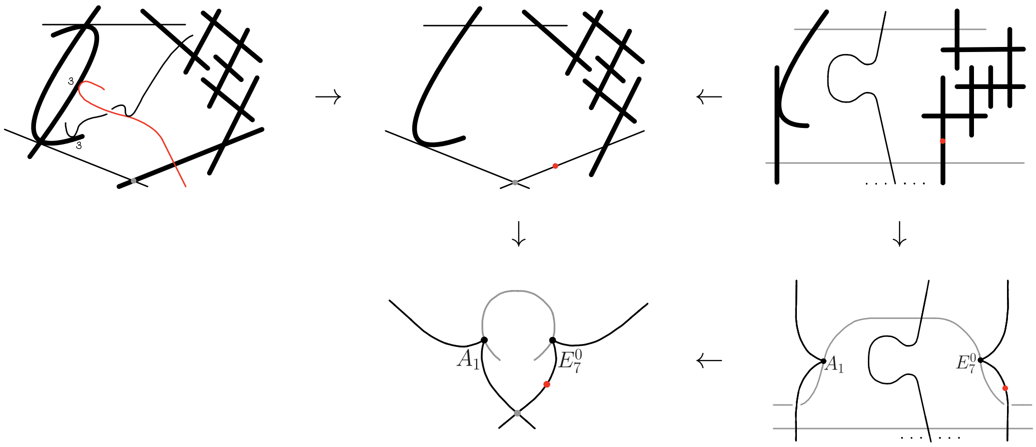


Figure 10. Non-Jacobian and Jacobian fibrations with global vector fields originating from $\widetilde{X}$ of type $1C$ ($p = 3$)

COROLLARY 4.16. *The unique non-Jacobian rational (quasi-)elliptic surface $\widetilde{Z}$ of Corollary 4.14 contains nine (-2)-curves with dual graph of type $\widetilde{E}_7 + \widetilde{A}_1$ forming configurations* III^* *and* I_2. *Moreover,* III^* *is the unique multiple fiber and of multiplicity* $m = 3$.

4.4. Family $1D$. This family exists only if $\mathrm{char}(k) = p \neq 2, 3$.

PROPOSITION 4.17. *Let $\widetilde{X}$ be a weak del Pezzo surface of degree 1 of type $1D$. Let $\widetilde{P} \in \widetilde{X}$ be a point which is a non-trivial torsion point of order m on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$.*

(0) If $p \neq 5$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 = \{*\}$.

(1) If $p = 5$, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \neq \{*\}$ if and only if C is of type II^*. *Moreover, then $(\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}))^0 \cong \mu_5$ and $m = 5$.*

PROOF. As explained in Strategy 4.2 for such a $\widetilde{P} \in \widetilde{X}$, we can compute $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P})$ as $\mathrm{Stab}_{\mathrm{Aut}^0_{\widetilde{X}}}(\widetilde{P}) \cong \mathrm{Stab}_{\mathbb{G}_m}(P)$, where P is the image of $\widetilde{P}$ in X under the minimal resolution and where X is explicitly given as a sextic hypersurface in $\mathbb{P}(1,1,2,3)$ by

$$y^2 = x^3 + st^5 \tag{4.4}$$

with $\mathrm{Aut}^0_{\widetilde{X}} \cong \mathbb{G}_m$ acting as $[s:t:x:y] \mapsto [\lambda^5 s : \lambda^{-1} t : x : y]$ (see Table 2). We note that the E_8-singularity is at $[1:0:0:0]$. To find points $P = [s:t:x:y] \in X$ with non-trivial $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$, which are distinct from the base point of $|-K_X|$ (i.e., with s and t not both zero), we consider the following cases:

(a) If $s = 0$, we can assume $t = 1$. For the action $[0 : 1 : x : y] \mapsto [0 : 1 : \lambda^2 x : \lambda^3 y]$ to fix P, we have two possibilities: either $x, y \neq 0$ and $\lambda = 1$, or $x = y = 0$. But the point $[0 : 1 : 0 : 0]$ corresponds to the singular point of a $C \in |-K_{\widetilde{X}}|$ of type II, hence does not lie in the smooth C^0.

(b) Thus, we can assume $s = 1$. If $t \neq 0$, then for the action $[1 : t : x : y] \mapsto [1 : \lambda^{-6} t : \lambda^{-10} x : \lambda^{-15} y]$ to fix P, we see that $\lambda^6 = 1$ must hold. Since $p \neq 2, 3$, this implies $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 = \{*\}$.

  (a) So, we can assume $t = 0$ and see that for the above action to fix P we get either $x = y = 0$, in which case P would be the E_8-singularity, or $x, y \neq 0$ and $\lambda^5 = 1$. Thus $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0$ is non-trivial if and only if $p = 5$ and

$$P = [1 : 0 : x : y] \quad \text{with} \quad x, y \neq 0 \ \text{ and } \ y^2 = x^3.$$

  In this case, $(\mathrm{Stab}_{\mathbb{G}_m}(P))^0 \cong \mu_5$. Moreover, since P and the E_8-singularity lie on the same fiber of the projection $\mathbb{P}(1, 1, 2, 3) \supseteq X \dashrightarrow \mathbb{P}^1$ onto s and t, $\widetilde{P}$ lies on the identity component of a curve $C \in |-K_{\widetilde{X}}|$ of type II^*. Since P lies on the cuspidal curve $X \cap \{t = 0\}$, we have $C^0 \cong \mathbb{G}_a$ and thus $m = 5$ by Corollary 3.2.

□

COROLLARY 4.18. *Let $\widetilde{Z}$ be a non-Jacobian rational (quasi-)elliptic surface that is a blow-up of a weak del Pezzo surface of type 1D in a point $\widetilde{P}$ on the identity component C^0 of a curve $C \in |-K_{\widetilde{X}}|$. Assume that $h^0(\widetilde{Z}, T_{\widetilde{Z}}) \neq 0$. Then, $p = 5$, the surface $\widetilde{Z}$ is unique up to isomorphism, has one multiple fiber $5\mathrm{II}^*$, and $\mathrm{Aut}^0_{\widetilde{Z}} \cong \mu_5$.*

PROOF. Everything except the uniqueness follows by combining Corollary 3.2 with Proposition 4.17. To show that $\widetilde{Z}$ is unique up to isomorphism, it suffices to observe that all points $\widetilde{P} \in C^0$ where $C \in |-K_{\widetilde{X}}|$ is the curve of type II^* are conjugate under $\mathrm{Aut}(\widetilde{X})$. This follows from our description of the $\mathbb{G}_m$-action on the Weierstraß model in Table 2: The points on C^0 which do not lie on the zero section are of the form $[1 : 0 : x : y]$ with $x, y \neq 0, y^2 = x^3$, and $\mathbb{G}_m$ sends such a point to $[1 : 0 : \lambda^{-10} x : \lambda^{-15} y]$, so all such points are in the same $\mathbb{G}_m$-orbit. □

DISCUSSION 4.19. By [Lan94, Theorem 4.1.] and [MP86], $\widetilde{Y}$ contains singular fibers II^* and II, the Mordell–Weil group of $\widetilde{Y}$ is trivial by [OS91], and thus there are no other (-1)-curves on $\widetilde{Y}$ besides the zero-section. By the computation in the proof of Proposition 4.17(1) $\widetilde{P}$ lies on this (-1)-curve as well. So, $\widetilde{Z}$ contains configuration II^* and by Lemma 4.3 these are all (-2)-curves on $\widetilde{Z}$. The situation is summarized in the following Figure 11 and Corollary 4.20.

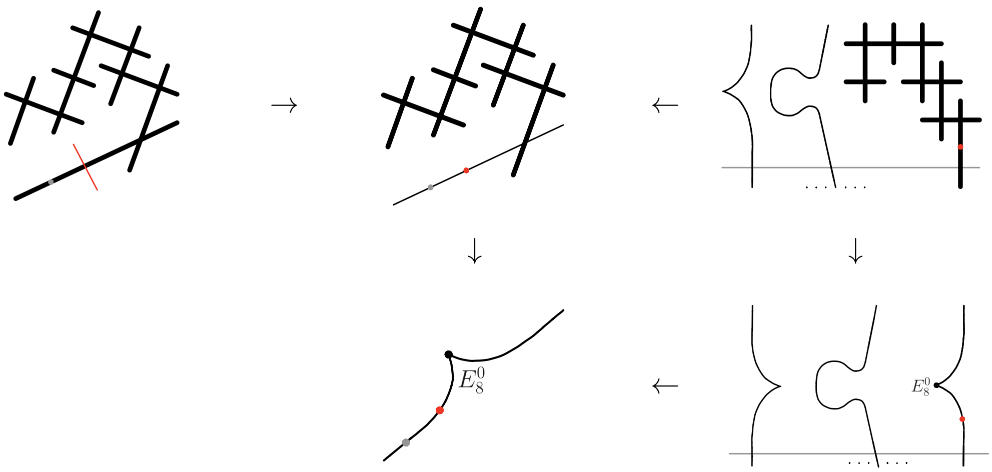


Figure 11. Non-Jacobian and Jacobian fibrations with global vector fields originating from $\widetilde{X}$ of type $1D$ ($p = 5$)

COROLLARY 4.20. *The unique non-Jacobian rational (quasi-)elliptic surface $\widetilde{Z}$ of Corollary 4.18 contains nine (-2)-curves with dual graph of type $\widetilde{E}_8$ forming configuration* II^*. *Moreover,* II^* *is the unique multiple fiber and of multiplicity $m = 5$.*

Appendix: Collection of all classification tables

0. Deformation spaces of weak del Pezzo surfaces

$K_{\widetilde{X}}^2$	$\chi(T_{\widetilde{X}})$	$h^0(\widetilde{X}, T_{\widetilde{X}})$ $= \dim(T_{\mathrm{id}}(\mathrm{Aut}_{\widetilde{X}}))$	$h^1(\widetilde{X}, T_{\widetilde{X}})$ $= \dim(\mathrm{Def}_{\widetilde{X}})$	Case(s)
9	8	8	0	$\mathbb{P}^2$
8	6	6	0	$\mathbb{P}^1 \times \mathbb{P}^1, 8A$
		7	1	$\mathbb{F}_2$
7	4	4	0	$7A$
		5	1	$7B$
6	2	2	0	$6A$
		3	1	$6B, 6C$
		4	2	$6D, 6E$
		5	3	$6F$
5	0	0	0	all other $\widetilde{X}$
		1	1	$5A$
		2	2	$5B, 5C$
		3	3	$5D, 5E$
		4	4	$5F$
4	-2	0	2	all other $\widetilde{X}$
		1	3	$4A, 4B, 4C, 4D, 4E$
		2	4	$4F, 4G, 4H, 4I, 4J, 4M, 4N$
		3	5	$4K, 4L, 4O, 4P$
		4	6	$4Q$
3	-4	0	4	all other $\widetilde{X}$
		1	5	$3A, 3B, 3C, 3D, 3E, 3F, 3G, 3N, 3O$
		2	6	$3H, 3I, 3J, 3K, 3L, 3P, 3Q$
		3	7	$3M, 3R$
2	-6	0	6	all other $\widetilde{X}$
		1	7	$2A, 2B, 2C, 2D, 2E, 2F, 2G, 2H, 2I, 2J,$ $2K, 2L, 2N, 2O, 2P, 2Q, 2R, 2T, 2W$
		2	8	$2M, 2S, 2U, 2V, 2X$
		3	9	$2Y$
1	-8	0	8	all other $\widetilde{X}$
		1	9	$1A, 1B, 1C, 1D, 1E, 1F, 1G, 1H, 1J, 1K,$ $1L, 1M, 1N, 1O, 1Q, 1S$
		2	10	$1I, 1P, 1R$
		3	11	$1T$

Table 0. Dimensions of $H^i(\widetilde{X}, T_{\widetilde{X}})$ for all weak del Pezzo surfaces $\widetilde{X}$

I. Which rational double points occur on del Pezzo surfaces?

$\Gamma' \hookrightarrow E_8$	occurs if		$\Gamma' \hookrightarrow E_8$	occurs if		$\Gamma' \hookrightarrow E_8$	occurs if	
	$p \neq 2$	$p = 2$		$p \neq 2$	$p = 2$		$p \neq 2$	$p = 2$
A_1	✓	✓	$A_3 + 3A_1$	✓	✓	$A_5 + A_2$	✓	✓
$2A_1$	✓	✓	$3A_2$	✓	✓	$D_5 + A_2$	✓	✓
A_2	✓	✓	$A_3 + A_2 + A_1$	✓	✓	$A_6 + A_1$	✓	✓
$3A_1$	✓	✓	$A_4 + 2A_1$	✓	✓	$D_6 + A_1$	✓	✓
$A_2 + A_1$	✓	✓	$D_4 + 2A_1$	✓	✓	$E_6 + A_1$	✓	✓
A_3	✓	✓	$2A_3$	✓	✓	A_7	✓	✓
$4A_1$	✓	✓	$A_4 + A_2$	✓	✓	D_7	✓	✓
$A_2 + 2A_1$	✓	✓	$D_4 + A_2$	✓	✓	E_7	✓	✓
$2A_2$	✓	✓	$A_5 + A_1$	✓	✓	$8A_1$	×	✓
$A_3 + A_1$	✓	✓	$D_5 + A_1$	✓	✓	$D_4 + 4A_1$	×	✓
A_4	✓	✓	A_6	✓	✓	$4A_2$	✓	✓
D_4	✓	✓	D_6	✓	✓	$2A_3 + 2A_1$	✓	×
$5A_1$	✓	✓	E_6	✓	✓	$A_5 + A_2 + A_1$	✓	✓
$A_2 + 3A_1$	✓	✓	$7A_1$	×	✓	$D_6 + 2A_1$	✓	✓
$2A_2 + A_1$	✓	✓	$A_3 + 4A_1$	✓	×	$2A_4$	✓	✓
$A_3 + 2A_1$	✓	✓	$3A_2 + A_1$	✓	✓	$2D_4$	✓	✓
$A_3 + A_2$	✓	✓	$A_3 + A_2 + 2A_1$	✓	✓	$D_5 + A_3$	✓	✓
$A_4 + A_1$	✓	✓	$D_4 + 3A_1$	✓	✓	$E_6 + A_2$	✓	✓
$D_4 + A_1$	✓	✓	$2A_3 + A_1$	✓	✓	$A_7 + A_1$	✓	✓
A_5	✓	✓	$A_4 + A_2 + A_1$	✓	✓	$E_7 + A_1$	✓	✓
D_5	✓	✓	$A_5 + 2A_1$	✓	✓	A_8	✓	✓
$6A_1$	✓	×	$D_5 + 2A_1$	✓	✓	D_8	✓	✓
$A_2 + 4A_1$	✓	✓	$A_4 + A_3$	✓	✓	E_8	✓	✓
$2A_2 + 2A_1$	✓	✓	$D_4 + A_3$	✓	✓			

Table 1. $\Gamma' \subseteq E_8$ occurring on weak del Pezzo surfaces

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$	$\Delta =$	$j =$	Miranda's & Persson's type
$\mathbf{E_8}$				
E_8^0	$y^2 = x^3 + t^5 s$	$-2t^{10}s^2$	0	X_{22}
E_8^1	$y^2 = x^3 + t^4 x + t^5 s$	$t^{10}(t^2 - 2s^2)$	$\frac{3t^{12}}{\Delta}$	X_{211}

Table 2. E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 5$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{E_6}$				
E_6^0	$y^2 = x^3 + t^4x + t^4s^2$	$-t^{12}$	0	6C	ell
E_6^0	$y^2 = x^3 + t^3sx + a_{6,5}t^5s + t^4s^2$	$-t^9s^3$	0	6A	ell
$+ \quad A_1$	if $a_{6,5} = 0$			6A 5.	ell
$E_6^0 \quad + \quad A_2$	$y^2 = x^3 + t^4s^2$	0		3.3(2)	q-ell
E_6^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s + t^4s^2$	$-t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	6B	ell
$+ \quad A_1$	if $a_{6,5} = 0$			6B 2.	ell
E_6^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3$	$-t^9s(a_{6,5}t^2 + a_{6,4}ts + s^2)$	$\frac{t^{12}}{\Delta}$	6A	ell
$+ \quad A_1$	if $a_{6,4} \neq 0$ and ($a_{6,5} = 0$ or $a_{6,5} = a_{6,4}^2$)			6A 2.	ell
$+ \quad A_2$	if $a_{6,5} = 0$ and $a_{6,4} = 0$			6A 1.	ell
	$\mathbf{E_7}$				
E_7^0	$y^2 = x^3 + t^3sx + t^5s$	$-t^9s^3$	0	7	ell
$E_7^0 \quad + \quad A_1$	$y^2 = x^3 + t^3sx$	$-t^9s^3$	0	7 5.	ell
E_7^1	$y^2 = x^3 + t^2x^2 + a_{6,5}t^5s - t^4s^2 + t^3s^3$	$-t^9s(a_{6,5}t^2 - ts + s^2)$	$\frac{t^{12}}{\Delta}$	7	ell
$+ \quad A_1$	if $a_{6,5} \in \{0, 1\}$			7 2.	ell
	$\mathbf{E_8}$				
E_8^0	$y^2 = x^3 + t^5s$	0		3.3(1)	q-ell
E_8^1	$y^2 = x^3 + t^4x + t^5s$	$-t^{12}$	0	8B	ell
E_8^2	$y^2 = x^3 + t^2x^2 + t^5s$	$-t^{11}s$	$-\frac{t}{s}$	8A 1.	ell

Table 3. E_6-, E_7- and E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 3$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{D_4}$				
D_4^0	$y^2+t^3y = x^3+(a_{4,2}s^2+a_{4,3}ts+a_{4,4}t^2)t^2x+s^3t^3$	t^{12}	0	12B	ell
D_4^0	$y^2+t^2sy = x^3+a_{2,1}tsx^2+a_{4,3}t^3sx+a_{6,6}t^6+t^3s^3$	t^8s^4	0	12A	ell
$+\ A_1$	if $a_{6,6}=0$ and $a_{4,3}\neq 0$			12A 10A	ell
$+\ A_2$	if $a_{6,6}=a_{4,3}=0$			12A 11	ell
$D_4^0\ +\ 3A_1$	occurs only in degree 2 (see Proposition 3.2 (C.))				
$D_4^0\ +\ 4A_1$	$y^2 = x^3+(t^3s+a_{4,2}t^2s^2+ts^3)x$ with $a_{4,2}\neq 0$	0		5.2.(f)	q-ell
$D_4^0\ +\ D_4^0$	$y^2 = x^3+a_{4,2}t^2s^2x+t^3s^3$	0		5.2.(d)	q-ell
D_4^1	$y^2+txy = x^3+a_{2,1}tsx^2+a_{6,5}t^5s+a_{6,4}t^4s^2+t^3s^3$	$t^9s(a_{6,5}t^2+a_{6,4}ts+s^2)$	$\frac{t^{12}}{\Delta}$	4B.	ell
$+\ A_1$	if ($a_{6,5}=0$ and $a_{6,4}\neq 0$) or ($a_{6,5}\neq 0$ and $a_{6,4}=0$)			4B. 2.	ell
$+\ A_2$	if $a_{6,5}=a_{6,4}=0$			4B. 3.	ell
D_4^1	$y^2+txy = x^3+a_{2,1}tsx^2+a_{6,5}t^5s+a_{6,4}t^4s^2+a_{6,3}t^3s^3+t^2s^4$ with $a_{2,1}+a_{6,3}\neq 0$	$t^8s(a_{6,5}t^3+a_{6,4}t^2s+a_{6,3}ts^2+s^3)$	$\frac{t^{12}}{\Delta}$	4A.	ell
$+\ A_1$	if ($a_{6,5}=0$ and $a_{6,3}\neq 0$) or $a_{6,5}=a_{6,4}a_{6,3}\neq 0$			4A. 2.	ell
$+\ 2A_1$	if $a_{6,5}=a_{6,3}=0$ and $a_{6,4}\neq 0$			4A. 4.	ell
$+\ A_2$	if ($a_{6,5}=a_{6,4}=0$ and $a_{6,3}\neq 0$) or ($a_{6,3}^2=a_{6,4}$ and $a_{6,3}^3=a_{6,5}\neq 0$)			4A. 3.	ell
$+\ A_3$	if $a_{6,5}=a_{6,4}=a_{6,3}=0$			4A. 5.	ell
	$\mathbf{D_5}$				
D_5^0	$y^2+t^2sy = x^3+(a_{2,2}t^2+ts)x^2+a_{6,5}t^5s$	t^8s^4	0	13A	ell
$+\ A_1$	if $a_{6,5}=0$ and $a_{2,2}\neq 0$			13A 10A	ell
$+\ A_2$	if $a_{6,5}=a_{2,2}=0$			13A 11	ell
D_5^1	$y^2+txy = x^3+a_{2,1}tsx^2+a_{6,5}t^5s+a_{6,4}t^4s^2+a_{6,3}t^3s^3+t^2s^4$ with $a_{2,1}=a_{6,3}$	$t^8s(a_{6,5}t^3+a_{6,4}t^2s+a_{6,3}ts^2+s^3)$	$\frac{t^{12}}{\Delta}$	5A.	ell
$+\ A_1$	if ($a_{6,5}=0$ and $a_{6,3}\neq 0$) or $a_{6,5}=a_{6,4}a_{6,3}\neq 0$			5A. 2.	ell
$+\ 2A_1$	if $a_{6,5}=a_{6,3}=0$ and $a_{6,4}\neq 0$			5A. 4.	ell
$+\ A_2$	if ($a_{6,5}=a_{6,4}=0$ and $a_{6,3}\neq 0$) or ($a_{6,3}^2=a_{6,4}$ and $a_{6,3}^3=a_{6,5}\neq 0$)			5A. 3.	ell
$+\ A_3$	if $a_{6,5}=a_{6,4}=a_{6,3}=0$			5A. 5.	ell

Table 4. D_4- and D_5-singularities on del Pezzo surfaces in $\mathrm{char}(k)=2$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{D_6}$				
$D_6^0 \quad + \quad A_1$	occurs only in degree 2 (see Proposition 3.2 (B.))				
$D_6^0 \quad + \quad 2A_1$	$y^2 = x^3 + (t^3s + t^2s^2)x$	0		5.2.(e)	q-ell
D_6^1	$y^2 + t^3y = x^3 + (a_{2,2}t^2 + a_{2,1}ts)x^2 + t^3sx$	t^{12}	0	13B	ell
D_6^2	$y^2 + txy = x^3 + a_{2,1}tsx^2 + a_{6,5}t^5s + t^4s^2$ with $a_{2,1} \neq 0$	$t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	5B.	ell
$+ \quad A_1$	if $a_{6,5} = 0$			5B. 2.	ell
	$\mathbf{D_7}$				
D_7^1	$y^2 + t^3y = x^3 + tsx^2$	t^{12}	0	13C	ell
D_7^2	$y^2 + txy = x^3 + a_{2,1}tsx^2 + t^5s$ with $a_{2,1} \neq 0$	$t^{11}s$	$\frac{t}{s}$	5C.	ell
	$\mathbf{D_8}$				
D_8^0	$y^2 = x^3 + t^2s^2x + t^5s$	0		5.2.(b)	q-ell
D_8^3	$y^2 + txy = x^3 + tsx^2 + a_{6,6}t^6$ with $a_{6,6} \neq 0$	$a_{6,6}t^{12}$	$\frac{1}{a_{6,6}}$	5D.	ell

Table 5. D_6-, D_7- and D_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 2$

RDP configuration	Weierstraß equation of X in $\mathbb{P}(1,1,2,3)$ / condition for extra RDPs	$\Delta =$	$j =$	Lang's / Ito's type	ell / q-ell
	$\mathbf{E_6}$				
E_6^0	$y^2 + t^2sy = x^3 + a_{2,2}t^2x^2 + a_{6,5}t^5s$	t^8s^4	0	14	ell
$+ \quad A_1$	if $a_{6,5} = 0$ and $a_{2,2} \neq 0$			14 10A	ell
$+ \quad A_2$	if $a_{6,5} = a_{2,2} = 0$			14 11	ell
E_6^1	$y^2 + txy = x^3 + tsx^2 + a_{6,5}t^5s + a_{6,4}t^4s^2 + t^3s^3 + t^2s^4$	$t^8s(a_{6,5}t^3 + a_{6,4}t^2s + ts^2 + s^3)$	$\frac{t^{12}}{\Delta}$	6.	ell
$+ \quad A_1$	if $(a_{6,5} = 0$ and $a_{6,4} \neq 0)$ or $a_{6,5} = a_{6,4} \notin \{0,1\}$			6. 2.	ell
$+ \quad A_2$	$a_{6,5} = a_{6,4} \in \{0,1\}$			6. 3.	ell
	$\mathbf{E_7}$				
E_7^0	occurs only in degree 2 (see Proposition 3.2 (A.))				
$E_7^0 \; + \quad A_1$	$y^2 = x^3 + t^3sx$	0		5.2.(c)	q-ell
E_7^2	$y^2 + t^3y = x^3 + t^3sx$	t^{12}	0	15	ell
E_7^3	$y^2 + txy = x^3 + a_{6,5}t^5s + t^4s^2$	$t^{10}s(a_{6,5}t + s)$	$\frac{t^{12}}{\Delta}$	7.	ell
$+ \quad A_1$	if $a_{6,5} = 0$			7. 2.	ell
	$\mathbf{E_8}$				
E_8^0	$y^2 = x^3 + t^5s$	0		5.2.(a)	q-ell
E_8^3	$y^2 + t^3y = x^3 + t^5s$	t^{12}	0	16	ell
E_8^4	$y^2 + txy = x^3 + t^5s$	$t^{11}s$	$\frac{t}{s}$	8.	ell

Table 6. E_6-, E_7- and E_8-singularities on del Pezzo surfaces in $\mathrm{char}(k) = 2$

II. Weak del Pezzo surfaces with global vector fields

Case	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}}$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\emptyset$	0	$\mathrm{PGL}_2 \times \mathrm{PGL}_2$	6	✓	$\{\text{pt}\}$	any
$\mathbb{F}_2$	A_1	0	$(\mathrm{Aut}_{\mathbb{P}(1,1,2)})_{\mathrm{red}}$ $= (\mathbb{G}_a^3 \rtimes \mathrm{GL}_2)/\mu_2$	7	✓	$\{\text{pt}\}$	any

Table 1. Weak del Pezzo surfaces of degree 8 that are not blow-ups of $\mathbb{P}^2$

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
degree 9								
$9A$		$\emptyset$	0	PGL_3	8	✓	$\{\mathrm{pt}\}$	any
degree 8								
$8A$	Fig. 5	$\emptyset$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & h & i \end{pmatrix}$	6	✓	$\{\mathrm{pt}\}$	any
degree 7								
$7A$	Fig. 4	$\emptyset$	3	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	$\{\mathrm{pt}\}$	any
$7B$	Fig. 26	A_1	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	$\{\mathrm{pt}\}$	any
degree 6								
$6A$	Fig. 3	$\emptyset$	6	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$6B$	Fig. 24	A_1	4	$\begin{pmatrix} 1 & & c \\ & e & \\ & & i \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	any
$6C$	Fig. 2	A_1	3	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & i \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	any
$6D$	Fig. 25	$2A_1$	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & i \end{pmatrix}$	4	✓	$\{\mathrm{pt}\}$	any
$6E$	Fig. 51	A_2	2	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^2 \end{pmatrix}$	4	✓	$\{\mathrm{pt}\}$	any
$6F$	Fig. 52	$A_2 + A_1$	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & i \end{pmatrix}$	5	✓	$\{\mathrm{pt}\}$	any
degree 5								
$5A$	Fig. 1	A_1	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$5B$	Fig. 22	$2A_1$	5	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$5C$	Fig. 18	A_2	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$5D$	Fig. 23	$A_2 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	any
$5E$	Fig. 50	A_3	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	any
$5F$	Fig. 60	A_4	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix}$	4	✓	$\{\mathrm{pt}\}$	any

Table 2. Weak del Pezzo surfaces with global vector fields of degree ≥ 5 that are blow-ups of $\mathbb{P}^2$

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$4A$	Fig. 13	$2A_1$	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$4B$	Fig. 14	$3A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$4C$	Fig. 15	$A_2 + A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$4D$	Fig. 17	A_3	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$4E$	Fig. 42	A_3	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$\neq 2$
$4F$	Fig. 21	$4A_1$	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$4G$	Fig. 20	$A_2 + 2A_1$	4	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$4H$	Fig. 43	$A_3 + A_1$	3	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$4I$	Fig. 49	A_4	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	any
$4J$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & e & \\ & & e^2 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$\neq 2$
$4K$	Fig. 48	$A_3 + 2A_1$	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & i \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	any
$4L$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	$\neq 2$
$4M$	Fig. 42	A_3	4	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	2	$\times$	$\{\mathrm{pt}\}$	$= 2$
$4N$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$= 2$
$4O$	Fig. 59	D_4	2	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^2 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	$= 2$
$4P$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & b & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	3	✓	$\{\mathrm{pt}\}$	$= 2$
$4Q$	Fig. 65	D_5	1	$\begin{pmatrix} 1 & b & c \\ & e & f \\ & & e^3 \end{pmatrix}$	4	✓	$\{\mathrm{pt}\}$	$= 2$

Table 3. Weak del Pezzo surfaces of degree 4 with global vector fields

Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$3A$	Fig. 10	$2A_2$	7	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$3B$	Fig. 16	D_4	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$3C$	Fig. 11	$2A_2 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$3D$	Fig. 12	$A_3 + 2A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$3E$	Fig. 41	$A_4 + A_1$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	{pt}	any
$3F$	Fig. 46	A_5	3	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$\neq 3$
$3G$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	{pt}	$\neq 2$
$3H$	Fig. 19	$3A_2$	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & i \end{pmatrix}$	2	✓	{pt}	any
$3I$	Fig. 47	$A_5 + A_1$	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	{pt}	any
$3J$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & e & \\ & & e^3 \end{pmatrix}$	2	✓	{pt}	$\neq 2, 3$
$3K$	Fig. 46	A_5	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, e^3 = 1$	2	×	{pt}	$= 3$
$3L$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & 1 & f \\ & & 1 \end{pmatrix}$	2	✓	{pt}	$= 3$
$3M$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & & c \\ & e & f \\ & & e^3 \end{pmatrix}$	3	✓	{pt}	$= 3$
$3N$	Fig. 33	A_4	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	{pt}	$= 2$
$3O$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	{pt}	$= 2$
$3P$	Fig. 58	D_5	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}$	2	✓	{pt}	$= 2$
$3Q$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & b & c \\ & 1 & b^2+b \\ & & 1 \end{pmatrix}$	2	✓	{pt}	$= 2$
$3R$	Fig. 66	E_6	1	$\begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix}$	3	✓	{pt}	$= 2$

Table 4. Weak del Pezzo surfaces of degree 3 with global vector fields

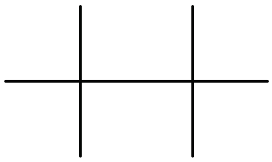
Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$2A$	Fig. 7	$2A_3$	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	1 dim	any
$2B$	Fig. 39	$D_5 + A_1$	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$2C$	Fig. 64	E_6	4	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2$
$2D$	Fig. 8	$2A_3 + A_1$	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$2E$	Fig. 9	$D_4 + 3A_1$	4	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$2F$	Fig. 40	$A_5 + A_2$	3	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}$	1	✓	$\{\text{pt}\}$	any
$2G$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2$
$2H$	Fig. 56	A_7	2	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2$
$2I$	Fig. 67	E_7	1	$\begin{pmatrix}1&&\\&e&\\&&e^3\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$\neq 2, 3$
$2J$	Fig. 45	A_6	4	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3 = 1$	1	×	$\{\text{pt}\}$	$= 3$
$2K$	Fig. 54	D_6	3	$\begin{pmatrix}1&&\\&e&\\&&e^2\end{pmatrix}, e^3 = 1$	1	×	$\{\text{pt}\}$	$= 3$
$2L$	Fig. 67	E_7	1	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$= 3$
$2M$	Fig. 67	E_7	1	$\begin{pmatrix}1&&\\&e&f\\&&e^3\end{pmatrix}$	2	✓	$\{\text{pt}\}$	$= 3$
$2N$	Fig. 30	A_5	7	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2 = 1$	1	×	1 dim	$= 2$
$2O$	Fig. 38	D_5	8	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2 = 1$	1	×	$\{\text{pt}\}$	$= 2$
$2P$	Fig. 32	$A_5 + A_1$	6	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2 = 1$	1	×	$\{\text{pt}\}$	$= 2$
$2Q$	Fig. 31	$A_5 + A_1$	5	$\begin{pmatrix}1&&\\&1&\\&&i\end{pmatrix}, i^2 = 1$	1	×	$\{\text{pt}\}$	$= 2$
$2R$	Fig. 54	D_6	3	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	1 dim	$= 2$
$2S$	Fig. 64	E_6	4	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, f^2 = 0$	2	×	$\{\text{pt}\}$	$= 2$
$2T$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix}1&&\\&1&f\\&&1\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$= 2$
$2U$	Fig. 57	$D_6 + A_1$	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}$	2	✓	$\{\text{pt}\}$	$= 2$
$2V$	Fig. 56	A_7	2	$\begin{pmatrix}1&&\\&e&f\\&&e^2\end{pmatrix}, e^4 = 1$	2	×	$\{\text{pt}\}$	$= 2$
$2W$	Fig. 67	E_7	1	$\begin{pmatrix}1&&c\\&1&\\&&1\end{pmatrix}$	1	✓	$\{\text{pt}\}$	$= 2$
$2X$	Fig. 67	E_7	1	$\begin{pmatrix}1&b&c\\&1&b^2\\&&1\end{pmatrix}$	2	✓	$\{\text{pt}\}$	$= 2$
$2Y$	Fig. 67	E_7	1	$\begin{pmatrix}1&b&c\\&e&b^2e\\&&e^3\end{pmatrix}$	3	✓	$\{\text{pt}\}$	$= 2$

Table 5. Weak del Pezzo surfaces of degree 2 with global vector fields

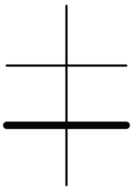
Case	Figure	(-2)-curves	$\#\{\text{lines}\}$	$\mathrm{Aut}^0_{\widetilde{X}} \subseteq \mathrm{PGL}_3$	$h^0(\widetilde{X}, T_{\widetilde{X}})$	$\mathrm{Aut}^0_{\widetilde{X}}$ smooth?	Moduli	$\mathrm{char}(k)$
$1A$	Fig. 6	$2D_4$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	1 dim	any
$1B$	Fig. 37	$E_6 + A_2$	4	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	any
$1C$	Fig. 63	$E_7 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$\neq 2$
$1D$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^3 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$\neq 2, 3$
$1E$	Fig. 53	D_7	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^3 = 1$	1	×	$\{\mathrm{pt}\}$	$= 3$
$1F$	Fig. 62	E_7	5	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	×	$\{\mathrm{pt}\}$	$= 3$
$1G$	Fig. 44	A_8	3	$\begin{pmatrix} 1 & & \\ & e & \\ & & e^2 \end{pmatrix}, e^3 = 1$	1	×	$\{\mathrm{pt}\}$	$= 3$
$1H$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$= 3$
$1I$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^3 \end{pmatrix}$	2	✓	$\{\mathrm{pt}\}$	$= 3$
$1J$	Fig. 35	E_6	13	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	1 dim	$= 2$
$1K$	Fig. 34	$E_6 + A_1$	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	$\{\mathrm{pt}\}$	$= 2$
$1L$	Fig. 27	A_7	8	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	1 dim	$= 2$
$1M$	Fig. 61	E_7	5	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}, f^2 = 0$	1	×	$\{\mathrm{pt}\}$	$= 2$
$1N$	Fig. 29	$D_6 + 2A_1$	6	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	$\{\mathrm{pt}\}$	$= 2$
$1O$	Fig. 28	$A_7 + A_1$	5	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & i \end{pmatrix}, i^2 = 1$	1	×	$\{\mathrm{pt}\}$	$= 2$
$1P$	Fig. 63	$E_7 + A_1$	3	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, f^2 = 0$	2	×	$\{\mathrm{pt}\}$	$= 2$
$1Q$	Fig. 55	D_8	2	$\begin{pmatrix} 1 & & \\ & 1 & f \\ & & 1 \end{pmatrix}$	1	✓	1 dim	$= 2$
$1R$	Fig. 55	D_8	2	$\begin{pmatrix} 1 & & \\ & e & f \\ & & e^2 \end{pmatrix}, e^4 = 1$	2	×	$\{\mathrm{pt}\}$	$= 2$
$1S$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix}$	1	✓	$\{\mathrm{pt}\}$	$= 2$
$1T$	Fig. 68	E_8	1	$\begin{pmatrix} 1 & b & c \\ & e & b^2e \\ & & e^3 \end{pmatrix}, b^4 = 0$	3	×	$\{\mathrm{pt}\}$	$= 2$

Table 6. Weak del Pezzo surfaces of degree 1 with global vector fields

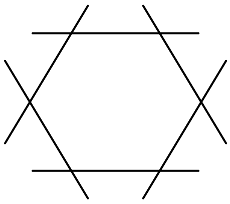
$8A$



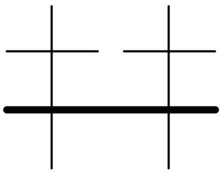
$7A$



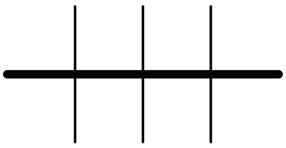
$7B$



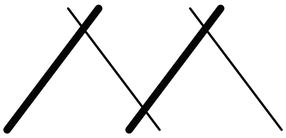
$6A$



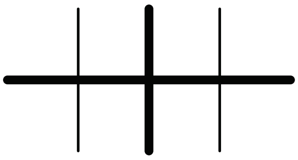
$6B$



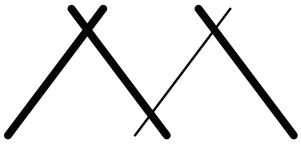
$6C$



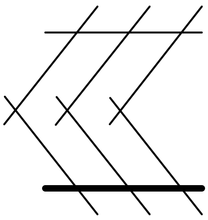
$6D$



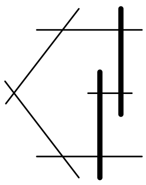
$6E$



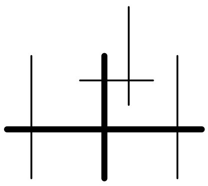
$6F$



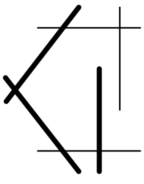
$5A$



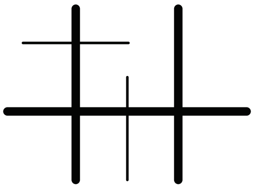
$5B$



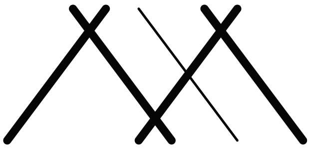
$5C$



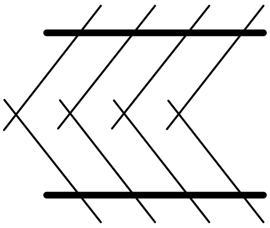
$5D$



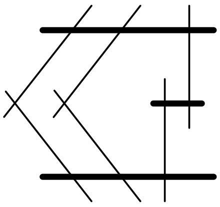
$5E$



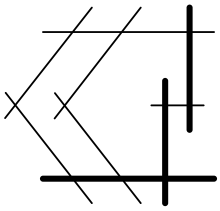
$5F$



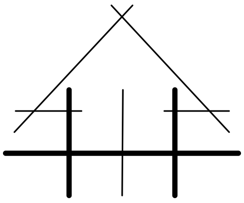
$4A$



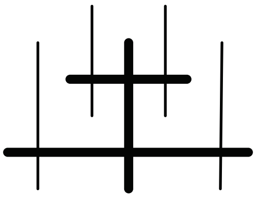
$4B$



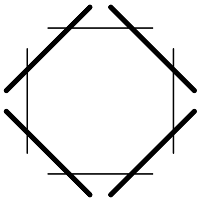
$4C$



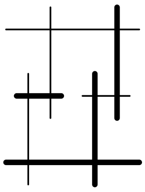
$4D$



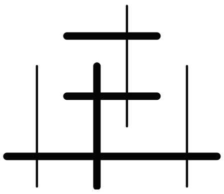
$4E, 4M$



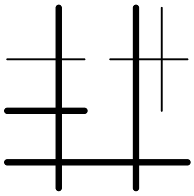
$4F$



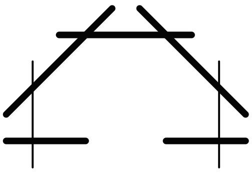
$4G$



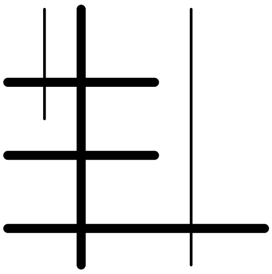
$4H$



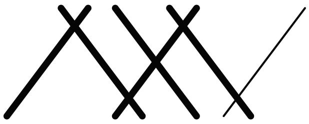
$4I$



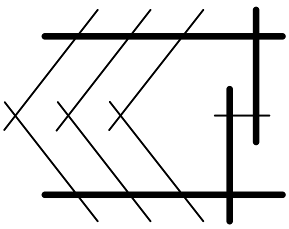
$4K$



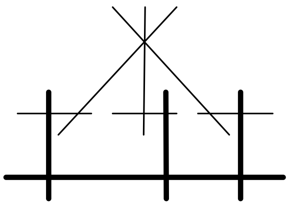
$4N, 4J, 4O$



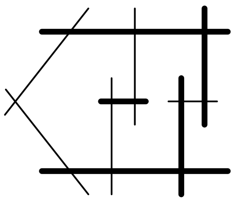
$4P, 4Q, 4L$



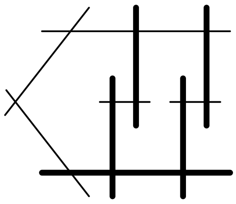
$3A$



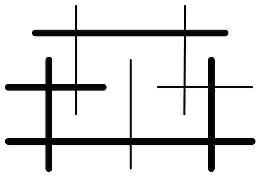
$3B$



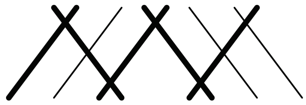
$3C$



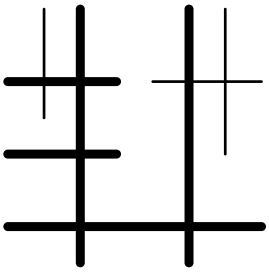
$3D$



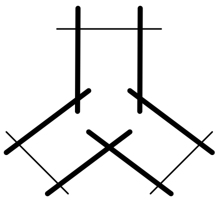
$3E$



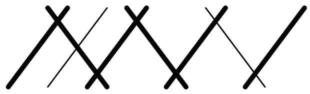
$3F, 3K$



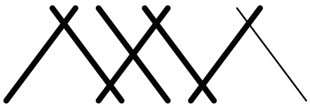
$3G, 3O, 3P$



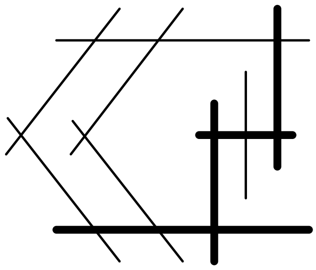
$3H$



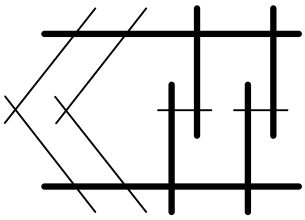
$3I$



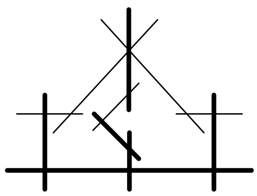
$3J, 3L, 3M,$
$3R, 3Q$



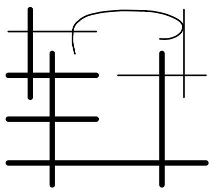
$3N$



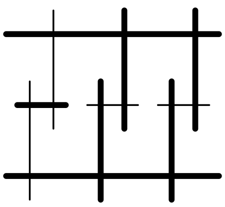
$2A$



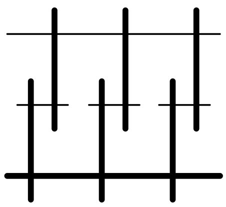
$2B$



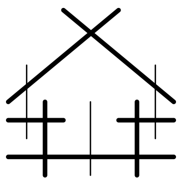
$2C, 2S$



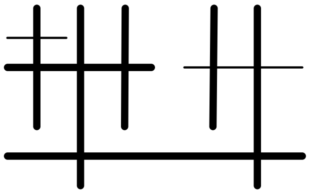
$2D$



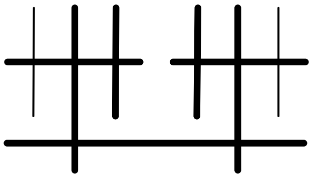
$2E$



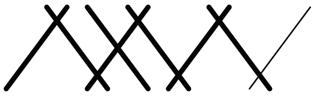
$2F$



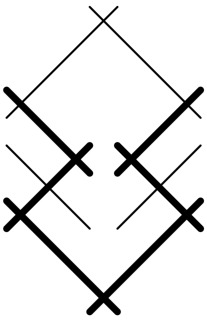
$2G, 2T, 2U$



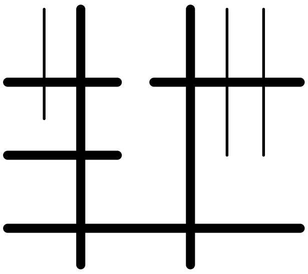
$2H, 2V$



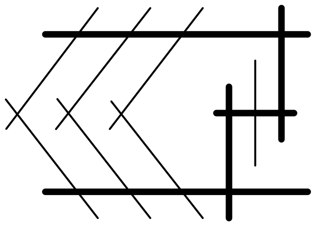
$2I, 2L, 2M,$
$2W, 2X, 2Y$



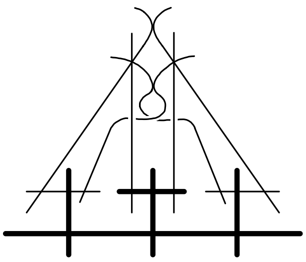
$2J$



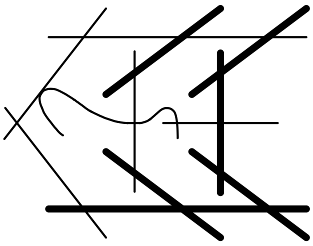
$2K, 2R$



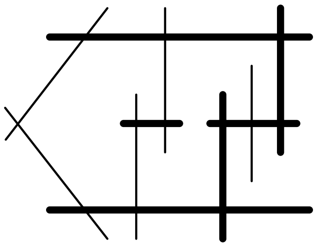
$2N$



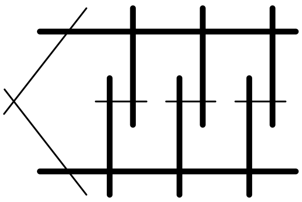
$2O$



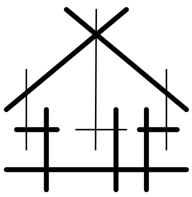
$2P$



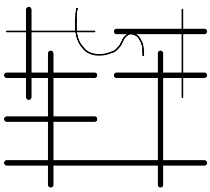
$2Q$



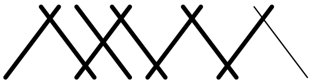
$1A$



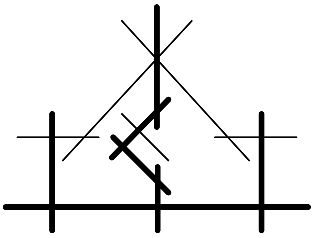
$1B$



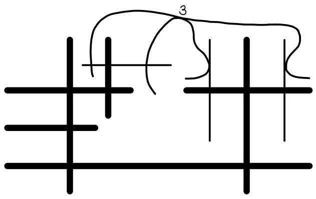
$1C, 1P$



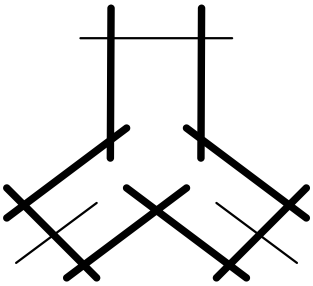
$1D, 1H, 1I,$
$1S, 1T$



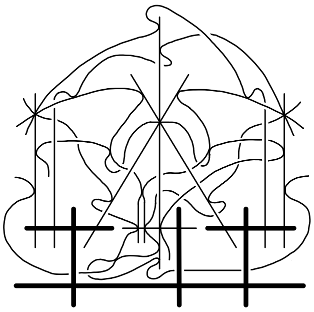
$1E$



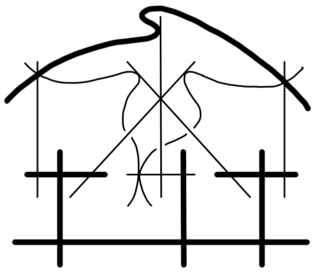
$1F$



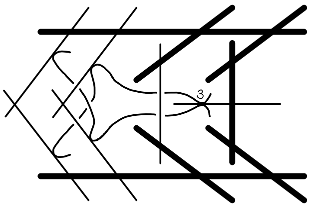
$1G$



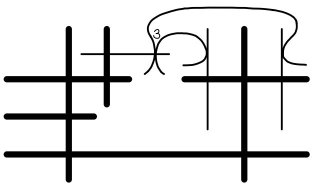
$1J$



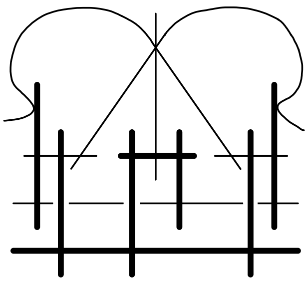
$1K$



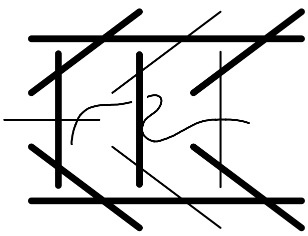
$1L$



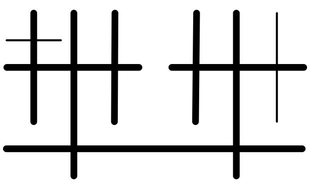
$1M$



$1N$



$1O$



$1Q, 1R$

III. RDP del Pezzo surfaces with global vector fields in odd characteristic

d	Γ	$\subseteq \langle k_{9-d}\rangle^{\perp}$
2	A_6	$\subseteq E_7$
1	$A_6, \quad A_6 + A_1$	$\subseteq E_8$

Table 7. Non-equivariant RDP configurations in characteristic 7

d	singularities	equation of X	Aut_X^0
2	A_6	$w^2 = x^3y + y^3z + z^3x$	$\mu_7 : [\lambda x : \lambda^4 y : \lambda^2 z : w]$
1	$A_6 + A_1$	$y^2 = x^3 + ts^3x + t^5 s$	$\mu_7 : [\lambda s : \lambda^4 t : x : y]$

Table 1. Non-equivariant RDP del Pezzo surfaces with global vector fields in characteristic 7

d	Γ	$\subseteq \langle k_{9-d}\rangle^{\perp}$
5	A_4	$\subseteq A_4$
4	A_4	$\subseteq D_5$
3	$A_4, \quad A_4 + A_1$	$\subseteq E_6$
2	$A_4, \quad A_4 + A_1, \quad A_4 + A_2$	$\subseteq E_7$
1	$A_4, \quad A_4 + A_1, \quad A_4 + 2A_1, \quad A_4 + A_2,$ $A_4 + A_2 + A_1, \quad A_4 + A_3, \quad 2A_4, \quad E_8^0$	$\subseteq E_8$

Table 8. Non-equivariant RDP configurations in characteristic 5

d	RDPs	equation(s) of X	Aut_X^0
5	A_4	$\begin{aligned} x_0x_2 - x_1^2 &= 0 \\ x_0x_3 - x_1x_4 &= 0 \\ x_2x_4 - x_1x_3 &= 0 \\ x_1x_2 + x_4^2 + x_0x_5 &= 0 \\ x_2^2 + x_3x_4 + x_1x_5 &= 0 \end{aligned}$	$\langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$ with $\alpha_5 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2\varepsilon^2 & 2\varepsilon^3 & \varepsilon & 2\varepsilon^4 \\ 0 & 0 & 1 & 2\varepsilon & 0 & -\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & \varepsilon & \varepsilon^2 & 1 & -2\varepsilon^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_4	$\begin{aligned} x_0x_1 - x_2x_3 &= 0 \\ x_0x_4 + x_1x_2 + x_3^2 &= 0 \end{aligned}$	$\langle \alpha_5, \mathrm{Aut}_{\widetilde{X}}^0 \rangle$ with $\alpha_5 : \begin{pmatrix} 1 & -\varepsilon^3 & -2\varepsilon & 2\varepsilon^2 & 2\varepsilon^4 \\ 0 & 1 & 0 & 0 & 2\varepsilon \\ 0 & -\varepsilon^2 & 1 & -2\varepsilon & \varepsilon^3 \\ 0 & \varepsilon & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
3	A_4	$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_0 = 0$	$\mu_5 : [x_0 : \lambda x_1 : \lambda^4 x_2 : \lambda^3 x_3]$
	$A_4 + A_1$	$x_0x_1x_3 + x_0x_2^2 + x_1^2x_2 = 0$	$\alpha_5 \rtimes \mathbb{G}_m$ with $\alpha_5 : \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 & -2\varepsilon^3 \\ 0 & 1 & 2\varepsilon & -\varepsilon^2 \\ 0 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m : [x_0 : \lambda x_1 : \lambda^2 x_2 : \lambda^3 x_3]$
2	$A_4 + A_1$	$w^2 = x^4 + xy^2z + yz^3$	$\mu_5 : [x : \lambda y : \lambda^3 z : w]$
	$A_4 + A_2$	$w^2 = xy^3 + yz^3 + x^2z^2$	$\mu_5 : [\lambda^2 x : \lambda y : \lambda^3 z : w]$
1	$A_4 + A_2 + A_1$	$y^2 = x^3 + s^3tx + s^2t^4$	$\mu_5 : [s : \lambda t : \lambda^3 x : \lambda^2 y]$
	$2A_4$	$y^2 = x^3 + t^4x + s^5t$	$\alpha_5 \rtimes \mu_5 : [\lambda s + \varepsilon t : t : x : y]$
	E_8^0	$y^2 = x^3 + s^5t$	$\alpha_5 \rtimes \mathbb{G}_m : [\lambda s + \varepsilon t : \lambda^{-5} t : x : y]$

Table 2. Non-equivariant RDP del Pezzo surfaces with global vector fields in characteristic 5

d	Γ	$\subseteq \langle k_{9-d} \rangle^{\perp}$
6	$A_2, \quad A_2 + A_1$	$\subseteq A_2 + A_1$
5	$A_2, \quad A_2 + A_1$	$\subseteq A_4$
4	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1$	$\subseteq D_5$
3	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad 2A_2 + A_1, \quad A_5,$ $3A_2, \quad A_5 + A_1, \quad E_6^0, \quad E_6^1$	$\subseteq E_6$
2	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad A_2 + 3A_1, \quad 2A_2 + A_1,$ $A_3 + A_2, \quad (A_5)', \quad 3A_2, \quad A_3 + A_2 + A_1, \quad A_4 + A_2,$ $(A_5 + A_1)', \quad E_6^0, \quad E_6^1, \quad A_5 + A_2, \quad E_7^0$	$\subseteq E_7$
1	$A_2, \quad A_2 + A_1, \quad A_2 + 2A_1, \quad 2A_2, \quad A_2 + 3A_1, \quad 2A_2 + A_1,$ $A_3 + A_2, \quad A_5, \quad A_2 + 4A_1, \quad 2A_2 + 2A_1, \quad 3A_2, \quad A_3 + A_2 + A_1,$ $A_4 + A_2, \quad D_4 + A_2, \quad (A_5 + A_1)', \quad E_6^0, \quad E_6^1,$ $3A_2 + A_1, \quad A_3 + A_2 + 2A_1, \quad A_4 + A_2 + A_1, \quad A_5 + 2A_1,$ $A_5 + A_2, \quad D_5 + A_2, \quad E_6^0 + A_1, \quad E_6^1 + A_1, \quad E_7^0,$ $4A_2, \quad A_5 + A_2 + A_1, \quad E_6^0 + A_2, \quad E_6^1 + A_2, \quad A_8, \quad E_8^0, \quad E_8^1$	$\subseteq E_8$

Table 9. Non-equivariant RDP configurations in characteristic 3

d	RDPs	equation(s) of X	Aut^0_X
6	A_2	$\begin{aligned} x_0x_5 - x_3x_4 &= 0 \\ x_0x_6 - x_1x_4 &= 0 \\ x_0x_6 - x_2x_3 &= 0 \\ x_3x_6 - x_1x_5 &= 0 \\ x_4x_6 - x_2x_5 &= 0 \\ x_1x_6 + x_3^2 + x_3x_4 &= 0 \\ x_2x_6 + x_3x_4 + x_4^2 &= 0 \\ x_6^2 + x_3x_5 + x_4x_5 &= 0 \\ x_1x_2 + x_0x_3 + x_0x_4 &= 0 \end{aligned}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & -\varepsilon & 0 & 1 & 0 & 0 & 0 \\ -\varepsilon^2 & 0 & \varepsilon & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\varepsilon^2 & -\varepsilon^2 & -\varepsilon & \varepsilon & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{aligned} x_0^2 - x_1x_5 &= 0 \\ x_0x_2 - x_1x_4 &= 0 \\ x_0x_3 - x_2x_4 &= 0 \\ x_0x_4 - x_2x_5 &= 0 \\ x_0x_5 - x_2x_6 &= 0 \\ x_1x_3 - x_2^2 &= 0 \\ x_3x_5 - x_4^2 &= 0 \\ x_3x_6 - x_4x_5 &= 0 \\ x_4x_6 - x_5^2 &= 0 \end{aligned}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
5	A_2	$\begin{aligned} x_0x_2 - x_1x_5 &= 0 \\ x_0x_2 - x_3x_4 &= 0 \\ x_0x_3 + x_1^2 + x_1x_4 &= 0 \\ x_0x_5 + x_1x_4 + x_4^2 &= 0 \\ x_3x_5 + x_1x_2 + x_2x_4 &= 0 \end{aligned}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & \varepsilon & 0 & -\varepsilon^2 & -\varepsilon & -\varepsilon^2 \\ 0 & 1 & -\varepsilon^2 & \varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 1 & 0 & 0 \\ 0 & 0 & -\varepsilon^2 & 0 & 1 & -\varepsilon \\ 0 & 0 & -\varepsilon & 0 & 0 & 1 \end{pmatrix}$
	$A_2 + A_1$	$\begin{aligned} x_0^2 - x_1x_4 &= 0 \\ x_0x_2 - x_1x_3 &= 0 \\ x_0x_3 - x_2x_4 &= 0 \\ x_0x_4 - x_2x_5 &= 0 \\ x_3x_5 - x_4^2 &= 0 \end{aligned}$	$\langle \alpha_3, \mathrm{Aut}^0_{\widetilde{X}} \rangle$ with $\alpha_3 : \begin{pmatrix} 1 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 1 & 0 & 0 \\ \varepsilon & \varepsilon^2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
4	A_2	$\begin{aligned} x_0x_1 + x_2x_4 + x_3x_4 &= 0 \\ x_0x_4 + x_1x_4 + x_2x_3 &= 0 \end{aligned}$	$\mu_3 : [x_0 : x_1 : \lambda x_2 : \lambda x_3 : \lambda^2 x_4]$
	$A_2 + A_1$	$\begin{aligned} x_0x_1 - x_2x_3 &= 0 \\ x_1x_2 + x_2x_4 + x_3x_4 &= 0 \end{aligned}$	$\alpha_3 \rtimes \mathbb{G}_m$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\varepsilon^2 & 1 & \varepsilon & -\varepsilon & 0 \\ -\varepsilon & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & 0 \\ -\varepsilon^2 & 0 & -\varepsilon & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m : [\lambda^2 x_0 : x_1 : \lambda x_2 : \lambda x_3 : x_4]$
	$A_2 + 2A_1$	$\begin{aligned} x_0^2 - x_3x_4 &= 0 \\ x_0x_3 - x_1x_2 &= 0 \end{aligned}$	$\alpha_3 \rtimes \mathbb{G}_m^2$ with $\alpha_3 : \begin{pmatrix} 1 & 0 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \varepsilon & 0 & 0 & 1 & \varepsilon^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ $\mathbb{G}_m^2 : [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : \lambda_1\lambda_2 x_3 : (\lambda_1\lambda_2)^{-1} x_4]$

Table 3. Non-equivariant RDP del Pezzo surfaces of degree at least 4 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
3	A_2	$x_0^2x_1 + x_0x_1^2 + x_2^2x_3 + x_2x_3^2 = 0$	$\mu_3 : [x_0 : x_1 : \lambda x_2 : \lambda x_3]$
	$A_2 + 2A_1$	$x_0^2x_1 + x_0^2x_2 + x_0x_3^2 + x_1x_2x_3 = 0$	$\mu_3 : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$2A_2$	$x_0^3 + x_1x_2x_3 + x_0x_1^2 + ax_0^2x_1 = 0$ with $a^2 \neq 1$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_2 : x_1 : x_2 : a\varepsilon x_0 - \varepsilon x_1 - a\varepsilon^2 x_2 + x_3]$ $\alpha_3 : [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 - \varepsilon x_1 + x_2 - a\varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m : [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$2A_2 + A_1$	$x_0^3 + x_1x_2x_3 + x_0^2x_1 = 0$	$\langle \alpha_3, \alpha_3, \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_2 : x_1 : x_2 : \varepsilon x_0 - \varepsilon^2 x_2 + x_3]$ $\alpha_3 : [x_0 + \varepsilon x_3 : x_1 : a\varepsilon x_0 + x_2 - \varepsilon^2 x_3 : x_3]$ $\mathbb{G}_m : [x_0 : x_1 : \lambda x_2 : \lambda^{-1} x_3]$
	$3A_2$	$x_0^3 + x_1x_2x_3 = 0$	$\alpha_3^3 \rtimes \mathbb{G}_m^2$ with $\alpha_3^3 : [x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 : x_1 : x_2 : x_3]$ $\mathbb{G}_m^2 : [x_0 : \lambda_1 x_1 : \lambda_2 x_2 : (\lambda_1\lambda_2)^{-1} x_3]$
	A_5	$x_0^3 + x_0x_2x_3 + x_1^2x_2 + x_2^3 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mu_3 \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mu_3 : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	$A_5 + A_1$	$x_0^3 + x_0x_2x_3 + x_1^2x_2 = 0$	$\langle \alpha_3, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$ with $\alpha_3 : [x_0 + \varepsilon x_1 - \varepsilon^2 x_3 : x_1 + \varepsilon x_3 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 : \varepsilon x_0 + x_1 : x_2 : -\varepsilon^2 x_0 + \varepsilon x_1 + x_3]$ $\mathbb{G}_m : [x_0 : \lambda x_1 : \lambda x_2 : \lambda^2 x_3]$
	E_6^0	$x_0^3 + x_1^2x_2 + x_2^2x_3 = 0$	$\langle G, \mathbb{G}_a^2 \rtimes \mathbb{G}_m \rangle$ with $\mathbb{G}_a : [x_0 + \varepsilon x_2 : x_1 : x_2 : -\varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a : [x_0 : x_1 + \varepsilon x_2 : x_2 : \varepsilon^3 x_1 - \varepsilon^2 x_2 + x_3]$ $\mathbb{G}_m : [x_0 : \lambda x_1 : x_2 : \lambda^{-2} x_2 : \lambda^4 x_3]$ and G non-commutative, $\lvert G \rvert = 27$, acting as $[x_0 + \varepsilon_1 x_1 + \varepsilon_2 x_3 : x_1 + \varepsilon_1^3 x_3 : -\varepsilon_1^3 x_1 + x_2 + \varepsilon_1^6 x_3 : x_3]$ where $\varepsilon_1^9 = \varepsilon_2^3 = 0$
	E_6^1	$x_0^3 + x_1^3 + x_0x_1x_2 + x_2^2x_3 = 0$	$\langle \mu_3, \mathbb{G}_a^2 \rangle$ with $\mu_3 : [\lambda x_0 : \lambda^2 x_1 : x_2 : x_3]$ $\mathbb{G}_a : [x_0 - \varepsilon x_2 : x_1 : x_2 : \varepsilon x_1 + \varepsilon^3 x_2 + x_3]$ $\mathbb{G}_a : [x_0 : x_1 - \varepsilon x_2 : x_2 : \varepsilon x_0 + \varepsilon^3 x_2 + x_3]$

Table 4. Non-equivariant RDP del Pezzo surfaces of degree 3 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
2	A_2+3A_1	$w^2=z(xy(x+y)+z^3)$	$\mu_3:[x:y:\lambda z:\lambda^{-1}w]$
	A_2+A_3	$w^2=x^4+a^3x^2yz+xy^3+y^2z^2$ with $a^2\neq 1$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$A_2+A_3+A_1$	$w^2=x^2yz+xy^3+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_2+A_4	$w^2=(xz+y^2)^2+y^3z$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	$2A_2$	$w^2=x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $(b^3-a^2b^2)^2\neq a^3b^3, b\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$2A_2$	$w^2=(xz+y^2)^2+x^3z+a^6z^4$ with $a\neq 0$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	$2A_2+A_1$	$w^2=ax^2yz+xy^3+xz^3+y^2z^2$ with $a\neq 0,1$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	$3A_2$	$w^2=y^4+x^2y^2+xz^3$	$\alpha_3^2\rtimes\mu_3:[x:y:\varepsilon_1x+\varepsilon_2y+\lambda z:w]$
	A_5	$w^2=x^4+xy^3+xz^3+ax^2yz+by^2z^2$ with $(b^3-a^2b^2)^2=a^3b^3, b\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2=x^4+ax^2yz+xy^3+xz^3$ with $a\neq 0$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5	$w^2=(xz+y^2)^2+x^3z$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	A_5	$w^2=z(z(xz+y^2)+x^3)$	$\alpha_3:[x+\varepsilon y-\varepsilon^2z:y+\varepsilon z:z:w]$
	A_5+A_1	$w^2=x^2yz+xy^3+xz^3$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_1	$w^2=x^2yz+xy^3+xz^3+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	A_5+A_2	$w^2=x^2y^2+xz^3$	$\alpha_3^2\rtimes\mathbb{G}_m:[x:\lambda^3y:\varepsilon_1x+\varepsilon_2y+\lambda^2z:\lambda^3w]$
	E_6^0	$w^2=y^4+xz^3$	$\langle G,\mathbb{G}_m\rangle$ with $\mathbb{G}_m:[x:\lambda^3y:\lambda^4z:\lambda^6w]$ and G non-commutative, $\lvert G\rvert=27$, acting as $[x:y-\varepsilon_1^3x:\varepsilon_2x+\varepsilon_1y+z:w]$ where $\varepsilon_1^9=\varepsilon_2^3=0$
	E_6^1	$w^2=(y^3+z^3)x+y^2z^2$	$\mu_3:[x:\lambda y:\lambda^{-1}z:w]$
	E_7^0	$w^2=x^3y+xz^3$	$\langle\alpha_3,\mathbb{G}_a\rtimes\mathbb{G}_m\rangle$ with $\alpha_3:[x:y:z+\varepsilon y:w]$ $\mathbb{G}_a:[x:y+\varepsilon^3x:z-\varepsilon x:w]$ $\mathbb{G}_m:[x:\lambda^6y:\lambda^2z:\lambda^3w]$

Table 5. Non-equivariant RDP del Pezzo surfaces of degree 2 with global vector fields in characteristic 3

d	RDPs	equation(s) of X	Aut_X^0
1	$A_2 + D_4$	$y^2 = x^3 + stx^2 + a^3s^6 + s^3t^3$ with $a \neq 0$	$\mu_3 : [\lambda s : \lambda^{-1}t : x : y]$
	$A_2 + D_4$	$y^2 = x^3 + s^2t^2x + t^6$	$\mu_3 : [\lambda s : \lambda^{-1}t : x : y]$
	$2A_2$	$y^2 = x^3 + stx^2 + a^3s^6 + b^3s^3t^3 + t^6$ with $a \neq 0, b^2 \neq a$	$\mu_3 : [\lambda s : \lambda^{-1}t : x : y]$
	$2A_2$	$y^2 = x^3 + s^2t^2x + a^3s^6 + t^6$ with $a \neq 0$	$\mu_3 : [\lambda s : \lambda^{-1}t : x : y]$
	$3A_2$	$y^2 = x^3 + s^2x^2 + st^3x + a^3s^3t^3 + b^3t^6$ with $a \notin \{0, (b-1)^2\}, b \neq 0$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$3A_2$	$y^2 = x^3 + s^2x^2 + a^3s^3t^3 + t^6$ with $a \neq 0$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$3A_2 + A_1$	$y^2 = x^3 + s^2x^2 + st^3x + a^3s^3t^3$ with $a \notin \{0, 1\}$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$4A_2$	$y^2 = x^3 + s^4t^2 + s^2t^4$	$\alpha_3^3 \rtimes \mu_3 : [\lambda s : \lambda t : x + \varepsilon_1 s^2 + \varepsilon_2 st + \varepsilon_3 t^2 : y]$
	A_5	$y^2 = x^3 + stx^2 + b^6s^6 + b^3s^3t^3 + t^6$ with $b \neq 0$	$\mu_3 : [\lambda s : \lambda^{-1}t : x : y]$
	$A_5 + A_2$	$y^2 = x^3 + s^2x^2 + st^3x + b^3t^6$ with $b \neq 0, 1$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$A_5 + A_2$	$y^2 = x^3 + s^2x^2 + t^6$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$A_5 + A_2 + A_1$	$y^2 = x^3 + s^2x^2 + st^3x$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	E_6^0	$y^2 = x^3 + st^3x + as^3t^3 + t^6$ with $a \neq 0$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	E_6^0	$y^2 = x^3 + s^4x + t^6$	$\mu_3 : [s : \lambda t : x : y]$
	$E_6^0 + A_1$	$y^2 = x^3 + st^3x + s^3t^3$	$\alpha_3 \rtimes \mu_9 : [\lambda^6 s : \varepsilon s + \lambda t : x + (1-\lambda^3)s^2 : y]$
	$E_6^0 + A_2$	$y^2 = x^3 + s^4t^2$	$\langle G, \mathbb{G}_m \rangle$ with $\mathbb{G}_m : [\lambda s : \lambda^{-2}t : x : y]$ and G non-commutative, $\lvert G \rvert = 81$, acting as $\mathbb{G}_m : [s : t - \varepsilon_2^3 s : x + \varepsilon_1 s^2 + \varepsilon_2 st + \varepsilon_3 t^2 : y]$ with $\varepsilon_1^3 = \varepsilon_2^9 = \varepsilon_3^3 = 0$
	$E_6^1 + A_2$	$y^2 = x^3 + s^2x^2 + s^3t^3$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	E_7^0	$y^2 = x^3 + st^3x + t^6$	$\alpha_3 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x : y]$
	$E_7^0 + A_1$	$y^2 = x^3 + st^3x$	$\alpha_3 \rtimes \mathbb{G}_m : [\lambda^{-3}s : \varepsilon s + \lambda t : x : y]$
	A_8	$y^2 = x^3 + s^2x^2 + st^3x + t^6$	$\alpha_9 \rtimes \mu_3 : [s : \varepsilon s + \lambda t : x + \varepsilon^3 s^2 : y]$
	E_8^0	$y^2 = x^3 + s^5t$	$\langle \alpha_3^2, \mathbb{G}_a \rtimes \mathbb{G}_m \rangle$ with $\mathbb{G}_a : [s : t - a^3 s : x + as^2 : y]$ $\mathbb{G}_m : [\lambda s : \lambda^{-5}t : x : y]$ $\alpha_3^2 : [s : t : x + \varepsilon_1 st + \varepsilon_2 t^2 : y]$
	E_8^1	$y^2 = x^3 + s^4x + s^3t^3$	$\mathbb{G}_a \rtimes \mu_3$ with $\mathbb{G}_a : [s : t - (a^3 + a)s : x + as^2 : y]$ $\mu_3 : [s : \lambda t : x : y]$

Table 6. Non-equivariant RDP del Pezzo surfaces of degree 1 with global vector fields in characteristic 3

Type	blow up $\widetilde{X}$ in	Jac. or non-Jac.	Multiple fiber	Reducible fibers	$\mathrm{Aut}^0_{\widetilde{Z}}$	$h^0(\widetilde{Z}, T_{\widetilde{Z}})$	Moduli of $\widetilde{Z}$	$\mathrm{char}(k) = p$
$1A$	base pt.	Jac.	none	$\mathrm{I}_0^* + \mathrm{I}_0^*$	$\mathbb{G}_m$	1	1 dim	any
$1A$	2-tors. pt.	non-Jac.	2II	I_4^*	μ_2	1	1 dim	$= 2$
$1B$	base pt.	Jac.	none	$\mathrm{IV}^* + \mathrm{IV}$	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	any
$1B$	3-tors. pt.	non-Jac.	3II	II^*	μ_3	1	$\{\mathrm{pt}\}$	$= 3$
$1B$	2-tors. pt.	non-Jac.	$2\mathrm{IV}^*$	$\mathrm{IV}^* + \mathrm{I}_3$	μ_2	1	$\{\mathrm{pt}\}$	$= 2$
$1C$	base pt.	Jac.	none	$\mathrm{III}^* + \mathrm{III}$	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	$\neq 2$
$1C$	3-tors. pt.	non-Jac.	$3\mathrm{III}^*$	$\mathrm{III}^* + \mathrm{I}_2$	μ_3	1	$\{\mathrm{pt}\}$	$= 3$
$1D$	base pt.	Jac.	none	II^*	$\mathbb{G}_m$	1	$\{\mathrm{pt}\}$	$\neq 2, 3$
$1D$	5-tors. pt.	non-Jac.	$5\mathrm{II}^*$	II^*	μ_5	1	$\{\mathrm{pt}\}$	$= 5$

Table 1. Rational (quasi-)elliptic surfaces $\widetilde{Z}$ with global vector fields that are blow-ups of weak del Pezzo surfaces $\widetilde{X}$ of types $1A$, $1B$, $1C$ and $1D$

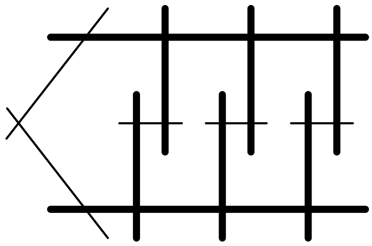
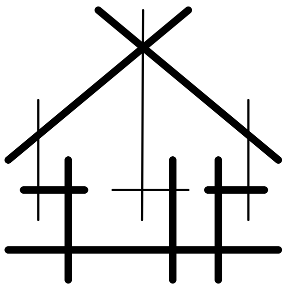
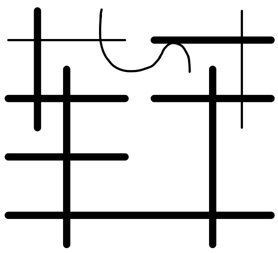
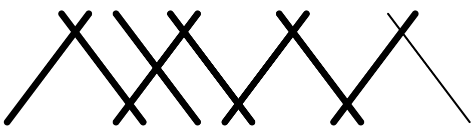
Case	(-2)-curves	Configuration of negative curves on $\widetilde{X}$	Action of $\mathrm{Aut}^0_{\widetilde{X}}$ on the Weierstraß equation of X	$\mathrm{char}(k)$
$1A$	$2D_4$		$\mathbb{G}_m : [\lambda s : \lambda^{-1}t : x : y]$ acting on $y^2 = x^3 + astx^2 + s^2t^2x$ where $a \in k$ and $a^2 \neq 4$	any
$1B$	$E_6 + A_2$		$\mathbb{G}_m : [\lambda^2 s : \lambda^{-1}t : x : y]$ acting on $y^2 + st^2 = x^3$	any
$1C$	$E_7 + A_1$		$\mathbb{G}_m : [\lambda^3 s : \lambda^{-1}t : x : y]$ acting on $y^2 = x^3 + st^3x$	$\neq 2$
$1D$	E_8		$\mathbb{G}_m : [\lambda^5 s : \lambda^{-1}t : x : y]$ acting on $y^2 = x^3 + st^5$	$\neq 2, 3$

Table 2. Four families of weak del Pezzo surfaces of degree 1 with global vector fields

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