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Basic mathematical properties of a vector Preisach operator in magnetic hysteresis modeling

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Abstract. This paper discusses the basic mathematical properties of the vector Preisach operator recently introduced by Della Torre, Pinzaglia and Cardelli. This includes an investigation of isotropy with a necessary and a sufficient isotropy condition, derivation of the neutral initial state and a "demagnetization process". Further, we examine the saturation behaviour with memory deletion, output alignment and a bound on the output. We show that periodic input results in periodic output and prove loop congruency. Finally, we study lag angles and losses, derive their formulas for isotropic Preisach densities and uniformly rotating input in $\mathbb{R}^2$ and show $\boldsymbol{\mathcal{P}}$ to be dissipative on closed loop inputs.

1. Introduction

The vectorial modeling of magnetic hysteresis is an important issue in the framework of electromagnetic field simulations and poses a challenge that has not yet been resolved to full satisfaction. For scalar hysteresis, the Preisach model [1, 2] is well established. The question of how to extend this model to the n-dimensional setup was first answered by Mayergoyz [3] and Damlamian and Visintin [4], who presented a vectorization by superposing scalar Preisach operators in all vectorial directions. Recently, a different vector Preisach concept was published by Della Torre, Pinzaglia and Cardelli [5, 6], which consists of a vectorization of the relay underlying the Preisach operator. It constitutes a promising new approach in the modeling of vectorial hysteresis.

In this paper, we investigate mathematically some of the basic properties of this operator. These include isotropy and the choice of a neutral memory state, the saturation behaviour, loop congruency and periodicity as well as lag angles and dissipation. Some of the results in this paper were previously mentioned in [7], but are formally discussed and proved here.

We use the following notation: $\mathrm{Map}(X;Y)$ represents the space of all functions mapping X to Y, and $\mathrm{C}(X;Y)$ (resp. $\mathrm{C}^1(X;Y)$) the subspace of continuous (resp. continuously differentiable) functions. We denote by $B_{\boldsymbol{x},r} = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y}-\boldsymbol{x}\| < r\}$ the ball of radius r centered at $\boldsymbol{x}$, and $\partial B_{\boldsymbol{x},r}$ its boundary. The *orthogonal group* of degree n, $O(n) = \{Q \in \mathbb{R}^{n\times n} \mid Q^T Q = \mathcal{I}d\}$, constitutes the symmetries of $\mathbb{R}^n$ preserving the origin. The action of $O(n)$ on $\mathbb{R}^n$ induces obvious actions on different sets of functions: $(Q\boldsymbol{u})(t) = Q(\boldsymbol{u}(t))$ for $\boldsymbol{u} \in \mathrm{Map}([0,T];\mathbb{R}^n)$; $(Q\omega)(\boldsymbol{x},r) = \omega(Q^{-1}\boldsymbol{x},r)$ for $\omega \in \mathrm{Map}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R})$; and $(Q\boldsymbol{\xi})_{(\boldsymbol{x},r)} = Q(\boldsymbol{\xi}_{(Q^{-1}\boldsymbol{x},r)})$ for $\boldsymbol{\xi} \in \mathrm{Map}(\mathbb{R}^n \times \mathbb{R}_+;\partial B_{\mathbf{0},1})$, where the evaluation of $\boldsymbol{\xi}$ at $(\boldsymbol{x},r)$ is denoted $\boldsymbol{\xi}_{(\boldsymbol{x},r)}$.

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Mathematically, an operator $\boldsymbol{\mathcal{W}} :\ \text{Map}([0,T];\mathbb{R}^n) \to \text{Map}([0,T];\mathbb{R}^n)$ is called a (vectorial) hysteresis operator if it is rate-independent and satisfies the Volterra property.

2. Vector relay operator

The n-dimensional vector relay $\boldsymbol{h}_{(\boldsymbol{x},r)}$ associated with a tuple $(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R}_+$ is represented by the open ball $B_{\boldsymbol{x},r}$. It maps a continuous input function $\boldsymbol{u}$ to an output function $\boldsymbol{w}$, the *relay state*, taking unit vectors as value. Setting $X_t := \{\tau \in [0,t] \mid \|\boldsymbol{u}(\tau) - \boldsymbol{x}\| \geq r\}$, the vector relay is defined by

$$\boldsymbol{h}_{(\boldsymbol{x},r)} :\ \text{C}([0,T];\mathbb{R}^n) \times \partial B_{\mathbf{0},1} \to \text{Map}([0,T];\partial B_{\mathbf{0},1}),$$

$$\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_{(\boldsymbol{x},r)}](t) = \begin{cases} \frac{\boldsymbol{u}(t)-\boldsymbol{x}}{\|\boldsymbol{u}(t)-\boldsymbol{x}\|} & \text{if } \|\boldsymbol{u}(t)-\boldsymbol{x}\| \geq r, \\ \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_{(\boldsymbol{x},r)}](\max X_t) & \text{if } \|\boldsymbol{u}(t)-\boldsymbol{x}\| < r \text{ and } X_t \neq \varnothing, \\ \boldsymbol{\xi}_{(\boldsymbol{x},r)} & \text{otherwise.} \end{cases} \quad (1)$$

Here, like for the scalar relay, an initial value $\boldsymbol{\xi}_{(\boldsymbol{x},r)} \in \partial B_{\mathbf{0},1}$ must be given, in case $\boldsymbol{u}(0)$ lies inside $B_{\boldsymbol{x},r}$ and thus leaves $\boldsymbol{w}(0)$ undetermined. For ease of notation, when $\boldsymbol{\xi}_{(\boldsymbol{x},r)}$ is irrelevant or clear from context, we will omit it and just write $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}]$.

Figure 1 gives an illustration of $\boldsymbol{h}_{(\boldsymbol{x},r)}$. Note that for $n = 1$, (2) results in the definition of the scalar relay operator, as exposed in detail in [7]. It is quickly seen that $\boldsymbol{h}_{(\boldsymbol{x},r)}$, like its scalar analogue, is rate-independent and satisfies the Volterra property, and is thus a hysteresis operator.

For $\tau \in [0,T]$, define the *shift* $\boldsymbol{u}^\tau$ of a function $\boldsymbol{u} : [0,T] \to \mathbb{R}^n$ by

$$\boldsymbol{u}^\tau(t) = \boldsymbol{u}(t+\tau), \quad t \in [0, T-\tau].$$

The proofs of the following two lemmas on basic properties of $\boldsymbol{h}_{(\boldsymbol{x},r)}$ are straightforward.

Lemma 1 (Semigroup property)**.** *For any* $(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_{(\boldsymbol{x},r)}](t_2) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}^{t_1},\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_{(\boldsymbol{x},r)}](t_1)](t_2-t_1) \quad \textit{for all } t_1,t_2 \in [0,T] :\ t_1 \leq t_2.$$

Lemma 2 (Rotation and reflection)**.** *For any* $(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$Q\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_{(\boldsymbol{x},r)}] = \boldsymbol{h}_{(Q\boldsymbol{x},r)}[Q\boldsymbol{u},Q\boldsymbol{\xi}_{(\boldsymbol{x},r)}] \quad \textit{for all } Q \in O(n).$$

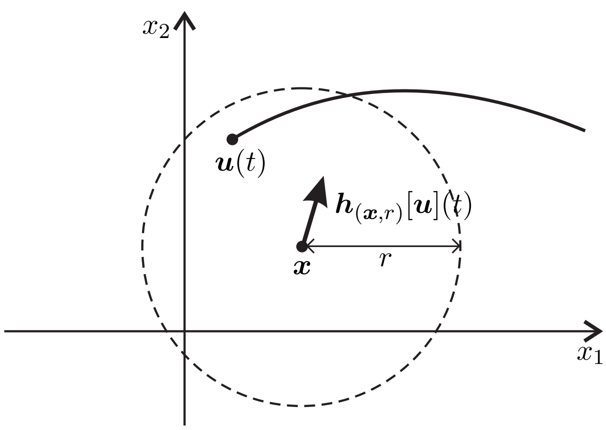

Figure 1. Vector relay operator $\boldsymbol{h}_{(\boldsymbol{x},r)}$: If $||\boldsymbol{u}(t)-\boldsymbol{x}|| \geq r$ then $\boldsymbol{w}(t) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}](t)$ is the unit vector based at $\boldsymbol{x}$ pointing at $\boldsymbol{u}(t)$. As $\boldsymbol{u}$ enters the relay and $||\boldsymbol{u}(t)-\boldsymbol{x}|| < r$, the output "freezes" the moment $\boldsymbol{u}$ crosses the relay boundary and does not vary until $\boldsymbol{u}$ leaves the relay again.

We say that a function $\boldsymbol{u} \in \mathrm{C}([0,T];\mathbb{R}^n)$ has *bounded oscillation* if, for any $(\boldsymbol{x}, r)$, the set $u([0,T]) \cap \partial B_{\boldsymbol{x},r}$ is finite. The input functions of interest to us (e.g. piecewise linear functions) all have bounded oscillation. We can show that each relay satisfies a dissipation condition on closed paths of bounded oscillation. Related investigations were done in [8] considering an elliptic relay and rectangular sample paths. For simplicity, we confine the statement to piecewise differentiable inputs. Integral (2) is to be interpreted as a Riemann-Stieltjes integral.

Lemma 3 (Dissipation)**.** *Assume* $\boldsymbol{u} \in \mathrm{C}([0,T];\mathbb{R}^n)$ *is piecewise* C^1 *and has bounded oscillation. Let* $\boldsymbol{w} = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}, \boldsymbol{\xi}]$*. If* $\boldsymbol{u}(0) = \boldsymbol{u}(T)$ *and* $\boldsymbol{w}(0) = \boldsymbol{w}(T)$*, then* $\boldsymbol{h}_{(\boldsymbol{x},r)}$ *satisfies the dissipation property*

$$\int_0^T \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{w} \geq 0. \tag{2}$$

Proof. Set $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{x}$. By the assumption that $\boldsymbol{w}(0) = \boldsymbol{w}(T)$ we have

$$\int_0^T \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{w} = \int_0^T \boldsymbol{x} \cdot \mathrm{d}\boldsymbol{w} + \int_0^T \widetilde{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{w} = \int_0^T \widetilde{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{w}. \tag{3}$$

By the bounded oscillation property, $\boldsymbol{u}$ intersects the relay boundary only finitely many times. For any interval $[t_1, t_2] \subseteq [0, T]$ on which $\|\widetilde{\boldsymbol{u}}(t) - \boldsymbol{x}\| \geq r$, the relay output $\boldsymbol{w}(t) = \widetilde{\boldsymbol{u}}(t)/\|\widetilde{\boldsymbol{u}}(t)\|$ is continuous and piecewise differentiable. Verifying that $\widetilde{\boldsymbol{u}}^T\, [\partial(\widetilde{\boldsymbol{u}}/\|\widetilde{\boldsymbol{u}}\|)/\partial\widetilde{\boldsymbol{u}}] = 0$, we obtain

$$\int_{t_1}^{t_2} \widetilde{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{w} = \int_{t_1}^{t_2} \widetilde{\boldsymbol{u}}(t) \cdot \boldsymbol{w}'(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \widetilde{\boldsymbol{u}}(t)^T\,[\partial\boldsymbol{w}/\partial\widetilde{\boldsymbol{u}}]\,\widetilde{\boldsymbol{u}}'(t)\,\mathrm{d}t = 0.$$

As well, for any interval $[t_1, t_2] \subseteq [0, T]$ on which $\|\boldsymbol{u}(t) - \boldsymbol{x}\| < r$, $\boldsymbol{w}$ is constant and thus implies

$$\int_{t_1}^{t_2} \widetilde{\boldsymbol{u}} \cdot \mathrm{d}\boldsymbol{w} = 0.$$

The only case where a non-zero contribution to the integral is made is when $\widetilde{\boldsymbol{u}}$ touches the relay boundary and leaves the relay, i.e. $\|\boldsymbol{u}(t) - \boldsymbol{x}\| = r$ and $\|\boldsymbol{u}(t-\delta) - \boldsymbol{x}\| < r$ for some $\varepsilon > 0$ and all $0 < \delta < \varepsilon$. That contribution is

$$\widetilde{\boldsymbol{u}}(t) \cdot \left(\boldsymbol{w}(t) - \lim_{\tau \to t^-} \boldsymbol{w}(\tau)\right) = \widetilde{\boldsymbol{u}}(t) \cdot \left(\frac{\widetilde{\boldsymbol{u}}(t)}{\|\widetilde{\boldsymbol{u}}(t)\|} - \lim_{\tau \to t^-} \frac{\widetilde{\boldsymbol{u}}(\tau)}{\|\widetilde{\boldsymbol{u}}(\tau)\|}\right) \geq 0. \qquad \square$$

The proof exposes the significance of the closed loop assumption $\boldsymbol{w}(0) = \boldsymbol{w}(T)$. If it is not satisfied, the integral of $\boldsymbol{x}$ in (3) does not cancel but gives the term $\boldsymbol{x} \cdot (\boldsymbol{w}(T) - \boldsymbol{w}(0))$, which may be negative and thus make the dissipation expression (2) negative.

3. Vector Preisach operator

The scalar Preisach operator arises as a weighted superposition of scalar relay operators [1, 2]. The n-dimensional vector Preisach operator $\boldsymbol{\mathcal{P}} : \mathrm{C}([0,T];\mathbb{R}^n) \to \mathrm{C}([0,T];\mathbb{R}^n)$ is constructed in exact analogy as the superposition of vector relays $\boldsymbol{h}_{(\boldsymbol{x},r)}$, $(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R}_+$:

$$\boldsymbol{w}(t) = \boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}](t) := \int_0^\infty \int_{\mathbb{R}^n} \omega(\boldsymbol{x}, r)\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}, \boldsymbol{\xi}_{(\boldsymbol{x},r)}](t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r, \tag{4}$$

with a Lebesgue-integrable Preisach density function $\omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$. The measurable function $\boldsymbol{\xi} : \mathbb{R}^n \times \mathbb{R}_+ \to \partial B_{\boldsymbol{0},1}$, where $\boldsymbol{\xi}_{(\boldsymbol{x},r)}$ denotes the evaluation of $\boldsymbol{\xi}$ at a point $(\boldsymbol{x}, r)$, represents the initial states of the relays. Replacing the Lebesgue measure by a finite Borel measure in (4) gives a larger class of operators. As $\boldsymbol{\mathcal{P}}$ is a linear superposition of relays, it is a hysteresis operator.

Note that, since $\boldsymbol{\mathcal{P}}$ is invariant under changes of ω on sets of measure 0, in all subsequent statements any assumption on ω will suffice to be satisfied for a function $\overline{\omega}$ such that

$$\omega = \overline{\omega} \quad \text{a.e. on } \mathbb{R}^n \times \mathbb{R}_+.$$

In [7], we presented a geometric visualization of the memory evolution of $\boldsymbol{\mathcal{P}}$ for $n = 2$, discussing that at current input $\boldsymbol{u}(t)$, the relays in a "frozen" state are exactly those forming the cone

$$\mathcal{C}_{\boldsymbol{u}(t)} := \{(\boldsymbol{x}, r) \mid \|\boldsymbol{u}(t) - \boldsymbol{x}\| < r\} \subset \mathbb{R}^n \times \mathbb{R}_+.$$

The remaining relays $(\boldsymbol{x}, r) \notin \mathcal{C}_{\boldsymbol{u}(t)}$ take the state $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}](t) = (\boldsymbol{u}(t) - \boldsymbol{x}(t))/\|\boldsymbol{u}(t) - \boldsymbol{x}(t)\|$.

The following two lemmas form the extension of Lemmas 1 and 2 to $\boldsymbol{\mathcal{P}}$.

Lemma 4 (Generalized semigroup property)**.** *Given* $t_1, t_2 \in [0, T]$*,* $t_1 \leq t_2$*, and* $\boldsymbol{\xi} \in \partial B_{\boldsymbol{0},1}$*, define* $\boldsymbol{\xi}(t)$ *by* $\boldsymbol{\xi}_{(\boldsymbol{x},r)}(t) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}, \boldsymbol{\xi}_{(\boldsymbol{x},r)}](t)$*. Then the following holds true:*

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}](t_2) = \boldsymbol{\mathcal{P}}[\boldsymbol{u}^{t_1}, \boldsymbol{\xi}(t_1)](t_2 - t_1).$$

Proof. By Lemma 1, at t_2 all the relay states are equal, so the statement follows. □

The behaviour of $\boldsymbol{\mathcal{P}}$ when the input $\boldsymbol{u}$ is subjected to rotations and reflections involves transformations of ω. In this context, we will index $\boldsymbol{\mathcal{P}}$ with ω where necessary and write $\boldsymbol{\mathcal{P}}_\omega$.

Lemma 5 (Rotation and reflection)**.** *For* $Q \in O(n)$*, the vector Preisach operator satisfies*

$$Q\boldsymbol{\mathcal{P}}_\omega[\boldsymbol{u}, \boldsymbol{\xi}] = \boldsymbol{\mathcal{P}}_{Q\omega}[Q\boldsymbol{u}, Q\boldsymbol{\xi}].$$

Proof. Straightforward using (4) and Lemma 2. □

4. Isotropy and neutral memory state

A vectorial hysteresis operator $\boldsymbol{\mathcal{W}}$ is called *isotropic* if its input-output behaviour is the same in all directions. Mathematically, this means that $\boldsymbol{\mathcal{W}}$ satisfies

$$\boldsymbol{\mathcal{W}}[Q\boldsymbol{u}] = Q\boldsymbol{\mathcal{W}}[\boldsymbol{u}] \quad \text{for all } \boldsymbol{u} \in \mathrm{C}([0, T]; \mathbb{R}^n) \text{ and } Q \in O(n).$$

The vector Preisach operator $\boldsymbol{\mathcal{P}}$ does not only depend on $\boldsymbol{u}$, but also on the initial state $\boldsymbol{\xi}$. We need to extend the notion of isotropy to take the effect of $\boldsymbol{\xi}$ into consideration. We have by Lemma 5

$$\boldsymbol{\mathcal{P}}[Q\boldsymbol{u}, \boldsymbol{\xi}](t) = \int_0^\infty \int_{\mathbb{R}^n} \omega(\boldsymbol{x}, r) \boldsymbol{h}_{(\boldsymbol{x},r)}[Q\boldsymbol{u}, \boldsymbol{\xi}_{(\boldsymbol{x},r)}](t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}r, \tag{5}$$

$$Q\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}](t) = \int_0^\infty \int_{\mathbb{R}^n} Q\omega(\boldsymbol{x}, r) \boldsymbol{h}_{(\boldsymbol{x},r)}[Q\boldsymbol{u}, (Q\boldsymbol{\xi})_{(\boldsymbol{x},r)})](t) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}r. \tag{6}$$

Assume ω satisfies $\omega = Q\omega$. Then obviously $\boldsymbol{\xi}$ can cause these integrals to differ if we have that $(Q\boldsymbol{\xi})_{(\boldsymbol{x},r)} \neq \boldsymbol{\xi}_{(\boldsymbol{x},r)}$ on some subset of $\mathbb{R}^n \times \mathbb{R}_+$ of non-zero measure. Thus, defining the set

$$\Xi = \{\boldsymbol{\xi} \mid \boldsymbol{\xi} = Q\boldsymbol{\xi} \text{ for all } Q \in O(n)\},$$

the natural definition of isotropy seems to be:

Definition 6 (Isotropy)**.** We call the vector Preisach operator $\boldsymbol{\mathcal{P}}$ *isotropic* if and only if for any $\boldsymbol{u} \in \mathrm{C}([0, T]; \mathbb{R}^n)$ and any rotation $Q \in O(n)$, it satisfies

$$\boldsymbol{\mathcal{P}}[Q\boldsymbol{u}, \boldsymbol{\xi}] = Q\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}] \quad \text{for all } \boldsymbol{\xi} \in \Xi. \tag{7}$$

The following auxiliary lemma is a result from linear algebra and will be used in showing the subsequent statements. Its proof is basic.

Lemma 7. *$\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ satisfy $\|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\|$ if and only if $\boldsymbol{x}_1 = Q\boldsymbol{x}_2$ for some $Q \in O(n)$.*

We can state the following isotropy conditions:

Lemma 8 (Necessary isotropy condition)**.** *If $\boldsymbol{\mathcal{P}}$ is isotropic, then ω satisfies*

$$\int_0^\infty \int_{\mathbb{R}^n} \omega(\boldsymbol{x}, r) \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}r = 0. \tag{8}$$

Proof. Let $\boldsymbol{u} = 0$ and $\boldsymbol{\xi}$ be given by $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$, so $\boldsymbol{\xi} \in \Xi$ as shown later in Lemma 11. We have that $Q\boldsymbol{u} = \boldsymbol{u} = 0$ for all $Q \in O(n)$, and therefore (7) holds by Lemma 7 for all $Q \in O(n)$ only if $\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}] = 0$. On the other hand, $\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}](t)$ is equal to the left-hand side of (8). □

Lemma 9 (Sufficient isotropy condition)**.** *If there exists a $\tilde{\omega} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that $\omega(\boldsymbol{x}, r) = \tilde{\omega}(||\boldsymbol{x}||, r)$, then $\boldsymbol{\mathcal{P}}$ is isotropic.*

Proof. This is obvious from comparing Equations (5) and (6), as for any $Q \in O(n)$ we have that $\omega(\boldsymbol{x}, r) = \tilde{\omega}(||\boldsymbol{x}||, r) = \tilde{\omega}(||Q^{-1}\boldsymbol{x}||, r) = \omega(Q^{-1}\boldsymbol{x}, r) = Q\omega(\boldsymbol{x}, r)$. □

For ease of reference, we will call $\omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ *isotropic* if and only if it satisfies the assumption of Lemma 9. We can use Lemma 7 to show that ω is isotropic exactly if it is invariant under rotation and reflection about the r-axis:

Lemma 10. *We have that ω is isotropic if and only if $\omega(\boldsymbol{x}, r) = Q\omega(\boldsymbol{x}, r)$ for all $Q \in O(n)$.*

Proof. Using Lemma 7, it is quickly shown that $\{Q\boldsymbol{x} \mid Q \in O(n)\} = \{\boldsymbol{y} \mid \|\boldsymbol{y}\| = \|\boldsymbol{x}\|\}$. Since $Q\omega(\boldsymbol{x}, r) = \omega(Q^{-1}\boldsymbol{x}, r)$, the statement follows. □

Assume now that ω is isotropic. An interesting question that has so far remained unaddressed is that of an appropriate *neutral memory state*, or, in the terminology of magnetic hysteresis, a "demagnetized state" of $\boldsymbol{\mathcal{P}}$. In the case of scalar hysteresis, such a neutral state $\xi^0 \in \mathrm{Map}\,(\mathbb{R} \times \mathbb{R}_+; \{-1, 1\})$ is characterized by giving Preisach output 0 for input $u = 0$ and symmetry with respect to input reflections,

$$\mathcal{P}[0, \xi^0] = 0 \quad \text{and} \quad \mathcal{P}[-u, \xi^0](t) = -\mathcal{P}[u, \xi^0](t) \quad \text{for all } u \in \mathrm{C}([0, T]; \mathbb{R}). \tag{9}$$

It is quickly seen the scalar Preisach operator $\mathcal{P}$ meets these conditions for arbitrary ω only if $\xi_{(x,r)} = -\xi_{(-x,r)}$ a.e. The only attainable memory state satisfying this condition is that represented by the staircase curve $\psi(r) \equiv 0$, so the neutral state ξ^0 is equal to [1, 2]

$$\xi^0_{(x,r)} = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases} \tag{10}$$

In the vectorial case, we are looking for the $\boldsymbol{\xi}$ in Definition 6 that do not affect isotropy, that is, $\boldsymbol{\xi} \in \Xi$. If ω is isotropic, $\boldsymbol{\xi} \in \Xi$ immediately implies the vectorial counterparts of (9),

$$\boldsymbol{\mathcal{P}}[\boldsymbol{0}, \boldsymbol{\xi}] = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\mathcal{P}}[Q\boldsymbol{u}, \boldsymbol{\xi}] = Q\boldsymbol{\mathcal{P}}[\boldsymbol{u}, \boldsymbol{\xi}] \quad \text{for all } Q \in O(n).$$

It turns out that the elements of Ξ have a very specific form:

Lemma 11. *A measurable $\boldsymbol{\xi} \in \mathrm{Map}(\mathbb{R}^n \times \mathbb{R}_+; \partial B_{\boldsymbol{0},1})$ is in Ξ if and only if, for some function $\alpha : \mathbb{R}_+ \to \{1, -1\}$, we have $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = \alpha(\|\boldsymbol{x}\|) \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ for all $(\boldsymbol{x}, r) \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \times \mathbb{R}_+$.*

Proof. We just outline the argument. Assume first $\boldsymbol{\xi} \in \Xi$. For $\boldsymbol{x} = (\lambda, 0, \ldots, 0)^T$, we can show that $\boldsymbol{x} = Q\boldsymbol{x}$ holds exactly for the subgroup $O^1(n) \subset O(n)$ containing all those $Q = (q_{ij})_{i,j=1,\ldots,n}$ whose first column is given by $(q_{11}, \ldots, q_{n1}) = (1, 0, \ldots, 0)$. Using that because of $\boldsymbol{\xi} \in \Xi$ this implies $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = (Q\boldsymbol{\xi})_{(\boldsymbol{x},r)} = Q(\boldsymbol{\xi}_{(\boldsymbol{x},r)})$ for all $Q \in O^1(n)$, we can conclude that $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = (\beta, 0, \ldots, 0)^T$, $\beta = \pm 1$. For arbitrary $\boldsymbol{x}$, we use that by Lemma 7 there exists a $Q \in O(n)$ such that $\boldsymbol{x} = Q(\|\boldsymbol{x}\|, 0, \ldots, 0)^T$, so $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = (Q\boldsymbol{\xi})_{(\boldsymbol{x},r)} = Q(\boldsymbol{\xi}_{(Q^{-1}\boldsymbol{x},r)}) = Q(\beta, 0, \ldots, 0) = \beta\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. Set $\alpha(\|\boldsymbol{x}\|) = \beta$. The converse is quickly computed. □

The fact that all relays outside $\mathcal{C}_{\boldsymbol{u}(0)}$ at $\boldsymbol{u}(0) = \mathbf{0}$ are in this state and that it represents the vectorial analogue of (10) suggests that the neutral initial state of $\boldsymbol{\mathcal{P}}$ is $\boldsymbol{\xi}^0$,

$$\boldsymbol{\xi}^0_{(\boldsymbol{x},r)} := -\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}.$$

In fact, for $n = 2$ it can be shown that $\boldsymbol{\xi}^0$ results asymptotically from a "demagnetization" process via a rotating field of decreasing amplitude. Define $\boldsymbol{u}_k : [0, 1] \to \mathbb{R}^2$ to be the spiral curve

$$\boldsymbol{u}_k(t) = R(1-t)\begin{pmatrix}\cos(2k\pi t)\\ \sin(2k\pi t)\end{pmatrix}$$

starting at $\boldsymbol{u}_k(0) = (R, 0)$ and rotating with uniformly decreasing amplitude to $\boldsymbol{u}_k(1) = (0, 0)$. The parameter $k \in \mathbb{N}$ represents the number of rotations of the spiral.

Lemma 12. *For all $(\boldsymbol{x}, r) \in \mathbb{R}^n \times \mathbb{R}_+$ such that $r < R$ and $\|\boldsymbol{x}\| > 0$, and for all initial states $\boldsymbol{\xi}$,*

$$\lim_{k\to\infty} \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_k, \boldsymbol{\xi}](1) = \boldsymbol{\xi}^0_{(\boldsymbol{x},r)}.$$

Proof. Note that any ring $B_{\mathbf{0},d+\frac{R}{k}} \setminus B_{\mathbf{0},d}$ contains exacty one revolution of the spiral.

If $||\boldsymbol{x}|| \geq r$, the statement is obviously true as $(\boldsymbol{x}, r) \notin \mathcal{C}_{\boldsymbol{u}(1)}$. Assume $||\boldsymbol{x}|| < r < R$. For $d < r$, the circle $\partial B_{\mathbf{0},d}$ intersects the boundary of the relay $\partial B_{\boldsymbol{x},r}$ not at all for $d < r - ||\boldsymbol{x}||$, exactly once for $d = r - ||\boldsymbol{x}||$, and twice for $d > r - ||\boldsymbol{x}||$. Let P denote the intersection point for $d = r - ||\boldsymbol{x}||$. It is quickly geometrically verified that $(P - \boldsymbol{x})/\|P - \boldsymbol{x}\| = -\boldsymbol{x}/\|\boldsymbol{x}\|$. Let Q_k be the point at which $\boldsymbol{u}_k(t)$ last intersects $\partial B_{\boldsymbol{x},r}$. This implies that

$$\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_k](t)\mid_{t=1} = \frac{Q_k - \boldsymbol{x}}{\|Q_k - \boldsymbol{x}\|}. \tag{11}$$

With the remark in the beginning and k large enough so $\frac{R}{k} < \|\boldsymbol{x}\|$, the point Q_k lies in the intersection of the ring $B_{\mathbf{0},r-||\boldsymbol{x}||+\frac{R}{k}} \setminus B_{\mathbf{0},r-||\boldsymbol{x}||}$ with $\partial B_{\boldsymbol{x},r}$. As $k \to \infty$, this intersection lies inside an arbitrarily small neighbourhood of P, so we have $Q_k \to P$. Then (11) results in the claim. □

This is the two-dimensional analogue of the standard uniaxial demagnetization process by an alternating field of decreasing amplitude, which results in ξ^0 [1].

5. Saturation

Suppose ω has bounded support, and let $\mathcal{K} := \{(\boldsymbol{x}, r) \mid ||\boldsymbol{x}|| + r \leq R\}$ be the minimal cone such that $\omega(\boldsymbol{x}, r) = 0$ a.e. outside $\mathcal{K}$. For any $\boldsymbol{u}(t)$ such that $||\boldsymbol{u}(t)|| > R$,

$$\mathcal{C}_{\boldsymbol{u}(t)} \cap \mathcal{K} = \varnothing. \tag{12}$$

In other words, all hysteresis memory is erased.

Lemma 13 (Memory deletion)**.** *Assume* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathrm{C}([0,T];\mathbb{R}^n)$ *satisfy* $\boldsymbol{u}_1^t = \boldsymbol{u}_2^t$ *for some* $t \in [0,T]$*. If* $\|\boldsymbol{u}_1(t)\| \geq R$*, then for arbitrary initial states* $\boldsymbol{\xi}_1$*,* $\boldsymbol{\xi}_2$*,*

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u}_1,\boldsymbol{\xi}_1](\tau) = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_2,\boldsymbol{\xi}_2](\tau) \quad \textit{for all } t \leq \tau \leq T.$$

Proof. For all $(\boldsymbol{x}, r) \in \mathcal{K}$, by (12) we have that

$$\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_1,\boldsymbol{\xi}_1](t) = \frac{\boldsymbol{u}_1(t)-\boldsymbol{x}}{\|\boldsymbol{u}_1(t)-\boldsymbol{x}\|} = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_2,\boldsymbol{\xi}_2](t).$$

Thus, for all $(\boldsymbol{x}, r) \in \mathcal{K}$ the semigroup property for vector relays at t results in $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_1,\boldsymbol{\xi}_1](\tau) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_2,\boldsymbol{\xi}_2](\tau)$ for all $t \leq \tau \leq T$, and the statement follows. □

In particular, there is a function $\boldsymbol{p} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t) = \boldsymbol{p}(\boldsymbol{u}(t))$ for any initial state $\boldsymbol{\xi}$ if $\|\boldsymbol{u}(t)\| \geq R$. It is given by

$$\boldsymbol{p}(\boldsymbol{v}) = \int_{\mathcal{K}} \omega(\boldsymbol{x},r)\frac{\boldsymbol{v}-\boldsymbol{x}}{\|\boldsymbol{v}-\boldsymbol{x}\|}\,\mathrm{d}(\boldsymbol{x},r), \qquad \boldsymbol{v} \in \mathbb{R}^n.$$

That is, the hysteresis output for any $\boldsymbol{u}(t)$ satisfying $\|\boldsymbol{u}(t)\| \geq R$ is not multivalued. As a consequence of Lemma 13, the initial state and $\boldsymbol{u}$ in $[0,t)$ are completely deleted from the memory.

If ω is isotropic then in addition $\boldsymbol{u}(t)$ and $\boldsymbol{w}(t) = \boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t)$ are aligned.

Lemma 14 (Alignment)**.** *Assume* ω *is isotropic. If* $\|\boldsymbol{u}(t)\| \geq R$*, then there exists a* $\lambda \in \mathbb{R}$ *such that*

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t) = \lambda\boldsymbol{u}(t). \tag{13}$$

Proof. By Lemma 13, we can assume $\boldsymbol{\xi} = \boldsymbol{\xi}^0$. Let first $\boldsymbol{u}(t) = (u(t), 0, \ldots, 0)$. In the proof of Lemma 11, we stated the group $O^1(n) \subset O(n)$ fixing $\boldsymbol{u}(t)$. By Lemma 5,

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t) = \boldsymbol{\mathcal{P}}_\omega[\boldsymbol{u},\boldsymbol{\xi}](t) = \boldsymbol{\mathcal{P}}_{Q\omega}[Q\boldsymbol{u},Q\boldsymbol{\xi}](t) = Q\boldsymbol{\mathcal{P}}_\omega[\boldsymbol{u},\boldsymbol{\xi}](t) = Q\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t) \tag{14}$$

for all $Q \in O^1(n)$. Thus, $\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)$ must be of the form $\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t) = \lambda(u(t), 0, \ldots, 0)$. For arbitrary $\boldsymbol{u}(t)$, there is a $Q \in O(n)$ such that $\boldsymbol{u}(t) = Q(\|\boldsymbol{u}(t)\|, 0, \ldots, 0)$. As the third equality in (14) holds for all $Q \in O(n)$, there is a λ such that $\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t) = Q\lambda(u(t), 0, \ldots, 0)$, resulting in (13). □

In general, independent of the symmetry properties of ω, the Preisach output $\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)$ asymptotically aligns with large $\boldsymbol{u}(t)$:

$$\lim_{\|\boldsymbol{v}\|\to\infty} \boldsymbol{p}(\boldsymbol{v}) = \lim_{\|\boldsymbol{v}\|\to\infty} \int_{\mathcal{K}} \omega(\boldsymbol{x},r)\frac{\boldsymbol{v}-\boldsymbol{x}}{\|\boldsymbol{v}-\boldsymbol{x}\|}\,\mathrm{d}(\boldsymbol{x},r) = \int_{\mathcal{K}} \omega(\boldsymbol{x},r)\,\mathrm{d}(\boldsymbol{x},r)\,\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}. \tag{15}$$

The relays inside $\mathcal{K}$ keep varying while $||\boldsymbol{u}(t)|| \geq R$. Thus, for large $\boldsymbol{u}(t)$, $\|\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)\|$ is not constant but varies. In this, the vector Preisach operator $\boldsymbol{\mathcal{P}}$ for $n \geq 2$ differs from the scalar Preisach operator ($n = 1$), which gives constant output as soon as $|u(t)| \geq R$.

It is obvious from (15) that the saturation limit

$$\int_0^\infty \int_{\mathbb{R}^n} \omega(\boldsymbol{x},r)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r \tag{16}$$

that $\boldsymbol{p}(\boldsymbol{v})$ attains as $\|\boldsymbol{v}\| \to \infty$ is independent of the direction $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$. Using that $\|\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}](t)\| = 1$, we can further derive the following bound on the Preisach output:

$$\|\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)\| \leq \int_0^\infty \int_{\mathbb{R}^n} \|\omega(\boldsymbol{x},r)\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}](t)\|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r \leq \int_0^\infty \int_{\mathbb{R}^n} |\omega(\boldsymbol{x},r)|\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r.$$

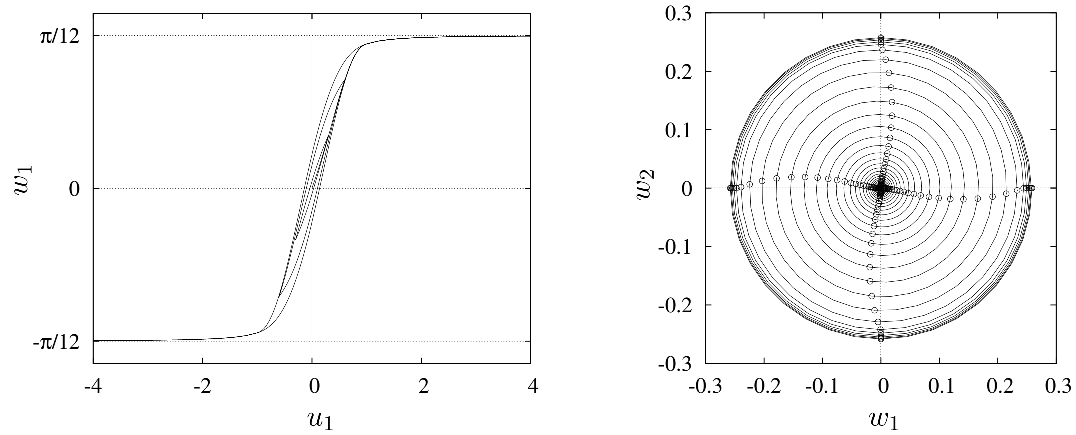

Figure 2. Saturation behaviour: Let $n = 2$ and the continuous isotropic ω be defined by $\omega(\boldsymbol{x}, r) = 1 - \|\boldsymbol{x}\| - r$ if $\|\boldsymbol{x}\| + r < 1$, and 0 otherwise. On the left, the uniaxial hysteresis loops show the merging of the loop into the function $\boldsymbol{p}$ at $R = 1$ and the curve approaching the assymptotic limit (16) equal to $\frac{\pi}{12}$. On the right, the output $\boldsymbol{w} = (w_1, w_2)$ resulting from the spiral input $\boldsymbol{u}(t) = \lambda^{t/(2\pi)} Q(t)\boldsymbol{u}_0$, with $\lambda \in \mathbb{R}$ and $Q(t)$ as defined in (17), is displayed. The marked points correspond to the input points $\boldsymbol{u}(t)$ lying on the coordinate axes and highlight well the hysteresis lag, which is 0 for $\|\boldsymbol{u}(t)\| \geq 1$ as predicted in Lemma 14.

Therefore, if $\omega \geq 0$, the saturation limit (16) represents an upper bound on $\|\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)\|$. Figure 2 illustrates the saturation behaviour on an example.

The derived behaviour of $\boldsymbol{\mathcal{P}}$ agrees with that observed in measurements for hysteretic materials: In saturation, magnetic field $\boldsymbol{H}$ and $\boldsymbol{M}$ are aligned for the isotropic model [9], and align asymptotically for the anisotropic model [10]. In either case the saturation limit is independent of the direction [11]. Curves reported from measurements (e.g. [12, 13, 14]) show the merging of the uniaxial $\boldsymbol{M}(\boldsymbol{H})$ hysteresis loop into a single curve before zero slope, or slope μ_0 in the case of $\boldsymbol{B}(\boldsymbol{H})$ curves (cf. Equation (20)), is attained. This is related to the existence of a reversible component in $\boldsymbol{\mathcal{P}}$. The lack of a reversible contribution is one of the shortcomings of the classical scalar Preisach model frequently addressed by model extensions [15, 9].

Remark. If ω has bounded support $\mathcal{K}$, then $\boldsymbol{\mathcal{P}}$ is not equal to a Mayergoyz' vector Preisach model [1]. This is seen comparing the output behaviour of both operators for large uniformly rotating input $\boldsymbol{u}$. For $\boldsymbol{\mathcal{P}}$, by Lemma 14, the output $\boldsymbol{\mathcal{P}}[\boldsymbol{u}](t)$ aligns with $\boldsymbol{u}(t)$ as soon as $\|\boldsymbol{u}(t)\| > R$. For any Mayergoyz' vector Preisach model $\boldsymbol{\mathcal{W}}$, the output always follows the input at a non-zero lag angle, as demonstrated in [1] for $n = 2$ in the proof of the "Rotational Symmetry Property".

6. Congruency and periodic behaviour

This section investigates the output periodicity and congruency properties of $\boldsymbol{\mathcal{P}}$. In particular, we investigate uniformly rotating input in $\mathbb{R}^2$ and show that for isotropic ω, it results in uniformly rotating output.

In the first two lemmas, we show that for periodic input the Preisach output is periodic and satisfies congruency of the vectorial loops. That is, if after an arbitrary initial variation, two functions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are equal and periodic, then from the second cycle onwards the outputs are periodic and congruent. Here, congruency of two curves in $\mathbb{R}^n$ means that they are equal up to translation. The transient phase in the first period results from the deletion of differing relay states in the course of this period. The congruency property of $\boldsymbol{\mathcal{P}}$ has been previously observed in computer simulations [16].

Lemma 15 (Periodicity, congruency for differing initial states)**.** *Suppose* $\boldsymbol{u} \in \mathrm{C}([0,T];\mathbb{R}^n)$ *is periodic, i.e.* $\boldsymbol{u}(t+\lambda) = \boldsymbol{u}(t)$ *for some* $\lambda < T$ *and any* $t \in [0, T-\lambda]$*. Then the following holds:*

(a) $\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}]$ *is periodic with period* λ *on* $[\lambda, T]$*, i.e.*

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t+\lambda) = \boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}](t) \quad \text{for all } t:\ \lambda \le t \le T-\lambda \text{ and } \boldsymbol{\xi}:\mathbb{R}^n\times\mathbb{R}_+ \to \partial B_{\mathbf{0},1}.$$

(b) There exists a $\boldsymbol{v} \in \mathbb{R}^n$ *such that*

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}_1](t) = \boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}_2](t) + \boldsymbol{v} \quad \text{for all } t:\ \lambda \le t \le T \text{ and } \boldsymbol{\xi}_1,\boldsymbol{\xi}_2:\mathbb{R}^n\times\mathbb{R}_+ \to \partial B_{\mathbf{0},1}.$$

Proof. (a) It is quickly verified that $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_1](t+\lambda) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_1](t)$ for $t \ge \lambda$ is satisfied by each relay. The periodicity carries over to $\boldsymbol{\mathcal{P}}$ in the obvious way.

(b) Clearly, $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_1](t) = \boldsymbol{\xi}_1$ as well as $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_2](t) = \boldsymbol{\xi}_2$ for all t if and only if $\|\boldsymbol{u}(t)-\boldsymbol{x}\| < r$ for all $t \in [0,\lambda)$. Otherwise, $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_1](t) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u},\boldsymbol{\xi}_2](t)$ for any $t \in [\lambda, T]$. Therefore, for all $t \in [\lambda, T]$,

$$\boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}_1](t) - \boldsymbol{\mathcal{P}}[\boldsymbol{u},\boldsymbol{\xi}_2](t) = \int_{\{(\boldsymbol{x},r) \mid \|\boldsymbol{u}(t)-\boldsymbol{x}\|<r\ \forall t\in[0,\lambda)\}} \omega(\boldsymbol{x},r)(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}r =: \boldsymbol{v}. \qquad \square$$

Lemma 16 (Congruency of vectorial loops)**.** *Suppose* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathrm{C}([0,T];\mathbb{R}^n)$ *satisfy* $\boldsymbol{u}_1^{t_0} = \boldsymbol{u}_2^{t_0}$ *for some* $t_0 \in [0,T]$*. Suppose further there exists a* $\lambda > 0$ *such that* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *are periodic on* $[t_0, T]$*, i.e.* $\boldsymbol{u}_1(t+\lambda) = \boldsymbol{u}_1(t)$ *for all* $t \in [t_0, T-\lambda]$*. Set* $\boldsymbol{w}_1 = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_1,\boldsymbol{\xi}_1]$ *and* $\boldsymbol{w}_2 = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_2,\boldsymbol{\xi}_2]$*, where* $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2:\mathbb{R}^n\times\mathbb{R}_+ \to \partial B_{\mathbf{0},1}$*. Then:*

(a) $\boldsymbol{w}_1$ *and* $\boldsymbol{w}_2$ *are periodic with period* λ *on* $[t_0+\lambda, T]$*, i.e.*

$$\boldsymbol{w}_i(t+\lambda) = \boldsymbol{w}_i(t) \quad \text{for all } t:\ t_0+\lambda \le t \le T-\lambda \text{ and } i=1,2.$$

(b) There exists a $\boldsymbol{v} \in \mathbb{R}^n$ *such that*

$$\boldsymbol{w}_2(t) = \boldsymbol{w}_1(t) + \boldsymbol{v} \quad \text{for all } t:\ t_0+\lambda \le t \le T.$$

Proof. Define $\boldsymbol{\xi}_i(t_0)$ by $\boldsymbol{\xi}_{i,(\boldsymbol{x},r)}(t_0) = \boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_i,\boldsymbol{\xi}_{i,(\boldsymbol{x},r)}]$. By the generalized semigroup property, Lemma 4, we have that $\boldsymbol{\mathcal{P}}[\boldsymbol{u}_i,\boldsymbol{\xi}_i](t) = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_i^{t_0},\boldsymbol{\xi}_i(t_0)](t-t_0)$ for all $t \in [t_0,T]$, $i=1,2$. As $\boldsymbol{u}_1^{t_0} = \boldsymbol{u}_2^{t_0}$ periodic on $[t_0,T]$, statements (a) and (b) follow directly from Lemma 15 (a) and (b), respectively, for $\boldsymbol{\mathcal{P}}[\boldsymbol{u}_i^{t_0},\boldsymbol{\xi}_i(t_0)](t-t_0)$. $\square$

We now look at $n=2$ and the behaviour of $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}]$ for uniformly rotating $\boldsymbol{u}$. Due to the rate-independence of $\boldsymbol{\mathcal{P}}$, it suffices to examine $\boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\mathrm{rot}}]$ for the function u_{rot},

$$\boldsymbol{u}_{\mathrm{rot}}(t) = Q(t)\boldsymbol{u}_0, \qquad \boldsymbol{u}_0 \in \mathbb{R}^2,\ Q(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},\ t \in [0,T], \tag{17}$$

uniformly rotating clockwise about 0 at magnitude $\|\boldsymbol{u}_{\mathrm{rot}}(t)\| = \|\boldsymbol{u}_0\|$ and period 2π.

The statement of the following lemma has been experimentally confirmed in [6].

Lemma 17 (Uniformly rotating input)**.** *Assume* ω *is isotropic. For all* $t \ge 2\pi$*, the curve* $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\mathrm{rot}},\boldsymbol{\xi}]$ *describes a circle, i.e. there exist constant vectors* $\boldsymbol{v}, \boldsymbol{w}_0 \in \mathbb{R}^2$ *such that*

$$\boldsymbol{w}(t) = \boldsymbol{v} + Q(t)\boldsymbol{w}_0.$$

In particular, if $\boldsymbol{\xi}_{(\boldsymbol{x},r)} = \boldsymbol{\xi}^0_{(\boldsymbol{x},r)}$ *for all* $(\boldsymbol{x},r)$ *such that* $\|\boldsymbol{x}\| < r - \|\boldsymbol{u}_0\|$*, then* $\boldsymbol{v} = 0$ *and the circle is centered at* 0*.*

To show Lemma 17, we apply the following two lemmas, the proofs of which we omit:

Lemma 18. *The following statements are equivalent:*

(a) $\|\boldsymbol{x}\| < r - \|\boldsymbol{u}_0\|$,

(b) $\|\boldsymbol{u}_{\mathrm{rot}}(t) - \boldsymbol{x}\| < r$ *for all* $t \in [t_0, t_0 + 2\pi)$.

Lemma 19. *If* $\|\boldsymbol{x}\| \geq r - \|\boldsymbol{u}_0\|$, *then* $\boldsymbol{h}_{(Q(t)\boldsymbol{x},r)}[\boldsymbol{u}_{\mathrm{rot}}](t) = Q(t)\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_{\mathrm{rot}}](2\pi)$ *for all* $t \geq 2\pi$.

Proof of Lemma 17. Lemma 18 implies that $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_{\mathrm{rot}}] = \boldsymbol{\xi}_{(\boldsymbol{x},r)}$ if and only if $\|\boldsymbol{x}\| < r - \|\boldsymbol{u}_0\|$. Otherwise, $\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_{\mathrm{rot}}]$ is given by Lemma 19. Therefore,

$$\boldsymbol{w}(t) = \int_{\|\boldsymbol{x}\|<r-\|\boldsymbol{u}_0\|} \omega(\boldsymbol{x},r)\boldsymbol{\xi}_{(\boldsymbol{x},r)}\,\mathrm{d}(\boldsymbol{x},r) \;+\; Q(t)\int_{\|\boldsymbol{x}\|\geq r-\|\boldsymbol{u}_0\|} \omega(\boldsymbol{x},r)\boldsymbol{h}_{(\boldsymbol{x},r)}[\boldsymbol{u}_{\mathrm{rot}}](2\pi)\,\mathrm{d}(\boldsymbol{x},r). \quad (18)$$

The first integral gives $\boldsymbol{v}$, the second integral $\boldsymbol{w}_0$.

To show the second statement, set $\boldsymbol{\xi} = \boldsymbol{\xi}^0$. By Lemma 7, $\boldsymbol{v} = 0$ if and only if $Q\boldsymbol{v} = \boldsymbol{v}$ for all $Q \in O(n)$. This is quickly verified using the expression for $\boldsymbol{v}$ from (18), invariance of ω, $\boldsymbol{\xi}^0$ and $\|\boldsymbol{x}\|$ under $O(n)$ and change of variables in the Lebesgue integral with $\widetilde{\boldsymbol{x}} = Q\boldsymbol{x}$. □

7. Lag angles and dissipation

Two questions that are closely related are those of the hysteretic lag between $\boldsymbol{u}$ and $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}]$ and the dissipative properties of $\boldsymbol{\mathcal{P}}$. We start this section by investigating the lag angle $\alpha_{\mathrm{lag}}(t) = \measuredangle(\boldsymbol{u}(t), \boldsymbol{w}(t))$ for uniformly rotating input $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{rot}}$ in $\mathbb{R}^2$. Then we discuss dissipation under periodic input in general and for $\boldsymbol{u}_{\mathrm{rot}}$ in particular. On an example, we will exhibit that $\boldsymbol{\mathcal{P}}$ shows good qualitative correspondence to the behaviour of magnetic hysteresis observed in measurements.

As a consequence of Lemma 15, which states that $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\mathrm{rot}}]$ is periodic with the same period as $\boldsymbol{u}_{\mathrm{rot}}$, the lag angle $\alpha_{\mathrm{lag}}(t)$ varies periodically for uniformly rotating input $\boldsymbol{u}_{\mathrm{rot}}$. In computer simulations [6], it was observed that $\alpha_{\mathrm{lag}}(t)$ is 0 for the isotropic model and sufficiently large input $\boldsymbol{u}(t)$. We can now add the mathematical proof for this fact:

Lemma 20 (Lag angle for uniformly rotating input)**.** *Assume* ω *is isotropic. Let* $\boldsymbol{w}(t) = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\mathrm{rot}}, \boldsymbol{\xi}^0](t)$. *Then* $\alpha_{\mathrm{lag}}(t)$ *is constant for all* $t \geq 2\pi$ *and equal to* $\measuredangle(\boldsymbol{u}_0, \boldsymbol{w}_0)$.

Proof. We have $\boldsymbol{u}_{\mathrm{rot}} = Q(t)\boldsymbol{u}_0$ and, by Lemma 17, $\boldsymbol{w} = Q(t)\boldsymbol{w}_0$. Therefore we find $\alpha_{\mathrm{lag}}(t) = \measuredangle(\boldsymbol{u}_{\mathrm{rot}}(t), \boldsymbol{w}(t)) = \measuredangle(\boldsymbol{u}_0, \boldsymbol{w}_0)$, which is constant. □

Thus, for isotropic ω, we can compute a curve $\alpha_{\mathrm{lag}}(\|\boldsymbol{u}_0\|) = \measuredangle(\boldsymbol{u}_{\mathrm{rot}}(t), \boldsymbol{w}(t))$ describing how the lag angle depends on the amplitude $\|\boldsymbol{u}_0\|$ at which $\boldsymbol{u}_{\mathrm{rot}}$ rotates. If ω has bounded support $\mathcal{K}$, then the alignment of $\boldsymbol{u}_{\mathrm{rot}}(t)$ and $\boldsymbol{w}(t)$ in saturation by Lemma 14 implies that $\alpha_{\mathrm{lag}}(\|\boldsymbol{u}_0\|) = 0$ for all $\|\boldsymbol{u}_0\| \geq R$. If, in addition, $\omega \geq 0$, then $\boldsymbol{w}(t)$ always lags behind $\boldsymbol{u}_{\mathrm{rot}}(t)$,

$$0 \leq \alpha_{\mathrm{lag}}(\|\boldsymbol{u}_0\|) \leq \pi. \quad (19)$$

To outline how (19) can be derived, we consider without loss of generality the memory state at $t = 4\pi$ for $\boldsymbol{u}_0 = (\|\boldsymbol{u}_0\|, 0)$, so $\boldsymbol{u}_{\mathrm{rot}}(4\pi) = \boldsymbol{u}_0$. Splitting the memory space up appropriately, we obtain that the subset $\|\boldsymbol{x}\| < r - \|\boldsymbol{u}_0\|$ contributes 0, $\|\boldsymbol{x} - \boldsymbol{u}_0\| \geq r$ contributes $\lambda\boldsymbol{u}_0$, and the remaining subset satisfying both $\|\boldsymbol{x}\| \geq r - \|\boldsymbol{u}_0\|$ and $\|\boldsymbol{x} - \boldsymbol{u}_0\| < r$ contributes a vector with nonnegative second component to $\boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\mathrm{rot}}]$, resulting in sum in (19). For details, see [17].

If ω is sufficiently reasonable, it seems to be an inherent property of the vector Preisach operator $\boldsymbol{\mathcal{P}}$ that the resulting lag angle curves look like those observed for real magnetic materials [11]. The left panel of Figure 3 shows a lag angle curve computed for an exemplary Preisach density.

Remark. $\boldsymbol{\mathcal{P}}$ is not equal to a Mròz model of vector hysteresis [18, 19]. This can be seen because for uniformly rotating input $\boldsymbol{u}_{\text{rot}}$, the output of the Mròz model is aligned with the input (see e.g. [20, Example 5.1]), whereas $\boldsymbol{\mathcal{P}}$ results in a non-zero lag for $\boldsymbol{u}(t) \leq R$, cf. Figure 3.

The remainder of this section addresses the question of hysteresis dissipation under periodic input. This is relevant in magnetic hysteresis modeling to guarantee agreement with the laws of thermodynamics when $\boldsymbol{\mathcal{P}}$ is used to represent the relationship between magnetic field $\boldsymbol{H}$, magnetization $\boldsymbol{M}$ and magnetic flux $\boldsymbol{B}$ by

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}, \qquad \boldsymbol{M} = \boldsymbol{\mathcal{P}}[\boldsymbol{H}]. \tag{20}$$

The energy injected in the time interval $[t_1, t_2]$ of a magnetization process is given by [21]

$$\int_{t_1}^{t_2} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B}. \tag{21}$$

If a process is periodic, say with period λ, since the internal states at t and $t + \lambda$ are equal, the energy dissipated in the course of $[t, t + \lambda]$ can be computed via (21) and must be non-negative:

$$\int_{t}^{t+\lambda} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B} \geq 0. \tag{22}$$

With (20), the energy balance (22) can be rephrased in terms of $\boldsymbol{H}$ and $\boldsymbol{M}$, that is, $\boldsymbol{\mathcal{P}}[\boldsymbol{H}]$, as

$$\int_{t}^{t+\lambda} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{B} = \int_{t}^{t+\lambda} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{M}.$$

In other words, the energy balance (22) is satisfied exactly if $\boldsymbol{\mathcal{P}}$ satisfies

$$\int_{t}^{t+\lambda} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{w} \geq 0 \tag{23}$$

for all λ-periodic continuous functions $\boldsymbol{u}$, $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}]$, and $t \geq \lambda$ to guarantee periodicity of $\boldsymbol{\mathcal{P}}$ by Lemma 15. By the definition of $\boldsymbol{\mathcal{P}}$, the dissipation results as weighted superposition of the losses of the single relays $\boldsymbol{h}_{(\boldsymbol{x},r)}$. Therefore, the following proposition is an immediate consequence of Lemma 3.

Proposition 21 (Dissipation)**.** *If* $\omega \geq 0$ *and* $\boldsymbol{u}$ *has bounded oscillation and is piecewise* C^1*, then* $\boldsymbol{\mathcal{P}}$ *satisfies the energy balance* (23)*.*

In particular, we again look at uniformly rotating input $\boldsymbol{u}_{\text{rot}}$ in $\mathbb{R}^2$. For this setup, dissipation was the subject of measurements since early in the history of magnetism research and results in a typical curve shape [22, 23, 24, 11]. If ω is isotropic then $\boldsymbol{w} = \boldsymbol{\mathcal{P}}[\boldsymbol{u}_{\text{rot}}]$, given in Lemma 17, is continuously differentiable. The dissipation rate is therefore equal to $\boldsymbol{u}_{\text{rot}}(t) \cdot \boldsymbol{w}'(t)$. A quick computation shows that for isotropic ω and $t \geq 2\pi$, we have $\boldsymbol{u}_{\text{rot}}(t) \cdot \boldsymbol{w}'(t) = \boldsymbol{u}_0^T Q(t)^T Q'(t) \boldsymbol{w}_0 = \det(\boldsymbol{u}_0, \boldsymbol{w}_0)$, and

$$\int_{t}^{t+2\pi} \boldsymbol{u}_{\text{rot}} \cdot \mathrm{d}\boldsymbol{w} = \int_{t}^{t+2\pi} \boldsymbol{u}_{\text{rot}}(\tau) \cdot \boldsymbol{w}'(\tau)\, \mathrm{d}\tau = 2\pi \det(\boldsymbol{u}_0, \boldsymbol{w}_0). \tag{24}$$

For arbitrary $\omega \geq 0$, we obtain that $\boldsymbol{u}_{\text{rot}}(t) \cdot \boldsymbol{w}'(t)$ is nonnegative and 0 if $||\boldsymbol{u}_0|| = 0$ or $||\boldsymbol{u}_0|| \geq R$. Here, non-negativity is a consequence of Proposition 21. (Note that $\boldsymbol{u}_{\text{rot}}$ has bounded oscillation.) If $||\boldsymbol{u}_0|| = 0$, then $\boldsymbol{u}_{\text{rot}} = 0$. If $||\boldsymbol{u}_0|| \geq R$, then the alignment of $\boldsymbol{u}_{\text{rot}}(t)$ and $\boldsymbol{w}(t)$ implies $\boldsymbol{w}_0 = \lambda \boldsymbol{u}_0$, $\lambda \geq 0$, and thus $\boldsymbol{u}_{\text{rot}}(t) \cdot \boldsymbol{w}'(t) = 0$.

The right panel of Figure 3 shows an example of a loss curve computed with the formula in Equation (24). We can see that the curve shape resembles that expected for magnetic materials.

A more general approach to the question of the dissipated energy requires an expression for the energy stored in the magnetic field, as represented by hysteresis potentials in the discussion in [2] for the scalar Preisach operator.

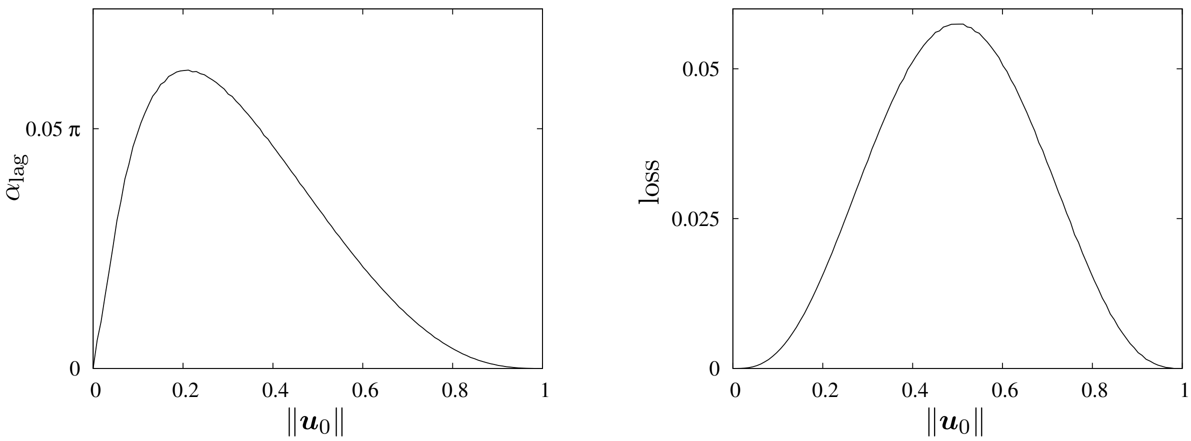

Figure 3. Lag angle curve (left panel) and loss curve (right panel) for the isotropic Preisach density ω defined by $\omega(\boldsymbol{x}, r) = 1 - \|\boldsymbol{x}\| - r$ if $\|\boldsymbol{x}\| + r < 1$, and 0 otherwise.

8. Conclusion

We have accumulated a number of properties of the vector Preisach operator $\boldsymbol{\mathcal{P}}$. The discussion comprised isotropy and the choice of a neutral memory state, the saturation behaviour, loop congruency and periodicity as well as lag angles and dissipation. Many questions remain open for investigation, most importantly that of the operator continuity of $\boldsymbol{\mathcal{P}}$.

Acknowledgments

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