Trajectory tracking control of a four mecanum wheeled mobile platform: an extended state observer-based sliding mode approach

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\textbf{Abstract:} This study proposes an extended state observer-based sliding mode control (ESO-SMC) strategy for trajectory tracking of a four mecanum wheeled mobile platform (FMWMP) with unknown disturbances and model uncertainties (UDMU) considered. Especially, the extended state observer (ESO) is designed to estimate not only the UDMU but also the unmeasured velocities of FMWMP. Based on the designed ESO, a sliding mode control (SMC) scheme is utilised to ensure the tracking performance as expected. By using Lyapunov synthesis, it is shown that all the signals of the whole system can be guaranteed to be uniformly ultimately bounded. To verify the effectiveness of the proposed control strategy, simulations and experiments are carried out with two different kinds of reference trajectories. Furthermore, a comparative work is done to show that the ESO-SMC controller has better control performance than traditional proportional–integral–derivative controller.

\section{1 Introduction}

Recently, with the increasing demands for autonomous manoeuvrability, wheeled mobile robots have attracted much attention and been broadly applied in many aspects of our society [1–4]. Owing to their enhanced mobility compared with conventional mobile robots, omnidirectional wheeled mobile robots, which are capable to move in confined spaces easily, have been extensively investigated [5–7]. Four mecanum wheeled mobile platform (FMWMP) is one kind of the omnidirectional mobile robot consisting of a rectangular configuration and four mecanum wheels. As shown in Fig. 1, the mecanum wheel is actually a conventional wheel with a series passive rollers attached to its circumference. These rollers normally have a axis of rotation at 45° to the axis of rotation of the wheel. Due to this special structure of mecanum wheels, FMWMP have an extra degree of freedom with respect to conventional differential-driven wheeled robots, i.e. FMWMP can move in any direction without reorientation.

As the manufacturing industry developed rapidly these years, mobile manipulator has become a widespread term for human assistance [8]. To enhance the mobility and flexibility of a mobile manipulator, some researchers intend to design an FMWMP-based manipulator to accomplish tele-operation tasks [9, 10]. It should be stressed that in such situations, trajectory tracking accuracy of FMWMP is the key point to ensure the performance of tele-operation. Thus, a high-precision controller which guarantees the FMWMP tracking the desired trajectory accurately is required. However, in practice, sometimes the FMWMP may enter the region of non-linear behaviours because of the unknown disturbances and model uncertainties (UDMU) [11], leading the FMWMP system to unstable. Therefore, there are still lots of challenges to propose a high-accuracy control strategy for FMWMP.

To overcome the control difficulties mentioned above, first, the kinematic and dynamic models of an FMWMP have been investigated in several literature [12–14]. Since FMWMP is a complex non-linear system, non-linear control strategies, such as fuzzy control, backstepping control and sliding mode control (SMC), perform better than traditional linear control strategy like proportional–integral–derivative (PID) controller [15–17]. In [15], a fuzzy wavelet networks approach is proposed to approximate some uncertain non-linear terms in controller design; in [16], a backstepping controller combining Neural-Networks approximation is investigated and applied for FMWMP system; in [17], in face of the external disturbances, SMC is presented to deal with the disturbances. It should be noted that although those non-linear strategies have been implemented on FMWMP, only the simulation results show the effectiveness and none of them have real experimental validation.

Owing to its insensitivity to disturbances, SMC has been gaining more and more attention for suppressing disturbances in complex non-linear systems [18–24]. It has been extensively employed in the controller design of mobile robots [25–27]. The main disadvantage of SMC is so-called chattering phenomenon, caused by discontinuity of the control law, which not only wastes energy but also reduces the trajectory smoothness [28]. Several methods, such as higher order SMC [29] and adaptive SMC [30] have been proposed to suppress chattering. Generally speaking, the UDMU are highly uncertain since the FMWMP usually works exposed to a dynamic environment. The SMC scheme may bring serious chattering if the lumped disturbances/uncertainties term goes too large, even leads the FMWMP system to unstable.

One approach to realise UDMU rejection is to design a disturbance-observer (DOB) estimating the UDMU, followed by the SMC controller design to compensate the UDMU [31, 32]. Such DOBs generally include high-gain observer (HGO) [33], sliding mode observer [34] and extended state observer (ESO) [35] etc. It is worth mentioning that, unlike traditional observers, ESO is actually one kind of state observer with the lumped disturbances/
Based on the estimations, an SMC controller is proposed to ensure the trajectory tracking errors to be uniformly ultimately bounded (UUB). To validate the effectiveness of the proposed control strategy, comparative studies with PID controller are carried out.

The rest of this paper is organised as follows: Section 2 presents the mathematical model of FMWMP. In Section 3, an ESO-SMC controller is designed to ensure all the signals of FMWMP system to be UUB. Sections 4 and 5 show the simulation and experiment results, respectively, followed by the conclusion in Section 6.

Nomenclature: Throughout this paper, some symbols representing constants or physical quantities are shown in Table 1.

Table 1 Nomenclature of this paper

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_i$ ($i = 1, \ldots, 4$)</td>
<td>Angular velocity of the $i$th wheel</td>
<td>rad/s</td>
</tr>
<tr>
<td>$\tau_i$ ($i = 1, \ldots, 4$)</td>
<td>External generalised force generated by the $i$th DC motor</td>
<td>N⋅m</td>
</tr>
<tr>
<td>$\phi_a$</td>
<td>Equals to $\phi + a/4$</td>
<td>—</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Half of width of the platform</td>
<td>m</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Half of length of the platform</td>
<td>m</td>
</tr>
<tr>
<td>$m$</td>
<td>Total mass of the platform</td>
<td>kg</td>
</tr>
<tr>
<td>$R$</td>
<td>Radius of mecanum wheel</td>
<td>m</td>
</tr>
<tr>
<td>$D_o$</td>
<td>Wheel's viscous friction coefficient</td>
<td>—</td>
</tr>
<tr>
<td>$f_i$ ($i = 1, \ldots, 4$)</td>
<td>Static friction of the $i$th wheel</td>
<td>N</td>
</tr>
<tr>
<td>$J_w$</td>
<td>Wheel's moment of inertia around the centre of the revolution</td>
<td>N⋅m⋅rad/s</td>
</tr>
<tr>
<td>$J_z$</td>
<td>Platform's moment of inertia around the centre of the revolution</td>
<td>N⋅m⋅rad/s</td>
</tr>
<tr>
<td>$A_j$</td>
<td>Constant equals to $\frac{m_i^2}{r^2}$</td>
<td>—</td>
</tr>
<tr>
<td>$B_j$</td>
<td>Constant equals to $\frac{1}{\mu_i^2}$</td>
<td>—</td>
</tr>
<tr>
<td>$R_a$</td>
<td>Armature resistance of a DC motor</td>
<td>Ω</td>
</tr>
<tr>
<td>$L_a$</td>
<td>Armature inductance of a DC motor</td>
<td>H</td>
</tr>
<tr>
<td>$K_b$</td>
<td>Back electromotive force (EMF) constant of a DC motor</td>
<td>V⋅s/rad</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Torque constant of a DC motor</td>
<td>N⋅m⋅A</td>
</tr>
</tbody>
</table>

(1) A novel FMWMP model considering the UDMU is reconstructed.

(2) An ESO-SMC scheme is implemented to an FMWMP where the ESO is designed to estimate both the unmeasured velocities and lumped disturbances/uncertainties term.

(3) Experimental validations on a real FMWMP are taken to show the effectiveness of the proposed control strategy. Furthermore, comparative studies with PID controller are carried out.

2 Mathematical model of FMWMP

As shown in Fig. 2, $X_a O_a Y_a$ and $X_O Y_O$ are defined as inertial frame and body frame, respectively. The origin of inertial frame represents the geometric centre of FMWMP at the initial place while the origin of body frame represents the real-time geometric centre of FMWMP. The states of the FMWMP in inertial frame and body frame can be described by $[x_q \ y_q \ \phi]$ and $[x_r \ y_r \ \phi]$, respectively.

2.1 Kinematics of FMWMP

The kinematic model of the FMWMP is introduced in [13], described by

$$
\begin{align*}
R \theta_1 &= x_r + y_r + \phi(a + b), \\
R \theta_2 &= \left(-x_r + y_r - \phi(a + b)\right), \\
R \theta_3 &= x_r + y_r - \phi(a + b), \\
R \theta_4 &= \left(-x_r + y_r + \phi(a + b)\right).
\end{align*}
$$

Define a matrix

$$
J = \begin{bmatrix}
1 & 1 & (a + b) \\
-1 & -1 & -(a + b) \\
1 & 1 & -(a + b) \\
-1 & 1 & (a + b)
\end{bmatrix} \in \mathbb{R}^{4 \times 4},
$$

then (1) can be rewritten as

$$
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3 \\
\dot{\theta}_4
\end{bmatrix} = \frac{J}{R} \begin{bmatrix}
x_r \\
y_r \\
\phi
\end{bmatrix}^T.
$$

It is noticed that although $J$ is a non-square matrix, there exists a new matrix

$$
J^* = \frac{1}{2} \begin{bmatrix}
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
a + b & a + b & -a + b & -a + b
\end{bmatrix} \in \mathbb{R}^{3 \times 4},
$$

which is the pseudo inverse matrix of $J$ and satisfying $J^* J = I \in \mathbb{R}^{3 \times 3}$. Note that the structure of matrix $J^*$ can also be explained by Fig. 3. Specially, in Fig. 3, Fig. 3a illustrates the velocities of the four rollers attached to the ground which are produced by the rotation of the wheels. The four velocities can be decomposed along the $X_r$ and $Y_r$ axes, as shown in Figs. 3b and 3c,
Furthermore, considering the body velocities in the inertial frame along $\omega$ axis which produce angular velocity along $\text{Fig. 3}$, where $\omega$ represents the rotation matrix of body frame with respect to the inertial frame.

The dynamic model of the FMWMP is derived by using Lagrange’s equation, which is proposed in [15], expressed as

\[ \text{eq} \]

Expressed as

2.2 Dynamics of FMWMP

The dynamic model of the FMWMP is derived by using Lagrange’s equation, which is proposed in [15], expressed as

\[ 2(\tau - F) = \frac{\partial}{\partial \theta} \left( m(x_1^2 + y_1^2) + J_1 \dot{\phi}^2 + J_0 \sum_{n=1}^{t} \theta_n^2 \right) \]

where

\[ \tau = [\tau_1, \tau_2, \tau_3, \tau_4]^T, \quad \theta = [\theta_1, \theta_2, \theta_3, \theta_4]^T, \]

\[ F = [f_1 \text{sgn}(\theta_1), f_2 \text{sgn}(\theta_2), f_3 \text{sgn}(\theta_3), f_4 \text{sgn}(\theta_4)]^T. \]

Substituting (5), (6) into (7), one obtains

\[ \tau = M \theta + D_\theta \dot{\theta} + F, \]

where

\[ M = \begin{bmatrix} A_1 + B_j + J_{\omega} & -B_j & B_j & A_j - B_j \\ -B_j & A_j + B_j + J_{\omega} & A_j - B_j & B_j \\ B_j & A_j - B_j & A_j + B_j + J_{\omega} & -B_j \\ A_j - B_j & B_j & -B_j & A_j + B_j + J_{\omega} \end{bmatrix}. \]

Remark 1: It is worth pointing out that in this model, two kinds of frictions are considered. $D_\theta \dot{\theta}$ denotes the viscous friction and $F$ represents the static friction. Generally speaking, the direction of static friction of a wheel is considered opposite to its moving direction, hence $F$ is modelled based on sign functions [41].

2.3 Unknown disturbances and model uncertainties

Considering the unknown dynamic disturbances and uncertainties, a new dynamic model is obtained as follows:

\[ \tau + \tau_d = (M + \Delta M) \dot{\theta} + (D_\theta + \Delta D_\theta) \dot{\theta} + F + \Delta F, \]

where $\tau_d$ is the unknown disturbances term.

Then, put the uncertain terms $\Delta M$, $\Delta D_\theta$ and $\Delta F$ on the left side and one can obtain

\[ \tau + \tau_d + H_d = M \dot{\theta} + D_\theta \dot{\theta} + F, \]

where $H_d = -\Delta M \dot{\theta} - \Delta D_\theta \dot{\theta} - \Delta F$.

Assumption 1: The unknown disturbance term $\tau_d(t)$ satisfies

\[ || \tau_d^j || \leq \mu_j, \ for \ j = 0, 1, 2, \ldots, n, \] where $\mu_i$ is an unknown positive number.

Assumption 2: The uncertain terms $\Delta M$, $\Delta D_\theta$ and $\Delta F$ are bounded. Moreover, the uncertain term $H_d$ satisfies

\[ || H_d^j || \leq \mu_j, \ for \ j = 0, 1, 2, \ldots, n, \] where $\mu_i$ is an unknown positive number.

Remark 2: In practice, the speed of revolution and its time derivatives of DC motors have the upper bounds, i.e. $|| \theta || \leq \bar{\theta}$, $|| \theta \dot{\theta} \dot{\theta} || \leq \bar{\theta}_f$. As $\Delta M$, $\Delta D_\theta$ and $\Delta F$ are related to $\theta$, it can be concluded that the time derivatives of $\Delta M$, $\Delta D_\theta$ and $\Delta F$ are also bounded. Assume that $|| \Delta M^j || \leq \bar{\Delta} M$, $|| \Delta D_\theta^j || \leq \bar{\Delta} D_\theta$ and $|| \Delta F^j || \leq \bar{\Delta} F$, then it can be known that

\[ || H_d^j || \leq || (\Delta M \dot{\theta})^j || + || (\Delta D_\theta \dot{\theta})^j || + || \Delta F^j || \leq \sum_{j=0}^{n} C_j (\bar{\Delta} M || \bar{\theta}_i \bar{\theta}_i \bar{\theta}_i || + \bar{\Delta} D_\theta || \bar{\theta}_i \bar{\theta}_i \bar{\theta}_i || + \bar{\Delta} F) \]

and this relaxes the Assumption 2.

2.4 State-space representation

Substituting (5), (6) into (10), yields

\[ \begin{bmatrix} x \phi \\ y \phi \\ \dot{x} \phi \\ \dot{y} \phi \end{bmatrix} = -j \gamma \text{sgn}(\theta) \dot{\theta} \phi + D_\theta \text{sgn}(\phi) \text{F}(\phi) \text{M}^{-1}(\phi) \begin{bmatrix} x \phi \\ y \phi \end{bmatrix} \]

where

\[ \begin{bmatrix} x \phi \\ y \phi \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \text{sgn}(\theta) \text{M}^{-1}(\phi) \text{F}(\phi) \text{M}^{-1}(\phi) \text{F}(\phi) \text{M}^{-1}(\phi) \]

\[ + R \text{sgn}(\phi) \text{M}^{-1}(\phi) \text{F}(\phi) \text{M}^{-1}(\phi) \text{F}(\phi) \text{M}^{-1}(\phi) \]
Define disturbances/uncertainties term be obtained from its catalogue or experiment \[ 43 \]. Moreover, the controller design based on ESO corresponding mecanum wheel, model uncertainties torque generated by the DC motor and the voltage applied to the implemented through voltage inputs regulated via PWM technique. Thus it is reasonable to regard \[ H \]
the state equation of the FMWMP can be derived as follows:

\[
\dot{x} = f(x, u, h, \tau_d, t) + B\tau_d + D\psi(x)\phi^T + h^T \phi \]

(9)

\[
J(\phi) = \left[ \begin{array}{cccc}
\sqrt{2} \cos(\phi_0) & -\sqrt{2} \sin(\phi_0) & \sqrt{2} \cos(\phi_0) & -\sqrt{2} \sin(\phi_0) \\
{\sqrt{2} \sin(\phi_0)} & \sqrt{2} \cos(\phi_0) & {\sqrt{2} \sin(\phi_0)} & \sqrt{2} \cos(\phi_0) \\
{1 \over a+b} & {1 \over a+b} & {1 \over a+b} & {1 \over a+b} \\
0 & 0 & 0 & 0
\end{array} \right]
\]

Define

\[
Z_t = \left[ \begin{array}{c}
x_q \\
y_q \\
\phi_0 \\
\omega
\end{array} \right]
\]

(11)

\[
Z_c = \left[ \begin{array}{c}
x_q \\
y_q \\
\phi_0 \\
\omega
\end{array} \right]
\]

(12)

\[
H = R^T(\phi)(M)^{-1}(H_q + \tau_d) 
\]

(15)

\[
f(Z_t, Z_c) = (J^T(\phi)f(\phi) + D_0 J^T(\phi)(M)^{-1}f(\phi))[i_\phi \dot{\phi}]^T + R^T \phi^T(\phi)(M)^{-1} H
\]

(16)

the state equation of the FMWMP can be derived as follows:

\[
Z_1 = Z_2
\]

\[
Z_2 = -f(Z_1, Z_2) + R^T(\phi)(M)^{-1} \tau + H 
\]

(17)

Remark 3: From (5) and (6), the values of \( \theta_i, i = 1, 2, 3, 4 \), as well as \( \text{sgn}(\theta_i), i = 1, 2, 3, 4\), are definitely up to \( Z_2 = [x_q \ y_q \ \phi_0]^T \). Thus it is reasonable to regard \( R^T(\phi)(M)^{-1} F \) as a part of \( f(Z_t, Z_c) \).

Remark 4: Based on the Assumptions 1 and 2, the lumped disturbances/uncertainties term \( H(t) \) and its time derivatives satisfy \( H^{(j)} \leq \delta, j = 0, 1, 2, \ldots, n \), where \( \delta \) is also an unknown positive constant [42].

Remark 5: Note that in practice, the control laws are often implemented through voltage inputs regulated via PWM technique for convenience. Generally speaking, the relationship between the torque generated by the DC motor and the voltage applied to the DC motor can be roughly expressed by \( \tau = ua_t - \rho v_a \), where \( u \) is the voltage applied to the DC motor, \( v_a \) the linear velocity of the corresponding mecanum wheel, \( a \) and \( \beta \) are motor characteristic coefficients depending on the parameters of the DC motor that can be obtained from its catalogue or experiment [43]. Moreover, the relationship between the torque and the voltage can also be modelled in detail, which will be discussed in the Appendix.

### 3 Controller design based on ESO

To achieve accurate tracking control performance, in this section, an ESO-SMC scheme will be proposed to ensure the tracking performance based on the kinematic and dynamic models introduced in Section 2. The control structure is shown in Fig. 4. The control objective in this paper is to synthesise a control algorithm for \( \tau_r, \tau_c, \tau_t, \tau_m \) with the unknown disturbances \( \tau_d \) and model uncertainties \( \Delta M, \Delta D_0 \) and \( \Delta F \) considered, such that \( Z_c(t) \) tracks the reference trajectory \( Z^*_c(t) \) accurately.

#### 3.1 Extended state observer design [44]

In this section, an ESO will be designed to derive the unmeasurable states and the lumped disturbances/uncertainties term. The state equation of the FMWMP can be rewritten as

\[
\begin{bmatrix}
Z_t \\
Z_c
\end{bmatrix} = \begin{bmatrix}
I_{3 \times 3} & I_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix} \begin{bmatrix}
Z_t \\
Z_c
\end{bmatrix} + \begin{bmatrix}
0_{3 \times 1} \\
R^T(\phi)(M)^{-1} \tau
\end{bmatrix} + \begin{bmatrix}
0_{3 \times 3} \\
0_{3 \times 3}
\end{bmatrix} H
\]

(18)

where

\[
0_{3 \times 3} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Based (18), define the extended state vector as \( Z = [Z_t^T \ Z_c^T \ H^T]^T = [x_q \ y_q \ \phi_0 \ \omega \ H_q \ H \ \phi_0 \ \omega]^T \), the lumped disturbances/uncertainties term \( H \) is defined as an extended state, and \( h \) is defined as the derivative of \( H \). Recall that \( || h \|| \leq \delta \), which has been discussed in Remark 4. Then (18) can be transformed into

\[
\dot{Z} = AZ + Br + B_f + B \tau
\]

(19)

where

\[
A = \begin{bmatrix}
0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix} \in \mathbb{R}^{3 \times 3}, \\
B = - \begin{bmatrix}
0_{3 \times 3} \\
0_{3 \times 3}
\end{bmatrix} \in \mathbb{R}^{3 \times 3},
\]

\[
B_f = \begin{bmatrix}
0_{3 \times 3} \\
0_{3 \times 3}
\end{bmatrix} \in \mathbb{R}^{3 \times 3},
\]

Note that the third partition of the matrix \( A \) in (19) is a null matrix, it implies that the lumped disturbances/uncertainties term \( H \) is generated by \( H = h \). Different from [31], where the \( H \) is generated by a exogenous system, here we use \( h \) to describe the time derivative of \( H \) since the \( H \) is mismatched. Specially, since \( h \) is bounded, \( H \) is aimed to be modelled as a varying vector whose time derivative is bounded with the upper bound unknown.

Then, the ESO is proposed as the following form:

\[
\dot{\hat{Z}} = A\hat{Z} + Br + B \dot{\hat{r}} + \alpha(Z_t - \hat{Z}_t)
\]

(20)

where

\[
\alpha = \begin{bmatrix}
\alpha_1 \ \alpha_2 \ \alpha_3 \\
\alpha_2 \ \alpha_3 \ \alpha_4 \\
\alpha_3 \ \alpha_4 \ \alpha_5
\end{bmatrix} \in \mathbb{R}^{3 \times 3}
\]

is the observer gain, and \( \alpha \) can be regarded as the bandwidth of the ESO. Define the estimation errors as follows:
Define the scaled estimation as

\[ \tilde{Z} = \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ H \end{bmatrix} = \begin{bmatrix} Z_1 - \tilde{Z}_1 \\ Z_2 - \tilde{Z}_2 \\ H - \tilde{H} \end{bmatrix}. \]

Then the estimation error dynamic equation is given by

\[ \dot{\tilde{Z}} = A\tilde{Z} + B_1\tilde{f} + B_2\tilde{h} - a\tilde{Z}, \]

where

\[ \tilde{f} = f(Z_1, Z_2) - f(\tilde{Z}_1, \tilde{Z}_2). \]

Define the scaled estimation as

\[ \xi = [\xi_1, \xi_2, \xi_3]^T = \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ H \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^T, \]

then the estimation error dynamics can be transformed into

\[ \dot{\xi} = a_0A_2\xi + B_2\tilde{f}/\omega_0 + B_2\tilde{h}/\omega_0, \]

where

\[ A_2 = \begin{bmatrix} -a_0I_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 1} \\ -a_0I_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 1} \\ -a_0J_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} \end{bmatrix} \in \mathbb{R}^{3 \times 9}. \]

Remark 6: Note that the ESO (20) proposed in this section combines the well-known HGO [45, Chapter 14.5] and ESO, i.e. the HGO is extended with a new state estimates the disturbances/uncertainties term.

**Lemma 1:** Considering the estimation error dynamics (22), the ESO (20) is bounded stable if only \( A_2 \) is Hurwitz and

\[ a_0 - 2\| P \| \cdot \frac{1}{\omega_0} \| \xi_1 + c_2a_0 \| - 1 > 0 \]

can be satisfied.

**Proof:** Since matrix \( A_2 \) is Hurwitz, there exists a matrix \( P \) which is symmetric and positive definite and it satisfies

\[ A_2^T P + PA_2 = I_{3 \times 3}. \]

Since \( \tilde{f} = f - \tilde{f} \), then according to the Lipschitz condition one can obtain that

\[ \| B_2\tilde{f} \| = \| f(Z_1, Z_2) - f(\tilde{Z}_1, \tilde{Z}_2) \| \leq \| c_1\tilde{Z}_1 + c_2\tilde{Z}_2 \| \]

\[ \leq c_1 \| \xi_1 \| + c_2a_0 \| \xi_3 \| \]

(23)

\[ \leq (c_1 + c_2a_0) \| \xi \|. \]

where \( c_1, c_2 \) are known positive constants.

Consider a Lyapunov function candidate as

\[ V_1 = \xi^T P \xi, \]

then the time derivative of \( V_1 \) can be written as

\[ \dot{V}_1 = \xi^T P \dot{\xi} + \xi^T \dot{P} \xi \]

\[ = a_0\xi^T A_2^T P \xi + \frac{(B_2\tilde{f})^T P \xi}{\omega_0} + \frac{(B_2\tilde{h})^T P \xi}{\omega_0} \]

\[ + a_0\xi^T P A_2 \xi + \frac{\xi^T P(B_2\tilde{f})}{\omega_0} + \frac{\xi^T P(B_2\tilde{h})}{\omega_0} \]

\[ = -a_0\xi^T \xi + 2\xi^T P B_2\tilde{f} + 2\xi^T P B_2\tilde{h} \]

\[ \leq \left( a_0 - 2\| P \| \cdot \frac{1}{\omega_0} \| c_1 + c_2a_0 \| \right) \| \xi \|^2 + 2\xi^T P B_2h \]

\[ \leq \left( a_0 - 2\| P \| \cdot \frac{1}{\omega_0} \| s \| \right) \| \xi \|^2. \]

By using Young's inequality, it can be easily known that

\[ 2\xi^T P B_2h \leq \| \xi \|^2 + \| P B_2 \| h \| h \| \]

Thus one can further obtain that

\[ V_1 \leq \left( a_0 - 2\| P \| \cdot \frac{1}{\omega_0} \| c_1 + c_2a_0 \| \right) \| \xi \|^2 + \| P B_2 \| h \| h \| \]

\[ + \| P B_2 \| h \| h \|. \]

Since the values of \( \| P B_2 \| \) and \( a_0 \) are depending on the design parameters and \( \delta \) is a bounded positive constant, the part \( ( \| P B_2 \| \cdot \delta )/\omega_0 \) in (26) is a bounded positive constant. It can be concluded that the proposed ESO is bounded stable if only the parameters satisfy

\[ a_0 - 2\| P \| \cdot \frac{1}{\omega_0} \| c_1 + c_2a_0 \| - 1 > 0. \]

This ends the proof. □

### 3.2 Sliding mode controller design [36]

To guarantee a satisfied tracking performance for the FMWM, an ESO-based SMC controller will be proposed in this section.

Define \( e = Z_1 - Z_1^d \) as the tracking error, where \( Z_1^d \) represents the desired trajectory. Then the sliding surface is designed as

\[ s(t) = \lambda e(t) + \epsilon(t), \]

where \( \lambda \) is a designed positive definite diagonal matrix. Employing the time derivative one can obtain that

\[ \dot{s}(t) = \lambda \dot{e}(t) + \dot{\epsilon}(t) \]

\[ = \lambda (\dot{Z}_1 - Z_1^d) + (\dot{Z}_2 - Z_2^d) \]

(28)

Thus the reaching control law is obtained as

\[ \tau_r(t) = \frac{1}{R} MJ(\dot{\phi})[\dot{\phi} - H - \dot{Z}_1 - \dot{Z}_2] \]

(29)

Design the switching control law as

\[ \tau_w(t) = \frac{1}{R} MJ(\dot{\phi})[ - k_2 s(t) - k_3 sgn(s(t)) ] \]

(30)

where \( k_2, k_3 \) is the positive definite switching gain matrices. Hence, the total input torque vector can be obtained
\[
\tau(t) = r(t) + r_u(t)
\]
\[
= \frac{1}{R}MJ(\phi)[\dot{j} - \dot{H} + Z_2(t) - Z_1(t)]
- k_s(s(t) - k_s\text{sgn}(s(t))].
\]

\textbf{Lemma 2:} Let function \(V(t) \geq 0\) be a continuous function defined \(\forall t \geq 0\) and bounded, and \(V(t) \leq -\rho V(t) + \kappa\), where \(\rho\) and \(\kappa\) are positive constants \([46]\), then
\[
V(t) \leq V(0)e^{-\rho t} + \frac{\kappa}{\rho}(1 - e^{-\rho t}).
\]

\textbf{Theorem 1:} Considering the uncertain system (12), with the designed ESO (20) and the SMC controller (31). The estimation errors of proposed ESO and the FMWMP tracking errors will be guaranteed to be UUB.

\textbf{Proof:} Consider a Lyapunov function candidate as
\[
V = \frac{1}{2}x^TPx + \frac{1}{2}y^Ts.
\]

According to (26)–(31), one can obtain that
\[
\dot{V} \leq -\frac{1}{2}\left(\omega_0 - 2\parallel P \parallel \cdot \parallel c_1 + c_2\omega_0 \parallel + 1\right) \parallel \xi \parallel^2
+ \parallel PB_2 \parallel^2 \cdot \delta^2 + s\left(1 - k_s - k_s\text{sgn}(s)\right)
\leq -\frac{1}{2}\left(\omega_0 - 2\parallel P \parallel \cdot \parallel c_1 + c_2\omega_0 \parallel + 1\right) \parallel \xi \parallel^2
- \lambda_{\text{min}}(k_1) \parallel s \parallel^2 - \lambda_{\text{min}}(k_2) \parallel s \parallel
+ \parallel PB_1 \parallel^2 \cdot \delta^2
\leq -\eta^T\Lambda_1 \eta + \zeta,
\]

where
\[
\eta = [\xi, s]^T \in \mathbb{R}^{1 \times 1}, \quad \Lambda = \begin{bmatrix}
\Lambda_1 & 0_{x \times 1} \\
0_{1 \times x} & \Lambda_2
\end{bmatrix} \in \mathbb{R}^{x \times x},
\]
\[
\zeta = \frac{\parallel PB_2 \parallel^2 \cdot \delta^2}{2\alpha_0} - \lambda_{\text{min}}(k_2) \parallel s \parallel.
\]

In which \(\Lambda_1 = \text{diag}\{\chi, \chi, \chi\}\), with
\[
\chi = \frac{1}{2}\left(\omega_0 - 1 - 2\parallel P \parallel \cdot \parallel c_1 + c_2\omega_0 \parallel\right), \quad \Lambda_2 = \text{diag}\{\zeta, \zeta, \zeta\}.
\]

with \(\zeta = \lambda_{\text{min}}(k_2)\). According to Lemma 1, \(\Lambda_1\) and \(\Lambda_2\) are positive definite matrices, thus it satisfies that
\[
\dot{V} \leq -\lambda_{\text{min}}(\Lambda) \parallel \xi \parallel^2 + \parallel s \parallel^2 + \zeta
\leq -\lambda_{\text{min}}(\Lambda) \left(\frac{1}{\lambda_{\text{min}}(P)}s^TPx + s\dot{x}\right) + \zeta
\leq -\gamma V + \zeta,
\]

where
\[
\gamma = 2\lambda_{\text{min}}(\Lambda) \min\left\{1, \frac{1}{\lambda_{\text{min}}(P)}\right\}.
\]

Then based on Lemma 2, it can be obtained that
\[
V(t) \leq V(0)e^{-\gamma t} + \frac{\zeta}{\gamma}(1 - e^{-\gamma t}).
\]
trajectory, not only with the circle reference trajectory but also with the lemniscate reference trajectory, which verifies the closed-loop stability.

Figs. 11 and 12 illustrate the control inputs of the four DC motors. It can be seen that all the control inputs are continuous and vary between $-20 \text{ N} \cdot \text{m}$ and $20 \text{ N} \cdot \text{m}$. One can conclude that the magnitude of control commands of DC motors are always within a given range as expected.

5 Experiments and comparisons

To further validate the effectiveness of the proposed ESO-SMC strategy from a practical perspective, experimental studies will be developed via an FMWMP in this section, followed by the performance comparisons between ESO-SMC controller and PID controller.

5.1 Experiment setup

The FMWMP used for the experiments is shown in Fig. 13. It includes four mecanum wheels which generated by four separated DC motors, an STM32 chip, and a driving board. To measure the (x-y) position and rotation information, an Ultra-wideband (UWB) positioning system is built and an Inertial measurement unit is settled on STM32 chip, respectively. UWB positioning system is one kind of high-precision in-door locating technology which consists four anchors whose positions are known. The position of FMWMP is obtained through the different distances between the
FMWMP and four anchors. Compared with other positioning methods such as Bluetooth-based and Wi-Fi-based positioning systems, the UWB positioning system is more suitable for real-time locating due to its shorter communication time and higher degree of accuracy [48, Chapter 8]. Note that the measurement error of UWB in this experiment is less than 5 cm. The computing unit adopted in our experiments is an NVIDIA’s Jetson TX2, which has 64-bit Denver 2 and A57 CPUs with 8 GB RAM. Note that the output voltage of the battery in experiments is less than 25.2 V. The sample rate in this experimental situation is 200 Hz. The signal flow of the FMWMP control is shown in Fig. 14.

To show the ability for tracking different trajectories of the FMWMP, two experiments were given for the FMWMP, similar to the simulations, one is with the circle reference trajectory and the other is with the lemniscate reference. The trajectories are set as

(1) Circle reference trajectory

\[
Z_1(t) = \begin{bmatrix}
x_1(t) \\
y_1(t) \\
\phi_1(t)
\end{bmatrix} = \begin{bmatrix}
3\sin(0.1t) \text{ m} \\
3\cos(0.1t) \text{ m} \\
\frac{\pi}{2}\sin(0.1t) \text{ rad}
\end{bmatrix}.
\]  

(2) Lemniscate reference trajectory

\[
Z_1(t) = \begin{bmatrix}
x_1(t) \\
y_1(t) \\
\phi_1(t)
\end{bmatrix} = \begin{bmatrix}
5\cos(0.1t) \text{ m} \\
2.5\sin(0.2t) \text{ m} \\
\frac{\pi}{2}\sin(0.1t) \text{ rad}
\end{bmatrix}.
\]  

5.2 Experimental results

Experimental results are shown in Figs. 15–18 and Figs. 19–22 for circle reference trajectory and lemniscate reference trajectory, respectively. For case (1), Fig. 15 shows the tracking errors in the experiment. It can be noticed that errors converge to zero quickly and the steady-state errors are bounded and small-enough. Fig. 16 presents the estimation results of the lumped disturbances/
uncertainties term $H$. Fig. 17 illustrates the control inputs of four DC motors, it can be seen that the control inputs are always within the expected range. In Fig. 18, graphics of the reference trajectory and the actual trajectory are also plotted as functions of time. For case (2), similar to case (1), Figs. 19–22 show the experimental results for lemniscate curve reference trajectory, also indicate that the proposed controller has a good tracking performance. It is shown that the proposed ESO-SMC scheme has both good tracking performance and robustness.

It is worth mentioning that in Figs. 15 and 19, peaking phenomenon appears, which is caused by different initial values of the ESO. This kind of phenomenon sometimes may lead the ESO even the plant to be unstable if the peaking values go too large. Some approaches to alleviate the peaking phenomenon have been proposed in the previous literature [49–52]. In [49, 50], a globally bounded state feedback control law is designed. During the short transient period when the state estimates exhibit peaking, the controller saturates, thus preventing peaking from being transmitted to the plant. Cunha et al. [51] proposes a technique to avoid peaking phenomenon that does not rely on globally bounding of the control input as in [49, 50], which allows global exponential
stability. In [52], the bandwidth

\[ 424 \]

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Fig. 22 Tracking trajectory with lemniscate reference trajectory

![Tracking trajectory with lemniscate reference trajectory](image)

Recall that \( Z_0(0) - \dot{Z}_0(0) = \omega_0 \xi_1 \) and \( H(0) - \dot{H}(0) = \omega_0^2 \xi_0 \). With this approach, the values of \( Z_0(0) - \dot{Z}_0(0) \) and \( H(0) - \dot{H}(0) \) will not go very large and thus the peaking phenomenon can be removed. However, it should be mentioned that all the aforementioned approaches may slow down the converge rate of the plant. Therefore, in practice, the controller should be designed in accordance with both two conflictive aspects, i.e. avoid peaking phenomenon and achieve quick response.

It can be observed in Figs. 17 and 21, the chattering phenomenon is obvious when the input voltages are small. The reason is that when the command voltage applied to a DC motor is small, the DC motor leaves the low-speed region. However, at this moment, the actual input voltage is larger than the command input voltage and therefore the DC motor generates larger torque than expected. In consequence, the tracking errors would increase again which causes the decrease of the input voltage. Then the DC motor leaves the low-speed region. However, at this moment, the actual input voltage is larger than the command input voltage and therefore the DC motor generates larger torque than expected. In consequence, the tracking errors would increase again which causes the decrease of the input voltage. Therefore, when the command control inputs are small, as the process mentioned above takes place repeatedly, the actual input voltages switch fast and there comes the chattering.

5.3 Performance comparisons

In this section, the tracking performance between ESO-SMC controller and PID controller of FMWMP will be compared. The PID controller is designed as follows

\[
\begin{align*}
U_1 &= K_{p1}\Delta_1 + K_{i1}\int \Delta_1 + K_{d1}\frac{d\Delta_1}{dt}, \\
U_2 &= K_{p2}\Delta_2 + K_{i2}\int \Delta_2 + K_{d2}\frac{d\Delta_2}{dt}, \\
U_3 &= K_{p3}\Delta_3 + K_{i3}\int \Delta_3 + K_{d3}\frac{d\Delta_3}{dt}, \\
U_4 &= K_{p4}\Delta_4 + K_{i4}\int \Delta_4 + K_{d4}\frac{d\Delta_4}{dt},
\end{align*}
\]

(45)

where

\[
\begin{align*}
\Delta_1 &= -(x_d - x) + (y_d - y) - (\phi_d - \phi)(a + b), \\
\Delta_2 &= -(x_d - x) + (y_d - y) - (\phi_d - \phi)(a + b), \\
\Delta_3 &= -(x_d - x) + (y_d - y) + (\phi_d - \phi)(a + b), \\
\Delta_4 &= -(x_d - x) + (y_d - y) + (\phi_d - \phi)(a + b).
\end{align*}
\]

(46)

To illustrate the steady state tracking errors in a quantitative way, the following indices are used to evaluate steady-state tracking performance:

\[
\sum_{i = 1}^{N} \| e \|_{\text{RMS}} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \| e \|}.
\]

where \( N \) equals to the sampling steps. As illustrated in Figs. 23 and 24, the tracking errors enter the steady state after 12 s with circle reference, and 10 s with lemniscate reference. The values \( \| e \|_{\text{RMS}}, \| e \|_{\text{LMS}} \) and \( \| e \|_{\text{RM}} \) with the circle reference and the lemniscate reference are shown in Tables 3 and 4, respectively.

Comparative results in Tables 3 and 4 indicate that after the FMWMP system enters the steady-state, i.e. after the FMWMP system breaks away from the impact region of the peak overshoots, the proposed ESO-SMC controller has higher tracking accuracy than PID controller. The performance of ESO-SMC controller has an obvious improvement compared with PID controller.

Note that there are two main aspects influencing on the tracking performance:

(1) The FMWMP used in the experiment has four DC motors with different characteristics, such as respond time, dead-zone, which

\[
\alpha_0 = \begin{cases} 
200(\frac{t}{12})^4 & 0 \leq t \leq 10 \\
200 & t > 10.
\end{cases}
\]
influences on the tracking performance. Moreover, the chattering induced by the DC motors in case that the control inputs are small (DC motor enters low-speed region) also brings influences on the tracking performance.

(2) It is noted that mecanum wheel suffers slippage more frequently compared than common wheel [53]. As [53] concerns, the main sources of position errors are the slippage between the mecanum wheel and the floor surface. Hence the slippage also influences on the tracking performance.

Generally speaking, when an FMWMP is tracking circle or lemniscate trajectory, the DC motors need to change their rotation directions frequently and therefore often work in low-speed region. Moreover, since both circle and lemniscate trajectories are curves, the slippage may appear almost all the time. Thus, the RMS errors in Tables 3 and 4 are acceptable in practice. It would be difficult to implement errors reduction since the aforementioned reasons are inherent characteristics of mecanum wheel and DC motor. In practical application, the trajectory for which an FMWMP tracking should be planned with slowly varying curvature as far as possible so that the two sources of tracking error mentioned above can be reduced as far as possible.

6 Conclusion

This paper concerns about the tracking control problems of a FMWMP in face of the UDMU. To overcome the control challenges, we propose an ESO to estimate the unmeasurable states and the lumped disturbances/uncertainties term, based on which an SMC scheme is applied. Simulations show that the proposed ESO has a good estimation performance. To further validate the effectiveness of the proposed method, two experiments are given to demonstrate the ability of the proposed method for tracking different trajectories. The experimental results illustrate that the proposed ESO-SMC scheme can drive the FMWMP to the desired trajectory with the steady-state tracking errors bounded. Also, comparative studies indicate that ESO-SMC controller has better tracking performance than PID controller.

7 Acknowledgments

This work was supported in part by the National Natural Science Foundation of China (61525303, 41772377 and 61673130), the Top-Notch Young Talents Program of China (Ligang Wu), and the Self-Planned Task of State Key Laboratory of Robotics and System (HIT) (SKLRS201806B).

8 References

Appendix

Considering a DC motor [54], it satisfies

\[ L_i U + R_i I_i + K_i \theta = U_i, \]  

where \( U_i \) and \( I_i \) are vector of motor voltages and vector of motor currents, respectively. Note that we take the armature inductance \( L_a \) into consideration in (47) since we are trying to give an accurate mathematical model of DC motor. Generally speaking, it can be neglectible in practice. Then the vector of motor torques \( \tau \) is produced by the currents

\[ \tau = K_a I_a, \]  

Define

\[ \nu = \begin{bmatrix} \nu_1 \nu_2 \nu_3 \nu_4 \end{bmatrix} \]  

where

\[ \begin{bmatrix} A_j \ B_j \end{bmatrix} \]  

and

\[ A_j \in \mathbb{R}^{10 \times 10}, \]  

where

\[ B_j \in \mathbb{R}^{10 \times 1} \]  

and

\[ C_j = \begin{bmatrix} 1 & 0 \end{bmatrix} \]  

are

9 Appendix


