Vine copulas and their applications to financial data

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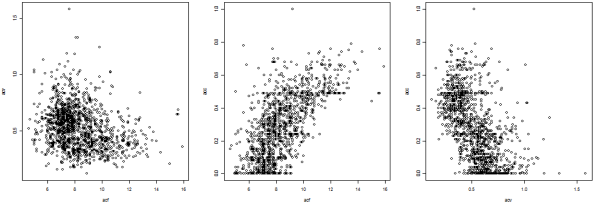
Why are copulas useful in financial modeling?



- Financial data has often complex dependency patterns, such as non symmetry and dependence in the extremes
- Cannot be captured by the multivariate normal distribution.
- The copula approach allows for these dependency patterns
- **Activity**: Google scholar with "copula statistics"

period	**60-69**	**70-79**	**80-89**	**90-99**	**00-10**	**10-12**
entries	106	360	710	1690	8920	5970

- **Areas:** Google books: medicine 638, biology 429, computer science 251, economics 1230 and finance 4100 entries.

Overview

Questions:

- **How to construct multivariate distributions with different margins?**
- **How to separate the dependency structure from the margins?**

Setup: Consider d random variables $\mathbf{X} = (X_1, \ldots, X_d)$ with

	density function	distribution function
marginal	$f_i(x_i),\ i = 1, \ldots, d$	$F_i(x_i),\ i = 1, \ldots, d$
joint	$f(x_1, \ldots, x_d)$	$F(x_1, \ldots, x_d)$
conditional	$f_{i\|j}(x_i\|x_j),\ i \neq j$	$F_{i\|j}(x_i\|x_j),\ i \neq j$

What are copulas and how it all started ...

Copula

A d-dimensional copula C is a multivariate distribution on $[0,1]^d$ with uniformly distributed marginals.

Copula density function: $c(u_1, ..., u_d) := \frac{\partial^d}{\partial u_1 ... \partial u_d} C(u_1, ..., u_d)$

Theorem (Sklar 1959)

$$F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$$
$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) f_1(x_1) ... f_d(x_d)$$

for some d-dimensional copula C.

$d = 2:$

$$f(x_1, x_2) = c_{12}(F_1(x_1), F_2(x_2)) f_1(x_1) f_2(x_2)$$
$$f_{2|1}(x_2|x_1) = c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2)$$

How to measure dependence?

Pearson correlation, Kendall's τ and Spearman's ρ:

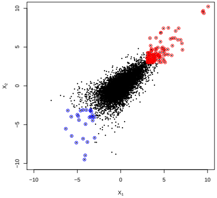
- Pearson correlation: linear dependence and not invariant
- Kendall's τ: probability of concordance minus discordance.
- Spearman's ρ: Pearson correlation of the ranked observations.
- τ and ρ are rank based and determined by the copula.

Tail dependence coefficients:

Upper (lower) tail dependence: probability of joint large (small) occurrences as one moves to the extremes.

$$\lambda^{upper} = \lim_{t \to 1^-} P(X_2 > F_2^{-1}(t) | X_1 > F_1^{-1}(t)) = \lim_{t \to 1^-} \frac{1 - 2t + C(t,t)}{1 - t}$$

$$\lambda^{lower} = \lim_{t \to 0^+} P(X_2 \leq F_2^{-1}(t) | X_1 \leq F_1^{-1}(t)) = \lim_{t \to 0^+} \frac{C(t,t)}{t}$$



What bivariate copula families are available?

Elliptical copula families

- Construction by inversion of Sklar's theorem:

$$C(u_1, u_2) = F(F_1^{-1}(u_1), F_2^{-1}(u_2)), \quad u_1, u_2 \in (0, 1),$$

  where F is elliptical.
- Gaussian (from bivariate normal with correlation ρ)
- Student's t (from bivariate Student's t with ν df and association ρ)

Archimedean copulae

- Construction through generator φ (McNeil and Nešlehová 2009):

$$C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2)), \quad u_1, u_2 \in (0, 1).$$

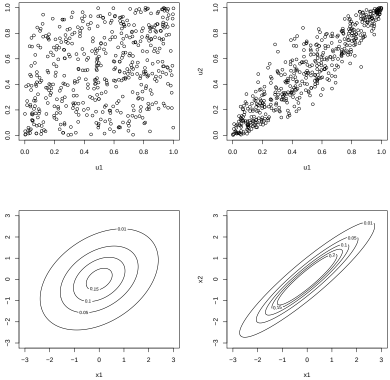
- Examples: Clayton, Gumbel, Frank, Joe ...

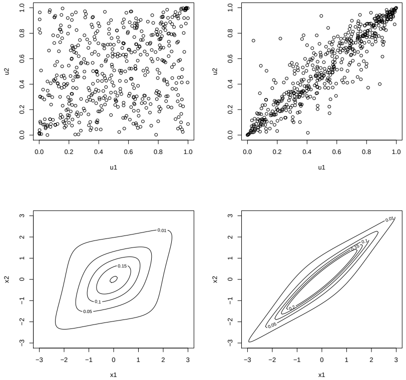
Books: Joe (1997) and Nelsen (2006)

Bivariate elliptical copula families

Gaussian copula
(left $\tau = .25$, right: $\tau = .75$)
symmetric dependence

t-copula with $df = 3$
(left $\tau = .25$, right: $\tau = .75$)
symmetric dependence

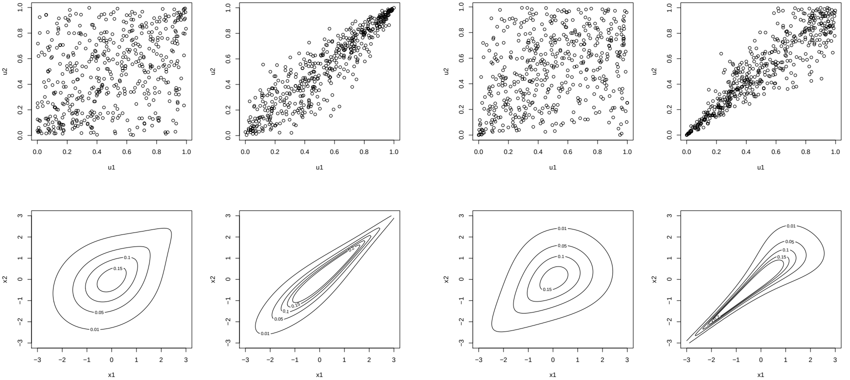




Bivariate Archimedian copula families

Gumbel copula
(left $\tau = .25$, right: $\tau = .75$)
upper tail dependent

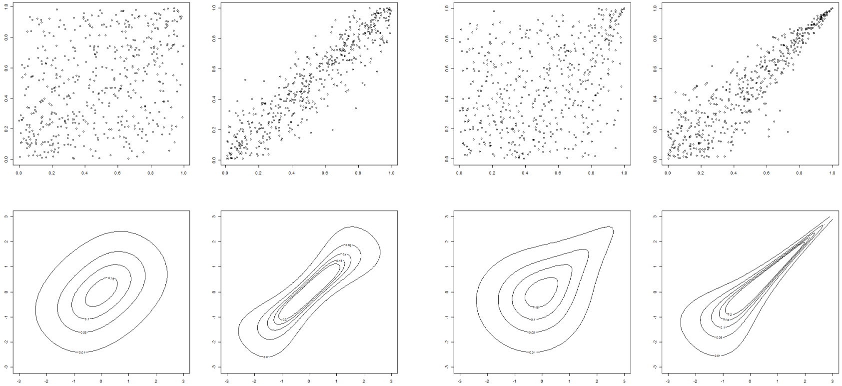
Clayton copula
(left $\tau = .25$, right: $\tau = .75$)
lower tail dependent



Bivariate Archimedian copula families

Frank copula
(left $\tau = .25$, right: $\tau = .75$)
symmetric dependence

Joe copula
(left $\tau = .25$, right: $\tau = .75$)
upper tail dependent



How does copula based modeling work?

- Consider a d dim. sample of size n (x_{ij})
- Copulas need to be estimated based on an random sample living on $[0,1]^d$, called copula data (u_{ij})
- If the marginal cdfs F_j are known, then $u_{ij} := F_j(x_{ij})$ (probability integral transform)
- If the marginal cdfs are unknown, then estimate them parametrically or non parametrically.
- Therefore often a two step approach is utilized.
- If each margin has time series or regression structure, then a copula model will be applied to the fitted standardized residuals.

What are these vine copulas?

- Vine copulas are multivariate copulas built out of bivariate copulas.
- A pair copula construction (PCC) is possible through conditioning. Joe (1996) gave a first example.
- Many PCC's are feasible. Bedford and Cooke (2002) introduced a graphical structure to help organize them.
- Gaussian vines were analyzed in Kurowicka and Cooke (2006) while ML estimation for Non Gaussian ones started with Aas et al. (2009).

How does this work in 3 dimensions?

$$f(x_1, x_2, x_3) = f_{3|12}(x_3|x_1, x_2) f_{2|1}(x_2|x_1) f_1(x_1)$$

Using Sklar for $f(x_1, x_2)$, $f(x_2, x_3)$ and $f_{13|2}(x_1, x_3|x_2)$ implies

$$\begin{aligned}
f_{2|1}(x_2|x_1) &= c_{12}(F_1(x_1), F_2(x_2)) f_2(x_2) \\
f_{3|12}(x_3|x_1, x_2) &= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) f_{3|2}(x_3|x_2) \\
&= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) c_{23}(F_2(x_2), F_3(x_3)) f_3(x_3)
\end{aligned}$$

$$\begin{aligned}
f(x_1, x_2, x_3) &= c_{13;2}(F_{1|2}(x_1|x_2), F_{3|2}(x_3|x_2)) c_{23}(F_2(x_2), F_3(x_3)) \\
&\times c_{12}(F_1(x_1), F_2(x_2)) \\
&\times f_3(x_3) f_2(x_2) f_1(x_1)
\end{aligned}$$

The copula corresponding to the distribution of (X_1, X_3) given $X_2 = x_2$ is denoted by $c_{13;2}$. Only bivariate copulas and univariate conditional cdf's are used. This can be easily generalized to d dimensions.

How do vines work in higher dimensions?

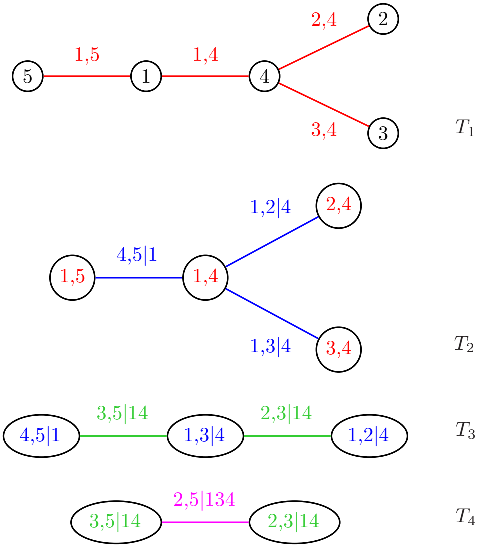
- Which pairs of variables are needed in the density expression?
- What are the conditioning variables for the term in density?

Components of a regular vine $R(\mathcal{V}, \mathcal{C}, \boldsymbol{\theta})$ distribution

1. Tree structure: The set of linked trees $\mathcal{V}$ identifies the pair of variables and the corresponding set of conditioning variables.
2. Parametric bivariate copulas $\mathcal{C} = \mathcal{C}(\mathcal{V})$ for each edge in the tree structure
3. Corresponding parameter value $\boldsymbol{\theta} = \boldsymbol{\theta}(\mathcal{C}(\mathcal{V}))$

- Conditional distribution functions can be computed recursively.

Can we see an example of a tree structure?



Density

$$f = f_1 \cdot f_2 \cdot f_3 \cdot f_4$$

- $c_{14} \cdot c_{15} \cdot c_{24} \cdot c_{34}$
- $c_{12;4} \cdot c_{13;4} \cdot c_{45;1}$
- $c_{23;14} \cdot c_{35;14}$
- $c_{25;134}$

How is a regular vine tree structure defined?

An d-dimensional vine tree structure $\mathcal{V} = \{T_1, \ldots, T_{d-1}\}$ is a sequence of $d-1$ linked trees with

Vine tree structure (Bedford and Cooke (2002))

- Tree T_1 is a tree on nodes 1 to d.
- Tree T_j has $d+1-j$ nodes and $d-j$ edges.
- Edges in tree T_j become nodes in tree T_{j+1}.
- **Proximity condition:** Two nodes in tree T_{j+1} can be joined by an edge only if the corresponding edges in tree T_j share a node.

Are there special cases?

- D-vines use only path like trees
- canonical (C)-vines use only star like tree

How do these C and D-vines look like?

C-vine: each tree has a unique node connected to $d-j$ edges

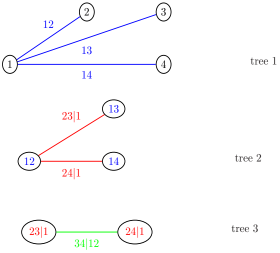
$$f_{1234} = [\prod_{i=1}^{4} f_i] \cdot c_{12} \cdot c_{13} \cdot c_{14} \cdot c_{23;1} \cdot c_{24;1} \cdot c_{34;12}$$

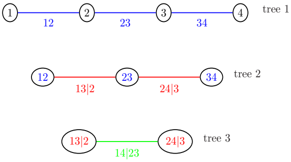
useful for ordering by importance

D-vine: no node is connected to more than 2 edges

$$f_{1234} = [\prod_{i=1}^{4} f_i] \cdot c_{12} \cdot c_{23} \cdot c_{34} \cdot c_{13;2} \cdot c_{24;3} \cdot c_{14;23}$$

useful for temporal ordering of variables

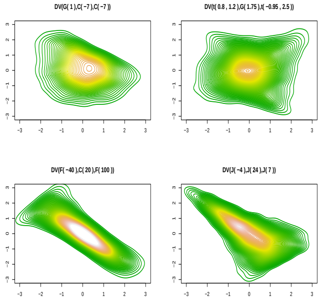




What is the scope of the vine models?

- Vine copulas classes
  - multivariate Gaussian copula
  - multivariate t copula
  - multivariate Clayton copula (Takahasi (1965), Stöber et al. (2012))
- The number of different vine tree structures is huge (see Morales-Nápoles et al. (2010)), additional flexibility through choice of copula families.

Contours of bivariate (1,3) margins with standard normal margins



(C=Clayton, G=Gumbel, t=Student, F=Frank, J=Joe)

Efficient estimation and model selection are vital

How can we estimate and model select PCCs ?

Three problems:

1. How to estimate the pair copula parameters for a given vine tree structure and the pair copula families for each edge?
2. How to select the pair copula families and estimate the corresponding parameters for a given vine tree structure?
3. How to select and estimate all components of a regular vine?

Problem 1: Parameter estimation for given tree structure and copula families

- **Sequential estimation:**
  - Parameters are sequentially estimated starting from the top tree until the last (Aas et al. (2009), Czado et al. (2012)).
  - Asymptotic theory available (Haff (2010)), however standard error estimates are difficult to compute.
  - Can be used as starting values for maximum likelihood.
- **Maximum likelihood estimation:**
  - Asymptotically efficient under regularity conditions, estimated standard errors numerically challenging (Stoeber and Schepsmeier (2012))
  - Uncertainty in value-at-risk (high quantiles) is difficult to assess.
- **Bayesian estimation:**
  - Posterior is tractable using Markov Chain Monte Carlo (Min and Czado (2011) for D-vines and Gruber et al. (2012) for R-vines)
  - Prior beliefs can be incorporated and credible intervals allow to assess uncertainty for all quantities.

How does sequential and ML estimation work ?

Parameters: $\Theta = (\theta_{12}, \theta_{23}, \theta_{13;2})$

Observations: $\{(x_{1t}, x_{2t}, x_{3t}), t = 1, \cdots, T\}$

Sequential estimates:

- Estimate θ_{12} from $\{(x_{1,t}, x_{2,t}), t = 1, \cdots, T\}$
- Estimate θ_{23} from $\{(x_{2,t}, x_{3,t}), t = 1, \cdots, T\}$.
- Define pseudo observations

$$\hat{v}_{1|2t} := F(x_{1t}|x_{2t}, \hat{\theta}_{12}) \text{ and } \hat{v}_{3|2t} := F(x_{2t}|x_{3t}, \hat{\theta}_{23})$$

Finally estimate $\theta_{13;2}$ from $\{(\hat{v}_{1|2t}, \hat{v}_{3|2t}), t = 1, \cdots, T\}$.

Maximum likelihood

$$\begin{aligned} L(\Theta|x) &= \sum_{t=1}^{T} [\log c_{12}(x_{1t}, x_{2t}|\theta_{12}) + \log c_{23}(x_{2t}, x_{3t}|\theta_{23}) \\ &+ \log c_{13;2}(F(x_{1t}|x_{2t}, \theta_{12}), F(x_{2t}|x_{3t}, \theta_{23})|\theta_{13;2})] \end{aligned}$$

Problem 2: Joint estimation of pair copula families and parameters

- **Classical approach:**
  - Restrict to a set of bivariate pair copula families and use AIC or Vuong test to select family
  - Check for truncation possibilities (Brechmann et al. (2012)) by using independence copulas in higher trees
- **Bayesian approach:**
  - Reversible jump (RJ) MCMC (Min and Czado (2011))
  - MCMC with model indicators (Smith et al. (2010)) choosing between an independence copula and a fixed copula family.

Only one more problem to go ...

sequential treewise approach
(see Dißmann et al. (2013))

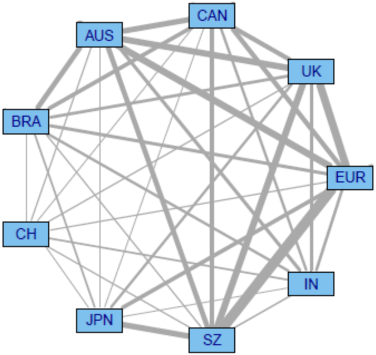
How does this treewise selection of R-vines work?

Idea: Capture strong pairwise dependencies first: **For Tree** $\ell = 1, \dots, d-1$

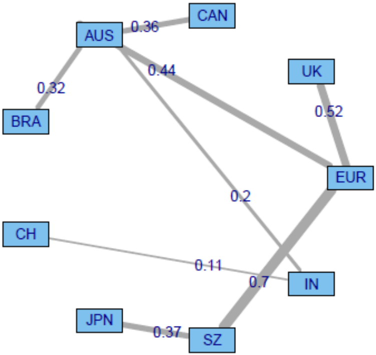
1. Calculate an empirical dependence measure $\hat{\delta}_{jk|D}$ for all variable pairs $\{jk|D\}$ ($\rightarrow$ edge weights: Kendall's τ, tail dependence coefficients) allowed by the proximity condition (D is empty for Tree 1).
2. Select the tree on all nodes that maximizes the sum of absolute empirical dependencies ($\rightarrow$ maximum spanning tree)
   Choose independence copula if possible.
3. For each selected edge $\{j, k\}$ ($\{j, k\}|D$) in Tree 1 (in Tree $\ell > 1$), select a copula and estimate the corresponding parameter(s).
4. Then transform to pseudo observations $F_{j|k \cup D}(u_{ij}|\mathbf{u}_{i,k\cup D}, \hat{\theta}_{j,k;D})$ and $F_{k|j\cup D}(u_{ik}|\mathbf{u}_{i,j\cup D}, \hat{\theta}_{j,k;D})$, $i = 1, ..., n$.

How does this look like for Tree 1?

(1) Pairwise dependencies.



(2) Maximum dependence tree.



Euro Stoxx 50:

- 50 large Eurozone companies.
- Major market indicator for the Eurozone.
- Underlying of many investment products.
- Brechmann and Czado (2012) consider 46 members from 5 countries (Germany, France, Italy, Spain and the Netherlands) together with their national indices.
- Daily log returns from May 2006 to April 2010 (985 obs.) are used.

Questions

- How do stock returns depend on the European and the national indices? Is dependence on national index dominant?
- Which dependencies are most important? Are they asymmetric and/or heavy-tailed?

Copula based models for Euro Stoxx 50 returns :

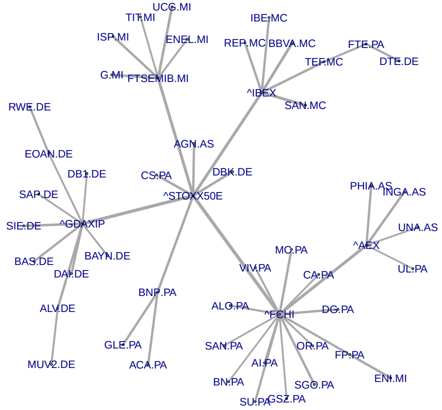
- Fit appropriate (ARMA-)GARCH models for return time series.
- Fit copula model such as R- and C-vine copulas as well as multivariate Student-t copula for comparison to standardized residuals

Results

Copula	Log likelihood	No. of param.	BIC
R-vine	30879.60	596	-57651.19
C-vine	30839.68	685	-56957.90
Student-t	30691.36	1327	-52236.18

R-vine > C-vine > Student-t

First tree of R-vine and C-vine order



Order	Root nodes
1^{st}	^STOXX50E
2^{nd}	GLE.PA
3^{rd}	^FCHI
4^{th}	^GDAXIP
5^{th}	^IBEX
6^{th}	INGA.AS
7^{th}	FTSEMIB.MI
⋮	⋮
13^{th}	^AEX
⋮	⋮

Factor models

- $r_{i,t}$ return of asset i at time t, $r_{M,t}$ market return at time t
- Gaussian $\varepsilon_{i,t}$ idiosyncratic error ind. of $\varepsilon_{i,t-1}$ and $\varepsilon_{j,t}$ $\forall j \neq i$

CAPM (Sharpe (1964), Lintner (1965))

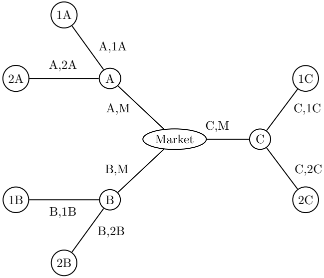
$$r_{i,t} = \beta_i r_{M,t} + \varepsilon_{i,t}.$$

Extension: The C-Vine Market Model (CVM) of Heinen and Valdesogo (2009b) loosens assumptions of Gaussianity and linearity:

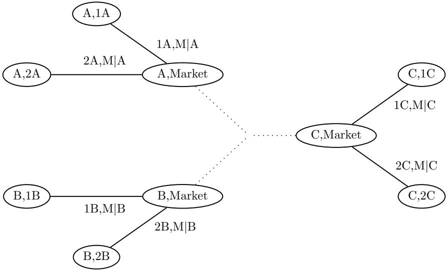
- GARCH-models for the marginal time series, and
- dependence between assets and the market modeled with bivariate copulas ($\rightarrow$ C-vine with 'market' as root node).
- Remaining (idiosyncratic) dependence captured with multivariate Gaussian copula.
- Including sectorial dependence gives C-Vine market sector model (CVMS)
- However CVMS needs restrictive independence assumptions.

Regular Vine Market Sector (RVMS) model

First tree:



Second tree:



- **Independence assumption:** conditionally on the market, sector returns are independent.
- Remaining pair-copulas of regular vine are modeled as Gaussian.

Model comparisons for Euro Stoxx 50

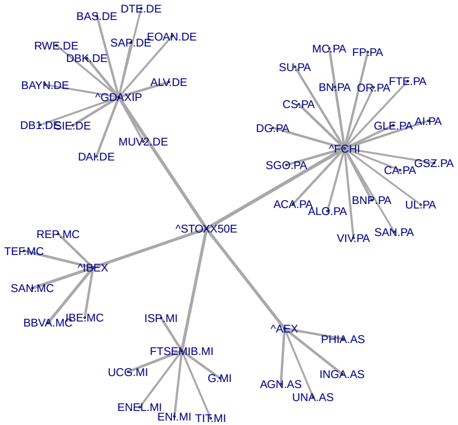
Independence assumptions...

- ...of the CVMS model are not satisfied.
- Those of the RVMS model approximately hold.

The **RVMS model** provides...

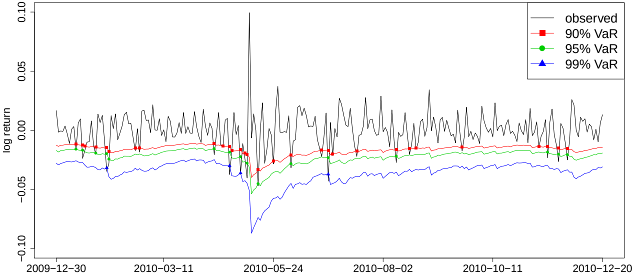
- ...a parsimonious fit,
- which is however close to full R-vine model and
- superior to CVMS model.

First tree of RVMS model:



R-vine $\gtrsim$ RVMS $>$ C-vine $>$ CVMS $\gtrsim$ Student-t

VaR backtests



- **Test period:** year 2010 (250 obs), moving window size 900
- **Backtests:** RVMS and DCC with Gaussian innovations perform similarly well.
- **Why using copula models then?**
  - Risk capital may be reduced!
  - Full DCC model requires a huge number of parameters!
- Simulations from RVMS are $\approx 40\%$ faster than from full R-vine

What is systemic risk and what are important criteria?

Systemic risk (FSB)

risk of disruption to financial services that is

- caused by an impairment of all or parts of the financial system and
- has the potential to have serious negative consequences for the real economy

Important criteria for systemic risk

- size (Too big to fail)
- lack of substitutability
- interconnectedness (Too interconnected to fail)

FSB already classified banks according to their systemic importance while this is expected also for insurers.

CDS spreads and credit worthiness

- CDS spreads of a company are often considered as a market based indicator of credit worthiness of the company.
- Interdependence among CDS spreads from different companies studied using copulas for first time in Brechmann, Hendrich, and Czado (2013).
- **Data base**: Senior CDS spreads with a 5 year maturity observed daily from Jan. 4, 2006 until Oct. 25, 2011 from 38 financial sector companies
  - 18 banks and 20 (re-)insurers
  - Among the 18 banks there are 15 systemically important and 3 are not
  - **Geographical regions**: Europe (EU) (11 banks/12 insurers), USA (US) (3 banks/6 insurers) and Asia Pacific (AP) (4 banks/2 insurers)

Vine based modeling of CDS spreads

- **Marginal models:**
  - Separate GARCH models are fitted to each time series of log returns.
  - Specification includes asymmetric exponential GARCH and GARCH-in-mean
  - Nonstandard innovations needed: generalized error, generalized hyperbolic and inverse Gaussian.
  - Exploratory analysis of pairwise empirical Kendall's tau shows strong clustering of sectors (bank, ins.) within region (AP, EU, US).
- **Copula models**:
  - C-vines are fitted using the selection method of Czado et al. (2012) allowing Gaussian, Student t, Clayton, Gumbel, Frank and rotations for pair copula families (C-vine)
  - Parameters in C-vine are reduced by independence tests.
  - **Other copulas**: multivariate Gaussian (Gaussian), multivariate Student t (Student's t) and multivariate exchangeable Gumbel (Gumbel)

How well do these multivariate copulas fit?

Copula	Max. log lik.	# Par.	AIC	BIC
Gumbel	8640.22	1	-17278.45	-17273.22
Gaussian	18326.53	703	-35247.07	-31575.09
Student's t	19915.88	704	-38423.76	-34746.56
C-vine	20393.29	488	-39810.58	-37261.61

- C-vine best and much fewer parameter than Student t.
- Gaussian is not sufficient and multivariate Gumbel does not fit.
- **C-vine**: 3 upper tail Gumbel pair copulas in first tree, overall nearly 50% of selected pair copulas are non-elliptical

What is copula based systemic stress testing?

Rationale

Severe drop in credit worthiness of a company $\rightarrow$ sharp increase of CDS spread $\rightarrow$ large standardized residual $\rightarrow$ large copula data value

- Therefore we study the impact of a large copula data value for a specified company on the copula data values of remaining companies
- For chosen company i we set $u_i = .99$ and then we need to

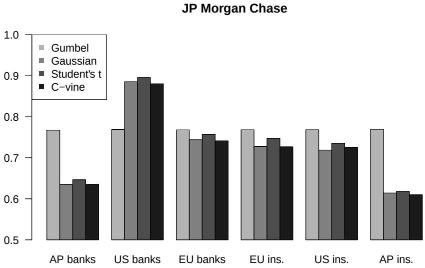
sample from the conditional distribution $\mathbf{U}_{-i}|U_i = u_i$

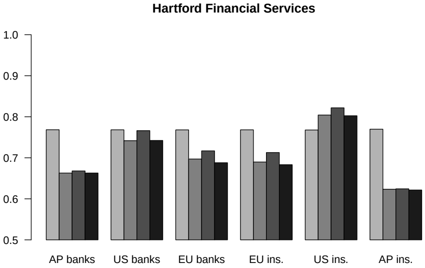
- Brechmann et al. (2013) develop the necessary conditional sampling algorithms.
- This sampling is repeated $N = 10000$ times for each company resulting in sampled values $\widetilde{u}_{\ell,j|i}$, $j \in \{1,...,38\} \setminus \{i\}$, $\ell = 1,...,N$.

What is the impact on sector (within region) of two chosen companies?

Impact of company i on sector (within region) s with members M_s:

$$\widetilde{\mu}_{s|i} := \frac{1}{N}\sum_{\ell=1}^{N} \widetilde{u}_{\ell,s|i}, \quad \text{where} \quad \widetilde{u}_{\ell,s|i} := \frac{1}{|M_s \setminus \{i\}|}\sum_{j\in M_s\setminus\{i\}} \widetilde{u}_{\ell,j|i}.$$





Impact of a stressed sector on another sector?

Mean impact of a stressed sector s_1 on another sector s_2

$$\widetilde{\mu}_{s_2|s_1} := \frac{1}{|M_{s_1}|} \sum_{i \in M_{s_1}} \widetilde{\mu}_{s_2|i}.$$

	Stress situation in					
Impact on	AP banks	US banks	EU banks	EU ins.	US ins.	AP ins.
AP banks	0.68	0.63	0.65	0.66	0.64	0.69
US banks	0.63	0.88	0.74	0.73	0.72	0.60
EU banks	0.65	0.73	0.87	0.83	0.67	0.62
EU ins.	0.66	0.72	0.83	0.87	0.68	0.64
US ins.	0.64	0.73	0.68	0.69	0.79	0.62
AP ins.	0.69	0.61	0.63	0.65	0.62	0.62

- Stress effect of a member on the other members of the sector highest compared to other sectors in US and EU.
- Stress effect in US banks on US ins., EU banks and EU ins. similar
- Stress effect in EU banks on EU ins. higher than on US banks and ins.
- Stress effect in EU ins. on EU banks higher than on US banks and ins.
- Stress effect in US ins. on US banks , EU banks and EU ins. similar

What else can we do?

- **Time varying R-vines:** (AR(1) copula dynamics (Almeida and Czado 2011),Almeida et al. (2012), regime switching (Chollete et al. 2008), (Stöber and Czado 2011))
- **Truncated and simplified R-vines**: (truncated C-vines (Heinen and Valdesogo 2009a), truncated R-vines (Brechmann et al. 2012))
- **Non Gaussian DAGs** (Bauer et al. 2012)
- **Discrete vines** (Panagiotelis et al. 2012)
- **Applications:** VaR, multivariate option pricing, genetics, hydrology, insurance

What have we learned?

- Standard multivariate copulas are less flexible, while PCC's such as C-, D- and R-vines are much more flexible.
- Sequential and MLE parameter estimation of C-, D- and R-vines are available in **R packages** `CDVine` and `VineCopula`.
- Sequential and full Bayesian estimation and Bayesian model selection of vine trees and copula families for regular vines available, but need further testing and development
- Pair copula constructions can be extended to mixed continuous and discrete data.
- Vine copulas are useful for financial risk management

What needs to be done?

- non parametric pair copulas, spatial vines, vines for data mining
- more applications in finance, insurance ...

Vine resource page:
`www-m4.ma.tum.de/forschung/vine-copula-models`
Vine workshop book: Kurowicka and Joe (2011)
Thanks to my collaborators (K. Aas, A. Frigessi , A. Min, E. Brechmann, C. Almeida, M. Smith, A. Panagiotelis, A. Bauer, T. Klein, M. Hofmann, J. Dißmann, H. Joe, J. Stöber, U. Schepsmeier, D. Kurowicka, L. Gruber, N. Krämer...)

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