Cooperative Control of Uncertain Multi-Agent Systems via Distributed Gaussian Processes

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***Abstract*— For single agent systems, probabilistic machine learning techniques such as Gaussian process regression have been shown to be suitable methods for inferring models of unknown nonlinearities, which can be employed to improve the performance of control laws. While this approach can be extended to the cooperative control of multi-agent systems, it leads to a decentralized learning of the unknown nonlinearity, i.e., each agent independently infers a model. However, a decentralized learning approach can potentially lead to poor control performance, since the models of individual agents are often accurate in merely a small region of the state space. In order to overcome this issue, we propose a novel method for the distributed aggregation of Gaussian process models, and extend probabilistic error bounds for Gaussian process regression to the proposed approach. Based on this distributed learning method, we develop a cooperative tracking control law for leader-follower consensus of multi-agent systems with partially unknown, higher-order, control-affine dynamics, and analyze its stability using Lyapunov theory. The effectiveness of the proposed methods is demonstrated in numerical evaluations.**

***Index Terms*— Cooperative control, machine learning, distributed learning, feedback linearization, Gaussian processes.**

I. Introduction

COOPERATIVE control of multi-agent systems has attracted much attention in the past two decades, see e.g., [1]–[3]. This is due to its broad applications such as formation control [4] and autonomous underwater vehicles [5]. However, common approaches for the control of multi-agent systems typically rely on accurate models of the system dynamics, which are often not available in practice, in particular when the agents operate in uncertain and dynamically changing environments.

To remove the dependency on accurate system models, neural network techniques are proposed in, e.g. [6], [7]. In [6], distributed adaptive control laws based on neural networks are proposed for unknown multi-agent systems with scalar nonlinear agent dynamics. The results are then generalized to unknown multi-agent systems with higher-order nonlinear agent dynamics [7]. However, these approaches often require huge amounts of data for learning, and the derived control error bounds depend crucially on accuracy guarantees for the neural networks, which are challenging to obtain in general.

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Probabilistic machine learning methods overcome these issues by representing epistemic uncertainty [8], thereby achieving high data efficiency and providing estimates for model errors. In recent years, these methods have been successfully applied to a variety of control problems, see, e.g., [9]–[11]. In particular, Gaussian process (GP) regression [12] has become a popular learning method for the control of single agent systems due to its stochastic foundations, which allow the derivation of probabilistic regression error bounds [13]. This has been exploited for the control of Euler-Lagrange systems [11], feedback linearization of affine systems [14], and model predictive control [9].

In recent years, several of these approaches have been extended to the control of multi-agent systems. For example, a distributed model predictive control method, which allows to cooperatively solve optimal control problems with Gaussian process models as dynamics, is developed in [15]. In [16], GP models are employed to design control laws for formation control, while a similar approach is used for deriving a distributed consensus control law in [17]. Even though these approaches follow cooperative control principles, they employ merely local data of each agent [15], [16] or from direct neighbors [17], but do not disseminate the information contained in the data through the communication network. Therefore, the control performance of an agent crucially depends on its local data set.

Analogously to the development of cooperative control methods, distributed learning techniques based on Gaussian process regression have also received increasing attraction. Addressing the issues faced when GPs are scaled to large data sets, early approaches only distribute the inference in local models to multiple agents but employ a central coordinator for their aggregation [18]. This limitation is overcome by computing various summary statistics for the individual GPs, which can be combined using consensus protocols [19]. Through an online optimization of hyperparameters and summary statistics, this can even be extended for implementing online learning [20]. While these distributed GP approaches have shown strong empirical performances, probabilistic regression error bounds similar to those for exact Gaussian process regression have not been derived. Due to this lack of theoretical guarantees, there exists, to the best of our knowledge, no approach safely combining cooperative control with distributed Gaussian process models.

A. Contribution and Structure

In this article, we address this gap in research by proposing distributed learning-based feedback linearizing tracking control laws for leader-follower consensus of multi-agent systems with partially unknown, higher-order, control-affine dynamics. For simplicity of exposition, we focus here on systems with a single control input, but the extension to systems with multiple control inputs is straightforward. To this end, we first present a novel, fully distributed Gaussian process approach for predicting unknown dynamics, which straightforwardly extends approaches requiring centralized coordinators [18] to arbitrary computation graphs by employing dynamic average consensus algorithms [21]. Based on GP error bounds [13], we derive

explicit, distributed regression error bounds for the proposed method, which are, to the best of our knowledge, the first of their kind for fully distributed GP predictions. We then make use of the proposed distributed learning approach to predict partially unknown dynamics shared by all agents. Subsequently, we present a novel cooperative tracking control law, which incorporates the proposed distributed learning approach, and show that the designed control law achieves tracking consensus with guaranteed control error bounds.

The remainder of this paper is structured as follows: After the problem statement is provided in Section II, a novel method for computing distributed GP predictions is introduced in Section III. Section IV presents the proposed cooperative tracking control law employing distributed predictions and provides the stability analysis. In Section V, numerical simulations are given to demonstrate the effectiveness of the proposed approaches, before Section VI concludes this paper.

B. Notation and Graph Theory

Vectors and matrices are denoted by bold lower and upper case symbols, respectively, real positive numbers without and with zero by $\mathbb{R}_+$ and $\mathbb{R}_{0,+}$, respectively, and positive integer numbers by $\mathbb{N}$. The $m \times m$ identity matrix is denoted by $\boldsymbol{I}_m$, and the $m \times 1$ vector with all one components by $\mathbf{1}_m$. If not stated otherwise, $|\cdot|$ denotes the absolute value, $\|\cdot\|$ the Euclidean norm and $\|\cdot\|_F$ the Frobenius norm. Minimum/maximum singular values of a matrix are denoted by $\underline{\sigma}(\cdot)/\bar{\sigma}(\cdot)$. The i-th element of a vector $\boldsymbol{a}$ is denoted by a_i, while a_{ij} denotes the element in the i-th row and j-th column of matrix $\boldsymbol{A}$. Matrix $\boldsymbol{A} \succ 0$, if $\boldsymbol{A}$ is a positive definite matrix.

Throughout this article, the topology of a network is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ denotes the set of agents, which correspond to the vertices of the graph, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. An edge $(n, n') \in \mathcal{E}$ represents that agent n' can receive information from agent n, and vice versa. The edges of a weighted graph can be compactly represented through the weighted adjacency matrix $\boldsymbol{A} \in \mathbb{R}^{N \times N}$, with elements $a_{nn'} > 0$ if $(n', n) \in \mathcal{E}$. The degree matrix of the graph $\mathcal{G}$ is a diagonal matrix with elements $D_{nn} = \sum_{i=1}^{N} a_{ni}$. The Laplacian matrix of the graph $\mathcal{G}$ is defined as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$. Throughout the paper we assume a fixed topology, i.e., $\boldsymbol{A}$ is constant, and the self-connectivity element $a_{nn} = 0$. For a connected graph, the eigenvalues of $\boldsymbol{L}$ are denoted by $\tilde{\lambda}_1, \dots, \tilde{\lambda}_N$, where $\tilde{\lambda}_1 = 0 < \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_N$.

II. Problem Statement

We consider a nonlinear multi-agent system with N possibly heterogeneous follower agents, for simplicity referred to as agents in the following, whose dynamics can be described by single-input systems in the controllable canonical form

$$\dot{x}_{n,1} = x_{n,2}, \quad \cdots \quad \dot{x}_{n,d} = f_n(\boldsymbol{x}_n) + g_n(\boldsymbol{x}_n)u_n + h(\boldsymbol{x}_n), \quad (1)$$

for $n = 1, \dots, N$, where $\boldsymbol{x}_n = [x_{n,1}\ x_{n,2}\ \dots\ x_{n,d}]^T \in \mathbb{R}^d$ denotes the state of agent n, $\boldsymbol{x}_n(0) = \boldsymbol{x}_n^0$ is the initial state, $u_n \in \mathbb{R}$ is the control input for agent n, $f_n : \mathbb{R}^d \to \mathbb{R}$ and $g_n : \mathbb{R}^d \to \mathbb{R}$ are known functions that describe the individual dynamics of agent n, and $h : \mathbb{R}^d \to \mathbb{R}$ represents an unknown nonlinearity shared by all agents such as, e.g., hydrodynamic forces due to ocean currents affecting the dynamics of a fleet of underwater vehicles [22]. In order to ensure global controllability of each agent, we require the following assumption on the functions $g_n(\cdot)$.

Assumption 1: All functions $g_n(\cdot)$ are non-singular for all $\boldsymbol{x}_n \in \mathbb{R}^d$.

The non-singularity of $g_n(\cdot)$ is a standard prerequisite for the design of feedback linearizing control laws [23, Defintion 13.1] and ensures that each agent's dynamics exhibits a relative degree d for all $\boldsymbol{x}_n$. Thereby, global controllability is ensured and the existence of internal dynamics is excluded. Although this assumption restricts the considered system class, the focus of this work is on the distributed learning for cooperative tracking control. Therefore, we leave the extension to larger system classes for future research.

The goal of the agents is to follow a virtual leader with state $\boldsymbol{x}_l \in \mathbb{R}^d$, which satisfies the following properties.

Assumption 2: The leader state $\boldsymbol{x}_l$ describes a reference trajectory for the agents, which has the form

$$x_{l,1} = x_r(t), \quad x_{l,2} = \dot{x}_{l,1}(t), \quad \cdots \quad x_{l,d} = \dot{x}_{l,d-1}(t), \quad (2)$$

where $x_r : \mathbb{R}_+ \to \mathbb{R}$ is an at least $d-1$ times continuously differentiable function, such that $\boldsymbol{x}_l(t) \in \Omega$ for all $t \geq 0$ and some compact set $\Omega \subset \mathbb{R}^d$, and $\|\dot{x}_{l,d}\| \leq \hat{\Delta}$ for some constant $\hat{\Delta} \in \mathbb{R}_+$.

Since the reference trajectory is a design choice, the assumption on differentiability is not restrictive in practice. Moreover, practical problems usually involve tasks in compact state spaces, such that the assumption on a bounded reference is typically not an issue. Therefore, this assumption can frequently be found in literature, e.g., [11], [14], in order to ensure that the agents are capable of following the leader.

As multi-agent systems may comprise a large number of agents in real-world applications, we consider a restricted communication between agents described by the graph $\mathcal{G}$ with adjacency matrix $\boldsymbol{A} \in \mathbb{R}^{N \times N}$. Moreover, we assume that only a few agents can communicate with the leader as expressed by the diagonal matrix $\boldsymbol{B} = \mathrm{diag}(b_1, \dots, b_N) \in \mathbb{R}^{N \times N}$, whose diagonal elements satisfy $b_n = 1$ if agent n has access to the leader state, and $b_n = 0$ otherwise. In order to allow a cooperative tracking control design, we require the following common assumptions on the communication topology.

Assumption 3: The communication graph $\mathcal{G}$ among the agents is undirected and connected. Moreover, at least one agent is connected to the leader, i.e., there exists $n = 1, \dots, N$ such that $b_n = 1$.

Assumption 4: The weighted adjacency matrix $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ of the graph $\mathcal{G}$ is weight balanced, i.e., $\mathbf{1}^T \boldsymbol{A} = \boldsymbol{A}\mathbf{1} = \mathbf{1}$.

Assumption 3 guarantees that information about the leader state is directly transmitted to some agents, and propagates through the network to all agents since they have at least indirect access through paths starting at the leader. This assumption is a crucial prerequisite to track the leader with all agents [7]. Finally, Assumption 4 ensures that information of each agent is equally treated in the network [1].

In order to be able to infer a model of the unmodeled nonlinearity $h(\cdot)$, we assume the availability of data as stated in the following.

Assumption 5: Each agent n has a training data set $\mathbb{D}_n = \{(\boldsymbol{x}_n^{(m)}, y_n^{(m)}), m = 1, \dots, M_n\}$ with $M_n \in \mathbb{N}$ data pairs, such that the noise ς_n of observations $y_n = h(\boldsymbol{x}_n) + \varsigma_n$ in each individual data set $\mathbb{D}_n$ follows an i.i.d. Gaussian distribution $\mathcal{N}(0, \sigma_o^2)$.

This assumption reflects the fact that agents are often able to collect data on their own, without sharing data amongst them. It also allows splitting a large data set into N subsets based on the states $\boldsymbol{x}_n$, but not if the samples are assigned using the observations y_n. While Assumption 5 requires exact state measurements $\boldsymbol{x}_n$, which is a necessary requirement for the feedback linearizing control law (3), it essentially allows noisy observations of the highest derivative $\dot{x}_{n,d}$ since $f_n(\cdot)$ and $g_n(\cdot)$ are known, which is a common assumption when dealing with unknown dynamics, see, e.g., [11], [14].

Based on these assumptions, this article considers the problem of designing a distributed, feedback linearizing controller of the form

$$u_n = \frac{1}{g_n(\boldsymbol{x}_n)}(-f_n(\boldsymbol{x}_n) - \hat{h}_n(\boldsymbol{x}_n) + \nu(\boldsymbol{\varepsilon}_n)), \quad n = 1, \dots, N, \quad (3)$$

where $\hat{h}_n : \mathbb{R}^d \to \mathbb{R}$ is a model of the unknown nonlinearity $h(\boldsymbol{x}_n)$ obtained from data which will be specified later, and $\nu : \mathbb{R}^d \to \mathbb{R}$ is a linear control law depending on the local consensus error

$$\boldsymbol{\varepsilon}_n = -\sum_{i=1}^{N} a_{n,i}(\boldsymbol{x}_n - \boldsymbol{x}_i) - b_n(\boldsymbol{x}_n - \boldsymbol{x}_l). \tag{4}$$

Remark 1: We consider a common control law $\nu(\cdot)$ in all agents merely for notational simplicity. An extension to heterogeneous linear control laws $\nu_n(\cdot)$, $n = 1, \dots, N$, is straightforward.

The goal of the distributed, feedback linearizing controllers is to track the state of the leader $\boldsymbol{x}_l$ with the agent states $\boldsymbol{x}_n$. Due to the effects of the unknown nonlinearity $h(\cdot)$, exact tracking cannot be achieved in general. Therefore, we want to guarantee a small tracking error, which we formalize using the following notion of stability.

Definition 1: The tracking error $\boldsymbol{e}_n(t) = \boldsymbol{x}_n(t) - \boldsymbol{x}_l(t)$ of agents $n = 1, \dots, N$ is ultimately bounded with time-varying ultimate bound $c_1 : \mathbb{R}_{0,+} \to \mathbb{R}_+$ if for every $c_2 \in \mathbb{R}_+$ there exists a $T = T(c_1, c_2) \in \mathbb{R}_+$ such that for all $n = 1, \dots, N$ it holds that

$$\|\boldsymbol{e}_n(0))\| \leq c_2 \quad \Rightarrow \quad \|\boldsymbol{e}_n(t)\| \leq c_1(t), \forall t \geq T. \tag{5}$$

While it would be straightforward to employ a model $\hat{h}_n(\cdot)$ based merely on the local data set $\mathcal{D}_n$ in the cooperative controller (3) and show this form of stability [16], such an approach crucially relies on training data being evenly-distributed among the agents. In practice, this requirement can possibly lead to a poor tracking performance as each agent typically has only data from a small portion of the state space, which we demonstrate in Section V. In order to overcome this issue, this article considers the problem of designing a cooperative control law of the form (3) with distributedly computed models $\hat{h}_n(\cdot)$, such that the tracking accuracy is improved and an ultimately bounded tracking error is guaranteed. For achieving this goal, we first derive a method for the distributed prediction of $h(\cdot)$ with bounded model errors. Based on the developed prediction scheme, we address the design of the cooperative control law and its stability guarantees.

III. Distributed Gaussian Process Regression for Dynamical Systems

In order to derive a method for the distributed prediction of an unmodeled nonlinearity, we employ Gaussian process regression [12] for learning individual models in each agent and aggregate the predictions in a distributed fashion using dynamic average consensus methods. For this purpose, we first introduce Gaussian process regression in Section III-A before deriving the dynamic average consensus aggregation scheme for predictions, which we prove to guarantee bounded regression errors, is presented in Section III-B.

A. Gaussian Process Regression

A Gaussian process defines a distribution over functions, expressed as $h(\boldsymbol{x}) \sim \mathcal{GP}\left(m(\boldsymbol{x}), k\left(\boldsymbol{x}, \boldsymbol{x}'\right)\right)$, where $m : \mathbb{R}^d \to \mathbb{R}$ is the prior mean function and $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{0,+}$ denotes the covariance function, also called kernel. The prior mean function is used to employ prior knowledge such as approximate models into the regression, while the kernel is used to encode abstract prior information such as periodicity and smoothness of the unknown function. A commonly used covariance function is the squared exponential kernel defined as

$$k(\boldsymbol{x}, \boldsymbol{x}') = \sigma_f^2 \exp\left(\sum_{i=1}^{d} \frac{(x_i - x_i')^2}{2l_i^2}\right), \tag{6}$$

where σ_f^2 and l_i^2 are so called hyperparameters, which are typically determined using log-likelihood maximization [12].

Given noisy observations $y^{(m)}$, $m = 1, \dots, M$ satisfying Assumption 5, we can compute the posterior Gaussian process by conditioning on the training data $\mathbb{D}$. This posterior distribution $g(\boldsymbol{x}^*)|\mathbb{D}, \boldsymbol{x}^*$ at a test point $\boldsymbol{x}^*$ is again Gaussian with mean and variance

$$\mu\left(\boldsymbol{x}^*\right) = \boldsymbol{k}\left(\boldsymbol{x}^*\right)^T \left(\boldsymbol{K} + \sigma_o^2 \boldsymbol{I}_M\right)^{-1} \boldsymbol{y} \tag{7}$$

$$\sigma^2\left(\boldsymbol{x}^*\right) = k\left(\boldsymbol{x}^*, \boldsymbol{x}^*\right) - \boldsymbol{k}\left(\boldsymbol{x}^*\right)^T \left(\boldsymbol{K} + \sigma_o^2 \boldsymbol{I}_M\right)^{-1} \boldsymbol{k}\left(\boldsymbol{x}^*\right), \tag{8}$$

where the elements of $\boldsymbol{k}\left(\boldsymbol{x}^*\right) \in \mathbb{R}^M$ and $\boldsymbol{K} \in \mathbb{R}^{M \times M}$ are defined through $k_m(\boldsymbol{x}^*) = k(\boldsymbol{x}^*, \boldsymbol{x}^{(m)})$ and $K_{mm'} = k(\boldsymbol{x}^{(m)}, \boldsymbol{x}^{(m')})$, respectively, and $\boldsymbol{y} = [y^{(1)} \ \dots \ y^{(M)}]^T$.

B. Consensus-Based Distributed Aggregation of Predictions

Our approach for the distributed computation of predictions is inspired by state-of-the-art aggregation approaches for Gaussian processes. These inference approximations have the goal of reducing the computational complexity by splitting the data into several subsets $\mathbb{D}_n$, $n = 1, \dots, N$ and training individual Gaussian process models with mean and variance defined via (7) and (8), respectively, on the subsets. Then, a central coordinator is employed to collect the individual predictions and aggregate them using schemes of the form

$$\tilde{\mu}(\boldsymbol{x}) = \phi\left(\sum_{n=1}^{N} w_n \boldsymbol{\psi}(\mu_n(\boldsymbol{x}), \sigma_n^2(\boldsymbol{x}))\right), \tag{9}$$

where $\phi : \mathbb{R}^q \to \mathbb{R}$, $\boldsymbol{\psi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^q$ can be arbitrary maps and $w_n \in \mathbb{R}_+$ are weights for the individual predictions. Due to the generality of the aggregation structure (9), it comprises many popular methods such as, e.g., product of experts (PoE) [18]. This aggregation scheme is realized by defining $w_n = 1$ for all $n = 1, \dots, N$ and

$$\boldsymbol{\psi}_{poe}\left(\mu_n(\boldsymbol{x}), \sigma_n^2(\boldsymbol{x})\right) = \begin{bmatrix} \mu_n(\boldsymbol{x})\sigma_n^{-2}(\boldsymbol{x}) \\ \sigma_n^{-2}(\boldsymbol{x}) \end{bmatrix}, \quad \phi_{poe}(\boldsymbol{\psi}) = \frac{\psi_1}{\psi_2}. \tag{10}$$

In order to overcome the weaknesses of such centralized schemes, we employ consensus methods to compute the aggregated prediction in a distributed fashion. This is possible since the argument of $\phi(\cdot)$ in (9) can be expressed as the dynamic average $\bar{\boldsymbol{p}}(t) = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{p}_n(t)$ of the local reference signals

$$\boldsymbol{p}_n(t) = w_n N \boldsymbol{\psi}(\mu_n(\boldsymbol{x}(t)), \sigma_n^2(\boldsymbol{x}(t))) \tag{11}$$

when time-varying inputs $\boldsymbol{x}(t)$ are considered. The dynamic average $\bar{\boldsymbol{p}}(t)$ can straightforwardly be determined in a distributed fashion using consensus algorithms such as the first-order method

$$\dot{\boldsymbol{\xi}}_n = -\kappa_p \sum_{i=1}^{N} a_{ni}(\boldsymbol{\xi}_n - \boldsymbol{\xi}_i) + \dot{\boldsymbol{p}}_n \tag{12a}$$

$$\boldsymbol{\xi}_n(0) = w_n N \boldsymbol{\psi}(\mu_n(\boldsymbol{x}(0)), \sigma_n^2(\boldsymbol{x}(0))), \tag{12b}$$

where $\boldsymbol{\xi}_n(t)$ denotes the local consensus state of agent n and $\kappa_p \in \mathbb{R}_+$ is a constant controlling the consensus convergence rate. We leave the extension to more sophisticated consensus algorithms open for future research. The local consensus state $\boldsymbol{\xi}_n$ provides each agent with a local estimate of the dynamic average consensus value $\bar{\boldsymbol{p}}(t)$. Therefore, each agent can directly approximate the aggregated mean (9) via

$$\hat{\mu}_n(\boldsymbol{x}) = \phi\left(\boldsymbol{\xi}_n\right). \tag{13}$$

Due to the strong theoretical foundation of consensus algorithms, the distributed prediction (13) inherits error bounds from GP regression. For their derivation, we introduce the following assumption.

Assumption 6: Assume the function $h(\cdot)$ with Lipschitz constant L_h is a sample from a Gaussian process $GP\left(0, k\left(\boldsymbol{x}, \boldsymbol{x}'\right)\right)$ with Lipschitz continuous kernel $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{0,+}$, which is used as prior distribution in all agents.

This assumption defines a prior probability distribution over the unknown functions, where the sample space depends on the choice of the kernel. Sample spaces of Gaussian processes are well-studied in the field of machine learning, and it has been shown that the sample

space of a GP with squared exponential kernel contains all analytic functions [24]. The hyperparameters of a kernel typically specify the shape of the corresponding probability distribution, e.g., large length scales l_i in (6) put a higher probability on slowly varying functions.

In addition to the assumption on the prior distribution, the aggregation method needs to satisfy the following consistency condition.

Assumption 7: There exists $\mathrm{w}_n\colon \mathbb{R}^d \to \mathbb{R}_{0,+}$, $\sum_{n=1}^N \mathrm{w}_n(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \mathbb{R}^d$, such that $\tilde{\mu}(\boldsymbol{x}) = \sum_{n=1}^N \mathrm{w}_n(\boldsymbol{x})\mu_n(\boldsymbol{x})$.

Many frequently used aggregation schemes satisfy this assumption. For example, we obtain for the PoE aggregation $\mathrm{w}_n(\boldsymbol{x}) = \sigma_n^{-2}(\boldsymbol{x})/(\sum_{n=1}^N \sigma_n^{-2}(\boldsymbol{x}))$, which implies that Assumption 7 is satisfied. Therefore, Assumption 7 is not restrictive in practice.

Based on these assumptions, we show a uniformly bounded error between the prediction $\hat{\mu}(\cdot)$ and the unknown function $h(\cdot)$.

Proposition 1: Consider an unknown function $h(\cdot)$ and N training data sets $\mathbb{D}_n$ satisfying Assumptions 5 and 6 on the compact set $\Omega \subset \mathbb{R}^d$. Moreover, assume that an aggregation method (9) is employed, which meets the conditions of Assumption 7 and is defined using continuous maps $\phi(\cdot)$ and $\boldsymbol{\psi}(\cdot)$ with Lipschitz constants L_ϕ and L_ψ, respectively. Assume the agents can communicate according to a topology satisfying Assumptions 3 and 4 and choose $\boldsymbol{p}(0) = [\boldsymbol{p}_1^T(0) \ \cdots \ \boldsymbol{p}_N^T(0)]^T$. Pick $\delta \in (0,1)$, $\tau \in \mathbb{R}_+$ such that $\min_{\boldsymbol{x}\in\Omega} \sigma_n^2(\boldsymbol{x}) \geq \gamma_n^2(\tau)/\beta(\tau,\delta)$ $\forall n = 1,\dots,N$ with

$$\beta(\tau,\delta) = 2d\log\left(r_\Omega\sqrt{d}\right) - \log(2\tau) - 2\log(\delta) \tag{14}$$

$$\gamma_n(\tau) = (L_h + L_{\mu_n})\tau + \sqrt{\beta(\tau,\delta)L_{\sigma_n^2}\tau}, \tag{15}$$

where $r_\Omega = \max_{\boldsymbol{x},\boldsymbol{x}'\in\Omega} \|\boldsymbol{x}-\boldsymbol{x}'\|$, L_{μ_n} and $L_{\sigma_n^2}$ are the Lipschitz constants of individual GP mean and variance functions, respectively. If $\boldsymbol{x}(t) \in \Omega$ for all $t \in \mathbb{R}_{0,+}$ and there exists a finite $\Delta \in \mathbb{R}_+$ such that

$$\sup_{t'\geq 0} \|\dot{\boldsymbol{x}}(t')\| < \Delta, \tag{16}$$

then, it holds with probability of at least $1-\delta$ that

$$|h(\boldsymbol{x}(t)) - \hat{\mu}_n(\boldsymbol{x}(t))| \leq \bar{\eta}(\boldsymbol{x}(t),\delta,t,\Delta), \tag{17}$$

for all $n = 1,\dots,N$, $t \in \mathbb{R}_{0,+}$, where

$$\bar{\eta}(\boldsymbol{x}(t),\delta,t,\Delta) = \eta_{\mathrm{GP}}(\boldsymbol{x}(t),\delta) + \eta_{\mathrm{tr}}(t) + \frac{L_{\tilde{\mu}}\Delta}{\tilde{\lambda}_2\kappa_p} \tag{18}$$

$$\eta_{\mathrm{GP}}(\boldsymbol{x},\delta) = 2\sqrt{\beta\left(\tau,\frac{\delta}{N}\right)}\sum_{n=1}^N \mathrm{w}_n(\boldsymbol{x})\sigma_n(\boldsymbol{x}) \tag{19}$$

$$\eta_{\mathrm{tr}}(t) = L_\phi \mathrm{e}^{-\tilde{\lambda}_2\kappa_p t}\left\|(\boldsymbol{I}_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T)\boldsymbol{p}(0)\right\| \tag{20}$$

$$L_{\tilde{\mu}} = L_\phi L_\psi \sqrt{\sum_{n=1}^N L_{\mu_n}^2 + L_{\sigma_n^2}^2}. \tag{21}$$

Proof: In order to prove this theorem, we apply the triangle inequality to the left hand side of (17), such that we obtain

$$|h(\boldsymbol{x}(t)) - \hat{\mu}(\boldsymbol{x}(t))| \leq |h(\boldsymbol{x}(t)) - \tilde{\mu}(\boldsymbol{x}(t))| + |\tilde{\mu}(\boldsymbol{x}(t)) - \phi(\boldsymbol{\xi}_n(t))|.$$

Due to Assumptions 6, 7, and $\min_{\boldsymbol{x}\in\Omega} \sigma_n^2(\boldsymbol{x}) \geq \gamma_n^2(\tau)/\beta(\tau,\delta)$, it follows from [25, Theorem 3.3] that the first summand satisfies

$$|h(\boldsymbol{x}(t)) - \tilde{\mu}(\boldsymbol{x}(t))| \leq \eta_{\mathrm{GP}}(\boldsymbol{x}(t),\delta). \tag{22}$$

By exploiting the Lipschitz continuity of $\phi(\cdot)$, we obtain

$$|\tilde{\mu}(\boldsymbol{x}(t)) - \phi(\boldsymbol{\xi}_n(t))| \leq L_\phi \left\|\boldsymbol{\xi}_n(t) - \sum_{n=1}^N w_n \boldsymbol{\psi}\left(\mu_n(\boldsymbol{x}(t)), \sigma_n^2(\boldsymbol{x}(t))\right)\right\|.$$

This bound corresponds to the consensus error, which is well known to be bounded for connected communication graphs with weight-balanced weighted adjacency matrix $\boldsymbol{A}$, see, e.g., [21]. More precisely, if $\|\dot{\boldsymbol{p}}(t)\| = \|[\dot{\boldsymbol{p}}_1^T(t) \ \cdots \ \dot{\boldsymbol{p}}_N^T(t)]^T\|$ is bounded for all $t \geq 0$, we obtain the consensus error bound

$$\left\|\boldsymbol{\xi}_n(t) - \sum_{n=1}^N w_n \boldsymbol{\psi}\left(\mu_n(\boldsymbol{x}(t)), \sigma_n^2(\boldsymbol{x}(t))\right)\right\| \leq \mathrm{e}^{-\tilde{\lambda}_2\kappa_p t}\left\|(\boldsymbol{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^T)\boldsymbol{p}(0)\right\| + \frac{1}{\tilde{\lambda}_2\kappa_p}\sup_{0\leq t'\leq t}\|\dot{\boldsymbol{p}}(t')\|, \tag{23}$$

where $\tilde{\lambda}_2$ is the smallest nonzero eigenvalue of $\boldsymbol{L}$. This condition is satisfied due to the definition of $\boldsymbol{p}_n(t)$ in (11), the Lipschitz continuity of $\boldsymbol{\psi}(\cdot)$ guaranteed by assumption, the Lipschitz continuity of $\mu_n(\cdot)$ and $\sigma_n^2(\cdot)$ following from the Lipschitz continuous kernel $k(\cdot,\cdot)$ [13], and the assumed boundedness of $\dot{\boldsymbol{x}}(t)$. Therefore, we have $\|\dot{\boldsymbol{p}}(t)\| \leq L_{\tilde{\mu}}\Delta/L_\phi$ for $L_{\tilde{\mu}}$ defined in (21), such that (23) yields

$$|\tilde{\mu}(\boldsymbol{x}(t)) - \phi(\boldsymbol{\xi}_n(t))| \leq \eta_{\mathrm{tr}}(t) + \frac{L_{\tilde{\mu}}\Delta}{\tilde{\lambda}_2\kappa_p}, \tag{24}$$

where $\eta_{\mathrm{tr}}(\cdot)$ is defined in (20). Finally, the summation of (22) and (24) concludes the proof. ∎

The factor $\sqrt{\beta(\tau,\delta/N)}$ in (19) is a common scaling factor for the posterior standard deviations $\sigma_n(\boldsymbol{x})$, and the condition $\min_{\boldsymbol{x}\in\Omega} \sigma_n^2(\boldsymbol{x}) \geq \gamma_n^2(\tau)/\beta(\tau,\delta)$ is necessary to ensure a sufficiently large scaling such that the prediction error bound holds jointly for all $\boldsymbol{x} \in \mathbb{X}$. While global Lipschitz continuity is a restrictive assumption violated, e.g., by the PoE aggregation, local Lipschitz continuity of $\phi(\cdot)$ and $\boldsymbol{\psi}(\cdot,\cdot)$ on their relevant input domains is sufficient for Proposition 1. This property is usually satisfied by commonly used aggregation schemes such as PoE, since the posterior variance $\sigma_n^2(\cdot)$ is guaranteed to be positive and bounded for GPs trained using finite data sets. Therefore, this assumption is not restrictive in practice. Similarly, the assumption on the boundedness of the temporal derivative $\dot{\boldsymbol{x}}(t)$ is not a severe restriction, since it merely requires $\boldsymbol{x}(t)$ to evolve continuously over time. When $\boldsymbol{x}(t)$ is the state of a control system, this assumption crucially depends on the specific control law. We will show in Section IV-B that the proposed cooperative tracking control law guarantees (16), and admits the straightforward computation of the constant Δ.

Based on these unrestrictive assumptions, Proposition 1 decouples the different components of the error bound in an intuitive way. In addition to the error bound resulting from a centralized aggregation $\eta_{\mathrm{GP}}(\cdot,\cdot)$, it considers the transient behavior of the consensus algorithm in $\eta_{\mathrm{tr}}(\cdot)$, and bounds the stationary consensus error using $L_{\tilde{\mu}}\Delta/(\tilde{\lambda}_2\kappa_p)$. While the transient error bound $\eta_{\mathrm{tr}}(\cdot)$ is almost negligible in practice due to the exponential decrease, the stationary error bound $L_{\tilde{\mu}}\Delta/(\tilde{\lambda}_2\kappa_p)$ can become large if Δ is large. In order to reduce this component of the bound, the connectivity of the communication graph $\mathcal{G}$ can be increased, such that $\tilde{\lambda}_2$ grows [1]. Analogously, one can increase the consensus gain κ_p, such that an arbitrarily small stationary consensus error can be guaranteed in principle. Due to [21, Theorem 2], it is in fact straightforward to see that $\bar{\eta}(\boldsymbol{x}(t),\delta,t,\Delta)$ converges to the centralized prediction error bound $\eta_{\mathrm{GP}}(\boldsymbol{x},\delta)$ if $\boldsymbol{x}(t)$ converges to a constant value.

Even though (12) admits a straightforward derivation of error bounds, it is impractical for an implementation as it requires the derivatives $\dot{\boldsymbol{p}}_n$ and consequently $\ddot{\boldsymbol{x}}(t)$, which can be a strong restriction. This can be overcome via a simple change of variables $\boldsymbol{\xi}_n = \boldsymbol{p}_n - \boldsymbol{\zeta}_n$, which directly yields the new consensus system

$$\dot{\boldsymbol{\zeta}}_n = \kappa_p \sum_{i=1}^N a_{ni}(\boldsymbol{p}_n - \boldsymbol{p}_i - \boldsymbol{\zeta}_n + \boldsymbol{\zeta}_i), \tag{25}$$

with initial state $\boldsymbol{\zeta}_n(0) = \mathbf{0}$. Based on the new consensus state $\boldsymbol{\zeta}_n$, the prediction can be computed using $\hat{\mu}_n(\boldsymbol{x}) = \phi\left(\boldsymbol{p}_n - \boldsymbol{\zeta}_n\right)$. Since merely a linear change of variables is employed for adapting the consensus algorithm, it is straightforward to see that Proposition 1 immediately extends to the consensus scheme (25). Thereby, this scheme combines practical applicability and strong theoretical guarantees.

IV. Cooperative Tracking Control using Distributed Gaussian Process Predictions

In order to mitigate the effect of unknown dynamics, we employ the previously introduced distributed prediction approach in a feedback linearizing controller to compensate unmodeled nonlinearities. This cooperative control design is presented in Section IV-A. By exploiting the probabilistic error bounds accompanying the distributed predictions, we derive a tracking error bound for the agents in Section IV-B.

A. Cooperative Tracking Control Design

Before starting with the derivation of the cooperative tracking control law using distributed GP predictions, we represent the multi-agent system in a compact form to simplify notation. The dynamics (1) of all agents can be jointly described by

$$\dot{\boldsymbol{x}}^1 = \boldsymbol{x}^2, \qquad \cdots \qquad \dot{\boldsymbol{x}}^d = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{x}), \tag{26}$$

where $\boldsymbol{x}^j = [x_{1,j} \ \dots \ x_{N,j}]^T$, $j = 1, \dots, d$, $\boldsymbol{x} = [\boldsymbol{x}_1^T \ \dots \ \boldsymbol{x}_N^T]^T$, $\boldsymbol{u} = [u_1 \ \dots \ u_N]^T$, $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}_1) \ \dots \ f_N(\boldsymbol{x}_N)]^T$, $\boldsymbol{G}(\boldsymbol{x}) = \operatorname{diag}\left(g_1(\boldsymbol{x}_1), \dots, g_N(\boldsymbol{x}_N)\right)$, and $\boldsymbol{h}(\boldsymbol{x}) = [h(\boldsymbol{x}_1) \ \dots \ h(\boldsymbol{x}_N)]^T$. Similarly, the leader dynamics (2) can be augmented, which yields

$$\boldsymbol{x}_l^1 = \boldsymbol{x}_r(t), \qquad \boldsymbol{x}_l^2 = \dot{\boldsymbol{x}}_l^1, \qquad \cdots \qquad \boldsymbol{x}_l^d = \dot{\boldsymbol{x}}_l^{d-1} \tag{27}$$

with $\boldsymbol{x}_r(t) = x_r(t)\mathbf{1}_N$, and we define the joint consensus error in the j-th dimension as $\boldsymbol{\varepsilon}^j = \left[\varepsilon_{1,j} \ \varepsilon_{2,j} \ \dots \ \varepsilon_{N,j}\right]^T \in \mathbb{R}^N$. Using this notation, we introduce the filtered state $\boldsymbol{r} = [r_1 \ \cdots \ r_N]^T$ in analogy to standard feedback linearizing control as $\boldsymbol{r} = \boldsymbol{\mathcal{E}}\left[\lambda_1 \ \cdots \ \lambda_{d-1} \ \ 1\right]^T \in \mathbb{R}^N$, where $\boldsymbol{\mathcal{E}} = \left[\boldsymbol{\varepsilon}^1 \ \cdots \ \boldsymbol{\varepsilon}^d\right] \in \mathbb{R}^{N \times d}$ and λ_i, $i = 1, \dots, d-1$, are filter coefficients defining a Hurwitz polynomial. It can easily be checked that the local filtered states r_n can be computed purely based on the local consensus error $\boldsymbol{\varepsilon}_n$, such that we can define the linear control law $\nu(\cdot)$ in (3) as

$$\nu(\boldsymbol{\varepsilon}_n) = \kappa_c r_n + \sum_{i=1}^{d-1} \lambda_i \varepsilon_{n,i+1}, \tag{28}$$

where $\kappa_c \in \mathbb{R}_+$ is a constant control gain. In order to completely specify the feedback linearizing control law (3), it remains to define the compensation $\hat{h}_n$ of the unmodeled nonlinearity $h(\boldsymbol{x}_n)$. For this purpose, we want to employ the distributed GP prediction scheme proposed in Section III-B yielding the control law

$$u_n = \frac{1}{g_n(\boldsymbol{x}_n)}\left(-f_n(\boldsymbol{x}_n) - \phi(\tilde{\boldsymbol{\xi}}_n) + \nu(\boldsymbol{\varepsilon}_n)\right) \tag{29a}$$

$$\dot{\tilde{\boldsymbol{\xi}}}_n = -\kappa_p \sum_{i=1}^{N} a_{ni}(\tilde{\boldsymbol{\xi}}_n - \tilde{\boldsymbol{\xi}}_i) + \dot{\tilde{\boldsymbol{p}}}_n \tag{29b}$$

$$\tilde{\boldsymbol{p}}_n(t) = w_n N \boldsymbol{\psi}(\mu_n(\boldsymbol{x}_n(t)), \sigma_n^2(\boldsymbol{x}_n(t))) \tag{29c}$$

$$\tilde{\boldsymbol{\xi}}_n(0) = w_n N \boldsymbol{\psi}(\mu_n(\boldsymbol{x}_n(0)), \sigma_n^2(\boldsymbol{x}_n(0))). \tag{29d}$$

Remark 2: While all agents employ local models $\mu_n(\cdot)$, $\sigma_n^2(\cdot)$ of the same function $h(\cdot)$ in (29c), each agent computes the prediction for its own local state $\boldsymbol{x}_n(t)$, which is generally not the same for all agents. However, using different states in the local predictions is not an issue, since its effect on the error bound (17) can be straightforwardly bounded as we will show in Lemma 1.

B. Stability Analysis

For proving an ultimately bounded tracking error of the controlled multi-agent system, inspired by [7], we define a Lyapunov function

$$V(\boldsymbol{r}, \boldsymbol{\mathcal{E}}_1) = \frac{1}{2}\boldsymbol{r}^T\boldsymbol{r} + \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\mathcal{E}}_1 \boldsymbol{P} \boldsymbol{\mathcal{E}}_1^T\right), \tag{30}$$

where $\boldsymbol{\mathcal{E}}_1 = \left[\boldsymbol{\varepsilon}^1 \ \boldsymbol{\varepsilon}^2 \ \cdots \ \boldsymbol{\varepsilon}^{d-1}\right] \in \mathbb{R}^{N \times (d-1)}$ and $\boldsymbol{P} \in \mathbb{R}^{(d-1)\times(d-1)}$ is a positive definite, symmetric matrix. The dynamics of the filtered state $\boldsymbol{r}$ can be described by

$$\dot{\boldsymbol{r}} = -\tilde{\boldsymbol{L}}\left(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{x}) - \frac{\mathrm{d}^d}{\mathrm{d}t^d}\boldsymbol{x}_r(t)\right) + \boldsymbol{\rho} \tag{31}$$

where $\boldsymbol{\rho} = [\rho_1 \ \cdots \ \rho_N]^T$, $\rho_n = \sum_{i=1}^{d-1} \lambda_i \varepsilon_{n,i+1}$, and $\tilde{\boldsymbol{L}} = \boldsymbol{L} + \boldsymbol{B}$. From the definition of the filtered state $\boldsymbol{r}$, it follows that

$$\dot{\boldsymbol{\mathcal{E}}}_1 = \boldsymbol{\mathcal{E}}_1 \boldsymbol{\Lambda}^T + \boldsymbol{r}\boldsymbol{l}^T, \tag{32}$$

where $\boldsymbol{l} = [0 \ 0 \ \cdots \ 1]^T \in \mathbb{R}^{d-1}$, and

$$\boldsymbol{\Lambda} = \left[\begin{array}{c|c} \mathbf{0}_{(d-2)\times 1} & \boldsymbol{I}_{d-2} \\ \hline -\lambda_1 & -\lambda_2 \cdots - \lambda_{d-1} \end{array}\right]. \tag{33}$$

Since the second summand of the Lyapunov function (30) depends on $\boldsymbol{\mathcal{E}}_1$, (32) allows us to relate the decrease of the Lyapunov function along system trajectories to the weights of the linear control law (28) described by $\boldsymbol{\Lambda}$. Analogously to the augmented Laplacian $\tilde{\boldsymbol{L}}$, the matrix $\boldsymbol{\Lambda}$ can be employed to define a positive definite matrix. For this definition, we employ the fact that $\boldsymbol{\Lambda}$ is a Hurwitz matrix, such that $\boldsymbol{P}$ is chosen to be the solution to the Lyapunov equation

$$\boldsymbol{\Lambda}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{\Lambda} = -\boldsymbol{Q}, \tag{34}$$

where $\boldsymbol{Q}$ is an arbitrary, positive definite, symmetric matrix.

Before starting with the stability analysis of the multi-agent system controlled by (29), we introduce the following lemma, which extends the prediction error bound in Proposition 1 by expressing the impact of the usage of local states $\boldsymbol{x}_n$ in (29c) in terms of $\boldsymbol{r}$ and $\boldsymbol{\mathcal{E}}_1$.

Lemma 1: Consider an unknown function $h(\cdot)$ and N training data sets $\mathbb{D}_n$ satisfying Assumptions 5 and 6 on the compact set $\Omega \subset \mathbb{R}^d$. Moreover, assume that an aggregation method (9) is employed, which meets the conditions of Assumption 7 and is defined using maps $\phi(\cdot)$, $\boldsymbol{\psi}(\cdot)$ with Lipschitz constants L_ϕ, L_ψ, respectively. Assume the agents can communicate according to a topology satisfying Assumptions 3 and 4. If there exists a finite $\tilde{\Delta} \in \mathbb{R}_+$ such that

$$\max_{n=1,\dots,N} \sup_{t' \geq 0} \|\dot{\boldsymbol{x}}_n(t')\| < \tilde{\Delta}, \tag{35}$$

then it holds with probability of at least $1 - \delta$ for all $t \in \mathbb{R}_{0,+}$, $\boldsymbol{x} \in \Omega^N = \Omega \times \cdots \times \Omega$ that

$$\begin{aligned}\|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{\phi}(\tilde{\boldsymbol{\xi}}(t))\| \leq \sqrt{N}\bar{\eta}(\boldsymbol{x}_l(t), \delta, t, \tilde{\Delta}) + \frac{\sqrt{N}L_{\tilde{\mu}} + L_h}{\underline{\sigma}(\tilde{\boldsymbol{L}})}\|\boldsymbol{r}(t)\| \\ + \frac{\sqrt{N}L_{\tilde{\mu}} + L_h}{\underline{\sigma}(\tilde{\boldsymbol{L}})}(1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}})\|\boldsymbol{\mathcal{E}}_1(t)\|_{\mathrm{F}},\end{aligned} \tag{36}$$

where $\boldsymbol{\phi}(\tilde{\boldsymbol{\xi}}(t)) = \left[\phi(\tilde{\boldsymbol{\xi}}_1^T(t)) \ \cdots \ \phi(\tilde{\boldsymbol{\xi}}_N^T(t))\right]^T$.

Proof: In order to prove this lemma, we first bound the difference between $\phi(\tilde{\boldsymbol{\xi}}_n(t))$ and the corresponding centralized aggregation, which yields analogously to Proposition 1

$$\left|\phi(\tilde{\boldsymbol{\xi}}_n(t)) - \breve{\mu}_n(\boldsymbol{x})\right| \leq \eta_{\mathrm{tr}}(t) + \frac{L_{\tilde{\mu}}\tilde{\Delta}}{\tilde{\lambda}_2 \kappa_p}, \tag{37}$$

where

$$\breve{\mu}_n(\boldsymbol{x}) = \phi\left(\sum_{n=1}^{N} w_n \boldsymbol{\psi}\left(\mu_n(\boldsymbol{x}_n(t)), \sigma_n^2(\boldsymbol{x}_n(t))\right)\right). \tag{38}$$

Due to Lipschitz continuity of $\phi(\cdot)$, $\boldsymbol{\psi}(\cdot,\cdot)$, $\mu_n(\cdot)$ and $\sigma_n^2(\cdot)$, we can bound the distance to the aggregated prediction for $\boldsymbol{x}_l$ by

$$|\check{\mu}_n(\boldsymbol{x}) - \tilde{\mu}(\boldsymbol{x}_l)|^2 \leq L_\phi^2 L_\psi^2 \sum_{i=1}^{N} (L_{\mu_i}^2 + L_{\sigma_i^2}^2) \|\boldsymbol{x}_l - \boldsymbol{x}_i\|^2. \quad (39)$$

For obtaining a bound in terms of $\boldsymbol{\mathcal{E}}(t)$ and $\boldsymbol{r}(t)$ we make use of the Cauchy-Schwarz and the triangle inequality to get

$$\|\check{\boldsymbol{\mu}}(\boldsymbol{x}) - \tilde{\boldsymbol{\mu}}(\boldsymbol{x}_l)\|^2 \leq N L_{\tilde{\mu}}^2 \sum_{i=1}^{N} \|\boldsymbol{x}_l - \boldsymbol{x}_i\|^2, \quad (40)$$

where $\check{\boldsymbol{\mu}}(\boldsymbol{x})$ and $\tilde{\boldsymbol{\mu}}(\boldsymbol{x}_l)$ are the concatenations of $\check{\mu}_n(\boldsymbol{x})$ and $\tilde{\mu}(\boldsymbol{x}_l)$, respectively. By reordering the summands, it can be seen that $\sum_{i=1}^{N} \|\boldsymbol{x}_l - \boldsymbol{x}_i\|^2 = \sum_{i=1}^{d} \|\boldsymbol{x}_l^i - \boldsymbol{x}^i\|^2$, such that we have

$$\|\check{\boldsymbol{\mu}}(\boldsymbol{x}) - \tilde{\boldsymbol{\mu}}(\boldsymbol{x}_l)\| \leq \frac{\sqrt{N} L_{\tilde{\mu}}}{\underline{\sigma}(\tilde{\boldsymbol{L}})} \left(\|\boldsymbol{r}\| + (1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}}) \|\boldsymbol{\mathcal{E}}_1\|_{\mathrm{F}} \right) \quad (41)$$

due to $\boldsymbol{\varepsilon}^j = -\tilde{\boldsymbol{L}}(\boldsymbol{x}^j - \boldsymbol{x}_l^j)$, $\sum_{i=1}^{d} \|\boldsymbol{\varepsilon}^i\|^2 = \|\boldsymbol{\varepsilon}^1\|^2 + \|\dot{\boldsymbol{\mathcal{E}}}_1\|_{\mathrm{F}}^2$ and (32). By applying the triangle inequality, we finally obtain

$$\|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{\phi}(\tilde{\boldsymbol{\xi}}(t))\| \leq \|\boldsymbol{h}(\boldsymbol{x}) - \tilde{\boldsymbol{\mu}}(\boldsymbol{x}_l)\| + \sqrt{N}\eta_{\mathrm{tr}}(t) + \frac{\sqrt{N} L_{\tilde{\mu}} \tilde{\Delta}}{\tilde{\lambda}_2 \kappa_p} + \frac{\sqrt{N} L_{\tilde{\mu}}}{\underline{\sigma}(\tilde{\boldsymbol{L}})} \left(\|\boldsymbol{r}\| + (1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}}) \|\boldsymbol{\mathcal{E}}_1\|_{\mathrm{F}} \right), \quad (42)$$

which directly yields the result due to [25, Theorem 3.3]. ∎

Lemma 1 theoretically justifies our approach of employing the local agent states for computing the local predictions: the smaller the consensus error is, the lower is the impact on the prediction error bound with local states. This is particularly beneficial since the feedback control $\nu(\cdot)$ dominates the control law (29) for large consensus errors regardless of the prediction error, such that a small consensus error can typically be ensured. In fact, due to the linear relationship between consensus error and prediction error bound, it is even possible to recover Proposition 1 in the limit of vanishing consensus errors.

Note that Lemma 1 requires a bound $\tilde{\Delta}$ for the agents' state derivative, which depends on the control law (29) and consequently on the distributed predictions. Due to the boundedness of GP predictions, we can provide a closed-form bound $\tilde{\Delta}$ as shown in the following lemma.

Lemma 2: Consider a multi-agent system (1) with N agents satisfying Assumption 1 and an unmodeled nonlinearity $h(\cdot)$ satisfying Assumption 6. Consider a leader of the form (2). Moreover, assume that a control law (29) with a distributed prediction is employed, which meets the conditions of Assumption 7 and is defined using continuous maps $\phi(\cdot)$ and $\boldsymbol{\psi}(\cdot)$ with Lipschitz constants L_ϕ and L_ψ, respectively. If the state of every agent remains in a compact set Ω for all $t \geq 0$, then we have

$$\max_{n=1,\dots,N} \sup_{t' \geq 0} \|\dot{\boldsymbol{x}}_n(t')\| \leq \tilde{\Delta} \quad (43)$$

for

$$\tilde{\Delta} = \bar{h} + \bar{\mu} + r_\Omega \left(1 + (\kappa_c + 1) \sqrt{\|\boldsymbol{\Lambda}\|_F^2 - d + 3} \right) \quad (44)$$

where $\bar{h} = \max_{\boldsymbol{x} \in \Omega} |h(\boldsymbol{x})|$, and $\bar{\mu} = \max_{\boldsymbol{z} \in (\Psi_1 \times \cdots \times \Psi_q)} |\phi(\boldsymbol{z})|$ for bounded sets $\Psi_i = \{z \in \mathbb{R} \colon z = \psi_i(\mu_n(\boldsymbol{x}), \sigma_n^2(\boldsymbol{x})) \text{ for } \boldsymbol{x} \in \Omega\}$.

Proof: By substituting the control law (29a) into (1), we obtain $\dot{x}_{n,d} = \beth_n(\boldsymbol{x}) + \nu(\boldsymbol{\varepsilon}_n)$, where $\beth(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}(t)) - \boldsymbol{\phi}(\tilde{\boldsymbol{\xi}}(t))$. Due to the compactness of Ω and the linearity of $\nu(\cdot)$ in $\boldsymbol{\varepsilon}_n$, we have

$$|\nu(\boldsymbol{\varepsilon}_n)| \leq r_\Omega (\kappa_c + 1) \sqrt{\|\boldsymbol{\Lambda}\|_F^2 - d + 3}. \quad (45)$$

Moreover, $\boldsymbol{\psi}(\cdot,\cdot)$ is bounded on the compact set Ω due to Lipschitz continuity, such that $\boldsymbol{\xi}_n \in (\Psi_1 \times \cdots \times \Psi_q)$ is guaranteed. Due to Lipschitz continuity of $\boldsymbol{\psi}(\cdot)$, the sets Ψ_i are bounded, such that continuity of $\phi(\cdot)$ implies the existence of a $\bar{\mu}$, which upper bounds $\phi(\boldsymbol{\xi}_n(t))$. Due to continuity of $h(\cdot)$, there exists a finite upper bound $\bar{h}$ on the compact set Ω. Hence, it directly follows that $|\beth_n(\boldsymbol{x})| \leq \bar{h} + \bar{\mu}$. Exploiting the compactness of Ω one more time, we finally obtain $\sup_{t \geq 0} \|\dot{\boldsymbol{x}}_n\| \leq \tilde{\Delta}$ for $\tilde{\Delta}$ defined in (44), such that Lemma 1 holds. ∎

Since the derivative state not only depends on the distributed prediction, but also on the linear control law $\nu(\cdot)$, the bound (44) depends linearly on the control gain κ_c. This introduces a linear dependency of the prediction error bound in Lemma 1 on κ_c due to (18), which can be compensated by the reciprocal dependency on the prediction gain κ_p. Therefore, the prediction gain κ_p should be chosen greater than the control gain κ_c, which intuitively resembles the fact that convergence of the prediction is necessary before a consensus of the agent states can be achieved.

Using these auxiliary results, we quantify the tracking error bound for multi-agent systems controlled by (29) as shown in the following.

Theorem 1: Consider a multi-agent system (1) with N agents and a leader (2) satisfying Assumptions 1 and 2, which are connected via a communication topology, such that Assumptions 3-4 hold. Assume that the unmodeled nonlinearity $h(\cdot)$ and N training data sets satisfy Assumptions 5 and 6. Moreover, assume that a control law (29) with distributed prediction is employed, which meets the conditions of Assumption 7 and is defined using maps $\phi(\cdot)$, $\boldsymbol{\psi}(\cdot)$ with Lipschitz constants L_ϕ, L_ψ. Choose a control gain $\kappa_c \in \mathbb{R}_+$ such that $\boldsymbol{\Upsilon} \succ \boldsymbol{0}$, where

$$\boldsymbol{\Upsilon} = \begin{bmatrix} \kappa_c \underline{\sigma}(\tilde{\boldsymbol{L}}) - \iota & -\frac{\iota}{2}(1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}}) - \frac{1}{2}\bar{\sigma}(\boldsymbol{P}) \\ -\frac{\iota}{2}(1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}}) - \frac{1}{2}\bar{\sigma}(\boldsymbol{P}) & \frac{1}{2}\underline{\sigma}(\boldsymbol{Q}) \end{bmatrix} \quad (46)$$

$$\iota = \bar{\sigma}(\boldsymbol{I}_N - \tilde{\boldsymbol{L}}) \|\boldsymbol{\lambda}\| + \frac{(\sqrt{N} L_{\tilde{\mu}} + L_h) \bar{\sigma}(\tilde{\boldsymbol{L}})}{\underline{\sigma}(\tilde{\boldsymbol{L}})} \quad (47)$$

for $\boldsymbol{P}$, $\boldsymbol{Q}$ defined in (34) such that $\underline{\sigma}(\boldsymbol{P}) \geq 1$. Moreover, choose a compact set $\Omega \subset \mathbb{R}^d$ such that

$$\mathbb{B}(t) = \left\{ \boldsymbol{x} \in \mathbb{R}^d \colon \|\boldsymbol{x} - \boldsymbol{x}_l(t)\| \leq \upsilon(t) \right\} \subset \Omega \quad \forall t \geq 0. \quad (48)$$

then, with probability of at least $1 - \delta$, the tracking error $\boldsymbol{e} = [\boldsymbol{e}_1^T \ \cdots \ \boldsymbol{e}_N^T]^T$ is ultimately bounded with the ultimate bound

$$\upsilon(t) = \frac{\sqrt{N} \bar{\sigma}(\tilde{\boldsymbol{L}}) \sqrt{\bar{\sigma}(\boldsymbol{P})} (1 + \|\boldsymbol{\Lambda}\|_{\mathrm{F}})}{\underline{\sigma}(\boldsymbol{\Upsilon}) \underline{\sigma}(\tilde{\boldsymbol{L}})} \left(\bar{\eta}(\boldsymbol{x}_l, \delta, t, \tilde{\Delta}) + \hat{\Delta} \right). \quad (49)$$

Proof: We prove this theorem by showing that the temporal derivative of the Lyapunov function (30) is decreasing except for a small ball around $\boldsymbol{\varepsilon} = \boldsymbol{0}$. Let $V_1(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{r}^T\boldsymbol{r}$ denote the first summand of (30). Due to (31), the temporal derivative of $V_1(\cdot)$ is given by

$$\dot{V}_1 = \boldsymbol{r}^T \Big(-\tilde{\boldsymbol{L}} \Big(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{h}(\boldsymbol{x}) - \dot{\boldsymbol{x}}_l^d \Big) + \boldsymbol{\rho} \Big). \quad (50)$$

Substituting the control law (29a) yields

$$\dot{V}_1(\boldsymbol{r}) = -\kappa_c \boldsymbol{r}^T \tilde{\boldsymbol{L}} \boldsymbol{r} - \boldsymbol{r}^T \tilde{\boldsymbol{L}} \Big(\beth(\boldsymbol{x}) - \dot{\boldsymbol{x}}_l^d \Big) + \boldsymbol{r}^T \left(\boldsymbol{I}_N - \tilde{\boldsymbol{L}} \right) \boldsymbol{\rho}. \quad (51)$$

It can be easily seen that $\boldsymbol{\rho} = \dot{\boldsymbol{\mathcal{E}}}_1 \boldsymbol{\lambda}$ with $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_{d-1}]^T$. Therefore, we have due to Eq. (32) that

$$\boldsymbol{r}^T \left(\boldsymbol{I}_N - \tilde{\boldsymbol{L}} \right) \boldsymbol{\rho} = \boldsymbol{r}^T \left(\boldsymbol{I}_N - \tilde{\boldsymbol{L}} \right) \left(\boldsymbol{r}\boldsymbol{l}^T + \boldsymbol{\mathcal{E}}_1 \Lambda^T \right) \boldsymbol{\lambda}, \quad (52)$$

such that we can bound $\dot{V}_1(\boldsymbol{r})$ by

$$\begin{aligned} \dot{V}_1(\boldsymbol{r}) \leq & \left(-\kappa_c \underline{\sigma}(\tilde{\boldsymbol{L}}) + \bar{\sigma}(\boldsymbol{I}_N - \tilde{\boldsymbol{L}}) \|\boldsymbol{\lambda}\| \right) \|\boldsymbol{r}\|^2 \\ & + \bar{\sigma}(\boldsymbol{I}_N - \tilde{\boldsymbol{L}}) \|\boldsymbol{\lambda}\| \|\boldsymbol{\Lambda}\|_F \|\boldsymbol{r}\| \|\boldsymbol{\mathcal{E}}_1\|_F \\ & + \bar{\sigma}(\tilde{\boldsymbol{L}}) \left(\|\beth(\boldsymbol{x})\| + \left\| \dot{\boldsymbol{x}}_l^d \right\| \right) \|\boldsymbol{r}\|. \end{aligned} \quad (53)$$

Boundedness of $\|\dot{\boldsymbol{x}}_l^d\|$ holds by Assumption 2, such that $\|\dot{\boldsymbol{x}}_l^d\| \leq \sqrt{N}\hat{\Delta}$. For bounding $\|\boldsymbol{\beth}(\boldsymbol{x})\|$ we employ Lemma 1 and Lemma 2, such that we can substitute (36) in (53) to get

$$\dot{V}_1(\boldsymbol{r}) \leq \left(-\kappa_c \underline{\sigma}(\tilde{\boldsymbol{L}}) + \iota\right) \|\boldsymbol{r}\|^2 + \iota(1+\|\boldsymbol{\Lambda}\|_{\mathrm{F}})\|\boldsymbol{r}\|\|\boldsymbol{\mathcal{E}}_1\|_F + \bar{\sigma}(\tilde{\boldsymbol{L}})\sqrt{N}\left(\bar{\eta}(\boldsymbol{x}_l(t),\delta,t,\tilde{\Delta})\|+\hat{\Delta}\right)\|\boldsymbol{r}\|, \tag{54}$$

where ι is defined in (47). For bounding the temporal derivative of the second summand of (30) denoted as $V_2(\boldsymbol{\mathcal{E}}_1) = \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\mathcal{E}}_1 \boldsymbol{P} \boldsymbol{\mathcal{E}}_1^T\right)$ we proceed similarly, such that it follows from (32) and (34) that

$$\dot{V}_2(\boldsymbol{\mathcal{E}}_1) \leq -\frac{1}{2}\underline{\sigma}(\boldsymbol{Q})\|\boldsymbol{\mathcal{E}}_1\|_F^2 + \bar{\sigma}(\boldsymbol{P})\|\boldsymbol{r}\|\|\boldsymbol{\mathcal{E}}_1\|_F. \tag{55}$$

Combining (54) and (55), and writing it in a quadratic form yields

$$\dot{V}(\boldsymbol{r},\boldsymbol{\mathcal{E}}_1) \leq -\begin{bmatrix}\|\boldsymbol{r}\| & \|\boldsymbol{\mathcal{E}}_1\|_F\end{bmatrix} \boldsymbol{\Upsilon} \begin{bmatrix}\|\boldsymbol{r}\| \\ \|\boldsymbol{\mathcal{E}}_1\|_F\end{bmatrix} + \tilde{\boldsymbol{\upsilon}} \begin{bmatrix}\|\boldsymbol{r}\| \\ \|\boldsymbol{\mathcal{E}}_1\|_F\end{bmatrix} \tag{56}$$

where

$$\tilde{\boldsymbol{\upsilon}}(t) = \begin{bmatrix}\sqrt{N}\bar{\sigma}(\tilde{\boldsymbol{L}})\left(\bar{\eta}(\boldsymbol{x}(t),\delta,t,\tilde{\Delta})+\hat{\Delta}\right) & 0\end{bmatrix} \tag{57}$$

and $\boldsymbol{\Upsilon}$ is defined in (46). By employing Sylvester's criterion, it is straightforward to see that there exist a κ_c, such that $\boldsymbol{\Upsilon}$ is positive definite. Then, we have

$$\dot{V}(\boldsymbol{r},\boldsymbol{\mathcal{E}}_1) \leq 0, \quad \text{for all } \boldsymbol{r},\boldsymbol{\mathcal{E}}_1 : \left\|\begin{bmatrix}\|\boldsymbol{r}\| & \|\boldsymbol{\mathcal{E}}_1\|_F\end{bmatrix}\right\| \leq \frac{\|\tilde{\boldsymbol{\upsilon}}(t)\|}{\underline{\sigma}(\boldsymbol{\Upsilon})}. \tag{58}$$

As $V(\cdot)$ has a quadratic form and $\underline{\sigma}(\boldsymbol{P}) \geq 1$, (58) implies that

$$\left\|\begin{bmatrix}\|\boldsymbol{r}(t)\| & \|\boldsymbol{\mathcal{E}}_1(t)\|_F\end{bmatrix}\right\| \leq \frac{\sqrt{\bar{\sigma}(\boldsymbol{P})}\|\tilde{\boldsymbol{\upsilon}}(t)\|}{\underline{\sigma}(\boldsymbol{\Upsilon})}. \tag{59}$$

Due to $\sum_{i=1}^d \|\boldsymbol{\varepsilon}^i\|^2 = \|\boldsymbol{\varepsilon}^1\|^2 + \|\dot{\boldsymbol{\mathcal{E}}}_1\|_{\mathrm{F}}^2$ and (32), the total consensus error can be bounded in terms of $\boldsymbol{r}$ and $\boldsymbol{\mathcal{E}}_1$, which leads to

$$\|\boldsymbol{\varepsilon}(t)\| \leq \frac{\sqrt{\bar{\sigma}(\boldsymbol{P})}(1+\|\boldsymbol{\Lambda}\|_F)\|\tilde{\boldsymbol{\upsilon}}(t)\|}{\underline{\sigma}(\boldsymbol{\Upsilon})}. \tag{60}$$

Finally, the cooperative tracking error $\boldsymbol{\varepsilon}$ satisfies the identity $\boldsymbol{\varepsilon}^i = -\tilde{\boldsymbol{L}}(\boldsymbol{x}^i - \boldsymbol{x}_l^i)$, such that $\|\boldsymbol{x} - \boldsymbol{x}_l\| \leq \|\boldsymbol{\varepsilon}(t)\|/\underline{\sigma}(\tilde{\boldsymbol{L}})$. ■

Theorem 1 exhibits several intuitive properties. A high connectivity of the graph $\mathcal{G}$ augmented by the leader node, as measured through the singular values of $\tilde{\boldsymbol{L}}$, implies comparatively large singular values of $\boldsymbol{\Upsilon}$. This has the effect that a high connectivity allows a lower control gain κ_c for ensuring positive definiteness of $\boldsymbol{\Upsilon}$. Moreover, since the ultimate tracking error bound (49) is reciprocal to $\underline{\sigma}(\boldsymbol{\Upsilon})$, it can be reduced by increasing the connectivity of the graph $\mathcal{G}$ augmented by the leader node. In addition to an increase in connectivity, a large control gain is the main parameter for guaranteeing small tracking errors. In fact, arbitrarily small ultimate tracking error bounds $\upsilon(t)$ can be achieved through a suitable value of κ_c. In order to see this, note that $\underline{\sigma}(\boldsymbol{Q})$ can be chosen arbitrarily large as positive definiteness of $\boldsymbol{\Upsilon}$ can always be ensured through a suitable control gain κ_c. Moreover, it is trivial to check that $\underline{\sigma}(\boldsymbol{\Upsilon}) \to \frac{1}{2}\underline{\sigma}(\boldsymbol{Q})$ for $\kappa_c \to \infty$. Since the ultimate bound is reciprocal to $\underline{\sigma}(\boldsymbol{\Upsilon})$, this implies that arbitrarily small tracking errors can be guaranteed.

V. Numerical Evaluation

We evaluate our proposed approach in two simulations. In Section V-A, we illustrate the prediction errors and error bounds of the proposed distributed GP approach. In Section V-B, we demonstrate the effectiveness of employing distributed predictions in a cooperative control scheme applied to a system with unmodeled nonlinearities.

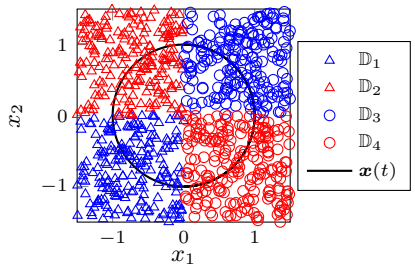

Fig. 1. Each Gaussian process is trained with data $\mathbb{D}_n$ from a single quadrant, while the trajectory $\boldsymbol{x}(t)$ passes through all quadrants.

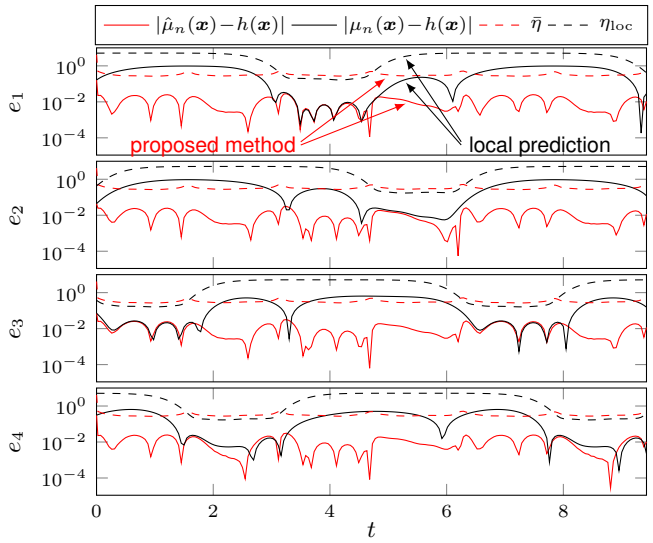

Fig. 2. The prediction errors $|\hat{\boldsymbol{\mu}}_{\boldsymbol{n}}(\boldsymbol{x}) - \boldsymbol{h}(\boldsymbol{x})|$ observed when using the proposed distributed prediction aggregation method (red full line) are small for the whole trajectory $\boldsymbol{x}(t)$, while the errors $|\boldsymbol{\mu}_{\boldsymbol{n}}(\boldsymbol{x}) - \boldsymbol{h}(\boldsymbol{x})|$ resulting from the standard approach based on local predictions (black full line) are only small for states $\boldsymbol{x}(t)$ close to the training data of the corresponding model. The theoretical error bounds for distributed predictions $\bar{\boldsymbol{\eta}}$ (red dashed line) and individual predictions $\boldsymbol{\eta}_{\mathbf{loc}}$ (black dashed line) exhibit the same behavior.

A. Distributed Predictions for Dynamical Systems

In this section, we investigate the performance of the distributed GP approach proposed in (12), (13) for learning the nonlinear function

$$h(\boldsymbol{x}) = \sin(x_1) + \frac{1}{2(1+\exp(\frac{x_2}{10}))}. \tag{61}$$

We consider a system with $N = 4$ GP models, each of which is based on a squared exponential kernel (6) with signal standard deviation $\sigma_f = 0.5$ and length scales $l_i = 0.5$, $i = 1, 2$. Training data is uniformly distributed on the domain $[-1, 1]\times[-1, 1]$, and training targets are perturbed by zero mean Gaussian noise with $\sigma_o = 0.1$. Each of the GP models is trained using $M = 500$ training samples from a single quadrant, as illustrated in Fig. 1. For the distributed aggregation of the predictions, we employ the PoE scheme (10), and assume a circular communication graph described by the adjacency matrix

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}. \tag{62}$$

The predictions are aggregated along the reference trajectory $\boldsymbol{x}(t) = \begin{bmatrix}x_r(t) & \dot{x}_r(t)\end{bmatrix}$ with $x_r(t) = \sin(t)$ using a prediction consensus gain $\kappa_p = 1000$. The prediction error is uniformly bounded with $\delta = 0.1$ and $\tau = 0.01$, and the Lipschitz constants for $\phi(\cdot)$ and $\boldsymbol{\psi}(\cdot,\cdot)$ required for (18) are numerically approximated along $\boldsymbol{x}(t)$.

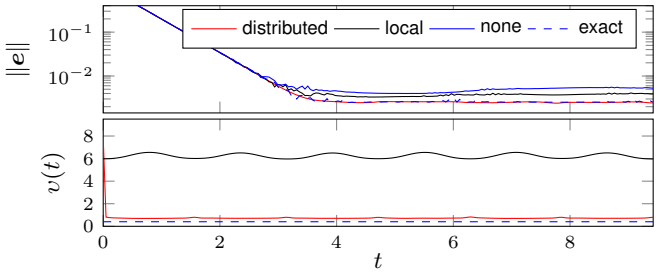

Fig. 3. Tracking errors $\|e\|$ (top) and ultimate error bounds $\upsilon(t)$ (bottom) for cooperative control with distributed learning are significantly smaller than with local learning and without learning.

We compare the prediction error $|\hat{\mu}_n(\boldsymbol{x}) - h(\boldsymbol{x})|$ and error bound $\bar{\eta}$ of the proposed method to the prediction error $|\mu_n(\boldsymbol{x}) - h(\boldsymbol{x})|$ and error bounds η_{loc} of the standard approach employing only local predictions. As depicted in Fig. 2, the distributed predictions yield errors, which are almost identical to the best available individual prediction. While the individual predictions achieve these small errors only for a small period of time, the distributed predictions constantly maintain this high accuracy. A similar behavior can be observed for the uniform prediction error bounds, where the distributed GP approach guarantees an almost constant prediction accuracy, while the local error bounds strongly vary over time. This clearly demonstrates the advantages of distributed predictions.

B. Cooperative Tracking Control with Unmodeled Nonlinearities

In order to demonstrate the efficiency of the cooperative control law proposed in (29), we extend the previously introduced simulation setting to the control of a multi-agent system. For this purpose, we define $f_n(\boldsymbol{x}) = 0$ and $g_n(\boldsymbol{x}) = 1$ for $n = 1, \dots, 4$ and $\boldsymbol{x} \in \mathbb{R}^2$. Moreover, we choose the diagonal matrix $\boldsymbol{B} = \text{diag}(1, 0, 1, 0)$, such that two follower agents are connected to the leader. In the feedback linearizing control law, we employ $\lambda = \frac{7}{4}$, $\kappa_c = 1000$, and use $\kappa_p = 50000$ as gain for the prediction consensus. Finally, we determine the ultimate tracking error bound $\upsilon(t)$ based on $q = 600$.

We compare the simulation results of our proposed cooperative tracking control law employing distributed GP predictions to the same control law using only the individual predictions of the agent, no compensation of the unknown nonlinearity, and an exact model of $h(\cdot)$. As illustrated in Fig. 3, employing only the local model of each agent in the control law yields an improvement compared to the absence of any model. However, it performs significantly worse then the proposed control law with distributed predictions, which even achieves tracking errors almost identical to those of the control law with exact knowledge of $h(\cdot)$. Even though the corresponding ultimate tracking error bounds $\upsilon(t)$, which can be obtained for local predictions and exact model knowledge through a straightforward adaptation of [7], are conservative, they analogously reflect this behavior. These results underline the improvement in control performance resulting from the application of distributed predictions in cooperative tracking control.

VI. Conclusion

This article proposes a novel cooperative tracking control law for leader-follower consensus of multi-agent systems with partially unknown, higher-order, control-affine dynamics. In order to compensate unmodeled nonlinearities with feedforward control, we derive a method for the distributed inference based on Gaussian process regression. We show probabilistic regression error bounds for the proposed distributed learning method, such that we can analyze the stability of the cooperative control law with distributed Gaussian process predictions using Lyapunov theory. The effectiveness of the proposed methods is demonstrated in simulations.

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