

Noam Berger · Nina Gantert · Yuval Peres

The speed of biased random walk on percolation clusters

Received: 20 November 2002 / Revised version: 17 January 2003

Published online: 15 April 2003 – © Springer-Verlag 2003

Abstract. We consider biased random walk on supercritical percolation clusters in \mathbb{Z}^2 . We show that the random walk is transient and that there are two speed regimes: If the bias is large enough, the random walk has speed zero, while if the bias is small enough, the speed of the random walk is positive.

1. Introduction

The following model is considered in the physics literature as a model for transport in an inhomogeneous medium. Let $p \in (p_c, 1)$, where $p_c = \frac{1}{2}$ is the critical probability for bond percolation on \mathbb{Z}^2 . We perform i.i.d. bond percolation with parameter p on \mathbb{Z}^2 . For convenience, we always condition on the event that the origin belongs to the infinite cluster. The corresponding measure on percolation configurations will be denoted by P_p^* . Let $\beta > 1$. Consider the random walk starting at the origin with transition probabilities defined as follows. Let $Z_n = (X_n, Y_n)$ be the location at time n . Let l_n be the number of neighbors Z_n has in the infinite cluster. If $\tilde{Z}_n = (X_n + 1, Y_n)$ is one of these neighbors, then $Z_{n+1} = \tilde{Z}_n$ with probability

$$\frac{\beta}{\beta + l_n - 1}$$

and Z_{n+1} is any of the other neighbors with probability

$$\frac{1}{\beta + l_n - 1}.$$

N. Berger, Y. Peres: University of California, Department of Statistics, 367 Evans Hall, CA 94720-3860, USA. e-mail: {noam, peres}@stat.berkeley.edu

Research supported by Microsoft Research graduate fellowship.
Research partially supported by the DFG under grant SPP 1033.

N. Gantert: University of Karlsruhe, Germany.
e-mail: n.gantert@math.uni-karlsruhe.de

Research partially supported by NSF grant #DMS-0104073 and by a Miller Professorship at UC Berkeley.

Mathematics Subject Classification (2000): 60K37; 60K35; 60G50

Key words or phrases: Percolation – Random walk

If \tilde{Z}_n is not a neighbor of Z_n (i.e. the edge (Z_n, \tilde{Z}_n) is closed) then Z_{n+1} is chosen among the neighbors of Z_n with equal probabilities. This is a random walk with bias to the right, where the strength of the bias is given by the parameter β . To our best knowledge, the first authors who considered this model are M. Barma and D. Dhar in [3].

Let ω be the percolation configuration. We write P_ω^β for the conditional law of the random walk given ω , and $\mathbb{P}^{\beta,*}$ for the joint distribution of $(\omega, (Z_n)_{n=1,2,\dots})$. $\mathbb{P}^{\beta,*}$ restricted to $(Z_n)_{n=1,2,\dots}$ is the law of the walk averaged over the realizations of the percolation configuration.

Our main result is the following theorem, which proves part of the predictions of [3].

Theorem 1. *For every $p \in (p_c, 1)$, there exist $1 < \beta_\ell \leq \beta_u < \infty$ such that if $1 < \beta < \beta_\ell$ then*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} > 0 \quad \mathbb{P}^{\beta,*}\text{-a.s.}$$

and if $\beta > \beta_u$ then

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0 \quad \mathbb{P}^{\beta,*}\text{-a.s.}$$

The following conjecture goes back to [3].

Conjecture 1. The statements of Theorem 1 hold with $\beta_{\text{crit}} := \beta_\ell = \beta_u$.

While there is a large physics literature on this model, as, for instance, [3, 9, 10], there are few mathematical results. The biased random walk on the percolation cluster is a random walk in a random environment on \mathbb{Z}^2 . There has been remarkable recent progress on laws of large numbers for random walk in dependent random environments, see [8, 20, 21]. However, in all of these papers, there are boundedness assumptions on the transition probabilities which are violated in our case.

In contrast to the biased case, simple random walks on percolation clusters were investigated in the probability literature for some time. The first work on the subject was done By Grimmett, Kesten and Zhang [14], where they proved that simple random walk on supercritical percolation clusters in \mathbb{Z}^d is transient for $d \geq 3$. Other papers include [6], [15], [16], [1], [5].

In order to prove that there is a positive speed regime, we first assume that p is close enough to 1 and show the following.

Proposition 2. *For every p close enough to 1, there exists $\beta_\ell > 1$ such that if $\beta < \beta_\ell$ then*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} > 0 \quad \mathbb{P}^{\beta,*}\text{-a.s.} \tag{1}$$

The paper is organized as follows. Sections 2, 3, 4, 5, 6, and 7 are devoted to the proof of Proposition 2. Using renormalization arguments we show in Section 8 that the statement of Proposition 2 holds for every $p > p_c$. In Section 9, we define β_u

and show that for $p > p_c$ and $\beta > \beta_u$, the speed is zero. In fact, our β_u is the predicted value of β_{crit} in Conjecture 1, see [3]. The proofs in this section carry over to the multidimensional case, i.e. to biased random walks on supercritical percolation clusters in \mathbb{Z}^d , $d \geq 2$.

While finishing this paper, we learned that A. S. Sznitman has independently obtained results similar to ours. In [22], he investigates biased random walks on supercritical percolation clusters in \mathbb{Z}^d , $d \geq 2$, where the transition probabilities correspond to weights given by scalar products with a direction vector. He shows the analogue of Theorem 1 and obtains a CLT in the positive speed regime. While both [22] and this paper use a regeneration structure to derive the main results, the techniques of the two papers are quite different. Sznitman uses very precise information about the random walk and its analytical properties, while our approach uses more detailed information about the percolation cluster.

2. A positivity criterion for the speed

In order to simplify the arguments, we will without loss of generality condition, throughout the proof, on the event that the origin is in the infinite cluster on the left half-plane, i.e. on the event that there is an infinite cluster on $\{(x, y) : x \leq 0\}$ and that the origin is in this infinite cluster. This event has positive probability: in fact, due to the results of [4], the probability that there is an infinite cluster on the left half-plane equals 1 whenever $p > p_c$ (see also [13]). Denote the corresponding probability measure on percolation configurations by $\widehat{\mathbb{P}}_p$, and the resulting joint law of $(\omega, (Z_n)_{n=1,2,\dots})$ by \mathbb{P}^β .

Remark 1. Assume that the origin is not in the infinite cluster on the left half-plane. Take an arbitrary vertex z which is in the infinite cluster to its left. Then there is a finite open path Γ connecting z to the origin. If the statements in Theorem 1 hold almost surely for the random walk starting from z , then they also hold almost surely for the random walk starting from the origin, since starting from z , we have a positive probability to go to the origin.

We give a criterion which will later be used to show that the speed is strictly positive for β small enough. We will prove in Lemma 6 and Lemma 13 that

$$\lim_{n \rightarrow \infty} X_n = \infty \quad \mathbb{P}^\beta\text{-a.s.} \tag{2}$$

We call $n > 0$ a **fresh epoch** if $X_n > X_k$ for all $k < n$ and we call n a **regeneration epoch** if, in addition, $X_k > X_n$ for all $k > n$. Let the regeneration epochs be $0 = R_0 < R_1 < R_2 < \dots$. Exactly as in [18], one shows that there are, \mathbb{P}^β -a.s., infinitely many regeneration epochs and that the time differences $(R_{i+1} - R_i)_{i=1,2,3,\dots}$ and the increments between regeneration epochs $(X_{R_{i+1}} - X_{R_i})_{i=1,2,3,\dots}$ are i.i.d. sequences under \mathbb{P}^β . Standard arguments then imply that if $E^\beta(R_2 - R_1) < \infty$, then

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = \frac{E^\beta(X_{R_2} - X_{R_1})}{E^\beta(R_2 - R_1)} > 0 \quad \mathbb{P}^\beta\text{-a.s.} \tag{3}$$

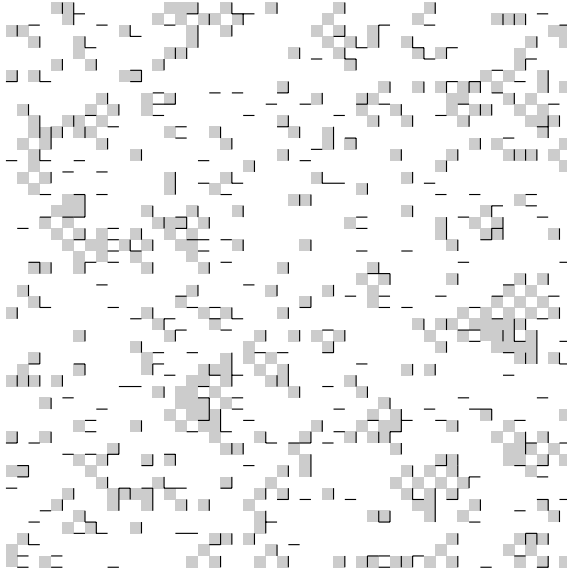


Fig. 1. Traps in the percolation cluster, $p = 0.9$. Vertices are represented by squares. Bad vertices are shaded. Marked are the closed dual bonds.

3. An exponential bound on the size of traps

We use the following decomposition of the percolation cluster into good and bad points. The definition of a good point might seem artificial at first sight, but the results of Sections 4 and 5 will clarify the choice of this definition.

Definition 1. A point $z = (x, y) \in \mathbb{Z}^2$ is **good** if there exists an infinite path $\{z_0 = z, z_1 = (x_1, y_1), z_2 = (x_2, y_2), \dots\}$ such that for $k = 1, 2, 3, \dots$,
 (A) $|y_k - y_{k-1}| = 1$ and $x_k - x_{k-1} = 1$.
 (B) The edges $\{(x_{k-1}, y_{k-1}), (x_k, y_{k-1})\}$ and $\{(x_k, y_{k-1}), (x_k, y_k)\}$ are open.

Denote the infinite cluster by I and the set of good vertices by J . A vertex z is **bad** if $z \in I$ and z is not good. Connected components of $I \setminus J$ will be called **traps** (see Figure 1). For a vertex v , let $C(v)$ be the trap containing v . ($C(v)$ is empty if v is a good point.) The length of a trap T is

$$L(T) = \sup\{|x_1 - x_2| : \exists y_1, y_2 \text{ such that } (x_1, y_1) \in T \text{ and } (x_2, y_2) \in T\}$$

and the width is

$$W(T) = \sup\{|y_1 - y_2| : \exists x_1, x_2 \text{ such that } (x_1, y_1) \in T \text{ and } (x_2, y_2) \in T\}$$

If T is empty, then we take $L(T) = W(T) = 0$. For convenience, we will use the notation $L(v)$ for $L(C(v))$ and $W(v)$ for $W(C(v))$.

Lemma 1. For every p close enough to 1, there exists $\alpha = \alpha(p) < 1$ such that $\widehat{P}_p(L(0) \geq n) \leq \alpha^n$ and $\widehat{P}_p(W(0) \geq n) \leq \alpha^n$ for every n . Further, $\lim_{p \rightarrow 1} \alpha(p) = 0$.

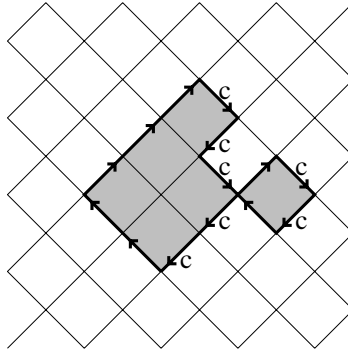


Fig. 2. The unique contour around an even trap in the even lattice. Vertices of the even lattice are represented as squares. Marked are the dual bonds.

Proof. Call two vertices **even-connected** if $\|u - w\|_1 = 2$. That is, (x, y) and (x', y') are even-connected if either $|x - x'| = |y - y'| = 1$ or $(|x - x'|, |y - y'|) = (0, 2)$ or $(|x - x'|, |y - y'|) = (2, 0)$. We define the **even trap** $C_e(v)$ of a point v as the even-connected component of bad points v' with

$$\|v'\|_1 \equiv \|v\|_1 \pmod{2},$$

containing v . In particular, all points v' in $C(v)$ with

$$\|v'\|_1 \equiv \|v\|_1 \pmod{2}$$

are also in the even trap of v (but the even trap may contain additional points not in $C(v)$). The following is obvious.

Fact 1. Every vertex in $C(0)$ is either an element of $C_e(0)$ or a neighbor of a vertex in $C_e(0)$. In particular, $L(0) \leq L(C_e(0)) + 2$ and $W(0) \leq W(C_e(0)) + 2$.

Thanks to Fact 1, we only need to give exponential bounds to $L(C_e(0))$ and to $W(C_e(0))$. Consider the following percolation model on the even lattice (i.e. the lattice whose vertices are $\{v \in \mathbb{Z}^2 : \|v\|_1 \text{ is even}\}$ and which has an undirected edge between every (x, y) and (x', y') such that $|x - x'| = |y - y'| = 1$): The bond between (x, y) and $(x + 1, y \pm 1)$ is open if and only if in the original model the edges $\{(x, y), (x + 1, y)\}$ and $\{(x + 1, y), (x + 1, y \pm 1)\}$ are open. This is a model of dependent oriented percolation, and we denote the corresponding probability measure by $P_{p,\text{oriented}}$.

Let p' be close to 1. By the results in [17], there exists $p < 1$ such that $P_{p,\text{oriented}}$ dominates i.i.d. bond percolation with parameter p' on the even lattice. Consider $C_e(0)$ in the even lattice. Let the outer boundary of a set of vertices be the set of all edges which have one vertex in the set and one in the complement. The outer boundary can be identified with a contour in the dual lattice (see Figure 2). Hence, the number of outer boundaries of size n is bounded by $\exp(O(n))$ (each contour, which needs not to be simply connected, can be identified with a random walk path).

By an argument similar to that of [12] p. 1026, at least half of the edges in the outer boundary of $C_e(0)$ are closed (in Figure 2, these are the boundary edges marked with a “C”). Therefore, if p' is close enough to 1, $L(C_e(0))$ and $W(C_e(0))$ have the desired exponential tail with respect to i.i.d. bond percolation with parameter p' on the even lattice, hence also with respect to $P_{p,\text{oriented}}$. \square

4. Bound for back-stepping from a good vertex

The following simple observation is essential to the proof. Let $\mathcal{H}(n)$ be the σ -field generated by the history of the random walk until time n , i. e. $\mathcal{H}(n) = \sigma(\{Z_0 = 0, Z_1, Z_2, \dots, Z_n\})$. Let $P_{\omega, \mathcal{H}(n)}^\beta$ be the conditional distribution of P_ω^β , given $\mathcal{H}(n)$, and $\mathbb{P}_{\mathcal{H}(n)}^\beta$ be the conditional distribution of \mathbb{P}^β , given $\mathcal{H}(n)$. Define $\tau_n(X) = \min\{i > n : X_n = X\}$. In order not to overload the notations, in many places throughout the section we chose to omit the integer brackets, e.g. we write $\ell/3$ instead of $\lceil \ell/3 \rceil$.

Lemma 2. *There exists $D' = D'(\beta)$ such that for every $\ell = 1, 2, 3, \dots$ and for every configuration ω such that $z = (x, y)$ is a good point,*

$$P_{\omega, \mathcal{H}(n)}^\beta(\tau_n(X_n - \ell) \leq \tau_n(X_n + \ell/3) | Z_n = z) < D' \beta^{-\ell/3}.$$

Proof. The transition probabilities can be described with the following electrical network: Give a weight to each open edge e : if $e = \{(x, y), (x + 1, y)\}$ then e has weight $w(e) = \beta^{x+1}$, and if $e = \{(x, y), (x, y \pm 1)\}$ then e has weight $w(e) = \beta^x$. If e is closed, then its weight is 0. The random walk (Z_n) has transition probabilities proportional to the weights of the edges from a vertex. For background on the description of reversible Markov chains as electrical networks, we refer to [11] and to [19].

The following fact is well known, but for the convenience of the reader we will recall its proof.

Fact 2. Let G be a finite electrical network, and let A and B be disjoint sets of vertices in G . Let z be a vertex in G , and let $\tau(z \rightarrow A)$ (resp. $\tau(z \rightarrow B)$) be the hitting time of A (resp. B) for a walk starting at z . Let $C_{z,A}$ (resp. $C_{z,B}$) be the effective conductance between z and A (resp. B). Then,

$$P(\tau(z \rightarrow B) < \tau(z \rightarrow A)) \leq \frac{C_{z,B}}{C_{z,A}}. \tag{4}$$

Proof. Let $\pi(z)$ be the sum of the weights of all edges containing z . Let u_j be the location of the walker at time j . Let k_i be the i -th time the walk returns to z (i.e. $k_0 = 0$, and $k_{i+1} = \tau_{k_i}(z)$). We call the interval $[k_{i-1}, k_i - 1]$ the i -th excursion. For a set $D \subseteq G$, let $V(i, D)$ be the event that the walker visits D during the i -th excursion. Then, for every i ,

$$P(V(i, D)) = \frac{C_{z,D}}{\pi(z)}. \tag{5}$$

(see e.g. equation (2.4) of [19]). By (5), for every i ,

$$P(V(i, B)|V(i, A \cup B)) = \frac{C_{z,B}}{C_{z,A \cup B}}.$$

In particular, decomposing the sequence of excursions according to the first visit to $A \cup B$ and using the fact that the excursions are i.i.d., we get

$$P(\tau(z \rightarrow B) < \tau(z \rightarrow A)) \leq \frac{C_{z,B}}{C_{z,A \cup B}} \leq \frac{C_{z,B}}{C_{z,A}}.$$

□

Consider the box $B = [x - \ell, x + \ell/3] \times [y - \beta^{2\ell/3}, y + \beta^{2\ell/3}]$. In view of Fact 2, we need to estimate the effective conductances between z and the face $B^+ = \{x + \ell/3\} \times [y - \beta^{2\ell/3}, y + \beta^{2\ell/3}]$ and between z and the rest of the boundary of the rectangle.

1. C_{z,B^+} is bounded from below by the conductance of the good path from z to B^+ , which is at least $D_1 \beta^x$ for some $D_1 = D_1(\beta)$.
2. Consider $B^- = \{x - \ell\} \times [y - \beta^{2\ell/3}, y + \beta^{2\ell/3}]$. The conductance C_{z,B^-} is bounded from above by the sum of the weights $w(u, u + (1, 0))$ for $u \in B^-$. But for every such u , $w(u, u + (1, 0)) \leq \beta^{x-\ell+1}$ (with inequality because the weight is zero if the edge is closed), and there are $2\beta^{2\ell/3}$ such edges. Therefore $C_{z,B^-} \leq D_2 \beta^x \cdot \beta^{-\ell/3}$ for some $D_2 = D_2(\beta)$.
3. Consider $B_1^* = [x - \ell, x + \ell/3] \times \{y - \beta^{2\ell/3}\}$ and $B_2^* = [x - \ell, x + \ell/3] \times \{y + \beta^{2\ell/3}\}$. By Nash–Williams’ inequality (equation (2.15) on page 38 of [19]),

$$C_{z,B_j^*} \leq \beta^{-2\ell/3} \sum_{i=x-\ell}^{x-1+\ell/3} \beta^{i+1} \leq D_3 \beta^x \cdot \beta^{-\ell/3}$$

for some $D_3 = D_3(\beta)$.

From 1., 2. and 3. we see, using (4), that the probability to exit B not through B^+ is at most $O(\beta^{-\ell/3})$. □

The following lemma gives a bound for the probability of back–stepping from a good point at a fresh epoch. Recall that $n > 0$ is a **fresh epoch** if $X_n > X_k$ for all $k < n$.

Lemma 3. *Assume that p is close enough to 1. Let $G(z)$ be the event that z is a good point and let $F(n)$ be the event that n is a fresh epoch. Then there exists $K = K(\beta, p)$ such that for every $\ell = 1, 2, \dots$,*

$$\begin{aligned} & \mathbb{P}_{\mathcal{H}(n)}^\beta(\text{there is an } m \geq n \text{ such that } X_m \leq x - \ell \mid Z_n = z, F(n), G(z)) \\ & \leq K \beta^{-\sqrt{\ell}/K}, \quad \mathbb{P}^\beta\text{-a.s.} \end{aligned}$$

To prove Lemma 3, we will use the following lemma:

Lemma 4. *In the notations of Lemma 2, let $\tau'_n(X)$ be the first fresh epoch, later than n , such that the random walk hits a good point whose first coordinate is larger or equal to X . Then, there exists a constant $D = D(\beta, p)$ such that for every $\ell = 1, 2, \dots$,*

$$\mathbb{P}_{\mathcal{H}(n)}^\beta \left(\tau_n(X_n - \ell) < \tau'_n(X_n + \ell/6) \mid Z_n = z, G(z), \max_{0 \leq i \leq n} X_i < X_n + \sqrt{\ell} \right) \leq D\beta^{-\sqrt{\ell}/D}, \quad \mathbb{P}^\beta\text{-a.s.}$$

In particular,

$$\mathbb{P}_{\mathcal{H}(n)}^\beta (\tau_n(X_n - \ell) < \tau'_n(X_n + \ell/6) \mid Z_n = z, F(n), G(z)) \leq D\beta^{-\sqrt{\ell}/D}, \quad \mathbb{P}^\beta\text{-a.s.} \tag{6}$$

Proof. For $i = 1, \dots, \lceil \sqrt{\ell}/6 \rceil$, let $t_i = \tau_n(X_n + i\sqrt{\ell})$. For convenience, if $t_i = \infty$ then we say that $Z_{t_i} = \infty$ and Z_{t_i} is not a good point. We define the **right hand trap** (resp. **right hand even trap**) of a bad point $z = (x, y)$ to be the connected component (resp. even connected component) of bad points $z' = (x', y')$ such that $x' \geq x$, containing z . The right hand even trap of a point z will be denoted by $\text{RT}(z)$. Let $L(\text{RT}(z))$ be the length of the (right hand even) trap $\text{RT}(z)$. If z is a good point then we say that $L(\text{RT}(z)) = 0$.

Claim 3. For $z = (x, y)$, let $\omega_l(z)$ be the configuration of all edges to the left of the line $L_x = \{(x, \tilde{y}) \mid \tilde{y} \in \mathbb{Z}\}$, and let $\omega_r(z)$ be the configuration of all edges to the right of L_x , including the vertical edges on the line L_x . For α as in Lemma 1, and $k = 1, 2, \dots$,

$$\widehat{P}_p (L(\text{RT}(z)) \geq k \mid \omega_l(z)) \leq \alpha^k, \quad \widehat{P}_p\text{-a.s.} \tag{7}$$

In particular,

$$\widehat{P}_p (G(z) \mid \omega_l(z)) \geq 1 - \alpha, \quad \widehat{P}_p\text{-a.s.} \tag{8}$$

Proof. Since we condition on the origin being in the infinite cluster on the left half-plane, the event $\{L(\text{RT}(z)) \geq k\}$ is independent of $\omega_l(z)$ and the claim follows from the proof of Lemma 1. \square

We want to estimate the probability of the following event: There exists some $1 \leq i \leq \lceil \sqrt{\ell}/6 \rceil$ such that $t_i < \tau_n(X_n - \ell)$ and the point Z_{t_i} is good. By (7), for every i , conditioned on $t_i < \infty$,

$$\mathbb{P}_{\mathcal{H}(t_i)}^\beta \left(L(\text{RT}(Z_{t_i})) \geq \frac{1}{2}\sqrt{\ell} \right) \leq \alpha^{\frac{1}{2}\sqrt{\ell}}, \quad \mathbb{P}^\beta\text{-a.s.} \tag{9}$$

Using (8), again conditioned on $t_i < \infty$, yields

$$\mathbb{P}_{\mathcal{H}(t_i)}^\beta \left(G(Z_{t_i}) \mid L(\text{RT}(Z_j)) < \frac{1}{2}\sqrt{\ell} \text{ for all } 1 \leq j < i \right) \geq 1 - \alpha, \quad \mathbb{P}^\beta\text{-a.s.} \tag{10}$$

since we condition on an event which is measurable with respect to ω_l . The lemma now follows from Lemma 2, (9) and (10). \square

Proof of Lemma 3. Lemma 3 now follows from (6) in Lemma 4 by iterating. \square

5. An a priori bound

In this section we show an a priori bound for the distance the random walk goes to the right.

Lemma 5. *If p is close enough to 1, then for $\beta > 1$ close enough to 1, there exists a constant C such that for every n large enough,*

$$\mathbb{P}^\beta(X_n \leq Cn^{\frac{1}{10}}) \leq n^{-2}.$$

In order to prove Lemma 5 we will give an estimate on the number of distinct sites visited by the random walk.

Definition 2. *For a trap T , the size of T is $S(T) = L(T) + W(T)$.*

Claim 4. Let T be a trap of size at most s , and let $z = (x, y) \in T$. Let

$$\begin{aligned} \phi(s) &= \beta^s \left(s^2 + \frac{2}{\beta - 1} \right) \cdot (3 + \beta) \\ &\leq C(\beta)\beta^{2s}. \end{aligned}$$

Then, for every m , and for every configuration ω with z and T as above,

$$\mathbb{P}_\omega^\beta (\#\{i : Z_i = z\} \geq m) \leq \left(1 - \phi(s)^{-1} \right)^m.$$

In particular, if z is a good point, then

$$\mathbb{P}_\omega^\beta (\#\{i : Z_i = z\} \geq m) \leq \left(1 - \phi(1)^{-1} \right)^m.$$

Proof. Recall the description of the transition probabilities with an electrical network. By equation (2.3) of [19], starting at z , the probability of ever hitting z again is

$$1 - \frac{C_{z,\infty}}{\pi(z)} \tag{11}$$

where $\pi(z)$ is the sum of the weights of all edges containing z . Clearly,

$$\pi(z) \leq \beta^x (3 + \beta). \tag{12}$$

We need to bound $C_{z,\infty}$ from below. In order to do that we will bound the resistance $R_{z,\infty} = 1/C_{z,\infty}$ from above. For a good point $z = (x, y)$, the resistance $R_{z,\infty}$ is bounded from above by the resistance of the good path which is

$$\frac{2\beta^{-x}}{\beta - 1}. \tag{13}$$

If z is in a trap T of size at most s , let z_0 be a good point on the boundary of T . Then,

$$R_{z,\infty} \leq R_{z,z_0} + R_{z_0,\infty}. \tag{14}$$

Let $q = (z, \dots, z_0)$ be a path in T from z to z_0 . Then, the resistance of q is bounded by the the product of the length of q and the maximal resistance of all bonds in q . Since q is in T , its length is bounded by s^2 and the maximal resistance of all bonds in q is bounded by the maximal resistance of all bonds in T which is at most β^{s-x} . Therefore,

$$R_{z,z_0} \leq s^2 \beta^{s-x}.$$

Further, $x_0 \geq x - s$. Hence, using (13) and (14),

$$\frac{1}{C_{z,\infty}} = R_{z,\infty} \leq \beta^{-x} \cdot \left(s^2 \beta^s + \frac{2\beta^s}{\beta - 1} \right) \tag{15}$$

The claim now follows from (11), (12) and (15). □

Proof of Lemma 5. Let p be close enough to 1 so that $P_p(S(0) \geq n) \leq \alpha^n$ for all n large enough, with some $\alpha < 1$. Let u be large enough so that

$$u \log \alpha < -4, \tag{16}$$

and $\beta > 1$ close enough to 1 so that

$$u < \frac{1}{200 \log \beta}. \tag{17}$$

By (17), for every large enough n and for every $s \leq u \log n$,

$$\phi(s) \leq n^{1/10}. \tag{18}$$

By the choice of u the probability that there exists a trap or an even trap of size bigger than $u \log n$ somewhere in the square $[-n, n] \times [-n, n]$ is smaller than $\frac{1}{2}n^{-2}$. We now condition on the event A_1 that there are no such traps. Since at times up to n the random walk cannot leave the cube $[-n, n]^2$, at any time before n we are either at a good vertex or in a trap of size at most $u \log n$.

Claim 5. Conditioned on A_1 , with probability larger than $1 - \exp(-\frac{1}{2}n^{1/5})$, the random walk visits at least $n^{7/10}$ points up to time n .

Proof. By (18) and by Claim 4, for every $z \in [-n, n] \times [-n, n]$, the probability that z is visited more than $n^{3/10}$ times is bounded by

$$\left(1 - n^{-1/10}\right)^{n^{3/10}} \leq \exp\left(-n^{2/10}\right).$$

Therefore, the probability that any point in $[-n, n] \times [-n, n]$ is visited more than $n^{3/10}$ times is bounded by

$$4n^2 \exp\left(-n^{2/10}\right) \leq \exp\left(-\frac{1}{2}n^{1/5}\right) \tag{19}$$

for n large enough. But if no point is visited more than $n^{3/10}$ times, then at least $n^{7/10}$ points are visited. □

Let B be the event that the random walk visits at least $n^{7/10}$ points up to time n .

Claim 6. Conditioned on B , with probability at least $1 - \exp(-\frac{1}{4}n^{1/5})$,

$$\max_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} X_i \geq n^{1/10}.$$

Proof. Recall the Varopoulos–Carne bound for the n -step transition probabilities of a reversible Markov chain with reversible measure π (see [7]):

$$P^n(a, b) \leq 2\sqrt{\pi(b)/\pi(a)} \exp\left(\frac{-d(a, b)^2}{2n}\right) \tag{20}$$

where $d(a, b)$ is the (graph) – distance between a and b . Using (20) and a union bound, for every $1 \leq i < j \leq n$, and all ω

$$P_\omega^\beta \left(X_i = X_j \text{ and } |Y_i - Y_j| \geq n^{6/10} \right) \leq Cn^4 \exp\left(-\frac{1}{2}n^{1/5}\right)$$

where $C = C(\beta)$ is a constant. Taking the union over all possible pairs i, j ,

$$\begin{aligned} P_\omega^\beta \left(\exists i, j \text{ such that } X_i = X_j \text{ and } |Y_i - Y_j| \geq n^{6/10} \right) &\leq Cn^6 \exp\left(-\frac{1}{2}n^{1/5}\right) \\ &\leq \exp\left(-\frac{1}{4}n^{1/5}\right) \end{aligned}$$

for n large enough. However, if $\max_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} X_i \leq n^{1/10}$ and at least $n^{7/10}$ points are visited, then there have to be i and j such that $X_i = X_j$ and $|Y_i - Y_j| \geq n^{6/10}$. \square

Claim 7. With probability at least $1 - \exp(-n^{1/30})$, for every $1 \leq i < j \leq n$

$$X_j - X_i \geq -n^{1/20}.$$

Proof. For $z = (x, y)$ and $z' = (x', y')$,

$$\frac{\pi(z')}{\pi(z)} \leq C\beta^{x'-x},$$

where $C = C(\beta)$ is a constant. Fix $i < j$ and z and z' in $[-n, n] \times [-n, n]$ such that $x - x' > n^{1/20}$. Then, again using (20),

$$P_\omega^\beta (Z_i = z \text{ and } Z_j = z') \leq 2\sqrt{\frac{\pi(z')}{\pi(z)}} \leq C\beta^{-n^{1/20}}$$

Summing over all of the possible values of i, j, z and z' , we get

$$\begin{aligned} P_\omega^\beta \left(\exists i < j \text{ such that } X_j - X_i < -n^{1/20} \right) &\leq Cn^6 \beta^{-n^{1/20}} \\ &\leq \exp\left(-n^{1/30}\right) \end{aligned}$$

for n large enough. \square

Hence, with \mathbb{P}^β -probability at least $1 - n^{-2}$, by Claim 7

$$\min_{1 \leq i \leq n} X_i \geq -n^{1/20}.$$

and, by Claim 6,

$$\max_{1 \leq i \leq n} X_i - \min_{1 \leq i \leq n} X_i \geq n^{1/10},$$

hence

$$\max_{1 \leq i \leq n} X_i \geq n^{1/10} - n^{1/20},$$

but, again due to Claim 7,

$$X_n - \max_{1 \leq i \leq n} X_i \geq -n^{1/20}.$$

Hence, with \mathbb{P}^β -probability at least $1 - n^{-2}$, for n large enough,

$$X_n \geq n^{1/10} - 2n^{1/20} \geq \frac{1}{2}n^{1/10}.$$

□

Lemma 6. *Let p be close enough to 1, and $\beta > 1$. Then*

$$\lim_{n \rightarrow \infty} X_n = \infty \quad \mathbb{P}^\beta\text{-a.s.} \tag{21}$$

Proof. We prove the lemma by iterating Lemma 4. Let $N > 1$ be an arbitrary positive integer. Let T be the even trap containing the origin. Let $\ell_0 = 2L(T)^2 + N$, and let $\ell_{i+1} = 13\ell_i/12$ for every $i = 0, 1, \dots$. Let τ_g be the first time in which the walker is in a good point. We define inductively the following times: $t_0 = \tau_g$, $t_{i+1} = \tau'_{t_i}(X_{t_i} + \ell_i/6)$. Let

$$A_i = \{ \tau'_{t_i}(X_{t_i} + \ell_i/6) < \tau_{t_i}(X_{t_i} - \ell_i) \}$$

Then by Lemma 4, for every i ,

$$\mathbb{P}^\beta(A_i) \geq 1 - D\beta^{-\sqrt{\ell_i}/D}.$$

(The first formula in Lemma 4 is needed since t_0 is not necessarily a fresh epoch). Therefore,

$$\mathbb{P}^\beta \left(\bigcap_{i=1}^{\infty} A_i \right) \geq 1 - 2C\beta^{-\sqrt{N}/D} \tag{22}$$

for some $C = C(\beta)$. Note that $X_{t_i} - \ell_i \geq X_{t_{i-1}} - \frac{11}{12}\ell_{i-1}$. Hence, if A_i occurs for every i , then $t_i < \infty$ for every i , and

$$X_s > X_{t_i} - \ell_i \geq X_{t_0} - \ell_0 + \frac{1}{12} \sum_{j=1}^{i-1} \ell_j \geq X_{t_0} + C\ell_0 \left(\frac{13}{12} \right)^i$$

for every $s > t_i$. In particular, if A_i occurs for every i then (21) holds. By (22), the event in (21) occurs with probability at least $1 - 2C\beta^{-\sqrt{N}/D}$ for every N . Therefore, it occurs a.s. □

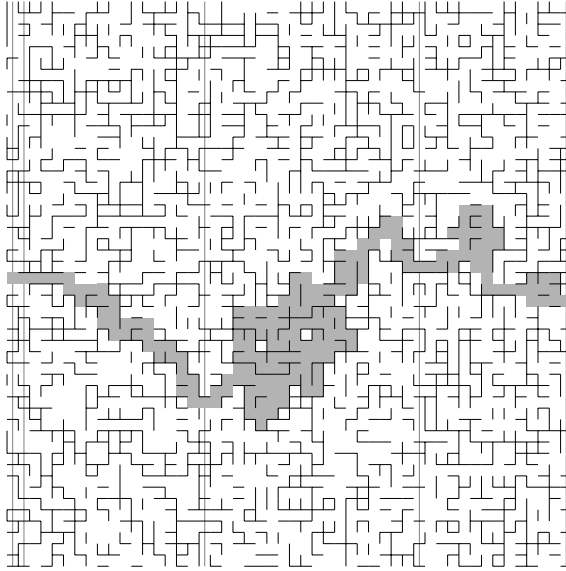


Fig. 3. A sample path, $p = 0.55$ and $\beta = 2.5$, gray lines are at regenerations.

6. The environment after a regeneration

As the reader recalls from Section 2, we say that $n > 0$ is a **fresh epoch** if $X_n > X_k$ for all $k < n$ and we say that a fresh epoch n is a **regeneration epoch** or **regeneration** if, in addition, $X_k > X_n$ for all $k > n$ (see Figures 3 and 4). In this section we consider the distribution of the percolation cluster to the right of $Z_n = z$, given that n is a regeneration.

For every $z = (x, y) \in \mathbb{Z}^2$, let S_z be the environment to the right of z , i.e. for every $\tilde{x} > 0$ and $\tilde{y} \in \mathbb{Z}$, the edge $S_z(\{(\tilde{x}, \tilde{y}), (\tilde{x}, \tilde{y} \pm 1)\})$ is open if and only if $\{(x + \tilde{x}, y + \tilde{y}), (x + \tilde{x}, y + \tilde{y} \pm 1)\}$ is open and $S_z(\{(\tilde{x}, \tilde{y}), (\tilde{x} + 1, \tilde{y})\})$ is open if and only if $\{(x + \tilde{x}, y + \tilde{y}), (x + \tilde{x} + 1, y + \tilde{y})\}$ is open. For every time n let (F_n) be the future of the walk after time n , i.e. $F_n(k) = Z_{n+k} - Z_n, k = 0, 1, 2, \dots$. Let $\mu = \mu^\beta$ be the distribution of $\{(F_0), S_0\}$ under \mathbb{P}^β , conditioned on the event $\{X_i \geq 1 \forall i \geq 1\}$. This is well defined because $\mathbb{P}^\beta(X_i \geq 1 \forall i \geq 1) > 0$.

Lemma 7. *Let R_n be the n -th regeneration. Then, for all n , the law of $\{(F_{R_n}), S_{Z_{R_n}}\}$ is μ .*

Lemma 7 is proved in the same way as Proposition 3.4 of [18]. Let $\zeta = \zeta(p, \beta) = \mathbb{P}^\beta(X_i \geq 1 \forall i \geq 1)$.

Corollary 3. *The law of $\{(F_{R_n}), S_{Z_{R_n}}\}$ is absolutely continuous with respect to \mathbb{P}^β . Furthermore, its Radon–Nikodym derivative with respect to \mathbb{P}^β is*

$$\frac{d\mu}{d\mathbb{P}^\beta} = \mathbf{I}_{\{X_i \geq 1 \forall i \geq 1\}} \cdot \zeta^{-1} \leq \zeta^{-1} < \infty.$$

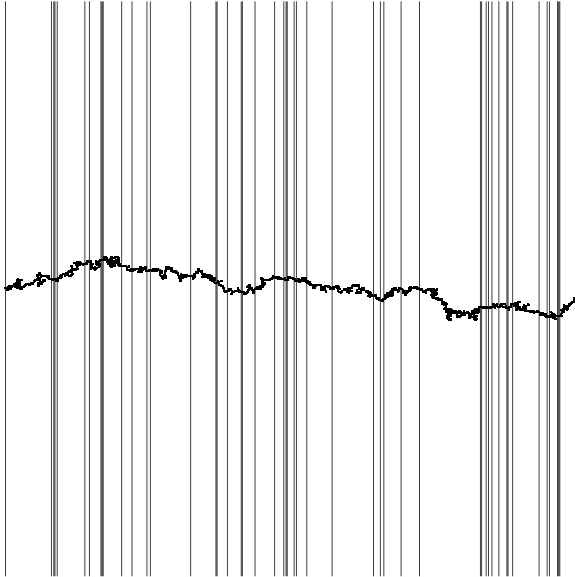


Fig. 4. A sample path, $p = 0.65$ and $\beta = 2.5$, gray lines are at regenerations.

7. Proof of Proposition 2

Proposition 2 is a consequence of the following lemma.

Lemma 8. *Let β and p be as in Lemma 5. Then, $E^\beta(R_2 - R_1) < \infty$.*

We will first show the following.

Lemma 9. *Let β and p be as in Lemma 5. Then, $E^\beta(R_1) < \infty$.*

Proof. We will show that

$$\sum_{n=1}^{\infty} \mathbb{P}^\beta(R_1 > n) < \infty. \tag{23}$$

We will estimate $\mathbb{P}^\beta(R_1 > n)$ for n large enough in order to show (23). Let u be as in the proof of Lemma 5. Let A_1 be the event that the even traps (as defined in Page 225) in $[-n, n]^2$ are of size not larger than $u \log n$. For n large enough, the probability of A_1 is at least $1 - n^{-2}$.

Let

$$\kappa = K^2 \max\left(\frac{3}{\log \beta}, 2u\right),$$

where K is the constant from Lemma 3. Let γ_n be the smallest even integer $\geq \kappa(\log n)^2$ and let

$$T_i = \inf\{k : X_k \geq i\gamma_n\}$$

Claim 8. Let $g(z)$ be $\mathbf{1}_{G(z)}$. Let $\eta = \eta(p) > 0$ be the probability of a vertex to be good. Then, there exists $\eta' > \eta/2$ and i.i.d. Bernoulli random variables $L_i, i = 1, 2, \dots$ where $L_i = 1$ with probability η' and $L_i = 0$ with probability $1 - \eta'$, such that the total variation distance between the conditional distribution of $(g(Z_{T_i}), 1 \leq i \leq n^{1/20})$, given A_1 , and the distribution of $(L_i, 1 \leq i \leq n^{1/20})$ is bounded by n^{-2} .

Proof. If we condition on nonexistence of even traps of size larger than $u \log n$, the point Z_{T_i} is good if and only if there exists a good path starting at Z_{T_i} and ending at the line

$$\{(i + 1)\kappa(\log n)^2, y) : y \in \mathbb{Z}\}.$$

Let η' be the \mathbb{P}^β -probability of the existence of such a path. We now define the random variables $L_i, i = 1, 2, \dots$: let L_i be the indicator of the event that there is a good path starting at Z_{T_i} and ending at the line $\{(i + 1)\kappa(\log n)^2, y) : y \in \mathbb{Z}\}$. Since we condition on the origin being in the infinite cluster in the left half-plane, the conditional probability of $\{L_n = 1\}$, given $\mathcal{H}(Z_{T_n})$ and the percolation configuration on $\{(x, y) | x \leq Z_{T_n}\}$, does not depend on $\mathcal{H}(Z_{T_n})$ and the percolation configuration on $\{(x, y) | x \leq Z_{T_n}\}$. Therefore, the random variables L_i are i.i.d. Since the conditional distribution of $(g(Z_{T_i}), 1 \leq i \leq n^{1/20})$, given A_1 , was obtained from the distribution of $(L_i, 1 \leq i \leq n^{1/20})$ by conditioning on an event of probability at least $1 - n^{-2}$, the total variation distance between the two distributions is bounded by n^{-2} . □

Let A_2 be the event that $T_i < n$ for every $1 \leq i \leq n^{1/20}$. By Lemma 5, for n large enough, $\mathbb{P}^\beta(A_2) \geq 1 - n^{-2}$.

Let A_3 be the event that there are at least $\frac{1}{4}\eta n^{1/20}$ values of i in $\{1, 2, \dots, [n^{1/20}]\}$ such that $g(Z_{T_i}) = 1$. By Claim 8, for n large enough, $\mathbb{P}^\beta(A_3|A_1) \geq 1 - 3n^{-2}$ and therefore $\mathbb{P}^\beta(A_3) \geq 1 - 4n^{-2}$.

Let ξ_j be the j -th value of T_i such that $g(Z_{T_i}) = 1$. We define

$$D(i) = \inf\{X_k - X_{\xi_i} : k > \xi_i\}$$

and

$$\tilde{D}(i) = \inf\{X_k - X_{\xi_i} : \xi_i < k < \xi_{i+1}\}.$$

Claim 9. There exists $\rho > 0$ such that

$$\mathbb{P}_{\mathcal{H}(\xi_i)}^\beta(D(i) = 1) \geq \rho \quad \mathbb{P}^\beta\text{-a.s.}$$

Proof. The claim is a direct consequence of Lemma 3: take ℓ such that $K\beta^{-\sqrt{\ell}/K} < 1$, then the probability of the event $\{D(i) = 1\}$ is bounded below by $(\beta + 3)^{-2\ell}(1 - K\beta^{-\sqrt{\ell}/K})$. □

Obviously, $D(i) \leq \tilde{D}(i)$ for every i . Therefore,

$$\mathbb{P}^\beta_{\mathcal{H}(\xi_i)}(\tilde{D}(i) = 1) \geq \rho \quad \mathbb{P}^\beta\text{-a.s.} \tag{24}$$

Note that, for all i , $\tilde{D}(i - 1)$ is $\mathcal{H}(\xi_i)$ -measurable. Therefore, by (24) and successive conditioning, for every k ,

$$\mathbb{P}^\beta(\tilde{D}(i) < 1 \text{ for all } i = 1, \dots, k) \leq (1 - \rho)^k.$$

Let A_4 be the event that there exists some $1 \leq i \leq \frac{1}{4}\eta n^{1/20}$ such that $\tilde{D}(i) = 1$. Then,

$$\mathbb{P}^\beta(A_4) \geq 1 - (1 - \rho)^{\frac{\eta n^{1/20}}{4}} = 1 - o(n^{-2}).$$

Let A_5 be the event that $D(i) = \tilde{D}(i)$ for every $1 \leq i \leq \frac{1}{4}\eta n^{1/20}$. For every i ,

$$D(i) = \min(\tilde{D}(i), D(i + 1) + X_{\xi_{i+1}} - X_{\xi_i}). \tag{25}$$

By Lemma 3,

$$\mathbb{P}^\beta(D(i + 1) \leq X_{\xi_i} - X_{\xi_{i+1}}) \leq \mathbb{P}^\beta(D(i + 1) \leq -\kappa \log n) \leq K\beta^{-3\frac{\log n}{\log \beta}} = Kn^{-3} \tag{26}$$

Combining (25) and (26), we get that $\mathbb{P}^\beta(D(i) \neq \tilde{D}(i)) \leq Kn^{-3}$ for every i . Therefore, $\mathbb{P}^\beta(A_5) = 1 - o(n^{-2})$.

Claim 10. If A_1, A_2, A_3, A_4 and A_5 all occur, then $R_1 \leq n$.

Proof. By the occurrence of A_2 and A_3 , $\xi_i < n$ for every $i \in B_n = [1, \dots, \frac{1}{4}\eta n^{1/20}]$. By the occurrence of A_4 , there exists $i_0 \in B_n$ such that $\tilde{D}(i_0) = 1$. By the occurrence of A_5 , $D(i_0) = 1$. Let $t = \xi_{i_0}$. Then $t < n$. By the definition of $\{\xi_i\}$, the epoch t is a fresh epoch. On the other hand, for every $k > t$,

$$X_k \geq X_t + \min(X_j - X_t : j > t) = X_t + D(i_0) = X_t + 1 > X_t$$

and therefore t is a regeneration epoch. □

Hence

$$\mathbb{P}^\beta(R_1 > n) \leq \mathbb{P}^\beta(A_1^c) + \mathbb{P}^\beta(A_2^c) + \mathbb{P}^\beta(A_3^c) + \mathbb{P}^\beta(A_4^c) + \mathbb{P}^\beta(A_5^c) = O(n^{-2}),$$

which yields (23). □

Proof of Lemma 8. For a random variable X and a distribution ν , we denote the expected value of X under ν by $E_\nu(X)$. We want to show that $E_{\mathbb{P}^\beta}(R_2 - R_1) < \infty$. Recall the distribution $\mu = \mu^\beta$ from Section 6. The distribution of $R_2 - R_1$ under \mathbb{P}^β is the same as the distribution of R_1 under μ . Therefore, all we need to show is that $E_\mu(R_1) < \infty$. But, using Corollary 3 and Lemma 9,

$$E_\mu(R_1) \leq E_{\mathbb{P}^\beta}(R_1) \cdot \sup\left(\frac{d\mu}{d\mathbb{P}^\beta}\right) < \infty.$$

□

8. Renormalization

In this section we show how to combine standard renormalization ideas with our arguments in order to carry over our results for every $p > p_c$. We use the renormalization scheme that is used in [6], [15] and [1]. Fix a value $p \in (p_c, 1)$.

Notice that everything we did so far is also valid when we consider site percolation with retention probability $\hat{p} < 1$ instead of bond percolation. We will assume that $\hat{p} < 1$ is close enough to 1 to apply our previous arguments (to be specified later). Let N be a (large) positive integer, divisible by 8.

For each $v \in N \cdot \mathbb{Z}^2$ define $Q_N(v)$ to be the square of side-length $5N/4$ centered at v . Let $p \in (p_c, 1)$. Consider i.i.d. bond percolation with parameter p on \mathbb{Z}^2 , and let A_p be the random set of vertices $v \in N \cdot \mathbb{Z}^2$ such that $Q_N(v)$ contains a connected open component which connects all 4 faces of $Q_N(v)$ but contains no other connected open component of diameter greater than $N/10$. It follows from Proposition 2.1 in [2] that if N is large enough then A_p dominates i.i.d. site percolation with parameter \hat{p} on \mathbb{Z}^2 . We choose N to be such a large enough value.

For p close to p_c it is possible to show that there is (a.s.) no point in the lattice that satisfies the definition of a good point (Definition 1 on page 224). Therefore, we need a new notion of a point being good. In order to avoid confusion, we will use the term p -good for the new definition.

Definition 3. We say that a square $Q_N(v)$ is p -good if $v \in A_p$ and there exist $v_1 = v = (x_1, y_1)$, $v_2 = (x_2, y_2)$, $v_3 = (x_3, y_3)$, $v_4 = (x_4, y_4)$, ... such that
 (A) For every k , $x_k - x_{k-1} = N$ and $y_k - y_{k-1} = \pm N$.
 (B) For every k , both v_k and $v_k + (N, 0)$ are in A_p .
 A square is considered p -bad if it is not p -good.

If z is a point in the p -good square $Q_N(v_1)$ that belongs to the big component in the square, then there exists an infinite path starting at z that is contained in the union of the squares $Q_N(v_1)$, $Q_N(v_1 + (N, 0))$, $Q_N(v_2)$, $Q_N(v_2 + (N, 0))$, $Q_N(v_3)$, $Q_N(v_3 + (N, 0))$, ... (This follows from the definition of A_p – note that a connected component crossing the overlapping part of two good squares has to cross both squares!) We call this path a p -good path starting at z .

Definition 4. We say that a point z is p -bad if it is in a p -bad square and belongs to the infinite cluster.

Definition 5. We say that a point z is p -good if
 (A) z is not p -bad.
 (B) There exists an (infinite) p -good path $z_1 = (x_1, y_1) = z$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$, $z_4 = (x_4, y_4)$, ... starting at z such that $x_k > x_1$ for every $k > 1$.

Definition 6. A p -trap is a connected component of p -bad points.

Remark 2. The reader is advised to notice that:

- (A) The squares are not disjoint. Therefore a point could belong to both a p -good square and a p -bad square. In this case, if it is connected to infinity then it is p -bad.
- (B) Not all of the points that are connected to infinity are p -good or p -bad.
- (C) A p -good path may also contain p -bad points.

(D) If \hat{p} is close enough to 1, a square has a positive probability of being p -good, and a vertex has a positive probability of being p -good.

In particular, a point at the boundary of a p -trap might not be a p -good point. Therefore, we also need the following weaker definition.

Definition 7. A p -OK point is a point that is not p -bad and is in the big cluster of a p -good square.

Once we defined a p -trap and a p -OK point, the argument for transience of the random walk follows the same lines as in the case where p is close enough to 1. More precisely, let $T_p(z)$ be the p -trap containing z , and let $L_p(z)$ and $W_p(z)$ be the length and the width of $T_p(z)$.

Lemma 10. There exists $\alpha < 1$ such that $\widehat{P}_p(L_p(0) \geq n) \leq \alpha^n$ and $\widehat{P}_p(W_p(0) \geq n) \leq \alpha^n$ for every n .

The proof is the same as the proof of Lemma 1, assuming \hat{p} is close enough to 1 and considering oriented percolation on the sublattice of the centers of squares.

Lemma 11. For a point $z = (x, y)$, let $OK(z)$ be the event that z is a p -OK point. Then, there exists a constant $K' = K'(p, \beta)$ such that for every $\ell = 1, 2, \dots$,

$$\begin{aligned} &\mathbb{P}_{\mathcal{H}(n)}^\beta (\text{there is an } m \geq n \text{ such that } X_m \leq x - \ell \mid Z_n = z, F(n), OK(z)) \\ &\leq K' \beta^{-\sqrt{\ell}/K'} . \end{aligned}$$

The proof, again, is similar to that of Lemma 3 since one can bound from below the conductance of every p -good path starting at z .

In order to prove the equivalents of Lemma 5 and Lemma 6 we need the following simple claim:

Claim 11. Let T be a p -trap. Every point at the boundary of T is p -OK.

Using Claim 11, Lemma 11 and Lemma 10 we can now prove the following two lemmas the same way Lemma 5 and Lemma 6 were proved.

Lemma 12. For $\beta > 1$ close enough to 1, there exists a constant C such that for every n large enough,

$$\mathbb{P}^\beta(X_n \leq Cn^{\frac{1}{10}}) \leq n^{-2}.$$

Lemma 13. Let $\beta > 1$, then

$$\lim_{n \rightarrow \infty} X_n = \infty \quad \mathbb{P}^\beta\text{-a.s.} \tag{27}$$

The proof of Theorem 1 now follows the same lines as the proof of Proposition 2 in Section 7, using the notions “ p -good” and “ p -trap” instead of “good” and “trap”.

9. Zero speed region

Theorem 4. *For every $p \in (p_c, 1)$, there exists a finite value $\beta_u = \beta_u(p) > 1$ such that for $\beta > \beta_u$,*

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = 0 \quad \mathbb{P}^\beta\text{-a.s.}$$

Further, $\lim_{p \searrow p_c} \beta_u(p) = 1$.

Proof. We will first define β_u and show that for $\beta > \beta_u$, the speed of the random walk is 0. For this purpose, we will consider configurations where the origin 0 is the beginning of a dead end. Call a vertex $z = (x, y)$ the **beginning of a dead end** if z is in the infinite cluster to its left, but in a finite cluster to its right. The dead end starting at z is the finite cluster to the right of z , containing z . We now consider a dead end A starting at the origin. Let $d(A) := \max\{x : (x, y) \in A\}$ denote the depth of A . Let $N(A)$ denote the number of vertices of A which are on the line $\mathcal{L} = \{(0, y) : y \in \mathbb{Z}\}$. Let E_A denote the set of edges of A and B_A denote the set of all edges which have at least one vertex in A , but are not in E_A . The probability of A under i.i.d. bond percolation is $p_A := p^{|E_A|}(1 - p)^{|B_A|}$. Let DE denote the set of all dead ends. The following claim is easy, we omit its proof.

Claim 12. Let $\{\omega_r = A\}$ denote the event that all the edges in E_A are open in ω and all the edges in B_A are closed in ω . Then,

$$\widehat{P}_p(\omega_r = A) \geq C(p)p_A$$

where $C(p)$ is a constant depending only on p .

Let

$$\Gamma(p, \beta) := \sum_{A \in DE} p_A \left(N(A)^{-1} \sum_{e=(z_1, z_2) \in E_A} \beta^{x_1 \vee x_2} \right) \tag{28}$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. We define $\beta_u = \beta_u(p)$ as the threshold value for convergence, i.e. such that $\Gamma(p, \beta) < \infty$ for $\beta < \beta_u$ and $\Gamma(p, \beta) = \infty$ for $\beta > \beta_u$. It is easy to see, giving a lower bound for $\Gamma(p, \beta)$, that $\beta_u < \infty$ for all p . Let $T_0 := \inf\{j > 1 : X_j = 0\}$, and let T_A be the time spent in the dead end A . Then, on $\{\omega_r = A\}$, $E_\omega^\beta(T_A) = E_\omega^\beta(T_0 | X_1 \geq 0)$.

Lemma 14. *For $\beta > \beta_u$,*

$$E^\beta(T_0) = \infty. \tag{29}$$

Proof. We will show that for $\beta > \beta_u$, the expected time spent in a dead end starting at 0 is infinite, giving a lower bound for the latter by considering the time spent in the dead end up to the first return to \mathcal{L} . Consider the random walk on A , starting from 0. Let $T_{A,0} := \inf\{j > 1 : Z_j \in \mathcal{L}\}$. We have, on $\{\omega_r = A\}$,

$$E_\omega^\beta(T_{A,0} | X_1 \geq 0) \geq \frac{2}{3 + \beta} N(A)^{-1} \sum_{e=(z_1, z_2) \in E_A} \beta^{x_1 \vee x_2} \tag{30}$$

This follows from the fact that for a recurrent Markov chain on A with invariant measure π , the expected return time to a vertex z is $\pi(A)/\pi(z)$. In our case, the invariant measure $\pi(z)$ is given by the sum of the weights of all edges $e = (z, \cdot)$ where the weight of an edge $e = (z_1, z_2)$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ is given by $\beta^{x_1 \vee x_2}$, hence $\pi(A) = 2 \sum_{e=(z_1, z_2) \in E_A} \beta^{x_1 \vee x_2}$. (30) now follows by merging all of the vertices of $A \cap \mathcal{L}$ into one vertex. □

Lemma 15. *For $\beta > \beta_u$, the speed of the random walk is zero.*

Proof. We define a sequence of **ladder times** L_1, L_2, \dots . Let L_1 be the first fresh epoch such that Z_{L_1} is the beginning of a dead end. Let A_{L_1} be the dead end starting at L_1 and $d(A_{L_1})$ its depth. Let L_2 be the first fresh epoch such that $X_{L_2} > X_{L_1} + d(A_{L_1})$ and Z_{L_2} is the beginning of a dead end, and continue the recursion. If n is a fresh epoch, the environment to the right of Z_n has the same distribution as the environment to the right of the origin under \widehat{P}_p . Therefore, the probability that the first hitting time of $\{(x, y) : x = X_{L_i} + d(A_{L_i}) + 1\}$ is a ladder time is strictly positive and does not depend on i . In particular, there are infinitely many ladder times. We will show that $X_{L_n}/L_n \rightarrow 0$, \mathbb{P}^β -a.s. for $n \rightarrow \infty$. Note that $L_{i+1} - L_i \geq T_{A_i}$ and the random variables (T_{A_i}) are i.i.d. under \mathbb{P}^β and have, due to Lemma 14, infinite expectation for $\beta > \beta_u$. This implies that $L_n/n \rightarrow \infty$, \mathbb{P}^β -a.s. for $n \rightarrow \infty$. On the other hand, the random variables $X_{L_{i+1}} - X_{L_i}$ are i.i.d. and we claim that they have exponential tails and, in particular, finite expectations. To see this, note that due to Lemma 1 and Lemma 10, the depth of a dead end has an exponential tail, i.e. $\widehat{P}_p(d(A_0) \geq s) \leq \exp(-c(p)s)$ for s large enough, where $c(p)$ is some constant depending only on p . For an integer t which is divisible by 20, we want to estimate the probability of the event

$$X_{L_{i+1}} - X_{L_i} > t.$$

Let $s = t/20$. Let $\tau_j := \inf\{k : X_k = X_{L_i} + 10j\}$, $j = 1, 2, \dots$. Let B denote the event that 0 is connected to $\{(10, y) : y \in \mathbb{Z}\}$ if we remove all the vertices on the line $\{(-1, y) : y \in \mathbb{Z}\}$, and let $\gamma = \widehat{P}_p(B)$. Then, conditioning on the event that the dead end beginning at L_i has depth at most $\frac{1}{2}t$, consider the fresh epochs τ_j , $j = 11s, \dots, 20s$. They have either to be beginnings of dead ends or they have to be connected to the next line at distance 10. Hence

$$\mathbb{P}^\beta(X_{L_{i+1}} - X_{L_i} \geq t) \leq \exp\left(-c(p)\frac{1}{2}t\right) + \gamma^s \leq \exp(-\tilde{c}(p)t)$$

for some constant $\tilde{c}(p)$.

Hence, $\limsup X_{L_n}/n < \infty$, \mathbb{P}^β -a.s. and we conclude that $X_{L_n}/L_n \rightarrow 0$, \mathbb{P}^β -a.s. Since $L_{n+1}/L_n \rightarrow 1$, \mathbb{P}^β -a.s. for $n \rightarrow \infty$, this suffices to prove that $X_n/n \rightarrow 0$, \mathbb{P}^β -a.s. for $n \rightarrow \infty$. □

Lemma 16. *We have $\beta_u(p) \rightarrow 1$ for $p \searrow p_c$.*

Proof. Fix $\beta > 1$. Let $\partial_+ B_n := \{(x, y) : x = n \text{ and } |y| \leq n\}$. Then, for every n , using the proof of Lemma 14,

$$E^\beta(T_0|X_1 \geq 0) \geq \beta^n \widehat{P}_p(0 \text{ is connected to a vertex } v \in \partial_+ B_n) \times \widehat{P}_p(\text{the clusters of all vertices } z \in \partial_+ B_n \text{ are finite}).$$

Now, since $p > p_c$,

$$\widehat{P}_p(0 \text{ is connected to a vertex } v \in \partial_+ B_n) \geq \mu_p > 0. \tag{31}$$

Let $\delta > 0$ be such that

$$W := \beta(1 - \delta)^4 > 1.$$

For p close enough to p_c , since $\theta(p_c) = 0$, $P_p(C_0 \text{ finite}) \geq 1 - \delta$ (where $\theta(p)$ denotes the probability that the origin belongs to an infinite open cluster, and we refer to [13] for the fact that $\theta(p_c) = 0$). Hence, using the FKG inequality, $P_p(\text{the clusters of all vertices } z \in \partial_+ B_n \text{ are finite})$ can be estimated as follows. For p close enough to p_c ,

$$P_p(\text{the clusters of all vertices } z \in \partial_+ B_n \text{ are finite}) \geq (1 - \delta)^{4n}.$$

We conclude that also

$$\widehat{P}_p(\text{the clusters of all vertices } z \in \partial_+ B_n \text{ are finite}) \geq c(1 - \delta)^{4n}.$$

for some constant $c = c(p)$. Thus, for every n ,

$$E^\beta(T_0|X_1 \geq 0) \geq \mu_p \beta^n (1 - \delta)^{4n} = \mu_p W^n. \tag{32}$$

Since $W > 1$ and (32) holds for every n , we conclude that $E^\beta(T_0|X_1 \geq 0) = \infty$. Recalling (28) and (30), we see that $\Gamma(p, \beta) = \infty$, hence $\beta \geq \beta_u$. \square

Theorem 4 now follows from Lemma 15 and Lemma 16. \square

Acknowledgements. We thank Alan Hammond and Manjunath Krishnapur for useful discussions. We also thank the referee for helpful comments on the first version of the paper.

References

[1] Angel, O., Benjamini, I., Berger, N., Peres, Y.: Transience of percolation clusters on wedges. Preprint 2002, available at <http://arxiv.org/abs/math.PR/0206130>

[2] Antal, P., Pisztora, A.: On the chemical distance in supercritical Bernoulli percolation. *Ann. Probab.* **24**, 1036–1048 (1996)

[3] Barma, M., Dhar, D.: Directed diffusion in a percolation network. *J. Phys. C: Solid State Phys.* **16**, 1451–1458 (1983)

[4] Barsky, D.J., Grimmett, G.R., Newman, C.M.: Percolation in half-spaces: equality of critical densities and continuity of the percolation probability. *Probab. Theory Related Fields* **90**, 111–148 (1991)

- [5] Benjamini, I., Mossel, E.: On the mixing time of simple random walk on the super critical percolation cluster. Preprint 2002, Available at <http://front.math.ucdavis.edu/math.PR/0011092>
- [6] Benjamini, I., Pemantle, R., Peres, Y.: Unpredictable paths and percolation. *Ann. Probab.* **26**(3), 1198–1211 (1998)
- [7] Carne, T.K.: A transmutation formula for Markov chains. *Bull. Sci. Math. (2)* **109**(4), 399–405 (1985)
- [8] Comets, F., Zeitouni, O.: A law of large numbers for random walks in random mixing environments. Preprint, 2002
- [9] Dhar, D.: Diffusion and drift on Percolation networks in an external field. *J. Phys A.* **17**, L257–L259 (1984)
- [10] Dhar, D., Stauffer, D.: Drift and trapping in biased diffusion on disordered lattices. *Int. J. Mod. Phys. C* **9**(2), 349–355 (1998)
- [11] Doyle, P.G., Snell, E.J.: *Random walks and Electrical Networks*. Carus Math. Monographs 22, Math. Assoc. Amer., Washington, D.C. 1984
- [12] Durrett, R.: Oriented percolation in two dimensions. *Ann. Probab.* **12**(4), 999–1040 (1984)
- [13] Grimmett, G.R.: *Percolation*, 2nd edition, Springer, New-York, 1999
- [14] Grimmett, G.R., Kesten, H., Zhang, Y.: Random walk on the infinite cluster of the percolation model. *Probab. Theory Related Fields* **96**(1), 33–44 (1993)
- [15] Häggström, O., Mossel, E.: Nearest-neighbor walks with low predictability profile and percolation in $2 + \epsilon$ dimensions. *Ann. Probab.* **26**, 1212–1231 (1998)
- [16] Hoffman, C., Hecklen, D.: Return probabilities of a simple random walk on percolation clusters. Preprint, 1999
- [17] Liggett, T.M., Schonmann, R.H., Stacey, A.M.: Domination by product measures. *Ann. Probab.* **25**(1), 71–95 (1997)
- [18] Lyons, R., Pemantle, R., Peres, Y.: Biased random walk on Galton-Watson trees. *Probab. Theory Related Fields* **106**, 249–264 (1996)
- [19] Lyons, R., Peres, Y.: Probability on trees and network. Available at <http://www.math.gatech.edu/~rdlyons/prbtree/prbtree.html>, 2002
- [20] Rassoul Agha, F.: Multi-dimensional random walks in mixing random environments. Preprint, 2002
- [21] Shen, L.: Asymptotic properties of certain anisotropic random walks in random media. *Ann. Appl. Probab.* **12**, 477–510 (2002)
- [22] Sznitman, A.S.: On the anisotropic random walk on the supercritical percolation cluster. Preprint, 2002