A branching random walk among disasters

Nina Gantert, Stefan Junk

October 11, 2016

Abstract:
We consider a branching random walk in a random space-time environment of disasters where each particle is killed when meeting a disaster. This extends the model of the "random walk in a disastrous random environment" introduced by [9]. We obtain a criterion for positive survival probability, see Theorem 1.

The proofs for the subcritical and the supercritical cases follow standard arguments, which involve moment methods and a comparison with an embedded branching process with i.i.d. offspring distributions. The proof of almost sure extinction in the critical case is more difficult and uses the techniques from [6]. We also show that, in the case of survival, the number of particles grows exponentially fast.

1 Introduction

In this work we introduce a branching random walk on $\mathbb{Z}^d$ in a killing random environment. The process consists of particles performing a branching random walk in continuous time. All particles jump independently at rate κ and give birth to children at rate λ. The jump rate κ, the birth rate λ and the distribution q of the number of children do not change over time and space, and are the parameters of the model.

We then consider this process in a random environment ω given by disasters in space-time, defined as follows: The environment ω consists of a collection $\left(\omega^{(x)}\right)_{x\in\mathbb{Z}^d}$ of i.i.d. random variables where $\omega^{(x)} = (\omega^{(x)}(t))_{t\geq 0}$ is a Poisson process of rate one. Whenever $\omega^{(x)}$ has a jump at time t, all the particles occupying x at time t are killed.

We give an answer to the following question:

For which values of λ, κ and q is the probability that the branching random walk survives strictly positive?

A priori, the answer might depend on the realization of the random environment, but we will see that the survival probability is either zero, for almost all environments, or strictly positive, for almost all environments.

Let us comment on the dependence on the parameters of the model: It is clear by a coupling argument that increasing λ will increase the probability of survival, simply because there are more particles. Similarly, replacing the distribution q of the number of descendants by some distribution $\widetilde{q}$ having a larger mean should also increase the chance of survival. The dependence on κ is more tricky: If the jump rate is small, the process is essentially frozen and remains concentrated on few sites, and can be killed quickly if the environment is particularly unfavorable in a small area. If we increase κ, the process will jump away from any small area that is atypical and see an environment that is more average. However even in the best case, particles will be killed at rate 1.

We will not fully resolve the dependence on κ, but instead connect the problem to the survival rate in the one-particle model, which was studied in [9]. This correspondence is similar to the connection between the random polymer model and branching random walks in random space-time-environments, as explained in Section 1.3 in [4]. The proof of extinction in the critical case borrows heavily from the proof given in [6], which confirmed Conjecture 1 in [4].

We point out that our model differs from the branching random walks considered in [4] not only because of continuous instead of discrete time, but also because disasters in the environment were excluded there (see formula (1.7) in [4]). The possibility of killing many particles at the same site at once makes our model interesting but also creates some technical difficulties.

The paper is organized as follows. In Sections 1.1-1.3 we define the process, recall some of the previous results about the one-particle model and state our results. Our main result is Theorem 1 which characterizes the set of parameters where the survival probability is strictly positive. The non-critical cases in Theorem 1 follow from standard arguments: In Section 2 we give the proof in the subcritical regime using the first moment method. For the supercritical regime in Section 3 we compare our process to an embedded Galton-Watson process with i.i.d. offspring distributions.

The critical case needs a longer argument, which we outline in Section 4.1. An important tool is an FKG-inequality which we state in Section 4.2. We prove it in Section 4.3. Next in Section 4.4 we state and prove some technical lemmas, which then in Section 4.5 allow us to construct local events characterizing the event of global survival. Finally Section 4.6 contains the proof of the main result.

A technical difficulty in the supercritical case is Proposition 2, where we show that the exponential decay rate from [9] does not change if instead we consider the probability to survive and return again to the origin. We do not have a simple proof for this, and instead give a proof using arguments from the field of random polymers in the appendix.

Acknowledgment: We thank Nobuo Yoshida for hosting SJ, for many inspiring and fruitful discussions and for helpful comments on an earlier version of this paper. SJ thanks the DAAD for financial support in the PROMOS scholarship.

1.1 Definition and Notation

We first define the branching random walk introduced above: We identify the nodes of a tree with the set

$$\mathbb{N}^* := \bigcup_{k=0}^{\infty} \mathbb{N}^k = \Big\{ x = (x_1, ..., x_k) \colon k \in \mathbb{N}, x_1, ..., x_k \in \mathbb{N} \Big\}.$$

We call $|(x_1, ..., x_k)| =: k$ the **height** of v and write $\emptyset$ for the unique element of height 0, which we call the **root**. Proceeding recursively we interpret $(x_1, ..., x_k)$ as the the x_k^{th} child of $(x_1, ..., x_{k-1})$, for $k \geq 1$. Fix now positive values κ and λ as well as a distribution $q = (q(k))_{k\in\mathbb{N}}$ on the natural numbers satisfying

$$m := \sum_{k=0}^{\infty} kq(k) < \infty \quad \text{and} \quad q(1) < 1. \tag{1}$$

We associate to every node an exponential clock of rate λ, and whenever a clock rings the node is removed and replaced by its children, where the number of children is distributed according to q. The clocks and the numbers of descendants are independent. We will write $V(t)$ for the set of nodes that are alive at time t, starting with $V(0) = \{\emptyset\}$.

Next, we extend this by associating to each node v alive at time t a position $X(t, v)$ in $\mathbb{Z}^d$. We let each particle perform a simple random walk in continuous time of jump rate κ between its birth and the time when it is replaced by its children, independently from everything else. The root initially starts in the origin, and all other nodes start at the position occupied by their parent node at the time of birth.

For $v \in V(t)$, it will be convenient to extend $X(t,v)$ to a function $X(\cdot, v)\colon [0,t] \to \mathbb{Z}^d$, where for $s \in [0,t]$ we set $X(s,v)$ equal to the position occupied at time s by the unique ancestor of v in $V(s)$.

The process described so far is well-studied, so we do not give a formal construction. Recall that the environment $\omega = \big(\omega^{(x)}\big)_{x\in\mathbb{Z}^d}$ consists of independent Poisson processes of rate 1 indexed by the sites of $\mathbb{Z}^d$ and independent of the random variables defined before. Let

$$\delta(t,x) := \omega^{(x)}(t) - \omega^{(x)}(t^-).$$

If $\delta(t,x) = 1$, we say that there is a disaster at time t at x. The process we are interested in is denoted $(Z(t))_{t\geq 0}$, with

$$Z(t) := \big\{v \in V(t)\colon \delta(s, X(s,v)) = 0 \text{ for all } 0 \leq s < t\big\} \subseteq V(t).$$

So $Z(t)$ contains all particles v where no disaster occurred along the trajectory of v until time t. Note that since we did not assume $q(0) = 0$ it is possible that a particle has zero children, and the process may die out even without the influence of the environment.

We will use Q to denote the law of the environment, and P for the law of the branching random walk. Typically we consider the processes $Z(t)$ for fixed realizations of ω, and then we write P_ω for the **conditional** or **quenched** law. The **annealed** or **averaged** law $\mathbb{P}$ is given by

$$\mathbb{P}(Z \in \cdot) := \int P_\omega(Z \in \cdot) Q(\,\mathrm{d}\omega).$$

We denote the corresponding expectation by $\mathbb{E}$. With a slight abuse of notation, we also use $\mathbb{E}$ for the expectation with respect to Q. Occasionally we want to stress the dependence on the parameters, in which case we write $\mathbb{P}^{\kappa,\lambda}$ and $P_\omega^{\kappa,\lambda}$.

1.2 Previous results about the one-particle model

There is a close relationship between our model and the model considered in [9]. There, the process consists of a single particle performing random walk at rate κ among disasters in the same way that particles in our model do. In this section we summarize some known results.

Let $(X(t))_{t\geq 0}$ be a simple random walk in continuous time, moving in $\mathbb{Z}^d$ at a jump rate $\kappa > 0$, with the corresponding probability measure denoted P. The environment $\omega = \big(\omega^{(x)}\big)_{x\in\mathbb{Z}^d}$ is the same as before. We let τ be the first time the random walk hits any of the disasters, that is

$$\tau := \inf\Big\{t \geq 0\colon \delta(s, X_s) > 0\Big\}.$$

We are interested in the probability to survive until time t for a fixed realization of the environment:

$$S(t) := P_\omega(\tau \geq t)$$

Note that by averaging over the environments one easily gets the annealed survival rate:

$$\mathbb{E}[S(t)] = \int S(t)\,\mathrm{d}Q = e^{-t}.$$

We summarize the results of [9] in the following

Theorem. *Define* $p\colon (0,\infty) \to (-\infty, 0)$ *by*

$$p(\kappa) := \lim_{t\to\infty} \frac{1}{t} \log S(t). \tag{2}$$

Then

(i) The limit in (2) *exists* Q*-almost surely and is deterministic, with*

$$p(\kappa) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\log S(t)]\,. \tag{3}$$

(ii) For all $\kappa > 0$ *we have* $p(\kappa) \le -1$*.*

(iii) For any d *we have* $\lim_{\kappa\to 0} p(\kappa) = -\infty$ *and* $\lim_{\kappa\to\infty} p(\kappa) = -1$*.*

(iv) There exists a critical rate $\kappa_c = \kappa_c(d) \in (0,\infty]$*, such that*

$$\begin{aligned} p(\kappa) &< -1 \quad \textit{if } \kappa < \kappa_c \\ p(\kappa) &= -1 \quad \textit{if } \kappa > \kappa_c \end{aligned}$$

(v) For $d \ge 3$ *we have* $\kappa_c(d) < \infty$*.*

1.3 The main result

We are interested in the event

$$\{Z \text{ survives}\} := \{|Z(t)| > 0,\, \forall t \ge 0\}\,. \tag{4}$$

Using the exponent $p(\kappa)$ we prove the following criterion:

Theorem 1.

$$P_\omega(Z \textit{ survives}) > 0 \quad Q\textit{-a.s.} \quad \Longleftrightarrow \lambda(m-1) + p(\kappa) > 0\,.$$

In analogy to classical branching processes, we define three regimes.

Definition. *We say that the process* $Z(t)$ *is*

$$\begin{array}{ccc} \textit{subcritical} & \textit{if} & \lambda(m-1) + p(\kappa) < 0, \\ \textit{critical} & \textit{if} & \lambda(m-1) + p(\kappa) = 0, \\ \textit{supercritical} & \textit{if} & \lambda(m-1) + p(\kappa) > 0. \end{array}$$

An easy corollary is

Corollary 1.

$$\mathbb{P}(Z \textit{ survives}) > 0 \iff \lambda(m-1) + p(\kappa) > 0\,.$$

We define the event of **local survival** to be

$$\{Z \text{ survives locally}\} := \{0 \text{ is occupied for arbitrarily large times}\}\,.$$

Clearly

$$\{Z \text{ survives }\} \subseteq \{Z \text{ survives locally}\}\,.$$

Our proof of Theorem 1 shows in fact that the process survives locally in the supercritical case, so that the following holds.

Corollary 2. *The process either has a positive probability to survive locally in almost every environment, or it dies out with probability* 1 *in almost all environments. Moreover*

$$P_\omega(Z \text{ survives locally}) > 0 \quad Q\text{-a.s.} \quad \Longleftrightarrow \lambda(m-1) + p(\kappa) > 0\,.$$

Corollary 3. *There exists $c > 0$ such that Q-almost surely*

$$\{Z \text{ survives }\} = \Big\{ \liminf_{t\to\infty} |Z_t| e^{-ct} > 0 \Big\}$$

Proof. For the proofs see the remark at the end of Section 3. □

Remark. *By an obvious truncation argument, the assumption $m < \infty$ can be dropped; if $m = \infty$, we are in the supercritical case.*

We do not make any assumption on the shape of p, so a priori it may be discontinuous or may not be increasing in κ. In Corollary 4.1 in [5] continuity of p is proven for a related class of models, but the relevant case of hard obstacles is excluded. However, if we interpret p as the free energy of a polymer in random environment as in Section 3 in [3], it is reasonable to conjecture that p is even concave. A proof might be attempted by showing the following

Conjecture. *Fix a branching mechanism with $m > 1$, and set*

$$U := \Big\{ (\kappa, \lambda) \colon \mathbb{P}^{\kappa,\lambda}\big(Z \text{ survives}\big) > 0 \Big\} \subseteq (0,\infty)^2.$$

Then U is a convex set.

2 Proof of Theorem 1: the subcritical case

Proof. Assume

$$-\varepsilon := \lambda(m-1) + p(\kappa) < 0.$$

For almost all environments ω, we can find $T = T(\omega)$ such that

$$S(t) = P_\omega(\tau \geq t) \leq e^{t(p(\kappa) + \frac{\varepsilon}{2})} \quad \forall t \geq T.$$

Then we have for $t \geq T(\omega)$

$$E_\omega[|Z(t)|] = E_\omega\Big[\sum_{v \in V(t)} \mathbb{1}_{\{v \text{ survives until } t\}} \Big]$$

$$= E[V(t)] S(t) = E[m^M] S(t) = e^{\lambda(m-1)t} S(t) \leq e^{-\frac{\varepsilon}{2} t}, \tag{5}$$

where M is a random variable whose law is Poisson with parameter λt. This implies $Z(t) \to 0$ for almost all environments. □

3 Proof of Theorem 1: the supercritical case

Proof. Assume

$$\lambda(m-1) + p(\kappa) > 0\,. \tag{6}$$

We will find a branching process with i.i.d. offspring distributions embedded in Z. More precisely, we introduce a process $(A(k))_{k\in\mathbb{N}}$ taking values in $\mathbb{N}$, such that for some $T > 0$ we have

$$A(k) \leq |Z(kT)| \quad \forall k \in \mathbb{N}\,.$$

The claim then follows by showing

$$Q\Big(P[A(k) > 0 \;\forall k \in \mathbb{N}] > 0\Big) = 1\,.$$

Fix some large T, and set $A(0) \coloneqq 1 = |Z_0|$ and

$$A(k) \coloneqq \Big|\Big\{v \in Z(kT)\colon X(iT, v) = 0 \text{ for all } i = 0, ..., k\Big\}\Big|$$

That is, for the process A we only consider particles that return to the origin at times $T, 2T, 3T, \ldots$. Note that every particle that contributes to $A(k)$ is the descendant of a particle that contributed to $A(k-1)$. To see that $(A(k))_k$ has i.i.d. offspring distributions, we introduce the notation

$$Z(t) \cap \{0\} \coloneqq \{v \in Z(t)\colon X(t, v) = 0\}. \tag{7}$$

Let $E(k)$ be the event that the process at time $(k-1)T$ consists of a single particle at the origin:

$$E(k) \coloneqq \Big\{1 = \big|Z((k-1)T) \cap \{0\}\big| = \big|Z((k-1)T)\big|\Big\}.$$

Then the sequence of environments $(q^{(k)})_{k\in\mathbb{N}}$ is given by

$$q^{(k)}(j) = P_\omega\Big(\big|Z(kT) \cap \{0\}\big| = j \Big| E(k)\Big) \quad \text{for } j \in \mathbb{N}.$$

Note that $q^{(k)}$ only depends on the environment in the interval $[(k-1)T, kT)$, and $(q^{(k)})_k$ is therefore an i.i.d. sequence in the space of probability measures on $\mathbb{N}$. We set

$$m^{(k)} \coloneqq \sum_{j=0}^{\infty} j q^{(k)}(j)\,.$$

By a well-known result on branching processes with i.i.d. offspring distributions, see [10, 11], the survival probability of $(A(k))_{k\in\mathbb{N}}$ is positive for almost all environments if

$$\mathbb{E}[\log(1 - q^{(1)}(0))] > -\infty \tag{8}$$

and

$$\mathbb{E}[\log(m^{(1)})] > 0\,. \tag{9}$$

We can write $m^{(1)}$ as

$$m^{(1)} = \sum_{j\in\mathbb{N}} j q^{(1)}(j) = \sum_j j P_\omega\big(\big|Z(T) \cap \{0\}\big| = j\big) = E_\omega\big[|Z(T) \cap \{0\}|\big]\,.$$

By the same computation as in (5) we get

$$m^{(1)} = e^{\lambda(m-1)T} \widetilde{S}(T) \tag{10}$$

where

$$\widetilde{S}(t) \coloneqq P_\omega(\tau \geq t, X_t = 0). \tag{11}$$

In order to estimate the expectation in (8), we compare the branching process to the random walk of a single particle. We choose a path by starting in the root, and whenever there is more than one child, we choose (say) the first child. Let $F(t)$ be the event that until time t, we never get zero children along this path. We have

$$E[F(t)] = E[(1 - q(0))^M] = \exp(-\lambda t q(0))$$

where M is the number of branching events along this path. Note that M has distribution Poisson(λt). Then

$$1-q^{(1)}(0) \geq E[F(T)]\widetilde{S}(T)$$

and by Proposition 2 we see that indeed

$$\mathbb{E}\left[\log(1-q^{(1)}(0))\right] \geq -\lambda T q(0) + \mathbb{E}[\log \widetilde{S}(T)] > -\infty.$$

Proposition 2 guarantees that the exponential decay rates of $(\widetilde{S}(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are the same.

Proposition 2. *Recall* (11)*. For any* $t \geq 0$*,*

$$\mathbb{E}[\log \widetilde{S}(t)] > -\infty. \tag{12}$$

Moreover we have

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log \widetilde{S}(t)] = p(\kappa). \tag{13}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \widetilde{S}(t) = p(\kappa) \quad \text{for } Q\text{-almost all } \omega. \tag{14}$$

For the proof, see the appendix. We can now conclude: By Proposition 2 and (10), there is for every $\varepsilon > 0$ some T large enough that

$$\mathbb{E}[\log(m^{(1)})] \geq T\big(\lambda(m-1) + (p(\kappa)-\varepsilon)\big)$$

By (6), we can satisfy (9) by choosing ε small enough. Moreover (8) always holds, so the process $(A(k))_{k\in\mathbb{N}}$ has a positive probability of survival for almost all realizations of the environment.

Remark. *The proof shows also that in the supercritical case, the process survives locally. Using results of [11] about branching processes with i.i.d. offspring distributions we also see that in the supercritical case, the number of particles grows exponentially fast.*

□

4 Proof of Theorem 1: the critical case

4.1 Outline of the proof

We follow the main ideas of the proof given in [6]. Fix κ and λ such that

$$\lambda(m-1) + p(\kappa) = 0. \tag{15}$$

Assume that

$$\mathbb{P}^{\kappa,\lambda}(Z \text{ survives}) > 0. \tag{16}$$

We will show that this is a contradiction. We introduce the rate at which disasters appear as a new parameter of the model (until now, it was set to be 1). Denote by Q^α the law such that $\big(\omega^{(x)}\big)_{x\in\mathbb{Z}^d}$ is a collection of independent Poisson processes of rate $\alpha > 0$, and write $\mathbb{P}^{\alpha,\kappa,\lambda}$ for the annealed measure $Q^\alpha \otimes P^{\kappa,\lambda}$. Denote by $p(\alpha,\kappa)$ the survival rate in this environment. We claim that

$$\lambda(m-1) + p(\alpha,\kappa) < 0 \quad \text{for any } \alpha > 1. \tag{17}$$

In fact, (17) follows from (15) since $\alpha \mapsto p(\alpha, \kappa)$ is strictly decreasing, see (49). Using the same arguments as in the proof of the subcritical part of Theorem 1, (17) implies that

$$\mathbb{P}^{\alpha,\kappa,\lambda}(Z \text{ survives}) = 0 \quad \text{ for any } \alpha > 1. \tag{18}$$

On the other hand we will show that (16) implies that there are events $A = A(L,T)$ such that for every $\varepsilon > 0$ we find L, T such that for $\alpha = 1$,

$$\mathbb{P}^{\alpha,\kappa,\lambda}(A(L,T)) > 1 - \varepsilon \tag{19}$$

and with the properties that

(1) $A(L,T)$ is a local event, that is it depends only on the space-time box $[0,T] \times \{-L, ..., L\}^d \subseteq \mathbb{R} \times \mathbb{Z}^d$.

(2) There exists $\varepsilon > 0$ such that for any α, (19) implies that $\mathbb{P}^{\alpha,\kappa,\lambda}(Z \text{ survives}) > 0$.

This is however a contradiction: Since $A(L,T)$ is a local event, (19) is satisfied with $\mathbb{P}^{\alpha,\kappa,\lambda}$ for all $\alpha > 1$ sufficiently close to 1. Then Property (2) implies that the survival probability under $\mathbb{P}^{\alpha,\kappa,\lambda}$ is strictly positive, which is a contradiction to (18). Therefore (16) cannot hold.

4.2 Space-time boxes and the FKG-inequality

We first extend the definition of Z to the case where we may have more than one particle in the beginning. We call $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ a **configuration**, and let Z^η denote the process as defined before, except that we start with $\eta(x)$ particles in x, all of which evolve independently but in the same environment. For a set $A \subseteq \mathbb{Z}^d$, the process Z^A is the process started in the configuration $\mathbb{1}_A$, so that initially there is exactly one particle on every site in A.

Moreover, for a set $B \subseteq \mathbb{Z}^d$, let $(Z_B(t))_{t\geq 0}$ be the **truncated process**, consisting of all particles that have never left B:

$$Z_B^A(t) := \Big\{ v \in Z^A(t) \colon X(s,v) \in B \ \text{ for all } s \in [0,t] \Big\} \tag{20}$$

In the simple case where $B = \{-L, ..., L\}^d$ for some $L \in \mathbb{N}$, we write $(Z_L(t))_{t\geq 0}$ and $(Z_L^A(t))_{t\geq 0}$. For $C \subseteq \mathbb{Z}^d$, we use the notation

$$\{C \subseteq Z(t)\} := \Big\{ \forall x \in C \ \exists v \in Z(t) \text{ such that } X(t,v) = x \Big\} \tag{21}$$

for the event that every site in C is occupied by at least one particle of $Z(t)$.

We can think of the process $(Z^A(t))_{0\leq t \leq T}$ as a process in space-time, which we denote by

$$[0,T\,] \times Z^A := \big\{ (t,v) \colon 0 \leq t \leq T, v \in Z^A(t) \big\} \subseteq [0,T] \times \mathbb{N}^*.$$

Consider now a **space-time box** $\mathbb{B} \subseteq \mathbb{R} \times \mathbb{Z}^d$ of the form

$$\mathbb{B} := [0,T] \times \{-L..., L\}^d \quad \text{for } L \in \mathbb{N} \text{ and } T > 0\,.$$

We denote the **top** of this box by

$$\mathbb{T}(L,T) := \{T\} \times \{-L..., L\}^d.$$

For simplicity, we define the sign of zero to be 1, that is sign $(x) := \mathbb{1}_{x\geq 0} - \mathbb{1}_{x<0}$. Let $\Delta \in \{+,-\}$, and divided $\mathbb{T}(L,T)$ in a **left** and **right** part with

$$\mathbb{T}_\Delta(L,T) \ \ := \ \ \big\{ (T,x) \in T(L,T) \colon \text{sign } x_1 = \Delta 1 \big\}\,. \tag{22}$$

The box also has **faces** $\mathbb{F}^i_+(L,T)$ and $\mathbb{F}^i_-(L,T)$ corresponding to the directions e_i and $-e_i$ for $i=1,...,d$:

$$\mathbb{F}^i_\Delta(L,T) \quad \coloneqq \quad [0,T]\times\{-L,...,L\}^{i-1}\times\{\Delta L\}\times\{-L,...,L\}^{d-i} \tag{23}$$

We will mostly be concerned with the faces in directions e_1 and $-e_1$, so we abbreviate

$$\mathbb{F}_+(L,T)\coloneqq\mathbb{F}^1_+(L,T) \quad \text{and} \quad \mathbb{F}_-(L,T)\coloneqq\mathbb{F}^1_-(L,T).$$

We need to introduce further subdivision of both the top and the sides: For every $\theta\in\{-1,1\}^{d-1}$ and $\Delta\in\{+,-\}$ we find an orthant

$$\mathbb{T}_\Delta(L,T,\theta)\coloneqq\{(T,x_1,...,x_d)\in\mathbb{T}_\Delta(L,T)\colon \text{sign } x_i=\theta_i\ \forall i=2,...,d\}$$

while the orthants of the face have the form

$$\mathbb{F}_\Delta(L,T,\theta)\coloneqq\{(t,\Delta L,x_2,...,x_d)\in\mathbb{F}_\Delta(L,T)\colon \text{sign } x_i=\theta_i\ \forall i=2,...,d\}$$

We further denote the **boundary** of $\mathbb{B}$ by $\partial\mathbb{B}$, that is

$$\partial\mathbb{B}(L,T)\coloneqq\mathbb{T}(L,T)\cup\bigcup_{i=1}^d\mathbb{F}^i_+(L,T)\cup\bigcup_{i=1}^d\mathbb{F}^i_-(L,T)$$

For all these quantities we sometimes omit the dependence on L and T if it is clear from the context.

Now for $\Delta\in\{+,-\}$ and $\theta\in\{-1,1\}^{d-1}$, let $N^A_\Delta(L,T,\theta)$ count the number of particles leaving $\mathbb{B}$ through $\mathbb{F}_\Delta(L,T,\theta)$, by which we mean the number of times such that a particle of Z^A hits $\partial\mathbb{B}$ for the first time at some $(t,x)\in\mathbb{F}_\Delta(L,T,\theta)$. Formally, $N^A_\Delta(L,T,\theta)$ is the cardinality of the set

$$\big\{(t,v)\in[0,T]\times Z^A\colon X(t,v)\in\mathbb{F}_\Delta(L,T,\theta)\ ,X(s,v)\notin\partial\mathbb{B}\ \forall s<t\big\}.$$

Furthermore, let $M^A_\Delta(L,T)$ count the particles exiting $\mathbb{B}$ through $\mathbb{T}_\Delta(L,T,\theta)$, that is

$$M^A_\Delta(L,T,\theta)\coloneqq\big|\big\{v\in Z^A(T)\colon X(T,v)\in\mathbb{T}_\Delta(L,T,\theta),X(s,v)\notin\partial\mathbb{B}\ \forall s<T\big\}\big|.$$

We will also consider

$$M^A(L,T)\coloneqq\sum_{\substack{\Delta\in\{+,-\}\\ \theta\in\{-1,1\}^{d-1}}}M^A_\Delta(L,T,\theta) \quad \text{and} \quad N^A(L,T)\coloneqq\sum_{\substack{\Delta\in\{+,-\}\\ \theta\in\{-1,1\}^{d-1}}}N^A_\Delta(L,T,\theta). \tag{24}$$

The important observation is that the family

$$(\widetilde{M}^A,\widetilde{N}^A)\coloneqq\Big(M^A_\Delta(\theta),N^A_\Delta(\theta)\Big)_{\Delta\in\{+,-\},\theta\in\{-1,1\}^{d-1}}$$

satisfies an FKG inequality, stated in the next theorem.

Theorem 3. *Let f and g be increasing, non-negative functions*

$$f,g\colon\mathbb{N}^{(2^{2d})}\times\mathbb{N}^{(2^{2d})}\to\mathbb{R}^+.$$

Then

$$\mathbb{E}\Big[f(\widetilde{M}^A,\widetilde{N}^A)g(\widetilde{M}^A,\widetilde{N}^A)\Big]\geq\mathbb{E}\Big[f(\widetilde{M}^A,\widetilde{N}^A)\Big]\mathbb{E}\Big[g(\widetilde{M}^A,\widetilde{N}^A)\Big] \tag{25}$$

We prove this in the next section. An intuitive explanation is that if a tree has many surviving particles occupying, say, the right side of the top, then this increases the chance that there are also many particles alive in the left side of the top, since they have common ancestors.

4.3 Proof of the FKG-inequality

Proof of Theorem 3. As before we will consider a space-time box $\mathbb{B} = [0,T] \times \Lambda$ where $\Lambda := \{-L, ..., L\}^d$.

First Step: We first show the FKG inequality for fixed environment. More precisely we show that for any ω, we have

$$E_\omega\Big[f(\widetilde{M}^A, \widetilde{N}^A) g(\widetilde{M}^A, \widetilde{N}^A)\Big] \geq E_\omega\Big[f(\widetilde{M}^A, \widetilde{N}^A)\Big] E_\omega\Big[g(\widetilde{M}^A, \widetilde{N}^A)\Big]. \tag{26}$$

We let K denote the number of disasters in $\mathbb{B}$, and write T_k for the time that the k^{th} disaster occurs. For $1 \leq k \leq K$ we let X_k be the site of the k^{th} disaster. For simplicity we also set $T_0 := 0$ and $T_{K+1} := T$.

Let now $(V^{x,k,i})_{x,k,i}$ be a family of independent copies of V indexed by $\Lambda \times \mathbb{N} \times \mathbb{N}$, shifted such that the root of $V^{x,k,i}$ is placed at time T_k at site x. We will interpret $V^{x,k,i}$ as the tree started from the i^{th} particle at (T_k, x), and use $X^{x,k,i}$ for the trajectories of the particles of $V^{x,k,i}$.

Given now this family together with $(\Delta_k)_{1\leq k \leq K}$, we introduce for $y \in \Lambda$ the random variables

$$R^{x,k,i}(y) := \sum_{v \in V^{x,k,i}(\Delta_k)} \mathbb{1}\{X^{x,k,i}(\Delta_k, v) = y, X^{x,k,i}(s,v) \notin \partial\Lambda \text{ for all } s \in [0, \Delta_k]\}.$$

In words, $R^{x,k,i}$ counts the number of descendants of the i^{th} particle at (T_k, x) that move from x to y in the time interval (T_k, T_{k+1}) without leaving Λ. For $\theta \in \{-1,1\}^{d-1}$ and $\Delta \in \{+,-\}$ we also define $S_\Delta^{x,k,i}(\theta)$ by

$$S_\Delta^{x,k,i}(\theta) := \sum_{v \in V^{x,k,i}(T_{k+1})} \mathbb{1}\big\{v \text{ leaves } \mathbb{B} \text{ through } \mathbb{F}_\Delta(\mathbb{B}, \theta) \cap [T_k, T_{k+1}) \times \Lambda\big\}.$$

Note that the vectors

$$\Big(R^{x,k,i}, S_+^{x,k,i}, S_-^{x,k,i}\Big)_{x,k,i} \in \Big(\mathbb{N}^\Lambda \times \mathbb{N}^{\{-1,1\}^{d-1}} \times \mathbb{N}^{\{-1,1\}^{d-1}}\Big)^{\Lambda\times\mathbb{N}\times\mathbb{N}} \tag{27}$$

are independent under P_ω. We now use those variables to construct a realization of $(\widetilde{M}^A, \widetilde{N}^A)$. For this we introduce processes

$$\big(\widetilde{Z}(k,x)\big) \quad \text{and} \quad \big(\widetilde{Y}_\Delta(k,\theta)\big)$$

indexed by $0 \leq k \leq K$, $x \in \Lambda$, $\theta \in \{1,-1\}^{d-1}$ and $\Delta \in \{+,-\}$. We use $\widetilde{Z}(k,x)$ to count the number of particles that occupy x at time T_k and that have never left $\mathbb{B}$, whereas $\widetilde{Y}_\Delta(k,\theta)$ counts the number of particles that have left $\mathbb{B}$ through $\mathbb{F}_\Delta(\mathbb{B},\theta) \cap [0, T_k) \times \Lambda$.

This is useful because now

$$\begin{aligned} N_\Delta^A(\theta) &= \widetilde{Y}_\Delta(K+1,\theta) \\ M_\Delta(\theta) &= \textstyle\sum_{x:\,(T,x)\in\mathbb{T}_\Delta(\theta)} \widetilde{Z}(K+1,x). \end{aligned} \tag{28}$$

To construct this sequence we start with

$$\widetilde{Z}(0,x) := \mathbb{1}_A(x) \quad \text{and} \quad \widetilde{Y}_+(0,\theta) = 0$$

and for $k \geq 0$ proceed by

$$\widetilde{Y}_\Delta(k+1,\theta) := \widetilde{Y}_\Delta(k,\theta) + \sum_{y \in \Lambda \setminus \{X_k\}} \sum_{i=1}^{\widetilde{Z}(k,y)} S_\Delta^{y,k,i}(\theta). \tag{29}$$

$$\widetilde{Z}(k+1,x) := \begin{cases} 0 & \text{if } x = X_k \text{ and } k > 0 \\ \sum_{y\in\Lambda} \sum_{i=1}^{\widetilde{Z}(k,y)} R^{y,k,i}(x) & \text{otherwise.} \end{cases} \tag{30}$$

We denote this process by $\big([\widetilde{Y},\widetilde{Z}]_k\big)_{0\leq k\leq K+1}$, with

$$\big[\widetilde{Y},\widetilde{Z}\,\big]_k := \Big(\big(\widetilde{Y}(l,\theta)\big)_{l=0,\dots,k,\theta\in\{1,-1\}^{2(d-1)}}, \big(\widetilde{Z}(l,x)\big)_{l=0,\dots,k,x\in\Lambda}\Big).$$

Then we can conclude by showing the following

Claim. *For any $k = 0,...,K+1$ the random variables $\big[\widetilde{Y},\widetilde{Z}\big]_k$ satisfy the FKG inequality. That is, for any increasing $\tilde{f}$, $\tilde{g}$ with*

$$\tilde{f},\tilde{g}\colon \quad \mathbb{N}^{(k+1)2^{d-1}} \times \mathbb{N}^{(k+1)(2L+1)} \to \mathbb{R}^+$$

we have

$$E_\omega\Big[\tilde{f}\big([\widetilde{Y},\widetilde{Z}]_k\big)\tilde{g}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big] \geq E_\omega\Big[\tilde{f}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big]E_\omega\Big[\tilde{g}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big]$$

Using (28), its is clear that (26) follows from the claim since $\widetilde{M}^A$ and $\widetilde{N}^A$ are increasing functions of $\big[\widetilde{Y},\widetilde{Z}\big]_{K+1}$.

Proof of the claim. For $k = 0$ the claim is trivial. Proceeding by induction, assume we have shown the claim up to $k-1$. Let $\mathcal{F}_k$ be the sigma algebra

$$\mathcal{F}_k := \sigma\Big(R^{x,l,i}, S_+^{x,l,i}, S_-^{x,l,i} \colon x \in \Lambda, i \in \mathbb{N}, 0 \leq l \leq k\Big).$$

From (30) and (29) we make the following observations:

- $(\widetilde{Z}(k,x))_{x\in\Lambda}$ and $(\widetilde{Y}(k,\theta))_{\theta\in\{1,-1\}^{2(d-1)}}$ are $\mathcal{F}_k$ -measurable.
- Given $\mathcal{F}_{k-1}$, both $(\widetilde{Z}(k,x))_{x\in\Lambda}$ and $(\widetilde{Y}(k,\theta))_{\theta\in\{1,-1\}^{2(d-1)}}$ are increasing functions of the independent family of vectors

$$\Big(R^{x,k,i}, S_+^{x,k,i}, S_-^{x,k,i}\Big)_{x\in\Lambda,i\in\mathbb{N}} \in \Big(\mathbb{N}^\Lambda \times \mathbb{N}^{\{-1,1\}^{d-1}} \times \mathbb{N}^{\{-1,1\}^{d-1}}\Big)^{\Lambda\times\mathbb{N}}.$$

- Moreover both $(\widetilde{Z}(k,x))_{x\in\Lambda}$ and $(\widetilde{Y}(k,\theta))_{\theta\in\{1,-1\}^{2(d-1)}}$ are increasing functions of $\big(\widetilde{Z}(k-1,x)\big)_{x\in\Lambda}$.

The first two observations together with the FKG inequality for independent vectors allows us to write

$$E_\omega\Big(\tilde{f}\big([\widetilde{Y},\widetilde{Z}]_k\big)\tilde{g}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big|\mathcal{F}_{k-1}\Big) \geq E_\omega\Big(\tilde{f}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big|\mathcal{F}_{k-1}\Big)E_\omega\Big(\tilde{g}\big([\widetilde{Y},\widetilde{Z}]_k\big)\Big|\mathcal{F}_{k-1}\Big).$$

Now from the first and the third observation we see that

$$E_\omega\big(\tilde{f}\big([\widetilde{Y},\widetilde{Z}]_k\big)\big|\mathcal{F}_{k-1}\big) \quad \text{and} \quad E_\omega\big(\tilde{g}\big([\widetilde{Y},\widetilde{Z}]_k\big)\big|\mathcal{F}_{k-1}\big)$$

are increasing functions in $\big[\widetilde{Y},\widetilde{Z}\big]_{k-1}$, so the claim follows by induction. □

Second Step: For the second part, we want to show that the conclusion also holds with respect to $\mathbb{P}$. Let $P^{\otimes 2}$ be the law of two independent copies V^1 and V^2, both starting with exactly one particle at every site in A and moving independently but in the same environment. We write Z^i, $\widetilde{M}^i$ and $\widetilde{N}^i$ $(i = 1, 2)$ for the respective copies of Z, $\widetilde{M}$ and $\widetilde{N}$.

Integrating (26) yields

$$\mathbb{E}\Big[f(\widetilde{M}^A, \widetilde{N}^A) g(\widetilde{M}^A, \widetilde{N}^A)\Big] \geq \int \Big(\int f(\widetilde{M}^1, \widetilde{N}^1) g(\widetilde{M}^2, \widetilde{N}^2) \, \mathrm{d}Q \Big) \, \mathrm{d}P^{\otimes 2}.$$

Hence it suffices to show that

$$\int f(\widetilde{M}^1, \widetilde{N}^1) g(\widetilde{M}^2, \widetilde{N}^2) \, \mathrm{d}Q \geq \int f(\widetilde{M}^1, \widetilde{N}^1) \, \mathrm{d}Q \int g(\widetilde{M}^2, \widetilde{N}^2) \, \mathrm{d}Q \tag{31}$$

holds for every fixed realization of V^1 and V^2. We can find $K \in \mathbb{N}$ and times

$$0 = U_0 < U_1 < \dots < U_K < U_{K+1} = T$$

such that both trees are constant on $[U_k, U_{k+1})$ for $0 \leq k \leq K$. That is, neither V^1 nor V^2 jumps or branches in $[0, T] \setminus \{U_1, \dots, U_K\}$.

We introduce the following family of random variables

$$\eta(k, x) := \mathbb{1}\big\{\text{no disaster occurs at } x \text{ in the interval } [U_k, U_{k+1})\big\}.$$

Clearly $\{\eta(k, x)) \colon 0 \leq k \leq K, x \in \Lambda\}$ is an independent family, and we denote its sigma algebra by $\mathcal{G}$. Note that for $j = 1$ or 2, $(\widetilde{M}^j, \widetilde{N}^j)$ is $\mathcal{G}$-measurable and increasing in η. Since f and g are increasing this means that both $f(\widetilde{M}^1, \widetilde{N}^1)$ and $g(\widetilde{M}^2, \widetilde{N}^2)$ are also increasing in η. Therefore (31) again follows from the FKG inequality for independent families. $\square$

4.4 Starting with many particles

From now on we fix λ, κ and q such that (16) holds. We first show that we can make the survival probability arbitrarily close to 1 by increasing the set of initially occupied sites. More precisely, we start with a box $D_n := \{-n, \dots, n\}^d$ of occupied sites for some large n. This is part (i) of Lemma 1 below.

Combining the notation of (20) and (7) we define

$$Z_B^\eta(t) \cap \{0\} := \Big\{ \text{ particles of } Z_B^\eta(t) \text{ occupying } 0 \text{ at time } t \Big\}.$$

Part (ii) concerns the number of particles that survive locally until time 1 by using only two sites. For this we let $Z_B^{(N)}$ denote the process started in the initial configuration that has N particles at the origin and zero particles everywhere else. We obtain that once we choose N large enough, the number of particles that are at 0 at time 1 while never leaving $\{0, e_1\}$ is large with high probability.

Part (iii) shows that with high probability, one finds at time 1 a configuration where D_n is fully occupied when starting with N particles occupying the origin. We need this to be a local event, so we restrict ourself to particles that do not leave certain boxes.

Lemma 1. *(i) For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ with*

$$\mathbb{P}\big(Z^{D_n} \textit{ survives}\big) > 1 - \varepsilon.$$

(ii) For every $\varepsilon > 0$ and $M \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that

$$\mathbb{P}\Big(\Big|Z^{(N)}_{\{0,e_1\}}(1) \cap \{0\}\Big| \geq M\Big) > 1 - \varepsilon.$$

(iii) Recall (21)*. For every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that*

$$\min\Big\{\mathbb{P}\Big(ne_1 + D_n \subseteq Z^{(N)}_{ne_1+D_n}(1)\Big), \mathbb{P}\Big(D_n \subseteq Z^{(N)}_{D_n}(1)\Big)\Big\} > 1 - \varepsilon\,.$$

Proof. **Part (i)**: Define a collection $(Y_x)_{x\in\mathbb{Z}^d}$ with $Y_x \coloneqq \mathbb{1}\{|Z^{\{x\}}(t)| > 0\ \forall t > 0\}$. We have

$$\mathbb{P}\big(|Z^{D_n}(t)| > 0\ \forall t\big) = \mathbb{P}\Big(\sum_{x\in D_n} Y_x > 0\Big) = \mathbb{E}\Big[P_\omega\Big(\sum_{x\in D_n} Y_x > 0\Big)\Big]$$

Writing $S_n \coloneqq \sum_{x\in D_n} Y_x$ we have

$$P_\omega\big(S_n = 0\big) \leq P_\omega\left(|S_n - E_\omega[S_n]| \geq E_\omega[S_n]\right) \leq \frac{\mathrm{Var}_\omega(S_n)}{\big(E_\omega[S_n]\big)^2} \tag{32}$$

Now, due to the spatial ergodic theorem we have $\frac{1}{|D_n|}E_\omega[S_n] \to \mathbb{E}\big[E_\omega[Y_0]\big]$ for almost all ω, while

$$\frac{1}{|D_n|}\mathrm{Var}_\omega\big(S_n\big) = \frac{1}{|D_n|}\sum_{x\in D_n}\mathrm{Var}_\omega(Y_x) \to \mathbb{E}\big[\mathrm{Var}_\omega(Y_0)\big] \quad Q\text{-a.s.},$$

where we used the fact that $\{Y_x, x \in \mathbb{Z}^d\}$ are independent with respect to P_ω. We conclude from (32) that $P_\omega(S_n = 0) \to 0$ almost surely and therefore $\mathbb{P}\left(S_n = 0\right) \to 0$ as well.

Part (ii): For $v \in \mathbb{N}^*$ let $B(v)$ denote the event that v

- does not branch until time 1
- satisfies $X([0,1], v) \subseteq \{0, e_1\}$ and $X(1, v) = 0$
- and is not killed by the environment.

For any $\alpha \in (0,1]$ we let $A(\alpha)$ be the event $A(\alpha) \coloneqq \{P_\omega(B) \geq \alpha\}$. Note that the events $A(\alpha)$ are increasing as $\alpha \downarrow 0$ with

$$Q\Big(\bigcup_{\alpha\in(0,1]\cap\mathbb{Q}} A(\alpha)\Big) = 1.$$

So for any $\eta > 0$ we can find $\alpha > 0$ small enough such that $Q(A(\alpha)) \geq 1 - \eta$.

Now starting with N initial particles at the origin in an environment $\omega \in A(\alpha)$, the number of particles v such that $B(v)$ occurs dominates the number of successes of a binomial random variable ξ_N with N trials and success probability α. Clearly we can choose N large enough such that

$$P(\xi_N \geq M) \geq 1 - \eta.$$

Then we can conclude since

$$\mathbb{P}\Big(\Big|Z^{(N)}_{\{0,e_1\}}(1) \cap \{0\}\Big| \geq M\Big) \geq Q(A(\alpha))P(\xi_N \geq M) \geq (1-\eta)^2 \geq 1 - \varepsilon$$

holds for η small enough.

Part (iii): Let $\widetilde{D}_n$ be equal to either D_n or $ne_1 + D_n$. We fix an enumeration $\widetilde{D}_n = \{x_1, ..., x_{(2n)^d}\}$ of the sites, and introduce the quantity

$$S(x) \coloneqq P_\omega\Big(\tau \geq 1, X(1) = x, X([0,1]) \subseteq \widetilde{D}_n\Big).$$

Here we use P_ω for the law of a single particle which does not branch and which is killed by the environment ω with τ denoting its extinction time. For $\alpha \in (0,1]$ we consider events

$$A(\alpha) := \Big\{ \min\{S(x) : x \in \widetilde{D}_n\} \geq \alpha \Big\}.$$

Fix $\eta > 0$. By the same argument as before we find that $Q(A(\alpha)) \geq 1-\eta$ holds for some $\alpha > 0$ small enough. We now choose $N := m(2n)^d$ for some large m. Letting $W \subseteq \mathbb{N}^*$ denote the set of initial particles, we partition it (deterministically) in such a way that

$$W = W_1 \uplus \cdots \uplus W_{(2n)^d} \qquad \text{with } |W_i| = m \ \forall i = 1, ..., (2n)^d.$$

Now for $w \in W_i$ consider the event $B_i(w)$ that the particle w

- does not branch until time 1
- satisfies $X([0,1], w) \subseteq \widetilde{D}_n$ and $X(1,w) = x_i$ and
- is not killed by the environment.

We set

$$B := \bigcap_{i=0}^{|\widetilde{D}_n|-1} \bigcup_{w \in W_i} B_i(w).$$

Noticing that $P(B_i(w)) = e^{-\lambda} S(x_i)$ we conclude that for $\omega \in A(\alpha)$ we have

$$P_\omega(B) = \prod_{i=0}^{|\widetilde{D}_n|} P_\omega\Big(\bigcup_{w \in W_i} B_i(w) \Big) \geq \big(1 - (1 - e^{-\lambda}\alpha)^m\big)^{(2n)^d}$$

But now we can choose m large enough that $P_\omega(B) \geq 1-\eta$, hence

$$\mathbb{P}\Big(\widetilde{D}_n \subseteq Z^{(N)}_{\widetilde{D}_n}(1) \Big) \geq \int_{A(\alpha)} P_\omega(B) \, \mathrm{d}Q \geq (1-\eta)^2 \geq 1 - \varepsilon$$

holds for η small enough. □

In the following, we think of $A \subseteq \mathbb{Z}^d$ as a large set, so that $\{Z^A \text{ dies out}\}$ is an event of small probability.

In the first part of the next lemma we recover the familiar property that survival is equivalent to the number of particles going to infinity. Looking at the process as a random tree embedded in space-time, this means that there are many particles occupying the top of a space-time box.

In the second part we extend this observation by showing that there will also be many particles occupying the sides of the box. In this context a particle v is said to occupy a side if v hits this side at some time $t < T$, and if v and all ancestors of v have never hit the boundary before. Recall from (24) the definitions of $N^A(L,T)$ and $M^A(L,T)$ denoting the total number of particles at the top and the sides of $\mathbb{B} = [0,T] \times \{-L, ..., L\}^d$.

Lemma 2. *(i) For every $A \subseteq \mathbb{Z}^d$ we have*

$$\mathbb{P}\big(Z^A \textit{ survives}\big) = \mathbb{P}\big(Z^A \textit{ survives}, \lim_{t\to\infty} |Z^A(t)| = \infty\big)$$

(ii) Let $(T_j)_j$ and $(L_j)_j$ be two sequences increasing to infinity. Then for any $K > 0$ we have

$$\limsup_{j\to\infty} \mathbb{P}(N^A(L_j, T_j) + M^A(L_j, T_j) < K) \leq \mathbb{P}\big(Z^A \textit{ dies out }\big)$$

Proof. **Part (i)**: Define constants

$$\alpha := Q\Big(\text{At least one disaster occurs at the origin until time } 1\Big) = 1 - e^{-1}$$
$$\beta := P\Big(\big(Z^{\{0\}}(t)\big)_{0\le t\le 1} \text{ stays at the origin and does not branch}\Big) = e^{-\lambda-\kappa}.$$

Let $\mathcal{F}_t$ be the sigma algebra generated by environment and tree up to time t, and write $\mathbb{P}(A|\mathcal{F}_t)$ for the conditional expectation $\mathbb{E}[\mathbb{1}_A|\mathcal{F}_t]$. Then for any t we have

$$\mathbb{P}(Z^A \text{ dies out}|\mathcal{F}_t) \ge \alpha^{|Z^A(t)|}\beta^{|Z^A(t)|}.$$

Letting t go to infinity, the left side converges to the indicator function $\mathbb{1}\{Z^A \text{ dies out}\} \in \{0,1\}$. However, if for some K we have $|Z^A(t)| < K$ for arbitrarily large t, the limit inferior of the right side will be bounded away from 0. Therefore the event

$$\big\{Z^A \text{ survives}, |Z^A(t)| < K \text{ for arbitrarily large } t\big\}$$

has probability 0. Now

$$\begin{aligned}&\mathbb{P}\big(Z^A \text{ survives}, \limsup_{t\to\infty} |Z^A(t)| < \infty\big)\\ &= \lim_{K\to\infty} \mathbb{P}\big(Z^A \text{ survives}, |Z^A(t)| < K \text{ for arbitrarily large } t\big) = 0.\end{aligned}$$

Part (ii) Denote by $\mathcal{F}_{L_j,T_j}$ the sigma algebra generated by the environment in $\mathbb{B}_j := [0,T_j]\times\Lambda_j$, where $\Lambda_j := \{-L_j+1,\cdots,L_j-1\}^d$. We will consider the process of particles in Z^A that have never left $\mathbb{B}_j$, which is $\mathcal{F}_{L_j,T_j}$-measurable.

$$\begin{aligned}E_j :=&\Big\{(s,v)\colon \|X(s,v)\|_\infty = L_j, \|X(r,v)\|_\infty < L_j \ \text{ for all } r<s\Big\}\\ &\cup \Big\{(T,v)\colon \|X(r,v)\| < L_j \ \text{ for all } r \le T\Big\} \qquad\qquad \subseteq [0,T]\times Z.\end{aligned}$$

Note that $(s,v)\in E_j$ implies that the particle v has just left $\mathbb{B}_j$ (for the first time) at time s, either through one of the sides or through the top. Clearly E_j is $\mathcal{F}_{L_j,T_j}$ measurable and we have $|E_j| = N^A(L_j,T_j) + M^A(L_j,T_j)$.

Now for $(s,v)\in[0,T]\times Z^A$ let $D(s,v)$ be the event that v is killed because

- there is a disaster at $X(s,v)$ in the interval $[s,s+1]$
- and v has no branching times and no jumps in $[s,s+1]$.

Then $\mathbb{P}(D(s,v)) = \alpha\beta$ with the same α and β as before. We can write

$$\mathbb{P}(Z^A \text{ dies out}|\mathcal{F}_{L_j,T_j}) \ge \mathbb{P}\Big(\bigcap_{(s,v)\in E_j} D(s,v)\Big|\mathcal{F}_{L_j,T_j}\Big) \ge \alpha^{|E_j|}\beta^{|E_j|}. \tag{33}$$

For the last estimate, note that for $(s,v)\in E_j$ the event $D(s,v)$ is independent of $\mathcal{F}_{L_j,T_j}$, and that

$$\mathbb{P}(D(s_1,v)\cap D(s_2,w)) \ge \mathbb{P}(D(s_1,v))\mathbb{P}(D(s_2,w)) \quad \text{ for } (s_1,v)\ne(s_2,w)\in E_j.$$

Now the same argument as in the proof of part (i) applies: For $j\to\infty$, the left hand side in (33) converges to $\mathbb{1}\{Z^A \text{ dies out}\}$, while the right side will be bounded away from zero whenever $|E_j|\le K$ for infinitely many j. Therefore we have

$$\limsup_{j\to\infty} \mathbb{P}(|E_j|<K) \le \mathbb{P}(|E_j|\le K \text{ i.o.}) \le \mathbb{P}(Z^A \text{ dies out}).$$

□

4.5 Approximating with local events

The following is the key proposition which will help to characterize survival as a local event depending only on a finite part of the environment. Recall $D_n = \{-n, \dots, n\}^d$.

Proposition 4. *Assume (16). For every ε there exist n, L and T such that*

$$\mathbb{P}\begin{pmatrix} \exists x \in \{L+n, ..., 2L+n\} \times \{-L, ..., L\}^{d-1}, t \in [T, 2T] \\ \text{such that } x + D_n \subseteq Z^{D_n}_{\{-L,...,3L\}\times\{-L,...,L\}^{d-1}}(t) \end{pmatrix} > 1 - \varepsilon. \tag{34}$$

The next proposition will be an extension of Proposition 4. We prove that the claim is true uniformly over all trees that are obtained by shifting the set of initially occupied sites inside a space-time box.

We use the notation $Z^{s,A}$ to denote the process started at time s with all sites of A occupied by one particle.

Proposition 5. *Assume (16). For every $\varepsilon' > 0$ there exists n', L', T' such that for any $y \in \{-L', ..., L'\}^d$ and $s \in [0, T']$:*

$$\mathbb{P}\begin{pmatrix} \exists x \in \{L', ..., 3L'\} \times \{-L', ..., L'\}^{d-1}, t \in [5T', 6T'] \\ \text{such that } x + D_n \subseteq Z^{s,y+D_n}_{\{-5L',...,5L'\}\times\{-3L',...,3L'\}^{d-1}}(t) \end{pmatrix} > 1 - \varepsilon'.$$

For the main theorem we only need Proposition 5, which is proved by repeatedly applying Proposition 4. For the proof of Proposition 4, we need to consider two cases depending on the value of ε. Since ε will depend in turn on ε', we state the two cases in terms of ε'. For this fix $\varepsilon' > 0$, and choose $\varepsilon > 0$ such that

$$(1-\varepsilon)^{10} \geq 1 - \varepsilon'. \tag{35}$$

With this value of ε, we find some $\delta > 0$ such that

$$\min\left\{ \left(1 - \delta^{2^{-d}}\right)^2 (1-\delta)^3, 1 - 3\delta \right\} \geq 1 - \varepsilon. \tag{36}$$

By Lemma 1 we can find $n \in \mathbb{N}$ such that

$$\mathbb{P}\big(Z^{D_n} \text{ survives}\big) \geq 1 - \delta^2. \tag{37}$$

Now one of the following two statements will be true, and we prove both propositions separately in each case:

$$\forall L \in \mathbb{N} \text{ we have } \mathbb{P}\big(Z^{D_n}_L \text{ survives}\big) < 1 - 2\delta. \tag{case 1}$$
$$\exists L \in \mathbb{N} \text{ such that } \mathbb{P}\big(Z^{D_n}_L \text{ survives}\big) \geq 1 - 2\delta. \tag{case 2}$$

4.5.1 Proof in case 1

Proof of Proposition 4 in case 1. We first have to find a number $R \in \mathbb{N}$ that is large enough for our purposes: We have

$$\alpha := \min\left\{ \mathbb{P}\Big(ne_1 + D_n \subseteq Z^{\{0\}}_{ne_1+D_n}(1)\Big), P\Big(D_n \subseteq Z^{\{0\}}_{D_n}(1)\Big) \right\} > 0. \tag{38}$$

Now choose R_1 such that $1 - (1-\alpha)^{R_1} > 1 - \delta$, and set $R_2 := R_1(4n)^d$. Note that this ensures that any set $A \subseteq \mathbb{Z}^d$ with $|A| \geq R_2$ contains a subset $A' \subseteq A$ with $|A'| \geq R_1$ and such that for every two sites $x \neq y \in A'$ we have $\|x - y\|_\infty \geq 4n$.

Next, recall the notation $Z^{(N)}$ from Lemma 1 for the process started with N particles at 0. We use part (iii) of Lemma 1 to find R_3 such that

$$\min\Big\{\mathbb{P}\Big(ne_1 + D_n \subseteq Z^{(R_3)}_{ne_1+D_n}(1)\Big), P\Big(D_n \subseteq Z^{(R_3)}_{D_n}(1)\Big)\Big\} > 1 - \delta.$$

And finally, let R_4 be the value obtained for N in part (ii) of Lemma 1, applied with M equal to R_3 and ε equal to δ. Now set

$$R \coloneqq \Big(\max\big\{R_1, R_2, nR_4\big\}\Big)^2.$$

The next step is to find the right values of L and T.

From part (i) of Lemma 2 and the definition of n we obtain, recalling (24),

$$\lim_{T\to\infty}\lim_{L\to\infty}\mathbb{P}(M^{D_n}(L,T) > 2^dR) = \lim_{T\to\infty}\mathbb{P}(|Z^{D_n}(T)| > 2^dR) \geq 1 - \delta^2.$$

We can rewrite this by saying that for all $T \geq T_0$ there exists $L(T)$ with

$$\mathbb{P}(M^{D_n}(L,T) > 2^dR) \geq 1 - \delta \quad \forall L \geq L(T). \tag{39}$$

That is, the probability that there are 2^dR particles at the top of a box $[0,T] \times \{-L,...,L\}^d$ can be made large by choosing some big L and T. We want a similar result for the number of particles leaving through the sides of $[0,T] \times \{-L,...,L\}^d$.

Using (39) and (case 1), we can define two increasing sequences $(L_k)_{k\geq 0}$ and $(T_k)_{k\geq 0}$: We start with $T_0 \coloneqq 1$ and $L_0 \coloneqq L(T_0) + 1$, and for $k \geq 1$ proceed by

$$\begin{aligned} L_{k+1} &\coloneqq \max\Big\{L_k + 1, L(T_k + 1)\Big\} \\ T_{k+1} &\coloneqq \inf\Big\{T > T_k : \mathbb{P}(M^{D_n}(L_{k+1}, T) > 2^dR) < 1 - 2\delta\Big\}. \end{aligned}$$

Note that $T \mapsto \mathbb{P}(M^{D_n}(L,T) > 2^dR)$ is continuous, and therefore

$$\mathbb{P}(M^{D_n}(L_k, T_k) > 2^dR) = 1 - 2\delta. \tag{40}$$

Now use part (ii) of Lemma 2 with the sequences $(L_k)_k$ and $(T_k)_k$ and with K equal to $2^{d+1}R+1$. By the conclusion of Lemma 2 we find k_0 such that for all $k \geq k_0$ we have

$$\mathbb{P}\Big(N^{D_n}(L_k, T_k) + M^{D_n}(L_k, T_k) \leq 2^{d+1}R\Big) \leq 2\mathbb{P}\big(Z^{D_n} \text{ dies out}\big) \leq 2\delta^2.$$

We set $L \coloneqq L_{k_0}$ and $T \coloneqq T_{k_0}$. Then we have

$$\begin{aligned} 2\delta^2 &\geq \mathbb{P}\Big(N^{D_n}(L,T) + M^{D_n}(L,T) \leq 2^{d+1}R\Big) \\ &\geq \mathbb{P}\Big(N^{D_n}(L,T) \leq 2^dR,\ M^{D_n}(L,T) \leq 2^dR\Big) \\ &\geq \mathbb{P}\Big(N^{D_n}(L,T) \leq 2^dR\Big)\mathbb{P}\Big(M^{D_n}(L,T) \leq 2^dR\Big), \end{aligned}$$

where we used the FKG-inequality (25) in the last step. Together with (40) we get

$$\mathbb{P}\big(N^{D_n}(L,T) > 2^dR\big) \geq 1 - \frac{2\delta^2}{\mathbb{P}(M^{D_n}(L,T) \leq 2^dR)} = 1 - \delta. \tag{41}$$

Using again the FKG-inequality we get, for any $\Delta \in \{+,-\}$ and $\theta \in \{1,-1\}^{d-1}$, that

$$\mathbb{P}\big(M^n_\Delta(L,T,\theta) > R\big) \geq 1 - \delta^{2^{-d}} \tag{42}$$

$$\mathbb{P}\big(N^n_\Delta(L,T,\theta) > R\big) \geq 1 - \delta^{2^{-d}}. \tag{43}$$

Remark. *Clearly the probabilities in (43) and (42) do not depend on the choice of θ and Δ, a fact that we will use in the proof of Proposition 5.*

Now we have to verify that L and T indeed satisfy the claim of the proposition, so that we find a shifted copy of D_n in the right location and where every site is occupied by at least one particle of the tree restricted to remain in a large box. This will consist of the following steps:

1. The tree Z^{D_n} has many particles leaving through $\mathbb{F}_+(L,T,\theta)$.
2. There exist $(t,x) \in \mathbb{F}_+(L,T,\theta)$ such that the particle occupying x at time t grows into a fully occupied copy $(t+1, x+ne_1) + D_n$ of D_n.
3. Let $\overline{\mathbb{B}}$ be the box $[0,T] \times \{-L,...,L\}^d + (t+1, x+ne_1)$. Then the tree growing from $(t+1, x+ne_1) + D_n$ has many descendants that leave through its top $\overline{\mathbb{T}}_+(-\theta)$.
4. There is one particle at $(\bar{t}, \bar{x}) \in \overline{\mathbb{T}}(-\theta)$ that grows into a new box $(\bar{t}+1, \bar{x}) + D_n$, which now satisfies the necessary conditions.

Remark. *The choice $-\theta$ in the last step will ensure that no matter where (t,x) is placed, the final copy $(\bar{t}, \bar{x}) + D_n$ is in the right location. The first choice of θ however is still arbitrary, and we will only need it in the proof of Proposition 5.*

First step: We have shown this in (43).

Second step: This will follow from our choice of R. We need to consider the geometry of the set

$$S(L,T,\theta) := \Big\{(t,v)\colon X(t,v) \in \mathbb{F}_+(L,T,\theta), X(s,v) \notin \partial\mathbb{B}\ \forall s < t\Big\}$$

which is the set of space-time-points where a particle leaves $[0,T] \times \{-L,...,L\}^d$ through the orthant $\mathbb{F}_+(L,T,\theta)$ for the first time, with $N_+^n(L,T,\theta) = |S(L,T,\theta)|$. Let I be the (finite) index set

$$I := \Big((\mathbb{N}) \times \{L\} \times (n\mathbb{Z}^{d-1})\Big) \cap \mathbb{F}_+(L,T,\theta).$$

Note that we now have a disjoint tiling

$$\mathbb{F}_+(L,T,\theta) \subseteq \bigcup_{(t_i,x_i)\in I} \big((t_i,x_i) + H\big) \quad \text{with} \quad H := [0,1] \times \{0\} \times \{0,...,n-1\}^{d-1}.$$

On $\{N_+^n(L,T,\theta) > R\}$ at least one of the following statements will be true:

- There exist at least $\sqrt{R}$ distinct indices $(t,x) \in I$ such that

$$S(L,T,\theta) \cap \big((t,x) + H\big) \neq \emptyset. \tag{case A}$$

- There exists $(t_0,x_0) \in I$ such that

$$\Big|S(L,T,\theta) \cap \big((t_0,x_0) + H\big)\Big| \geq \sqrt{R}. \tag{case B}$$

To treat both cases we consider the event $E_{t,v}$ that (t,v) grows into a shifted copy of D_n:

$$E_{t,v} := \Big\{X(t,v) + D_n + ne_1 \subseteq Z^{t,\{X(t,v)\}}_{x+ne_1+D_n}(1)\Big\} \quad \text{for } (t,v) \in S(L,T,\theta).$$

In **(case A)** note that $\sqrt{R} \geq (4n)^d R_1$, so we can find at least R_1 distinct indices (t_1,x_1), ..., $(t_{R_1}, x_{R_1}) \in I$ such that

$$|t_i - t_j| \geq 2 \quad \text{and} \quad \|x_i - x_j\|_\infty \geq 4n \quad \text{holds for all } i \neq j.$$

Choosing now in some deterministic way $(s_i, v_i) \in S(L,T,\theta)$ such that $(s_i, X(s_i,v_i)) \in (t_i, x_i) + H$, we find that because of the truncation the events E_{s_i,v_i} and E_{s_j,v_j} are independent for $i \neq j$. Moreover E_{s_i,v_i} happens with probability at least α, recalling (38). By our choice of R_1 we obtain

$$\mathbb{P}\Big(\bigcup_{(s,v)\in S(L,T,\theta)} E_{(s,v)}\Big) > 1-\delta.$$

In **(case B)** we find $\widetilde{x} \in x_0 + \{L\} \times \{0,...,n-1\}$ such that at least $\frac{\sqrt{R}}{n} \geq R_4$ particles arrive at $[t_0, t_0+1] \times \{\widetilde{x}\}$. Let G be the event that

- at least R_3 of those particles survive until time t_0+1
- while not leaving the set $\{\widetilde{x}, \widetilde{x}+e_1\}$,
- and occupying $\widetilde{x}$ at time t_0+1.

By our choice of R_4 and part (ii) of Lemma 1 we obtain

$$\mathbb{P}(G) \geq \mathbb{P}\Big(\Big|Z^{(R_4)}_{\{0,e_1\}}(1) \cap \{0\}\Big| \geq R_3\Big) \geq 1-\delta.$$

Letting now $\widetilde{G}$ be the event that at time t_0+2 the box $\widetilde{x} + ne_1 + D_n$ is occupied by the descendants of the particles at $(t_0+1, \widetilde{x})$, we find that on G we have

$$\mathbb{P}(\widetilde{G}|G) \geq \mathbb{P}\big(ne_1 + D_n \subseteq Z^{(R_3)}_{ne_1+D_n}(1)\big) \geq 1-\delta$$

by our choice of R_3 and part (iii) of Lemma 1. Now combining the two cases with (43) we find that

$$\mathbb{P}\left(\begin{array}{c} \exists x \in \{L+n\} \times \{-L,...,L\}^{d-1}, t \in [0, T+1] \\ \text{such that } x + D_n \subseteq Z^{D_n}_{\{-L,...,L+2n\}\times\{-L,...,L\}^{d-1}}(t) \end{array}\right) \geq (1-\delta^{2^{-d}})(1-\delta)^2 \tag{44}$$

Third step: We now write $\overline{\mathbb{P}}$ for $\mathbb{P}$ conditioned on the event in (44), and denote the first such pair by (t,x). From now on we consider the process

$$\Big(Z^{t,x+D_n}_{x+\{-L,...,L\}^d}(s)\Big)_{s\geq t}$$

started from $(t, x+D_n)$, which we abbreviate by $\big(\overline{Z}_L(s)\big)_{s\geq t}$. Observe that $\overline{Z}_L$ under the law $\overline{\mathbb{P}}$ is independent of the process up to time t. We denote the new space-time box by

$$\overline{\mathbb{B}} := (t,x) + [0,T] \times \{-L,...,L\}^d.$$

and let $\overline{M}_+(\theta)$ resp. $\overline{M}_-(\theta)$ count the number particles of $\overline{Z}_L$ that leave $\overline{\mathbb{B}}$ through $\overline{\mathbb{T}}_+(\theta)$ resp. $\overline{\mathbb{T}}_-(\theta)$. By (42) we have

$$\overline{\mathbb{P}}(\overline{M}_+(-\theta) \geq R) \geq 1-\delta^{2^{-d}}.$$

Fourth step: On $\{\overline{M}_+(-\theta) \geq R\}$ we want to argue that a suitable copy of the box D_n will be occupied with a high probability, and again one of the following two cases will occur:

$$\Big|\Big\{\overline{x} \in \overline{\mathbb{T}}(-\theta) : \big|\{\overline{x}\} \cap \overline{Z}_L(T)\big| > 0\Big\}\Big| \geq \sqrt{R} \tag{case A'}$$

$$\exists \overline{x_0} \in \overline{\mathbb{T}}(-\theta) \quad \text{such that} \quad \big|\{\overline{x_0}\} \cap \overline{Z}_L(T)\big| \geq \sqrt{R}. \tag{case B'}$$

In **(case A')** we note that $\sqrt{R} \geq (4n)^d R_1$, and thus we find at least R_1 sites $\overline{x}_1, ..., \overline{x}_{R_1}$ with the property that $\|\overline{x}_i - \overline{x}_j\|_\infty \geq 2n+1$ holds for all $i \neq j$. Considering now for $\overline{x} \in \overline{\mathbb{T}}(-\theta)$ the events $\overline{E}_x$ with

$$\overline{E}_{\overline{x}} := \Big\{\overline{x} + D_n \subseteq Z^{t+T,\{\overline{x}\}}_{\overline{x}+D_n}(t+T+1)\Big\}$$

we again find that because of the truncation, $\overline{E}_{\overline{x}_i}$ and $\overline{E}_{\overline{x}_j}$ are independent w.r.t. $\overline{\mathbb{P}}$ for all $i \neq j$. We have $\overline{\mathbb{P}}(\overline{E}_x) \geq \alpha$, so that the definition of R_1 implies

$$\overline{\mathbb{P}}\Big(\bigcup_{i=1}^{R_1} \overline{E}_{\overline{x}_i}\Big) \geq 1 - \delta.$$

Finally, in **(case B')** our choice of R_3 implies that the $\sqrt{R} \geq R_3$ particles at $(t+T, \overline{x})$ will occupy the box $(t+T+1, \overline{x}+D_n)$ with probability at least $1-\delta$. Altogether we showed

$$\overline{\mathbb{P}}\left(\exists \overline{x} \in \overline{\mathbb{T}}(-\theta) \text{ such that } \overline{x} + D_n \subseteq \overline{Z}_L(t+T+1)\right) \geq 1 - \delta. \tag{45}$$

Since $(t+T+1, \overline{x}) \in [T, 2T] \times \{L+n, ..., 2L+n\} \times \{-L, ..., L\}^{d-1}$, the claim is now proved by (44) and (45) together with (36). $\square$

Proof of Proposition 5 in case 1. Set $L' := 2L + n$ and $T' := 2T$. Recall that in the previous proof we chose θ and a sign $\Delta \in \{+,-\}$ and then bounded the probability of the event that

- we find R particles in the orthant $\mathbb{F}_\Delta(\theta)$ in (42).
- starting from those particles, we again find R particles in the orthant $\overline{\mathbb{T}}_\Delta(-\theta)$ of the top of a shifted box in (43).

We now repeatedly apply this result, each time making a convenient choice for θ and Δ.

We start with $\big(s^{(0)}, y^{(0)}\big) := (s, y)$ from the statement of the proposition. Having constructed $\big(s^{(0)}, y^{(0)}\big), ..., \big(s^{(k)}, y^{(k)}\big)$ we choose

$$\theta_{k+1} := -\big(\text{sign } y_2^{(k)}, ..., \text{sign } y_d^{(k)}\big) \in \{-1, 1\}^{d-1}$$

and Δ_{k+1} equal to "+" until the first k with $y_1^{(k)} \geq L' + L$, after which we set $\Delta_{i+1} := -\Delta_i$. By Proposition 4 we know that with probability at least $(1-\varepsilon)$ we find $\big(s^{(k+1)}, y^{(k+1)}\big)$ such that

$$y^{(k+1)} + D_n \subseteq Z^{s^{(k)}, y^{(k)}+D_n}_{y^{(k)}+\{-L,...,3L\}\times\{-L,...,L\}^{d-1}}\big(s^{(k+1)} + s^{(k)}\big)$$

Note that by our choice for θ_k and Δ_k, we have

- $|y_i^{(k)}| \leq 2L \leq L'$ for all $k \geq 0$ and $i = 2, ..., d$.
- $y_1^{(k)} \in \{L', ..., 3L'\}$ eventually: We archive $y_1^{(k)} \geq L' + L$ after at most 4 applications, and by alternating Δ_i for $i \geq k$ we ensure $L' \leq y_1^{(i)} \leq 3L'$ for all $i \geq k$.
- $s^{(i)} \in [5T', ..., 6T']$ for some $i \geq k$: After 4 applications we have $s^{(i)} \in [4T, ..., 8T] = [2T', ..., 4T']$. Since $y^{(i)}$ remains in the target area, we can repeat the procedure until $s^{(i)} \in [5T', ..., 6T']$.

Note that this requires between 4 and 10 applications of the proposition, so we have a success probability of at least $(1-\varepsilon)^{10} \geq 1 - \varepsilon'$. $\square$

4.5.2 Proof in case 2

Proof of Proposition 4 in case 2: Take $L \in 2\mathbb{N}$ large enough for (case 2) to hold and fix some large $t \in \mathbb{N}$. We introduce the two sites

$$z^1 := \Big(L+n, \frac{L}{2}, ..., \frac{L}{2}\Big) \quad \text{and} \quad z^2 := \Big(0, \frac{L}{2}, ..., \frac{L}{2}\Big).$$

In the case $d = 1$ we read this as $z^1 = L + n$ and $z^2 = 0$.

On the event $\{Z_L^{D_n} \text{ survives}\}$ we consider a random sequence $(v_k)_{k\in\mathbb{N}}$ of particles in $\mathbb{N}^*$ by choosing v_k from $Z_L^{D_n}(tk)$ in some deterministic way, say by choosing the minimal element in the lexicographical order. This sequence enables us to make infinitely many attempts to find a fully occupied box at the required position:

For every k, denote by $\big(Z^k(s)\big)_{s\geq tk}$ the process obtained by taking v_k as the new root and considering only its descendants. We define events

$$A_k^1 := \Big\{z^1 + D_n \subseteq Z^k(t(k+1))\Big\} \quad \text{and} \quad A_k^2 := \Big\{z^2 + D_n \subseteq Z^k(t(k+1))\Big\}.$$

We want to bound the probability of A_k^1 and A_k^2 from below and therefore introduce the quantity

$$M(z) := \min_{x\in\{-L,...,L\}^d} \Big\{P_\omega\Big(z + D_n \subseteq Z^{\{x\}}_{\{-L,...,3L\}\times\{-L,...,L\}^{d-1}}(t)\Big)\Big\}.$$

Setting now

$$\alpha := \min\{\mathbb{E}[M(z^1)], \mathbb{E}[M(z^2)]\} > 0 \tag{46}$$

we can choose k large enough for $(1-\alpha)^k \leq \delta$. For the claim observe that with $T := kt$ we get

$$A^1 \subseteq \left\{\begin{matrix} \exists x \in \{L+n, ..., 2L+n\} \times \{-L, ..., L\}^{d-1}, t \in [T, 2T] \\ \text{s.th. } x + D_n \subseteq Z^{D_n}_{\{-L,...,3L\}\times\{-L,...,L\}^{d-1}}(t) \end{matrix}\right\}$$

with $A^1 := \{Z_L^{D_n} \text{ survives}\} \cap \bigcup_{i=k}^{2k} A_i^1$, and that we have

$$\begin{aligned}\mathbb{P}(A^1) &= \mathbb{P}(Z_L^{D_n} \text{ survives}) - \mathbb{P}\Big(Z_L^{D_n} \text{ survives}, \bigcap_{i=k}^{2k} (A_i^1)^c\Big) \\ &\geq 1 - \delta^2 - (1-\alpha)^k \geq 1 - 3\delta \geq 1 - \varepsilon.\end{aligned}$$

□

Proof of Proposition 5 in case 2: For this we choose the same values of L and T, and observe that by symmetry the value of α does not change when we flip the sign of any coordinate in z^1 or z^2. So we choose them in such a way that

$$\text{sign } z_i^j = -\text{sign } y_i \quad \text{for all } i = 2, ..., d \text{ and } j = 1, 2.$$

where y appeared in the statement of the theorem. Now consider a sequence $(z^{(i)})_{i\in\mathbb{N}}$ with

$$z^{(1)} := y + z^1 \quad \text{and} \quad z^{(i)} = y + z^1 + \sum_{j=2}^{i} (-1)^j z^2 \text{ for } i \geq 2.$$

Note that we have chosen the signs in such a way that $z^{(i)} \in \{L+n\} \times \{-L, ..., L\}^{d-1}$ for all i. Let $\widetilde{A}_i^1$ be the same event as A_i^1 with z^1 replaced by $z^{(1)}$, and let $\widetilde{A}$ be defined as A with A_i^1 replaced by $\widetilde{A}_i^1$.

Then on $\widetilde{A}$ we find a minimal $K_1 \in \{k, ..., 2k\}$ such that $\widetilde{A}^1_{K_1}$ holds, so in particular

$$z^{(1)} + D_n \subseteq Z^{s,y+D_n}_{\{-L,...,3L\}\times\{-3L,...,3L\}^{d-1}}(tK_1). \tag{47}$$

We now have to improve (47) so that it holds for some time in $[5T, ..., 6T]$. For this we define the events

$$\widetilde{B}^i := \Big\{\exists\, j \in \{k, ...2k\} \colon z^{(i+1)} + D_n \subseteq Z^{t^{(i)}, z^{(i)}+D_n}_{\{-L,...,3L\}\times\{-3L,...,3L\}^{d-1}}(t^{(i)} + jt)\Big\}.$$

So $\widetilde{B}^i$ is (up to shifts) the same event as A^1 with z^1 replaced by z^2 and started from $z^{(i)} + D_n$ at some time $t^{(i)}$, which we did not specify yet. Note that a similar argument as before allows us bound $\mathbb{P}(\widetilde{B}^i)$ from below by the probability of having at least one success among k independent attempts that succeed with probability at least α. By our choice of α in (46) we now get

$$\mathbb{P}(\widetilde{B}^i) \geq 1 - 3\delta \geq 1 - \varepsilon.$$

Now on $\widetilde{B}^i$ we can proceed by finding a minimal value K_{i+1} such that $z^{(i+1)} + D_n$ is occupied by at time $t^{(i)} + tK_{i+1}$, and starting from $t^{(1)} := K_1 t$ we can define

$$t^{(i+1)} := t^{(i)} + K_{i+1}k \quad \text{ for } i \geq 1$$

which is well defined on $\widetilde{A} \cap \bigcap_{j=2}^{i+1} \widetilde{B}^j$. Since $t^{(i+1)} - t^{(i)} \in [T, ..., 2T]$ we have

$$\widetilde{A} \cap \bigcap_{j=2}^{6} \widetilde{B}_i \subseteq \left\{ \begin{array}{c} \exists x \in \{L, ..., 3L\} \times \{-L, ..., L\}^{d-1}, t \in [5T, 6T] \\ \text{such that } x + D_n \subseteq Z^{s,y+D_n}_{\{-L,...,5L\}\times\{-3L,...,3L\}^{d-1}}(t) \end{array} \right\}.$$

So the claim follows from our choice of ε in (35) and because $P(\widetilde{A} \cap \bigcap_{j=2}^{6} \widetilde{B}_i) \geq (1 - \varepsilon)^6$. $\square$

4.6 Proof of Theorem 1

Proof of Theorem 1. Recall from Section 4.1 that we assume (16) and (15) which we want to lead to a contradiction. We write Q^α for the law of the environment where disasters arrive at rate $\alpha > 0$.

For $(s, y) \in [0, T] \times \{-L, ..., L\}^d$, we now denote the event from Proposition 5 by $A^{s,y}(L, T, n)$, that is

$$A^{s,y}(L, T, n) := \left\{ \begin{array}{c} \exists x \in \{L, ..., 3L\} \times \{-L, ..., L\}^{d-1}, t \in [5T, 6T] \\ \text{such that } x + D_n \subseteq Z^{s,y+D_n}_{\{-5L,...,5L\}\times\{-3L,...,3L\}^{d-1}}(t) \end{array} \right\}$$

Note that $A^{s,y}$ is a local event, i.e. it depends only on the process in some finite space-time box. Therefore the following lemma is easy, and we omit its proof.

Lemma 3. *For every $\varepsilon > 0$ there exists $L, T, n \in \mathbb{N}$ and $\delta > 0$ such that for any $(s, y) \in [0, T] \times \{-L, ..., L\}^d$ we have*

$$\mathbb{P}^{1+\delta,\kappa,\lambda}(A^{s,y}(L, T, n)) \geq 1 - \varepsilon\,. \tag{48}$$

Recall that τ is the extinction time of a single particle, and that we denote the survival rate by

$$p(\alpha, \kappa) = \lim_{t\to\infty} \frac{1}{t} \int \log P_\omega(\tau \geq t)\, \mathrm{d}Q^\alpha.$$

We claim now that

$$\alpha \mapsto p(\alpha, \kappa) \text{ is strictly decreasing for any } \kappa > 0. \tag{49}$$

Proof of (49). Let ω_α and ω_β be independent environments of disaster rates α and β, respectively. We write τ_α and τ_β for the extinction time in ω_α and ω_β and get, using (3), that

$$\begin{aligned}
p(\alpha+\beta,\kappa) &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\big[\log P_{\omega_\alpha+\omega_\beta}(\tau_\alpha \wedge \tau_\beta \geq t)\big] \\
&= p(\alpha,\kappa) + \lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log P_{\omega_\alpha+\omega_\beta}(\tau_\beta \geq t|\tau_\alpha \geq t)] \\
&\leq p(\alpha,\kappa) + \lim_{t\to\infty} \frac{1}{t}\log \mathbb{E}\big[P_{\omega_\alpha+\omega_\beta}(\tau_\beta \geq t|\tau_\alpha \geq t)\big] = p(\alpha,\kappa) - \beta,
\end{aligned}$$

where for the last line we used Jensen's inequality. □

The theorem now follows from

Proposition 6. *There exists some $\varepsilon > 0$ such the conclusion of Lemma 3 implies*

$$\mathbb{P}^{1+\delta,\kappa,\lambda}(\ Z^{D_n} \textit{ survives }) > 0,$$

where $\delta > 0$ is the value given by Lemma 3.

For the proof, observe that Lemma 3 rephrases Proposition 2.22 in [7] about the contact process to the context of our model. The same argument as in Theorem 2.23 in [7] proves this proposition.

Now however since we assume (15), and because $\alpha \mapsto p(\alpha,\kappa)$ is strictly decreasing, we must have

$$\lambda(m-1) + p(1+\delta,\kappa) < 0 \qquad \text{for any } \delta > 0.$$

By the same arguments as in the subcritical case we must therefore have

$$\mathbb{P}^{1+\delta,\kappa,\lambda}(Z \text{ survives }) = 0 \qquad \text{for all } \delta > 0$$

which is a contradiction to Proposition 6, so that (16) cannot hold. □

5 Appendix: Proof of Proposition 2

In this section we give a proof for Proposition 2 using techniques which are well established in the field of random polymers and not related to those in the proof of the main result. The following uniform moment bound is key to proving a concentration inequality:

Lemma 4. *For every $\delta \in (0,1)$ there is some $C > 0$ such that*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E}\Big[P_\omega(\tau \geq 1|X(1) = x)^{-\delta}\Big] < C < \infty.$$

For the proof we use the equivalence relation $\equiv$ on $\mathbb{Z}^d$ defined by

$$(y_1,\dots,y_d) \equiv (z_1,\dots,z_d) \qquad \Longleftrightarrow \qquad y_1 = z_1 \mod 2.$$

We will identify $\mathbb{Z}^d/_\equiv$ with $\mathbb{Z}_2 = \{0,1\}$, and we use $\pi\colon \mathbb{Z}^d \to \{0,1\}$ to denote the projection map. Let $\widetilde{\omega}$ be an environment on $\{0,1\}$, consisting as usual of two independent Poisson processes $\widetilde{\omega}^{(0)}$ and $\widetilde{\omega}^{(1)}$ of rate 1. We write $\pi^{-1}(\widetilde{\omega})$ for the environment on $\mathbb{Z}^d$ given by

$$(\pi^{-1}(\widetilde{\omega}))^{(y)} = \omega^{(\pi(y))} \qquad \text{for } y \in \mathbb{Z}^d.$$

Note that this is a degenerate environment on $\mathbb{Z}^d$, where all sites that share an equivalence class in $\equiv$ experience the same disasters. We will slightly abuse notation by writing $\mathbb{E}$ for the law of $\widetilde{\omega}$ as well.

Lemma 5. *Let $(\widetilde{Y}(t))_{t\in[0,1]}$ be simple random walk on $\{0,1\}$ of jump rate κ. Then for any $\delta \in (0,1)$ we have*

$$\sup_{i=0,1} \mathbb{E}\big[P^{\kappa}_{\widetilde{\omega}}\big(\tau \geq 1, \widetilde{Y}(1) = i\big)^{-\delta}\big] < \infty.$$

Proof. The proof is a modification of the proof of Lemma 2.4 in [9], where the integrability of $P_{\widetilde{\omega}}(\tau \geq 1)^{-\delta}$ is proven. □

Proof of Lemma 4. By Lemma 2.2 in [9] we have

$$\mathbb{E}\big[(P^{\kappa}_{\omega}(\tau \geq 1, X(1) = x))^{-\delta}\big] \leq \mathbb{E}\big[(P^{\kappa}_{\pi^{-1}(\widetilde{\omega})}(\tau \geq 1, X(1) = x))^{-\delta}\big]$$

and dividing both sides by $\big(P^{\kappa}(X(1) = x)\big)^{-\delta}$ gives

$$\mathbb{E}\big[(P^{\kappa}_{\omega}(\tau \geq 1|X(1) = x))^{-\delta}\big] \leq \mathbb{E}\big[(P^{\kappa}_{\pi^{-1}(\widetilde{\omega})}(\tau \geq 1|X(1) = x))^{-\delta}\big]$$

Moreover by Lemma 5 we have

$$\sup_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\big(P^{\frac{\kappa}{2}}_{\pi^{-1}(\widetilde{\omega})}(\tau \geq 1|Y(1) \equiv x)\big)^{-\delta}\Big] < \infty.$$

So the lemma is proven once we show that

$$\mathbb{E}\Big[\big(P^{\kappa}_{\pi^{-1}(\widetilde{\omega})}(\tau \geq 1|X(1) = x)\big)^{-\delta}\Big] \leq \mathbb{E}\Big[\big(P^{\frac{\kappa}{2}}_{\pi^{-1}(\widetilde{\omega})}(\tau \geq 1|Y(1) \equiv x)\big)^{-\delta}\Big] \tag{50}$$

For simplicity we only treat the case where $x \equiv 0$, noting that the case $x \equiv (1, 0, ..., 0)$ is similar. For a fixed environment $\widetilde{\omega}$, let N be the number of disasters in $[0, 1]$. We write $T_1, ..., T_N$ for the disaster times in increasing order, and $E_1, ..., E_N$ for their locations. One can define a configuration $I(\widetilde{\omega}) = \big(I(\widetilde{\omega})(i)\big)_{i=0}^{N}$ by

$$I(\widetilde{\omega}) = \big(\mathbb{1}\{E_1 = 0\}, \mathbb{1}\{E_2 \neq E_1\}, ..., \mathbb{1}\{E_N \neq E_{N-1}\}, \mathbb{1}\{E_N = 0\}\big) \in \{0,1\}^{N+1}.$$

The intuition is that if $\{I(\widetilde{\omega})(i) = 1\}$ for some $i \in \{0, ..., N\}$, the process has to switch sites in $[T_i, T_{i+1})$ in order to survive and end up in a location equivalent to 0.

Let Ω be the set of all I of this form having an even number of ones, and note that there is a one-to-one correspondence between $(E_1, ..., E_N) \in \{0,1\}^N$ and Ω. We can therefore sample $(E_1, ..., E_N)$ by drawing $I(\widetilde{\omega})$ uniformly from Ω.

On the other hand, we define for a càdlàg process Z on $\mathbb{Z}^d$ a random variable

$$\mathcal{I}^Z := \Big(\pi\big(Z(T_1)\big), \pi\big(Z(T_2) - Z(T_1)\big), ..., \pi\big(Z(T_N) - Z(T_{N-1})\big), \pi\big(Z(1) - Z(T_N)\big)\Big).$$

Notice that $P(\mathcal{I}^Z \in \Omega) = P(Z(1) \equiv 0)$, and therefore we can define two probability measures μ and ν on Ω by setting

$$\mu(I) := P^{\kappa}\big(\mathcal{I}^X = I\big|X(1) = x\big) \quad \text{and} \quad \nu(I) := P^{\frac{\kappa}{2}}\big(\mathcal{I}^Y = I\big|Y(1) \equiv x\big) \tag{51}$$

Our new notation now lets us write

$$P^{\kappa}_{\pi^{-1}(\widetilde{\omega})}(\tau_X \geq 1|X(1) = x) = \mu(I(\widetilde{\omega})) \quad \text{and} \quad P^{\frac{\kappa}{2}}_{\pi^{-1}(\widetilde{\omega})}(\tau_Y \geq 1|X(1) \equiv x) = \nu(I(\widetilde{\omega})).$$

Recall at this point that for two probability measures on Ω we say that μ is majorized by ν, denoted $\mu \preceq_M \nu$, if

$$\sum_{i=1}^{k} \mu(a_i) \leq \sum_{i=1}^{k} \nu(b_i) \quad \text{for all } k = 1, ..., 2^N,$$

where $\Omega = \{a_1, ..., a_{2^N}\} = \{b_1, ..., b_{2^N}\}$, and the ordering is chosen in such a way that

$$\mu(a_1) \geq ... \geq \mu(a_{2^N}) \quad \text{and} \quad \nu(b_1) \geq ... \geq \nu(b_{2^N}).$$

For the conclusion, let us write $\mathbb{P}^{T_1,...,T_N}$ resp. $\mathbb{E}^{T_1,...,T_N}$ for the law resp. expectation of $E_1, ..., E_N$ conditioned on N and $T_1, ..., T_N$. We have already argued before that $I(\widetilde{\omega})$ under $\mathbb{P}^{T_1,...,T_N}$ is uniformly distributed on Ω. In Lemma 7 we show that $\mu \preceq_M \nu$ holds, so that Corollary 1.5.37 in [8] implies

$$\mathbb{E}^{T_1,...,T_N}\big[f\big(\mu(I(\widetilde{\omega}))\big)\big] \leq \mathbb{E}^{T_1,...,T_N}\big[f\big(\nu(I(\widetilde{\omega}))\big)\big]$$

for all convex functions $f : (0, 1] \to \mathbb{R}$. Inserting $f : x \mapsto x^{-\delta}$, this in particular shows (50) by taking expectations. $\square$

For a càdlàg process Z on $\mathbb{Z}^d$, let us call t a jump time of Z if $Z(t) \not\equiv Z(t^-)$, and write R_Z for the number of jumps times of Z in $[0, 1]$.

Lemma 6. *Let X resp. Y be simple random walks on $\mathbb{Z}^d$ with jump rate κ resp. $\frac{\kappa}{2}$. Then*

$$R_Y\big|\{Y(1) \equiv x\} \quad \preceq_{st} \quad R_X\big|\{X(1) = x\}$$

where $\preceq_{st}$ denotes stochastic domination.

Proof. It is easier to show that

$$R_Y\big|\{Y(1) \equiv x\} \quad \preceq_{lr} \quad R_X\big|\{X(1) = x\}$$

where $\preceq_{lr}$ denotes domination in the likelihood ratio order, see for example chapter 1.4 in [8], where it is also shown that $\preceq_{lr}$ is stronger than $\preceq_{st}$.

We have to check that for $k, l \in \mathbb{N}$ of the same parity as x_1 and such that $|x_1| \leq k \leq l$, the following holds:

$$P^\kappa(R_X = k|X(1) = x)P^{\frac{\kappa}{2}}(R_Y = l|Y(1) \equiv x) \leq P^\kappa(R_X = l|X(1) = x)P^{\frac{\kappa}{2}}(R_Y = k|Y(1) \equiv x)$$

We apply the definition of conditional probability and cancel the terms that appear on both sides, and can rewrite the equation as

$$\frac{P(Z_k = x_1)}{P(Z_l = x_1)} \leq \frac{P(A = l)P(A' = k)}{P(A = k)P(A' = l)} = 2^{l-k}.$$

Here $(Z_i)_{i\in\mathbb{N}}$ is a simple random walk on $\mathbb{Z}$ in discrete time, and A resp. A' is a Poisson random variable of parameter $\frac{\kappa}{d}$ resp. $\frac{\kappa}{2d}$. But this inequality holds, since by the Markov Property

$$P(Z_l = x_1) \geq P(Z_k = x_1)P(Z_{l-k} = 0) \geq P(Z_k = x_1)2^{-(l-k)}.$$

$\square$

Lemma 7. *Let μ and ν be defined as in* (51)*. Then $\mu \preceq_M \nu$.*

Proof. Let us define weights $p_0, ..., p_N$ by $p_0 := T_1$, $p_N := 1 - T_N$ and $p_i := T_{i+1} - T_i$ for all other values of i. We note that μ and ν do not depend on the order of $T_1, ..., T_N$, and therefore we can rearrange them to satisfy

$$p_0 \leq p_1 \leq ... \leq p_N. \tag{52}$$

Now for $k \in \mathbb{N}$, let $M_k = (M_k(0), ..., M_k(N))$ denote a random variable having the multinomial distribution with k trials, and write P_k for its law. That is, k indistinguishable balls are thrown in bins numbered $0, ..., N$ such that each balls independently lands in bin i with probability p_i, and $M_k(i)$ is the final number of balls in bin i. We define

$$\mathcal{I}_k := \big(\mathbb{1}\{M_k(0) \text{ is odd}\}, ..., \mathbb{1}\{M_k(N) \text{ is odd}\}\big) \in \{0,1\}^{N+1}. \tag{53}$$

We will often use $\mathcal{I}_k$ interchangeably with the set $\{i : M_k(i) \text{ is odd}\} \subseteq [\![N]\!]$, where $[\![N]\!] = \{0, ..., N\}$. Observe that

$$\mu(I) = E[P_L(\mathcal{I}_L = I)] \quad \text{and} \quad \nu(I) = E[P_K(\mathcal{I}_K = I)]$$

where K and L are random variables with

$$P(L = l) = P^{\kappa}(R_X = l | X(1) = x) \quad \text{and} \quad P(K = k) = P^{\frac{\kappa}{2}}(R_Y = k | Y(1) \equiv x).$$

Indeed, conditional on a random walk having K jumps in $[0, 1]$, each jumps occurs in $[T_i, T_{i+1})$ with probability p_i, independently of the other jump times, and the process switches sites between T_i and T_{i+1} exactly if there is an odd number of jumps in $[T_i, T_{i+1})$.

Now it is easy to see that $P_l(\mathcal{I}_l \in \cdot) \preceq_M P_k(\mathcal{I}_k \in \cdot)$ holds for any $k \le l$: The distribution of $\mathcal{I}_l$ can be obtained from the distribution of $\mathcal{I}_k$ by the application of a doubly stochastic matrix, and this is an equivalent characterization of $\preceq_M$, see for example Theorem 1.5.34 in [8]. Moreover from Lemma 6 we know that there exists a coupling between K and L such that $K \le L$ holds with probability one. Unfortunately this is not enough to conclude $\mu \preceq_M \nu$, and more work is necessary.

For this we define a partial order $\preceq$ on Ω by

$$(x_0, ..., x_N) \preceq (y_0, ..., y_N) \quad \iff \quad \sum_{i=0}^{k} x_i \le \sum_{i=0}^{k} y_i \quad \text{for all } k = 0, ..., N.$$

Some properties of $\preceq$ and of $P_k(\mathcal{I}_k \in \cdot)$ are collected in Lemma 8 below.

We see from (54) and the fact that K and L are supported on the even numbers, that both μ and ν are decreasing in $\preceq$. Moreover we can couple K and L such that $K \le L$ holds with probability one, so from (55) we get for all decreasing sets A that

$$\mu(A) \le \nu(A).$$

We have checked conditions (1) and (2) from [1], and $\mu \preceq_M \nu$ now follows from Theorem 3 in that work. □

Lemma 8. *(i) $P_{2k}(\mathcal{I}_{2k} \in \cdot)$ is decreasing in $\preceq$. That is, for all $I, J \in \Omega$ we have*

$$I \preceq J \quad \implies \quad P_{2k}(\mathcal{I}_{2k} = I) \ge P_{2k}(\mathcal{I}_{2k} = J). \tag{54}$$

(ii) Let $A \subseteq \mathcal{P}(\Omega)$ be a decreasing set, i.e. $J \in A$ implies $I \in A$ for all I with $I \preceq J$. Then

$$P_{2k+2}(\mathcal{I}_{2k+2} \in A) \le P_{2k}(\mathcal{I}_{2k} \in A). \tag{55}$$

Proof. For $S \subseteq [\![N]\!]$ we write $M_k(S) := \sum_{i \in S} M_k(i)$. We recall the following fact about a binomial random variable $B_{n,p}$ with n trials and success probability p:

$$P(B_{n,p} \text{ is even}) = \frac{1}{2}\big(1 + (1 - 2p)^n\big). \tag{56}$$

Part (i): For $S, T \subseteq [\![N]\!]$ disjoint, we consider the function

$$f_T^S(r) := P\big(M_k(i) \text{ is even } \forall i \in S, M_k(j) \text{ is odd } \forall j \in T \big| M_k(S \cup T) = r\big).$$

Whenever S or T is the empty set, we drop it from the notation and just write f_T or f^S. We first show (54) in two special cases:

Assume that $I \subseteq J$, with $J \setminus I =: \{a_1, ..., a_{2m}\}$. Letting A be the event $A := \{I \subseteq \mathcal{I}_{2k} \subseteq J\}$ and setting $S_j := \{a_{2j-1}, a_{2j}\}$, we have

$$P_{2k}(\mathcal{I}_{2k} = I) = P_{2k}(A) E\Big[\prod_{j=1}^{m} f^{S_j}(M_{2k}(S_j)) \Big| A\Big].$$

Clearly $f^{S_j}(m)$ is only positive if m is even, and in this case $f^{S_j}(m) \geq f_{S_j}(m)$ follows from (56). But this means

$$P_{2k}(\mathcal{I}_{2k} = J) = P_{2k}(A) E\Big[\prod_{i=1}^{m} f_{S_j}(M_{2k}(S_j)) \Big| A\Big] \leq P_{2k}(\mathcal{I}_{2k} = I).$$

Next we assume that $|I| = |J|$ and that I and J only differ in one coordinate, that is $I = I_0 \cup \{a\}$ and $J = I_0 \cup \{b\}$ for some $b < a$. Let B be the event $B := \big\{I_0 \subseteq \mathcal{I}_{2k} \subseteq I_0 \cup \{a, b\}\big\}$. Then

$$\begin{aligned} P_{2k}(\mathcal{I}_{2k} = I) =& P(B) E\Big[f_a^b\big(M_{2k}(\{a,b\})\big) \Big| B\Big] \\ \geq& P(B) E\Big[f_b^a\big(M_{2k}(\{a,b\})\big) \Big| B\Big] = P_{2k}(\mathcal{I}_{2k} = J). \end{aligned}$$

For the inequality we have used that for m odd we have

$$f_a^b(m) = P(B_{m,p} \text{ is even}) \geq P(B_{m,p} \text{ is odd}) = f_b^a(m)$$

for $p = \frac{p_b}{p_a + p_b}$, noting that $a > b$ and (52) imply $p \leq \frac{1}{2}$.

Now the general case follows from the observation that for any $I \preceq J$ we can find $I_0 \preceq ... \preceq I_r$ such that $I_0 = I$ and $I_r \subseteq J$, and with the property that I_{i+1} and I_i only differ in one coordinate.

Part (ii): We do this by constructing a coupling $(\mathcal{I}_{2k}, \mathcal{I}_{2k+2})$ with the property that $\mathcal{I}_{2k} \preceq \mathcal{I}_{2k+2}$ holds with probability one, which clearly implies (55).

For this, let (B, A) be chosen from $\{0 \leq b \leq a \leq N\}$ such that

$$P\big((B, A) = (b, a)\big) = 2p_a p_b \mathbb{1}\{b < a\} + p_a^2 \mathbb{1}\{a = b\}$$

and let M be independently sampled with the multinomial distribution with $2k$ trials. On the event $\{A = B\}$ we define $\mathcal{I}_{2k}$ from M according to the definition, and set $\mathcal{I}_{2k+2}$ equal to $\mathcal{I}_{2k}$.

In the case where $B < A$, we first fix the coupling on $[\![N]\!] \setminus \{A, B\}$ by

$$\mathcal{I}_l(i) := \mathbb{1}\{M(i) \text{ is odd}\} \quad \text{ for } i \notin \{A, B\} \text{ and } l \in \{2k, 2k+2\}.$$

We set $R := M(A) + M(B)$ and $p := \frac{p_B}{p_A + p_B}$, and introduce an independent random variable U distributed uniformly in $[0, 1]$, so that we can define

$$\begin{aligned} \mathcal{I}_{2k}(B) &:= \mathbb{1}\{U \leq P(B_{R,p} \text{ is odd})\} \\ \mathcal{I}_{2k+2}(B) &:= \mathbb{1}\{U \leq P(B_{R,p} \text{ is even})\} \end{aligned}$$

Finally, for l equal to $2k$ or $2k+2$, we set

$$\mathcal{I}_l(A) := R - \mathcal{I}_l(B) \mod (2). \tag{57}$$

We claim that this is indeed the desired coupling. First note that we can sample a realization of the multinomial distribution M_{2k+2} with $2k+2$ trials by sampling M together with two additional balls A and B as described above. If the extra balls end up in the same bin, then the parity of all coordinates of M and M_{2k+2} will agree, and we can take $\mathcal{I}_{2k} = \mathcal{I}_{2k+2}$.

Otherwise adding A and B will flip the parity of $M(A)$ and $M(B)$. So conditionally on $\{M(A)+M(B)=R\}$ we have sampled $\mathcal{I}_{2k}(B)$ and $\mathcal{I}_{2k+2}(B)$ with the correct laws, which then forces us to choose $\mathcal{I}_{2k}(A)$ and $\mathcal{I}_{2k+2}(A)$ as in (57).

But now (52) and $B<A$ imply $p = \frac{p_B}{p_A+p_B} \leq \frac{1}{2}$, so from (56) we obtain

$$P(B_{R,p} \text{ is odd}) = \frac{1}{2} - \frac{1}{2}(1-2p)^R \leq \frac{1}{2} \leq P(B_{R,p} \text{ is even}).$$

Therefore $\mathcal{I}_{2k}(B) \leq \mathcal{I}_{2k+2}(B)$, which implies $\mathcal{I}_{2k} \preceq \mathcal{I}_{2k+2}$.

More precisely, if R is even we have

$$(\mathcal{I}_{2k}(B), \mathcal{I}_{2k}(A)) = (1,1) \implies (\mathcal{I}_{2k+2}(B), \mathcal{I}_{2k+2}(A)) = (1,1)$$

so that $\mathcal{I}_{2k} \subseteq \mathcal{I}_{2k+2}$ with probability one. If R is odd, we note that

$$(\mathcal{I}_{2k}(B), \mathcal{I}_{2k}(A)) = (1,0) \quad\implies\quad (\mathcal{I}_{2k+2}(B), \mathcal{I}_{2k+2}(A)) = (1,0)$$

while on $\{(\mathcal{I}_{2k}(B), \mathcal{I}_{2k}(A)) = (0,1)\}$ we can have either $\mathcal{I}_{2k+2} = \mathcal{I}_{2k}$ or $(\mathcal{I}_{2k+2}(B), \mathcal{I}_{2k+2}(A)) = (1,0)$. In the second case $\mathcal{I}_{2k} \prec \mathcal{I}_{2k+2}$ holds. □

We write

$$S(t,x) \coloneqq P_\omega(\tau \geq t, X(t) = x).$$

With the previous moment bound at hand, we can now proceed to prove a concentration inequality for the sequences $\big(S(t,x)\big)_{t\geq 0}$ where the bounds do not depend on x. We follow the proof of Proposition 3.2.1 in [4].

Proposition 7. *There exist $c>0$ and $C>0$, such that $\varepsilon \in (0,c)$ implies*

$$Q\Big(\big|\log S(t,x) - \mathbb{E}[\log S(t,x)]\big| > \varepsilon t\Big) \leq 2\exp(-C\varepsilon^2 t) \tag{58}$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{Z}^d$.

Proof. For simplicity we assume $t \in \mathbb{N}$. We will drop the dependence on t and x in the notation, and only write S for $S(t,x)$. Let ω_i be the environment that contains all disasters (t,y) of ω except for those with $t \in [i-1,i)$. We now consider the filtration $\big(\mathcal{F}_i\big)_{i=0}^t$ with

$$\mathcal{F}_i \coloneqq \sigma\big(\omega^{(y)}(s)\colon s<i, y \in \mathbb{Z}^d\big)$$

and the random variables $\big(S_i\big)_{i=1}^t$ given by

$$S_i \coloneqq P_{\omega_i}(\tau \geq t, X(t) = x).$$

Notice that $\mathbb{E}[S_i|\mathcal{F}_i] = \mathbb{E}[S_i|\mathcal{F}_{i-1}]$. Now by Lemma 5.2.1 in [4], we obtain

$$Q\big(\big|\log S - \mathbb{E}[\log S]\big| > \varepsilon t\big) \leq 2\exp(-C\varepsilon^2 t)$$

for some explicit constant $C>0$ once we have shown that

$$\mathbb{E}\Big[e^{\delta|\log S - \log S_i|}\Big|\mathcal{F}_{i-1}\Big] \leq A$$

holds for some $\delta > 0$ and for some $A > 0$ not depending on i, t or x. In this case $S \leq S_i$, and therefore

$$e^{\delta|\log S - \log S_i|} = \Big(\frac{S}{S_i}\Big)^{-\delta} = \Big(\sum_{y,z} \alpha_{y,z}\eta_{y,z}\Big)^{-\delta}$$

where

$$\alpha_{y,z} \coloneqq P_{\omega_i}\big(X(i-1) = y, X(i) = z \big| \tau \geq t\big)$$

and

$$\eta_{y,z} \coloneqq P_\omega^{(i-1,y),(i,z)}\big(\tau \geq i\big)$$

Here $P^{(r,y),(s,z)}$ is the law of a random walk starting at time r in y and conditioned to end up in z at time s. To compute the expectation, consider the sigma algebra

$$\mathcal{F}^* \coloneqq \sigma\big(\omega_i^{(y)}(s) \colon s < t, y \in \mathbb{Z}^d\big)$$

From our choice of ω_i we clearly have $\mathcal{F}_{i-1} \subseteq \mathcal{F}^*$, and $\eta_{y,z}$ is independent of $\mathcal{F}^*$ while $\alpha_{y,z}$ is $\mathcal{F}^*$ measurable. So using Jensen's inequality we obtain

$$\mathbb{E}\Big[\Big(\frac{S}{S_i}\Big)^{-\delta}\Big|\mathcal{F}^*\Big] \leq \sum_{y,z} \alpha_{y,z}\mathbb{E}\big[\eta_{y,z}^{-\delta}\big] = \sum_{y,z} \alpha_{y,z}\mathbb{E}\Big[P_\omega(\tau \geq 1 | X(1) = x - y)^{-\delta}\Big]$$

By Lemma 4 we have

$$\sup_{y,z} \mathbb{E}\Big[P_\omega\big(\tau \geq 1 \big| X(1) = y - z\big)^{-\delta}\Big] = c < \infty$$

and therefore

$$\mathbb{E}\Big[\Big(\frac{S}{S_i}\Big)^{-\delta}\Big|\mathcal{F}_{i-1}\Big] \leq c\,\mathbb{E}\Big[\sum_{y,z} \alpha_{y,z}\Big|\mathcal{F}_{i-1}\Big] = c.$$

□

Equipped with this concentration inequality we can now prove Proposition 2. We follow the proof of Proposition 2.4 in [2].

Proof of Proposition 2. For (12) we again assume $t \in \mathbb{N}$, so that

$$\mathbb{E}[\log \widetilde{S}(t)] \geq t \log P(X(1) = 0) + t\,\mathbb{E}[\log P_\omega(\tau \geq 1 | X(1) = 0)].$$

Then for all $\delta \in (0, 1)$ we have

$$\begin{aligned}\mathbb{E}[\log P_\omega(\tau \geq 1 | X(1) = 0)] &= -\frac{1}{\delta}\mathbb{E}\big[\log\big(P_\omega(\tau \geq 1 | X(1) = 0)^{-\delta}\big)\big] \\ &\geq -\frac{1}{\delta}\log \mathbb{E}\big[P_\omega(\tau \geq 1 | X(1) = 0)^{-\delta}\big] > -\infty,\end{aligned}$$

where we used Jensen's inequality, and the integrability follows from Lemma 4.

From (13) and the concentration inequality (58) we obtain (14) by a simple Borel-Cantelli argument. Now to prove (13), we remark that the existence of the limit

$$\widetilde{p}(\kappa) = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log \widetilde{S}(t)]$$

can be shown by subadditivity as usual, but this is not even necessary for our claim. Clearly we have

$$\limsup_{t\to\infty} \frac{1}{t}\mathbb{E}[\log \widetilde{S}(t)] \leq p(\kappa).$$

We now prove the other direction. Note that for any $x \in \mathbb{Z}^d$ we have

$$P_\omega(\tau \geq 2t, X(2t) = 0) \geq P_\omega(\tau \geq t, X(t) = x) P^{t,x}(\tau \geq 2t, X(2t) = 0)$$

Since $P^{t,x}(\tau \geq 2t, X(2t) = 0)$ has the same law as $P_\omega(\tau \geq t, X(t) = x)$, we conclude that

$$\mathbb{E}[\log S(2t, 0)] \geq 2\mathbb{E}[\log S(t, x)]. \tag{59}$$

For $\gamma > 0$ we consider a box $B_t := \{x \in \mathbb{Z}^d \colon \|x\| \leq \gamma t\}$ and the event $A_t := \{X(t) \in B_t\}$. Using standard large deviation techniques, we can choose γ large enough such that

$$\log P(A_t^c) < t p(\kappa) \quad \forall t \geq t_0.$$

Consequently we have

$$p(\kappa) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\log P(\tau \geq t)] = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\log P(\tau \geq t, A_t)]\,. \tag{60}$$

Take now $\varepsilon := t^{-\frac{3}{4}}$ and apply the fractional moments method:

$$\begin{aligned}
\mathbb{E}[\log P_\omega(\tau \geq t, A_t)] &= \frac{1}{\varepsilon} \mathbb{E}\big[\log\big(P_\omega(\tau \geq t, A_t)^\varepsilon\big)\big] \\
&\leq \frac{1}{\varepsilon} \log \mathbb{E}\big[P_\omega(\tau \geq t, A_t)^\varepsilon\big] = \frac{1}{\varepsilon} \log \mathbb{E}\Big[\Big(\sum_{x\in B_t} S(t,x)\Big)^\varepsilon\Big] && (61) \\
&\leq \frac{1}{\varepsilon} \log \mathbb{E}\Big[\sum_{x\in B_t} S(t,x)^\varepsilon\Big] && (62) \\
&= \frac{1}{\varepsilon} \log \sum_{x\in B_t} \mathbb{E}\big[e^{\varepsilon(\log S(t,x) - \mathbb{E}[\log S(t,x)])}\big] e^{\varepsilon \mathbb{E}[\log S(t,x)]}, && (63)
\end{aligned}$$

where we get (61) from Jensen's inequality, and the inequality in (62) comes from the general estimate $\left(\sum_{j=1}^N a_j\right)^\varepsilon \leq \sum_{j=1}^N a_j^\varepsilon$ for nonnegative $a_1, \ldots, a_N$ and $0 < \varepsilon < 1$. For the left factor of the summands in (63) we compute, using (58),

$$\begin{aligned}
&\mathbb{E}\Big[\exp\big(\varepsilon(\log S(t,x) - \mathbb{E}[\log S(t,x)])\big)\Big] \\
&\leq 1 + \int_1^\infty Q\Big(\big|\log S(t,x) - \mathbb{E}[\log S(t,x)]\big| > t^{\frac{3}{4}} \log u\Big)\, \mathrm{d}u \\
&\leq 1 + 2\int_1^\infty e^{-Ct^{\frac{1}{2}}(\log u)^2}\, \mathrm{d}u := c(t).
\end{aligned}$$

Then we are left with

$$\begin{aligned}
\mathbb{E}[\log P_\omega(\tau \geq t, A_t)] &\leq \frac{1}{\varepsilon} \log c(t) + \frac{1}{\varepsilon} \log \sum_{x\in B_t} e^{\varepsilon \mathbb{E}[\log S(t,x)]} \\
&\leq \frac{1}{\varepsilon} \log c(t) + \frac{1}{\varepsilon} \log |B_t| + \frac{1}{2} \mathbb{E}[\log S(2t, 0)],
\end{aligned}$$

where we have used (59). Dividing by t and taking limits, taking into account (60), we obtain

$$\liminf_{t\to\infty} \frac{1}{2t} \mathbb{E}[\log S(2t, 0)] \geq p(\kappa).$$

□

References

[1] Richard A. Brualdi and Geir Dahl. Majorization for partially ordered sets. *Discrete Mathematics*, 313(22):2592 – 2601, 2013.

[2] Philippe Carmona and Yueyun Hu. Fluctuation exponents and large deviations for directed polymers in a random environment. *Stochastic Processes and their Applications*, 112(2):285–308, 2004.

[3] Francis Comets and Nobuo Yoshida. Directed polymers in random environment are diffusive at weak disorder. *Annals of Probability*, 34(5):1746–1770, 09 2006.

[4] Francis Comets and Nobuo Yoshida. Branching random walks in space–time random environment: Survival probability, global and local growth rates. *Journal of Theoretical Probability*, 24(3):657–687, 2011.

[5] Tasuku Furuoya and Tokuzo Shiga. Sample Lyapunov exponent for a class of linear Markovian systems over $\mathbb{Z}^d$. *Osaka Journal of Mathematics*, 35:35–72, 1998.

[6] Olivier Garet and Régine Marchand. The critical branching random walk in a random environment dies out. *Electronic Communications in Probabilitys*, 18:no. 9, 1–15, 2013.

[7] Thomas M. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1999.

[8] A. Müller and D. Stoyan. *Comparison Methods for Stochastic Models and Risks.* Wiley Series in Probability and Statistics. Wiley, 2002.

[9] Tokuzo Shiga. Exponential decay rate of survival probability in a disastrous random environment. *Probability Theory and Related Fields*, 108(3):417–439, 1997.

[10] Walter L. Smith. Necessary conditions for almost sure extinction of a branching process with random environment. *The Annals of Mathematical Statistics*, 39:2136–2140, 1968.

[11] David Tanny. Limit theorems for branching processes in a random environment. *Annals of Probability*, 5(1):100–116, 1977.

Authors addresses:
Nina Gantert, Stefan Junk
Fakultät für Mathematik
Technische Universität München
85748 Garching, Germany
email: gantert@ma.tum.de, junk@tum.de