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AN INVERSION OF STRASSEN'S LAW OF THE ITERATED LOGARITHM FOR SMALL TIME

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We prove a local version of Strassen's law of the iterated logarithm. Instead of shrinking larger and larger pieces of a Brownian path and letting time go to infinity, we look at a sequence of functions we get by blowing up smaller and smaller pieces and we investigate the asymptotic behaviour of this sequence as time goes to zero. It turns out that this sequence of functions is a relatively compact subset of $C[0,1]$ with probability 1, and the set of its limit points is the same as in Strassen's theorem.

Let $(X(t))_{t\geq 0}$ be a real-valued Brownian motion on a probability space $(\Omega, \mathscr{A}, P)$ with $X_0 = 0$. For $n \geq 3$, let

$$\xi_n(t) := \frac{X(nt)}{\sqrt{2n\log\log n}}, \qquad 0 \leq t \leq 1.$$

$(\xi_n)_{n\geq 3} \subseteq C[0,1]$ is a sequence of functions we get from the Brownian path, rescaling larger and larger pieces to the unit interval. The asymptotic behaviour of $(\xi_n)_{n\geq 3}$ can be described as follows. Let $C[0,1]$ be equipped with the supremum norm $\|f\| = \sup_{0\leq t\leq 1}|f(t)|$. Let H be the space of $\psi \in C[0,1]$ with the property that $\psi(t) = \int_0^t \dot{\psi}(s)\,ds$, $0 \leq t \leq 1$, for some $\dot{\psi} \in L^2([0,1])$ and set $\|\psi\|_H := \|\dot{\psi}\|_{L^2([0,1])}$ for $\psi \in H$. Set

$$K = \{\psi \in H \mid \|\psi\|_H \leq 1\}.$$

Strassen's law of the iterated logarithm tells us that $\{\xi_n | n \geq 3\}$ is relatively compact in $C[0,1]$ with probability 1 and the set of its limit points is K. In particular, for every continuous function $F\colon C[0,1] \to \mathbb{R}$, we have

$$P\left[\limsup_{n\to\infty} F(\xi_n) = \sup_{\psi\in K} F(\psi)\right] = 1.$$

Taking $F(X) = X(1)$, Strassen's theorem yields the usual law of the iterated logarithm,

$$P\left[\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = 1\right] = 1.$$

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It is well known that there is a *local* law of the iterated logarithm, too:

$$P\left[\limsup_{n\to\infty}\frac{X(1/n)}{\sqrt{(2/n)\log\log n}}=1\right]=1.$$

Therefore, the following question comes to mind: Is it possible to give a *local* version of Strassen's functional limit theorem, that is, to blow up small pieces of the path instead of shrinking large pieces to the unit interval? In analogy to Strassen's theorem, we can state:

THEOREM 1. *For $n \geq 3$, define*

$$\xi_{1/n}(t)=\frac{X(t/n)}{\sqrt{(2/n)\log\log n}},\qquad 0\leq t\leq 1$$

and set

$$K=\{\psi\in H \mid \|\psi\|_H\leq 1\}.$$

Then, for P-almost every X, the sequence $\{\xi_{1/n}\}_{n=3}^{\infty}$ has the following properties:

(i) $\{\xi_{1/n}\}_{n=3}^{\infty}$ *is relatively compact in* $C[0,1]$ *and every limit point is an element of* K.

(ii) *For every $\psi \in K$ there is a subsequence of $\{\xi_{1/n}\}_{n=3}^{\infty}$ which converges to ψ.*

In particular, for every continuous F: $C[0,1]\to\mathbb{R}$, we have

$$P\left[\limsup_{n\to\infty}F(\xi_{1/n})=\sup_{\psi\in K}F(\psi)\right]=1.$$

As an application of Theorem 1, we get statements about the asymptotic behaviour of Brownian increments in small time.

Strassen's law can be proved as an application of Schilder's theorem on large deviations for a Brownian motion with small variance. This goes back to Stroock and Varadhan (1972); see also Stroock (1984). It is possible to give a proof of Theorem 1 along the same lines, but, as Stroock pointed out to me, it can also be derived directly from an extension of Strassen's theorem "for the whole time axis" [cf. Theorem 1.4.1 in Deuschel and Stroock (1989)]. We now give a proof in this second way.

Let $\Theta := \{\theta \in C([0,\infty);\ \mathbb{R}^d) \mid \theta(0) = 0$ and $\lim_{t\to\infty}(|\theta(t)|/t) = 0\}$. For $\theta\in\Theta$ define $\|\theta\| := \sup_{t\geq 0}(|\theta(t)|/(1+t))$. Then $(\Theta, \|\cdot\|_\Theta)$ is a separable real Banach space. Let $H^1 = H^1([0,\infty);\mathbb{R}^d)$ be the space of $\psi\in\Theta$ with the property that $\psi(t) = \int_0^t\dot\psi(s)\,ds$, $t\geq 0$, for some $\dot\psi \in L^2([0,\infty);\mathbb{R}^d)$ and set $\|\psi\|_{H^1} := \|\dot\psi\|_{L^2([0,\infty);\mathbb{R}^d)}$ for $\psi\in H^1$. Let P denote Wiener measure on Θ. Then the following form of Strassen's theorem holds:

THEOREM 2. *For $n\geq 3$, define*

$$\xi_n(t,\theta)=\frac{\theta(nt)}{\sqrt{2n\log\log n}},\qquad (t,\theta)\in[0,\infty)\times\Theta$$

and set

$$K = \{\psi \in H^1 | \, \|\psi\|_{H^1} \le 1\}.$$

Then, for P-almost every $\theta \in \Theta$, the sequence $\{\xi_n(\theta)\}_{n=3}^\infty$ has the following properties:

(i) $\{\xi_n(\theta)\}_{n=3}^\infty$ *is relatively compact in* Θ *and every limit point is an element of* K.

(ii) *For every* $\psi \in K$ *there is a subsequence of* $\{\xi_n(\theta)\}_{n=3}^\infty$ *which converges in* Θ *to* ψ.

In particular, for every $F \in C(\Theta; \mathbb{R})$,

$$P\left[\limsup_{n\to\infty} F(\xi_n(\theta)) = \sup_{\psi \in K} F(\psi)\right] = 1.$$

See Deuschel and Stroock (1989) for the proof.

Let us state the corresponding "small time" theorem:

THEOREM 3. *For* $n \ge 3$, *define*

$$\xi_{1/n}(t, \theta) = \frac{\theta(t/n)}{\sqrt{(2/n)\log\log n}}, \qquad (t, \theta) \in [0, \infty) \times \Theta$$

and set

$$K = \{\psi \in H^1 | \, \|\psi\|_{H^1} \le 1\}.$$

Then, for P-almost every $\theta \in \Theta$, the sequence $\{\xi_{1/n}(\theta)\}_{n=3}^\infty$ has the following properties:

(i) $\{\xi_{1/n}(\theta)\}_{n=3}^\infty$ *is relatively compact in* Θ *and every limit point is an element of* K.

(ii) *For every* $\psi \in K$ *there is a subsequence of* $\{\xi_{1/n}(\theta)\}_{n=3}^\infty$ *which converges in* Θ *to* ψ. *In particular, for every* $F \in C(\Theta; \mathbb{R})$,

$$P\left[\limsup_{n\to\infty} F\big(\xi_{1/n}(\theta)\big) = \sup_{\psi \in K} F(\psi)\right] = 1.$$

PROOF. Let T denote the time inversion transformation on Θ, that is,

$$(T\theta)(t) = \begin{cases} 0, & t = 0, \\ t\theta\left(\dfrac{1}{t}\right), & t > 0. \end{cases}$$

Note that T is an isometry from Θ onto Θ and $T|_{H^1}$ is an isometry from H^1 onto H^1: An easy computation shows that $\|T\theta\|_\Theta = \|\theta\|_\Theta$ and $\|T\psi\|_{H^1} = \|\psi\|_{H^1}$. Further, P is invariant under T. We have

$$\xi_{1/n}(t, \theta) = \frac{\theta(t/n)}{\sqrt{(2/n)\log\log n}} = T\left(\frac{T\theta(n\,\cdot)}{\sqrt{2n\,\log\log n}}\right)(t).$$

Since P is invariant under T, Theorem 2 tells us that $\{\bar{\xi}_n(\theta)\}_{n=3}^{\infty}$, defined by

$$\bar{\xi}_n(t,\theta) = \frac{T\theta(nt)}{\sqrt{2n\log\log n}},$$

satisfies (i) and (ii). We now use the above properties of T: If the subsequence $\{\bar{\xi}_{n_k}(\theta)\}_{k=1}^{\infty}$ of $\{\bar{\xi}_n(\theta)\}_{n=3}^{\infty}$ converges to $\psi \in K$, then $\{T\bar{\xi}_{n_k}(\theta)\}_{k=1}^{\infty}$ converges to $T\psi \in K$; since T is bijective on K, we conclude that $\{\xi_{1/n}(\theta)\}_{n=3}^{\infty}$, defined by $\xi_{1/n}(\theta) = T\bar{\xi}_n(\theta)$, satisfies (i) and (ii). □

Of course, Theorem 1 follows from Theorem 3.

Remarks. Let us consider the sequences $\{\xi_{h_n}(\theta)\}_{n=1}^{\infty}$ with $h_n \in (e,\infty)$, $h_n \to_{n\to\infty} \infty$, instead of $\{\xi_n(\theta)\}_{n=3}^{\infty}$.

Remark 1. The proof of Theorem 2 in Deuschel and Stroock (1989) shows that $\{\xi_{h_n}(\theta)\}_{n=1}^{\infty}$ still satisfies (i); one can even show

$$P\Big[\sup_{s\ge h}\|\xi_s - K\| \to_{h\to\infty} 0\Big] = 1$$

and conclude

$$P\Big[\limsup_{h\to\infty} F(\xi_h) = \sup_{\psi\in K} F(\psi)\Big] = 1.$$

The same is true for $\{\xi_{1/h_n}(\theta)\}_{n=1}^{\infty}$ and we get the corresponding "small time" statement

$$P\Big[\limsup_{h\to\infty} F(\xi_{1/h}) = \sup_{\psi\in K} F(\psi)\Big] = 1.$$

Remark 2. If we want the sequence $\{\xi_{h_n}(\theta)\}_{n=1}^{\infty}$ to "generate" the whole set K as limit points, we cannot do without a condition saying that $\{h_n\}_{n\ge1}$ does not go to ∞ too fast. [It is possible to choose $\{h_n\}_{n\ge1}$ such that the function ψ with $\psi(t) = 0$, $t \ge 0$, is, for P-almost every θ, the only limit point of $\{\xi_{h_n}(\theta)\}_{n=1}^{\infty}$.] Careful examination of the proof of Theorem 2 in Deuschel and Stroock (1989) shows that the following condition is sufficient: Assume for each $C > 0$ there is a subsequence $\{h_{n_k}\}_{k\ge1}$ of $\{h_n\}_{n\ge1}$ satisfying $h_{n_{k+1}}/h_{n_k} \ge C$, $\forall\, k$ and

$$\sum_k \frac{1}{\log(h_{n_k})^{\gamma}} = \infty \quad \text{for each } \gamma < 1.$$

Then, for P-almost all θ, the sequence $\{\xi_{h_n}(\theta)\}_{n=1}^{\infty}$, and therefore also the sequence $\{\xi_{1/h_n}(\theta)\}_{n=1}^{\infty}$, satisfy (ii).

Examples. Of course, the local law of the iterated logarithm follows from Theorem 1. Take $F(X) = X(1)$ to get the following statement:

$$P\left[\limsup_{n\to\infty} \frac{X(1/n)}{\sqrt{(2/n)\log\log n}} = 1\right] = 1$$

and, taking into account Remark 1,

$$P\left[\limsup_{h\to 0} \frac{X(h)}{\sqrt{2h \log\log(1/h)}} = 1\right] = 1.$$

In the same way, we can take $F(X) = \sup_{0\le t\le 1}|X(t)|$ to obtain

$$P\left[\limsup_{n\to\infty} \sup_{0\le t\le 1/n} \frac{|X(t)|}{\sqrt{(2/n)\log\log n}} = 1\right] = 1.$$

Consider $F(X) = \sup_{0\le t\le 1-c}|X(t+c) - X(t)|$, where $0 < c < 1$. We have $\sup_{\psi\in K} F(\psi) = \sqrt{c}$, hence Theorem 1 yields

$$P\left[\limsup_{n\to\infty} \sup_{0\le t\le 1/n - c/n} \frac{|X(t+c/n) - X(t)|}{\sqrt{(2/n)\log\log n}} = \sqrt{c}\right] = 1$$

and, taking into account Remark 1,

$$P\left[\limsup_{h\to 0} \sup_{0\le t\le h-hc} \frac{|X(t+hc) - X(t)|}{\sqrt{2h \log\log(1/h)}} = \sqrt{c}\right] = 1.$$

In the same way, taking $F(X) = \sup_{0\le t\le 1-c}\sup_{0\le s\le c}|X(t+s) - X(t)|$, we get

$$P\left[\limsup_{n\to\infty} \sup_{0\le t\le 1/n-c/n} \sup_{0\le s\le c/n} \frac{|X(t+s) - X(t)|}{\sqrt{(2/n)\log\log n}} = \sqrt{c}\right] = 1$$

and

$$P\left[\limsup_{h\to 0} \sup_{0\le t\le h-hc} \sup_{0\le s\le hc} \frac{|X(t+s) - X(t)|}{\sqrt{2h \log\log(1/h)}} = \sqrt{c}\right] = 1.$$

The last two examples are the "small time statements" which correspond to Corollary 1.2.2 in Csörgő and Révész (1981).

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