BIASED RANDOM WALKS ON GALTON–WATSON TREES WITH LEAVES

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We consider a biased random walk X_n on a Galton–Watson tree with leaves in the sub-ballistic regime. We prove that there exists an explicit constant $\gamma = \gamma(\beta) \in (0, 1)$, depending on the bias β, such that $|X_n|$ is of order n^γ. Denoting Δ_n the hitting time of level n, we prove that $\Delta_n/n^{1/\gamma}$ is tight. Moreover, we show that $\Delta_n/n^{1/\gamma}$ does not converge in law (at least for large values of β). We prove that along the sequences $n_\lambda(k) = \lfloor \lambda\beta^{\gamma k} \rfloor$, $\Delta_n/n^{1/\gamma}$ converges to certain infinitely divisible laws. Key tools for the proof are the classical Harris decomposition for Galton–Watson trees, a new variant of regeneration times and the careful analysis of triangular arrays of i.i.d. heavy-tailed random variables.

1. Introduction and statement of the results. Consider a supercritical Galton–Watson branching process with generating function $\mathbf{f}(z) = \sum_{k\geq 0} p_k z^k$, that is, the offspring of all individuals are i.i.d. copies of Z, where $\mathbf{P}[Z = k] = p_k$. We assume that the tree is supercritical and has leaves, that is, $\mathbf{m} := \mathbf{E}[Z] = \mathbf{f}'(1) \in (1, \infty)$ and $p_0 > 0$. We denote by $q \in (0, 1)$ the extinction probability, which is characterized by $\mathbf{f}(q) = q$. Starting from a single progenitor called root and denoted by 0, this process yields a random tree T. We will always condition on the event of nonextinction, so that T is an infinite random tree. We denote $(\Omega, \mathbf{P})$ the associated probability space: $\mathbf{P}$ is the law of the original tree, conditioned on nonextinction. For a vertex $u \in T$, we denote by $|u| = d(0, u)$ the distance of u to the root.

For $\omega \in \Omega$, on the infinite Galton–Watson tree $T(\omega)$, we consider the β-biased random walk as in [16]. More precisely, we define, for $\beta > 1$, a Markov chain $(X_n)_{n\in\mathbb{N}}$ on the vertices of T, such that if $u \neq 0$ and u has k children $v_1, \ldots, v_k$ and parent $\overleftarrow{u}$, then:

(1) $P[X_{n+1} = \overleftarrow{u} | X_n = u] = \frac{1}{1+\beta k}$,

(2) $P[X_{n+1} = v_i | X_n = u] = \frac{\beta}{1+\beta k}$, for $1 \leq i \leq k$,

and from 0 all transitions to its children are equally likely. This is a reversible Markov chain, and as such, can be described as an electrical network with conductances $c(\overleftarrow{x}, x) := \beta^{|x|-1}$ on every edge of the tree (see [17] for background on electrical networks).

We always take $X_0 = 0$, that is, we start the walk from the root of the tree. We denote by $P^\omega[\cdot]$ the law of $(X_n)_{n=0,1,2,\ldots}$ and we define the averaged law as the semidirect product $\mathbb{P} = \mathbf{P} \times P^\omega$.

Many interesting facts are known about this walk (see [16]). As one might expect, it is transient. It is known that $\mathbb{P}$-a.s., $|X_n|/n$ converges to a deterministic limit v. Moreover, the random walk is *ballistic*, that is, its limiting velocity $v > 0$, if and only if $\beta < \beta_c = 1/\mathbf{f}'(q)$. In the *sub-ballistic* regime, that is, if $\beta \geq \beta_c$, we have $v = 0$. The reason for the sub-ballistic regime is that the walk loses time in traps of the tree, from where it cannot go to infinity without having to go for a long time against the drift which keeps it into the trap. The hypothesis $p_0 > 0$ is crucial for this to happen.

As in all sub-ballistic models, a natural question comes up: what is the typical distance of the walker from the root after n steps? This is the question we address in this paper. We always assume that

$$\mathbf{E}[Z^2] < \infty$$

and

$$\beta > 1/\mathbf{f}'(q),$$

recalling that $1/\mathbf{f}'(q) > 1$. We introduce the exponent

$$(1.1) \qquad \gamma := \frac{-\ln \mathbf{f}'(q)}{\ln \beta} = \frac{\ln \beta_c}{\ln \beta} < 1$$

so that $\beta^\gamma = 1/\mathbf{f}'(q)$.

Let Δ_n be the hitting time of the nth level:

$$\Delta_n = \inf\{i \geq 0 : |X_i| = n\}.$$

THEOREM 1.1.   (i) *The laws of $(\Delta_n/n^{1/\gamma})_{n \geq 0}$ under $\mathbb{P}$ are tight.*
  (ii) *The laws of $(|X_n|/n^\gamma)_{n \geq 0}$ under $\mathbb{P}$ are tight.*
  (iii) *We have*

$$(1.2) \qquad \lim_{n \to \infty} \frac{\ln |X_n|}{\ln n} = \gamma, \qquad \mathbb{P}\text{-a.s.}$$

Of course, this raises the question of convergence in distribution of the sequence $(\Delta_n/n^{1/\gamma})_{n \geq 0}$. The next theorem gives a negative answer.

THEOREM 1.2.   *For β large enough, the sequence $(\Delta_n/n^{1/\gamma})_{n \geq 0}$ does not converge in distribution.*

However, we can establish convergence in distribution along certain subsequences.

THEOREM 1.3. *For any $\lambda > 0$, denoting $n_\lambda(k) = \lfloor \lambda \mathbf{f}'(q)^{-k} \rfloor$, we have*

$$\frac{\Delta_{n_\lambda(k)}}{n_\lambda(k)^{1/\gamma}} \xrightarrow{d} Y_\lambda,$$

where the random variable Y_λ has an infinitely divisible law μ_λ.

We now describe the limit laws μ_λ. For some constants ρ and C_a [the constant ρ is defined in (2.2), the constant C_a in Lemma 6.1], we have

$$Y_\lambda = (\rho C_a \lambda)^{1/\gamma} \widetilde{Y}_{(\rho C_a \lambda)^{1/\gamma}},$$

where

$$\widetilde{Y}_\lambda \text{ has the law } \mathfrak{I}(d_\lambda, 0, \mathcal{L}_\lambda).$$

The infinitely divisible law $\mathfrak{I}(d_\lambda, 0, \mathcal{L}_\lambda)$ is given by its Lévy representation (see [19], page 32). More precisely, the characteristic function of $\mathfrak{I}(d_\lambda, 0, \mathcal{L}_\lambda)$ can be written in the form

$$\mathbb{E}[e^{it\widetilde{Y}_\lambda}] = \int e^{itx} \mathfrak{I}(d_\lambda, 0, \mathcal{L}_\lambda)(dx)$$

$$= \exp\left(id_\lambda t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) d\mathcal{L}_\lambda(x)\right),$$

where d_λ is a real constant and $\mathcal{L}_\lambda$ a real function which is nondecreasing on the interval $(0, \infty)$ and satisfies $\mathcal{L}_\lambda(x) \to 0$ for $x \to \infty$ and $\int_0^a x^2 d\mathcal{L}_\lambda(x) < \infty$ for every $a > 0$. Comparing to the general representation formula in [19], page 32, we here have that the Gaussian part vanishes and $\mathcal{L}_\lambda(x) = 0$ for $x < 0$. The function $\mathcal{L}_\lambda$ is called the Lévy spectral function. Note that $\mathcal{L}_\lambda$ is not a Lévy–Khintchine spectral function.

In order to describe $\mathcal{L}_\lambda$, define the random variable

$$(1.3) \qquad \mathcal{Z}_\infty = \frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\mathrm{Bin}(W_\infty, p_\infty)} \mathbf{e}_i,$$

where $p_\infty = 1 - \beta^{-1}$ is the escape probability of a β-biased random walk on $\mathbb{N}$. Further, the random variables $\mathbf{e}_i$ in (1.3) are i.i.d. exponential random variables of parameter 1 and the nonnegative random variables $(\mathbf{e}_i)$, W_∞ and S_∞ in (1.3) are independent. The random variables S_∞ and W_∞ will be described in (3.6) and Proposition 6.1, respectively. The random variable $\mathrm{Bin}(W_\infty, p_\infty)$ has a Binomial law with parameters W_∞ and p_∞. Now, denoting by $\overline{F}_\infty(x) = \mathbb{P}[\mathcal{Z}_\infty > x]$ the tail function of $\mathcal{Z}_\infty$, we have:

THEOREM 1.4.   *For all $\lambda > 0$, the following statements hold*:

(i) *The Lévy spectral function $\mathcal{L}_1$ is given as follows*:

$$\mathcal{L}_1(x) = \begin{cases} 0, & \text{if } x < 0, \\ -(1 - \beta^{-\gamma}) \sum_{k \in \mathbb{Z}} \beta^{\gamma k} \overline{F}_\infty(x\beta^k), & \text{if } x > 0. \end{cases}$$

In particular, $\mathcal{L}_1(x) = \beta^\gamma \mathcal{L}_1(\beta x)$.

(ii) *For all $x \in \mathbb{R}$, $\mathcal{L}_\lambda(x) = \lambda^\gamma \mathcal{L}_1(\lambda x)$. In particular, $\mathcal{L}_{\beta^j}(x) = \mathcal{L}_1(x)$, for all integers j.*

(iii) *d_λ is given by*

$$d_\lambda = \lambda^{1+\gamma}(1 - \beta^{-\gamma}) \sum_{k \in \mathbb{Z}} \beta^{(1+\gamma)k} E\left[\frac{\mathcal{Z}_\infty}{(\lambda\beta^k)^2 + \mathcal{Z}_\infty^2}\right].$$

(iv) *$\mathcal{L}_\lambda$ is absolutely continuous.*

(v) *The following bounds hold*:

(1.4) $$\frac{1}{\beta^\gamma}\mathbb{E}[\mathcal{Z}_\infty^\gamma]\frac{1}{x^\gamma} \le -\mathcal{L}_1(x) \le \mathbb{E}[\mathcal{Z}_\infty^\gamma]\frac{1}{x^\gamma}.$$

(vi) *The measure μ_λ is absolutely continuous with respect to Lebesgue measure and has a moment of order α if and only if $\alpha < \gamma$.*

(vii) *When β is large enough, $x^\gamma \mathcal{L}_\lambda(x)$ is not a constant.*

(viii) *The random variable $\mathcal{Z}_\infty$ has an atom at 0 and a smooth density ψ on $(0, \infty)$. Further, $\mathcal{Z}_\infty$ has finite expectation.*

REMARK 1.1.   We believe that Theorem 1.2 holds true for all values $\beta > \beta_c$. The proof would amount to showing that the function $x^\gamma \mathcal{L}_1(x)$, with $\mathcal{L}_1(x)$ given in Theorem 1.4, is not a constant.

Next we explain briefly, using a toy example, the reason for the nonconvergence of $(\Delta_n/n^{1/\gamma})_{n\ge 0}$ and the convergence of subsequences in Theorems 1.2 and 1.3. The reasons lie in the classical theory of sums of i.i.d. random variables. Consider a sequence of i.i.d. random variables G_i, geometrically distributed with parameter a. Let

$$S_n = \sum_{i=1}^n \beta^{G_i}.$$

It is easy to see, using classical results about triangular arrays of i.i.d. random variables (cf. [19]), that for $\alpha = \frac{-\log(1-a)}{\log \beta}$ and $n_\lambda(k) = \beta^{-\alpha k}$, the distributions of

$$\frac{1}{n_\lambda(k)^{1/\alpha}} S_{n_\lambda(k)} \text{ converge to an infinitely divisible law}$$

(see Theorem 10.1 for a more general result). But obviously here $S_n/n^{1/\alpha}$ cannot converge in law, if $\alpha < 2$, because one easily checks that the distribution of β^{G_1} does not belong to the domain of attraction of any stable law. This is the basis of our belief that Theorem 1.2 should be valid for any $\beta > \beta_c$.

We now discuss the motivation for this work. If one considers a biased random walk on a supercritical percolation cluster on $\mathbb{Z}^d$ (introduced by [8]), it is known that, at low bias, the random walk is ballistic (i.e., has a positive velocity) and has Gaussian fluctuations; see [6] and [21]. It is also known that, at strong bias, the random walk is sub-ballistic (i.e., the velocity vanishes). It should be noted that, in contrast to the Galton–Watson tree, the existence of a critical value separating the two regimes is not established for supercritical percolation clusters. The behavior of the (law of) the random walk in the sub-ballistic regime is a very interesting open problem. It was noted in [22] that the behavior of the random walk in this regime is reminiscent of trap models introduced by Bouchaud (see [4] and [7]). Our work indeed substantiates this analogy in the simpler case of supercritical random trees. We show that most of the time spent by the random walk before reaching level n is spent in deep traps. These trapping times are roughly independent and are heavy-tailed. However, their distribution does not belong to the domain of attraction of a stable law, which explains the nonconvergence result in Theorem 1.2.

We note that it is possible to obtain convergence results to stable laws if one gets rid of the inherent lattice structure. One way to do this is to randomize the bias β. This is the approach of the forthcoming papers [2, 3].

For other recent interesting works about random walks on trees, we refer to [1, 13] and [18].

There is also an analogy with the one-dimensional random walk in an i.i.d. random environment (RWRE). This model also shows a ballistic and a sub-ballistic regime, explicitly known in terms of the parameters of the model. We refer to [25] for a survey. In the sub-ballistic regime, it was shown in [14] that depending on a certain parameter $\kappa \in (0, 1]$, and under a nonlattice assumption, $\frac{X_n}{n^\kappa}$ converges to a functional of a stable law, if $\kappa < 1$, and $\frac{X_n}{n/\ln n}$ converges to a functional of a stable law, if $\kappa = 1$. Recently, using a precise description of the environment, [10] and [9] refined this last theorem by describing all the parameters of the stable law, in the case $\kappa < 1$.

Our method has some similarity to the one used in [9]. In comparison to [9], an additional difficulty arises from the fact the traps met depend not only on the environment but also on the walk. Moreover, one has to take into account the number of times the walker enters a trap, which is a complicated matter because of the inhomogeneity of the tree. This major technical difficulty can be overcome by decomposing the tree and the walk into independent parts, which we do using a new variant of regeneration times.

The paper is organized as follows: in Sections 2 and 3, we explain how to decompose the tree and the walk. In Section 4, we give a sketch of the proof of

Theorem 1.3. Sections 5–9 prepare the proof of Theorem 1.3 and explain why the hitting time of level n is comparable to a sum of i.i.d. random variables. Section 10 is self-contained and its main result, Theorem 10.1, is a classical limit theorem for sums of i.i.d. random variables which is tailored for our situation. In Section 11, we finally give the proofs of the results. In Section 11.1, we apply Theorem 10.1 to prove Theorem 1.3. Section 11.2 is devoted to the proof of Theorem 1.2, Section 11.3 gives the proof of Theorem 1.1 and Section 11.4 the proof of Theorem 1.4.

Let us give some conventions about notation. The parameters β and $(p_k)_{k \geq 0}$ will remain fixed so we will usually not point out that constants depend on them. Most constants will be denoted c or C and their value may change from line to line to ease notation. Specific constants will have a subscript as, for example, C_a. We will always denote by $G(a)$ a geometric random variable of parameter a, with law given by $P[G_a \geq k] = (1 - a)^{k-1}$ for $k \geq 1$.

2. Constructing the environment and the walk in the appropriate way. In order to understand properly the way the walk is slowed down, we need to decompose the tree; see Figure 1. Set

$$(2.1) \qquad \mathbf{g}(s) = \frac{\mathbf{f}((1-q)s + q) - q}{1 - q} \quad \text{and} \quad \mathbf{h}(s) = \frac{\mathbf{f}(qs)}{q}.$$

It is known (see [15]) that a $\mathbf{f}$-Galton–Watson tree (with $p_0 > 0$) can be generated by:

(i) growing a $\mathbf{g}$-Galton–Watson tree $T_{\mathbf{g}}$ called the backbone, where all vertices have an infinite line of descent,
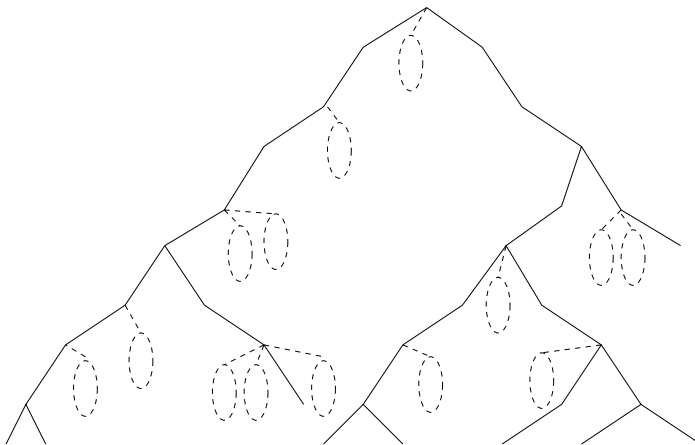

FIG. 1. *The Galton–Watson tree is decomposed into the backbone (solid lines) and the traps (dashed lines).*

(ii) attaching on each vertex x of $T_{\mathbf{g}}$ a random number N_x of $\mathbf{h}$-Galton–Watson trees, acting as traps in the environment T,

where N_x has a distribution depending only on $\deg_{T_{\mathbf{g}}}(x)$ and given $T_{\mathbf{g}}$ and N_x the traps are i.i.d., see [15] for details.

We will refer to any vertex at distance exactly one of the backbone as a bud. It is important to consider the backbone together with the buds to understand the number of visits to traps.

It will be convenient to consider the attached Galton–Watson trees together with the edge which connects them to the backbone. We define a *trap* to be a graph $(x \cup V, [x, y] \cup E)$, where x is a vertex of the backbone, y is a bud adjacent to x and V (resp., E) are the vertices (resp., edges) of the descendants of y. The traps can themselves be decomposed in a portion of $\mathbb{Z}$ called the spine, to which smaller trees called subtraps are added: this construction is presented in detail in Section 3.

Let us now construct the random walk. We need to consider the walk on the backbone and on the buds, to this end we introduce:

(1) $\sigma_0 = \sigma_0' = 0$,
(2) $\sigma_{n+1} = \inf\{i > \sigma_n | X_{i-1}, X_i \in \text{backbone}\}$,
(3) $\sigma_{n+1}' = \inf\{i > \sigma_n' | X_{i-1}, X_i \in \text{backbone} \cup \text{buds}\}$,

and we define $Y_n = X_{\sigma_n}$ the embedded walk on the backbone, respectively, $Y_n' = X_{\sigma_n'}$ the embedded walk on the backbone and the buds.

Moreover define $\Delta_n^Y = \text{card}\{i \geq 0 : \sigma_i \leq \Delta_n\}$ the time spent on the backbone to reach level n and similarly $\Delta_n^{Y'} = \text{card}\{i \geq 0 : \sigma_i' \leq \Delta_n\}$.

Denote, for a set A in the tree, $T_A^+ = \min\{n \geq 1 | X_n \in A\}$, $T_y^+ := T_{\{y\}}^+$, $T_A := \min\{n \geq 0 | X_n \in A\}$ and $T_y := T_{\{y\}}$.

Note that the process $(Y_n)_{n \geq 0}$ is a Markov chain on the backbone, which is independent of the traps and the time spent in the traps. Here one has to be aware that visits to *root* do not count as "time spent in a trap," precise definitions will follow below. Hence, in order to generate Y_n we use a sequence of i.i.d. random variables U_i uniformly distributed on $[0, 1]$. If $Y_j = w$ with Z_1^* children on the backbone, then:

(1) $Y_{i+1} = \overleftarrow{w}$, if $U_i \in [0, \frac{1}{Z_1^* \beta + 1}]$,
(2) $Y_{i+1} =$ the jth-child of w, if $U_i \in [1 - \frac{j\beta}{Z_1^* \beta + 1}, 1 - \frac{(j-1)\beta}{Z_1^* \beta + 1}]$.

For background on regeneration times we refer to [23] or [25]. In the case of a β-biased random walk $\widetilde{Y}_n$ on $\mathbb{Z}$, a time t is a regeneration time if

$$\widetilde{Y}_t > \max_{s < t} \widetilde{Y}_s \quad \text{and} \quad \widetilde{Y}_t < \min_{s > t} \widetilde{Y}_s.$$

DEFINITION 2.1.   A time t is a super-regeneration time for Y_n, if t is a regeneration time for the corresponding β-biased random walk $\widetilde{Y}_n$ on $\mathbb{Z}$ defined by:

(i) $\widetilde{Y}_0 = 0$,
(ii) $\widetilde{Y}_{n+1} = \widetilde{Y}_n - 1$, if $U_n \in [0, \frac{1}{\beta+1}]$,
(iii) $\widetilde{Y}_{n+1} = \widetilde{Y}_n + 1$ otherwise.

We denote t-SR the event that t is a super-regeneration time for Y_n.

It is obvious that a super-regeneration time for Y_n is a regeneration time for Y_n in the usual sense (the converse is false).

The walk can then be decomposed between the successive super-regeneration times:

(i) $\tau_0 = 0$,
(ii) $\tau_{i+1} = \inf\{j \geq \tau_i : j\text{-SR}\}$.

Since the regeneration times of a β-biased random walk on $\mathbb{Z}$ have some exponential moments, there exists $a > 1$ such that $\mathbb{E}[a^{\tau_2 - \tau_1}] < \infty$ and $\mathbb{E}[a^{\tau_1}] < \infty$.

REMARK 2.1. The advantage of super-regeneration times compared to classical regeneration times is that the presence of a super-regeneration time does not depend on the environment, but only the on the sequence $(U_i)_{i \geq 0}$.

REMARK 2.2. The drawback of super-regeneration times is that the event that k is a super-regeneration time depends on the random variables $(U_i)_{i \geq 0}$ and not only on the trajectory of the random walk $(Y_n)_{n \geq 0}$.

Denoting for $k \geq 1$, the σ-field

$$\mathcal{G}_k = \sigma\big(\tau_1, \ldots, \tau_k, (Y_{n \wedge \tau_k})_{n \geq 0}, \{x \in T(\omega), x \text{ is not a descendant of } Y_{\tau_k}\}\big).$$

We have the following proposition.

PROPOSITION 2.1. *For $k \geq 1$,*

$$\mathbb{P}[(Y_{\tau_k+n} - Y_{\tau_k})_{n \geq 0} \in \cdot, \{x \in T(\omega), x \text{ is a descendant of } Y_{\tau_k}\} \in \cdot | \mathcal{G}_k]$$
$$= \mathbb{P}[(Y_n)_{n \geq 0} \in \cdot, T(\omega) \in \cdot | 0\text{-SR}].$$

REMARK 2.3. The conditioning 0-SR refers only to the walk on the backbone, hence it is obvious that the behavior of the walk in the traps and the number of times the walker enters a trap is independent of that event.

We skip the proof of this proposition since it is standard. A consequence of the proposition is that the environment and the walk can be subdivided into super-regeneration blocks which are i.i.d. (except for the first one). As a consequence, we have that

(2.2)
$$\rho_n := \frac{\operatorname{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}}{n} \quad \text{satisfies}$$

$$\rho_n \to \rho := \frac{\mathbb{E}[\operatorname{card}\{Y_{\tau_1}, \ldots, Y_{\tau_2-1}\}]}{\mathbb{E}[\tau_2 - \tau_1]}, \qquad \mathbb{P}\text{-a.s.},$$

which is the average number of vertices per level visited by Y_n. This quantity is finite since it is bounded above by than $1/v(\beta)$, where $v(\beta)$ is the speed of $|Y_n|$ which is strictly positive by a comparison to the β-biased random walk on $\mathbb{Z}$.

When applying the previous proposition, it will be convenient to use the time-shift for the random walk, which we will denote by θ.

3. Constructing a trap. In the decomposition theorem for Galton–Watson trees, we attach to the vertices of the backbone a (random) number of **h**-Galton–Watson trees. We will denote their distribution with **Q**, hence $\mathbf{Q}[Z = k] = q_k :=$ $p_k q^{k-1}$, where Z denotes the number of children of a given vertex. As stated before the object, we will denote a trap has an additional edge: to describe a trap ℓ we take a vertex called *root* [or *root*(ℓ) to emphasize the trap], link it to another vertex [denoted $\overrightarrow{root}(\ell)$], which is the actual root of a random **h**-Galton–Watson tree.

When we use random variables associated to a trap, we refer to the random part of that trap (the **h**-Galton–Watson tree). For example, the notation Z_n is the number of children at the generation n with $\overrightarrow{root}$ being generation 0. In particular, we introduce the height of a trap

$$(3.1) \qquad\qquad H = \max\{n \geq 0, Z_n > 0\},$$

and we say a trap has height k if $H(\ell) = k$, that is, the distance between $\overrightarrow{root}$ and the bottom point of the trap is k.

This way of denoting the random variables has the advantage that Z_n (resp., H) are distributed under **Q**, as the number of children at generation n (resp., the height) of an **h**-Galton–Watson tree.

The biggest traps seen up to level n are of size $-\ln n/\ln \mathbf{f}'(q)$, therefore a trap will be considered big if its height is greater or equal to

$$(3.2) \qquad\qquad h_n = \left\lceil (1-\varepsilon)\frac{\ln n}{-\ln \mathbf{f}'(q)} \right\rceil$$

for some $\varepsilon > 0$ which will eventually be chosen small enough. Such a trap will be called an h_n-trap or a big trap. It is in those traps that the walker will spend the majority of his time and therefore it is important to have a good description of them.

The traps are (apart from the additional edge) subcritical Galton–Watson trees, as such, they can be grown from the bottom following a procedure described in [11], that we recall for completeness. We will denote by δ the starting point of the procedure, corresponding to the leftmost bottom point of the trap, this last notation will be kept for the whole paper.

With a slight abuse of notation, we will denote by **Q** a probability measure on an enlarged probability space containing the following additional information.

We denote by (ϕ_{n+1}, ψ_{n+1}) with $n \geq 0$, a sequence of i.i.d. pairs of random variables with joint law given by

$$(3.3) \quad \mathbf{Q}[\phi_{n+1} = j, \psi_{n+1} = k] = c_n q_k \mathbf{Q}[Z_n = 0]^{j-1} \mathbf{Q}[Z_{n+1} = 0]^{k-j},$$

$$1 \leq j \leq k, k \geq 1,$$

where $c_n = \frac{\mathbf{Q}[H=n]}{\mathbf{Q}[H=n+1]}$.

Set $\mathcal{T}_0 = \{\delta\}$. Construct $\mathcal{T}_{n+1}$, $n \geq 0$, inductively as follows:

(1) let the first generation size of $\mathcal{T}_{n+1}$ be ψ_{n+1},

(2) let $\mathcal{T}_n$ be the subtree founded by the ϕ_{n+1}th first generation vertex of $\mathcal{T}_{n+1}$,

(3) attach independent **h**-Galton–Watson trees which are conditioned on having height strictly less than n to the $\phi_{n+1} - 1$ siblings to the left of the distinguished first generation vertex,

(4) attach independent **h**-Galton–Watson trees which are conditioned on having height strictly less than $n + 1$ to the $\psi_{n+1} - \phi_{n+1}$ siblings to the right of the distinguished first generation vertex.

Then $\mathcal{T}_{n+1}$ has the law of an **h**-Galton–Watson tree conditioned to have height $n + 1$ (see [11]).

We denote $\mathcal{T}$ the infinite tree asymptotically obtained by this procedure; from this tree we can obviously recover all $\mathcal{T}_n$. If we pick independently the height H of an **h**-Galton–Watson tree and the infinite tree $\mathcal{T}$ obtained by the previous algorithm, then $\mathcal{T}_H$ has the same law as an **h**-Galton–Watson tree.

We will call the spine of this Galton–Watson tree the ancestors of δ. If $y \neq \delta$ is in the spine, $\overrightarrow{y}$ denotes its only child in the spine. We define a subtrap to be a graph $(x \cup V, [x, y] \cup E)$, where x is a vertex of the spine, y is a descendant of x not on the spine and V (resp., E) are the vertices (resp., edges) of the descendants of y. The vertex x is called the root of the subtrap, and we denote

$$(3.4) \qquad S_x \text{ the set of all subtraps rooted at } x.$$

See Figure 2 for an illustration.

We denote by $S_{n+1}^{i,j,k}$ and $\Pi_{n+1}^{i,j,k}$ with $n, i, j \geq 0$ and $k = 1, 2$, two sequences of independent random variables, which are independent of $(\phi_n, \psi_n)_{n \geq 0}$ and given by:

(1) $S_n^{n+1,j,1}$ (resp., $S_n^{n,j,2}$) is the jth subtrap conditioned to have height less than n added on the left (resp., right) of the $(n + 1)$th (resp., nth) ancestor of δ,

(2) $\Pi_n^{i,j,k}$ is the weight of $S_n^{i,j,k}$ under the invariant measure associated to the conductances β^{i+1} between the level i and $i + 1$, the root of $S_n^{i,j,k}$ being counted as level 0.

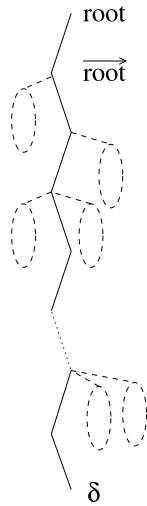These random variables describe the subtraps and their weights.

FIG. 2. *The trap is decomposed into the spine (*solid lines*) and the subtraps (*dashed lines*).*

We denote $\Pi_{-1}^{i,j,k} = 0$ and

$$(3.5) \qquad \Lambda_i(\omega) = \sum_{j=1}^{\phi_i - 1} \Pi_{i-1}^{i,j,1} + \sum_{j=1}^{\psi_i - \phi_i} \Pi_i^{i,j,2},$$

which is the weight of the subtraps added to the ith ancestor of δ.

Due to the next lemma, the random variables Λ_i will be important to describe the time spent in traps. We recall that a reversible Markov chain can be described as an electrical network. An electrical network is a connected graph $G = (V, E)$ with positive conductances $c(e)_{e \in E}$ on its edges. It defines a Markov chain with transition probabilities $p(x, y) = c([x, y]) / \sum_{z : [x,z] \in E} c([x, z])$, where we denote $[x, y]$ the edge connecting x and y [and $p(x, y) = 0$ if there is no such edge]. This Markov chain has $\hat{\pi}(x) = \sum_{[x,y] \in E} c([x, y])$, $x \in V$, as an invariant measure. If it is positive recurrent, the unique invariant probability measure is given by $\hat{\pi}(x) / \hat{\pi}(V)$, where $\hat{\pi}(V) = \sum_{y \in V} \hat{\pi}(y)$. We refer to [17] for background on electrical networks, and we recall the following lemma.

LEMMA 3.1. *Let $(G, c(e)_{e \in E})$ be a positive recurrent electrical network, $x \in V$ and P_x the law of the Markov chain started at x. If $\sum_{z : [x,z] \in E} c([x, z]) = 1$, then*

$$E_x[T_x^+] = 2 \sum_{e \in E} c(e).$$

PROOF. The expected number of visits to y before returning to x is $\widehat{\pi}(y)/\widehat{\pi}(x)$. Hence,

$$E_x[T_x^+] = \frac{\widehat{\pi}(V)}{\widehat{\pi}(x)} = \widehat{\pi}(V),$$

since $\widehat{\pi}(x) = \sum_{z:[x,z]\in E} c([x,z]) = 1$. Then we simply notice that $\widehat{\pi}(V) = 2\sum_{e\in E} c(e)$. $\square$

Let us introduce another important random variable

$$(3.6) \qquad S_\infty = 2\sum_{i=0}^{\infty} \beta^{-i}(1 + \Lambda_i),$$

which appears in the statement of Theorem 1.4. It is the mean return time to δ of the walk on the infinite tree $\mathcal{T}$ described in the algorithm following (3.3).

LEMMA 3.2. *There exists a constant C_ψ depending on $(p_k)_{k\geq 0}$, such that for $n \geq 0$ and $k \geq 0$,*

$$\mathbf{Q}[\psi_{n+1} = k] \leq C_\psi k q^k.$$

In particular, for another constant $\tilde{C}_\psi$, $\sup_{i\in\mathbb{N}} E_{\mathbf{Q}}[\psi_i] \leq \tilde{C}_\psi < \infty$.

PROOF. Recalling (3.3), we get

$$\mathbf{Q}[\psi_{n+1} = k] = \sum_{j=1}^{k} \mathbf{Q}[\phi_{n+1} = j, \psi_{n+1} = k]$$

$$= c_n q_k \sum_{j=1}^{k} \mathbf{Q}[Z_n = 0]^{j-1} \mathbf{Q}[Z_{n+1} = 0]^{k-j}$$

$$\leq c_n k q_k.$$

It is enough to show that the sequence $(c_n)_{n\geq 0}$ is bounded from above. A Galton–Watson tree of height $n + 1$ can be obtained as *root* having j children, one of which produces a Galton–Watson tree of height n, the others having no children of their own. Thus,

$$1/c_n = \mathbf{Q}[H = n + 1]/\mathbf{Q}[H = n] \geq q_j q_0^{j-1}$$

for any $j \geq 1$. We fix $j_0 \geq 1$ so that $q_{j_0} > 0$ and we get

$$\mathbf{Q}[\psi_{n+1} = k] \leq \frac{1}{q_{j_0} q_0^{j_0-1}} k q^{k-1},$$

where we used $q_k = p_k q^{k-1} \leq q^{k-1}$. $\square$

Using this lemma, we can get a tail estimate for the height of traps.

LEMMA 3.3.   *There exists $\alpha > 0$ such that*

$$\mathbf{Q}[H \geq n] \sim \alpha \mathbf{f}'(q)^n.$$

PROOF.   It is classical (see [12]) that for any Galton–Watson tree of law $\widetilde{Q}$ with $E_{\widetilde{Q}}[Z_1] = m < 1$ expected number of children, we have

$$\lim_{n \to \infty} \frac{\widetilde{Q}[Z_n > 0]}{m^n} > 0 \quad \Longleftrightarrow \quad E_{\widetilde{Q}}[Z_1 \log^+ Z_1] < \infty.$$

The integrability condition is satisfied for $\mathbf{Q}$ since $q_k = p_k q^{k-1} \leq q^{k-1}$, and the result follows.   □

We also recall the following classical upper bound

$$(3.7) \qquad \mathbf{Q}[H \geq n] = \mathbf{Q}[Z_n > 0] = \mathbf{Q}[Z_n \geq 1] \leq E_{\mathbf{Q}}[Z_n] = \mathbf{f}'(q)^n.$$

The following lemma seems obvious, but not standard, so we include its proof for the convenience of the reader.

LEMMA 3.4.   *We have for $k \geq 0$,*

$$\mathbf{Q}[Z_1 \leq k | Z_n = 0] \geq \mathbf{Q}[Z_1 \leq k].$$

In particular $E_{\mathbf{Q}}[Z_i | Z_n = 0] \leq \mathbf{f}'(q)^i$, for any $i \geq 0$ and $n \geq 0$.

PROOF.   Denoting D_n a geometric random variable of parameter $1 - \mathbf{Q}[Z_{n-1} = 0]$ which is independent of Z_1, we have $\mathbf{Q}[Z_1 \leq k | Z_n = 0] = \mathbf{Q}[Z_1 \leq k | Z_1 < D_n]$. Then compute

$$\mathbf{Q}[Z_1 \leq k | Z_1 < D_n] = \frac{\sum_{j=0}^{k} \mathbf{Q}[D_n > j] \mathbf{Q}[Z_1 = j]}{\sum_{j=0}^{\infty} \mathbf{Q}[D_n > j] \mathbf{Q}[Z_1 = j]}$$

$$= \left(1 + \frac{\sum_{j=k+1}^{\infty} \mathbf{Q}[D_n > j] \mathbf{Q}[Z_1 = j]}{\sum_{j=0}^{k} \mathbf{Q}[D_n > j] \mathbf{Q}[Z_1 = j]}\right)^{-1},$$

now use that for all $j' < k < j$ we have $\mathbf{Q}[D_n > j] \leq \mathbf{Q}[D_n > k] \leq \mathbf{Q}[D_n > j']$, yielding

$$\mathbf{Q}[Z_1 \leq k | Z_1 < D_n] \geq \frac{\sum_{j=0}^{k} \mathbf{Q}[Z_1 = j]}{\sum_{j=0}^{\infty} \mathbf{Q}[Z_1 = j]} = \mathbf{Q}[Z_1 \leq k]. \qquad \qquad \square$$

We can now estimate $E_{\mathbf{Q}}[\Lambda_i]$.

LEMMA 3.5.   *For all $i \geq 0$,*

$$E_{\mathbf{Q}}[\Lambda_i] \leq \frac{\widetilde{C}_\psi}{1 - (\beta \mathbf{f}'(q))^{-1}} (\mathbf{f}'(q)\beta)^i.$$

PROOF. Using (3.5), Lemmas 3.2 and 3.4, we get

$$E_{\mathbf{Q}}[\Lambda_i] = E_{\mathbf{Q}}[\phi_i - 1]E_{\mathbf{Q}}[\Pi_{i-1}] + E_{\mathbf{Q}}[\psi_i - \phi_i]E_{\mathbf{Q}}[\Pi_i]$$

$$\leq E_{\mathbf{Q}}[\Pi_i]\sup_{i\in\mathbb{N}} E_{\mathbf{Q}}[\psi_i] \leq \widetilde{C}_\psi \sum_{j=1}^{i} \mathbf{f}'(q)^j \beta^j,$$

and the result follows immediately, since $\beta\mathbf{f}'(q) > 1$. $\square$

Finally, we get the following proposition.

PROPOSITION 3.1. *We have*

$$E_{\mathbf{Q}}[S_\infty] \leq \frac{2\widetilde{C}_\psi}{1 - (\beta\mathbf{f}'(q))^{-1}}\left(\frac{\beta}{\beta - 1} + \frac{1}{1 - \mathbf{f}'(q)}\right) < \infty.$$

PROOF. Recalling Lemma 3.5, we get

$$E_{\mathbf{Q}}[S_\infty] \leq 2\sum_{i=0}^{\infty} \beta^{-i} E_{\mathbf{Q}}[1 + \Lambda_i] \leq \frac{2\widetilde{C}_\psi}{1 - (\beta\mathbf{f}'(q))^{-1}} \sum_{i=0}^{\infty} \beta^{-i}\left(1 + (\beta\mathbf{f}'(q))^i\right) < \infty,$$

the last term being finite since $\mathbf{f}'(q) < 1$. $\square$

4. Sketch of the proof.

In the first step, we show (see Theorem 5.1) that the time is essentially spent in h_n-traps.

Then we show that these h_n-traps are far away from each other, and thus the correlation between the time spent in different h_n-traps can be neglected. Moreover, the number of h_n-traps met before level n is roughly $\rho C_a n^\varepsilon$. Let

(4.1) $\qquad \chi_0(n) = $ the time spent in the first h_n-trap met,

where we point out that there can be several visits to this trap. At this point, we have reduced our problem to estimating

$$\Delta_n \approx \chi_1(n) + \cdots + \chi_{\rho C_a n^\varepsilon}(n),$$

where $\chi_i(n)$ are i.i.d. copies of $\chi_0(n)$.

Now we decompose the time spent in the first h_n-trap according to the number of excursions in it starting from the *root*

$$\chi_1(n) = \sum_{i=1}^{W_n} T_0^{(i)},$$

where W_n denotes the number of visits of the trap until time n and $T_0^{(i)}$ an i.i.d. sequence of random variables measuring the time spent during an excursion in a big trap. It is important to notice that the presence of an h_n-trap at a vertex gives information on the number of traps at this vertex, and thus on the geometry of

the backbone. So the law of W_n depends on n. Nevertheless, we show that this dependence can be asymptotically neglected, and that for large n, W_n is close to some random variable W_∞ (Proposition 6.1).

Now we have essentially no more correlations between what happens on the backbone and on big traps. The only thing left to understand is the time spent during an excursion in an h_n-trap from the *root*. To simplify if the walker does not reach the point δ in the trap (this has probability $\approx 1 - p_\infty$), the time in the trap can be neglected. Otherwise, the time spent to go to δ, and to go directly from δ back to the *root* of the trap can also be neglected, in other words, only the successive excursions from δ contribute to the time spent in the trap. This is developed in Section 8, and we have

$$(4.2) \qquad \chi_1(n) \approx \sum_{i=1}^{\mathrm{Bin}(W_\infty, p_\infty)} \sum_{j=0}^{G^{(i)}-1} T_{\mathrm{exc}}^{(i,j)},$$

where $T_{\mathrm{exc}}^{(i,j)}$ are i.i.d. random variables giving the lengths of the excursions from δ to δ. Further, $G^{(i)}$ is the number of excursions from δ during the ith excursion in the trap: it is a geometric random variable with a parameter of order β^{-H}. Since β^{-H} is very small (H being conditioned to be big), the law of large numbers should imply that

$$\sum_{j=0}^{G^{(i)}-1} T_{\mathrm{exc}}^{(i,j)} \approx G^{(i)} E^\omega[T_{\mathrm{exc}}^{(i,j)}] \approx G^{(i)} S_\infty,$$

and also we should have $G^{(i)} - 1 \approx \beta^H \mathbf{e}_i$. This explains why, recalling (1.3),

$$\chi_1(n) \approx \beta^H \mathcal{Z}_\infty.$$

We are then reduced to considering sums of i.i.d. random variables of the form $Z_i \beta^{X_i}$ with X_i integer-valued. This is investigated in Section 10. We then finish the proof of Theorem 1.3 in Section 11.

REMARK 4.1. The reasoning fails in the critical case $\gamma = 1$, indeed in this case we have to consider a critical height h_n which is smaller. This causes many problems, in particular in big traps there can be big subtraps and so, for example, the time to go from the top to the bottom of a trap cannot be neglected anymore.

5. The time is essentially spent in big traps. We recall that $h_n = \lceil -(1 - \varepsilon) \ln n / \ln \mathbf{f}'(q) \rceil$. Lemma 3.3 gives the probability that a trap is an h_n-trap:

$$(5.1) \qquad \eta_n := \mathbf{Q}[H \geq h_n] \sim \alpha \mathbf{f}'(q)^{h_n}.$$

For $x \in$ backbone, we denote

$$(5.2) \qquad L_x \text{ the set of traps rooted at } x$$

(if x is not in the backbone then $L_x = \varnothing$). Let us denote the vertices in big traps by $L(h_n) = \{y \in T(\omega) : y$ is in an h_n-trap$\}$.

Our aim in this section is to show the following proposition.

PROPOSITION 5.1. *For $\varepsilon > 0$, we have*

$$\text{for all } t \geq 0 \qquad \mathbb{P}\left[\left|\frac{\Delta_n - \chi(n)}{n^{1/\gamma}}\right| \geq t\right] \to 0,$$

where

(5.3) $$\chi(n) = \text{card}\{1 \leq i \leq \Delta_n : X_{i-1}, X_i \in L(h_n)\}$$

is the time spent in big traps up to time Δ_n.

Define:

(i) $A_1(n) = \{\Delta_n^Y \leq C_1 n\}$,
(ii) $A_2(n) = \{\text{card} \bigcup_{i=1}^{\Delta_n^Y} L_{Y_i} \leq C_2 n\}$,
(iii) $A_3(n) = \{\max_{\ell \in L_{Y_i}, i \leq \Delta_n^Y} \text{card}\{0 \leq i \leq \Delta_n^Y : Y_i \in \ell, X_{\sigma_i+1} \in \ell\} \leq C_3 \ln n\}$,
(iv) $A(n) = A_1(n) \cap A_2(n) \cap A_3(n)$.

The following lemma tells us that typically the walk spends less than $C_1 n$ time units before reaching level n, sees less than $C_2 n$ traps and enters each trap at most $C_3 \ln n$ times.

LEMMA 5.1. *For appropriate constants C_1, C_2 and C_3, we have*

$$\mathbb{P}[A_1(n)^c] = o(n^{-2}) \quad and \quad \mathbb{P}[A(n)^c] \to 0.$$

PROOF. By a comparison to the β-biased random walk on $\mathbb{Z}$, standard large deviations estimates yields

$$\mathbb{P}[A_1(n)^c] = o(n^{-2})$$

for C_1 large enough.

On $A_1(n)$, the number of different vertices visited by $(Y_i)_{i \geq 0}$ up to time Δ_n^Y is at most $C_1 n$. The descendants at each new vertex are drawn independently of the preceding vertices. Moreover, at each vertex the mean number of traps is at most the mean number of children, thus $\mathbf{E}[\text{card } L_0] \leq \mathbf{m}/(1-q)$. The law of large numbers yields for $C_2 > C_1 \mathbf{m}/(1-q)$ that

$$\mathbb{P}[A_2(n)^c] \leq \mathbb{P}\left[\sum_{i=0}^{C_1 n} \text{card } L_0^{(i)} > C_2 n\right] + \mathbb{P}[A_1(n)^c] \to 0,$$

where $\text{card } L_0^{(i)}$ are i.i.d. random variables with the law of $\text{card } L_0$. This yields the second part.

For $A_3(n)$, we want, given a vertex x in the backbone and any $\ell \in L_x$ to give an upper bound on the number of transitions from x to y, where y is the bud associated to ℓ. Let z be an offspring of x in the backbone. Then, at each visit to x, either the walker does not visit y or z, or it has probability $1/2$ to visit y first (or z first). Hence:

  (i) the number of transitions from x to y before reaching z is dominated by a geometric random variable of parameter $1/2$,

  (ii) the number of transitions from x to z is dominated by a geometric random variable of parameter p_∞, since the escape probability from z is at least p_∞.

Consequently the number of transitions from x to y is dominated by a geometric random variable of parameter $p_\infty/2$. Thus,

$$\mathbb{P}[A_3(n)^c \cap A_2(n)] \leq C_2 n \mathbb{P}[G(p_\infty/2) \geq C_3 \ln n]$$
$$\leq C n^{C_3 \ln(1-p_\infty/2)+1},$$

and if we take C_3 large enough we get the result. $\quad\square$

PROOF OF PROPOSITION 5.1. Now we can start proving Proposition 5.1. Decompose Δ_n into

(5.4)
$$\Delta_n = \Delta_n^Y + \chi(n) + \sum_{\ell \in \bigcup_{i=0}^{\Delta_n^Y} L_{Y_i} \setminus L(h_n)} N(\ell),$$

where $N(\ell) = \mathrm{card}\{1 \leq i \leq \Delta_n : X_{i-1} \in \ell,\, X_i \in \ell\}$.

The distribution of $N(\ell)$ conditioned on the backbone, the buds and $(Y_i')_{i \leq \Delta_n^{Y'}}$, the walk on the backbone and the buds, is $\sum_{i=1}^{E_\ell} R_\ell^{(i)}$. Here we denoted E_ℓ the number of visits to ℓ and $R_\ell^{(i)}$ is the return time during the ith excursion from the top of ℓ. These quantities are considered for traps ℓ, conditioned to have height at most h_n.

Obviously, we get from (5.4) that

(5.5)
$$\Delta_n \geq \chi(n).$$

From (5.4), we get for $t > 0$,

(5.6)
$$\mathbb{P}\left[\frac{\Delta_n - \chi(n)}{n^{1/\gamma}} > t\right] \leq \mathbb{P}[A(n)^c] + \mathbb{P}\left[C_1 n + \sum_{i=0}^{C_2 n} \sum_{j=0}^{C_3 \ln n} R_\ell^{(i)} \geq t n^{1/\gamma}\right]$$

$$\leq o(1) + \mathbb{P}\left[\sum_{i=0}^{C_2 n} \sum_{j=0}^{C_3 \ln n} R_\ell^{(i)} \geq \frac{t}{2} n^{1/\gamma}\right],$$

where we used Lemma 5.1.

Chebyshev's inequality yields

$$\mathbb{P}\left[\sum_{i=0}^{C_2 n}\sum_{j=0}^{C_3 \ln n} R_\ell^{(i)} \ge \frac{t}{2}n^{1/\gamma}\right] \le \frac{2}{tn^{1/\gamma}}\mathbb{E}\left[\sum_{i=0}^{C_2 n}\sum_{j=0}^{C_3 \ln n} R_\ell^{(i)}\right]$$

$$\le \frac{2C_2 C_3 n^{1-1/\gamma}\ln n}{t}\mathbb{E}\big[R_1^{(1)}\big].$$

Using Lemmas 3.4 and 3.1, we have

$$\mathbb{E}\big[R_1^{(1)}\big] = E_{\mathbf{Q}}\big[E_{\text{root}}^{\omega}[T_{\text{root}}^{+}]|H < h_n\big] = 2\sum_{i=0}^{h_n-1}\beta^i E_{\mathbf{Q}}[Z_n|H < h_n]$$

$$\le 2\sum_{i=0}^{h_n-1}(\beta\mathbf{f}'(q))^i \le Cn^{(1-\varepsilon)(-1+1/\gamma)}.$$

Plugging this into the previous inequality, we get for any $\varepsilon > 0$ and $t > 0$

$$\mathbb{P}\left[\sum_{i=0}^{C_2 n}\sum_{j=0}^{C_3 \ln n} R_\ell^{(i)} \ge \frac{t}{2}n^{1/\gamma}\right] = o(1),$$

thus recalling (5.6) and (5.5) we have proved Proposition 5.1. □

6. Number of visits to a big trap. We denote $K_x = \max_{\ell \in L_x} H(\ell)$, the height of the biggest trap rooted at x for $x \in$ backbone, where we recall that H denotes the height of the trap from the bud and not from the *root*.

LEMMA 6.1. *We have*

$$\mathbf{P}[K_0 \ge h_n] \sim C_a \mathbf{f}'(q)^{h_n},$$

where $C_a = \alpha q\frac{\mathbf{m}-\mathbf{f}'(q)}{1-q}$, recalling Lemma 3.3 for the definition of α.

PROOF. We denote Z the number of children of the root and Z^* the number of children with an infinite line of descent. Let P be the law of a $\mathbf{f}$-Galton–Watson tree which is not conditioned on nonextinction and E the corresponding expectation. Recall (5.1) and let $H^{(i)}, i = 1, 2, \ldots$, be i.i.d. random variables which have the law of the height of an $\mathbf{h}$-Galton–Watson tree, and are independent of $Z - Z^*$. Then

$$\mathbf{P}[K_0 \ge h_n] = \mathbf{P}\left[\max_{i=1,\ldots,Z-Z^*} H^{(i)} \ge h_n\right] = 1 - \frac{E[(1-\eta_n)^{Z-Z^*}(1-\mathbf{1}\{Z^*=0\})]}{1-q},$$

where the indicator function comes from the conditioning on nonextinction, which corresponds to $Z^* \ne 0$.

Hence,

$$\mathbf{P}[K_0 \geq h_n] = 1 - \frac{E[(1-\eta_n)^{Z-Z^*}] - E[(1-\eta_n)^{Z-Z^*}0^{Z^*}]}{1-q},$$

and using $E[s^{Z-Z^*}t^{Z^*}] = \mathbf{f}(sq + t(1-q))$ (see [15]) we get

$$\mathbf{P}[K_0 \geq h_n] = 1 - \frac{\mathbf{f}((1-\eta_n)q + 1 - q) - \mathbf{f}((1-\eta_n)q)}{1-q}.$$

Now, using (5.1) and the expansion $\mathbf{f}(z-x) = \mathbf{f}(z) - \mathbf{f}'(z)x + o(x)$ for $z \in \{q, 1\}$, we get the result. $\square$

Define the first time when we meet the root of an h_n-trap using the clock of Y_n,

(6.1) $$K(n) = \inf\{i \geq 0 | K_{Y_i} \geq h_n\}.$$

We also define $\ell(n)$ to be an h_n-trap rooted at $Y_{K(n)}$, if there are several possibilities we choose one trap according to some predetermined order. We denote $\mathbf{b}(n)$ the associated bud.

We describe, on the event 0-SR, the number of visits to $\ell(n)$, by the following random variable:

(6.2) $$W_n = \mathrm{card}\{i : X_i = Y_{K(n)}, X_{i+1} = \mathbf{b}(n)\},$$

where ω is chosen under the law $\mathbf{P}[\cdot]$ and X_n under $P_0^\omega[\cdot|0\text{-SR}]$. We will need the following bounds for the random variables $(W_n)_{n \geq 1}$.

LEMMA 6.2. *We have $W_n \preceq G(p_\infty/3)$ for $n \in \mathbb{N}$, that is, the random variables W_n are stochastically dominated by a geometric random variable with parameter $p_\infty/3$.*

PROOF.   For $n \in \mathbb{N}$, starting from any point x of the backbone, the walker has probability at least $1/3$ to go to an offspring y of x on the backbone before going to $\mathbf{b}(n)$ or $\overleftarrow{x}$. But the first hitting time of y has probability at least p_∞ to be a super-regeneration time. The result follows as in the proof of Lemma 5.1. $\square$

PROPOSITION 6.1. *There exists a random variable W_∞ such that*

$$W_n \xrightarrow{d} W_\infty,$$

where we recall that for the law of W_n, ω is chosen under the law $\mathbf{P}[\cdot]$ and X_n under $P_0^\omega[\cdot|0\text{-SR}]$.

REMARK 6.1.   It follows from Lemma 6.2 that $W_\infty \preceq G(p_\infty/3)$.

Fix $n \in \mathbb{N}^*$ and set $m \geq n$. We aim at comparing the law of W_m with that of W_n and to do that we want to study the behavior of the random walk starting from the last super-regeneration time before an h_n-trap (resp., h_m-trap) is seen. This motivates the definition of the last super-regeneration time seen before time n,

$$\Sigma(n) := \max\{0 \leq i \leq n : i\text{-SR}\}.$$

For our purpose it is convenient to introduce a modified version of W_m, which will coincide with high probability with it. For $m \geq n$, recall that θ denotes the time-shift for the walk and set

$$\overline{K}(m, n) = \inf\{j \geq 0 : K_{Y_j} \geq h_m, \ell(m) \circ \theta_{\Sigma(j)} = \ell(n) \circ \theta_{\Sigma(j)}\},$$

the first time the walker meets a h_m-trap which is the first h_n-trap of the current regeneration block and we denote by $b(m, n)$ the associated bud. Set

$$\overline{W}_{m,n} = \text{card}\{i : X_i = Y_{\overline{K}(m,n)}, X_{i+1} = b(m, n)\},$$

where ω is chosen under the law $\mathbf{P}[\cdot]$ and $(U_i)_{i \leq \overline{K}(m,n)}$ under $P_0^\omega[\cdot | 0\text{-SR}]$.

LEMMA 6.3.   *For $m \geq n$, we have that*

$$\overline{W}_{m,n} \overset{d}{=} W_n.$$

PROOF.   To reach a vertex where an h_m-trap is rooted, the walker has to reach a vertex where an h_n-trap is rooted. Two cases can occur: either the first h_n-trap met is also a h_m-trap or it is not. In the former case, which has probability $\eta_m/\eta_n > 0$, since the height of the first h_n-trap met is independent of the sequence $(U_i)_{i \leq K(n)}$, the random variables $\overline{W}_{m,n}$ and W_n coincide. In the latter case, by its definition, $\overline{K}(m, n)$ cannot occur before the next super-regeneration time, hence $\overline{K}(m, n) \geq \tau_1 \circ \theta_{K(n)}$. In this case $\overline{W}_{m,n} = \overline{W}_{m,n} \circ \theta_{K(n)}$ and then by Proposition 2.1,

$$\overline{W}_{m,n} \circ \theta_{\tau_1 \circ \theta_{K(n)}} \overset{d}{=} \overline{W}_{m,n},$$

and $\overline{W}_{m,n} \circ \theta_{\tau_1 \circ \theta_{K(n)}}$ is independent of $(U_i)_{i \leq \tau_1 \circ \theta_{K(n)} - 1}$.

The scenario repeats itself until the h_n-trap reached is in fact a h_m-trap, the number of attempts necessary to reach this h_m-trap is a geometric random variable of parameter η_m/η_n which is independent of the (U_i)'s.

This means that there is a family $(W_n^{(i)})_{i \geq 1}$ of i.i.d. random variables with the same law as W_n such that

$$\overline{W}_{m,n} = W_n^{(G)},$$

where G is a geometric random variable independent of the $(W_n^{(i)})_{i \geq 1}$. Then, note that we have

$$\overline{W}_{m,n} = W_n^{(G)} \overset{d}{=} W_n. \qquad \square$$

Now we need to show that $\overline{W}_{m,n}$ and W_m coincide with high probability, so we introduce the event

$$A_{m,n} = \left\{ \boldsymbol{\ell}(m) = \boldsymbol{\ell}(n) \circ \theta_{\Sigma(K(m))} \right\}$$

on which clearly $\overline{W}_{m,n}$ and W_m are equal.

LEMMA 6.4.

$$(6.3) \qquad \sup_{m \geq n} \mathbb{P}[A^c_{m,n} | 0\text{-}SR] \to 0 \qquad for\ n \to \infty.$$

PROOF.   Let us denote, recalling (5.2),

$$V_j^i = \left\{ \operatorname{card} \bigcup_{k=0}^{\tau_1} \{ \ell \in L_{Y_k}, \ell \text{ is a } h_j\text{-trap} \} = i \right\}$$

and

$$V_j^{i,+} = \left\{ \operatorname{card} \bigcup_{k=0}^{\tau_1} \{ \ell \in L_{Y_k}, \ell \text{ is a } h_j\text{-trap} \} \geq i \right\}.$$

Then we have

$$(6.4) \qquad \mathbb{P}[A^c_{m,n} | 0\text{-}SR] \leq \mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0\text{-}SR].$$

Let us denote card Trap the number of traps seen before τ_1,

$$\operatorname{card} \operatorname{Trap} = \operatorname{card} \left\{ \ell : \ell \in \bigcup_{i=0}^{\tau_1} L_{Y_i} \right\},$$

and its generating function by

$$\varphi(s) := \mathbb{E}[s^{\operatorname{card} \operatorname{Trap}} | 0\text{-}SR].$$

The probability of $A_{m,n}$ can be estimated with the following lemma, whose proof is deferred.

LEMMA 6.5.   *We have*

$$\forall m \geq n \qquad \mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0\text{-}SR] \leq \varphi'(1) - \varphi'(1 - \eta_n).$$

Now we have $\mathbb{E}[\operatorname{card} \operatorname{Trap} | 0\text{-}SR] \leq \mathbb{E}[\tau_1 | 0\text{-}SR] \mathbb{E}[\operatorname{card} L_0] < \infty$ because of Remark 2.1 and hence φ' is continuous at 1, and (6.3) follows from (6.4).   □

Applying Lemmas 6.5, 6.3 and (6.3) we get

$$\mathbb{P}[W_m \geq y | 0\text{-}SR] = \mathbb{P}[A_{m,n}, \overline{W}_{m,n} \geq y | 0\text{-}SR] + o(m, n)$$
$$= \mathbb{P}[\overline{W}_{m,n} \geq y | 0\text{-}SR] + o(m, n)$$
$$= \mathbb{P}[W_n \geq y | 0\text{-}SR] + o(m, n),$$

where $\sup_{m \geq n} o(m, n) \to 0$ as n goes to infinity.

The law of a random variable W_∞ can be defined as a limit of the laws of some subsequence of (W_m), since the family $(W_m)_{m \geq 0}$ is tight by Lemma 6.2. Then taking m to infinity along this subsequence in the preceding equation yields

$$\forall t > 0 \qquad \mathbb{P}[W_\infty \leq t | 0\text{-SR}] = \mathbb{P}[W_n \leq t | 0\text{-SR}] + o(1).$$

This proves Proposition 6.1.

It remains to show Lemma 6.5.

PROOF.   Note that for $i \geq 1$,

$$\begin{aligned}
\mathbb{P}[V_n^i | V_m^{1,+}, 0\text{-SR}] &= \frac{\mathbb{P}[V_n^i | 0\text{-SR}]}{\mathbb{P}[V_m^{1,+} | 0\text{-SR}]} \mathbb{P}[V_m^{1,+} | V_n^i, 0\text{-SR}] \\
&\leq \frac{\mathbb{P}[V_n^i | 0\text{-SR}]}{\mathbb{P}[V_m^{1,+} | 0\text{-SR}]} i \mathbf{Q}[H \geq h_m | H \geq h_n] \\
&= i \frac{\mathbb{P}[V_n^i | 0\text{-SR}]}{\eta_n} \frac{\eta_m}{\mathbb{P}[V_m^{1,+} | 0\text{-SR}]} \\
&\leq i \frac{\mathbb{P}[V_n^i | 0\text{-SR}]}{\eta_n}.
\end{aligned}$$

(6.5)

Then we have

$$\begin{aligned}
\sum_{i \geq 2} i \mathbb{P}[V_n^i | 0\text{-SR}] &= \sum_{i \geq 2} \sum_{j \geq i} \mathbb{P}[\text{card Trap} = j | 0\text{-SR}] i \binom{j}{i} \eta_n^i (1 - \eta_n)^{j-i} \\
&= \sum_{j \geq 0} j \mathbb{P}[\text{card Trap} = j | 0\text{-SR}] \sum_{i=2}^{j} \binom{j-1}{i-1} \eta_n^i (1 - \eta_n)^{j-i} \\
&= \eta_n \sum_{j \geq 0} j \mathbb{P}[\text{card Trap} = j | 0\text{-SR}] \sum_{i=1}^{j-1} \binom{j-1}{i} \eta_n^i (1 - \eta_n)^{(j-1)-i} \\
&= \eta_n \sum_{j \geq 0} j \mathbb{P}[\text{card Trap} = j | 0\text{-SR}] (1 - (1 - \eta_n)^{j-1}) \\
&= \eta_n (\varphi'(1) - \varphi'(1 - \eta_n)).
\end{aligned}$$

Inserting this in (6.5) we get

$$\begin{aligned}
\mathbb{P}[V_n^{2,+} | V_m^{1,+}, 0\text{-SR}] &= \sum_{i=2}^{\infty} \mathbb{P}[V_n^i | V_m^{1,+}, 0\text{-SR}] \\
&\leq \frac{1}{\eta_n} \sum_{i \geq 2} i \mathbb{P}[V_n^i | 0\text{-SR}] = \varphi'(1) - \varphi'(1 - \eta_n),
\end{aligned}$$

which concludes the proof of Lemma 6.5. $\square$

We will need the following lower bound for the random variable W_∞.

LEMMA 6.6.   *There exists a constant $c_W > 0$ depending only on $(p_i)_{i \geq 0}$, such that*

$$\mathbb{P}[W_\infty \geq 1] \geq c_W.$$

PROOF.   By Proposition 6.1, it is enough to show the lower bound for all W_n. First let us notice that

(6.6)
$$\mathbb{P}[W_n \geq 1|0\text{-SR}] \geq \mathbb{E}\left[\left(\frac{1}{Z(K(n))+1}\right)^2 p_\infty\right]$$
$$\geq (1 - \mathbf{f}'(q))\mathbb{E}\left[\frac{1}{(Z(K(n))+1)^2}\right],$$

where $Z(K(n))$ is the number of offspring of $Y_{K(n)}$. To show (6.6), note that the particle has probability at least $\beta/(\beta Z(K(n)) + 1) \geq 1/(Z(K(n)) + 1)$ of going from $Y_{K(n)}$ to $\mathbf{b}(n)$ and when it comes back to $Y_{K(n)}$ again there is probability at least $1/(Z(K(n)) + 1)$ to go from $Y_{K(n)}$ to one of its descendants on the backbone and then there is a probability of at least p_∞ that a super-regeneration occurs. The event we just described is in $\{W_n \geq 1\} \cap \{0\text{-SR}\}$. For the second inequality in (6.6), use $\beta > \beta_c = \mathbf{f}'(q)^{-1}$ hence $p_\infty = 1 - \beta^{-1} \geq 1 - \mathbf{f}'(q)$.

Now, we notice that the law of the $Z(K(n))$ is that of Z_1 conditioned on the event $\{$an h_n-trap is rooted at $0\}$. Denote j_0 the smallest index such that $j_0 > 1$ and $p_{j_0} > 0$ (which exists since $\mathbf{m} > 1$) and Z_1^* the number of descendants of 0 with an infinite line of descent. All $Z_1 - Z_1^*$ traps rooted at 0 have, independently of each other, probability η_n of being h_n-traps, so that

$$\mathbb{P}[Z(K(n)) = j_0] = \mathbf{P}[Z_1 = j_0|\text{an } h_n\text{-trap is rooted at } 0]$$
$$= \frac{\mathbf{P}[Z_1 = j_0, \text{Bin}(Z_1 - Z_1^*, \eta_n) \geq 1]}{\mathbf{P}[\text{an } h_n\text{-trap is rooted at } 0]}$$
$$\geq \frac{\eta_n \mathbf{P}[Z_1 = j_0, Z_1 - Z_1^* \geq 1]}{\mathbf{P}[\text{an } h_n\text{-trap is rooted at } 0]}.$$

Further, since $P[\text{Bin}(Z_1 - Z_1^*, \eta_n) \geq 1] \leq Z_1 \eta_n$, we have

$$\mathbf{P}[\text{an } h_n\text{-trap is rooted at } 0] \leq \sum_{j=0}^{\infty} \mathbf{P}[Z_1 - Z_1^* = j]j\eta_n \leq \mathbf{m}\eta_n.$$

Putting these equations together, we get that

$$\mathbb{P}[Z(K(n)) = j_0] \geq \frac{\mathbf{P}[Z_1 = j_0, Z_1 - Z_1^* \geq 1]}{\mathbf{m}}.$$

The last equation and (6.6) yield a lower bound for $\mathbb{P}[W_\infty \geq 1]$ which depends only on $(p_k)_{k \geq 0}$. $\square$

7. The time spent in different traps is asymptotically independent. In order to show the asymptotic independence of the time spent in different big traps we shall use super-regeneration times. First, we show that the probability that there is an h_n-trap in the first super-regeneration block goes to 0 for $n \to \infty$.

Define:

(i) $B_1(n) = \{\forall i \in [1, n], \operatorname{card}\{Y_{\tau_i}, \ldots, Y_{\tau_{i+1}}\} \leq n^\varepsilon\}$,

(ii) $B_2(n) = \{\forall i \in [0, \tau_1], \operatorname{card} L_{Y_i} \leq n^{2\varepsilon}\}$,

(iii) $B_3(n) = \{\forall i \in [0, \tau_1], \forall \ell \in L_{Y_i}, \ell$ is not an h_n-trap$\}$,

(iv) $B_4(n) = \{\forall i \in [2, n], \operatorname{card}\{j : j \in [\tau_i, \tau_{i+1}], L_{Y_j}$ contains an h_n-trap$\} \leq 1\}$,

(v) $B(n) = B_1(n) \cap B_2(n) \cap B_3(n) \cap B_4(n)$.

LEMMA 7.1. *For $\varepsilon < 1/4$, we have*

$$\mathbb{P}[B_1(n)^c] = o(n^{-2}) \quad and \quad \mathbb{P}[B(n)^c] \to 0.$$

PROOF. Since $\tau_2 - \tau_1$ (resp., τ_1) has some positive exponential moments and $B_1(n)^c \subseteq \bigcup_{i=1}^n \{\tau_{i+1} - \tau_i \geq n^\varepsilon\}$,

$$\mathbb{P}[B_1(n)^c] = o(n^{-2}).$$

Using the fact that the number of traps at different vertices has the same law,

$$\mathbb{P}[B_2(n)^c] \leq \mathbb{P}[B_1(n)^c] + n^\varepsilon \mathbf{P}[\operatorname{card} L_0 \geq n^{2\varepsilon}] \leq o(1) + n^{-\varepsilon} \frac{\mathbf{m}}{1-q} = o(1),$$

where we used Chebyshev's inequality and $\mathbf{E}[\operatorname{card} L_0] \leq \mathbf{E}[Z_1] \leq \mathbf{m}/(1-q)$.

Then we have

$$\mathbb{P}[B_3(n)^c] \leq \mathbb{P}[B_2(n)^c] + n^{3\varepsilon} \eta_n = o(1),$$

yielding the result using (5.1), since $\varepsilon < 1/4$.

Finally, up to time n we have at most n super-regeneration blocks, on $B_1(n)$ they contain at most n^ε visited vertices. But the probability that among the n^ε first visited vertices after a super-regeneration time, two of them are adjacent to a big trap is bounded above by $n^{2\varepsilon} \mathbf{P}[K_0 \geq h_n]^2$ (here we implicitly used Remark 2.1). Hence, we get

$$\mathbb{P}[B_4(n)^c] \leq \mathbb{P}[B_1(n)^c] + n n^{2\varepsilon}(C n^{\varepsilon-1})^2 = O(n^{4\varepsilon-1}),$$

yielding the result for $\varepsilon < 1/4$. $\square$

We define $R(n) = \operatorname{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}$ and l_n the number of vertices where an h_n-trap is rooted:

(7.1) $$l_n = \operatorname{card}\{i \in [0, \Delta_n^Y] : L_{Y_i} \text{ contains an } h_n\text{-trap}\}.$$

Recall (2.2) and define

$$(7.2) \quad C_1(n) = \{(1 - n^{-1/4})\rho n \leq R(n) \leq (1 + n^{-1/4})\rho n\},$$

$$(7.3) \quad C_2(n) = \{(1 - n^{-\varepsilon/4})\rho C_a n \mathbf{f}'(q)^{h_n} \leq l_n \leq (1 + n^{-\varepsilon/4})\rho C_a n \mathbf{f}'(q)^{h_n}\},$$

$$(7.4) \quad C_3(n) = \{\forall 1 \leq i \leq \Delta_n^Y, \text{card}\{\ell \in L_{Y_i} : \ell \text{ is an } h_n\text{-trap}\} \leq 1\}$$

and $C(n) = C_1(n) \cap C_2(n) \cap C_3(n)$.

LEMMA 7.2. *For $\varepsilon < 1/4$, we have*

$$\mathbb{P}[C(n)^c] \to 0.$$

PROOF. First, we notice that for $i \geq 0$ and with the convention $\tau_0 := 0$ we have

$$Z_i := \text{card}\{Y_{\tau_i+1}, \ldots, Y_{\tau_{i+1}}\} \preceq G^{(i)}(p_\infty),$$

where the geometric random variables $G^{(i)}$ are i.i.d. Indeed, at each new vertex visited we have probability at least p_∞ to have a super-regeneration time. Let us denote by n_0 the smallest integer such that $\Delta_n^Y \leq \tau_{n_0}$, which satisfies $n_0 \leq n$ since $|X_{\tau_{i+1}}| - |X_{\tau_i}| \geq 1$.

Now, since the random variables Z_i are i.i.d. and $\sum_{i=1}^{n_0-1} Z_i \leq \text{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\} \leq \sum_{i=1}^{n_0} Z_i$, we have

$$\mathbb{P}\left[\left|\frac{\text{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}}{n} - \rho\right| \geq n^{-1/4}\right]$$

$$\leq n^{1/2} \text{Var}\left(\frac{\text{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}}{n}\right)$$

$$= n^{1/2}\left(\mathbb{E}\left[\left(\frac{\text{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}}{n}\right)^2\right] - \mathbb{E}\left[\frac{\text{card}\{Y_1, \ldots, Y_{\Delta_n^Y}\}}{n}\right]^2\right)$$

$$\leq n^{-3/2}\left(\mathbb{E}\left[\left(\sum_{i=1}^{n_0} Z_i\right)^2\right] - \mathbb{E}\left[\sum_{i=1}^{n_0-1} Z_i\right]^2\right)$$

$$\leq n^{-1/2}(E[G(p_\infty)^2] + E[G(p_\infty)]^2),$$

yielding $\mathbb{P}[C_1(n)^c] \to 0$.

On $C_1(n)$ we know that there are $R(n) \in [\rho n(1 - n^{-1/4}), \rho n(1 + n^{-1/4})]$ vertices where we have independent trials to have h_n-traps. Hence l_n has the law $\text{Bin}(R(n), \mathbf{P}[K_0 \geq h_n])$, where the success probability satisfies $\mathbf{P}[K_0 \geq h_n] \leq Cn^{\varepsilon-1}$ has asymptotics given by Lemma 6.1. Now, standard estimates for Binomial distributions imply that $\mathbb{P}[C_2(n)^c \cap C_1(n)] \to 0$.

On $C_2(n)$, there are at most Cn^ε vertices where (at least) one h_n-trap can be rooted, we only need to prove that, with probability going to 1, those vertices do

not contain more than two h_n-traps. Using the same reasoning as in Lemma 6.5, we get

$$\mathbf{P}[0 \text{ has at least two } h_n\text{-traps}|0 \text{ has at least one } h_n\text{-trap}, 0\text{-SR}]$$

$$\leq \mathbf{f}'(1) - \mathbf{f}'(1 - \eta_n) \leq C\eta_n,$$

where we used that $\mathbf{E}[Z^2] < \infty$, which implies that $\mathbf{f}''(1) < \infty$.

The result follows from the fact that $\eta_n = o(n^{-\varepsilon})$ for $\varepsilon < 1/4$. $\square$

Let us denote, recalling (3.1),

$$D(n) = \left\{ \max_{\ell \in \bigcup_{i=0,\dots,\Delta_n^Y} L_{Y_i}} H(\ell) \leq \frac{2\ln n}{-\ln \mathbf{f}'(q)} \right\}.$$

LEMMA 7.3.   *We have*

$$\mathbb{P}[D(n)^c] \to 0.$$

PROOF.   Due to (3.7), we know that $\mathbf{Q}[H \geq \frac{2\ln n}{-\ln \mathbf{f}'(q)}] \leq n^{-2}$, so using Lemma 5.1

$$\mathbb{P}[D(n)^c] \leq \mathbb{P}[A_2(n)^c] + \mathbb{P}[A_2(n) \cap D(n)^c] \leq o(1) + C_2 n^{-1} = o(1),$$

which concludes the proof. $\square$

On $B(n)$ there is no big trap in the first super-regeneration block, on $B(n) \cap C(n)$ all big traps are met in distinct super-regeneration blocks and $C_2(n)$ tells us the asymptotic number of such blocks. Moreover on $D(n)$, we know that to cross level n on a trap, it has to be rooted after level $n - (-2\ln n/\ln \mathbf{f}'(q))$. Hence using Lemmas 7.1, 7.2, 7.3, Proposition 2.1 and Remark 2.3, we get:

PROPOSITION 7.1.   *Let* $\chi_i(n), i \geq 1$, *be i.i.d. copies of* $\chi_0(n)$, *see* (4.1), *and* $\tilde{n} = n - (-2\ln n/\ln \mathbf{f}'(q))$. *Then we have*

$$o_1(n) + \sum_{i=1}^{(1-\tilde{n}^{-\varepsilon/4})\rho_n C_a \tilde{n} \mathbf{f}'(q)^{h_{\tilde{n}}}} \chi_i(n) \leq \chi(n) \leq \sum_{i=1}^{(1+n^{-\varepsilon/4})\rho_n C_a n \mathbf{f}'(q)^{h_n}} \chi_i(n) + o_2(n),$$

where $\lim_{n\to\infty} o_1(n) = \lim_{n\to\infty} o_2(n) = 0$.

In light of Proposition 5.1, our problem reduces to understanding the convergence in law of a sum of i.i.d. random variables. The aim of the next section is to reduce $\chi_1(n)$ to a specific type of random variable for which limit laws can be derived (see Section 10).

8. The time is spent at the bottom of the traps.   We denote by $\ell_j(n)$ the jth h_n-trap seen by the walker, and by $\delta_j(n)$ [resp., $root_j(n)$, $\mathbf{b}_j(n)$] the leftmost bottom point (the root, the bud) of $\ell_j(n)$. Further, $\chi_j(n)$ denotes the time spent in the jth h_n-trap met. Then, $\chi_i(n)$, $i \geq 1$, are i.i.d. copies of $\chi_0(n)$ as in Proposition 7.1.

We want to show that the time spent in the big traps is essentially spent at the bottom of them, that is, during excursions from the the bottom leftmost point δ. In order to prove our claim, we introduce

$$\chi_j^*(n) = \operatorname{card}\{k \geq 0 : X_k \in \ell_j(n), k \geq T_{\delta_j(n)}, T_{\delta_j(n)} \circ \theta_k < \infty\},$$

the time spent during excursions from the bottom in the jth h_n-trap met. It is obvious that

$$\chi_j(n) \geq \chi_j^*(n).$$

We prove the following proposition.

PROPOSITION 8.1.   *For $\varepsilon < 1/4$, we have, recalling* (7.1),

$$\textit{for all } t > 0 \qquad \mathbb{P}\left[\frac{1}{n^{1/\gamma}}\left|\sum_{j=1}^{l_n}(\chi_j(n) - \chi_j^*(n))\right| \geq t\right] \to 0.$$

In order to prove the preceding proposition, we mainly need to understand $\chi_1(n)$ and $\chi_1^*(n)$. Note that $\chi_1(n)$ is a sum of W_n successive i.i.d. times spent in $\ell_1(n)$ and $\chi_1^*(n)$ is a sum of W_n successive i.i.d. times spent during excursions from the bottom of $\ell_1(n)$. We can rewrite the proposition as follows:

$$(8.1) \qquad \textit{for all } t > 0 \qquad \mathbb{P}\left[\frac{1}{n^{1/\gamma}}\left|\sum_{i=1}^{l_n}\sum_{j=1}^{W_n^{(i)}}(T_{\operatorname{root}_i(n)}^j - T_{\operatorname{root}_i(n)}^{*,j})\right| \geq t\right] \to 0,$$

where

$$T_{\operatorname{root}_i(n)}^j = \operatorname{card}\{k \geq 0 : X_k \in \ell_i(n),$$
$$\operatorname{card}\{\widetilde{k} \leq k : X_{\widetilde{k}+1} = \mathbf{b}_i(n), X_{\widetilde{k}} = root_i(n)\} = j\}$$

and

$$T_{\operatorname{root}_i(n)}^{*,j} = \operatorname{card}\{k \geq 0 : X_k \in \ell_i(n),$$
$$\operatorname{card}\{\widetilde{k} \leq k : X_{\widetilde{k}+1} = \mathbf{b}_i(n), X_{\widetilde{k}} = root_i(n)\} = j,$$
$$k \geq T_{\delta_j(n)}, T_{\delta_j(n)} \circ \theta_k < \infty\},$$

and $(W_n^{(i)})$, $i \geq 1$, are i.i.d. copies of W_n.

Consequently, in this section, we mainly investigate the walk on a big trap, which is a random walk in a finite random environment. Recall that *root* is the vertex $Y_{K(n)}$ on the backbone where $\ell(n)$ is attached. Moreover, set $\mathbf{Q}_n[\cdot] = \mathbf{Q}[\cdot | H \geq h_n]$, $E_{\mathbf{Q}_n}[\cdot] = E_{\mathbf{Q}}[\cdot | H \geq h_n]$, $E^\omega[\cdot] := E_{\operatorname{root}}^\omega[\cdot]$ and $\mathbb{E}_{\mathbf{Q}_n}[\cdot] = E_{\mathbf{Q}_n}[E^\omega[\cdot]]$.

REMARK 8.1. To ease notation, we add to all these probability spaces an independent random variable W_n whose law is given by (6.2), under the law $\mathbb{P}[\cdot|0\text{-SR}]$ for $n \in \mathbb{N} \cup \{\infty\}$.

We will extensively use the description of Section 3, in particular we recall that a trap is composed of *root* which is linked by an edge to an **h**-Galton–Watson tree.

We want to specify what $\ell(n)$ looks like. Denoting

$$h_n^+ = \left\lceil \frac{(1+\varepsilon)\ln n}{-\ln \mathbf{f}'(q)} \right\rceil,$$

consider:

   (i) $\widetilde{A}_1(n) = \{H \le h_n^+\}$,
  (ii) $\widetilde{A}_2(n) = \{\text{there are fewer than } n^\varepsilon \text{ subtraps}\}$,
 (iii) $\widetilde{A}_3(n) = \{\text{all subtraps of } \ell(n) \text{ have height} \le h_n\}$,
 (iv) $\widetilde{A}(n) = \widetilde{A}_1(n) \cap \widetilde{A}_2(n) \cap \widetilde{A}_3(n)$.

LEMMA 8.1. *For $\varepsilon < 1/4$, we have*

$$\mathbf{Q}_n[\widetilde{A}(n)^c] = o(n^{-\varepsilon}).$$

PROOF. First,

$$\mathbf{Q}_n[\widetilde{A}_1(n)^c] \le \frac{\mathbf{Q}[H \ge h_n^+]}{\mathbf{Q}[H \ge h_n]} \le Cn^{-2\varepsilon} = o(n^{-\varepsilon}).$$

Furthermore using Lemmas 3.2, 3.4 and (3.7), we get

$$\mathbf{Q}_n[\widetilde{A}_2(n)^c] \le \mathbf{Q}_n[\widetilde{A}_1(n)^c]$$
$$+ \mathbf{Q}_n[\widetilde{A}_1(n), \text{there are } n^\varepsilon/h_n^+ \text{ subtraps on a vertex of the spine}]$$
$$\le o(n^{-\varepsilon}) + h_n^+ C_\psi \frac{n^\varepsilon}{h_n^+} q^{n^\varepsilon/h_n^+}$$
$$= o(n^{-\varepsilon}).$$

Finally,

$$\mathbf{Q}_n[\widetilde{A}_3(n)^c] \le \mathbf{Q}_n[\widetilde{A}_2(n)^c]$$
$$+ \mathbf{Q}_n[\widetilde{A}_2(n), \text{there exists a subtrap of height} \ge h_n]$$
$$\le o(n^{-\varepsilon}) + n^\varepsilon \eta_n$$
$$= o(n^{-\varepsilon}),$$

where we used (5.1) and $\varepsilon < 1/4$. □

Using Chebyshev's inequality we get, recalling (7.3),

$$\mathbb{P}\left[\frac{1}{n^{1/\gamma}}\left|\sum_{i=1}^{l_n}\sum_{j=1}^{W_n^{(i)}}(T_{\mathrm{root}_i(n)}^j - T_{\mathrm{root}_i(n)}^{*,j})\right| \ge t\right]$$

$$\le \frac{1}{tn^{1/\gamma}}\mathbb{E}\left[\mathbf{1}\{C_2(n)\}\mathbf{1}\{\widetilde{A}(n)\}\sum_{i=1}^{l(n)}\sum_{j=1}^{W_n^{(i)}}(T_{\mathrm{root}_i(n)}^j - T_{\mathrm{root}_i(n)}^{*,j})\right]$$

$$+ \mathbb{P}[C_2(n)^c] + \mathbf{Q}_n[\widetilde{A}(n)^c]$$

$$\le \mathbb{P}[C_2(n)^c] + \mathbf{Q}_n[\widetilde{A}(n)^c]$$

$$+ \frac{2\rho C_a c_\beta n^\varepsilon}{tn^{1/\gamma}}\mathbb{E}[\mathbf{1}\{\widetilde{A}(n)\}(T_{\mathrm{root}_1(n)}^1 - T_{\mathrm{root}_1(n)}^{*,1})],$$

where $c_\beta = E[G(p_\infty/3)]$, implying $\mathbb{E}[W_n^{(j)}] \le c_\beta$. Hence using Lemmas 8.1 and 7.2, with

$$a_n := \mathbb{P}\left[\frac{1}{n^{1/\gamma}}\left|\sum_{i=1}^{l_n}\sum_{j=1}^{W_n^{(i)}}(T_{\mathrm{root}_i(n)}^j - T_{\mathrm{root}_i(n)}^{*,j})\right| \ge t\right]$$

and

$$b_n := \frac{C}{t}n^{\varepsilon-1/\gamma}E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}(T_{\mathrm{root}_1(n)}^1 - T_{\mathrm{root}_1(n)}^{*,1})],$$

we have

(8.2) $$\limsup_{n\to\infty}(a_n - b_n) \le 0.$$

We have to estimate this last expectation. Consider an h_n-trap. Each time the walker enters the h_n-trap two cases can occur: either the walker will reach δ, or he will not reach δ before he comes back to *root*. In the former case, $T_{\mathrm{root}}^+ - T_{\mathrm{root}}^{*,+}$ is the time spent going from *root* to δ for the first time plus the time coming back from δ to *root* for the last time (starting from δ and going back to *root* without returning to δ). In the latter case, $T_{\mathrm{root}}^+ - T_{\mathrm{root}}^{*,+}$ equals T_{root}^+. This yields the following upper bound

(8.3)
$$\mathbb{E}[\mathbf{1}\{\widetilde{A}(n)\}(T_{\mathrm{root}}^+ - T_{\mathrm{root}}^{*,+})]$$
$$\le E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}E_{\mathrm{root}}^\omega[\mathbf{1}\{\widetilde{A}(n)\}T_\delta^+|T_\delta^+ < T_{\mathrm{root}}^+]]$$
$$+ E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_{\mathrm{root}}^+|T_{\mathrm{root}}^+ < T_\delta^+]]$$
$$+ E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}E_{\mathrm{root}}^\omega[T_{\mathrm{root}}^+|T_{\mathrm{root}}^+ < T_\delta^+]].$$

To tackle the conditionings that appear, we shall use h-processes; see [9] and [25] for further references. For a given environment ω let us denote h^ω the following function on the trap: $h^\omega(z) = P_z^\omega[T_{\mathrm{root}}^+ < T_\delta^+]$, with $h^\omega(root) = 1$ and

$h^\omega(\delta) = 0$. Then we have the following formula for the transition probabilities of the conditioned Markov chain:

$$(8.4) \qquad P_y^\omega[X_1 = z | T_{\text{root}}^+ < T_\delta] = \frac{h^\omega(z)}{h^\omega(y)} P_y^\omega[X_1 = z]$$

for y, z in the trap. Due to the Markov property, h^ω is harmonic except on δ and *root*. It can be computed easily here (note that this is a one-dimensional calculation involving only the spine):

$$h^\omega(y) = h^\omega(y \wedge \delta) = \beta^{-d(\text{root}, y \wedge \delta)} \frac{1 - \beta^{-(H+1-d(\text{root}, y \wedge \delta))}}{1 - \beta^{-(H+1)}}.$$

In particular, comparing the walk conditioned on the event $\{T_\delta^+ > T_{\text{root}}^+\}$ to the original walk, we have the following:

(1) the walk remains unchanged on the subtraps,
(2) for y on the spine and z a descendant of y not on the spine, we have

$$P_y^\omega[X_1 = \overleftarrow{y} | T_{\text{root}}^+ < T_\delta] > P_y^\omega[X_1 = z | T_{\text{root}}^+ < T_\delta],$$

(3) for $y \notin \{\delta, \text{root}\}$ on the spine, we have

$$P_y^\omega[X_1 = \overleftarrow{y} | T_{\text{root}}^+ < T_\delta] > \beta P_y^\omega[X_1 = \overrightarrow{y} | T_{\text{root}}^+ < T_\delta].$$

The points (2) and (3) state, respectively, that the conditioned walk is more likely to go toward *root* than to go to a given vertex of a subtrap and that restricted to the spine the conditioned walk is more than β-drifted toward *root*.

LEMMA 8.2. *For $z \in \{\delta, \text{root}\}$, we have*

$$E_{\mathbf{Q}_n}[\mathbf{1}\{\tilde{A}(n)\} E_z^\omega[T_{\text{root}}^+ | T_{\text{root}}^+ < T_\delta^+]] \leq C(\ln n)n^{(1-\varepsilon)(1/\gamma-1)+\varepsilon}.$$

PROOF. First, let us show that the walk cannot visit too often a vertex of the spine. Indeed let y be a vertex of the spine, using fact (3), we have $P_y^\omega[T_y^+ > T_{\text{root}}^+ | T_{\text{root}}^+ < T_\delta] \geq p_\infty$. Hence, the random variable $N(y) = \text{card}\{n \leq T_{\text{root}}^+ : X_n = y\}$ with (X_n) conditioned on $\{T_{\text{root}}^+ < T_\delta\}$ is stochastically dominated by $G(p_\infty)$, a geometric random variable with parameter p_∞.

Furthermore, we cannot visit often a given subtrap $s(y) \in S_y$ [recall (3.4)]. Indeed, if we denote the number of visits to $s(y)$ by $N(s(y)) = \text{card}\{n \leq T_{\text{root}}^+ : X_n = y, X_{n+1} \in s(y)\}$, using fact (2) and a reasoning similar to the one for the asymptotics on $A_3(n)$ in Lemma 5.1 we have that $N(s(y))$ with (X_n) conditioned on $\{T_{\text{root}}^+ < T_\delta\}$ is stochastically dominated by $G(p_\infty/2)$.

Let us now consider the following decomposition:

$$T_{\text{root}}^+ = T_{\text{spine}} + \sum_{s \in \text{subtraps}} \sum_{j=1}^{N(s)} R_s^j,$$

where $T_{\text{spine}} = \text{card}\{n \leq T_{\text{root}}^+ : X_n \text{ is in the spine}\} = \sum_{x \in \text{spine}} N(x)$ and R_s^j is the time spent in the subtrap s during the jth excursion in it. Moreover, on $\widetilde{A}(n)$, the law of any subtrap s is that of a Galton–Watson tree conditioned to have height strictly less than $i + 1$ for some $i \leq h_n$. Then Lemma 3.1 implies that for such a subtrap, $E^\omega[R_s^j]$ has the same law as $2\Pi_{i+1}^{i,1,1}$ which satisfies using Lemma 3.4 that $E_{\mathbf{Q}_n}[\Pi_j^i] \leq C(\beta\mathbf{f}'(q))^i$. Moreover, on $\widetilde{A}(n)$, there are at most h_n^+ vertices in the spine and at most n^ε subtraps, hence

$$E_{\mathbf{Q}_n}\big[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_{\text{root}}^+|T_{\text{root}}^+ < T_\delta^+]\big]$$

$$\leq h_n^+ E[G(p_\infty)] + h_n^+ n^\varepsilon E[G(p_\infty/2)]C(\beta\mathbf{f}'(q))^{h_n},$$

and using $(\beta\mathbf{f}'(q))^{h_n} \leq Cn^{(1-\varepsilon)(1/\gamma-1)}$ we get

$$E_{\mathbf{Q}_n}\big[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_{\text{root}}^+|T_{\text{root}}^+ < T_\delta^+]\big] \leq C(\ln n)n^{(1-\varepsilon)(1/\gamma-1)+\varepsilon}. \qquad \square$$

The previous proof is mainly based on the three statements preceding the statement of Lemma 8.2. Similarly, one can show the following lemma.

LEMMA 8.3.   *For $z \in \{\delta, \text{root}\}$, we have*

$$E_{\mathbf{Q}_n}\big[\mathbf{1}\{\widetilde{A}(n)\}E_z^\omega[T_\delta^+|T_\delta^+ < T_{\text{root}}^+]\big] \leq C(\ln n)n^{(1-\varepsilon)(1/\gamma-1)+\varepsilon}.$$

PROOF.   To apply the same methods as in the proof of Lemma 8.2, we only need that the h-process corresponding to the conditioning on the event $\{T_\delta^+ < T_{\text{root}}^+\}$ satisfies that:

(1) the walk remains unchanged on the subtraps,

(2) for y on the spine and z a descendant of y not on the spine, we have $P_y^\omega[X_1 = \overrightarrow{y}|T_\delta^+ < T_{\text{root}}^+] > P_y^\omega[X_1 = z|T_\delta^+ < T_{\text{root}}^+]$,

(3) for $y \neq \{\delta, \text{root}\}$ on the spine, we have $P_y^\omega[X_1 = \overrightarrow{y}|T_\delta^+ < T_{\text{root}}^+] > \beta P_y^\omega[X_1 = \overleftarrow{y}|T_\delta^+ < T_{\text{root}}^+]$.

This immediately follows from the computation of the function $\widehat{h}^\omega$, given by $\widehat{h}^\omega(z) = P_z^\omega[T_\delta^+ < T_{\text{root}}^+]$, with $\widehat{h}^\omega(\text{root}) = 0$ and $\widehat{h}^\omega(\delta) = 1$. A computation gives

$$(8.5) \qquad \widehat{h}^\omega(y) = \widehat{h}^\omega(d(y \wedge \delta, \delta)) = \frac{\beta^{H+1} - \beta^{d(y\wedge\delta,\delta)}}{\beta^{H+1} - 1}. \qquad \square$$

From (8.3), Lemmas 8.3 and 8.2, we deduce that

$$(8.6) \qquad \mathbb{E}[\mathbf{1}\{\widetilde{A}(n)\}(T_{\text{root}}^+ - T_{\text{root}}^{*,+})] \leq C(\ln n)n^{(1-\varepsilon)(1/\gamma-1)+\varepsilon}.$$

Now using (8.6) and (8.2) we prove (8.1), more precisely

$$\text{for all } t > 0 \qquad \mathbb{P}\left[\left|\frac{\sum_{i=1}^{l_n}\chi_j(n) - \chi_j^*(n)}{n^{1/\gamma}}\right| \geq t\right] \leq o(1) + C(\ln n)n^{2\varepsilon-1-\varepsilon(1/\gamma-1)},$$

and thus Proposition 8.1 follows for $\varepsilon < 1/4$.

9. Analysis of the time spent in big traps. Let us denote $\overline{\mathbf{Q}}_n := \mathbf{Q}[\cdot | H = h_n^0]$ where

$$h_n^0 = \lceil \ln n / -\ln \mathbf{f}'(q) \rceil.$$

Note that

(9.1) $$p_1(H) := P_\delta^\omega[T_\delta^+ < T_{\text{root}}^+] = \frac{1 - \beta^{-1}}{1 - \beta^{-(H+1)}},$$

where we recall that the distance between *root* and δ is $1 + H$. Moreover, let us denote

(9.2) $$p_2(H) := P_\delta^\omega[T_{\text{root}}^+ < T_\delta^+] = \frac{1 - \beta^{-1}}{\beta^H - \beta^{-1}}.$$

We have the following decomposition:

(9.3) $$\chi_1^*(n) = \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))^{(i)} - 1} T_{\text{exc}}^{(i,j)},$$

where $T_{\text{exc}}^{(i,j)}$ is the time spent during the jth excursion in the ith trap, which is distributed under $\mathbf{Q}_n$ as T_δ^+ under $P_\delta^\omega[\cdot | T_\delta^+ < T_{\text{root}}^+]$ with ω chosen according to $\mathbf{Q}_n$, for all (i, j). The $T_{\text{exc}}^{(i,j)}$ are independent with respect to P^ω and for $i_1 \neq i_2$ $(T_{\text{exc}}^{(i_1,j)})_{j \geq 1}$ and $(T_{\text{exc}}^{(i_2,j)})_{j \geq 1}$ are independent with respect to $\mathbf{Q}_n$. For $k \in \mathbb{Z}$ and n large enough, let $\mathcal{Z}_n^k$ be a random variable with the law of $\chi_1^*(n)/\beta^H$ under $\mathbf{Q}_{n+k}$ and $\overline{\mathcal{Z}}_n^k$ be a random variable with the law of $\chi_1^*(n)/\beta^H$ under $\overline{\mathbf{Q}}_{n+k}$. Furthermore, we define $\mathcal{Z}_\infty := \frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \mathbf{e}_i$ [see (1.3)] where $(\mathbf{e}_i)_{i \geq 1}$ is a family of i.i.d. exponential random variables of parameter 1, chosen independently of the (independent) random variables S_∞ and W_∞. Our aim is to show the following proposition.

PROPOSITION 9.1. *We have*

$$\overline{\mathcal{Z}}_n^k \xrightarrow{d} \mathcal{Z}_\infty.$$

Moreover there exists a random variable $\mathcal{Z}_{\text{sup}}$ such that

$$E[\mathcal{Z}_{\text{sup}}^{1-\varepsilon}] < \infty \qquad \text{for any } \varepsilon > 0$$

and

for $n \in \mathbb{N}$ and $k > -n$ $\qquad \overline{\mathcal{Z}}_n^k \preceq \mathcal{Z}_{\text{sup}}.$

PROOF.   Let us start by proving the convergence in law. The decomposition (9.3) for $\chi_1^*(n)$ can be rewritten using (9.2),

(9.4)
$$\chi_1^*(n) = \beta^H \sum_{i=1}^{\mathrm{Bin}(W_n, p_1(H))} \frac{1-\beta^{-H-1}}{1-\beta^{-1}} \frac{\beta^H-1}{\beta^H-\beta^{-1}} \frac{p_2(H)}{1-p_2(H)}$$
$$\times \sum_{j=1}^{G(p_2(H))^{(i)}-1} T_{\mathrm{exc}}^{(i,j)},$$

which yields an explicit expression of $\mathcal{Z}_n^k$. We point out that $E[G(p_2(H)) - 1] = (1 - p_2(H))/p_2(H)$. The convergence in law is due to the following facts (more precise statements follow below):

(1) For H large,
$$\frac{(1-\beta^{-H-1})(\beta^H-1)}{(1-\beta^{-1})(\beta^H-\beta^{-1})} \approx \frac{1}{1-\beta^{-1}}.$$

(2) By the law of large numbers, we can expect
$$\sum_{j=1}^{G(p_2(H))^{(i)}-1} T_{\mathrm{exc}}^{(i,j)} \approx \big(G(p_2(H))^{(i)} - 1\big) E_\delta^\omega[T_{\mathrm{exc}}].$$

(3) Since $p_2(H)$ is small, $(G(p_2(H))^{(i)} - 1)/E[G(p_2(H))^{(i)} - 1] \approx \mathbf{e}_i$.
(4) $E_\delta^\omega[T_{\mathrm{exc}}^{(1,1)}] \approx S_\infty$ for H large enough.

(5) $\mathrm{Bin}(W_n, p_1(H)) \approx \mathrm{Bin}(W_\infty, p_\infty)$ since $W_n \xrightarrow{d} W_\infty$ by Proposition 6.1 and $p_1(H) \to p_\infty$ as H goes to infinity.

Fact (1) is easily obtained, since for $\xi > 0$

(9.5) $$\mathbf{Q}_n\left[(1-\xi)\frac{1}{1-\beta^{-1}} \le \frac{1-\beta^{-H+1}}{1-\beta^{-1}}\frac{\beta^H-1}{\beta^H-\beta^{-1}} \le \frac{1}{1-\beta^{-1}}\right] = 1$$

for n large enough.

We start by computations with the measure $\mathbf{Q}_n$ and we will be able to come back to $\overline{\mathbf{Q}}_{n+k}$.

For (2) and (4), we need to understand $P_\delta^\omega[\cdot | T_\delta^+ < T_{\mathrm{root}}^+]$ and to this end we will consider the h-process associated with this conditioning. Recall the function $\widehat{h}^\omega$ given as $\widehat{h}^\omega(z) = P_z^\omega[T_\delta < T_{\mathrm{root}}]$, with $\widehat{h}^\omega(\delta) = 1$ and $\widehat{h}^\omega(root) = 0$, see (8.5).

We shall enumerate the vertices of the backbone from 0 to $H + 1$, starting from δ up to $root$. With these new notations, formula (8.5) becomes

(9.6) $$\widehat{h}^\omega(y) = \widehat{h}^\omega(y \wedge \delta) = \frac{\beta^{H+1} - \beta^{y \wedge \delta}}{\beta^{H+1} - 1},$$

where $y \wedge \delta$ is identified to its number which is $d(y \wedge \delta, \delta)$ as it is a vertex of the backbone.

The transition probabilities are then given as in (8.4). Obviously, they arise from conductances, we may take:

(i) $\widehat{c}(0, 1) = 1$,

(ii) $\widehat{c}(i, i+1) = \widehat{c}(i-1, i) \frac{P_i^\omega[X_1 = i+1 | T_\delta^+ < T_{\text{root}}^+]}{P_i^\omega[X_1 = i-1 | T_\delta^+ < T_{\text{root}}^+]}$, for $1 \leq i \leq H$,

(iii) $\widehat{c}(i, z) = \widehat{c}(i, i-1) \frac{P_i^\omega[X_1 = z | T_\delta^+ < T_{\text{root}}^+]}{P_i^\omega[X_1 = i-1 | T_\delta^+ < T_{\text{root}}^+]}$, for $i \neq 0$ on the spine and z one of its descendants which is not on the spine,

(iv) $\widehat{c}(y, z) = \beta \widehat{c}(\overleftarrow{y}, y)$ for any vertex y not on the spine and z one of its descendants.

We can easily deduce from this that for $y \neq root$ in the trap and denoting $z_0 = \delta, \ldots, z_n = y$ the geodesic path from δ to y:

$$\widehat{c}(z_{n-1}, y) = \prod_{j=1}^{n-1} \frac{P_{z_j}^\omega[X_1 = z_{j+1} | T_\delta^+ < T_{\text{root}}^+]}{P_{z_j}^\omega[X_1 = z_{j-1} | T_\delta^+ < T_{\text{root}}^+]},$$

which gives using (9.6) that

(9.7) $\quad \widehat{c}(i, i+1) = \beta^{-i} \frac{\widehat{h}(i+1)\widehat{h}(i)}{\widehat{h}(1)\widehat{h}(0)} = \beta^{-i} \frac{(1 - \beta^{i-H})(1 - \beta^{i-(H+1)})}{(1 - \beta^{-H})(1 - \beta^{-(H+1)})}.$

For a vertex z not on the spine, we have

(9.8)
$$\widehat{c}(z, \overleftarrow{z}) = \beta^{d(\overleftarrow{z}, z \wedge \delta)} \frac{\widehat{h}(z \wedge \delta)}{\widehat{h}(z \wedge \delta - 1)} c(z \wedge \delta, z \wedge \delta - 1)$$

$$= \beta^{d(\overleftarrow{z}, z \wedge \delta)} \frac{1 - \beta^{z \wedge \delta - (H+1)}}{1 - \beta^{z \wedge \delta - 1 - (H+1)}} c(z \wedge \delta, z \wedge \delta - 1).$$

Together with Lemma 3.1, this yields, with T_{exc} a generic random variable with the law of $T_{\text{exc}}^{(1,1)}$,

(9.9)
$$E_\delta^\omega[T_{\text{exc}}] = 2 \sum_{i=0}^{H-1} \beta^{-i} \frac{(1 - \beta^{i-H})(1 - \beta^{i-(H+1)})}{(1 - \beta^{-H})(1 - \beta^{-(H+1)})}$$

$$\times \left(1 + \frac{1 - \beta^{i-(H+1)}}{1 - \beta^{(i-1)-(H+1)}} \Lambda_i(\omega) \right),$$

where Λ_i was defined in (3.5).

We see that the random variable S_∞ is the limit of the last quantity as H goes to infinity. More precisely, using (9.9) we have $0 \leq S_\infty - E_\delta^\omega[T_{\text{exc}}]$ and for n large enough such that

for all $k \leq h_n/2$ $\quad \frac{(1 - \beta^{k-h_n})(1 - \beta^{k-(h_n+1)})}{(1 - \beta^{-h_n})(1 - \beta^{-(h_n+1)})} \geq 1 - 2\beta^{h_n}$,

we get

$$S_\infty - E_\delta^\omega[T_{\mathrm{exc}}] \leq 2\left(\sum_{i=0}^{h_n/2} \beta^{-i}\left(1 - \frac{(1 - \beta^{i-H})(1 - \beta^{i-(H+1)})}{(1 - \beta^{-H})(1 - \beta^{-(H+1)})}\right)(1 + \Lambda_i)\right)$$

$$+ 2\left(\sum_{i=h_n/2+1}^{\infty} \beta^{-i}(1 + \Lambda_i)\right)$$

$$\leq 4\beta^{-h_n/2}\left(\sum_{i=0}^{h_n/2} \beta^{-i}(1 + \Lambda_i)\right) + 2\left(\sum_{i=h_n/2+1}^{\infty} \beta^{-i}(1 + \Lambda_i)\right).$$

Hence, since $S_\infty \geq 1$ and using Chebyshev's inequality, we get

$$\mathbf{Q}_n\big[(1 - \xi)S_\infty < E_\delta^\omega[T_{\mathrm{exc}}] < S_\infty\big]$$

$$\geq 1 - \mathbf{Q}_n\big[S_\infty - E_\delta^\omega[T_{\mathrm{exc}}] \geq \xi\big]$$

(9.10)

$$\geq 1 - \frac{1}{\xi}\beta^{-h_n/2}\frac{10}{1 - \beta^{-1}}\sup_{i \geq 0} E_{\mathbf{Q}}[1 + \Lambda_i]$$

$$= 1 + o(1),$$

where we used Lemma 3.5 and the fact that $\varepsilon < 1/4$, this proves (4).

In order to prove (2), we have to bound $E_\delta^\omega[T_1^2]$ from above. This is not possible for all ω, but we consider the event

$$A_4(n) = \big\{E_\delta^\omega[T_1^2] \leq n^{(1-2\varepsilon)/\gamma}\big\}$$

and show that it satisfies the following lemma.

LEMMA 9.1.   *For $0 < \varepsilon < \min(1/3, 2\gamma/3)$, we have*

$$\mathbf{Q}_n[A_4(n)^c] \to 0.$$

PROOF.   In this proof we denote for y in the trap, $N(y)$ the number of visits to y during an excursion from δ, which is distributed as $\mathrm{card}\{0 \leq n \leq T_\delta^+ : X_n = y\}$ under $P_\delta^\omega[\cdot|T_\delta^+ < T_{\mathrm{root}}^+]$. We have, using the Minkowski inequality,

$$E_\delta^\omega[T_{\mathrm{exc}}^2] = E_\delta^\omega\left[\left(\sum_{y \in \mathrm{trap}} N(y)\right)^2\right]$$

$$\leq \sum_{y,z \in \mathrm{trap}} E_\delta^\omega[N(y)^2]^{1/2}E_\delta^\omega[N(z)^2]^{1/2}$$

$$= \left(\sum_{y \in \mathrm{trap}} E_\delta^\omega[N(y)^2]^{1/2}\right)^2.$$

Now fix y in the trap, denote $q_1 = P_\delta^\omega[T_y^+ < T_\delta^+ | T_\delta^+ < T_{\text{root}}^+]$ and $q_2 = P_y^\omega[T_\delta^+ < T_y^+ | T_\delta^+ < T_{\text{root}}^+]$. Then we have

$$\forall k \geq 1 \qquad P_\delta^\omega[N(y) = k] = q_1(1 - q_2)^{k-1} q_2.$$

Hence,

$$E_\delta^\omega[N(y)^2] = \sum_{n \geq 1} n^2 q_1 (1 - q_2)^{n-1} q_2 = q_1 \frac{2 - q_2}{q_2^2} \leq \frac{2q_1}{q_2^2}.$$

Then by reversibility of the walk, if $\widehat{\pi}$ is the invariant measure associated with the conductances $\widehat{c}$, we get $q_1 = \widehat{\pi}(\delta) q_1 = \widehat{\pi}(y) q_2$. This yields

$$(9.11) \qquad E_\delta^\omega[N(y)^2] \leq \frac{2\widehat{\pi}(y)}{q_2}.$$

Furthermore, we have

$$(9.12) \qquad q_2 \geq \big(1/(Z_1(y)\beta + 1)\big) p_\infty \beta^{-d(\delta, \delta \wedge y)}/2.$$

Indeed suppose that y is not on the spine, otherwise the bound is simple. Starting from y, we reach the ancestor of y with probability at least $(1/(Z_1(y)\beta + 1))$ then the walker has probability at least $\beta^{-d(y, y \wedge \delta)}$ to reach $y \wedge \delta$ before y, next he has probability at least $1/2$ to go to $\overrightarrow{y \wedge \delta}$ before going to z, where z is the first vertex on the geodesic path from $y \wedge \delta$ to y. Finally, from $\overrightarrow{y \wedge \delta}$, the walker has probability at least p_∞ to go to δ before coming back to $\overrightarrow{y \wedge \delta}$.

We denote by π the invariant measure associated with the β-biased random walk (i.e., not conditioned on $T_\delta^+ < T_{\text{root}}^+$), normalized so as to have $\pi(\delta) = 1$. Then we have:

(1) for any y in the trap, $\widehat{\pi}(y) \leq \pi(y)$ because of (9.7) and (9.8),
(2) and by definition of the invariant measure $(Z_1(y)\beta + 1)\beta^{d(\delta, \delta \wedge y) - d(y, \delta \wedge y)} = \pi(y)$.

Now plugging (2) in (9.12) yields a lower bound on q_2 which can be used together with (1) in (9.11) to get

$$E_\delta^\omega[N(y)^2] \leq C\beta^{d(\delta, \delta \wedge y)} \pi(y)^2$$

and

$$E_\delta^\omega[T_{\text{exc}}^2]^{1/2} \leq C \sum_{y \in \text{trap}} \beta^{d(\delta, \delta \wedge y)/2} \pi(y).$$

As a consequence, with $\widetilde{A}(n)$ as in Lemma 8.1 we get

$$E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_{\text{exc}}^2]^{1/2}] \le C E_{\mathbf{Q}_n}\left[\mathbf{1}\{\widetilde{A}(n)\}\sum_{i=0}^{h_n^+}\beta^{-i/2}\Lambda_i\right]$$

$$\le C\sum_{i=0}^{h_n^+}(\beta^{1/2}\mathbf{f}'(q))^i$$

$$\le C\max(1, (\beta^{1/2}\mathbf{f}'(q))^{h_n^+}),$$

where we used Lemmas 3.1 and 3.4 for the first inequality.

Since $(\beta^{1/2}\mathbf{f}'(q))^{h_n^+} = n^{(1+\varepsilon)(1/2\gamma-1)}$, we get by Chebyshev's inequality that

$$\mathbf{Q}_n\big[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_{\text{exc}}^2]^{1/2} \ge n^{(1-2\varepsilon)/(2\gamma)}\big] \le \frac{1}{n^{(1-2\varepsilon)/(2\gamma)}}E_{\mathbf{Q}_n}[\mathbf{1}\{\widetilde{A}(n)\}E_\delta^\omega[T_1^2]^{1/2}]$$

$$\le C\max\big(n^{-(1-2\varepsilon)/(2\gamma)}, n^{3\varepsilon/(2\gamma)-1-\varepsilon}\big).$$

The conditions on ε ensure that this last term goes to 0 for $n \to \infty$. Hence,

$$\mathbf{P}[\widetilde{A}(n) \cap A_4(n)^c] \to 0$$

and the result follows using Lemma 8.1. □

We now turn to the study of

$$\frac{p_2(H)}{1-p_2(H)}\sum_{i=1}^{G(p_2(H))-1}T_{\text{exc}}^{(i)}.$$

Consider the random variable

(9.13)
$$N_g = \left\lfloor \frac{-1}{\ln(1-p_2(H))}\mathbf{e} \right\rfloor,$$

where $\mathbf{e}$ is an exponential random variable of parameter 1. A simple computation shows that N_g has the law of $G(p_2(H)) - 1$.

Set $\xi > 0$; we have, using Chebyshev's inequality,

$$\mathbf{Q}_n\left[(1-\xi)N_g E_\delta^\omega[T_{\text{exc}}] \le \sum_{i=1}^{N_g}T_{\text{exc}}^{(i)} \le (1+\xi)N_g E_\delta^\omega[T_{\text{exc}}]\right]$$

$$\ge 1 - \mathbf{Q}_n\left[\left|\frac{\sum_{i=1}^{N_g}T_{\text{exc}}^{(i)}}{N_g} - E_\delta^\omega[T_{\text{exc}}]\right| > \xi E_\delta^\omega[T_{\text{exc}}],\right.$$

$$\left. N_g \ne 0, E_\delta^\omega[T_{\text{exc}}^2] \le n^{(1-2\varepsilon)/\gamma}\right]$$

$$- \mathbf{Q}_n\big[E^\omega[T_{\text{exc}}^2] \ge n^{(1-2\varepsilon)/\gamma}\big] - \mathbf{Q}_n[N_g = 0]$$

$$\geq E_{\mathbf{Q}_n}\left[\frac{n^{(1-2\varepsilon)/\gamma}}{N_g}\mathbf{1}\{N_g \neq 0\}\frac{1}{\xi^2}\right] - \mathbf{Q}_n\big[E^\omega[T_{\text{exc}}^2] \geq n^{(1-2\varepsilon)/\gamma}\big]$$

$$- \mathbf{Q}_n[N_g = 0].$$

We have $\mathbf{Q}_n[N_g = 0] = p_2(H) \leq p_2(h_n) \leq Cn^{-(1-\varepsilon)/\gamma}\ln n$, and hence

$$E_{\mathbf{Q}_n}\left[\frac{\mathbf{1}\{N_g \neq 0\}}{N_g}\right] = \mathbb{E}\left[-\frac{p_2(H)}{1 - p_2(H)}\ln p_2(H)\right] \leq Cn^{-(1-\varepsilon)/\gamma}(\ln n)^2.$$

Putting together the two previous equations, using Lemma 9.1, we get for $\xi < 1$

$$(9.14) \quad \mathbf{Q}_n\left[(1-\xi)N_g E_\delta^\omega[T_{\text{exc}}] < \sum_{i=1}^{N_g} T_{\text{exc}}^{(i)} < (1+\xi)N_g E_\delta^\omega[T_{\text{exc}}]\right] \to 1.$$

This shows (2). Turning to prove (3), we have

$$\mathbf{Q}_n\left[(1-\xi)\left\lfloor\frac{1}{-\ln(1 - p_2(H))}\mathbf{e}\right\rfloor\right.$$

$$\left.\leq \frac{1 - p_2(H)}{p_2(H)}\mathbf{e} \leq (1+\xi)\left\lfloor\frac{1}{-\ln(1 - p_2(H))}\mathbf{e}\right\rfloor\right]$$

$$\geq 1 - \mathbf{Q}_n\left[\left(\frac{1 - p_2(H)}{p_2(H)} - \frac{1-\xi}{-\ln(1 - p_2(H))}\right)\mathbf{e} < 1\right]$$

$$- \mathbf{Q}_n\left[\left(\frac{1 - p_2(H)}{p_2(H)} - \frac{1+\xi}{-\ln(1 - p_2(H))}\right)\mathbf{e} > -2\right],$$

furthermore since $|\frac{1-p}{p} - \frac{1}{-\ln(1-p)}|$ is bounded on $(0, \varepsilon_1)$ by a certain $M > 0$ so that for n large enough with $p_2(h_n) < \varepsilon_1$, we get

$$\mathbf{Q}_n\left[(1-\xi)\left\lfloor\frac{1}{-\ln(1 - p_2(H))}\mathbf{e}\right\rfloor\right.$$

$$\left.\leq \frac{1 - p_2(H)}{p_2(H)}\mathbf{e} \leq (1+\xi)\left\lfloor\frac{1}{-\ln(1 - p_2(H))}\mathbf{e}\right\rfloor\right]$$

$$\geq 1 - \mathbf{Q}_n\left[\left(\frac{\xi}{-\ln(1 - p_2(H))} - M\right)\mathbf{e} < 1\right]$$

(9.15)

$$- \mathbf{Q}_n\left[\left(-\frac{\xi}{-\ln(1 - p_2(H))} + M\right)\mathbf{e} > -2\right]$$

$$\geq \exp\left(-\frac{2}{\xi/(-\ln(1 - p_2(h_n))) - M}\right)$$

$$\geq 1 - (C/\xi)p_2(h_n),$$

which shows (3).

As a consequence of (9.10), (9.14) and (9.15), we see that for all $\xi \in (0, 1)$,

$$(9.16) \quad \mathbf{Q}_n\left[(1 - \xi)S_\infty \mathbf{e} \leq \frac{p_2(H)}{1 - p_2(H)} \sum_{i=1}^{N_g} E^\omega[T_{\text{exc}}^{(i)}] \leq (1 + \xi)S_\infty \mathbf{e}\right] \to 1$$

for $n \to \infty$. Using (9.2), (9.16) and (9.5) we get

$$(9.17) \quad \mathbf{Q}_n\left[(1 - \xi)\frac{S_\infty \mathbf{e}}{1 - \beta^{-1}} \leq \frac{1}{\beta^H} \sum_{i=1}^{N_g} E^\omega[T_{\text{exc}}^{(i)}] \leq (1 + \xi)\frac{S_\infty \mathbf{e}}{1 - \beta^{-1}}\right] \to 1$$

for $n \to \infty$, which sums up (2), (3) and (4). For any $k > -n$, the equation (9.17) obviously holds replacing n with $n + k$, and since $\mathbf{Q}_n[H = \lceil \ln(n + k)/\ln \mathbf{f}'(q)\rceil] \geq c_k > 0$ (this follows from Lemma 3.3), we have

$$(9.18) \quad \overline{\mathbf{Q}}_{n+k}\left[(1 - \xi)\frac{S_\infty \mathbf{e}}{1 - \beta^{-1}} \leq \frac{1}{\beta^H} \sum_{i=1}^{N_g} E^\omega[T_{\text{exc}}^{(i)}] \leq (1 + \xi)\frac{S_\infty \mathbf{e}}{1 - \beta^{-1}}\right] \to 1.$$

Only part (5) remains to be shown. Coupling $\text{Bin}(W_n, p_\infty)$ and $\text{Bin}(W_n, p_1(H))$ in the standard way,

$$\overline{\mathbf{Q}}_{n+k}[\text{Bin}(W_n, p_\infty) \neq \text{Bin}(W_n, p_1(H))]$$
$$\leq \sum_{j \geq 0} \mathbb{P}[W_n = j]\overline{\mathbf{Q}}_{n+k}[\text{Bin}(j, p_\infty) \neq \text{Bin}(j, p_1(H))]$$
$$\leq \sum_{j \geq 0} \mathbb{P}[W_n = j]j(p_1(h_{n+k}^0) - p_\infty)$$
$$\leq \mathbb{E}[W_n](p_1(h_{n+k}^0) - p_\infty)$$
$$\leq C(p_1(h_{n+k}^0) - p_\infty) \to 0 \qquad \text{for } n \to \infty,$$

where $C := E[G(p_\infty/3)] \geq \mathbb{E}[W_n]$ by Lemma 6.2. Hence,

$$\overline{\mathbf{Q}}_{n+k}\left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t\right]$$
$$- \overline{\mathbf{Q}}_{n+k}\left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t\right] \to 0.$$

For $\varepsilon_1 > 0$, introduce $N(\varepsilon_1)$ such that $\max_{n \leq \infty} \mathbb{P}[W_n \geq N(\varepsilon_1)] \leq (1 - p_\infty/3)^{N(\varepsilon_1)} \leq \varepsilon_1$ and using the independence of W_n (for $n \in \mathbb{N} \cup \{\infty\}$) of the

trap and the walk on the trap, we get, for any $\varepsilon_1 > 0$,

$$\left| \overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right] \right.$$

$$\left. - \overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right] \right|$$

$$\leq \left| \sum_{j \geq 0} (\mathbb{P}[W_n = j] - \mathbb{P}[W_\infty = j]) \right.$$

$$\left. \times \overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(j, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right] \right|$$

$$\leq \left| \sum_{j \in [0, N(\varepsilon_1)]} (\mathbb{P}[W_n = j] - \mathbb{P}[W_\infty = j]) \right.$$

$$\left. \times \overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(j, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right] \right| + \varepsilon_1,$$

and the right-hand side goes to ε_1 as n goes to infinity since

$$\max_{j \leq N(\varepsilon_1)} |\mathbb{P}[W_n = j] - \mathbb{P}[W_\infty = j]| \to 0$$

by Proposition 6.1. So letting ε_1 go to 0, we see that

(9.19)
$$\overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right]$$

$$- \overline{\mathbf{Q}}_{n+k} \left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t \right] \to 0.$$

Let us introduce

$$A(\xi) = \left\{ \text{for all } i \in [1, \text{Bin}(W_\infty, p_\infty)], \right.$$

$$\left. \frac{1}{\beta^H} \sum_{j=1}^{G^{(i)}(p_2(H))-1} T_{\text{exc}}^{(i,j)} \in \left[(1-\xi)\frac{S_\infty}{1-\beta^{-1}}\mathbf{e}_i, (1+\xi)\frac{S_\infty}{1-\beta^{-1}}\mathbf{e}_i \right] \right\},$$

where $(\mathbf{e}_i)_{i \geq 1}$ is a sequence of i.i.d. exponential random variables of parameter 1 which satisfy

$$G^{(i)}(p_2(H)) - 1 = \left\lfloor \frac{-1}{\ln(1-p_2(H))}\mathbf{e}_i \right\rfloor.$$

We have, denoting $o_1(1)$ the left-hand side of (9.19)

$$\overline{\mathbf{Q}}_{n+k}[A(\xi)] \geq \sum_{i \geq 0} \mathbb{P}[\text{Bin}(W_\infty, p_\infty) = i](1 - o_1(1))^i$$

$$\geq \sum_{i \geq 0} \mathbb{P}[\text{Bin}(W_\infty, p_\infty) = i](1 - i o_1(1))$$

$$= 1 - \mathbb{E}[W_\infty] o_1(1) \to 1 \qquad \text{for } n \to \infty.$$

Hence, for any $\xi > 0$, we get

$$\overline{\mathbf{Q}}_{n+k}\left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t\right]$$

$$- \mathbb{P}\left[\frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \mathbf{e}_i \geq \frac{t}{1 + \xi}\right] \to 0$$

and

$$\overline{\mathbf{Q}}_{n+k}\left[\frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \geq t\right]$$

$$- \mathbb{P}\left[\frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \mathbf{e}_i \geq \frac{t}{1 - \xi}\right] \to 0.$$

Concluding by using the two previous equations with ξ going to 0, we have the following convergence in law:

$$\overline{\mathbb{Z}}_n^k = \frac{1}{\beta^H} \sum_{i=1}^{\text{Bin}(W_n, p_1(H))} \sum_{j=1}^{G(p_2(H))-1} T_{\text{exc}}^{(i,j)} \xrightarrow{d} \frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \mathbf{e}_i,$$

where we recall that $\overline{\mathbb{Z}}_n^k$ has the law of $\chi_1^*(n)/\beta^H$ under $\overline{\mathbf{Q}}_{n+k}$ and the $\mathbf{e}_i$ are i.i.d. exponential random variables of parameter 1. This shows the first part of Proposition 9.1.

Now let us prove the stochastic domination part. First, notice that

$$\text{Bin}(W_n, p_1(H)) \preceq G(p_\infty/3) \quad \text{and} \quad E_\delta^\omega[T_1] \preceq T_{\text{exc}}^\infty,$$

where T_{exc}^∞ is distributed as the return time to δ, starting from δ, on an infinite trap. Hence, for $k > -n$,

$$\overline{\mathbb{Z}}_n^k \preceq \frac{1}{\beta^{h_{n+k}^0}} \sum_{i=1}^{G(p_\infty/3)} \sum_{j=1}^{G(p_2(h_{n+k}^0))} T_{\text{exc}}^{\infty,(i,j)},$$

where $(T_{\text{exc}}^{\infty,(i,j)})_{i,j\geq1}$ are i.i.d. copies of T_{exc}^{∞}. Now recalling that $\sum_{i=1}^{G(a)} G(b)^{(i)}$ has the same law as $G(ab)$, where all geometric random variables are independent, and using the fact that

$$\beta^{h_{n+k}^0} \geq cE\big[G(p_\infty p_2(h_{n+k}^0)/3)\big]$$

for some $c = c(\beta) > 0$, we get

$$\overline{\mathbb{Z}}_n^k \preceq \frac{C}{E[G(p_\infty p_2(h_{n+k}^0)/3)]} \sum_{i=1}^{G(p_\infty p_2(h_{n+k}^0)/3)} T_{\text{exc}}^{\infty,(i)}.$$

Now, we prove the following technical lemma.

LEMMA 9.2. *Let $(X_i)_{i\geq0}$ a sequence of i.i.d. nonnegative random variables such that $E[X_1] < \infty$ and set $Y_i := (X_1 + \cdots + X_i)/i$. Then there exists a random variable $Y_{\sup}$ such that*

$$\text{for all } i \geq 0 \qquad Y_i \preceq Y_{\sup}$$

and $E[Y_{\sup}^{1-\varepsilon}] < \infty$ for all $\varepsilon > 0$.

PROOF. Using Chebyshev's inequality, we get that for any $i \geq 0$,

$$\text{for all } t \geq 0 \qquad P[Y_i > t] \leq \frac{1}{t}E[X_1].$$

If we choose $Y_{\sup}$ such that $P[Y_{\sup} > t] = \min(1, E[X_1]/t)$ for $x \geq 0$, then $Y_{\sup}$ stochastically dominates all Y_n and has a finite $(1-\varepsilon)$th moment for all $\varepsilon > 0$. $\quad\square$

Now we apply this lemma to the random variables $T_{\text{ext}}^{\infty,(i)}$ which are integrable under $\mathbb{P}$ and we get a certain random variable $T_{\sup}$. We add to our probability spaces a copy of $T_{\sup}$ which is independent of all other random variables. Then for any $t \geq 0$,

$$\mathbb{P}[\overline{\mathbb{Z}}_n^k \geq t] \leq \mathbb{P}\Bigg[\frac{C}{E[G(p_\infty p_2(h_{n+k}^0)/3)]} \sum_{i=1}^{G(p_\infty p_2(h_{n+k}^0)/3)} T_{\text{exc}}^{\infty,(i)} \geq t\Bigg]$$

$$\leq \sum_{k\geq0} \mathbb{P}\big[G(p_\infty p_2(h_{n+k}^0)/3) = k\big]$$

$$\times \mathbb{P}\Bigg[\frac{C}{E[G(p_\infty p_2(h_{n+k}^0)/3)]} \sum_{i=1}^{k} T_{\text{exc}}^{\infty,(i)} \geq x\Bigg]$$

$$\leq \sum_{k\geq0} \mathbb{P}[G(p_\infty p_2(h_{n+k}^0)/3) = k]\mathbb{P}\Bigg[C\frac{k}{E[G(p_\infty p_2(h_{n+k}^0)/3)]}T_{\sup} \geq t\Bigg]$$

$$\leq \mathbb{P}\Bigg[C\frac{G(p_\infty p_2(h_{n+k}^0)/3)}{E[G(p_\infty p_2(h_{n+k}^0)/3)]}T_{\sup} \geq t\Bigg],$$

and since $p_\infty p_2(h_{n+k}^0)/3 < 1/3$, we can use the fact that for any $a < 1/3$ we have $G(a)/E[G(a)] \preceq 3/2\mathbf{e}$. This shows that

$$\text{for all } n \geq 0 \text{ and } k > -n \qquad \overline{\mathcal{Z}}_n^k \preceq C\mathbf{e}T_{\text{sup}},$$

where $\mathbf{e}$ and T_{sup} are independent, so that the right-hand side has finite $(1-\varepsilon)$th moment for all $\varepsilon > 0$. This finishes the proof of the second part in Proposition 9.1.

$\square$

10. Sums of i.i.d. random variables. This section is completely self-contained and the notation used here are not related to those used previously.

Set $\beta > 1$ and let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. integer-valued nonnegative random variables such that

$$(10.1) \qquad\qquad P[X_1 \geq n] \sim C_X \beta^{-\gamma n}$$

for $C_X \in (0, \infty)$ and $\gamma > 0$.

Let $(X_i^{(l)})_{i\geq 0}$ be a sequence of i.i.d. integer-valued nonnegative random variables with the law of X_i conditioned on $X_i \geq f(l)$, where $f : \mathbb{N} \to \mathbb{N}$ is such that $l - f(l) \to \infty$.

Let $(Z_i^{(l)})_{i\geq 0, l\geq 0}$ be another sequence of i.i.d. nonnegative random variables and let $Z_i^{(l),(k)}$ have the law of $Z_i^{(l)}$ under $P[\cdot | X_i^{(l)} = l + k]$, if this last probability is well defined, and as $Z_i^{(l),(k)} = 0$ otherwise. Define

$$(10.2) \qquad\qquad \text{for } k \in \mathbb{Z}, l \geq 0 \qquad \overline{F}_k^{(l)}(x) := P[Z_i^{(l),(k)} > x].$$

We introduce the following assumptions:

(1) There exists a certain random variable Z_∞ such that

$$\text{for all } k \in \mathbb{Z} \text{ and } l \geq 0 \qquad Z_i^{(l),(k)} \xrightarrow{d} Z_\infty.$$

(2) There exists a random variable Z_{sup} such that

$$\text{for all } l \geq 0, k \geq -(l - f(l)) \text{ and } i \geq 0 \qquad Z_i^{(l),(k)} \preceq Z_{\text{sup}},$$

and $E[Z_{\text{sup}}^{\gamma+\varepsilon}] < \infty$ for some $\varepsilon > 0$.

Moreover, set

$$Y_i^{(l)} = Z_i^{(l)} \beta^{X_i^{(l)}} \quad \text{and} \quad S_n^{(l)} = \sum_{i=1}^n Y_i^{(l)},$$

and for $\lambda > 0$, $(\lambda_l)_{l\geq 0}$ converging to λ and $l \in \mathbb{N}$, define

$$N_l^{(\lambda)} = \lfloor \lambda_l^\gamma \beta^{\gamma(l-f(l))} \rfloor,$$

$$K_l^{(\lambda)} = \lambda \beta^l.$$

Finally, we denote by $\overline{F}_\infty(x) = P[Z_\infty > x]$ the tail function of Z_∞.

THEOREM 10.1. *Suppose that $\gamma < 1$ and assumptions* (1) *and* (2) *hold true. Then we have*

$$\text{for all } \lambda > 0 \text{ and } (\lambda_l)_{l \geq 0} \text{ going to } \lambda \qquad \frac{S^{(l)}_{N_l^{(\lambda)}}}{K_l^{(\lambda)}} \to \Im(d_\lambda, 0, \mathcal{L}_\lambda),$$

where $\Im$ is an infinitely divisible law. The Lévy spectral function $\mathcal{L}_\lambda$ satisfies

(10.3) $\qquad$ *for all $\lambda > 0$ and $x \in \mathbb{R}$* $\qquad \mathcal{L}_\lambda(x) = \lambda^\gamma \mathcal{L}_1(\lambda x)$

and

(10.4) $\qquad \mathcal{L}_1(x) = \begin{cases} 0, & \text{if } x < 0, \\ -(1 - \beta^{-\gamma}) \sum_{k \in \mathbb{Z}} \beta^{\gamma k} \overline{F}_\infty(x \beta^k), & \text{if } x > 0. \end{cases}$

In particular, $\Im(d_\lambda, 0, \mathcal{L}_\lambda)$ is continuous. Moreover, d_λ is given by

(10.5) $\qquad d_\lambda = \lambda^{1+\gamma} (1 - \beta^{-\gamma}) \sum_{k \in \mathbb{Z}} \beta^{(1+\gamma)k} E\left[\frac{Z_\infty}{(\lambda \beta^k)^2 + Z_\infty^2} \right].$

The fact that the quantities appearing above are well defined will be established in the course of the proof.

In order to prove Theorem 10.1, we will apply Theorem 4 in [5], which is itself a consequence of Theorem IV.6 (page 77) in [19].

THEOREM 10.2. *Let $n(t): [0, \infty) \to \mathbb{N}$ and for each t let $\{Y_k(t): 1 \leq k \leq n(t)\}$ be a sequence of independent identically distributed random variables. Assume that for every $\varepsilon > 0$, it is true that*

(10.6) $$\lim_{t \to \infty} P[Y_1(t) > \varepsilon] = 0.$$

Now let $\mathcal{L}(x): \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a Lévy spectral function, $d \in \mathbb{R}$ and $\sigma > 0$. Then the following statements are equivalent:

(i)

$$\lim_{t \to \infty} \sum_{k=1}^{n(t)} Y_k(t) \xrightarrow{d} X_{d, \sigma, \mathcal{L}} \qquad \text{for } t \to \infty,$$

where $X_{d, \sigma, \mathcal{L}}$ has law $\Im(d, \sigma, \mathcal{L})$.

(ii) *Define for $\tau > 0$ the random variable $Z_\tau(t) := Y_1(t) \mathbf{1}\{|Y_1(t)| \leq \tau\}$. Then if x is a continuity point of $\mathcal{L}$,*

$$\mathcal{L}(x) = \begin{cases} \lim_{t \to \infty} n(t) P[Y_1(t) \leq x], & \text{for } x < 0, \\ -\lim_{t \to \infty} n(t) P[Y_1(t) > x], & \text{for } x > 0, \end{cases}$$

$$\sigma^2 = \lim_{\tau \to 0} \lim_{t \to \infty} \sup(n(t) \, \text{Var}(Z_\tau(t))),$$

and for any $\tau > 0$ which is a continuity point of $\mathcal{L}(x)$,

$$d = \lim_{n \to \infty} n(t) E[Z_\tau(t)] + \int_{|x| > \tau} \frac{x}{1 + x^2} d\mathcal{L}(x) - \int_{\tau \geq |x| > 0} \frac{x^3}{1 + x^2} d\mathcal{L}(x).$$

Condition (10.6) is verified in the course of the proof; in our case $n(t)$ goes to infinity.

10.1. *Computation of the Lévy spectral function.* Fix $\lambda \in [1, \beta)$ and assume that $x > 0$ is a continuity point of $\mathcal{L}_\lambda$. We want to show that

$$(10.7) \qquad -\lim_{l \to \infty} N_l^{(\lambda)} P\left[\frac{Y_1^{(l)}}{K_l^{(\lambda)}} > x \right] = \mathcal{L}_\lambda(x).$$

The discontinuity points of $\mathcal{L}_\lambda$ are exactly $\mathcal{C}_\lambda = \{(\beta^k y_n)/\lambda, k \in \mathbb{Z}, n \in \mathbb{N}\}$ where $\{y_n, n \in \mathbb{N}\}$ are the discontinuity points of $\overline{F}_\infty$ (these sets are possibly empty).

Let us introduce

$$(10.8) \qquad \text{for } k \in \mathbb{Z} \qquad a_k^{(l)} := P[X_1^{(l)} \geq l + k].$$

Since $N_l^{(\lambda)} \sim (\lambda \beta^{l - f(l)})^\gamma$, we can write, recalling (10.2),

$$\beta^{\gamma(l - f(l))} P\left(\frac{Y_1^{(l)}}{K_l^{(\lambda)}} > x \right)$$
$$= \sum_{k \in \mathbb{Z}} \mathbf{1}\{k \geq -(l - f(l))\} \overline{F}_k^{(l)}(\lambda x \beta^{-k}) \beta^{\gamma(l - f(l))} (a_k^{(l)} - a_{k+1}^{(l)}).$$

Now recalling (10.1) and (10.8), we see that for $l \to \infty$,

$$(10.9) \qquad \beta^{\gamma(l - f(l))} a_k^{(l)} \to \beta^{-\gamma k}$$

using $l - f(l) \to \infty$, the fact that $\lambda x \beta^k$ is a continuity point of $\overline{F}_\infty$ (because $x > 0$ is a continuity point of $\mathcal{L}_\lambda$) for any k and assumption (1), we see that for all $k \in \mathbb{Z}$

$$(10.10) \qquad \begin{aligned} &\mathbf{1}\{k \geq -(l - f(l))\} \overline{F}_k^{(l)}(\lambda x \beta^{-k}) \beta^{\gamma(l - f(l))} (a_k^{(l)} - a_{k+1}^{(l)}) \\ &\to \overline{F}_\infty(\lambda x \beta^{-k}) \beta^{-\gamma k} (1 - \beta^{-\gamma}) \qquad \text{for } l \to \infty. \end{aligned}$$

In order to exchange limit and summation, we need to show that the terms of the sum are dominated by a function which does not depend on l and is summable. Recalling assumption (2) and using (10.1), we see that $\beta^{\gamma(l - f(l))} a_k^{(l)} \leq C_1 \beta^{-\gamma k}$ and

$$\sum_{k \in \mathbb{Z}} \mathbf{1}\{k \geq -(l - f(l))\} \overline{F}_k^{(l)}(\lambda x \beta^{-k}) \beta^{\gamma(l - f(l))} (a_k^{(l)} - a_{k+1}^{(l)})$$
$$\leq C \sum_{k \in \mathbb{Z}} \overline{F}_{\sup}(\lambda x \beta^{-k}) \beta^{-\gamma k},$$

where $\overline{F}_{\sup}(x) = P[Z_{\sup} > x]$. This last sum converges clearly for $k \to \infty$, and to show that it converges for $k \to -\infty$ we simply notice that for any $y > 0$

$$\sum_{k>0} E[\mathbf{1}\{Z_{\sup} > y\beta^k\}]\beta^{\gamma k} = E\left[\sum_{0<k\leq\lfloor\ln(Z_{\sup}/y)/\ln\beta\rfloor} \beta^{\gamma k}\right]$$

$$\leq (1-\beta^{-\gamma})^{-1} E[\beta^{\gamma \ln(Z_{\sup}/y)/\ln\beta}]$$

$$< \infty,$$

since we assume that $E[Z_{\sup}^{\gamma+\varepsilon}] < \infty$.

Hence, we can exchange limit and sum. Using (10.10) and the fact that $l - f(l) \to \infty$, we get

$$-\lim_{l\to\infty} N_l^{(\lambda)} P\left[\frac{Y_1^{(l)}}{K_l^{(\lambda)}} > x\right] = -\lambda^{\gamma}(1-\beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \overline{F}_{\infty}(\lambda x\beta^k)\beta^{\gamma k}$$

and, taking into account (10.3), this proves (10.7).

10.2. *Computation of d_λ.* Fix $\lambda \in [1, \beta)$. Since the integral $\int_0^\tau x\, d\mathcal{L}_\lambda$ is well defined, it suffices to show that for all $\tau \in \mathcal{C}_\lambda$, $\tau > 0$

$$d_\lambda = \lim_{l\to\infty} \frac{N_l^{(\lambda)}}{K_l^{(\lambda)}} E[Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} < \tau K_l^{(\lambda)}\}]$$

(10.11)

$$-\int_0^\tau x\, d\mathcal{L}_\lambda + \int_0^\infty \frac{x}{1+x^2}\, d\mathcal{L}_\lambda.$$

First, let us notice that $N_l^{(\lambda)}/K_l^{(\lambda)} \sim (\lambda\beta^l)^{\gamma-1}\beta^{-\gamma f(l)}$. We introduce

(10.12)   for all $u > 0$ $G_k^{(l)}(u) = E[Z_1^{(l)}\mathbf{1}\{Z_1^{(l)} \leq u\}|X_1^{(l)} = k+l]$.

Considering the first term in (10.11), we compute

$$\beta^{(\gamma-1)l-\gamma f(l)} E[Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} < \tau\lambda\beta^l\}]$$

$$= \sum_{k\in\mathbb{Z}} \mathbf{1}\{k \geq -(l-f(l))\}[(a_k^{(l)} - a_{k+1}^{(l)})\beta^{\gamma(l-f(l))}]\beta^k G_k^{(l)}(\tau\lambda\beta^{-k}).$$

Using $l - f(l) \to \infty$, (10.9) and assumption (1), we see that for all $k \in \mathbb{Z}$ and $\tau \in \mathcal{C}_\lambda$,

(10.13)

$$\mathbf{1}\{k \geq -(l-f(l))\}[(a_k^{(l)} - a_{k+1}^{(l)})\beta^{\gamma(l-f(l))}]\beta^k G_k^{(l)}(\tau\lambda\beta^{-k})$$

$$\to (1-\beta^{-\gamma})\beta^{(1-\gamma)k} G_{\infty}(\tau\lambda\beta^{-k}),$$

where

(10.14) $G_{\infty}(x) = E[Z_{\infty}\mathbf{1}\{Z_{\infty} \leq x\}].$

Once again we need to show that we can exchange limit and sum, which amounts to find a summable dominating function which does not depend on l. Using the fact that for $u > 0$

$$G_k^{(l)}(u) \le u \quad \text{and} \quad \beta^{-(\gamma-1)k} G_k^{(l)}(u\beta^{-k}) \le \beta^{\varepsilon k} u^{1-\gamma-\varepsilon} E[Z_{\sup}^{\gamma+\varepsilon}]$$

(to see the second inequality, use $E[Y\mathbf{1}\{Y \le s\}] \le s^a E[Y^{1-a}\mathbf{1}\{Y \le s\}]$ with $a = 1 - \gamma - \varepsilon$), we get that

$$\sum_{k\in\mathbb{Z}} \mathbf{1}\{k \ge -(l - f(l))\}[(a_k^{(l)} - a_{k+1}^{(l)})\beta^{\gamma(l-f(l))}]\beta^k G_k^{(l)}(\tau\lambda\beta^{-k})$$

$$\le C\left(\tau\lambda \sum_{k\ge 0} \beta^{-\gamma k} + (\tau\lambda)^{1-\gamma-\varepsilon} E[Z_{\sup}^{\gamma+\varepsilon}] \sum_{k<0} \beta^{\varepsilon k}\right) < \infty,$$

due to assumption (2). Hence, recalling (10.13), we get that for $\tau \in \mathcal{C}_\lambda$

(10.15)
$$\lim_{l\to\infty} \frac{N_l^{(\lambda)}}{K_l^{(\lambda)}} E[Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} < \tau K_l^{(\lambda)}\}]$$
$$= \lambda^{\gamma-1}(1 - \beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \beta^{k(\gamma-1)} G_\infty(\tau\lambda\beta^k).$$

Furthermore, recalling (10.3) and (10.4), we get for $\tau \in \mathcal{C}_\lambda$

$$\int_0^\tau x \, d\mathcal{L}_\lambda = \lambda^\gamma(1 - \beta^{-\gamma}) \int_{x\le\tau} x \sum_{k\in\mathbb{Z}} \beta^{\gamma k} \, d(-\overline{F}_\infty)(\lambda x\beta^k)$$

$$= \lambda^{\gamma-1}(1 - \beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \beta^{(\gamma-1)k} \int_{\lambda x\beta^k \le \lambda\tau\beta^k} \lambda x\beta^k \, d(-\overline{F}_\infty)(\lambda x\beta^k)$$

$$= \lambda^{\gamma-1}(1 - \beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \beta^{(\gamma-1)k} G_\infty(\tau\lambda\beta^k),$$

and this term exactly compensates for (10.15). Hence, we are left to compute in a similar fashion

$$d_\lambda = \int_0^\infty \frac{x}{1 + x^2} \, d\mathcal{L}_\lambda$$

$$= \lambda^\gamma(1 - \beta^{-\gamma}) \int_0^\infty \frac{x}{1 + x^2} \sum_{k\in\mathbb{Z}} \beta^{\gamma k} \, d(-\overline{F}_\infty)(\lambda x\beta^k)$$

$$= \lambda^{1+\gamma}(1 - \beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \beta^{(1+\gamma)k} \int_0^\infty \frac{\lambda x\beta^k}{(\lambda\beta^k)^2 + (\lambda x\beta^k)^2} \, d(-\overline{F}_\infty)(\lambda x\beta^k)$$

$$= \lambda^{1+\gamma}(1 - \beta^{-\gamma}) \sum_{k\in\mathbb{Z}} \beta^{(1+\gamma)k} E\left[\frac{Z_\infty}{(\lambda\beta^k)^2 + Z_\infty^2}\right].$$

This sum is finite since the terms in the sum can be bounded from above by $C_1(\lambda)\beta^{-\varepsilon k}E[Z_{\sup}^{\gamma+\varepsilon}]$ and $C_2(\lambda)\beta^{\gamma k}$, where $C_1(\lambda) = \max_{x\geq 0}(x^{1-\gamma}/(\lambda^2 + x^2))$ and $C_2(\lambda) = \max_{x\geq 0} x/(\lambda^2 + x^2)$. The first upper bound is summable for $k \to \infty$, the other for $k \to -\infty$ and so d_λ is well defined.

10.3. *Computation of the variance.* We show that for any $\lambda \in [1, \beta)$ we have

$$(10.16) \qquad \sigma^2 = \lim_{\tau\to 0}\limsup_{l\to\infty}\frac{N_l^{(\lambda)}}{(K_l^{(\lambda)})^2}\mathrm{Var}(Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} \leq \tau K_l^{(\lambda)}\}) = 0.$$

First, using (10.15), let us notice that

$$(10.17) \qquad \lim_{l\to\infty}\frac{N_l^{(\lambda)}}{(K_l^{(\lambda)})^2}E[Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} \leq \tau K_l^{(\lambda)}\}] = 0.$$

Further, we have $N_l^{(\lambda)}/(K_l^{(\lambda)})^2 \sim (\lambda\beta^l)^{\gamma-2}\beta^{-\gamma f(l)}$. Define

$$\text{for all } u \geq 0 \qquad H_k^{(l)}(u) = E[(Z_1^{(l)})^2\mathbf{1}\{Z_1^{(l)} \leq u\}|X_1^{(l)} = k+l].$$

We compute

$$\beta^{(\gamma-2)l-\gamma f(l)}E[(Y_1^{(l)})^2\mathbf{1}\{Y_1^{(l)} < \tau\lambda\beta^l\}]$$
$$= \sum_{k\in\mathbb{Z}}\mathbf{1}\{k \geq -(l-f(l))\}\beta^{(\gamma-2)l-\gamma f(l)}\beta^{2(k+l)}H_k^{(l)}(\tau\lambda\beta^{-k})(a_k^{(l)} - a_{k+1}^{(l)}).$$

By (10.1) we have $a_k^{(l)}\beta^{\gamma(l-f(l))} \leq C_1\beta^{-\gamma k}$, hence the terms of our sum are bounded above by $C_1\beta^{(2-\gamma)k}H_k^{(l)}(\tau\lambda\beta^{-k})$. Note that $H_k^{(l)}(u) \leq u^2$, so that

$$\beta^{(2-\gamma)k}H_k^{(l)}(\tau\lambda\beta^{-k}) \leq \beta^{-\gamma k}(\tau\lambda)^2,$$

which gives an upper bound for $k \geq 0$. On the other hand, assumption (2) implies that

$$\beta^{(2-\gamma)k}H_k^{(l)}(\tau\lambda\beta^{-k}) \leq \beta^{\varepsilon k}(\tau\lambda)^{2-\gamma-\varepsilon}E[Z_{\sup}^{\gamma+\varepsilon}].$$

These inequalities imply that

$$\limsup_{l\to\infty}\frac{N_l^{(\lambda)}}{(K_l^{(\lambda)})^2}\mathrm{Var}(Y_1^{(l)}\mathbf{1}\{Y_1^{(l)} \leq \tau K_l^{(\lambda)}\}) \leq C_2\tau^{2-\gamma-\varepsilon},$$

where C_2 is finite and depends on ε and λ. Hence, letting τ go to 0 yields the result, since in assumption (2) we can assume ε to be as small as we need in particular it can be chosen such that $2 - \gamma - \varepsilon > 0$.

11. Limit theorems.

11.1. *Proof of Theorem* 1.3. Assume $\varepsilon < \min(1/4, 2\gamma/3)$. For $\lambda > 0$, we will study the limit distributions of the hitting time properly renormalized along the

subsequences defined as follows:

$$\text{for } k \in \mathbb{N} \qquad n_\lambda(k) = \lfloor \lambda \mathbf{f}'(q)^{-k} \rfloor.$$

First, recalling (9.3), using Proposition 9.1 and Lemma 3.3, we can apply Theorem 10.1 to get

$$(11.1) \quad \text{for any } (\lambda_l)_{l \geq 0} \text{ going to } \lambda \qquad \frac{1}{\lambda \beta^k} \sum_{i=1}^{\lfloor \lambda_{n_\lambda(k)}^\gamma \beta^{\gamma(k-f(k))} \rfloor} \chi_i^*(n_\lambda(k)) \xrightarrow{d} Y_{d_\lambda, 0, \mathcal{L}_\lambda},$$

where $f(k) := h_{n_\lambda(k)} = \lceil -(1 - \varepsilon) \ln(n_\lambda(k)) / \ln \mathbf{f}'(q) \rceil$ and $Y_{d_\lambda, 0, \mathcal{L}_\lambda}$ is a random variable whose law $\mathfrak{I}(d_\lambda, 0, \mathcal{L}_\lambda)$ is the infinitely divisible law characterized by (10.3), (10.4) and (10.5), where $\mathcal{Z}_\infty$ is given by (1.3).

Using Proposition 8.1, (11.1) still holds if we replace $\chi_i^*(n)$ by $\chi_i(n)$.

Recalling Proposition 7.1, we have

$$\sum_{i=1}^{\lfloor (1+o_1(1))\lambda \rho C_a \mathbf{f}'(q)^{-(k-h_{n_\lambda(k)})} \rfloor} \chi_i(n_\lambda(k))$$

$$\preceq \chi_{n_\lambda(k)} \preceq \sum_{i=1}^{\lfloor (1+o_2(1))\lambda \rho C_a \mathbf{f}'(q)^{-(k-h_{n_\lambda(k)})} \rfloor} \chi_i(n_\lambda(k)),$$

where

$$1 + o_1(1) = (1 - \tilde{n}^{-\varepsilon/4}) \frac{\rho_n}{\rho} \frac{n}{\lambda \mathbf{f}'(q)^{-k}} \frac{\mathbf{f}'(q)^{h_{\tilde{n}}}}{\mathbf{f}'(q)^{h_n}}$$

and

$$1 + o_2(1) = (1 + 2n^{-\varepsilon/4}) \frac{\rho_n}{\rho} \frac{n}{\lambda \mathbf{f}'(q)^{-k}},$$

writing n for $n_\lambda(k) = \lfloor \lambda \mathbf{f}'(q)^{-k} \rfloor$ and $\tilde{n} = n - (-2 \ln n / \ln \mathbf{f}'(q))$.

Hence, both sides of the previous equation, properly renormalized, converge in distribution to the same limit law, implying that (the law of) $\chi(n)$ converges to the same law as well. Recalling (5.3), this yields for any $\lambda > 0$

$$\frac{\chi(n_\lambda(k))}{(\rho C_a n_\lambda(k))^{1/\gamma}} \xrightarrow{d} Y_{d_{(\rho C_a \lambda)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a \lambda)^{1/\gamma}}},$$

where $Y_{d_{(\rho C_a \lambda)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a \lambda)^{1/\gamma}}}$ is a random variable with law $\mathfrak{I}(d_{(\rho C_a \lambda)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a \lambda)^{1/\gamma}})$ and we used that $\beta^\gamma = 1/\mathbf{f}'(q)$.

Then by Proposition 5.1, we get that

$$\frac{\Delta_{n_\lambda(k)}}{(\rho C_a n_\lambda(k))^{1/\gamma}} \xrightarrow{d} Y_{d_{(\rho C_a \lambda)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a \lambda)^{1/\gamma}}},$$

which proves Theorem 1.3.

We note for further reference that

(11.2) $\Im(d_\lambda, 0, \mathcal{L}_\lambda)$ is continuous.

This follows from Theorem III.2 (page 43) in [19], since, due to (10.4),

$$\lim_{x \to 0} \mathcal{L}_1(x) = -\infty.$$

11.2. *Proof of Theorem* 1.2. In order to prove Theorem 1.2, assume that $(\Delta_n/n^{1/\gamma})_{n \geq 0}$ converges in law. It follows that all subsequential limits are the same, so that

for all $\lambda \in [1, \beta)$ and $x \in \mathbb{R}^+$ $\mathcal{L}_1(x) = \lambda^\gamma \mathcal{L}_1(\lambda x).$

Plugging in the values $\lambda = \beta^{1/3}$ and $x = \beta^{-2/3}$ gives

(11.3)
$$\sum_{k \in \mathbb{Z}} \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-2/3}]$$
$$= \mathbf{f}'(q)^{-1/3} \sum_{k \in \mathbb{Z}} \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-1/3}].$$

We will show that for $\beta \to \infty$, the right-hand side and the left-hand side of (11.3) have different limits. First, for $k \geq 1$ using Remark 6.1, we see that

(11.4)
$$\mathbb{P}[\mathcal{Z}_\infty > \beta^{k-1/3}] \leq \beta^{1/3-k}\mathbb{E}[\mathcal{Z}_\infty] \leq \beta^{1/3-k}\mathbb{E}[S_\infty]E[G(p_\infty/3)]$$
$$= \beta^{-(k-1)}O(\beta^{-1/3})$$

for $\beta \to \infty$ where $O(\beta^{-1/3}) = \beta^{-2/3}\mathbb{E}[S_\infty]E[G(p_\infty/3)]$ does not depend on k (recall Proposition 3.1 to see that $\mathbb{E}[S_\infty]$ is bounded in β). In the same way,

(11.5) $\mathbb{P}[\mathcal{Z}_\infty > \beta^{k-2/3}] = \beta^{-(k-1)}O(\beta^{-1/3})$

for $O(\cdot)$ independent of $k \geq 1$.

Hence,

(11.6)
$$\lim_{\beta \to \infty} \sum_{k=1}^\infty \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-2/3}]$$
$$= 0 = \lim_{\beta \to \infty} \sum_{k=1}^\infty \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-1/3}].$$

For $k \leq 0$, we have

$$\mathbb{P}[\mathcal{Z}_\infty > \beta^{k-1/3}] \leq \mathbb{P}[\mathcal{Z}_\infty > 0] \leq \mathbb{P}[\mathrm{Bin}(W_\infty, p_\infty) > 0],$$

further, since $S_\infty \geq 1$, we see that

on $\{\mathcal{Z}_\infty > 0\}$ $\mathcal{Z}_\infty \geq S_\infty \mathbf{e}_1 \geq \mathbf{e}_1,$

where $\mathbf{e}_1$ is independent of the event $\{\mathcal{Z}_\infty > 0\} = \{\mathrm{Bin}(W_\infty, p_\infty) > 0\}$. Hence,

$$\mathbb{P}[\mathcal{Z}_\infty > \beta^{k-1/3}] = 1 - \mathbb{P}[\mathcal{Z}_\infty \leq \beta^{k-1/3}]$$
$$\geq 1 - \mathbb{P}[\mathrm{Bin}(W_\infty, p_\infty) = 0] - P[\mathbf{e}_1 \leq \beta^{-1/3}]$$
$$= \mathbb{P}[\mathrm{Bin}(W_\infty, p_\infty) > 0] + o(1)$$

for $\beta \to \infty$ and hence

$$(11.7) \qquad \mathbb{P}[\mathcal{Z}_\infty > \beta^{k-1/3}] = \mathbb{P}[\mathrm{Bin}(W_\infty, p_\infty) > 0] + o(1),$$

where $o(\cdot)$ does not depend on k.

In the same way,

$$(11.8) \qquad \mathbb{P}[\mathcal{Z}_\infty > \beta^{k-2/3}] = \mathbb{P}[\mathrm{Bin}(W_\infty, p_\infty) > 0] + o(1)$$

for $\beta \to \infty$. Plugging (11.7) and (11.8) in equation (11.3) and taking into account (11.6), we see that

$$\lim_{\beta \to \infty} \sum_{k \in \mathbb{Z}} \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-2/3}] = \lim_{\beta \to \infty} \frac{1}{1 - \mathbf{f}'(q)} P[\mathrm{Bin}(W_\infty, p_\infty) > 0]$$

and

$$\lim_{\beta \to \infty} \sum_{k \in \mathbb{Z}} \mathbf{f}'(q)^{-k} P[\mathcal{Z}_\infty > \beta^{k-1/3}] = \lim_{\beta \to \infty} \frac{1}{1 - \mathbf{f}'(q)} P[\mathrm{Bin}(W_\infty, p_\infty) > 0].$$

Hence, we would have

$$\lim_{\beta \to \infty} \frac{1}{1 - \mathbf{f}'(q)} P[\mathrm{Bin}(W_\infty, p_\infty) > 0]$$
$$= \lim_{\beta \to \infty} \mathbf{f}'(q)^{-1/3} \frac{1}{1 - \mathbf{f}'(q)} P[\mathrm{Bin}(W_\infty, p_\infty) > 0].$$

This could only be possible if $P[\mathrm{Bin}(W_\infty, p_\infty) > 0] \to 0$ for $\beta \to \infty$, but we know that

$$P[\mathrm{Bin}(W_\infty, p_\infty) > 0] > p_\infty \mathbb{P}[W_\infty \geq 1] > c > 0,$$

where c does not depend on β, see Lemma 6.6. This proves Theorem 1.2.

In particular, if β is large enough, $\mathfrak{I}(d_1, 0, \mathcal{L}_1)$ is not a stable law and this implies (vii) in Theorem 1.4.

11.3. *Proof of Theorem* 1.1. We will show that

$$(11.9) \qquad \lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[\frac{\Delta_n}{n^{1/\gamma}} \notin [1/M, M] \right] = 0.$$

This implies in particular that the family $(\Delta_n/n^{1/\gamma})_{n \geq 0}$ is tight. We will then prove

$$(11.10) \qquad \lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \notin [1/M, M] \right] = 0,$$

which implies that the family $(|X_n|/n^\gamma)_{n \geq 0}$ is tight. Equation (1.2) will then shown to be a consequence of (11.9) and (11.10).

To show (11.9), note that for $n \in [\mathbf{f}'(q)^{-k}, \mathbf{f}'(q)^{-(k+1)})$

$$\mathbb{P}\left[\frac{\Delta_n}{n^{1/\gamma}} \notin [1/M, M]\right]$$

$$\leq \mathbb{P}\left[\frac{\Delta_{\mathbf{f}'(q)^{-k}}}{(\mathbf{f}'(q)^{-k}\rho C_a)^{1/\gamma}} < \frac{M}{(\mathbf{f}'(q)\rho C_a)^{1/\gamma}}\right]$$

$$+ \mathbb{P}\left[\frac{\Delta_{\mathbf{f}'(q)^{-(k+1)}}}{(\mathbf{f}'(q)^{k+1}\rho C_a)^{1/\gamma}} > \frac{1}{M(\mathbf{f}'(q)^{-1}\rho C_a)^{1/\gamma}}\right].$$

Using Theorem 1.3, we get

$$\limsup_n \mathbb{P}\left[\frac{\Delta_n}{n^{1/\gamma}} \notin [1/M, M]\right]$$

$$\leq \mathbb{P}\left[Y_{d_{(\rho C_a)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a)^{1/\gamma}}} \notin \left[\frac{1}{M(\mathbf{f}'(q)^{-1}\rho C_a)^{1/\gamma}}, \frac{M}{(\mathbf{f}'(q)\rho C_a)^{1/\gamma}}\right]\right],$$

where $Y_{d_{(\rho C_a)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a)^{1/\gamma}}}$ is a random variable with law $\mathfrak{I}(d_{(\rho C_a)^{1/\gamma}}, 0, \mathcal{L}_{(\rho C_a)^{1/\gamma}})$. Here we used that the limiting law $\mathfrak{I}(d_x, 0, \mathcal{L}_x)$ is continuous, see (11.2), and has in particular no atom at 0, so we get that

$$\lim_{M \to \infty} P[Y_{d_x, 0, \mathcal{L}_x} \notin [1/M, M]] = 0,$$

which proves (11.9).

Let us prove (11.10). Let $n \geq 0$ and write $n^\gamma = \lambda_0 \mathbf{f}'(q)^{-i_0}$ for some $i_0 \in \mathbb{N}$ and $\lambda_0 \in [1, 1/\mathbf{f}'(q))$. Let $i \in \mathbb{N}$. To control the probability that $|X_n|$ is be much larger than n^γ, note that

$$\mathbb{P}\left[\frac{|X_n|}{n^\gamma} \geq \lambda_0^{-1}\mathbf{f}'(q)^{-i}\right] \leq \mathbb{P}\left[\Delta_{\lfloor(\lambda_0^{-1}\mathbf{f}'(q)^{-i})(\lambda_0\mathbf{f}'(q)^{-i_0})\rfloor} < (\lambda_0\mathbf{f}'(q)^{-i_0})^{1/\gamma}\right]$$

$$= \mathbb{P}\left[\frac{\Delta_{\lfloor\mathbf{f}'(q)^{-i-i_0}\rfloor}}{(\rho C_a\mathbf{f}'(q)^{-i-i_0})^{1/\gamma}} < (\lambda_0\rho C_a\mathbf{f}'(q)^{-i})^{-1/\gamma}\right].$$

Hence, for any $\varepsilon > 0$, and i large enough such that $(\rho C_a\mathbf{f}'(q)^{-i})^{-1/\gamma} < \varepsilon$,

$$\mathbb{P}\left[\frac{|X_n|}{n^\gamma} \geq \mathbf{f}'(q)^{-i-1}\right] \leq \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \geq \lambda_0^{-1}\mathbf{f}'(q)^{-i}\right] \leq \mathbb{P}\left[\frac{\Delta_{\lfloor\mathbf{f}'(q)^{-i-i_0}\rfloor}}{(\rho C_a\mathbf{f}'(q)^{-i-i_0})^{1/\gamma}} < \varepsilon\right].$$

Now, using Theorem 1.3, taking n (i.e., i_0) to infinity, we get that for any $\varepsilon > 0$,

$$\text{for } i \text{ large enough} \quad \limsup_n \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \geq \mathbf{f}'(q)^{-i-1}\right] \leq P[Y_{d_1, 0, \mathcal{L}_1} \leq \varepsilon],$$

using (11.2) and hence

$$(11.11) \quad \limsup_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \ge M\right] \le \limsup_{\varepsilon \to 0} P[Y_{d_1,0,\mathcal{L}_1} \le \varepsilon] = 0.$$

Next, we will consider the probability that $|X_n|$ is much smaller than n^γ. Let us denote

$$\mathrm{Back}(n) = \max_{i < j \le n} (|X_i| - |X_j|),$$

the maximal backtracking of the random walk. It is easy to see that

$$\mathrm{Back}(n) \le \max_{2 \le i \le n} (\tau_i - \tau_{i-1}) \vee \tau_1.$$

Hence, since τ_1 and $\tau_2 - \tau_1$ have exponential moments

$$(11.12) \qquad \mathbb{P}[\mathrm{Back}(n) \ge n^{\gamma/2}] \le Cn \exp(-cn^{\gamma/2}).$$

If the walk is at a level inferior to $(1/M)n^\gamma$ at time n and has not backtracked more than $n^{\gamma/2}$, it has not reached $(2/M)n^\gamma$. This implies that for all $M > 0$,

$$\mathbb{P}\left[\frac{|X_n|}{n^\gamma} < 1/M\right] \le \mathbb{P}[\mathrm{Back}(n) \ge n^{\gamma/2}] + \mathbb{P}\left[\frac{\Delta_{\lfloor (2/M)n^\gamma \rfloor}}{(\rho C_a 2/M)^{1/\gamma} n} > (\rho C_a 2/M)^{1/\gamma}\right].$$

Hence, using a reasoning similar to the proof of (11.11), we have

$$(11.13) \quad \lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} < 1/M\right] \le \liminf_{M \to \infty} P[Y_{d_1,0,\mathcal{L}_1} \ge M] = 0.$$

Using (11.11) and (11.13), we get

$$(11.14) \qquad \lim_{M \to \infty} \limsup_{n \to \infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \notin [1/M, M]\right] = 0,$$

which shows (11.10) in Theorem 1.1.

Let us prove (iii) in Theorem 1.1. We have

$$\mathbb{P}\left[\lim_{n \to \infty} \frac{\ln |X_n|}{\ln n} \ne \gamma\right] \le \mathbb{P}\left[\limsup_{n \to \infty} \frac{\ln |X_n|}{\ln n} > \gamma\right] + \lim_{M \to \infty} \mathbb{P}\left[\liminf_{n \to \infty} \frac{|X_n|}{n^\gamma} \le \frac{1}{M}\right].$$

Using Fatou's lemma,

$$\mathbb{P}\left[\liminf_{n \to \infty} \frac{|X_n|}{n^\gamma} < \frac{1}{M}\right] \le \liminf_{n \to \infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} < \frac{1}{M}\right],$$

and taking M to infinity we get

$$\mathbb{P}\left[\lim_{n \to \infty} \frac{\ln |X_n|}{\ln n} \ne \gamma\right] \le \mathbb{P}\left[\limsup_{n \to \infty} \frac{\ln |X_n|}{\ln n} > \gamma\right].$$

Set $\varepsilon > 0$, we have

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\ln|X_n|}{\ln n} > (1+2\varepsilon)\gamma\right] \leq \mathbb{P}\left[\limsup_{n\to\infty} \frac{|X_n|}{n^{(1+\varepsilon)\gamma}} \geq 1\right]$$
$$\leq \mathbb{P}\left[\limsup_{n\to\infty} \frac{\sup_{i\leq n}|X_i|}{n^{(1+\varepsilon)\gamma}} \geq 1\right].$$

Define

$$D'(n) = \left\{\max_{\ell\in\cup_{i=0,\ldots,\Delta_n^Y} L_{Y_i}} H(\ell) \leq \frac{4\ln n}{-\ln \mathbf{f}'(q)}\right\}.$$

Denoting $t(n)$ such that $\sigma_{\tau_{t(n)}} \leq n < \sigma_{\tau_{t(n)+1}}$, we have

$$(11.15) \quad \text{for } \omega\in D'(n) \qquad |X_{\sigma_{\tau_{t(n)}}}| \leq |X_n| \leq |X_{\sigma_{\tau_{t(n)+1}}}| + \frac{4\ln n}{-\ln \mathbf{f}'(q)},$$

and since using $B_1(n)$ defined right above Lemma 7.1, we get

$$(11.16) \qquad \text{for } \omega\in B_1(n) \qquad |X_{\sigma_{\tau_{t(n)+1}}}| \leq |X_{\sigma_{\tau_{t(n)}}}| + n^\varepsilon.$$

We have using Lemma 5.1 and (3.7)

$$\mathbb{P}[D'(n)^c] \leq \mathbb{P}[A_1(n)^c] + \mathbb{P}\left[A_1(n), \operatorname{card}\bigcup_{i=1}^{\Delta_n^Y} L_{Y_i} > n^2\right]$$

$$+ \mathbb{P}\left[\operatorname{card}\bigcup_{i=1}^{\Delta_n^Y} L_{Y_i} \leq n^2, D'(n)^c\right]$$

$$\leq O(n^{-2}) + \mathbb{P}\left[\sum_{i=0}^{C_1 n} \operatorname{card} L_0^{(i)} > n^2\right] + n^2 \mathbf{Q}\left[H \geq \frac{4\ln n}{-\ln \mathbf{f}'(q)}\right]$$

$$\leq O(n^{-2}) + n^{-4}\operatorname{Var}\left(\sum_{i=0}^{C_1 n} \operatorname{card} L_0^{(i)}\right) + n^2 n^{-4} = O(n^{-2}),$$

where we used that $\operatorname{card} L_0^{(i)}$ are i.i.d. random variables which are L^2 since they are stochastically dominated by the number of offspring Z which is L^2 by our assumption.

By Lemma 7.1, the previous estimate and Borel–Cantelli we have $\omega\in B_1(n)\cap D(n)$ asymptotically, we get, recalling (11.15) and (11.16), that for $\varepsilon < \gamma$

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\sup_{i\leq n}|X_i|}{n^{(1+\varepsilon)\gamma}} \geq 1\right] \leq \mathbb{P}\left[\liminf_{n\to\infty}\left(\frac{|X_{\sigma_{\tau_{t(n)}}}|}{n^{(1+\varepsilon)\gamma}} + o(1)\right) \geq 1\right].$$

Since $|X_{\sigma_{\tau_t(n)}}| \le |X_n|$, we have

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{\sup_{i \le n} |X_i|}{n^{(1+\varepsilon)\gamma}} \ge 1\right] \le \mathbb{P}\left[\liminf_{n\to\infty} \frac{|X_{\sigma_{\tau_t(n)}}|}{n^{(1+\varepsilon)\gamma}} \ge 1\right]$$

$$\le \liminf_{n\to\infty} \mathbb{P}\left[\frac{|X_n|}{n^{(1+\varepsilon)\gamma}} \ge 1\right]$$

$$\le \liminf_{M\to\infty} \liminf_{n\to\infty} \mathbb{P}\left[\frac{|X_n|}{n^\gamma} \ge M\right]$$

$$= 0,$$

where we used Fatou's lemma and (ii) in Theorem 1.1.

Now since this result is true for all $\varepsilon > 0$ small enough we get

$$\mathbb{P}\left[\lim_{n\to\infty} \frac{\ln |X_n|}{\ln n} \ne \gamma\right] \le \mathbb{P}\left[\limsup_{n\to\infty} \frac{\ln |X_n|}{\ln n} > \gamma\right]$$

$$\le \liminf_{\varepsilon\to 0} \mathbb{P}\left[\limsup_{n\to\infty} \frac{\ln |X_n|}{\ln n} > (1+2\varepsilon)\gamma\right]$$

$$= 0,$$

which finishes the proof of (1.2).

11.4. *Proof of Theorem* 1.4.   It remains to show (iv), (v), (vi) and (viii) in Theorem 1.4.

PROOF OF THEOREM 1.4.   We start by proving (viii). Recall

$$\mathcal{Z}_\infty = \frac{S_\infty}{1 - \beta^{-1}} \sum_{i=1}^{\text{Bin}(W_\infty, p_\infty)} \mathbf{e}_i$$

and in particular the fact that S_∞, W_∞ and the i.i.d. exponential random variables $\mathbf{e}_i$ are independent. Let $\widetilde{S}_\infty = S_\infty / p_\infty$ and denote its law by ν_∞. Further, let $\alpha_k = \mathbb{P}[\text{Bin}(W_\infty, p_\infty) = k]$, $k = 0, 1, 2, \ldots$. Conditioned on $\widetilde{S}_\infty$ and $\text{Bin}(W_\infty, p_\infty)$, the law of $\mathcal{Z}_\infty$ is a Gamma distribution. More precisely, for any test function φ,

$$\mathbb{E}[\varphi(\mathcal{Z}_\infty)] = \alpha_0 \varphi(0) + \sum_{k=1}^\infty \int_0^\infty \left(\int_0^\infty \varphi(su)e^{-u} \frac{u^{k-1}}{(k-1)!} \, du\right) \nu_\infty(ds)\alpha_k$$

$$= \alpha_0 \varphi(0) + \sum_{k=1}^\infty \int_0^\infty \left(\int_0^\infty \varphi(v)e^{-v/s} \frac{v^{k-1}}{(k-1)!} \frac{1}{s^k} \, dv\right) \nu_\infty(ds)\alpha_k$$

$$= \alpha_0 \varphi(0) + \int_0^\infty \varphi(v) \sum_{k=1}^\infty \alpha_k \frac{v^{k-1}}{(k-1)!} E_{\mathbf{Q}}[e^{-v/\widetilde{S}_\infty}(\widetilde{S}_\infty)^{-k}] \, dv.$$

We point out that, due to Lemma 6.6, we have $0 < \alpha_0 < 1$. Hence, $\mathcal{Z}_\infty$ has an atom of mass α_0 at 0 and the conditioned law of $\mathcal{Z}_\infty$, conditioned on $\mathcal{Z}_\infty > 0$, has the density ψ, where

$$\psi(v) = \sum_{k=1}^\infty \alpha_k \frac{v^{k-1}}{(k-1)!} E_{\mathbf{Q}}[e^{-v/\widetilde{S}_\infty}(\widetilde{S}_\infty)^{-k}]$$

$$= E_{\mathbf{Q}}\left[\frac{1}{\widetilde{S}_\infty} e^{-v/\widetilde{S}_\infty} \sum_{k=1}^\infty \frac{\alpha_k}{(k-1)!}\left(\frac{v}{\widetilde{S}_\infty}\right)^{k-1}\right].$$

Using the fact that $S_\infty \geq 2$ and $\limsup \frac{1}{k}\log \alpha_k < 0$ (see Lemma 6.2), we see that ψ is bounded and C_∞. Note that since S_∞ and W_∞ have finite expectation, $\mathcal{Z}_\infty$ has also finite expectation and in particular

$$(11.17) \qquad \int_0^\infty v\psi(v)\,dv < \infty.$$

This shows (viii) in Theorem 1.4. We will later need that

$$(11.18) \qquad \int_0^\infty v^{1+\gamma}|\psi'(v)|\,dv < \infty.$$

To show (11.18), note that $\psi'(v)$ equals

$$E_{\mathbf{Q}}\left[\frac{-1}{(\widetilde{S}_\infty)^2}e^{-v/\widetilde{S}_\infty}\sum_{k=1}^\infty \frac{\alpha_k}{(k-1)!}\left(\frac{v}{\widetilde{S}_\infty}\right)^{k-1}\right.$$

$$\left. + \frac{1}{(\widetilde{S}_\infty)^2}e^{-v/\widetilde{S}_\infty}\sum_{k=2}^\infty \frac{\alpha_k}{(k-2)!}\left(\frac{v}{\widetilde{S}_\infty}\right)^{k-2}\right],$$

which implies, with $\alpha := \limsup(\alpha_k^{1/k}) < 1$,

$$|\psi'(v)| \leq C_1 E_{\mathbf{Q}}\left[\frac{1}{(\widetilde{S}_\infty)^2}e^{-(1-\alpha)v/\widetilde{S}_\infty}\right].$$

But, for $\delta \in (0, 1)$,

$$E_{\mathbf{Q}}\left[\frac{1}{(\widetilde{S}_\infty)^2}e^{-(1-\alpha)v/\widetilde{S}_\infty}\right]$$

$$\leq E_{\mathbf{Q}}[e^{-(1-\alpha)v/\widetilde{S}_\infty}\mathbf{1}\{\widetilde{S}_\infty < v^{1-\delta}\}] + E_{\mathbf{Q}}\left[\frac{1}{(\widetilde{S}_\infty)^2}\mathbf{1}\{\widetilde{S}_\infty \geq v^{1-\delta}\}\right]$$

$$\leq e^{-(1-\alpha)v^\delta} + \frac{1}{v^{3-3\delta}}E_{\mathbf{Q}}[\widetilde{S}_\infty].$$

Now, choosing δ small enough such that $3\delta + \gamma < 1$ yields (11.18).

We next show that the function $\mathcal{L}_1$ is absolutely continuous. Recalling (i) in Theorem 1.4 we see that, for $x > 0$,

$$-(1 - \beta^{-\gamma})^{-1}\mathcal{L}_1(x) = \sum_{k \in \mathbb{Z}} \beta^{\gamma k} \overline{F}_\infty(x\beta^k)$$

$$= \sum_{k \in \mathbb{Z}} \beta^{\gamma k} \int_{x\beta^k}^\infty \psi(v)\,dv$$

$$= \int_0^\infty \left(\sum_{k \in \mathbb{Z}} \beta^{\gamma k} \mathbf{1}\{v \geq x\beta^k\}\right) \psi(v)\,dv.$$

Now,

$$\sum_{k \in \mathbb{Z}} \beta^{\gamma k} \mathbf{1}\{v \geq x\beta^k\} = \sum_{k \leq K(v/x)} \beta^{\gamma k} =: g\left(\frac{v}{x}\right),$$

where, setting $u = \frac{v}{x}$, $K(u) = \lfloor \frac{\log u}{\log \beta} \rfloor$. An easy computation gives

$$(11.19) \qquad\qquad g(u) = \frac{\beta^{\gamma(K(u)+1)}}{\beta^\gamma - 1}.$$

Hence, for $x > 0$,

$$-(1 - \beta^{-\gamma})^{-1}\mathcal{L}_1(x) = \int_0^\infty g\left(\frac{v}{x}\right) \psi(v)\,dv$$

$$(11.20)$$

$$= x \cdot \int_0^\infty g(u)\psi(xu)\,du.$$

The last formula shows, noting that $g(u)$ is of order u^γ for $u \to \infty$ and recalling (11.17) and (11.18), that $\mathcal{L}_1$ is C_1 and in particular absolutely continuous. Due to the scaling relation (ii), the same holds true for $\mathcal{L}_\lambda$. This shows (iv) in Theorem 1.4.

Due to (11.19), we have

$$\frac{1}{\beta^\gamma - 1}u^\gamma \leq g(u) \leq \frac{\beta^\gamma}{\beta^\gamma - 1}u^\gamma.$$

Plugging this into the first equality in (11.20) yields (1.4). This proves (v) in Theorem 1.4. To show (vi), we use a result of [24] which says that an infinite divisible law is absolutely continuous if the absolutely continuous component $\mathcal{L}^{ac}$ of its Lévy spectral function satisfies $\int_{-\infty}^\infty d\mathcal{L}^{ac}(x) = \infty$, see also [19], page 37. In our case, this is satisfied since $\mathcal{L}_1^{ac}(x) = \mathcal{L}_1(x)$ and $\lim_{x \to 0}\mathcal{L}_1(x) = -\infty$. Further, the statement about the moments of μ_λ follows from the corresponding statement about the moments of $\mathcal{L}_\lambda$; see [20] or [19], page 36. $\quad\square$

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