New Journal of Physics
The open access journal at the forefront of physics
Deutsche Physikalische Gesellschaft DPG | IOP Institute of Physics

Controlling several atoms in a cavity

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Received 13 January 2014
Accepted for publication 21 March 2014
Published 17 June 2014
New Journal of Physics **16** (2014) 065010
doi:10.1088/1367-2630/16/6/065010

Abstract
We treat control of several two-level atoms interacting with one mode of the electromagnetic field in a cavity. This provides a useful model to study pertinent aspects of quantum control in infinite dimensions via the emergence of infinite-dimensional system algebras. Hence we address problems arising with infinite-dimensional Lie algebras and those of unbounded operators. For the models considered, these problems can be solved by splitting the set of control Hamiltonians into two subsets: the first obeys an Abelian symmetry and can be treated in terms of infinite-dimensional Lie algebras and strongly closed subgroups of the unitary group of the system Hilbert space. The second breaks this symmetry, and its discussion introduces new arguments. Yet, full controllability can be achieved in a strong sense: e.g., in a time dependent Jaynes–Cummings model we show that, by tuning coupling constants appropriately, every unitary of the coupled system (atoms and cavity) can be approximated with arbitrarily small error.

Keywords: quantum control, cavity QED, infinite dimensions

1. Introduction

Exploiting controlled dynamics of quantum systems is becoming of increasing importance not only for solving computational tasks or quantum-secured communication, but also for simulating other physical systems [1–6]. An interesting direction in quantum simulation applies many-body correlations to create 'quantum matter', e.g., ultra-cold atoms in optical lattices are versatile models for studying large-scale correlations [6, 7]. Tunability and control over the system parameters of optical lattices allows for switching between several low-energy states of different quantum phases [8, 9] or in particular for following real-time dynamics such as the quantum quench from the super-fluid to the Mott-insulator regime [10].

Thus manipulating several atoms in a cavity is a key step to this end [11] at the same time posing challenging infinite-dimensional control problems. While in finite dimensions controllability can readily be assessed by the Lie-algebra rank condition [12–16], infinite-dimensional systems are more intricate [17]. As exact controllability in infinite dimensions seemed daunting in earlier work [18–21], it took a while before approximate control paved the way to more realistic assessment [22–24], for a recent (partial) review see, e.g., also [25] and references therein.

Here we explore systems and control aspects for systems consisting of several two-level atoms coupled to a cavity mode, i.e. the *Jaynes–Cummings model* [26–29]. We build upon our previous symmetry arguments [30, 31] and moreover, we apply appropriate operator topologies for addressing two controllability problems in particular: (i) to which extent can pure states be interconverted and (ii) can unitary gates be approximated with arbitrary precision. In particular by treating the latter, we go beyond previous work, which started out by a finite-dimensional truncation of a two-level atom coupled to an oscillator [32] followed by generalizations to infinite dimensions [33–35] both being confined to establishing criteria of pure-state controllability. Note that [35] also treats one atom coupled to several oscillators. Yuan and Lloyd [34] show that by induction over an appropriate finite-dimensional truncation, one obtains full controllability, where the dimensions can be made arbitrarily large. To complete the argument, however, here we provide explicit convergence analysis in the strong operator topology.

The general aim of this paper is twofold: on the one hand we study control problems for atoms interacting with electromagnetic fields in cavities. On the other hand, we address quantum control in infinite dimensions. Therefore, the purpose of section 2 is to provide enough material for a non-technical overview on the second subject in order to understand the results on the first (where the difficulties come from). Mathematical details are postponed to sections 4 and 5, while results on cavity systems are presented in overview in section 3.

2. Controllability

The control of quantum systems poses considerable mathematical challenges when applied to infinite dimensions. Basically, they arise from the fact that anti-self-adjoint operators (recall that according to Stone's theorem (see VIII.4 in [36]), they are generators of strongly continuous, unitary one-parameter groups) do neither form a Lie algebra nor even a vector space. Or seen on the group level, the group of unitaries equipped with the strong operator topology is a topological group yet not a Lie group. So whenever strong topology has to be invoked, controllability cannot be assessed via a system Lie algebra. Thus in these cases we address the challenges on the group level by employing the controlled time evolution of the quantum system in order to approximate unitary operators, the action of which is measured with respect to arbitrary, but finite sets of vectors. This is formalized in the notion of *strong controllability* (see section 2.3) introduced here as a generalization of pure-state controllability already discussed in the literature. Central to our discussion are Abelian symmetries. Assuming that all but one of our Hamiltonians observe such an Abelian symmetry, we systematically analyze the infinite-dimensional control system in its block-diagonalized basis. We obtain strong controllability (beyond pure-state controllability) if one of the Hamiltonian breaks this Abelian symmetry and some further technical conditions are fullfilled.

2.1. Time evolution

We treat control problems of the form

$$\dot{\psi}(t) = \sum_k u_k(t) H_k \psi(t) = H(t)\psi(t), \tag{1}$$

where the H_k with $k \in \{1, \ldots, d\}$ are self-adjoint control Hamiltonians on an infinite-dimensional, separable Hilbert space $\mathcal{H}$ and the controls u_k: $\mathbb{R} \to \mathbb{R}$ are piecewise-constant control functions. Since $\mathcal{H}$ is infinite-dimensional, the operators H_k are usually only defined on a dense subspace $D(H_k) \subset \mathcal{H}$ called the *domain* of H_k, the only exceptions being those H_k which are bounded. However, in this context, control problems where all H_k are bounded are not very interesting from a physical point of view. In other words, there is no way around considering those domains and many difficulties of control theory in infinite dimensions arises from this fact[3].

We will also assume that equation (1) will have unique solutions for all initial states $\psi_0 \in \mathcal{H}$ and all times t. So for each pair of times $t_1 < t_2$ there is a unitary propagator $U(t_1, t_2)\psi_0 = \mathcal{T}\int_{t_1}^{t_2} \mathrm{d}t \exp(-itH(t))\psi_0$, where $\mathcal{T}$ denotes time ordering. Observe that this condition is usually *not* satisfied, not even if the H_k share a joint domain of essential self-adjointness. Fortunately, the systems we are going to study do not show such pathological behavior. Yet, a minimalistic way to avoid this problem would be to restrict to control functions where only one u_k is different from 0 at each time t. In this case the propagator $U(t_1, t_2)$ is just a concatenation of unitaries $\exp(itH_k)$ which are guaranteed to exist due to self-adjointness of the H_k.

2.2. Pure-state controllability

A key-issue in quantum control theory is *reachability*: given two pure states $\psi_0, \psi \in \mathcal{H}$, we are looking for a time $T > 0$ and control functions u_k such that $\psi = U(0, T)\psi_0$. In infinite dimensions, however, this condition is too strong, since there might be states which can be reached only in infinite time, or not at all. Yet, one may find a reachable state 'close by' with arbitrary small control error. Therefore we will call ψ reachable from ψ_0 if for all $\epsilon > 0$ there is a *finite* time $T > 0$ and control functions u_k such that $\| \psi - U(0, T)\psi_0 \| < \epsilon$ holds. Accordingly, we will call the system (1) *pure-state controllable*, if each pure state ψ can be reached from one ψ_0 (and, by unitarity, also vice versa).

[3] Note that domains of unbounded operators are not just a mathematical pedantism. The domain is a crucial part of the definition of an operator and contains physically relevant information. A typical example is the Laplacian in a box which requires *boundary conditions* for a complete description. Up to a certain degree, domains can be regarded as an abstract form of boundary conditions (possibly at infinity).

Since pure states are described by one-dimensional projections, two state vectors describe the same state if they differ only by a global phase. Hence the definition just given is actually a bit too strong. There are several ways around this problem, like using the trace norm distance of $|\psi\rangle\langle\psi|$ and $|\psi_0\rangle\langle\psi_0|$ rather then the norm distance of ψ and ψ_0. For our purposes, however, the most appropriate method is to assume that the unit operator $\mathbb{1}$ on $\mathcal{H}$ is always among the control Hamiltonians. This may appear somewhat arbitrary, but it helps to avoid problems with determinants and traces on infinite-dimensional Hilbert spaces, which otherwise would arise.

2.3. Strong controllability

Next, the analysis shall be lifted to the level of operators, i.e. to unitaries U from the group $\mathcal{U}(\mathcal{H})$ of unitary operators on the Hilbert space $\mathcal{H}$ such that a time $T > 0$ and control functions u_k exist with $U = U(0, T)$. As in the last paragraph, this has to be generalized to an approximative condition again. The best choice—mathematically as well as from a practical point of view—is approximation in the *strong sense*: we look for unitaries U such that for each set of (not necessarily orthonormal or linearly independent) vectors $\psi_1, \ldots, \psi_f \in \mathcal{H}$ and each $\epsilon > 0$, there exists a time $T > 0$ and control functions u_k such that

$$\| [U - U(0, T)] \psi_k \| < \epsilon \quad \text{for all } k \in \{1, \ldots, f\}. \tag{2}$$

In other words, we are comparing U and $U(0, T)$ only on a finite set of states, and the worst-case error one can get here is bounded by ϵ. We will call the control system (1) *strongly controllable* if each unitary U can be approximated that way. (NB: in strong controllability, one again has the choice of one single joint global phase factor.)

Clearly, strong controllability implies pure-state controllability. To see this, choose an arbitrary but fixed $\psi_0 \in \mathcal{H}$. For each $\psi \in \mathcal{H}$, there is a unitary U with $U\psi_0 = \psi$. Hence strong controllability implies $\| \psi - U(0, T)\psi_0 \| = \| [U - U(0, T)] \psi_0 \| < \epsilon$.

2.4. The dynamical group $\mathcal{G}$

Strong controllability is concept-wise related to the strong operator topology (see VI.1 in [36]) on the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$. To this end, consider the sets

$$\mathcal{N}(U; \psi_1, \ldots, \psi_f; \epsilon) = \{ V \in \mathcal{U}(\mathcal{H}) \,|\, \|(V - U)\psi_k \| < \epsilon \ \text{ for all } \ k \in \{1, \ldots, f\} \}. \tag{3}$$

They form a neighborhood base for the strong topology, and we will call them *(strong) ϵ-neighborhoods*. The condition in equation (2) can now be restated as: any ϵ-neighborhood of U contains a time-evolution operator $U(0, T)$ for appropriate time T and control functions u_k. In turn, this can be reformulated as: U is an *accumulation point* of the set $\tilde{\mathcal{G}}$ of all unitaries $U(0, T)$. The set of all accumulation points of $\tilde{\mathcal{G}}$ (which contains $\tilde{\mathcal{G}}$ itself) is a strongly closed subgroup[4] of

[4] There is a subtle point here: the group $\mathcal{U}(\mathcal{H})$ is not strongly closed as a subset of the bounded operators $\mathcal{B}(\mathcal{H})$. Actually its strong closure is the set of all isometries; cf Prob. 225 of [37]. Hence whenever we talk about strongly closed groups of unitaries, this has to be understood as the closure in the restriction of the strong topology to $\mathcal{U}(\mathcal{H})$ (which coincides with the restriction of the weak topology).

$\mathcal{U}(\mathcal{H})$, which we will call the *dynamical group* $\mathcal{G}$ generated by control Hamiltonians H_k with $k \in \{1, \ldots, d\}$. If we choose the controls as described in subsection 2.1 (i.e. piecewise constant and only one u_k different from zero at each time), $\mathcal{G}$ is just the smallest strongly closed subgroup of $\mathcal{U}(\mathcal{H})$ that contains all $\exp(itH_k)$ for all $k \in \{1, \ldots, d\}$ and all $t \in \mathbb{R}$. Note that it contains in particular all unitaries that can be written as a strong limit $\text{s-lim}_{T \to \infty} U(0, T)$. In finite dimensions, $\mathcal{G}$ can be calculated via its system algebra, i.e. the Lie algebra $\mathfrak{l}$ generated by the iH_k, since each $U \in \mathcal{G}$ can be written as $U = \exp(H)$ for an $H \in \mathfrak{l}$.

In infinite dimensions, however, several difficulties can occur. First, unbounded operators H_k are only defined on a dense domain $D(H_k) \subset \mathcal{H}$. The sum $H_k + H_j$ is therefore only defined on the intersection $D(H_k) \cap D(H_j)$ and the commutator even only on a subspace thereof. There is no guarantee that $D(H_k) \cap D(H_j)$ contains more than just the zero vector. In this case, the Lie algebra cannot even be defined.

The minimal requirement to get around this difficulty is the existence of a joint dense domain D, i.e. $D \subset D(H_j)$ and $H_j D \subset D$ for all j. However, even then we do not know whether $\mathcal{G}$ can be generated from $\mathfrak{l}$ in terms of exponentials. In general, it is impossible to define some $\exp(H)$ for all $H \in \mathfrak{l}$.

There are several ways to deal with these problems. One is to consider cases where the H_k generate (i) a finite-dimensional Lie algebra and admit (ii) a common, invariant, dense domain consisting of analytic vectors [18, 20]. In this case the exponential function is defined on all of $\mathfrak{l}$, and we can proceed in analogy to the finite-dimensional case. The problem is that the group $\mathcal{G}$ will become a finite-dimensional Lie group and its orbits through a vector $\psi \in \mathcal{H}$ are finite-dimensional as well. Hence, we never can achieve full controllability. This approach is well studied; cf [18, 20] and references therein.

Another possibility which includes the possibility to study an infinite-dimensional Lie algebra $\mathfrak{l}$ is to restrict to bounded generators H_k. In this case, one can define $\mathfrak{l}$ as a norm-closed subalgebra of the Lie algebra $\mathcal{B}(\mathcal{H})$ of bounded operators, and one ends up with a Banach-space theory which works almost in the same way as the finite-dimensional analog; cf [38] for details. Although this is a perfectly reasonable approach from the mathematical point of view, it is not very useful for physical applications, since in most cases at least some of the H_k are unbounded.

In this paper, we will thus consider a different approach which splits the generators into two classes. The first $d-1$ generators $H_1, \ldots, H_{d-1}$ admit an Abelian symmetry and can be treated —with Lie-algebra methods—along the lines outlined in the next subsection. Secondly, the last generator H_d breaks this symmetry and achieves full controllability with a comparably simple argument. The details will be explained in sections 4 and 5.

2.5. Abelian symmetries

One way to avoid the problem arising from unboundedness of the control Hamiltonians (as described in the last subsection) is to study control systems admitting symmetries. In this section, we will only sketch the structure, while the details are postponed to section 4.

Let us consider the case of a U(1)-symmetry[5], i.e. a (strongly continuous) unitary representation $z \mapsto \pi(z) \in \mathcal{U}(\mathcal{H})$ of the Abelian group U(1) on $\mathcal{H}$ where $\mathcal{U}(\mathcal{H})$ denotes the

[5] The generalization to multiple charges, i.e. a $\mathrm{U}(1)^N$, is straightforward.

group of unitaries on $\mathcal{H}$. It can be written in terms of a self-adjoint operator X with pure point spectrum consisting of (a subset of) $\mathbb{Z}$ as $\mathrm{U}(1) \ni z = e^{i\alpha} \mapsto \pi(z) = \exp(i\alpha X) \in \mathcal{U}(\mathcal{H})$. If we denote the eigenprojection of X belonging to the eigenvalue $\mu \in \mathbb{Z}$ as $X^{(\mu)}$ (allowing the case $X^{(\mu)} = 0$ if μ is not an eigenvalue of X) we get a block-diagonal decomposition of $\mathcal{H}$ in the symmetry-adapted basis as

$$\mathcal{H} = \bigoplus_{\mu=-\infty}^{\infty} \mathcal{H}^{(\mu)} \text{ with } \mathcal{H}^{(\mu)} = X^{(\mu)}\mathcal{H}, \tag{4}$$

and we can rewrite $\pi(z)$ again as $\mathrm{U}(1) \ni z = e^{i\alpha} \mapsto \pi(z) = \sum_{\mu=-\infty}^{\infty} e^{i\alpha\mu} X^{(\mu)} \in \mathcal{U}(\mathcal{H})$. Here we will make two assumptions representing substantial restrictions of generality:

(i) All eigenvalues of X are of finite multiplicity, i.e. the $\mathcal{H}^{(\mu)}$ are finite-dimensional. This is crucial for basically everything we will discuss in this paper.

(ii) All eigenvalues of X are non-negative. This assumption can be relaxed at certain points (e.g. all material in section 4.1 can be easily generalized). However, it helps to simplify the discussion at a technical level and all examples we are going to consider in the next section are of this form.

The first important consequence of (i) concerns the space of *finite particle vectors*

$$D_X = \{\psi \in \mathcal{H} \mid X^{(\mu)}\psi = 0 \text{ for all but finitely many } \mu\}, \tag{5}$$

since it becomes (due to finite-dimensionality of $\mathcal{H}^{(\mu)}$) a 'good' domain for basically all unbounded operators appearing in this paper. Moreover one gets the following theorem:

Theorem 2.1. *Consider a strongly continuous representation π of $\mathrm{U}(1)$ on $\mathcal{H}$ and the corresponding charge-type operator X. Then the following statements hold:*

(i) A self-adjoint operator H commuting with X admits D_X as an invariant domain, i.e. $D_X \subset D(H)$ and $HD_X = D_X$. Hence the space $\mathfrak{u}(X) = \{iH \mid H = H^* \text{ commuting with } X\}$ is a Lie algebra with the commutator as its Lie bracket.

(ii) The exponential map is well defined on $\mathfrak{u}(X)$ and maps it onto the strongly closed subgroup $\mathcal{U}(X) = \{U \in \mathcal{U}(\mathcal{H}) \mid [U, \pi(z)] = 0 \text{ for all } z \in \mathrm{U}(1)\}$ of $\mathcal{U}(\mathcal{H})$, i.e. the centralizer of $\{\exp(i\alpha X) \mid \alpha \in \mathbb{R}\}$ in $\mathcal{U}(\mathcal{H})$.

(iii) The subalgebra $\mathfrak{l} \subset \mathfrak{u}(X)$ generated by a family of Hamiltonians $iH_1, \ldots, iH_d \in \mathfrak{u}(X)$ is mapped by the exponential map into the dynamical group $\mathcal{G}$ of the corresponding control problem. The strong closure of $\exp(\mathfrak{l})$ coincides with $\mathcal{G}$.

The basic idea behind this theorem, is that one can cut off the decomposition (4) at a sufficiently high μ without sacrificing strong approximations as described in subsection 2.3. One only has to take into account that the cut-off on μ has to become higher when the approximation error decreases. This strategy allows for tracing a lot of calculations back to finite-dimensional Lie algebras. We will postpone a detailed discussion of this topic—including the proof of theorem 2.1—to section 4.

The only additional material one needs at this point, since it is of relevance for the next section, is a subgroup of $\mathcal{U}(X)$ and its corresponding Lie algebra which relates unitaries with determinant one and their traceless generators. Since the $iH \in \mathfrak{u}(X)$ are unbounded and not necessarily positive, it is difficult to give a reasonable definition of tracelessness, and the determinant of $U \in \mathcal{U}(X)$ runs into similar problems. However, the elements of $U \in \mathcal{U}(X)$ and $iH \in \mathfrak{u}(X)$ are block diagonal with respect to the decomposition of $\mathcal{H}$ given in (4). In other words $U = \Sigma_\mu U^{(\mu)}$ and $H = \Sigma_\mu H^{(\mu)}$ are infinite sums of operators[6], where $U^{(\mu)} = X^{(\mu)} U X^{(\mu)} \in \mathcal{U}(\mathcal{H}^{(\mu)})$, $H^{(\mu)} = X^{(\mu)} H X^{(\mu)} \in \mathcal{B}(\mathcal{H}^{(\mu)})$, and $X^{(\mu)}$ denotes the projection onto the X-eigenspace $\mathcal{H}^{(\mu)}$. Since all the $U^{(\mu)}$ and $H^{(\mu)}$ are operators on finite-dimensional vector spaces, one can define

$$\mathcal{SU}(X) := \{U \in \mathcal{U}(X) \,|\, \det U^{(\mu)} = 1 \ \text{ for all } \ \mu \in \mathbb{Z}\}, \tag{6}$$

$$\mathfrak{su}(X) := \{iH \in \mathfrak{u}(X) \,|\, \operatorname{tr}(H^{(\mu)}) = 0 \ \text{ for all } \ \mu \in \mathbb{Z}\}. \tag{7}$$

Obviously, $\mathcal{SU}(X)$ is a (strongly closed) subgroup of $\mathcal{U}(X)$ and $\mathfrak{su}(X)$ is a Lie subalgebra of $\mathfrak{u}(X)$. The image of $\mathfrak{su}(X)$ under the exponential map therefore coincides with $\mathcal{SU}(X)$. Note that $\mathcal{SU}(X)$ is effectively an infinite direct product of groups $\mathrm{SU}(d^{(\mu)})$, if $d^{(\mu)} = \dim\mathcal{H}^{(\mu)}$ and not the 'special' subgroup of $\mathcal{U}(X)$.

2.6. Breaking the symmetry

To get a fully controllable system, one has to leave the group $\mathcal{U}(X)$, which can be thought of as being represented as block diagonal, see figure 1(a). To this end, we have to add control Hamiltonians that break the symmetry. There are several ways of doing so, and a successful strategy depends on the system in question (beyond the treatment of the symmetric part of the dynamics captured in theorem 2.1). Here, we will present a special result which covers the examples discussed in the next section. The first step is another direct sum decomposition of $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$, where $\mathcal{H}_\alpha = E_\alpha \mathcal{H}$, with $\alpha \in \{+, 0, -\}$ are projections onto the subspaces $\mathcal{H}_\alpha$ and should satisfy $[E_\alpha, X^{(\mu)}] = 0$. Let in the following $\mathbb{N} := \{1, 2, 3, \ldots\}$ denote the set of positive integers and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Hence for $\mu \in \mathbb{N}_0$ we can introduce the projections $X_\pm^{(\mu)} = X^{(\mu)} E_\pm$ which we require to be non-zero. For the exceptional case $\mu = 0$ the relation $X_-^{(0)} = X^{(0)} E_- = X^{(0)}$ should hold. Futhermore we write $X_0^{(\mu)} = X^{(\mu)} E_0$ for the overlap of $X^{(\mu)}$ and E_0 which can (in contrast to $X_\pm^{(\mu)}$) be equal to zero for all μ. The $X_\alpha^{(\mu)}$ are projections onto the subspaces $\mathcal{H}_\alpha^{(\mu)} := X_\alpha^{(\mu)} \mathcal{H}$ satisfying $X^{(\mu)} = X_-^{(\mu)} \oplus X_0^{(\mu)} \oplus X_+^{(\mu)}$.

Definition 2.2. A self-adjoint operator H with domain $D(H)$ is called complementary to X, if there exists a decompositon $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$ as defined above such that:

(i) $\mathcal{H}_0 \subset D(X)$ and $H\psi = 0$ for all $\psi \in \mathcal{H}_0$.

(ii) $D_X \subset D(H)$ and for all $\mu > 1$ we have $HX_+^{(\mu+1)}\psi = X_-^{(\mu)} H\psi$. The corresponding operator $X_-^{(\mu)} H X_+^{(\mu+1)} \in \mathcal{B}(\mathcal{H})$ is a partial isometry with $X_+^{(\mu+1)}$ as its source and $X_-^{(\mu)}$ as its target projection.

[6] Two small remarks are in order here: (i) infinite sums require a proper definition of convergence in an appropriate topology. In section 4, this will be made precise. (ii) Operator products of the form $X^{(\mu)} H X^{(\mu)}$ are potentially problematic if H is unbounded and therefore only defined on a domain. In our case, however, $X^{(\mu)}$ projects onto $\mathcal{H}^{(\mu)}$, which is a subspace of the domain D_X on which H is defined.

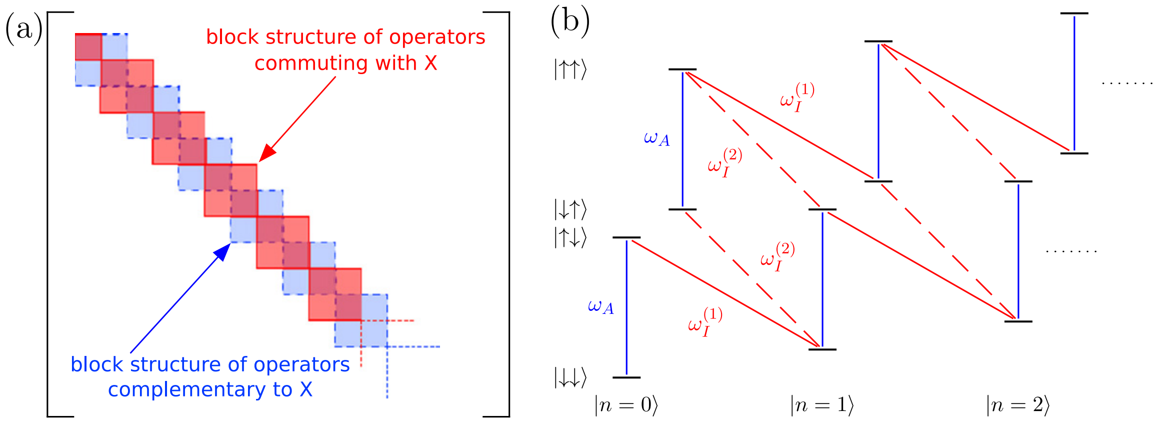

Figure 1. (a) Block structure of operators in $\mathfrak{u}(X)$ (red) and of operators complementary to X (blue) in the case where the projection E_0 vanishes. (b) Energy diagram for the Jaynes–Cummings model (here two atoms in a cavity under individual controls $\omega_I^{(1)}$ and $\omega_I^{(2)}$) with combined atom–cavity transitions matching the block structure of (a) given in red (see equations (10, 18)) since commuting with X_1 or X_M of equations (12, 20), and complementary transitions solely within the atoms given in blue (see equations (15, 22)).

(iii) Given the projection $F_{[0]} = X^{(0)} \oplus X_{-}^{(1)}$ and the corresponding subspace $\mathcal{H}_{[0]} = F_{[0]}\mathcal{H}$. The group generated by $\exp(itH)$ with $t \in \mathbb{R}$ and those $U \in \mathcal{SU}(X)$ which commute with $F_{[0]}$ acts transitively on the space of one-dimensional projections in $\mathcal{H}_{[0]}$.

At first sight, the definition may look somewhat clumsy, but it allows for proving a controllability result which covers all examples we are going to present in the next section. We will state them here without a proof and postpone the latter to section 5.

Theorem 2.3. *Consider a strongly continuous representation* π: $\mathrm{U}(1) \to \mathcal{U}(\mathcal{H})$ *with charge operator X and a family of self-adjoint operators* $H_1,\ldots, H_d$ *on* $\mathcal{H}$. *Assume that the following conditions hold:*

(i) $H_1,\ldots, H_{d-1}$ *commute with X.*

(ii) The dynamical group generated by $H_1,\ldots, H_{d-1}$ *contains* $\mathcal{SU}(X)$.

(iii) The operator H_d *is complementary to X.*

Then the control system equation (1) with Hamiltonians $H_0 = \mathbb{1}, H_1,\ldots, H_d$ *is pure-state controllable.*

Theorem 2.4. *The control system (1) is even strongly controllable if in addition to the assumptions of theorem 2.3 the condition* $\dim \mathcal{H}^{(\mu)} > 2$ *holds for at least one* $\mu \in \mathbb{N}_0$.

3. Atoms in a cavity

An important class of examples that can be treated along the lines described in the last section are atoms interacting with the light field in a cavity. We will discuss the case of M two-level atoms interacting with one mode in detail and consider three particular scenarios: one atom in section 3.1, individually controlled atoms in section 3.2, and atoms under collective control in section 3.3.

3.1. One atom

Let us start with the special case $M = 1$, i.e. one atom and one mode as discussed in a number of previous publications mostly on pure-state controllability [34, 35, 39]. Our results go beyond this, in particular because we are considering strong controllability not just pure-state controllability. The Hilbert space of the system is given by

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathrm{L}^2(\mathbb{R}) \tag{8}$$

and the dynamics is described by the well known Jaynes–Cummings Hamiltonian [26]:

$$H_{\mathrm{JC}} := \omega_A H_{\mathrm{JC},1} + \omega_I H_{\mathrm{JC},2} + \omega_C H_{\mathrm{JC},3} \text{ with} \tag{9}$$

$$H_{\mathrm{JC},1} := (\sigma_3 \otimes \mathbb{1})/2,\ \ H_{\mathrm{JC},2} := (\sigma_+ \otimes a + \sigma_- \otimes a^*)/2,\ H_{\mathrm{JC},3} := \mathbb{1} \otimes N, \tag{10}$$

where σ_α with $\alpha \in \{1, 2, 3\}$ are the Pauli matrices ($\sigma_\pm = \sigma_1 \pm i\sigma_2$), a, a^* denote the annihilation and creation operator, and $N = a^*a$ is the number operator. The joint domain of all these Hamiltonians is the space

$$D = \operatorname{span}\left\{|\nu\rangle \otimes |n\rangle \,|\, \nu \in \{0, 1\} \text{ and } n \in \mathbb{N}_0\right\}, \tag{11}$$

with $\nu \in \mathbb{C}^2$ as canonical basis and $|n\rangle \in \mathrm{L}^2(\mathbb{R})$ as number basis (Hermite functions).

We will assume that the frequencies ω_A, ω_I and ω_C can be controlled independently (or at least two of them) such that we get a control system with control Hamiltonians $H_{\mathrm{JC},j}$ where $j \in \{1, 2, 3\}$ corresponding to the lower half (1 atom) of the energy diagram in figure 1(b), where we adopt the widely used convention of forcing the atom (spin) state $|\uparrow\rangle$ to be of 'higher' energy than $|\downarrow\rangle$ to compensate for negative Larmor frequencies, see, e.g., the note on p 144 in [11]. The task is to determine the dynamical group $\mathcal{G}$. To this end, we use the strategy described in subsection 2.5, which follows in this particular case closely the exact solution of the Jaynes–Cummings model [26]. The charge-type operator X_1 (determining the block structure) then takes the form

$$X_1 = \sigma_3 \otimes \mathbb{1} + \mathbb{1} \otimes N, \tag{12}$$

again with D from (11) as its domain, which in this case turns out to be identical to the space D_{X_1} of finite-particle vectors. The operator X_1 is diagonalized by the basis $|\nu\rangle \otimes |n\rangle$. It is convenient to relabel these vectors in order to get

$$|\mu, \nu\rangle = |\nu\rangle \otimes |\mu - \nu\rangle \in \mathcal{H} \quad \text{with } \mu = n + \nu \geqslant 0. \tag{13}$$

In this basis, we have $X_1 |\mu, \nu\rangle = \mu |\mu, \nu\rangle$ and the subspaces $\mathcal{H}^{(\mu)}$ from (4) become

$$\mathcal{H}^{(\mu)} = \operatorname{span}\left\{|\mu, 0\rangle,\ \ |\mu, 1\rangle\right\}, \tag{14}$$

for $\mu > 0$ and $\mathcal{H}^{(0)} = \mathbb{C}\,|0,\, 0\rangle$ for $\mu = 0$. The space $D_{X_1} \subset \mathcal{H}$ of finite-particle vectors turns out to be identical with the domain D from (11).

It is easy to see that the operators $H_{\mathrm{JC},j}$ from equation (10) commute with X_1, and therefore we get $iH_{\mathrm{JC},j} \in \mathfrak{u}(X_1)$. A more detailed analysis, as will be given in section 4, shows that $iH_{\mathrm{JC},1}$ and $iH_{\mathrm{JC},2}$ generate $\mathfrak{su}(X_1)$, and therefore we get according to theorem 2.1:

Theorem 3.1. *The dynamical group $\mathcal{G}$ generated by $H_{\mathrm{JC},j}$ with $j \in \{1, 2\}$ from equation (10) coincides with the group $\mathcal{SU}(X_1)$ defined in (6).*

To get a fully controllable system, apply theorem 2.4 to see that one has to add a Hamiltonian which breaks the symmetry. A possible candidate is

$$H_{\mathrm{JC},4} = \sigma_1 \otimes \mathbb{1} \in \mathcal{B}(\mathcal{H}), \tag{15}$$

so that transitions within the two-level system can be driven by $\omega_x(t)\, H_{\mathrm{JC},4}$ in the sense of x-pulses. If we define the spaces $\mathcal{H}_\alpha$ as $\mathcal{H}_- = \mathrm{span}\,\{|\mu,\, 0\rangle \mid \mu \in \mathbb{N}_0\}$, $\mathcal{H}_0 = \{0\}$, and $\mathcal{H}_+ = \mathrm{span}\,\{|\mu,\, 1\rangle \mid \mu \in \mathbb{N}\}$ the operator $H_{\mathrm{JC},4}$ becomes complementary to X_1, which can be easily seen since $\mathcal{H}_+^{(\mu)} = \mathbb{C}\,|\mu,\, 1\rangle$, $\mathcal{H}_-^{(\mu)} = \mathbb{C}\,|\mu,\, 0\rangle$, and $\mathcal{H}_0^{(\mu)} = \{0\}$. Hence, according to theorem 2.3, the control system with Hamiltonians of equations (9, 10)

$$H_0 = \mathbb{1}, \;\; H_1 = H_{\mathrm{JC},1}, \;\; H_2 = H_{\mathrm{JC},2}, \;\; H_3 = H_{\mathrm{JC},4} \tag{16}$$

is pure-state controllable[7], and we are recovering a previous result from [34, 35, 39]. However, with our methods we can go beyond this and prove even strong controllability. Theorem 2.4 cannot be applied since $\dim \mathcal{H}^{(\mu)} \leqslant 2$ for all μ, but the analysis of section 5 will lead to an independent argument.

Theorem 3.2. *The control problem (1) with Hamiltonians H_j and $j \in \{0, \ldots, 3\}$ from equation (16) is strongly controllable.*

Hence any unitary U on $\mathcal{H}$ can be approximated by varying the control amplitudes $u_1 = \omega_A$ and $u_2 = \omega_I$ in the Hamiltonian H_{JC} of (9) plus flipping ground and excited state of the atom in terms of $H_{\mathrm{JC},4}$ (with strength u_3)—both in an appropriate time-dependent manner. The approximation has to be understood in the strong sense as described in equation (2).

Finally, note that theorem 3.2 implies that one can simulate (again in the sense of strong approximations) any unitary $V \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}))$ operating on the cavity mode alone. One only has to find controls u_j such that $U(0,\, T)\, \phi \otimes \psi_k \approx \phi \otimes V\psi_k$ for a finite set of states ψ_k of the cavity (and an arbitrary auxiliary state ϕ of the atom).

3.2. Many atoms with individual control

Next, consider the case of many atoms interacting with the same mode, and under the assumption that each atom (including the coupling with the cavity) can be controlled individually. Such a scenario is relevant for experiments with ion traps, if the number of ions is

[7] We have omitted the Hamiltonian $H_{\mathrm{JC},3}$ since it is not needed for the result. However, it can be added as a drift term without changing the result.

not too big as have been studied since [40–42]. The Hilbert space of the system is

$$\mathcal{H} = (\mathbb{C}^2)^{\otimes M} \otimes \mathrm{L}^2(\mathbb{R}), \tag{17}$$

where M denotes the number of atoms. We define the basis $|b\rangle \otimes |n\rangle \in \mathcal{H}$ where $n \in \mathbb{N}_0$, $|b\rangle = |b_1\rangle \otimes \ldots \otimes |b_M\rangle$, $b = (b_1, \ldots, b_M) \in \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 = \mathbb{Z}_2^M$, and the canonical basis $|b_j\rangle \in \mathbb{C}^2$ with $b_j \in \{0, 1\}$. The control Hamiltonians become

$$H_{\mathrm{IC},j} = \sigma_{3,j} \otimes \mathbb{1} \text{ and } H_{\mathrm{IC},M+j} = \sigma_{+,j} \otimes a + \sigma_{-,j} \otimes a^*, \tag{18}$$

where $j \in \{1, \ldots, M\}$ and $\sigma_{\alpha,j} = \mathbb{1}^{\otimes(j-1)} \otimes \sigma_\alpha \otimes \mathbb{1}^{\otimes(M-j)}$. As before, a and a^* denote annihilation and creation operator. The joint domain of all these operators is

$$D = \operatorname{span}\left\{|b\rangle \otimes |n\rangle \mid b \in \mathbb{Z}_2^M \text{ and } n \in \mathbb{N}_0\right\}, \tag{19}$$

with the basis $|b\rangle \otimes |n\rangle$ as defined above. As depicted by the red parts in figure 1, all the $H_{\mathrm{IC},k}$ are invariant under the symmetry defined by the charge operator

$$X_M = S_3 \otimes \mathbb{1} + \mathbb{1} \otimes N \text{ with } S_3 = \sum_{j=1}^{N} \sigma_{3,j}, \tag{20}$$

where $N = a^*a$ denotes again the number operator and D from (19) is the domain of X_M. The eigenvalues of X_M are $\mu \in \mathbb{N}_0$ and the eigenbasis is given by

$$|\mu, b\rangle = |b\rangle \otimes |\mu - |b|\rangle \text{ for } |b| = \sum_{j=1}^{M} b_j \leqslant \mu. \tag{21}$$

In this basis, X_M becomes $X_M|\mu, b\rangle = \mu|\mu, b\rangle$ and the eigenspaces $\mathcal{H}^{(\mu)}$ are $\mathcal{H}^{(\mu)} = \operatorname{span}\{|\mu, b\rangle \,|\, b \in \mathbb{Z}_2^M$ with $|b| \leqslant \mu\}$. From now on, one may readily proceed as for one atom to arrive at the following analogy to theorem 3.1:

Theorem 3.3. *The dynamical group $\mathcal{G}$ generated by $H_{\mathrm{IC},k}$ with $k \in \{1, \ldots, 2M\}$ from equation (18) coincides with the group $\mathcal{SU}(X_M)$ of unitaries commuting with X_M.*

To get strong controllability, one has to add again one Hamiltonian. As before a σ_1-flip of one atom is sufficient (see the blue parts in figure 1), and

$$H_{\mathrm{IC},2M+1} = \sigma_{1,1} \otimes \mathbb{1}. \tag{22}$$

is complementary to X_M with $\mathcal{H}_\alpha$ given by $\mathcal{H}_0 = \{0\}$, $\mathcal{H}_- = \operatorname{span}\{|\mu; 0, b_2, \ldots, b_M\rangle \,|\, \mu \in \mathbb{N}_0, |(b_2, \ldots, b_M)| \leqslant \mu\}$, $\mathcal{H}_+ = \operatorname{span}\{|\mu; 1, b_2, \ldots, b_M\rangle \mid \mu \in \mathbb{N}, |(b_2, \ldots, b_M)| < \mu\}$. Obviously, all the conditions of theorem 2.4 are satisfied such that one gets

Theorem 3.4. *The control problem (1) with $H_{\mathrm{IC},k}$ and $k \in \{1, \ldots, 2M + 1\}$ from (18) and (22) is strongly controllable.*

As a special case of this theorem, one can approximate any unitary U acting on the atoms alone, i.e. $U \in \mathcal{U}((\mathbb{C}^2)^{\otimes M})$, by applying theorem 3.4 to $U \otimes \mathbb{1}$. That is, one can simulate U only by operations on one atom and the interactions with the harmonic oscillator. This is used in ion-trap experiments and is known as 'phonon bus'.

3.3. Many atoms under collective control

Now one may modify the setup from the last section by considering again M atoms interacting with one mode, but assuming that one can control the atoms only collectively rather than individually. In other words instead of the Hamiltonians $H_{\mathrm{IC},j}$ and $H_{\mathrm{IC},M+j}$ with $j \in \{1, \ldots, M\}$ from equation (18) one only has their sums

$$H_{\mathrm{TC},1} = S_3 \otimes \mathbb{1} \text{ and } H_{\mathrm{TC},2} = S_+ \otimes a + S_- \otimes a^*, \tag{23}$$

where $S_\alpha = \sum_{j=1}^{M} \sigma_{\alpha,j}$ and $\alpha \in \{1, 2, 3, \pm\}$, combinded with the free evolution

$$H_{\mathrm{TC},3} = \mathbb{1} \otimes N, \tag{24}$$

of the cavity. As before, all operators are defined on the domain D from (19). Note that one readily recovers the original setup from subsection 3.1 with Pauli operators σ_α replaced by pseudo-spin operators S_α. The multi-atom analogue of the Jaynes–Cummings Hamiltonian, which can be formed from the $H_{\mathrm{TC},j}$ just defined, is called Tavis–Cummings Hamiltonian [27, 28].

All the Hamiltonians in equations (23) and (24) are invariant under the U(1)-action generated by X_M of equation (20). However, this is not the only symmetry, since all these $H_{\mathrm{TC},j}$ are also invariant under the permutation of the atoms. Therefore, one may no longer exhaust the group $\mathcal{SU}(X_M)$ as in theorem 3.3 (since the following operators cannot be reached: those commuting only with X_M but not also with permutations of the atoms). A minimal modification is to restrict the states of the atoms to spaces on which permutation-invariant unitaries operate transitively[8]. The most natural choice is the symmetric tensor product $(\mathbb{C}^2)^{\otimes M}_{\mathrm{sym}} \subset (\mathbb{C}^2)^{\otimes M}$, i.e. the Bose subspace of $(\mathbb{C}^2)^{\otimes M}$. The preferred basis of $(\mathbb{C}^2)^{\otimes M}_{\mathrm{sym}}$ is $|\nu\rangle = \mathrm{Sym}_M(|1\rangle^{\otimes \nu} \otimes |0\rangle^{\otimes (M-\nu)})$ with $\nu \in \{0, \ldots, M\}$ and the projection Sym_M from $(\mathbb{C}^2)^{\otimes M}$ onto the symmetric subspace $(\mathbb{C}^2)^{\otimes M}_{\mathrm{sym}}$. In other words $|\nu\rangle$ is the unique, pure, permutation-invariant state with ν atoms in the excited state $|1\rangle$ and $M - \nu$ ones in the ground state $|0\rangle$. Therefore, $(\mathbb{C}^2)^{\otimes M}_{\mathrm{sym}}$ can be identified with the Hilbert space $\mathbb{C}^{M+1}$ of a (pseudo-)spin-$M/2$ system. Its basis $|\nu\rangle$, with $\nu \in \{0, \ldots, M\}$ becomes the canonical basis. Combining this with $\mathrm{L}^2(\mathbb{R})$ for the cavity one gets $\mathcal{H}_{\mathrm{sym}} = \mathbb{C}^{M+1} \otimes \mathrm{L}^2(\mathbb{R})$ as the new Hilbert space of the system.

All the operators defined above ($H_{\mathrm{TC},1}$, $H_{\mathrm{TC},2}$, $H_{\mathrm{TC},3}$ and X_M) can be restricted to $\mathcal{H}_{\mathrm{sym}}$ (and in slight abuse of notation we will re-use the symbols after restriction) and their domain becomes

$$D_{\mathrm{sym}} = \mathrm{span}\left\{ |\nu\rangle \otimes |n\rangle \,|\, \nu \in \{0, \ldots, M\} \text{ and } n \in \mathbb{N}_0 \right\}, \tag{25}$$

which is just the projection of D from (19), i.e. $D_{\mathrm{sym}} = \mathrm{Sym}_M D$. The eigenbasis of X_M now takes the form $|\mu, \nu\rangle = |\nu\rangle \otimes |\mu - \nu\rangle$ where $\mu \in \mathbb{N}_0$ and $\nu < d_\mu = \min(\mu, M + 1)$. For the X_M-eigenspaces, we get again $X_M |\mu, \nu\rangle = \mu |\mu, \nu\rangle$ and

[8] An alternative strategy would be to treat permutation symmetry in the same way as U(1)-symmetry. However, already the restriction to permutation-invariant states will turn out to be difficult enough.

$$\mathcal{H}_{\text{sym}}^{(\mu)} = \text{span}\left\{ |\mu,\, \nu\rangle \,|\nu \in \{0,\ldots,\, d_\mu\} \right\}. \tag{26}$$

Now one can proceed as in the previous cases: The operators $H_{\text{TC},1}$, $H_{\text{TC},2}$, $H_{\text{TC},3}$ are (as operators on $\mathcal{H}_{\text{sym}}$) invariant under the action generated by X_M and therefore elements of $\mathfrak{u}(X_M)$. However, one still cannot exhaust all of $\mathcal{U}(X_M)$ (or $\mathcal{SU}(X_M)$). One only gets:

Theorem 3.5. *The dynamical group $\mathcal{G}$ generated by the operators $H_{\text{TC},1}$, $H_{\text{TC},2}$, $H_{\text{TC},3}$ from equations (23) and (24) is a strongly closed subgroup of $\mathcal{U}(X_M)$. For each unitary $V \in \mathcal{U}(X_M)$ and each $\mu \in \mathbb{N}_0$ we can find an element $U \in \mathcal{G}$ such that $U\psi^{(\mu)} = V\psi^{(\mu)}$ holds for all $\psi^{(\mu)} \in \mathcal{H}_{\text{sym}}^{(\mu)}$.*

In other words: as long as the charge μ is fixed, one can still approximate any $V \in \mathcal{U}(X_M)$, but if one considers superpositions of different charges this is no longer the case, i.e. there are $\psi \in D_{X_M}$ and $V \in \mathcal{U}(X_M)$ such that $U\psi \neq V\psi$ holds for all $U \in \mathcal{G}$. We have checked the latter explicitly with the computer algebra system Magma [43] for the case $M = 2$. To circumvent this problem, one has to add control Hamiltonians. Unfortunately, it seems that one has to add quite a lot. The best result we have got so far is to replace the operators from equations (23) and (24) by

$$\begin{aligned} H_{\text{CC},k} &= (|k\rangle\langle k| - |k-1\rangle\langle k-1|) \otimes \mathbb{1} \text{ with } k \in \{1,\ldots,\, M\}, \\ H_{\text{CC},M+1} &= H_{\text{TC},2} = S_+ \otimes a + S_- \otimes a^* \text{ and } H_{\text{CC},M+2} = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes \mathbb{1}. \end{aligned} \tag{27}$$

The operators $H_{\text{CC},k}$ with $k \in \{1,\ldots,\, M+1\}$ commute with X_M and generate (as we will see in section 4.4) the Lie algebra $\mathfrak{su}(X_M)$. In addition we have $H_{\text{CC},M+2}$ which is complementary to X_M with Hilbert spaces $\mathcal{H}_+ = \text{span}\{|\mu;\, 0\rangle \,|\mu \in \mathbb{N}_0\}$, $\mathcal{H}_- = \text{span}\{|\mu;\, 1\rangle \mid \mu \in \mathbb{N}\}$, as well as $\mathcal{H}_0 = \text{span}\{|\mu,\, \nu\rangle \mid \mu \in \mathbb{N},\, \mu > 2,\; \nu \in \{3,\ldots,\, \min(M,\, \mu)\}\}$. Note that we get an example for definition 2.2 with a non-trivial $\mathcal{H}_0$. Now one can apply theorems 2.1 and 2.4 to get the analogues of theorems 3.1 and 3.2:

Theorem 3.6. *The dynamical group $\mathcal{G}$ generated by $H_{\text{CC},k}$ with $k \in \{1,\ldots,\, M+1\}$ from equation (27) coincides with the group $\mathcal{SU}(X_M)$ of unitaries commuting with X_M.*

Theorem 3.7. *The control problem (1) with $H_0 = \mathbb{1}$ and $H_{\text{CC},k}$ for $k \in \{1,\ldots,\, M+2\}$ from (27) is strongly controllable.*

To be able to control all diagonal traceless operators $H_{\text{CC},k}$, with $k \in \{1,\ldots,\, M\}$ is a very strong assumption. Unfortunately, a detailed analysis including computer algebra indicates that we cannot recover theorem 3.7 with fewer resources.

4. A Lie algebra of block-diagonal operators

The purpose of this section is to re-discuss Abelian symmetries and to provide technical details (in particular proofs) we omitted in sections 2 and 3. To this end, we re-use the notations already introduced in section 2.5. In particular, the Abelian symmetry induces a block-diagonal decomposition which, in infinite dimensions, allows for defining a block-diagonal Lie algebra and its exponential map onto a block-diagonal Lie group; see propositions 4.1 and 4.2. We identify the set of all block-diagonal unitaries reachable by block-diagonal time evolutions in proposition 4.4 as the strong closure of exponentials of block-diagonal Lie algebra elements. A central result is corollary 4.6, in which the question of controllability for the block-diagonal system of infinite dimensions is reduced to analyzing controllability for all finite-dimensional blocks. Using finite-dimensional commutator calculations one can now establish controllability on the infinite-dimensional but block-diagonal space for each of the three control systems analyzed.

4.1. Commuting operators

The first step is a closer look at the Lie algebra $\mathfrak{u}(X)$ and the corresponding group $\mathcal{U}(X)$ introduced in theorem 2.1 (which we will prove in this context). To this end, let us start with a unitary U commuting with the representatives $\pi(z)$, i.e. $[\pi(z), U] = 0$ for all $z \in \mathrm{U}(1)$. This is equivalent to $U\psi = \sum_{\mu=0}^{\infty} U^{(\mu)}\psi^{(\mu)}$ for all $\psi \in \mathcal{H}$ with $\psi^{(\mu)} := X^{(\mu)}\psi \in \mathcal{H}^{(\mu)}$ given a sequence of unitaries $U^{(\mu)}$ on the μ-eigenspaces $\mathcal{H}^{(\mu)}$ of X. Similarly one can consider a self-adjoint H with domain $D(H)$ commuting with X. By definition[9] this means the spectral projections of H commute with the $X^{(\mu)}$, which is equivalent to

$$D_X \subset D(H), \;\; HD_X \subset D_X \;\; \text{and} \;\; H\psi = \sum_{\mu=0}^{\infty} H^{(\mu)}\psi^{(\mu)} \;\text{for}\; \psi \in D_X \tag{28}$$

with a sequence of self-adjoint operators $H^{(\mu)}$ on the eigenspaces $\mathcal{H}^{(\mu)}$ and the $\psi^{(\mu)}$ as defined above. The $\mathcal{H}^{(\mu)}$ are finite-dimensional, and therefore the $H^{(\mu)}$ are bounded. Hence the unboundedness of H is inherited only from the unboundedness of the sequence of norms $\|H^{(\mu)}\|$. So it is easy to see that all elements of D_X are analytic for H and therefore D_X becomes a domain of essential self-adjointness for H (i.e., H is uniquely determined by its restriction to D_X as a consequence of Nelson's analytic vector theorem, see theorem X.39 of [44]). Accordingly, we will denote (in slight abuse of notation) the self-adjoint operator H and its restriction to D_X by the same symbol. This proves very handy when introducing, on the set $\mathfrak{u}(X)$ of anti-self-adjoint operators commuting with X, the structure of a Lie algebra by $(\lambda Q_1 + Q_2)\psi = \lambda Q_1\psi + Q_2\psi$, $[Q_1, Q_2]\psi = Q_1Q_2\psi - Q_2Q_1\psi$ for $Q_1, Q_2 \in \mathfrak{u}(X)$, $\lambda \in \mathbb{R}$, and $\psi \in D_X$. The linear combination $\lambda Q_1 + Q_2$ and the commutator $[Q_1, Q_2]$ are defined only on the joint domain D_X but since they are essentially self-adjoint on it, their self-adjoint extensions exist and are uniquely determined. This proves the first statement of theorem 2.1, which we restate as follows:

Proposition 4.1. *A self-adjoint operator H commuting with X admits D_X as an invariant domain of essential self-adjointness. The space*

$$\mathfrak{u}(X) = \{iH \mid H = H^* \text{ commuting with } X\} \tag{29}$$

[9] Note that the identity $[X, Y]\,\psi = 0$ for all ψ on a common dense domain is—in contrast to popular belief—*not a proper definition* for two commuting self-adjoint operators; cf the discussion in VIII.6 of [36]. Fortunately, such pathological cases do not occur in our set-up.

$$= \{iH \mid H\psi = \sum_{\mu} H^{(\mu)}\psi^{(\mu)},\ \ \psi \in D_X,\ \ H^{(\mu)} = (H^{(\mu)})^* \in \mathcal{B}(\mathcal{H}^{(\mu)})\}, \tag{30}$$

becomes a Lie algebra with the commutator as its Lie bracket.

Since all $iH \in \mathfrak{u}(X)$ are anti-self-adjoint, they admit a well-defined exponential map $\exp(iH)$. Boundedness of the $H^{(\mu)}$ together with equation (28) allows to express $\exp(iH)$ very explicitly. More precisely one has

$$\exp(iH)\psi = \sum_{\mu=-\infty}^{\infty} \exp(iH^{(\mu)})\psi^{(\mu)} \ \text{ where } \psi^{(\mu)} = X^{(\mu)}\psi \in \mathcal{H}^{(\mu)}, \tag{31}$$

and $\exp(iH^{(\mu)}) = \sum_{n=0}^{\infty}(iH^{(\mu)})^n/(n!)$. This shows that $\exp\colon \mathfrak{u}(X) \to \mathcal{U}(X)$ is well-defined and onto as stated in theorem 2.1, which we are now ready to prove:

Proposition 4.2. *The exponential map on* $\mathfrak{u}(X)$ *is well-defined and given in terms of equation (31). It maps* $\mathfrak{u}(X)$ *onto the strongly closed subgroup*

$$\mathcal{U}(X) = \left\{U \in \mathcal{U}(\mathcal{H}) \mid [U, \pi(z)] = 0 \ \text{ for all } \ z \in \mathrm{U}(1)\right\} \tag{32}$$

$$= \left\{U \mid U\psi = \sum_{\mu} U^{(\mu)}\psi^{(\mu)},\ \ \psi \in \mathcal{H},\ \ U^{(\mu)} \in \mathcal{U}(\mathcal{H}^{(\mu)})\right\} \text{ of } \mathcal{U}(\mathcal{H}). \tag{33}$$

Proof. The only statement not yet proven is the closedness of $\mathcal{U}(X)$. To this end, we have to show that for any net $(U_\lambda)_{\lambda\in I}$ strongly converging to a bounded operator U we have $U \in \mathcal{U}(X)$. As $U_\lambda \in \mathcal{U}(X)$ we have $[\pi(z), U_\lambda] = 0$ for all λ. Due to strong continuity of the map $A \mapsto [\pi(z), A]$ and the convergence of the U_λ to U it follows that $[\pi(z), U] = 0$. Hence U decomposes into a strongly converging series $U = \sum_\mu U^{(\mu)}$ with $U^{(\mu)} \in \mathcal{B}(\mathcal{H}^{(\mu)})$, and for each fixed μ we get $\lim_\lambda U_\lambda^{(\mu)} = U^{(\mu)}$. Since $\mathcal{H}^{(\mu)}$ is finite-dimensional, the nets $(U_\lambda^{(\mu)})_{\lambda\in I}$ converge in norm and therefore $U^{(\mu)} \in \mathcal{U}(\mathcal{H}^{(\mu)})$ which implies $U \in \mathcal{U}(X)$. □

Note that we actually proved more than what we stated. A strongly convergent sequence (or net) of elements of $\mathcal{U}(X)$ cannot converge to an isometry which is not unitary as well. Hence $\mathcal{U}(X)$ is strongly closed as a subset of $\mathcal{B}(\mathcal{H})$—and not only as a subset of $\mathcal{U}(\mathcal{H})$ as generally is the case (cf corresponding remarks in section 2.4).

The remaining statements in this subsection are devoted to the dynamical group $\mathcal{G}$ generated by a family of self-adjoint operators $H_1, \ldots, H_d$. Recall that we have introduced it as the smallest strongly closed subgroup of $\mathcal{U}(\mathcal{H})$ containing all unitaries of the form $\exp(itH_k)$. If the H_k are commuting with X, i.e. $iH_k \in \mathfrak{u}(X)$, then the group $\mathcal{G}$ is a subgroup of $\mathcal{U}(X)$, and the simple structure of the latter makes explicit calculations at least feasible. In the following, we show how $\mathcal{U}(X)$ is related to the Lie algebra $\mathfrak{l}$ generated by the iH_k. To this end, we need some additional notations. For each $K \in \mathbb{N}$, $U \in \mathcal{U}(X)$, and $iH \in \mathfrak{u}(X)$, let us consider

$$U^{[K]} = \sum_{\mu=0}^{K} U^{(\mu)}, \qquad H^{[K]} = \sum_{\mu=0}^{K} H^{(\mu)}, \qquad \mathcal{H}^{[K]} = \bigoplus_{\mu=0}^{K} \mathcal{H}^{(\mu)}. \tag{34}$$

The operators $U^{[K]}$ and $H^{[K]}$ act on the finite-dimensional Hilbert space $\mathcal{H}^{[K]}$. Therefore all operator topologies coincide and we can apply the well-known finite-dimensional theory. The dynamical group $\mathcal{G}^{[K]}$ (generated by $H_k^{[K]}$ with $k \in \{1, \dots, d\}$) becomes a closed subgroup of the unitary group $\mathcal{U}(\mathcal{H}^{[K]})$, which is a Lie group. Hence $\mathcal{G}^{[K]}$ is a Lie group, too, and its Lie algebra $\mathfrak{l}^{[K]}$ is generated by $iH_k^{[K]}$ with $k \in \{1, \dots, d\}$. Now, the crucial point is that one can approximate the infinite-dimensional objects $\mathcal{G}$ and $\mathfrak{l}$ by the finite-dimensional $\mathcal{G}^{[K]}$ and $\mathfrak{l}^{[K]}$. To see this, the first step is the following lemma.

Lemma 4.3. *Consider the Lie algebras $\mathfrak{l} \subset \mathfrak{u}(X)$ and $\mathfrak{l}^{[K]} \subset \mathcal{B}(\mathcal{H}^{[K]})$ (with $K \in \mathbb{N}$) generated by $iH_1, \dots, iH_d$ and $iH_1^{[K]}, \dots, iH_d^{[K]}$, respectively. Each element $\tilde{Q} \in \mathfrak{l}^{[K]}$ can be written as $\tilde{Q} = Q^{[K]}$ for some element $Q \in \mathfrak{l}$.*

Proof. Since $\tilde{Q} \in \mathfrak{l}^{[K]}$, it is equal to a linear combination $\sum_\ell c_\ell C_\ell(iH_{j_1}^{[K]}, \dots, iH_{j_n}^{[K]})$ of repeated commutators $C_\ell(iH_{j_1}^{[K]}, \dots, iH_{j_n}^{[K]})$ containing the elements $\{iH_{j_1}^{[K]}, \dots, iH_{j_n}^{[K]}\}$ with $j_k \in \{1, \dots, d\}$. However, $\mathfrak{l}$ is generated by $iH_1, \dots, iH_k$ and it contains the same commutators $C_\ell(iH_{j_1}, \dots, iH_{j_n})$ yet with $H_j^{[K]}$ replaced by H_j. Hence one can form a linear combination Q such that $Q^{[K]} = \tilde{Q}$ as stated. □

Moreover, we now have the tools to prove the relation between the Lie algebra $\mathfrak{l}$ and the dynamical group $\mathcal{G}$ already stated in theorem 2.1.

Proposition 4.4. *Consider again $iH_1, \dots, iH_d \in \mathfrak{u}(X)$ and the Lie algebra $\mathfrak{l}$ generated by them. Then the corresponding dynamical group $\mathcal{G}$ coincides with the strong closure of* $\exp(\mathfrak{l}) \subset \mathcal{U}(X)$.

Proof. Each $U \in \mathcal{G}$ can be written as the limit of a net $(U_\lambda)_{\lambda \in \mathcal{I}}$ of operators U_λ, which are monomials in $\exp(it_k H_k)$ with $k \in \{1, \dots, d\}$ with appropriate times t_k. This implies in particular that the U_λ commute with $\pi(z)$ for all z, and, by continuity, the same is true for U. Hence $U \in \mathcal{U}(X)$, and for each $K \in \mathbb{N}$ we can define $U^{[K]}$ which is the limit of the net $(U_\lambda^{[K]})_{\lambda \in \mathcal{I}}$. The latter converges in norm (since $\mathcal{H}^{[K]}$ is finite-dimensional), and therefore $U^{[K]} \in \mathcal{G}^{[K]}$. This implies $U^{[K]} = \exp(Q_K)$ with $Q_K \in \mathfrak{l}^{[K]}$ as $\mathcal{G}^{[K]}$ is a Lie group and $\mathfrak{l}^{[K]}$ its Lie algebra.

For U to be in the strong closure of $\exp(\mathfrak{l})$, each strong ϵ-neighborhood of U, i.e. the sets $\mathcal{N}(U; \psi_1, \dots, \psi_f; \epsilon)$ introduced in equation (3), should contain an element of $\exp(\mathfrak{l})$ for all $\psi_1, \dots, \psi_f$ and all $\epsilon > 0$. However, the unitary group is contained in the unit ball of $\mathcal{B}(\mathcal{H})$, and thus it is sufficient to consider only those $\mathcal{N}(U; \psi_1, \dots; \psi_f, \epsilon)$ with vectors $\psi_1, \dots, \psi_f$ from a dense subspace of $\mathcal{H}$; cf I.3.1.2 in [45]. Hence, in turn, it is sufficient to consider only neighborhoods with $\psi_j \in D_X$. But then there is a $K \in \mathbb{N}$ such that $\psi_j \in \mathcal{H}^{[K]}$ for all $j \in \{1, \dots, f\}$. Now take the operator Q_K from the last paragraph and $\tilde{Q}_K \in \mathfrak{l}$ with $\tilde{Q}_K^{[K]} = Q_K$, which exists due to lemma 4.3. By construction we have $\|[U - \exp(\tilde{Q}_K)]\psi_j\| = \|[U^{[K]} - \exp(\tilde{Q}_K)^{[K]}]\psi_j\| = \|[U^{[K]} - \exp(\tilde{Q}_K^{[K]})]\psi_j\| = \|[U^{[K]} - \exp(Q_K)]\psi_j\| = 0$ since $U^{[K]} = \exp(Q_K)$, as was also seen in the previous paragraph. Hence $\exp(\tilde{Q}_K) \in \mathcal{N}(U; \psi_1, \ldots, \psi_f; \epsilon)$ which shows that U is in the strong closure of $\exp(\mathfrak{l})$. This shows that the dynamical group $\mathcal{G}$ is contained in the strong closure of $\exp(\mathfrak{l})$.

Conversely, consider $\exp(Q)$ for $Q \in \mathfrak{l}$. We have to show that $\exp(Q)$ is in the dynamical group $\mathcal{G}$. To this end we observe, for each $K \in \mathbb{N}$, that $\exp(Q^{[K]}) = \exp(Q)^{[K]}$, which is obviously in $\mathcal{G}^{[K]}$. Hence there is a $U_K = \exp(iH_{j_1}^{[K]}) \cdots \exp(iH_{j_n}^{[K]})$ with $j_k \in \{1, \ldots, d\}$ which is ϵ-close (in norm) to $\exp(Q^{[K]})$. As in the last paragraph, this implies that $\tilde{U} = \exp(iH_{j_1}) \cdots \exp(iH_{j_n})$ is in $\mathcal{N}(\exp(Q); \psi_1, \ldots, \psi_f; \epsilon)$ provided $\psi_j \in \mathcal{H}^{[K]}$ for all $j \in \{1, \ldots, f\}$. Hence $\exp(Q)$ is in the strong closure of the group of monomials in the $\exp(iH_j)$, but this is just the dynamical group $\mathcal{G}$. Since $\mathcal{G}$ is strongly closed, the strong closure of $\exp(\mathfrak{l})$ is contained in $\mathcal{G}$, too. Since we have shown the other inclusion before, the entire proposition is proven. □

Moreover, with this proposition the proof of theorem 2.1 is complete. The rest of this subsection is devoted to analyzing a related question: if, in finite dimension, two Lie algebras $\mathfrak{l}_1$, $\mathfrak{l}_2$ generate the same group, then they are actually identical. However, in infinite dimensions this is no longer true. Therefore, the next proposition is meant to decide if dynamical groups generated by two different sets of Hamiltonians do in fact coincide.

Proposition 4.5. *Consider two Lie algebras $\mathfrak{l}_1, \mathfrak{l}_2 \subset \mathfrak{u}(X)$. Assume that for each $Q \in \mathfrak{l}_1$ and each $K \in \mathbb{N}$, there is a $\tilde{Q} \in \mathfrak{l}_2$ such that $Q^{[K]} = \tilde{Q}^{[K]}$ holds (note that we can have different $\tilde{Q}$ for the same Q but different K). Then $\exp(\mathfrak{l}_1)$ is contained in the strong closure of $\exp(\mathfrak{l}_2)$.*

Proof. One may readily use the same strategy as in the proof of proposition 4.4: if the given condition holds, one can find in each neighborhood $\mathcal{N}(\exp(Q); \psi_1, \ldots, \psi_f; \epsilon)$ of $\exp(Q)$ with $\psi_1, \ldots, \psi_f \in D_X$ an $\exp(\tilde{Q})$ with $\tilde{Q} \in \mathfrak{l}_2$. Hence $\exp(Q)$ is in the strong closure of $\exp(\mathfrak{l}_2)$. □

Inserting $\mathfrak{su}(X)$ for $\mathfrak{l}_2$ provides a useful criterion to check whether the dynamical group $\mathcal{G}$ generated by $H_1, \ldots, H_d \in \mathfrak{su}(X)$ is as large as possible in the sense that $\mathcal{G} = \mathcal{SU}(X)$. To this end, let us introduce the truncated versions

$$\begin{aligned} \mathfrak{su}^{[K]}(X) &= \{Q^{[K]} \mid Q \in \mathfrak{su}(X)\} = \oplus_{\mu=0}^{K} \ \mathfrak{su}(\mathcal{H}^{(\mu)}), \\ \mathcal{SU}^{[K]}(X) &= \{U^{[K]} \mid U \in \mathcal{SU}(X)\} = \oplus_{\mu=0}^{K} \ \mathcal{SU}(\mathcal{H}^{(\mu)}), \end{aligned} \tag{35}$$

where we have used for any finite-dimensional subspace $\mathcal{K}$ of $\mathcal{H}$ the notations $\mathfrak{su}(\mathcal{K})$ for the Lie algebra of traceless operators on $\mathcal{K}$ and similarly $\mathcal{SU}(\mathcal{K})$ for the Lie group of unitaries on $\mathcal{K}$ with determinant 1. Note that elements of $\mathfrak{su}(\mathcal{K})$ and $\mathcal{SU}(\mathcal{K})$ have—as operators on $\mathcal{H}$—a finite rank and their support and range are both contained in $\mathcal{K}$.

Corollary 4.6. *Consider Hamiltonians* $iH_1, \ldots, iH_d \in \mathfrak{su}(X)$*, the corresponding dynamical group* $\mathcal{G}$ *and the generated Lie algebra* $\mathfrak{l}$*. If* $\mathfrak{su}^{[K]}(X) = \mathfrak{l}^{[K]}$ *holds for all* $K \in \mathbb{N}$*, then one finds* $\mathcal{G} = \mathcal{SU}(X)$*.*

Proof. Simple application of propositions 4.4 and 4.5. □

4.2. One atom

The material just introduced readily applies to the systems studied in section 3. This includes in particular the proofs of theorems 3.1, 3.3, 3.5 and 3.6. The first step is again one atom interacting with a cavity (section 3.1). Hence the Hilbert space is $\mathcal{H} = \mathbb{C}^2 \otimes \mathrm{L}^2(\mathbb{R})$ and the U(1)-symmetry under consideration is generated by the operator $X_1 = \sigma_3 \otimes \mathbb{1} + \mathbb{1} \otimes N$ already defined in (12). The domain of X_1 is D from equation (11), which is identical to D_{X_1} introduced in (5).

The next step is to characterize the Lie algebra $\mathfrak{l}$ generated by the control Hamiltonians $H_{\mathrm{JC},1}$ and $H_{\mathrm{JC},2}$ as defined in (10). They admit $D = D_{X_1}$ as a joint common domain, and it is easy to see that they commute with X_1 (in the sense introduced in the previous subsection). Hence $\mathfrak{l} \subset \mathfrak{u}(X_1)$, and all the machinery from subsection 4.1 applies. This includes in particular the block-diagonal decomposition of operators $A \in \mathfrak{u}(X_1)$ given in equation (28). In our case the subspaces $\mathcal{H}^{(\mu)}$ with $\mu \in \mathbb{N}$ are given by (cf equation (14)) $\mathcal{H}^{(\mu)} = \mathrm{span}\,\{|\mu, 0\rangle, \; |\mu, 1\rangle\}$ using the basis $|\mu, \nu\rangle \in \mathcal{H}$ introduced in (13). For $\mu = 0$, we get the one-dimensional space $\mathcal{H}^{(0)} = \mathbb{C}\,|0, 0\rangle$. The restrictions $H^{(\mu)}_{\mathrm{JC},j}$ of the operators $H_{\mathrm{JC},j}$ to the subspaces $\mathcal{H}^{(\mu)}$ are given by (for $\mu \geqslant 1$):

$$H^{(\mu)}_{\mathrm{JC},1} = -\varsigma^{(\mu)}_3/2, \qquad H^{(\mu)}_{\mathrm{JC},2} = \sqrt{\mu}\,\varsigma^{(\mu)}_1, \qquad H_{\mathrm{JC},3} = \left(\mu + 1/2\right)\varsigma^{(\mu)}_0 - \varsigma^{(\mu)}_3/2, \quad (36)$$

where we have introduced the operators $\varsigma_\alpha = \sum_\mu \varsigma^{(\mu)}_\alpha$ with $\alpha \in \{0, \ldots, 3\}$ via their projections $\varsigma^{(\mu)}_0 = \mathbb{1}^{(\mu)} = X^{(\mu)} = |\mu, 0\rangle\langle\mu, 0| + |\mu, 1\rangle\langle\mu, 1|$, $\varsigma^{(\mu)}_1 = |\mu, 0\rangle\langle\mu, 1| + |\mu, 1\rangle\langle\mu, 0|$, $\varsigma^{(\mu)}_2 = i\left(|\mu, 1\rangle\langle\mu, 0| - |\mu, 0\rangle\langle\mu, 1|\right)$, and $\varsigma^{(\mu)}_3 = |\mu, 0\rangle\langle\mu, 0| - |\mu, 1\rangle\langle\mu, 1|$. Hence, for each fixed μ, the operator $\varsigma^{(\mu)}_\alpha$ is just the corresponding Pauli operator on $\mathcal{H}^{(\mu)}$ given in the basis $|\mu, 0\rangle$, $|\mu, 1\rangle$. We have used the core symbol ς rather than σ in order to avoid confusion with the operators $\sigma_\alpha \otimes \mathbb{1}$ acting only on the atom. In addition we introduce the operators $A_{\alpha,k} \in \mathfrak{u}(X_1)$ with $\alpha \in \{0, \ldots, 3\}$ and $k \in \mathbb{N}_0$ by

$$A_{\alpha,k} = \sqrt{X_1}\,X_1^k \varsigma_\alpha \text{ for } \alpha \in \{1, 2\}, \; A_{3,k} = X_1^k \varsigma_3, \; A_{0,k} = X_1^k. \quad (37)$$

In terms of the $A_{\alpha,k}$, now the $H_{\mathrm{JC},j}$ can readily be re-expressed as

$$H_{\mathrm{JC},1} = -A_{3,0}/2, \; H_{\mathrm{JC},3} = A_{1,0}/2, \; H_{\mathrm{JC},2} = A_{0,1} + (A_{0,0} - A_{3,0})/2. \quad (38)$$

The next lemma shows that the Lie algebra $\mathfrak{l}$ generated by the $H_{\mathrm{JC},j}$ is spanned as a vector space by a subset of the $A_{\alpha,k}$.

Lemma 4.7. *The Lie algebra* $\mathfrak{l}$ *generated by* $iH_{\mathrm{JC},j}$ *with* $j \in \{1, 2\}$ *is spanned as a vector space by the operators* $iA_{\alpha,k}$ *with* $\alpha \in \{1, 2, 3\}$ *and* $k \in \mathbb{N}_0$*.*

Proof. Obviously the operators $iA_{\alpha,k}$ are in $\mathfrak{su}(X_1)$. Hence, they span a subspace $\tilde{\mathfrak{l}} \subset \mathfrak{su}(X_1)$. To prove that $\tilde{\mathfrak{l}}$ is a Lie subalgebra of $\mathfrak{su}(X_1)$ one only has to check that $[A_{\alpha,k}, A_{\beta,j}] \in \tilde{\mathfrak{l}}$ for all $\alpha, \beta \in \{1, 2, 3\}$ and $j, k \in \mathbb{N}_0$. This follows easily, because the $A_{\alpha,k}$ are just products of powers of X_1 and the ς_α. But the latter are representatives of the Pauli operators. Hence

$$[A_{1,k}, A_{2,\ell}] = 2iA_{3,k+\ell+1}, \ [A_{3,k}, A_{1,\ell}] = 2iA_{2,k+\ell}, \ [A_{2,k}, A_{3,\ell}] = 2iA_{1,k+\ell}, \tag{39}$$

All operators vanish in the case of $\mu = 0$. Hence $\tilde{\mathfrak{l}}$ is a Lie algebra and equation (38) proves that $\mathfrak{l} \subset \tilde{\mathfrak{l}}$.

For proving $\tilde{\mathfrak{l}} = \mathfrak{l}$, one has to express the $A_{\alpha,k}$ for $\alpha \in \{1, 2, 3\}$ and $k \in \mathbb{N}_0$ in terms of repeated commutators of the $H_{\mathrm{JC},2}$ and $H_{\mathrm{JC},3}$. By the commutation relations in equation (39) it is obvious that $\tilde{\mathfrak{l}}$ is generated (as a Lie algebra) by $A_{\alpha,0}$ with $\alpha \in \{1, 2, 3\}$. Therefore, the statement follows from equation (38), which in turn shows that $A_{1,0}$ and $A_{3,0}$ are just $H_{\mathrm{JC},3}$ and $H_{\mathrm{JC},1}$, while $A_{2,0}$ can be derived from the commutator $[H_{\mathrm{JC},1}, H_{\mathrm{JC},3}]$. □

With this lemma and the material developed in the last subsection, one can proceed to determine the structure of the dynamical group generated by $H_{\mathrm{JC},1}$ and $H_{\mathrm{JC},2}$. This is the content of theorem 3.1, which is restated (and proven) here as a proposition.

Proposition 4.8. *The dynamical group generated by $H_{\mathrm{JC},1}$ and $H_{\mathrm{JC},2}$ is equal to $\mathcal{SU}(X)$.*

Proof. According to proposition 4.4 the dynamical group $\mathcal{G}$ is the strong closure of $\exp(H)$ with $H \in \mathfrak{l}$, i.e. the Lie algebra generated by $H_{\mathrm{JC},1}$ and $H_{\mathrm{JC},2}$, while $\mathcal{SU}(X)$ is the strong closure of $\exp(\mathfrak{su}(X))$. Hence, by corollary 4.6 we have to show that the truncated algebras $\mathfrak{l}^{[K]}$ and $\mathfrak{su}^{[K]}(X)$ are identical. The inclusion $\mathfrak{l}^{[K]} \subset \mathfrak{su}^{[K]}(X)$ is trivial, since all the blocks $H^{(\mu)}_{\mathrm{JC},j}$ with $j \in \{1, 2\}$ are traceless. To show the other inclusion, first note that $\mathfrak{l}^{[0]} = \mathfrak{su}(X)^{[0]} = \{0\}$. Hence it is sufficient to check that for each fixed $0 < \mu_0 \leqslant K$ and each $iH \in \mathfrak{su}^{[K]}(X)$ with $H^{(\mu)} = 0$ for $\mu \neq \mu_0$ there is an $iA \in \mathfrak{l}$ such that $iA^{(\mu_0)} = iH^{(\mu_0)}$ and $A^{(\mu)} = 0$ for all $0 < \mu \leqslant K$ with $\mu \neq \mu_0$. The rest follows by linearity.

For constructing such an A, recall from lemma 4.7 that $\mathfrak{l}$ is spanned (as a vector space) by the $A_{\alpha,k}$ with $\alpha \in \{1, 2, 3\}$ and $k \in \mathbb{N}_0$. Now consider a polynomial f in one real variable satisfying $f(\mu) = 0$ for all $0 < \mu \leqslant K$ with $\mu \neq \mu_0$ and $f(\mu_0) = 1$. The operators $B_{\alpha,f} = f(X)\sqrt{X}\varsigma_\alpha$ with $\alpha \in \{1, 2\}$ and $B_{3,f} = f(X)\varsigma_3$ are linear combinations of the $A_{\alpha,k}$, and they satisfy the condition $B^{(\mu)}_{\alpha,f} = 0$ for all $0 < \mu \leqslant K$ such that $\mu \neq \mu_0$ and $B^{(\mu_0)}_{\alpha,f} = c_\alpha \varsigma^{(\mu_0)}_\alpha$ for a constant c_α given by $c_1 = c_2 = \sqrt{\mu_0}$ and $c_3 = 1$. But all traceless operators $H^{(\mu_0)} \in \mathcal{B}(\mathcal{H}^{(\mu_0)})$ can be written as a linear combinations of the $\varsigma^{(\mu_0)}_\alpha$, which concludes the proof. □

Before proceeding to the next subsection, consider the free Hamiltonian of the cavity $H_{\mathrm{JC},3}$. We have omitted it from the discussion of the dynamical group, and the reason can be seen easily from (39): $H_{\mathrm{JC},2}$ differs from $\mathrm{H}_{\mathrm{JC},1}$ only by $X_1 + \mathbb{1}/2$ which commutes with all elements of $\mathfrak{su}(X)$. Hence adding $H_{\mathrm{JC},3}$ as a control Hamiltonian would just add a one-dimensional center to the dynamical group $\mathcal{G} = \mathcal{SU}(X)$. For the same reason, $H_{\mathrm{JC},3}$ could be easily added as a drift term. Any effect it may have can be undone by evolving the system with $H_{\mathrm{JC},1}$, and the remaining relative phase between sectors of different charge μ does not affect the discussion of strong controllability in section 5. Finally, let us remark that—due to the same reasons just discussed—we could exchange $H_{\mathrm{JC},1}$ and $H_{\mathrm{JC},3}$ almost without changes to the results of this subsection.

4.3. Many atoms with individual control

First, recall some notations from section 3.2. The Hilbert space is $\mathcal{H}_M = (\mathbb{C}^2)^{\otimes M} \otimes \mathrm{L}^2(\mathbb{R})$ using the distinguished basis $|\mu; \vec{b}\rangle$ with $\vec{b} \in \mathbb{Z}_2^M$ from equation (21). The charge operator is $X_M = S_3 \otimes \mathbb{1} + \mathbb{1} \otimes N$, cf equation (20), with domain D_M from equation (19). In addition, let us introduce the re-ordered tensor product (where $|\mu, b_1, \dots, b_M\rangle \in \mathcal{H}_M$ and $b \in \mathbb{Z}_2$)

$$|\mu, \vec{b}\rangle \,\hat{\otimes}_k\, |b\rangle = |\mu + b; b_1, \dots, b_{k-1}, b, b_k, \dots, b_M\rangle \in \mathcal{H}_{M+1}. \tag{40}$$

The key result of this section is split into the following three lemmas, which eventually will lead to a proof of theorem 3.3.

Lemma 4.9. *The complexification $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu)})$ of the real Lie algebra $\mathfrak{su}(\mathcal{H}_M^{(\mu)})$ is generated by elements $|\mu; \vec{b}\rangle\langle\mu; \vec{c}|$ with $\vec{b}, \vec{c} \in \mathbb{Z}_2^M$ satisfying $\vec{b} \neq \vec{c}$.*

Proof. $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu)})$ is isomorphic to the Lie algebra $\mathfrak{sl}(\mathcal{H}_M^{(\mu)})$ of traceless operators on $\mathcal{H}_M^{(\mu)}$. The $|\mu; \vec{b}\rangle\langle\mu; \vec{c}|$ with $\vec{b} \neq \vec{c}$ span the full vector space of all $A \in \mathcal{B}(\mathcal{H}_M^{(\mu)})$ satisfying $\langle\mu; \vec{b} \mid A \mid \mu; \vec{b}\rangle = 0$ for all $\vec{b} \in \mathbb{Z}_2^M$ i.e. all operators which are off-diagonal in the basis $|\mu; \vec{b}\rangle$. The smallest Lie algebra containing this space is $\mathfrak{sl}(\mathcal{H}_M^{(\mu)})$. □

Lemma 4.10. *The Lie algebra $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_{M+1}^{(\mu)})$ is generated by the union of the subalgebras $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu-b)} \hat{\otimes}_k |b\rangle)$ with $b \in \mathbb{Z}_2$ and $k \in \{1, \dots, M\}$.*

Proof. First of all, note that (by definition) $|\mu - b; \vec{b}\rangle \in \mathcal{H}_M^{(\mu-b)}$. Hence $|\mu - b; \vec{b}\rangle \,\hat{\otimes}_k\, |b\rangle \in \mathcal{H}_{M+1}^{(\mu)}$ which shows that all the Hilbert spaces $\mathcal{H}^{(\mu-b)} \hat{\otimes}_k |b\rangle$ are subspaces of $\mathcal{H}_M^{(\mu)}$. According to the previous lemma, we have to show that operators $A = |\mu; \vec{b}\rangle\langle\mu; \vec{c}|$ with $\vec{b}, \vec{c} \in \mathbb{Z}_2^{M+1}$ and $\vec{b} \neq \vec{c}$ can be written as commutators from operators in the Lie algebra $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu+b)} \hat{\otimes}_k |b\rangle)$. We have to distinguish two cases: In the first case, there is at least one $k \in \{1, \dots, M\}$ with $b_k = c_k = b$. If this holds, A can be written as $|\mu - b; b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_{M+1}\rangle\langle\mu - b; c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_{M+1}| \otimes |b\rangle\langle b| \in \mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu-b)} \hat{\otimes}_k |b\rangle)$. The second case arises if $b_k \neq c_k$ for all k. Now consider the commutator of the operators $B = |\mu; \vec{b}\rangle\langle\mu; b_1, c_2, \dots, c_{M+1}|$ and $C = |\mu; b_1, c_2, \dots, c_{M+1}\rangle\langle\mu; \vec{c}|$ obviously it follows that $A = [B, C]$, $B \in \mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu-b_1)} \hat{\otimes}_1 |b_1\rangle)$, and $C \in \mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu-c_k)} \hat{\otimes}_k |c_k\rangle)$ for $k > 1$. This concludes the proof. □

Lemma 4.11. *The Lie algebra* $\mathfrak{su}(\mathcal{H}_{M+1}^{(\mu)})$ *is contained in the Lie algebra* $\mathfrak{g}$ *generated by* $\mathfrak{su}(\mathcal{H}_M^{(\mu)})\,\hat{\otimes}_k \mathbb{1}$ *and* $\mathfrak{su}(\mathcal{H}_M^{(\mu-1)})\,\hat{\otimes}_k \mathbb{1}$.

Proof. First of all note that it is sufficient to prove the statement for the corresponding complexified Lie algebras $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_{M+1}^{(\mu)}) = \mathfrak{su}(\mathcal{H}_{M+1}^{(\mu)}) \oplus i(\mathcal{H}_{M+1}^{(\mu)})$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, since we get the original statement back by restricting the inclusion $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_{M+1}^{(\mu)}) \subset \mathfrak{g}_{\mathbb{C}}$ to anti-self-adjoint elements on both sides.

The elements of $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}^{(\mu)})\,\hat{\otimes}_k \mathbb{1}$ are of the form $A = a\hat{\otimes}_k |0\rangle\langle 0| + a\hat{\otimes}_k |1\rangle\langle 1|$ with $a \in \mathfrak{su}_{\mathbb{C}}(\mathcal{H}^{(\mu)})$. We will show that both summands are elements of $\mathfrak{g}_{\mathbb{C}}$, i.e. $a\hat{\otimes}_k |b\rangle\langle b| \in \mathfrak{g}_{\mathbb{C}}$ for $b \in \{0, 1\}$. The same holds for $\mu - 1$. The statement then follows from lemma 4.10.

Use again lemma 4.9 and choose $a = |\mu; \vec{b}\rangle\langle \mu; \vec{c}|$ with $\vec{b}, \vec{c} \in \mathbb{Z}_2^M$ and $\vec{b} \neq \vec{c}$. We rewrite $A = a\hat{\otimes}_k |0\rangle\langle 0| + a\hat{\otimes}_k |1\rangle\langle 1|$ as

$$\begin{aligned} &|\mu; (b_1, \ldots, b_k, 0, b_{k+1}, \ldots, b_M)\rangle\langle \mu; (c_1, \ldots, c_k, 0, c_{k+1}, \ldots, c_M)| \\ &\quad + |\mu + 1; (b_1, \ldots, b_k, 1, b_{k+1}, \ldots, b_M)\rangle\langle \mu + 1; (c_1, \ldots, c_k, 1, c_{k+1}, \ldots, c_M)|. \end{aligned} \tag{41}$$

Moreover, the notations $\vec{b}_0 := (b_2, \ldots, b_k, 0, b_{k+1}, \ldots, b_M)$, $\vec{b}_1 := (b_2, \ldots, b_k, 1, b_{k+1}, \ldots, b_M)$, $\vec{c}_0 := (c_2, \ldots, c_k, 0, c_{k+1}, \ldots, c_M)$, and $\vec{c}_1 := (c_2, \ldots, c_k, 1, c_{k+1}, \ldots, c_M)$ allow us to simplify

$$A = \Big(|\mu - b_1; \vec{b}_0\rangle\langle \mu - c_1; \vec{c}_0| + |\mu - b_1 + 1; \vec{b}_1\rangle\langle \mu - c_1 + 1; \vec{c}_1| \Big)\,\hat{\otimes}_1 |b_1\rangle\langle c_1|. \tag{42}$$

Next, consider a second operator $B = (|\mu - c_1; \vec{c}_0\rangle\langle \mu - c_1; \vec{c}_0| - |\mu - c_1; \vec{c}_1\rangle\langle \mu - c_1; \vec{c}_1|)\,\hat{\otimes}_1 \mathbb{1}$ and assume that $M > 1$ holds. Then there is a $\ell \in \{1, \ldots, M\}$ with $b_\ell \neq c_\ell$. Without loss of generality one can assume that $\ell \neq 1$ (otherwise rewrite A in (42) as $\tilde{A}\,\hat{\otimes}_j\, |b_j\rangle\langle c_j|$ with another index j). The commutator now equals the expression $[A, B] = |\mu - b_1; \vec{b}_0\rangle\langle \mu - c_1; \vec{c}_0|\,\hat{\otimes}_1\, |b_1\rangle\langle c_1| = a\,\hat{\otimes}_k\, |0\rangle\langle 0|$. If $M = 1$ one has two possible cases: either $b = 0$ and $c = 1$ or $b = 1$ and $c = 1$. In the first case choose $B = (|\mu - c; 0\rangle\langle \mu - c; 0| - |\mu - c; 1\rangle\langle \mu - c; 1|) \otimes \mathbb{1}$, and in the second case pick $B = (|\mu - b; 0\rangle\langle \mu - b; 0| - |\mu - b; 1\rangle\langle \mu - b; 1|) \otimes \mathbb{1}$. Then the commutator $[A, B]$ leads again to $\pm|\mu - b; 0\rangle\langle \mu - c; 0| \otimes |b\rangle\langle c|$.

Therefore, one can conclude that $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu)}\,\hat{\otimes}_k\, |0\rangle) \subset \mathfrak{g}_{\mathbb{C}}$ for all k. The same reasoning holds for $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_M^{(\mu-1)}\,\hat{\otimes}_k\, |1\rangle)$. Hence the statement follows from the previous lemma. □

Now let us consider the control Hamiltonians $H_{\mathrm{IC},j}$, $H_{\mathrm{IC},M+j}$ from equation (18). We will use lemma 4.11 and an induction in M to prove theorem 3.3, which we restate here as a proposition.

Proposition 4.12. *The dynamical group generated by the control Hamiltonians* $H_{\mathrm{IC},j}$ *with* $j \in \{1, \ldots, 2M\}$ *is identical to* $\mathcal{SU}(X_M)$.

Proof. According to corollary 4.6 we have to show that for each K, we find that $\mathfrak{l}_M^{[K]} = \mathfrak{su}^{[K]}(X_M)$, where $\mathfrak{l}_M$ denotes the Lie algebra generated by the $H_{\mathrm{IC},j}$ with $j \in \{1, \ldots, 2M\}$. Since $\mathfrak{l}_M \subset \mathfrak{su}(X_M)$ is trivial, only the other inclusion has to be shown. This will be done by induction. By proposition 4.8 the statement is true for $M = 1$. Now we assume it is true for M to show that it is true for $M+1$, too. To this end, consider for each $k \in \{1, \ldots, M+1\}$ the Hamiltonians $H_{\mathrm{IC},j}$, $H_{\mathrm{IC},M+1+j}$ with $j \in \{1, \ldots, M+1\}$ and $j \neq k$. They can be regarded as operators on the Hilbert space $\mathcal{H}_M$ and they generate a Lie algebra $\mathfrak{l}_M$ which satisfies by assumption

$$\mathfrak{l}_M^{[K]} = \mathfrak{su}^{[K]}(X_M) = \bigoplus_{\mu=1}^{K} \mathfrak{su}(\mathcal{H}_M^{(\mu)}) \tag{43}$$

for all K. As operators on $\mathcal{H}_{M+1}$, they generate the Lie algebra $\mathfrak{l}_M \hat{\otimes}_k \mathbb{1} \subset \mathfrak{l}_{M+1}$ and according to (43) one finds that $\mathfrak{su}(\mathcal{H}_M^{(\mu)}) \hat{\otimes}_k \mathbb{1} \subset \mathfrak{l}_{M+1}^{[K+1]}$ holds for all $\mu \leqslant K$ and $k \in \{1, \ldots, M+1\}$. Thus, we can apply lemma 4.11 and $\mathfrak{su}(\mathcal{H}_{M+1}^{(\mu)})$ is contained in the Lie algebra $\mathfrak{l}_{M+1}^{[K+1]}$ for all $\mu \leqslant K$. But since $\mathfrak{l}_{M+1}^{(K)} \subset \mathfrak{su}^{[K]}(X_{M+1}) = \mathfrak{su}(\mathcal{H}_{M+1}^{(K)})$, one even gets $\mathfrak{su}^{[K]}(X_{M+1}) \subset \mathfrak{l}_{M+1}^{[K]}$, just as was to be shown. □

4.4. Many atoms under collective control

As a last topic in this section, we provide proofs for theorems 3.5 and 3.6. To this end, recall the notation from section 3.3. The Hilbert space is $\mathcal{H}_{\mathrm{sym}} = \mathbb{C}^{M+1} \otimes \mathrm{L}^2(\mathbb{R})$ with basis

$$|\mu; \nu\rangle = |\nu\rangle \otimes |\mu - \nu\rangle \text{ where } \nu \in \{0, \ldots, d_\mu\} \text{ and } d_\mu = \min(\mu, M). \tag{44}$$

The charge operator is again $X_M = S_3 \otimes \mathbb{1} + \mathbb{1} \otimes N$ from equation (20) but now as an operator on $\mathcal{H}_{\mathrm{sym}}$ with domain D_{sym} defined in (25) and the μ-eigenspaces $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$ become $\mathcal{H}_{\mathrm{sym}}^{(\mu)} = \operatorname{span}\{|\mu; \nu\rangle \mid \nu \in \{0, \ldots, d_\mu\}\}$; cf equation (26). The control Hamiltonians are $H_{\mathrm{TC},j}$ with $j \in \{1, \ldots, 3\}$ defined in (23) and (24). In addition let us introduce the operators $Y_3, Y_\pm \in \mathfrak{su}_{\mathbb{C}}(X_M)$ (which denotes again the complexification of $\mathfrak{su}(X_M)$) given by

$$Y_3 |\mu; \nu\rangle = \nu |\mu; \nu\rangle, \quad Y_+^{(\mu)} = \sum_{\nu=0}^{d_\mu - 1} |\mu; \nu+1\rangle\langle\mu; \nu|, \quad Y_-^{(\mu)} = \sum_{\nu=1}^{d_\mu} |\mu; \nu-1\rangle\langle\mu; \nu|. \tag{45}$$

They are related to the $H_{\mathrm{TC},j}$ by

$$\begin{aligned} H_{\mathrm{TC},1} &= Y_3 - (M/2)\mathbb{1}, \quad H_{\mathrm{TC},3} = X_M - Y_3, \\ H_{\mathrm{TC},+} &= S_+ \otimes a = f(X_M, Y_3) Y_+, \qquad H_{\mathrm{TC},-} = S_- \otimes a^* = Y_- f(X_M, Y_3) \end{aligned} \tag{46}$$

where f is a function in two variables x, y given by

$$f(x, y) = h_1(x, y) h_2(y) \sqrt{y}, \quad h_1(x, y) = \sqrt{x + 1 - y}, \quad h_2(y) = \sqrt{M + 1 - y}, \tag{47}$$

and $f(X_M, Y_3)$ has to be understood in the sense of functional caculus (both operators commute). As operators on $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$ for fixed μ, the $Y_\pm$ satisfy

$$Y_+ Y_- = \mathbb{1} - |\mu, 0\rangle\langle\mu, 0|, \quad Y_- Y_+ = \mathbb{1} - |\mu, d_\mu\rangle\langle\mu, d_\mu|, \tag{48}$$

and for any function $g(y)$ which is continuous on the spectrum of Y_3, one finds

$$Y_+ g(Y_3) = g(Y_3 - \mathbb{1}) Y_+, \quad Y_- g(Y_3) = g(Y_3 + \mathbb{1}) Y_-. \tag{49}$$

We are now prepared for the first lemma.

Lemma 4.13. *The operators $H_{\mathrm{TC},1}$, $H_{\mathrm{TC},+} = S_+ \otimes a$, and $H_{\mathrm{TC},-} = S_- \otimes a^*$ satisfy the following commutation relations (as operators on $\mathcal{H}^{(\mu)}$) (i) $[Y_3^{n-1} H_{\mathrm{TC},+}, H_{\mathrm{TC},-}] = (X_M - Y_3)$*

$Y_3^n + (N\mathbb{1} - Y_3)\,Y_3^n - (X_M - Y_3)\,(N\mathbb{1} - Y_3)\sum_{k=0}^{n-1}\binom{n}{k}\,Y_3^k$ *and* *(ii)* $[Y_3^{n+1}, H_{\mathrm{TC},+}] = \sum_{k=0}^{n}\binom{n}{k}$ $(-1)^{n-k}Y_3^k H_{\mathrm{TC},+}$.

Proof. Using equation (46) to re-express $H_{\mathrm{TC},\pm}$ in terms of $Y_\pm$, Y_3 and X_N, we get for the first commutator

$$[Y_3^{n-1}H_{\mathrm{TC},+}, H_{\mathrm{TC},-}] = Y_3^{n-1}f(X_M, Y_3)\,Y_+Y_f(X_M, Y_3) - Y_-f^2(X_M, Y_3)\,Y_3^{n-1}Y_+. \tag{50}$$

It is easy to check that $f(X_M, Y_3)\,|\mu; 0\rangle = 0$ holds. Together with (48) this leads to

$$Y_3^{n-1}f(X_M, Y_3)\,Y_+Y_- = Y_3^{n-1}f(X_M, Y_3). \tag{51}$$

With (49) we get on the other hand $Y_-f^2(X_M, Y_3)Y_+ = f^2(X_M, Y_3 + \mathbb{1})\,(Y_3 + \mathbb{1})^{n-1}Y_-Y_+$. Now observe that $h_1^2(X_M, Y_3 + \mathbb{1})\,h_2^2(Y_3 + \mathbb{1})\,|\mu; d_\mu\rangle = 0$ and use again (48) to get

$$Y_-f^2(X_M, Y_3)\,Y_+ = f^2(X_M, Y_3 + \mathbb{1})\,(Y_3 + \mathbb{1})^{(n-1)}. \tag{52}$$

Inserting (51) and (52) into (50) leads to $[Y_3^{n-1}H_{\mathrm{TC},+}, H_{\mathrm{TC},-}] = Y_3^{n-1}f^2(X_M, Y_3) - f^2(X_M, Y_3 + \mathbb{1})$ $(Y_3 + \mathbb{1})^{n-1}$, where we have used the fact that $f(X_M, Y_3)$ and Y_3 commute. Inserting the definition of f in (47) and expanding $(Y_3 + \mathbb{1})^{n-1}$ leads to the first commutator. The second commutator follows similarly from $[Y_3^{n+1}, H_{\mathrm{TC},+}] = Y_3^{n+1}f(X_M, Y_3)\,Y_+ - f(X_M, Y_3)\,Y_+Y_3^{n+1}$ and applying (49) to commute Y_+ to the right. □

We are now ready to prove theorem 3.5. The statement about the dynamical group $\mathcal{G}$ as a subgroup of $\mathcal{U}(X_M)$ is an easy consequence of the discussion in section 4.1. The second statement in theorem 3.5 is rephrased in the following proposition.

Proposition 4.14. *Consider the Lie algebra* $\mathfrak{l}_{\mathrm{TC}} \subset \mathfrak{u}(X_M)$ *generated by the* $H_{\mathrm{TC},j}$ *with* $j \in \{1,\ldots, 3\}$ *and* $\mu \in \mathbb{N}$. *The restriction* $\mathfrak{l}_{\mathrm{TC}}^{(\mu)}$ *of* $\mathfrak{l}_{\mathrm{TC}}$ *to* $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$ *coincides with the Lie algebra* $\mathfrak{u}(\mathcal{H}_{\mathrm{sym}}^{(\mu)})$ *of anti-hermitian operators on* $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$.

Proof. We will prove the corresponding statements for the complexifications: $\mathfrak{l}_{\mathrm{TC},\mathbb{C}} = \mathfrak{l}_{\mathrm{TC}} \oplus i\mathfrak{l}_{\mathrm{TC}} = \mathcal{B}(\mathcal{H}_{\mathrm{sym}}^{(\mu)})$. The proposition then follows from taking only anti-hermitian operators on both sides. Now note that $H_{\mathrm{TC},\pm} \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}$ since we can express them as linear combinations of $H_{\mathrm{TC},2}$ with the commutator of $H_{\mathrm{TC},1}$ and $H_{\mathrm{TC},2}$. Furthermore, X_M act as $\mu\mathbb{1}$ on $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$. Hence, equation (46) shows that the restriction $\mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$ is generated by $\mathbb{1}$, Y_3 and $H_{\mathrm{TC},\pm}$ considered as operators on $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$. Note that all operators in this proof are operators on $\mathcal{H}_{\mathrm{sym}}^{(\mu)}$, and therefore we simplify the notation by dropping *temporarily* the superscript μ, when operators are concerned.

The first step is to show that Y_3^k, $Y_3^jH_{\mathrm{TC},\pm} \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$ holds for all $k, j \in \mathbb{N}_0$. This is done by induction. The statement is true for $k \in \{0, 1\}$ and $j = 0$. Now assume it holds for all $k \in \{0,\ldots, n\}$ and $j \in \{1,\ldots, n-1\}$. Lemma 4.13(i) shows that the commutator $[Y_3^{n-1}H_{\mathrm{TC},+}, H_{\mathrm{TC},-}]$ is a polynomial in Y_3 with $-(n+2)\,Y_3^{n+1}$ as leading term. Since $Y_3^j \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$ for $j \in \{0,\ldots, n\}$ we can subtract all lower order terms and get $Y_3^{n+1} \in \mathfrak{l}_{\mathrm{TC}}^{(\mu)}$. To handle $Y_3^nH_{\mathrm{TC},\pm}$

we use lemma 4.13(ii). The commutator $[Y_3^{n+1}, H_{\mathrm{TC},+}]$ is of the form $P(Y_3)\, H_{\mathrm{TC},+}$ with an n^{th}-order polynomial P. Since $Y_3^k H_{\mathrm{TC},+} \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$, we can subtract all terms of order $k < n$ and conclude that $Y_3^n H_{\mathrm{TC},+} \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$.

Now consider a polynomial P with $P(\nu) = 0$ for $\nu \neq \kappa$ and $P(\kappa) = 1$ with $\nu, \kappa \in \{0,\dots, d_\mu\}$. Since all Y_3^n are in $\mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$, we get $|\mu;\kappa\rangle\langle\mu;\kappa| = P(Y_3) \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$. Applying the same argument to $Y_3^n H_{\mathrm{TC},\pm}$, we also get $|\mu;\kappa\rangle\langle\mu;\kappa\pm 1| \in \mathfrak{l}_{\mathrm{TC},\mathbb{C}}^{(\mu)}$ and the general case $|\mu;\nu\rangle\langle\mu;\lambda|$ with $\mu \neq \lambda$ can be treated with repeated commutators of $|\mu;\kappa\rangle\langle\mu;\kappa\pm 1|$ for different values of κ. □

This proposition says that the control system with Hamiltonians $H_{\mathrm{TC},j}$ with $j \in \{1, 2, 3\}$ can generate any special unitary $U^{(\mu_0)}$ on $\mathcal{H}_{\mathrm{sym}}^{(\mu_0)}$ for any μ_0. However, some calculations using computer algebra, we have done for the case $M = 2$ indicate that we cannot exhaust all of $\mathcal{SU}(X_M)$. In other words: after $U^{(\mu_0)}$ is fixed, we loose the possibility to choose an *arbitrary* $U^{(\mu)} \in \mathcal{SU}(\mathcal{H}_{\mathrm{sym}}^{(\mu)})$ for another μ. Our analysis for two atoms suggests that the Lie algebra generated by the $H_{\mathrm{TC},j}$ is almost as big as $\mathfrak{su}(X_2)$, but does not contain operators of the form $A \otimes \mathbb{1}$ with a diagonal traceless operator A (except $H_{\mathrm{TC},1}$). This observation suggests the choice of the Hamiltonians $H_{\mathrm{CC},k}$ with $k \in \{1,\dots, M+1\}$ in equation (27), which lead to a dynamical group exhausting $\mathcal{SU}(X_M)$. This is shown in the next proposition, which completes the proof of theorem 3.6.

Proposition 4.15. *The dynamical group generated by $H_{\mathrm{CC},k}$ with $k \in \{1,\dots, M+1\}$ coincides with $\mathcal{SU}(X_M)$.*

Proof. Let us introduce the operators $\kappa(k, j) \in \mathfrak{u}_{\mathbb{C}}(X_M)$ (the complexification of $\mathfrak{u}(X_M)$) given by $\kappa(k, j)^{(\mu)} = |\mu; k\rangle\langle\mu; j|$ with $k, j \in \{0,\dots, M\}$ and $\kappa(k, j) = 0$ if $k \geqslant d_\mu$ and $j \leqslant d_\mu$, where $d_\mu = \min(\mu, M+1)$; cf equation (44). We can re-express $Y_\pm$ in terms of $\kappa(k, j)$ as $Y_+ = \sum_{k=0}^{M-1}\kappa(k+1, k)$, $Y_- = \sum_{k=1}^{M}\kappa(k-1, k)$. Compare this to the definition of $Y_\pm$ in (45). The truncation of the sums occuring for $\mu < M$ is now built into the definition of the $\kappa(k, j)$. Similarly we can write the $H_{\mathrm{CC},j}$ for $j \in \{1,\dots, M\}$ as $H_{\mathrm{CC},j} = \kappa(k, k) - \kappa(k-1, k-1)$. The $\kappa(k, j)$ are particularly useful because their commutator has the following simple form: $[\kappa(k, j), \kappa(p, q)] = \delta_{jp}\kappa(k, q) - \delta_{kq}\kappa(p, j)$. Note that all truncations for small μ are automatically respected. This can be used to calculate the commutator of $H_{\mathrm{CC},k}$ and $Y_\pm$. To this end we introduce the $M \times M$ matrix (A_{jk}) with $A_{jj} = 2$, $A_{j,k} = -1$ if $|j - k| = 1$ and $A_{jk} = 0$ otherwise. Using (A_{jk}) we can write $[H_{\mathrm{CC},j}, Y_+] = \sum_k A_{jk}\kappa(k, k-1)$. The matrix (A_{jk}) is tridiagonal, and therefore its determinant can be easily calculated and it equals $M + 1$. Hence (A_{jk}) is invertible, and we can express $\kappa(j, j-1)$ for $j \in \{1,\dots, M\}$ as a linear combination of the commutators $[H_{\mathrm{CC},k}, Y_+]$.

Now consider the Lie algebra $\mathfrak{l}_{\mathrm{CC}}$ generated by $H_{\mathrm{CC},k}$ with $k \in \{1,\dots, M+1\}$ and its complexification $\mathfrak{l}_{\mathrm{CC},\mathbb{C}}$. We have $H_{\mathrm{TC},1} \in \mathfrak{l}_{\mathrm{CC}}$ since it can be written as a linear combination of the

$H_{\mathrm{CC},j}$. In addition $H_{\mathrm{TC},3} = H_{\mathrm{CC},M+1} \in \mathfrak{l}_{\mathrm{CC}}$ and since $S_+ \otimes a, S_- \otimes a^*$ can be written as (complex) linear combinations of $H_{\mathrm{TC},3}$ and its commutator with $H_{\mathrm{TC},1}$, we get $S_+ \otimes a, S_- \otimes a^* \in \mathfrak{l}_{\mathrm{CC},\mathbb{C}}$. To calculate the commutators $[H_{\mathrm{CC},j}, S_+ \otimes a]$ note that according to (45) we have $S_+ \otimes a = f(X_M, Y_3)\, Y_+$ and $f(X_M, Y_+)$ commutes with $H_{\mathrm{CC},k}$. Hence $[H_{\mathrm{CC},j}, S_+ \otimes a] = [H_{\mathrm{CC},j}, f(X_M, Y_3)\, Y_+] = f(X_M, Y_3)\, [H_{\mathrm{CC},j}, Y_+] = \sum_k A_{jk} f(X_M, Y_3)\, \kappa(k, k-1)$. Using the reasoning from the last paragraph, we see that $f(X_M, Y_3)\, \kappa(k, k-1) \in \mathfrak{l}_{\mathrm{CC}}$. Similarly we can show by using commutators with $S_- \otimes a^*$ that all $\kappa(k, k+1) f(X_M, Y_3)$ are in $\mathfrak{l}_{\mathrm{CC}}$, too. By expanding the function f we see in this way that for $k \in \{1, \ldots, M\}$ the operators

$$A_+ = P(k)\ \kappa(k, k-1),\ \ A_- = P(k)\ \kappa(k-1, k),\ \ A_3 = \kappa(k, k) - \kappa(k-1, k-1) \tag{53}$$

with $P(k) := \sqrt{X_M + (1-k)\mathbb{1}}$ are elements of $\mathfrak{l}_{\mathrm{CC},\mathbb{C}}$.

To conclude the proof, we apply again corollary 4.6. Hence we have to consider the truncated algebra $\mathfrak{l}_{\mathrm{CC}}^{[K]}$. To this end, look at the subalgebra $\mathfrak{l}_{\mathrm{CC},k}$ of $\mathfrak{l}_{\mathrm{CC}}$ generated by the operators in (53). They are acting on the subspace generated by the basis vectors $|\mu; k\rangle, |\mu; k-1\rangle$ and if we write $A_1 = A_+ + A_-$, $A_2 = i(A_+ - A_-)$ we get (up to an additive shift in the operator X_M) the same structure as already analyzed in lemma 4.7 (cf also the operators $A_{\alpha,k}$ in equation (37)). Hence we can apply the method from section 4.2 to see that for all $\mu \in \{0, \ldots, K\}$ the operators $|\mu; k\rangle\langle\mu; k| - |\mu, k-1\rangle\langle\mu, k-1|$, $|\mu; k\rangle\langle\mu; k-1|$ and $|\mu; k-1\rangle\langle\mu; k|$ are elements of $\mathfrak{l}_{\mathrm{CC},\mathbb{C}}^{[K]}$ (provided $k \leqslant d_\mu$). Now we can generate all operators $|\mu; p\rangle\langle\mu, j|$ with $p, j \leqslant d_\mu$ by repeated commutators of $|k\rangle\langle k-1|$ and $|k-1\rangle\langle k|$ for different values of k. This shows that $\mathfrak{su}_{\mathbb{C}}(\mathcal{H}_{\mathrm{sym}}^{(\mu)}) \subset \mathfrak{l}_{\mathrm{CC},\mathbb{C}}^{[K]}$ for all $\mu \leqslant K$. By passing to anti-self-adjoint elements we conclude that $\mathfrak{l}_{\mathrm{CC}}^{[K]} = \mathfrak{su}(X_M)^{[K]}$ holds for all K. Hence the statement follows from corollary 4.6. □

5. Strong controllability

The purpose of this section is to show how one can complement the block-diagonal dynamical groups from the last section to get strong controllability. We add one generator which breaks the Abelian symmetry of the block-diagonal decomposition. The proofs for pure-state controllability and strong controllability are given in propositions 5.2 and 5.6, respectively. This completes the proof of theorems 3.2, 3.4 and 3.7.

5.1. Pure-state controllability

Consider a family $H_1, \ldots, H_n$ of control Hamiltonians on the Hilbert space $\mathcal{H}$ with joint domain $D \subset \mathcal{H}$ admitting a U(1)-symmetry defined by a charge operator X with the same domain. Since all the subspaces $\mathcal{H}^{(\mu)}$ are invariant under all time evolutions, which can be constructed from the H_k, pure-state controllability cannot be achieved. For rectifying this problem, we have to add a Hamiltonian that breaks this symmetry in a specific way. We will do so by using complementary operators as in definition 2.2. Hence in addition to the projections $X^{(\mu)}$, $\mu \in \mathbb{N}_0$ we have the mutually orthogonal projections E_α, $\alpha \in \{+, 0, -\}$ introduced in section 3 and the corresponding derived structures. This includes in particular the subprojections

$X_\alpha^{(\mu)} \leqslant X^{(\mu)}$, $\mu \in \mathbb{N}_0$ and the Hilbert spaces $\mathcal{H}_\alpha^{(\mu)}$ onto which they project. Recall, that they satisfy $X_\alpha^{(\mu)} = E_\alpha X^{(\mu)}$ and $X^{(\mu)} = X_-^{(\mu)} \oplus X_0^{(\mu)} \oplus X_+^{(\mu)}$, and that for $\mu > 0$ the $X_\pm^{(\mu)}$ are required to be non-zero. For the following discussion we need in addition the Hilbert spaces $\mathcal{H}_{[K]} = \mathcal{H}^{[K]} \oplus \mathcal{H}_-^{(K+1)}$, the projections $F_{[K]}$ onto them and the group $\mathcal{SU}(X, F_{[K]})$ of $U \in \mathcal{SU}(X)$ commuting with $F_{[K]}$. Furthermore we will indicate restrictions to the subspaces $\mathcal{H}_{[K]}$ by a subscript $[K]$, e.g. $\mathcal{SU}_{[K]}(X, F_{[K]})$ denotes the corresponding restriction of $\mathcal{SU}(X, F_{[K]})$ which has the form $\mathcal{SU}_{[K]}(X, F_{[K]}) = \mathcal{SU}^{[K]}(X) \oplus \mathcal{SU}(X_-^{(K+1)})$. Now one can prove the following lemma, which will be of importance in the subsequent subsections.

Lemma 5.1. *Consider a strongly continuous representation* $\pi\colon \mathrm{U}(1) \to \mathcal{U}(\mathcal{H})$ *with charge operator X, an operator H complementary to X, and the objects just introduced. For all* $K \in \mathbb{N}$, *introduce the Lie group* $\mathcal{G}_{X,F,K}$ *generated by* $\mathcal{SU}_{[K]}(X, F_{[K]})$, $\exp(itH)$, $t \in \mathbb{R}$ *and global phases* $\exp(i\alpha)\mathbb{1}$, $\alpha \in [0, 2\pi)$. *Then the group* $\mathcal{G}_{X,F,K}$ *acts transitively on the unit sphere of* $\mathcal{H}_{[K]}$.

Proof. Consider $\phi \in \mathcal{H}_{[K]}$ and choose $\tilde{U}_1 \in \mathcal{SU}_{[K]}(X, F_{[K]})$ such that $X_+^{(\mu)}\tilde{U}_1\phi = 0$ for all $\mu > 0$. This is possible, since $\mathcal{SU}(\mathcal{H}^{(\mu)})$ acts transitively (up to a phase) on the unit vectors of $\mathcal{H}^{(\mu)} = \mathcal{H}_-^{(\mu)} \oplus \mathcal{H}_0^{(\mu)} \oplus \mathcal{H}_+^{(\mu)}$. According to item (ii) of definition 2.2 we can find $t \in \mathbb{R}$ (e.g., $t = \pi/2$ will do) such that $\exp(itH)\mathcal{H}_+^{(K+1)} = \mathcal{H}_-^{(K)}$ holds. Hence $\exp(itH)\phi \in \mathcal{H}^{[K]}$ and we can find a $\tilde{U}_2 \in \mathcal{SU}_{[K]}(X, F_{[K]})$ with $\phi_1 = \tilde{U}_2 \exp(itH)\tilde{U}_1\phi \in \mathcal{H}_{[K-1]}$. Applying this procedure K times we get $\phi_K = U_K \cdots U_1\phi \in \mathcal{H}_{[0]}$ with $U_j \in \mathcal{G}_{X,F,k}$. Similarly we can find $V_1, \ldots, V_K \in \mathcal{G}_{X,F,k}$ with $\psi_K = V_k \cdots V_1\psi \in \mathcal{H}_{[0]}$.

Now note that the group $\mathcal{G}_{X,F,0}$ can be regarded as a subgroup of $\mathcal{G}_{X,F,k}$ (which acts trivially on the orthocomplement of $\mathcal{H}_{[0]}$ in $\mathcal{H}_{[K]}$). Hence, the statement of the lemma follows from the fact that, due to condition (iii) of definition 2.2, the group $G_{X,F,0}$ acts transitively on the unit vectors in $\mathcal{K}_{[0]} = F_{[0]}\mathcal{H}$. □

The first easy consequence of this lemma is the following result which is a proof of theorem 2.3 and we restate it here as a proposition.

Proposition 5.2. *Consider a strongly continuous representation* $\pi\colon \mathrm{U}(1) \to \mathcal{U}(\mathcal{H})$ *with charge operator X and a family of self-adjoint operators* $H_1, \ldots, H_d$ *on* $\mathcal{H}$. *Assume that the following conditions hold:*

(i) All eigenvalues μ *of X are greater than or equal to 0.*
(ii) $H_1, \ldots, H_{d-1}$ *commute with X.*
(iii) The dynamical group generated by $H_1, \ldots, H_{d-1}$ *contains* $\mathcal{SU}(X)$.
(iv) The operator H_d *is complementary to X.*

Then the system (1) with Hamiltonians $H_0 = \mathbb{1}, H_1, \ldots, H_d$ *is pure-state controllable.*

Proof. We have to show that for each pair of pure states $\psi, \phi \in \mathcal{H}$ and each $\epsilon > 0$ there is a finite sequence $U_k \in \mathcal{U}(\mathcal{H})$ with $k \in \{1, \ldots, N\}$ and either $U_k \in \mathcal{SU}(X)$, $U_k = \exp(itH_d)$, or $U_k = \exp(i\alpha)\mathbb{1}$ such that $\| \psi - U_N \cdots U_1 \phi \| < \epsilon$. To this end, first note that we can find $K \in \mathbb{N}$ such that $\| \psi - F_{[K]}\psi \| < \epsilon/3$ and $\| \phi - F_{[K]}\psi \| < \epsilon/3$, where $F_{[K]}$ is the projection defined in the first paragraph of this subsection. Therefore $\| \psi - U_N \cdots U_1 \phi \| \leqslant \| \psi - F_{[K]}\psi \| + \| F_{[K]}\psi - U_N \cdots U_1 F_{[K]}\phi \| + \| U_N \cdots U_1 F_{[K]}\phi - U_N \cdots U_1 \phi \| < \epsilon$ provided that $\| F_{[K]}\psi - U_N \cdots U_1 F_{[K]}\phi \| < \epsilon/3$. Hence we can assume that $\psi, \phi \in \mathcal{H}_{[K]}$ and apply lemma 5.1. This leads to a sequence $V_1, \ldots, V_N \in \mathcal{G}_{X,F,K}$ with $V_N \cdots V_1 \phi = \psi$. Now note that the dynamical group $\mathcal{G}$ generated by $H_0, \ldots, H_d$ contains by assumption the group $\mathcal{SU}(X)$, the unitaries $\exp(itH_d)$ and the global phases $\exp(i\alpha)\mathbb{1}$. Hence with the definition of $\mathcal{G}_{X,F,K}$, we get for $j \in \{1, \ldots, N\}$ a $W_j \in \mathcal{G}$ with $[W_j, F_{[K]}] = 0$ and $F_{[K]} W_j = V_j$, and therefore $\psi = W_N \cdots W_1 \phi$. But by definition the dynamical group is the strong closure of monomials $U_N \cdots U_1$ with $U_j = \exp[it_j H_{k_j}]$ for some $t_j \in \mathbb{R}$ and $k_j \in \{0, \ldots, d+1\}$. In other words for all $U \in \mathcal{G}$, $\xi \in \mathcal{H}$ and $\epsilon > 0$ we can find such a monomial satisfying $\| U_N \cdots U_1 \xi - U\xi \| < \epsilon$. Applying this statement to the operators W_j and the vectors $W_{j-1} \cdots W_1 \phi$ concludes the proof. □

This proposition can be applied to all systems studied in section 3. Therefore, *they are all pure-state controllable*. However, as already stated, one can even prove strong controllability, which is the next goal.

5.2. Approximating unitaries

Lemma 5.1 shows that the group $\mathcal{G}_{X,F,K}$ acts transitively on the pure states in the Hilbert space $\mathcal{H}_{[K]}$. This implies that there are only two possibilities for this group: either $\mathcal{G}_{X,F,K}$ coincides with the group of symplectic unitaries on $\mathcal{H}_{[K]}$ (which is only possible if the dimension of $\mathcal{H}_{[K]}$ is even), or it is the whole unitary group [46–48]. At the same time we have seen in proposition 5.2 that (under appropriate conditions on the control Hamiltonians) each $U \in \mathcal{G}_{X,F,K}$ admits an element W in the dynamical group satisfying $W\xi = U\xi$ for all $\xi \in \mathcal{H}_{[K]}$. Proving full controllability can therefore be reduced to two steps:

(i) Find arguments that for an infinite number of $K \in \mathbb{N}$, the group $\mathcal{G}_{X,F,K}$ cannot be unitary symplectic, such that it has to coincide with the full unitary group on $\mathcal{H}_{[K]}$.

(ii) Show that each unitary $U \in \mathcal{U}(\mathcal{H})$ can be approximated by a sequence W_K, $K \in \mathbb{N}$ of unitaries of the form $W_K = U_k \oplus V_k$, where $U_k \in \mathcal{U}(\mathcal{H}_{[K]})$ can be chosen arbitrarily, while V_K is a unitary on $(\mathbb{1} - F_{[K]})\mathcal{H}$ which is (at least partly) fixed by the choice of U_k.

The purpose of this subsection is to prove the second statement, while the first one is postponed to section 5.3. We start with the following lemma:

Lemma 5.3. *Consider a sequence $F_{[K]}$, $K \in \mathbb{N}$ of finite-rank projections converging strongly to $\mathbb{1}$ and satisfying $F_{[K]} \lneqq F_{[K+1]}$. For each unitary $U \in \mathcal{U}(\mathcal{H})$ there is a sequence $U_{[K]}$, $K \in \mathbb{N}$ of partial isometries, which converges strongly to U and satisfies $U_{[K]}^* U_{[K]} = U_{[U]} U_{[K]}^* = F_{[K]}$; i.e. $F_{[K]}$ is the source and the target projection of $U_{[K]}$.*

Proof. Let us start by introducing the space $D \subset \mathcal{H}$ of vectors $\xi \in \mathcal{H}$ satisfying $F_{[K]}\xi = \xi$ for a $K \in \mathbb{N}$. It is a dense subset of $\mathcal{H}$ and we can define the map $m : D \to \mathbb{N}$, $m(\xi) = \min \{K \in \mathbb{N} \mid F_{[K]}\xi = \xi\}$. All operators in this proof are elements of the unit ball $\mathcal{B}_1(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid \| A \| \leqslant 1\}$ in $\mathcal{B}(\mathcal{H})$. A sequence A_K of elements of $\mathcal{B}_1(\mathcal{H})$ converges to $A \in \mathcal{B}_1(\mathcal{H})$ iff $\lim_{K\to\infty} A_K\xi = A\xi$ holds for all $\xi \in D$; see I.3.1.2 in [45].

Now define $A_{[K]} = F_{[K]}UF_{[K]}$. For $\xi \in D$, we have $UF_{[K]}\xi = U\xi$ if $K > m(\xi)$ and $\lim_{K\to\infty} F_{[K]}U\xi = U\xi$ since $F_{[K]}$ converges strongly to $\mathbb{1}$. Hence the strong limit of the $A_{[K]}$ is U, similarly one can show that the strong limit of $A_{[K]}^*$ is U^*. The $A_{[K]}$ are not partial isometries. We will rectify this problem by looking at the polar decomposition. To this end, first consider $|A_{[K]}|^2 = A_{[K]}^* A_{[K]}$ and $\| A_{[K]}^* A_{[K]}\xi - \xi \| = \| A_{[K]}^* A_{[K]}\xi - U^*U\xi \| \leqslant \| A_{[K]}^* (A_{[K]} - U)\xi \| + \|(A_{[K]}^* - U^*)U\xi \| \leqslant \|(A_{[K]} - U)\xi \| + \|(A_{[K]}^* - U^*)U\xi \|$ where we have used that $\| A_{[K]}^* \| \leqslant 1$ holds. Strong convergence of $A_{[K]}$ and $A_{[K]}^*$ implies $\lim_{K\to\infty} \| A_{[K]}^* A_{[K]}\xi - \xi \| = 0$. Hence $|A_{[K]}|^2$ converges strongly to $\mathbb{1}$.

The operators $A_{[K]}$ are of finite rank with support and range contained in $\mathcal{H}_{[K]} = F_{[K]}\mathcal{H}$. Hence the $|A_{[K]}|$ have pure point spectrum and their spectral decomposition is $\sum_{\lambda\in\sigma}(|A_{[K]}|)\lambda P_\lambda$ with eigenvalues $0 \leqslant \lambda \leqslant 1$ and spectral projections P_λ satisfying $P_\lambda \leqslant F_{[K]}$ for $\lambda > 0$. Using the fact that the P_λ are mutually orthogonal, we get for $|A_{[K]}|^2$,

$$\| |A_{[K]}|^2\xi - \xi \| =$$

$$\| \sum_{\lambda\in\sigma(|A_{[K]}|)} (\lambda^2 - 1)P_\lambda\phi \| = \sum_{\lambda\in\sigma(|A_{[K]}|)} |\lambda^2 - 1| \| P_\lambda\xi \| = \sum_{\lambda\in\sigma(|A_{[K]}|)} |\lambda - 1|(\lambda + 1) \| P_\lambda\xi \|$$

$$\geqslant \sum_{\lambda\in\sigma(|A_{[K]}|)} |\lambda - 1| \| P_\lambda\xi \|.$$

Hence strong convergence of $|A_{[K]}|^2$ implies strong convergence of $|A_{[K]}|$.

Now we can look at the polar decomposition $A_{[K]} = W_{[K]}|A_{[K]}|$. The $W_{[K]}$ are partial isometries, and moreover, since support and range of the $A_{[K]}$ are contained in $\mathcal{K}_{[K]}$, they satisfy $W_{[K]}^* W_{[K]} \leqslant F_{[K]}$ and $W_{[K]} W_{[K]}^* \leqslant F_{[K]}$. In other words, we can look upon the $W_{[K]}$ as partial isometries on the finite dimensional Hilbert space $\mathcal{H}_{[K]}$. As such we can extend them to untaries $U_{[K]} \in \mathcal{U}(\mathcal{H}_{[K]})$ without sacrificing the relation to $A_{[K]}$, i.e. $A_{[K]} = U_{[K]}|A_{[K]}|$. As operators on $\mathcal{H}$, the $U_{[K]}$ are still partial isometries, but now with source and target projections equal to $F_{[K]}$ as stated in the lemma.

The only remaining point is to show that the $U_{[K]}$ converges strongly to U. This follows from $\| U_{[K]}\xi - U\xi \| \leqslant \| U_{[K]}\xi - A_{[K]}\xi \| + \| A_{[K]}\xi - U\xi \|$ and $\| U_{[K]}\xi - A_{[K]}\xi \| = \| U_{[K]}(\mathbb{1} - |A_{[K]}|)\xi \|$ since $A_{[K]}$ converges strongly to U and $|A_{[K]}|$ to $\mathbb{1}$. □

Now we come back to the case discussed in the beginning of this subsection under item (ii):

Lemma 5.4. *Consider U, $F_{[K]}$ and $U_{[K]}$ as in lemma 5.3, and an additional sequence of partial isometries $V_{[K]}$, $K \in \mathbb{N}$ with $V_{[K]}^* V_{[K]} = V_{[K]} V_{[K]}^* = \mathbb{1} - F_{[K]}$. The operators $W_{[K]} = U_{[K]} + V_{[K]}$ are unitary, and if U is the strong limit of the $U_{[K]}$, the same is true for the $W_{[K]}$.*

Proof. The kernels of $U_{[K]}$ and $V_{[K]}$ are $(\mathbb{1} - F_{[K]})\mathcal{H}$ and $\mathcal{H}_{[K]} = F_{[K]}\mathcal{H}$, respectively. These spaces are complementary, and therefore $W_{[K]} = U_{[K]} + V_{[K]}$ is unitary for all K. To show strong convergence, recall the space D and the function $D \ni \xi \mapsto m(\xi) \in \mathbb{N}$ introduced in the last proof. For $\xi \in F$ we have $W_{[K]}\xi = U_{[K]}\xi$ if $K > m(\xi)$. Hence by assumption $\lim_{K\to\infty} W_{[K]}\xi = \lim_{K\to\infty} U_{[K]}\xi = U\xi$, which implies strong convergence of $W_{[K]}$ to U. □

5.3. Strong controllability

We are now prepared to prove theorem 3.4. The first step is the following lemma announced already at the beginning of subsection 5.2.

Lemma 5.5. *Consider the group $\mathcal{G}_{X,F,K}$ introduced in lemma 5.1 and assume that there is a $\mu \leqslant K$ with $d^{(\mu)} = \dim(\mathcal{H}^{(\mu)}) > 2$. Then $\mathcal{G}_{X,F,K} = \mathcal{U}(\mathcal{H}_{[K]})$.*

Proof. Consider the group $\mathcal{SG}_{X,F,K}$ consisting of elements of $\mathcal{G}_{X,F,K}$ with determinant 1. By lemma 5.1 this group acts transitively on the set of pure states of the Hilbert space $\mathcal{H}_{[K]}$. Hence, there are only two possibilities left[10]: $\mathcal{SG}_{X,F,K}$ coincides either with the unitary symplectic group $\mathrm{USp}(\mathcal{H}_{[K]})$ or with the full unitary group $\mathcal{U}(\mathcal{H}_{[K]})$; cf [46–48]. Assume $\mathcal{SG}_{X,F,K} = \mathrm{USp}(\mathcal{K}_{[K]})$ holds. This would imply that $\mathcal{SG}_{X,F,K}$ is self-conjugate (or more precisely the representation given by the identity map on $\mathcal{SG}_{X,F,K} \subset \mathcal{B}(\mathcal{H}_{[K]})$ is self-conjugate). In other words, there would be a unitary $V \in \mathcal{U}(\mathcal{H}_{[K]})$ with $VUV^* = \bar{U}$ for all $U \in \mathcal{SG}_{X,F,K}$. Here $\bar{U}$ denotes complex conjugation in an arbitrary but fixed basis (cf footnote 9).

Now consider $\mathcal{SU}(\mathcal{H}^{(\mu)})$ with $d^{(\mu)} > 2$. It can be identified with $\mathrm{SU}(d)$ in its first fundamental representation λ_1 (i.e. the 'defining' representation). At the same time it is a subgroup of $\mathcal{SG}_{X,F,K}$ (one which acts non-trivially only on $\mathcal{H}^{(\mu)} \subset \mathcal{H}_{[K]}$). Existence of a V as in the last paragraph would imply that λ_1 is unitarily equivalent to its conjugate representation, which is the $d - 1^{\mathrm{st}}$ fundamental representation. This is impossible if $d^{(\mu)} > 2$ holds. Hence V with the described properties does not exist and $\mathcal{SG}_{X,F,k}$ has to coincide with $\mathcal{SU}(\mathcal{H}_{[K]})$ and therefore $\mathcal{G}_{X,F,K} = \mathcal{U}(\mathcal{H}_{[K]})$ as stated. □

Finally we can conclude the proof of theorem 3.4 which we restate here as the following proposition:

[10] Note that $\mathcal{H}_{[K]}$ is a finite-dimensional Hilbert space. Hence after fixing a basis $e_1, \ldots, e_d$ it can be identified with $\mathbb{C}^d$.

Proposition 5.6. *A control system (1) with control Hamiltonians $H_0 = \mathbb{1}, \ldots, H_d$ satisfying the conditions from proposition 5.2 is strongly controllable, if $d^{(\mu)} = \dim \mathcal{H}^{(\mu)} > 2$ for at least one $\mu \in \mathbb{N}$.*

Proof. Consider an arbitrary unitary $U \in \mathcal{U}(\mathcal{H})$. By lemma 5.3, there is a sequence of partial isometries $U_{[K]}$ converging strongly to U, and by lemma 5.5 we can assume that $U_{[K]} \in \mathcal{G}_{X,F,K}$. Now considering the dynamical group $\mathcal{G}$ generated by the H_j, define the subgroup $\mathcal{G}(F_{[K]})$ of $U \in \mathcal{G}$ commuting with $F_{[K]}$, and the restriction $\mathcal{G}_{[K]}$ of $\mathcal{G}(F_{[K]})$ to $\mathcal{H}_{[K]}$. The assumptions on the H_j imply that $\mathcal{G}_{[K]} = \mathcal{G}_{X,F,K} = \mathcal{U}(\mathcal{H}_{[K]})$. Hence there is a sequence W_K, $K \in \mathbb{N}$ of unitaries with $W_{[K]} \in \mathcal{G}(F_{[K]}) \subset \mathcal{G}$ and $F_{[K]} W_{[K]} = U_{[K]}$. Since $U_{[K]}$ converges to U strongly, lemma 5.4 implies that the strong limit of the $W_{[K]}$ is U, which was to show. □

This proposition shows strong controllability for all the systems studied in section 3. The only exception is one atom interacting with one harmonic oscillator (section 3.1). Here we have $d^{(\mu)} = \dim \mathcal{H}^{(\mu)} \leqslant 2$ and we *can* actually find a unitary V with $VUV = \bar{U}$ for all $U \in \mathcal{SU}_{[K]}(X_1, F_{[K]})$. However, the elements U of $\mathcal{SU}(X_1)$ are block diagonal where the blocks $U^{(\mu)} \in \mathcal{SU}(\mathcal{H}^{(\mu)})$ can be chosen independently. This implies $V \in \mathcal{SU}_{[K]}(X_1, F_{[K]})$, which is incompatible with $VH_{\mathrm{JC},4}V^* = -H_{\mathrm{JC},4}$ (cf equation (15) for the definition of $\mathcal{H}_{\mathrm{JC},4}$) which would be necessary for the group $\mathcal{G}_{X_1,F,K}$ to be self-conjugate. Hence we can proceed as in the proof of proposition 5.6 to prove theorem 3.2.

6. Conclusions and Outlook

Many of the difficulties of quantum control theory in infinite dimensions arise from the fact that, due to unbounded operators, the group $\mathcal{U}(\mathcal{H})$ of all unitaries on an infinite-dimensional separable Hilbert space $\mathcal{H}$ is in fact no Lie group as long as it is equipped with the strong topology, which inevitably is the correct choice when studying questions of quantum dynamics. Yet $\mathcal{U}(\mathcal{H})$ contains a plethora of subgroups which are still infinite-dimensional while admitting a proper Lie structure—including in particular a Lie algebra $\mathfrak{l}$ consisting of unbounded operators and a well-defined exponential map. An important example are those unitaries with an Abelian U(1)-symmetry, which in the Jaynes–Cummings model relates to a kind of particle-number operator.

As shown here, this infinite-dimensional system Lie algebra $\mathfrak{l}$ can be exploited for control theory in infinite dimensions in close analogy to the finite-dimensional case. Due to the in-born symmetry of $\mathfrak{l}$ and the corresponding Lie group $\mathcal{G}$, full controllability cannot be achieved that way. Yet we have also shown that this problem can readily be overcome by complementary methods directly on the group level.

Furthermore, our scheme is quite paradigmatic: It can be generalized in a natural and (mostly) straightforward way to other Abelian symmetries (i.e. $\mathrm{U}(1)^n$ and $\mathbb{R}^n$ representations with $n > 1$).

For several (i.e., as many as one can practically control separately) 2-level atoms interacting with one harmonic oscillator (e.g. a cavity mode or a phonon mode), these methods allowed us to extend previous results substantially, in particular in two aspects also summarized

Table 1. Controllability results for several 2-level atoms in a cavity as derived here.

System	Control Hamiltonians	Controllability: system algebra $\mathfrak{g}$, dynamic group $\mathcal{G}$	
One atom	$H_{\mathrm{JC},j}$, $j = 1, 2$, equation (9)	$\mathfrak{g} = \mathfrak{su}(X_1)$, $\mathcal{G} = \mathcal{SU}(X_1)$	[theorem 3.1]
	$H_{\mathrm{JC},j}$, $j = 1, 2$, equation (9) $H_{\mathrm{JC},4}$, equation (15)	Strongly controllable[a] with $\mathcal{G} = \mathcal{U}(\mathcal{H})$	[theorem 3.2]
M atoms	$H_{\mathrm{IC},j}$, $j = 1, \ldots, 2M$ with individual controls of equation (18)	$\mathfrak{g} = \mathfrak{su}(X_M)$ and $\mathcal{G} = \mathcal{SU}(X_M)$	[theorem 3.3]
	$H_{\mathrm{IC},j}$, $j = 1, \ldots, 2M + 1$ with individual controls of equations (18, 22)	Strongly controllable[a] with $\mathcal{G} = \mathcal{U}(\mathcal{H})$	[theorem 3.4]
M atoms	$H_{\mathrm{TC},j}$, $j = 1, 2, 3$ under collective control of equation (23)	$\mathfrak{g} \subset \mathfrak{u}(X_M)$ and $\mathcal{G} \subset \mathcal{U}(X_M)$	[theorem 3.5]
	$H_{\mathrm{CC},j}$, $j = 1,\ldots, M + 1$ under collective control of equation (27)	$\mathfrak{g} = \mathfrak{su}(X_M)$ and $\mathcal{G} = \mathcal{SU}(X_M)$	[theorem 3.6]
	$H_{\mathrm{CC},j}$, $j = 1,\ldots, M + 2$ under collective control of equation (27)	Strongly controllable[a] with $\mathcal{G} = \mathcal{U}(\mathcal{H})$	[theorem 3.7]

[a] Here in the strong topology, no system algebra or exponential map exists.

in table 1: (A) We have answered approximate control and convergence questions for asymptotically vanishing control error. (B) Our results include not only reachability of states, but also its operator lift, i.e. simulability of unitary gates. To this end, we have introduced the notion of *strong controllability*, and we have shown that all systems under consideration require only a fairly small set of control Hamiltonians for guaranteeing strong controllability, i.e. simulability. Thus we anticipate the methods introduced here will find wide application to systematically characterize experimental set-ups of cavity QED and ion-traps in terms of pure-state controllability and simulability.

Acknowledgments

This work was supported in part by the EU through the integrated programmes Q-ESSENCE and SIQS, and the EU-STREP COQUIT, and moreover by the Bavarian Excellence Network ENB via the international doctorate programme of excellence *Quantum Computing, Control, and Communication* (QCCC), by *Deutsche Forschungsgemeinschaft* (DFG) in the collaborative research centre SFB 631 as well as the international research group FOR 1482 through the grant SCHU 1374/2-1.

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