Variance-based reliability sensitivity analysis and the FORM α-factors

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Abstract

In reliability assessments, it is useful to compute importance measures that provide information on the influence of the input random variables on the probability of failure. Classical importance measures are the α-factors, which are obtained as a by-product of the first-order reliability method (FORM). These factors are the directional cosines of the most probable failure point in an underlying independent standard normal space. Alternatively, one might assess sensitivity by a variance decomposition of the indicator function, i.e., the function that indicates membership of the random variables to the failure domain. This paper discusses the relation of the latter variance-based sensitivity measures to the FORM α-factors and analytically shows that there exist one-to-one relationships between them for linear limit-state functions of normal random variables. We also demonstrate that these relationships enable a good approximation of variance-based sensitivities for general reliability problems. The derived relationships shed light on the behavior of first-order and total-effect indices of the failure event in engineering reliability problems.

Keywords: Reliability analysis, Sensitivity analysis, α-factors, FORM

1. Introduction

In reliability analysis, the interest is in the evaluation of the probability of failure of an engineering system. Let $\boldsymbol{X}$ denote a continuous random vector of dimension n modeling the uncertain system variables; it is described by a joint probability density function (PDF) $f(\boldsymbol{x})$. The failure event F can be defined as the collection of the outcomes of $\boldsymbol{X}$ for which the so-called limit-state function (LSF) $g(\boldsymbol{x})$ takes non-positive values, i.e., $F = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) \leq 0\}$ [1]. The probability of failure can be expressed as

$$p_F = \Pr(F) = \int_{g(\boldsymbol{x}) \leq 0} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \,. \tag{1}$$

The LSF often depends on a computationally intensive numerical model of the engineering system, which makes evaluation of p_F a nontrivial task. Standard Monte Carlo, although suitable for estimating high-dimensional integrals, becomes inefficient when F is a rare event and p_F is small, as is typically the case for failure probabilities. Various tailored methods have been developed to efficiently estimate the integral in Eq. (1) when p_F is small [1, 2]. These include approximation methods such as the first/second order reliability method (FORM/SORM) [3], simulation-based methods, e.g., [4, 5, 6, 7, 8], and methods based on surrogate modeling [9].

In many applications of reliability analysis, one is interested in understanding the influence of components of $\boldsymbol{X}$ or the parameters of their joint PDF on the probability of failure. The sensitivity to distribution parameters can be quantified through local reliability sensitivity analysis, which involves evaluating partial derivatives of the probability of failure at the nominal values of the parameters, e.g., [10, 11, 12, 13, 14]. Global reliability sensitivity analysis examines the average effects of the variables in $\boldsymbol{X}$ on the probability of failure [15, 16, 17, 18, 19, 20, 21, 22]. These sensitivity measures can be viewed as extensions of variance-based sensitivity analysis [23] and/or moment-independent sensitivity analysis [24].

The global sensitivity indices introduced in [18, 19] are based on the variance decomposition of the indicator function of the event F, i.e., the function that indicates membership of $\boldsymbol{X}$ to the failure domain $\Omega_F = \{g(\boldsymbol{x}) \leq 0\}$. The first-order indices [18] indicate the contribution of the variance of individual components of $\boldsymbol{X}$ to the variance of the indicator function and can be viewed as a modified version of the moment-independent sensitivity measure of [24] for the probability of failure. Higher-order and total-effect indices [19] respectively represent the combined contribution of collections of components of $\boldsymbol{X}$ and the total contribution of all variance terms that include a certain component. Several methods have been proposed to estimate the first- and higher- order indices, including single-loop sampling methods [19], the state dependent parameter method [18] and an approach that post-processes failure samples from sampling-based reliability methods [25, 26].

A more traditional sensitivity measure that is widely used in structural reliability is the α-factors, obtained as a by-product of FORM [3, 27]. FORM performs a first-order Taylor series approximation of the boundary of the LSF in an equivalent independent standard normal space, at the point of Ω_F with largest probability density value. The evaluation of this so-called most probable failure point (MPFP) (also known as design point) requires the solution of an optimization problem. The α-factors are the directional cosines of the MPFP and they can be interpreted as variance-based sensitivity measures of the linearized LSF [3]. Generalizations of the α-factors have been proposed for dependent inputs in [3] and for multimodal failure domains in [28].

This contribution explores the relation of the FORM α-factors and the variance-based sensitivities of the indicator function and analytically derives one-to-one relationships between them for linear LSFs of normal random variables. The derived relationships motivate an investigation of the first-order and total-effect indices of the failure event for linear problems from which one can draw conclusions on the behavior of these indices in general engineering reliability problems. We also demonstrate that these relationships provide good approximations of the variance-based indices of general problems with independent inputs.

The structure of the paper is as follows. In Section 2, we review global reliability sensitivity analysis and the variance-based sensitivity indices of the indicator function. Section 3 discusses FORM and the α-factors. Section 4 introduces approximations of the variance-based sensitivities with FORM and numerically investigates the relationship between the α-factors and the derived first-order and total-effect indices. Section 5 presents two numerical examples that test the accuracy of the FORM approximations. The paper closes with the conclusions in Section 6.

2. Variance-based reliability sensitivity measures

2.1. Variance-based sensitivity analysis

Variance-based sensitivity analysis aims at identifying the input random variables in $\boldsymbol{X}$ that have largest impact on the variance of a quantity of interest (QOI) $Q = h(\boldsymbol{X})$, where $h : \mathbb{R}^n \to \mathbb{R}$ defines an input-output relationship. It is based on the functional analysis of variance (ANOVA) decomposition of $h(\boldsymbol{x})$, also known as high dimensional model representation

(HDMR). Consider the case where the random vector $\boldsymbol{X}$ consists of statistically independent components, i.e., $f(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)$ with $f_i(x_i)$ denoting the marginal PDF of X_i. Also assume that $h(\boldsymbol{x})$ is square-integrable, i.e., $\mathrm{E}[h(\boldsymbol{X})^2] < \infty$. The functional ANOVA decomposition of $h(\boldsymbol{x})$ reads [29, 23]

$$h(\boldsymbol{x}) = h_{\emptyset} + \sum_{i=1}^{n} h_i(x_i) + \sum_{1 \leq i < j \leq n} h_{i,j}(x_i, x_j) + \cdots + h_{1,\ldots,n}(x_1, \ldots, x_n)\,. \quad (2)$$

The representation of Eq. (2) exists and is unique provided that

$$\mathrm{E}[h_{\boldsymbol{v}}(\boldsymbol{X}_{\boldsymbol{v}})|\boldsymbol{X}_{\boldsymbol{v}\setminus i}] = \int_{-\infty}^{\infty} h_{\boldsymbol{v}}(\boldsymbol{x}_{\boldsymbol{v}}) f_i(x_i) \mathrm{d}x_i = 0,\ \forall i \in \boldsymbol{v}, \forall \boldsymbol{v} \in \mathcal{P}(\{1, \ldots, n\})\,, \quad (3)$$

with $\boldsymbol{x}_{\boldsymbol{v}} = \{x_i, i \in \boldsymbol{v}\}$ and $\mathcal{P}(S)$ denoting the power set of S. From Eq. (3) it follows that $h_{\emptyset} = \mathrm{E}[h(\boldsymbol{X})]$ and that the summands in Eq. (2) are mutually orthogonal, i.e., it is $\mathrm{E}[h_{\boldsymbol{v}}(\boldsymbol{X}_{\boldsymbol{v}}) h_{\boldsymbol{w}}(\boldsymbol{X}_{\boldsymbol{w}})] = 0$ for $\boldsymbol{v} \neq \boldsymbol{w} \in \mathcal{P}(\{1, \ldots, n\})$. From the orthogonality property, one gets the following decomposition of the variance of Q in terms of the variances of the ANOVA summands:

$$\mathrm{Var}(Q) = \sum_{i=1}^{n} V_i + \sum_{1 \leq i < j \leq n} V_{i,j} + \cdots + V_{1,\ldots,n}\,, \quad (4)$$

where $V_{\boldsymbol{v}} = \mathrm{Var}(h_{\boldsymbol{v}}(\boldsymbol{X}_{\boldsymbol{v}}))$. The component variances can also be expressed in terms of variances of conditional expectations as

$$V_{\boldsymbol{v}} = \mathrm{Var}(\mathrm{E}[Q|\boldsymbol{X}_{\boldsymbol{v}}]) - \sum_{\boldsymbol{w} \in \mathcal{P}(\boldsymbol{v}) \setminus \{\emptyset, \boldsymbol{v}\}} V_{\boldsymbol{w}}\,. \quad (5)$$

Assume now that $\mathrm{Var}(Q) \neq 0$. The Sobol' sensitivity index associated to $\boldsymbol{v}$ is defined as [23]

$$S_{\boldsymbol{v}} = \frac{V_{\boldsymbol{v}}}{\mathrm{Var}(Q)}\,. \quad (6)$$

For $\boldsymbol{v} = \{i\}$, the first-order Sobol' index S_i measures the contribution of the main effect of X_i on the variance of Q, whereas for $|\boldsymbol{v}| > 1$ the index $S_{\boldsymbol{v}}$ measures the portion of $\mathrm{Var}(Q)$ due to the high-order interactions between variables $\boldsymbol{X}_{\boldsymbol{v}}$. It is

$$\sum_{\boldsymbol{v} \in \mathcal{P}(1,\ldots,n) \setminus \{\emptyset\}} S_{\boldsymbol{v}} = 1\,. \quad (7)$$

The first-order component indices S_i can be used for factor prioritization: the larger S_i, the higher the reduction in the output variance upon full knowledge of X_i. In other words, the indices S_i enable one to identify those variables that would have the greatest impact on the variance of Q, if their values become known or if their uncertainties are decreased.

The total-effect index, measuring the contribution due to variables $\boldsymbol{X}_{\boldsymbol{v}}$ and their interactions with all other variables in $\boldsymbol{X}$, is given by [30]

$$S_{\boldsymbol{v}}^T = \frac{\mathrm{E}[\mathrm{Var}(Q|\boldsymbol{X}_{\sim\boldsymbol{v}})]}{\mathrm{Var}(Q)} = 1 - \frac{\mathrm{Var}(\mathrm{E}[Q|\boldsymbol{X}_{\sim\boldsymbol{v}}])}{\mathrm{Var}(Q)}\,, \tag{8}$$

where "$\sim \boldsymbol{v}$" denotes the set $\{1,\ldots,n\}\backslash\boldsymbol{v}$. For $\boldsymbol{v}=\{i\}$, it is

$$S_i^T = \sum_{\boldsymbol{v}\in\mathcal{P}(1,\ldots,n),i\in\boldsymbol{v}} S_{\boldsymbol{v}}\,. \tag{9}$$

The total-effect indices S_i^T are used for factor fixing, i.e., to identify which variables, if fixed, will impact the variance of Q the least.

2.2. Variance-based reliability sensitivities

To apply variance-based sensitivity analysis to the reliability problem of Eq. (1), one needs to choose an appropriate QOI that describes the failure event F as a function of the input variables $\boldsymbol{X}$. Define the random variable $Z = I(g(\boldsymbol{X}) \leq 0)$, where $I(g(\boldsymbol{x}) \leq 0)$ is the indicator function that defines the failure domain. It is $I(g(\boldsymbol{x}) \leq 0) = 1$ if $g(\boldsymbol{x}) \leq 0$ and $I(g(\boldsymbol{x}) \leq 0) = 0$ otherwise. The variable Z follows the Bernoulli distribution with parameter p_F; it has mean $\mathrm{E}[Z] = p_F$ and variance $\mathrm{Var}(Z) = p_F(1-p_F)$. A decomposition of the variance of Z leads to the following Sobol' reliability sensitivity index:

$$S_{F,\boldsymbol{v}} = \frac{V_{F,\boldsymbol{v}}}{\mathrm{Var}(Z)} = \frac{V_{F,\boldsymbol{v}}}{p_F(1-p_F)}\,, \tag{10}$$

with

$$V_{F,\boldsymbol{v}} = \mathrm{Var}(\mathrm{E}[Z|\boldsymbol{X}_{\boldsymbol{v}}]) - \sum_{\boldsymbol{w}\in\mathcal{P}(\boldsymbol{v})\backslash\{\emptyset,\boldsymbol{v}\}} V_{F,\boldsymbol{w}}\,. \tag{11}$$

For a scalar $\boldsymbol{v} = \{i\}$, the first-order Sobol' index $S_{F,i}$ associated to random variable X_i is given as follows:

$$S_{F,i} = \frac{\mathrm{Var}(\mathrm{E}[Z|X_i])}{p_F(1-p_F)}\,. \tag{12}$$

The conditional expectation in the numerator of Eq. (12) can also be written as the conditional probability of F given X_i, i.e., $\mathrm{E}[Z|X_i] = \Pr(F|X_i)$. Therefore, the index $S_{F,i}$ is equivalent to the moment-independent importance measure δ_i^p of [18], since it holds

$$\delta_i^p = \mathrm{E}[(\Pr(F) - \Pr(F|X_i))^2] = \mathrm{Var}(\Pr(F|X_i)) = \mathrm{Var}(\mathrm{E}[Z|X_i]) \,. \quad (13)$$

The measure of Eq. (13) can also be extended for groups of variables $\boldsymbol{X_v}$ [18], in which case it corresponds to the first term of the right hand side of Eq. (11), which if normalized with the total variance is also known as the closed Sobol' index of the failure event [31].

The total-effect reliability sensitivity index is given by [19]

$$S_{F,\boldsymbol{v}}^T = 1 - \frac{\mathrm{Var}(\mathrm{E}[Z|\boldsymbol{X}_{\sim\boldsymbol{v}}])}{\mathrm{Var}(Z)} = 1 - \frac{\mathrm{Var}(\Pr(F|\boldsymbol{X}_{\sim\boldsymbol{v}}))}{p_F(1-p_F)} \,. \quad (14)$$

The first-order reliability component index $S_{F,i}$ can be used to identify which random variable X_i if learned (e.g., through investing in measurement campaigns) will increase the accuracy of p_F the most. The total-effect reliability component index $S_{F,i}^T$ can be used to identify the random variables with $S_{F,i}^T \approx 0$, which, if fixed, will not impact the prediction of p_F. Fixing variables with small $S_{F,i}^T$ can decrease the modeling and possibly the computational complexity of further analyses.

The Sobol' and total-effect reliability sensitivity indices of Eqs. (10) and (14) can be estimated by several sampling based approaches, e.g [18, 19, 25, 26]. Here, we derive approximations of these indices based on the FORM approach to reliability analysis. Before discussing these approximations, we review FORM and the classic related sensitivity indices, the so-called FORM α-factors.

3. FORM and the $\boldsymbol{\alpha}$-factors

FORM is an approximation method for solving the reliability problem of Eq. (1). It approximates the probability integral through linearizing the LSF at the most probable failure point (MPFP) in an equivalent standard normal space, the $\boldsymbol{U}$-space, where $\boldsymbol{U}$ is an n-dimensional vector of independent standard normal random variables. The first step of FORM is to transform the problem to the $\boldsymbol{U}$-space. The vector $\boldsymbol{U}$ can be expressed in terms of the original random vector $\boldsymbol{X}$ through an isoprobabilistic mapping $\boldsymbol{U} = \boldsymbol{T}(\boldsymbol{X})$

[32, 33]. For the case where the variables $\boldsymbol{X}$ are statistically independent and have strictly increasing marginal cumulative distribution functions (CDFs) $F_i(x_i), i = 1, \ldots, n$, this mapping is $\boldsymbol{T}(\boldsymbol{x}) = [\Phi^{-1}(F_1(x_1)); \cdots; \Phi^{-1}(F_n(x_n))]$, with Φ denoting the standard normal CDF. The LSF can be expressed in the $\boldsymbol{U}$-space as $G(\boldsymbol{u}) = g[\boldsymbol{T}^{-1}(\boldsymbol{u})]$ with $\boldsymbol{T}^{-1}$ denoting the inverse mapping $\boldsymbol{X} = \boldsymbol{T}^{-1}(\boldsymbol{U})$. The probability integral of Eq. (1) can then be transformed as

$$p_F = \Pr(F) = \int_{G(\boldsymbol{u})\leq 0} \varphi_n(\boldsymbol{u})\mathrm{d}\boldsymbol{u}\,, \tag{15}$$

where φ_n is the n-variate independent standard normal PDF. Next, the MPFP $\boldsymbol{u}^*$ is found as the point of the limit-state surface $G(\boldsymbol{u}) = 0$ that maximizes $\varphi_n(\boldsymbol{u})$, or equivalently that minimizes the distance to the origin $\|\boldsymbol{u}\|$, through solving

$$\boldsymbol{u}^* = \operatorname{argmin}\{\|\boldsymbol{u}\| \mid G(\boldsymbol{u}) = 0\}\,. \tag{16}$$

The program of Eq. (16) can be solved by a variety of optimization algorithms, e.g., [34]. Assuming that the LSF $G(\boldsymbol{u})$ is continuous and differentiable in the neighborhood of $\boldsymbol{u}^*$, we can approximate $G(\boldsymbol{u})$ in this neighborhood through its linearization at $\boldsymbol{u}^*$,

$$G(\boldsymbol{u}) \cong G_1(\boldsymbol{u}) = \nabla G(\boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*) = \|\nabla G(\boldsymbol{u}^*)\|(\beta - \boldsymbol{\alpha}\boldsymbol{u})\,. \tag{17}$$

Here $\nabla G(\boldsymbol{u}^*) = [\partial G/\partial u_1|_{\boldsymbol{u}=\boldsymbol{u}^*}, \ldots, \partial G/\partial u_n|_{\boldsymbol{u}=\boldsymbol{u}^*}]$ is the gradient row vector, $\boldsymbol{\alpha} = -\nabla G(\boldsymbol{u}^*)/\|\nabla G(\boldsymbol{u}^*)\|$ is the normalized negative gradient vector at the MPFP (directed towards the failure domain) and $\beta = \boldsymbol{\alpha}\boldsymbol{u}^*$ is the FORM reliability index. The linearization $G_1(\boldsymbol{u})$ is illustrated in Figure 1. The FORM approximation of the failure event is

$$F \cong F_1 = \{\boldsymbol{u} \in \mathbb{R}^n : G_1(\boldsymbol{u}) \leq 0\} = \{\boldsymbol{u} \in \mathbb{R}^n : \boldsymbol{\alpha}\boldsymbol{u} \geq \beta\}\,, \tag{18}$$

which leads to the following approximation of the probability of failure:

$$p_F \cong p_{F_1} = \Pr(\boldsymbol{\alpha}\boldsymbol{U} \geq \beta) = \Phi(-\beta)\,. \tag{19}$$

The latter follows from the fact that the random variable $Y = \boldsymbol{\alpha}\boldsymbol{U}$ follows the standard normal distribution.

The components of the vector $\boldsymbol{\alpha}$, also known as α-factors, are used in reliability analysis to assess the contribution of each random variable in $\boldsymbol{U}$

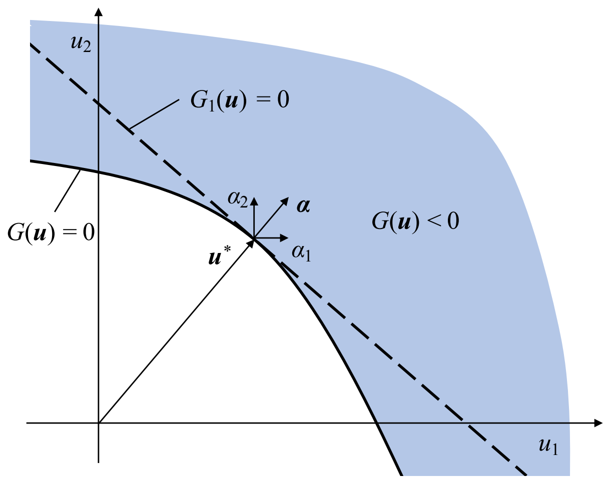

Figure 1: Illustration of the FORM approximation of the LSF $G_1(\boldsymbol{u})$ and the $\boldsymbol{\alpha}$-factors.

to the probability of failure. The α-factors are the directional cosines of the MPFP $\boldsymbol{u}^*$ (Figure 1). They can be evaluated in terms of the coordinates of the MPFP through[1] $\alpha_i = u_i^*/\beta$. The variance of the function $G_1(\boldsymbol{U})$ is given by

$$\mathrm{Var}(G_1(\boldsymbol{U})) = \|\nabla G(\boldsymbol{u}^*)\|^2 \sum_{i=1}^{n} \alpha_i^2 \mathrm{Var}(U_i) = \|\nabla G(\boldsymbol{u}^*)\|^2 . \tag{20}$$

Comparing Eq. (20) with Eq. (4) we see that the squares of the α-factors are proportional to the contributions of the variables $\boldsymbol{U}$ on the variance of $G_1(\boldsymbol{U})$. In fact it is easy to see that $V_{G_1,i} = \mathrm{Var}(\mathrm{E}[G_1(\boldsymbol{U})|U_i]) = \|\nabla G(\boldsymbol{u}^*)\|^2\alpha_i^2$, which gives

$$S_{G_1,i} = \frac{\mathrm{Var}(\mathrm{E}[G_1(\boldsymbol{U})|U_i])}{\mathrm{Var}(G_1(\boldsymbol{U}))} = \frac{\|\nabla G(\boldsymbol{u}^*)\|^2\alpha_i^2}{\|\nabla G(\boldsymbol{u}^*)\|^2} = \alpha_i^2 . \tag{21}$$

That is, the squared α-factors are the first-order Sobol' indices of the linearized LSF in $\boldsymbol{U}$-space. They are also identical to the total-effect indices of $G_1(\boldsymbol{U})$, since $G_1(\boldsymbol{u})$ is a linear function and, hence, the components of $\boldsymbol{u}$ enter only as main effects in $G_1(\cdot)$.

We note that for the case where the random variables $\boldsymbol{X}$ are statistically

[1]The reliability index β can also be evaluated in terms of the MPFP. It is $\beta = \|\boldsymbol{u}^*\|$ if $G(\boldsymbol{0}) > 0$ and $\beta = -\|\boldsymbol{u}^*\|$ if $G(\boldsymbol{0}) < 0$.

independent, the same decomposition as in Eq. (20) applies for the variance of the linearized LSF transformed back to the $\boldsymbol{X}$-space. However this does not apply if $\boldsymbol{X}$ consists of dependent random variables. Extensions of the α-factors for dependent inputs are given in [3, 28].

4. Variance-based reliability sensitivity analysis with FORM

We now derive expressions for the reliability sensitivity indices defined in Eq. (10) and Eq. (14) of the reliability problem described by the linearized LSF of Eq. (17). Define the Bernoulli random variable $Z_1 = I(G_1(\boldsymbol{U}) \leq 0)$, describing the geometry of the failure domain of the FORM approximation in $\boldsymbol{U}$-space. Z_1 has mean $\mathrm{E}[Z_1] = p_{F_1}$ and variance $\mathrm{Var}(Z_1) = p_{F_1}(1-p_{F_1})$. The Sobol' index of the indicator of the failure event F_1 associated to component indices $\boldsymbol{v}$ is given by Eq. (10) as

$$S_{F_1,\boldsymbol{v}} = \frac{V_{F_1,\boldsymbol{v}}}{\mathrm{Var}(Z_1)} = \frac{V_{F_1,\boldsymbol{v}}}{p_{F_1}(1-p_{F_1})}\,, \tag{22}$$

with

$$V_{F_1,\boldsymbol{v}} = \mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\boldsymbol{v}}]) - \sum_{\boldsymbol{w}\in\mathcal{P}(\boldsymbol{v})\setminus\{\emptyset,\boldsymbol{v}\}} V_{F_1,\boldsymbol{w}}\,. \tag{23}$$

Evaluating the indices of Eq. (22) amounts to evaluating the variances of conditional expectations in Eq. (23), $\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\boldsymbol{v}}])$, for all $\boldsymbol{v} \in \mathcal{P}(\{1,\ldots,n\})$. The main result of the paper is given by the following proposition.

Proposition 4.1. *The variance of conditional expectation of the random variable* $Z_1 = I(G_1(\boldsymbol{U}) \leq 0)$, $\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\boldsymbol{v}}])$, *with* $\boldsymbol{U}_{\boldsymbol{v}} = \{U_i, i \in \boldsymbol{v}\}$ *can be expressed through the following integral:*

$$\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\boldsymbol{v}}]) = \int_0^{\|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2} \varphi_2(-\beta,-\beta,r)\mathrm{d}r\,, \tag{24}$$

with $\varphi_2(\cdot,\cdot,r)$ *denoting the bivariate standard normal PDF with correlation parameter* r*,*

$$\varphi_2(-\beta,-\beta,r) = \frac{1}{2\pi\sqrt{1-r^2}}\exp\left(-\frac{\beta^2}{1+r}\right)\,. \tag{25}$$

Proof. The variance $\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U_v}])$ can be expanded as follows:

$$\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U_v}]) = \mathrm{Var}(\Pr(F_1|\boldsymbol{U_v})) = \mathrm{E}[\Pr(F_1|\boldsymbol{U_v})^2] - \mathrm{E}[\Pr(F_1|\boldsymbol{U_v})]^2 \,. \quad (26)$$

The mean of the conditional probability of F_1 is equal to the unconditional probability,

$$\mathrm{E}[\Pr(F_1|\boldsymbol{U_v})] = \Pr(F_1) = \Phi(-\beta) \,. \quad (27)$$

The conditional probability of F_1 given $\{\boldsymbol{U_v} = \boldsymbol{u_v}\}$ reads

$$\Pr(F_1|\boldsymbol{U_v} = \boldsymbol{u_v}) = \Pr(\boldsymbol{\alpha_{\sim v}}\boldsymbol{U_{\sim v}} \geq \beta - \boldsymbol{\alpha_v}\boldsymbol{u_v}) \,. \quad (28)$$

The random variable $\boldsymbol{\alpha_{\sim v}}\boldsymbol{U_{\sim v}}$ follows the normal distribution with zero mean and variance $\|\boldsymbol{\alpha_{\sim v}}\|^2$. Therefore,

$$\Pr(F_1|\boldsymbol{U_v} = \boldsymbol{u_v}) = \Phi\left(\frac{\boldsymbol{\alpha_v}\boldsymbol{u_v} - \beta}{\|\boldsymbol{\alpha_{\sim v}}\|}\right) = \Phi\left(\frac{\boldsymbol{\alpha_v}\boldsymbol{u_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right) \,. \quad (29)$$

We then have

$$\begin{aligned}
\mathrm{E}[\Pr(F_1|\boldsymbol{U_v})^2] &= \mathrm{E}\left[\Phi\left(\frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right)^2\right] \\
&= \mathrm{E}\left[\Pr\left(\tilde{U}_1 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}} \middle| \boldsymbol{U_v}\right) \Pr\left(\tilde{U}_2 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}} \middle| \boldsymbol{U_v}\right)\right] \\
&= \mathrm{E}\left[\Pr\left(\left\{\tilde{U}_1 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right\} \cap \left\{\tilde{U}_2 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right\} \middle| \boldsymbol{U_v}\right)\right] \\
&= \Pr\left(\left\{\tilde{U}_1 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right\} \cap \left\{\tilde{U}_2 \leq \frac{\boldsymbol{\alpha_v}\boldsymbol{U_v} - \beta}{\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2}}\right\}\right) \,,
\end{aligned} \quad (30)$$

where $\tilde{U}_1$ and $\tilde{U}_2$ are auxiliary independent standard normal random variables. Define the random variables $\tilde{Y}_i = \tilde{U}_i\sqrt{1 - \|\boldsymbol{\alpha_v}\|^2} - \boldsymbol{\alpha_v}\boldsymbol{U_v}, i = 1, 2$. The variables $\tilde{Y}_i, i = 1, 2$, have zero means, unit variances, correlation coefficient $\tilde{\rho}_{12} = \|\boldsymbol{\alpha_v}\|^2$, and, since they are linear functions of normal random variables, they follow the bivariate standard normal distribution. We have

$$\mathrm{E}[\Pr(F_1|\boldsymbol{U_v})^2] = \Pr\left(\left\{\tilde{Y}_1 \leq -\beta\right\} \cap \left\{\tilde{Y}_2 \leq -\beta\right\}\right) = \Phi_2(-\beta, -\beta, \|\boldsymbol{\alpha_v}\|^2) \,, \quad (31)$$

where $\Phi_2(\cdot, \cdot, r)$ is the bivariate standard normal CDF with correlation parameter r. The bivariate normal CDF can be expressed in terms of a single-fold integral (e.g., [35]), such that

$$\Phi_2(-\beta, -\beta, \|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2) = \Phi(-\beta)^2 + \int_0^{\|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2} \varphi_2(-\beta, -\beta, r)\mathrm{d}r \,. \tag{32}$$

Combining Eqs. (26), (27), (31) and (32), we arrive at the final result of Eq. (24).

□

Proposition 4.1 shows that the quantity $\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\boldsymbol{v}}])$ and, hence, the Sobol' index $S_{F_1,\boldsymbol{v}}$ can be computed as functions of the reliability index β and the factors $\boldsymbol{\alpha}_{\boldsymbol{v}}$. These functions are integrals of the bivariate standard normal PDF over the correlation parameter r, with both arguments set equal to $-\beta$ and can be evaluated efficiently through one-dimensional numerical integration. From Proposition 4.1, Eqs. (22) and (23), and setting $\boldsymbol{v} = \{i\}$, the first-order Sobol' index, $S_{F_1,i}$, takes the following form:

$$S_{F_1,i} = \frac{1}{p_{F_1}(1-p_{F_1})} \int_0^{\alpha_i^2} \varphi_2(-\beta, -\beta, r)\mathrm{d}r \,. \tag{33}$$

The total-effect index of Z_1 associated to component indices $\boldsymbol{v}$ is given by Eq. (14) as

$$S^T_{F_1,\boldsymbol{v}} = 1 - \frac{\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\sim\boldsymbol{v}}])}{\mathrm{Var}(Z_1)} = 1 - \frac{\mathrm{Var}(\Pr(F_1|\boldsymbol{U}_{\sim\boldsymbol{v}}))}{p_{F_1}(1-p_{F_1})} \,. \tag{34}$$

Corollary 4.1. *The total-effect index of Z_1, $S^T_{F_1,\boldsymbol{v}}$, associated to component indices $\boldsymbol{v}$ can be expressed through the following integrals:*

$$S^T_{F_1,\boldsymbol{v}} = 1 - \frac{1}{p_{F_1}(1-p_{F_1})} \int_0^{1-\|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2} \varphi_2(-\beta, -\beta, r)\mathrm{d}r \tag{35}$$

$$= \frac{1}{p_{F_1}(1-p_{F_1})} \int_{1-\|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2}^{1} \varphi_2(-\beta, -\beta, r)\mathrm{d}r \,. \tag{36}$$

Proof. From Proposition 4.1 and since $\|\boldsymbol{\alpha}\| = 1$ we have

$$\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{U}_{\sim\boldsymbol{v}}]) = \int_0^{\|\boldsymbol{\alpha}_{\sim\boldsymbol{v}}\|^2} \varphi_2(-\beta, -\beta, r)\mathrm{d}r = \int_0^{1-\|\boldsymbol{\alpha}_{\boldsymbol{v}}\|^2} \varphi_2(-\beta, -\beta, r)\mathrm{d}r \,. \tag{37}$$

Substitution to Eq. (34) gives the first result of Eq. (35). From Eq. (32) and since $\Phi_2(-\beta, -\beta, 1) = \Phi(-\beta)$, we get

$$\int_0^1 \varphi_2(-\beta, -\beta, r)\mathrm{d}r = \Phi(-\beta)[1 - \Phi(-\beta)] = p_{F_1}(1 - p_{F_1})\,. \qquad (38)$$

Combining Eqs. (38) and (35) gives the second result of Eq. (36). □

From Corollary 4.1, it follows that the total-effect index of Z_1 associated to random variable U_i, $S^T_{F_1,i}$, can be expressed as

$$S^T_{F_1,i} = \frac{1}{p_{F_1}(1 - p_{F_1})} \int_{1-\alpha_i^2}^1 \varphi_2(-\beta, -\beta, r)\mathrm{d}r\,. \qquad (39)$$

Hence, both the first-order and total-effect indices for an individual component U_i can be determined by knowledge of the reliability index β and the factor α_i. This is not surprising, since the linearized failure event is completely defined by β and the factors $\boldsymbol{\alpha}$, cf. Eq.(18). Also, through observing Eqs. (33) and (39) and noting that $\varphi_2(-\beta, -\beta, r)$ is a non-negative function, it is straightforward to see that $\alpha_i^2 < \alpha_j^2$ implies $S_{F_1,i} < S_{F_1,j}$ and $S^T_{F_1,i} < S^T_{F_1,j}$. That is, the ranking obtained by the squared α-factors is the same as the one obtained by both the first-order Sobol' indices and the total-effect indices of the failure event F_1.

Remark 4.1. *We remark that the results discussed here directly apply to the Sobol' and total-effect indices of Z_1 with respect to the original variables $\boldsymbol{X}$ for the case where $\boldsymbol{X}$ consists of statistically independent components. This can be understood by examining the variance of the conditional expectation of Z_1 given $\boldsymbol{X_v}$,* $\mathrm{Var}(\mathrm{E}[Z_1|\boldsymbol{X_v}]) = \mathrm{Var}(\Pr[F_1|\boldsymbol{X_v}])$. *Because of independence of the components, the conditional probability* $\Pr[F_1|\boldsymbol{X_v} = \boldsymbol{x_v}]$ *is evaluated under the probability measure of $\boldsymbol{X}_{\sim\boldsymbol{v}}$, i.e., the conditional density of $\boldsymbol{X}_{\sim\boldsymbol{v}}$ given $\boldsymbol{X_v} = \boldsymbol{x_v}$ is identical to the marginal density of $\boldsymbol{X}_{\sim\boldsymbol{v}}$. Since the transformation of each component $U_i = T_i(X_i)$ is one-to-one and probability-preserving, it is* $\Pr[F_1|\boldsymbol{X_v} = \boldsymbol{x_v}] = \Pr[F_1|\boldsymbol{U_v} = \boldsymbol{T_v}(\boldsymbol{x_v})]$, *where $\boldsymbol{T_v}(\cdot)$ collects the component-wise transformations. It directly follows that*

$$\mathrm{Var}(\Pr[F_1|\boldsymbol{X_v}]) = \mathrm{Var}(\Pr[F_1|\boldsymbol{U_v}])\,. \qquad (40)$$

Figure 2 shows the behavior of the first-order and total-effect indices $S_{F_1,i}$ and $S^T_{F_1,i}$ with changing α_i^2 for different values of the FORM probability approximation p_{F_1}. We see that $S_{F_1,i}$ differs significantly from $S^T_{F_1,i}$, indicating that the variance of the indicator function Z_1 of the linearized failure domain is dominated by high-order effects. This can be explained by the fact that the majority of the probability mass in the failure region of the independent standard normal joint PDF tends to concentrate in areas closer to the origin, i.e., in the vicinity of the MPFP. Therefore, even for the case where a single random variable dominates the linearized LSF, the probability mass in the failure domain will be at areas where the remaining variables are also important. This is illustrated in Figure 3, which shows the bulk of the probability mass in the failure domain of a two-dimensional linear LSF with $\boldsymbol{\alpha} = [0.44, 0.9]$ and $p_F = 10^{-3}$. In this example, it is clear that the variable U_2 is the dominating one, it has $\alpha_2^2 = 0.81$ whereas $\alpha_1^2 = 0.19$. In spite of this, one can see that the probability mass in the failure domain significantly increases for larger values of U_1.

The more the linearized LSF moves away from the origin the more probability mass concentrates around the MPFP (e.g, see the asymptotic results in [36]). Therefore, the difference between $S_{F_1,i}$ and $S^T_{F_1,i}$ increases as p_{F_1} becomes smaller. Consider for example the case where $p_{F_1} = 10^{-5}$. For $\alpha_i^2 = 0.9$, which indicates a high contribution of variable U_i, it is $S_{F_1,i} = 0.31$ and $S^T_{F_1,i} \approx 1$. We postulate that this result is generalizable to the first-order and total-effect indices of nonlinear problems for which FORM provides a good approximation to the probability of failure. This is verified in the numerical examples in Section 5. Similar observations can be found in [19, 37].

Variables with small $|\alpha_i|$ are often fixed at their mean values in further reliability assessments or simpler probablistic models are used to describe them, e.g., [38, 39, 40, 41]. This is consistent with the information obtained from the total-effect index. For example for $p_{F_1} = 10^{-5}$ and $|\alpha_i| = 0.01$ it is $S^T_{F_1,i} = 0.025$. Hence, one can conclude that the α-factors provide consistent information with the total-effect indices, and they can be reasonably applied for variable fixing. The threshold value for application of the α-factors for variable fixing depends on the value of β (or, equivalently, p_{F_1}), which can also be observed from Figure 2. Hence, if one wants to use a factor fixing threshold in terms of α_i^2, such a threshold is ideally determined by fixing a threshold on $S^T_{F_1,i}$ and translating it to α_i^2 for a specific β.

Figure 2 also shows that the α-factors are better behaved than both $S_{F_1,i}$ and $S^T_{F_1,i}$, which tend to take values close to 0 and 1, respectively, especially for small failure probabilities. We remark that a sensitivity factor that serves a similar purpose than the total-effect index is the omission sensitivity index [42]. This index measures the effect on the reliability index when a random

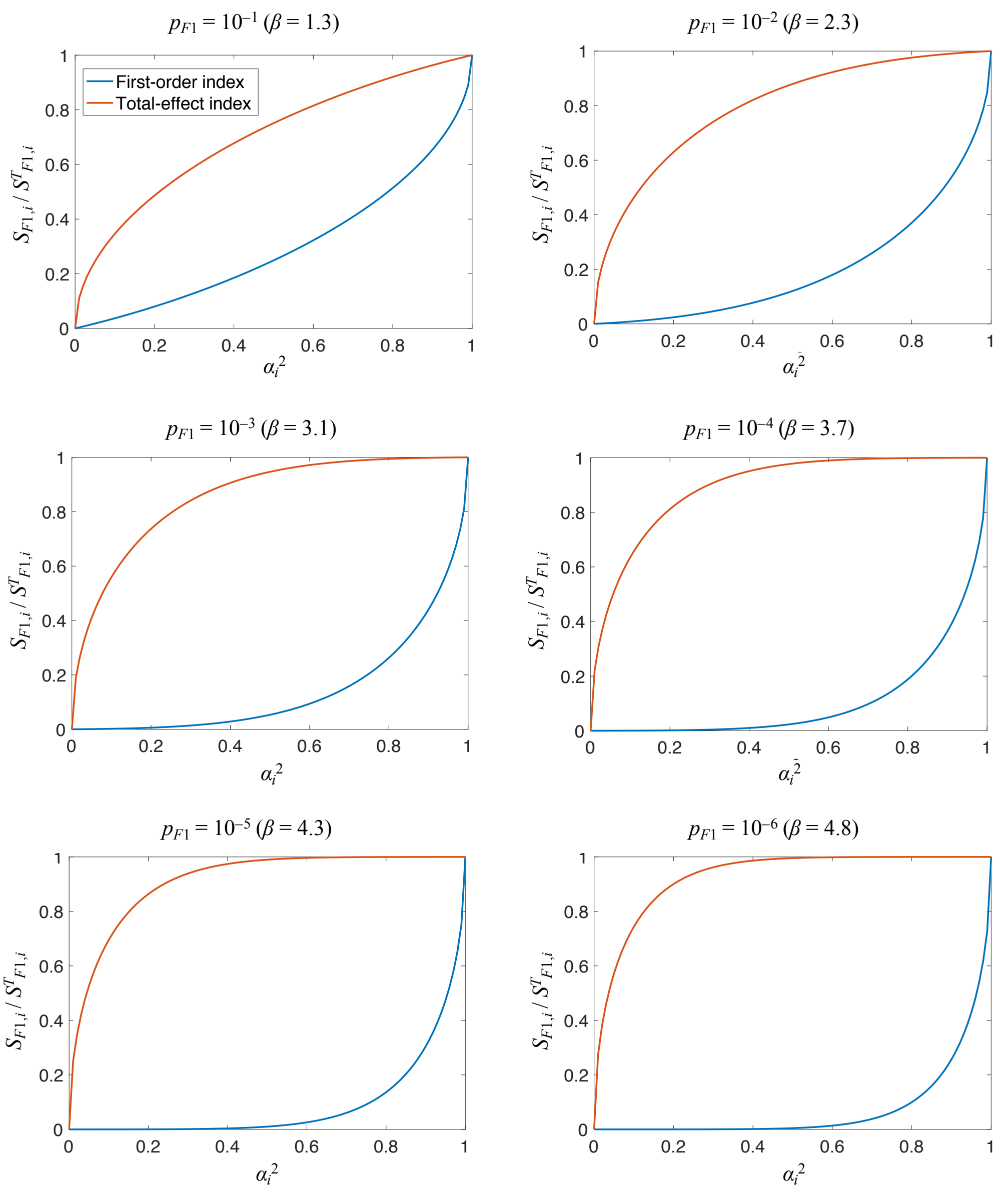

Figure 2: Illustration of the first-order and total-effect indices for different values of p_{F_1}.

variable is replaced by a deterministic value and is often used for variable fixing.

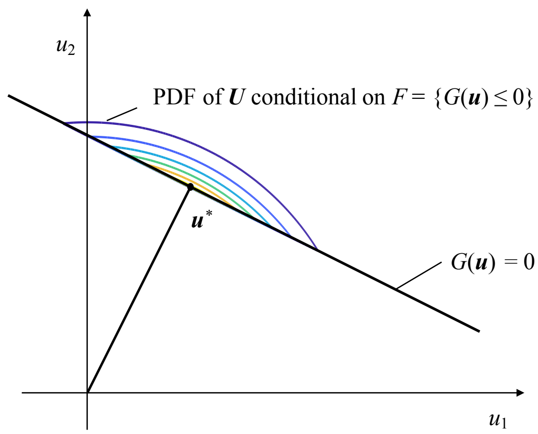

Figure 3: Probability density of $\boldsymbol{U}$ in the failure domain for a problem with linear LSF, $\boldsymbol{\alpha} = [0.44, 0.9]$ and $p_F = 10^{-3}$.

5. Numerical examples

We test the quality of the FORM approximation of the sensitivity indices of nonlinear reliability problems with two numerical examples. We consider examples where FORM is expected to give a reasonable approximation to the probability of failure. The goal is to assess whether the FORM approximation of the sensitivity indices can be applied to problems where FORM is routinely applied. The first example is concerned with the safety assessment of a steel column and the second with a deformation-sensitive elastic truss structure. The FORM estimates of the first-order and total-effect indices are compared with the estimates obtained by Monte Carlo (MC) simulation. MC estimates are obtained using the classical estimators (e.g., see [31]), which require $n_s(n+2)$ LSF evaluations, where n_s denotes the number of samples.

5.1. Steel column

The first example consists of a wide flange steel column that is simply supported, as depicted in Figure 4. It is based on the test bed example presented in [43]. The dimensions of the cross-section are taken as $b = h = 250$ mm, $t_\mathrm{b} = 15$ mm and $t_\mathrm{h} = 10$ mm. The column is subjected to a compressive load P consisting of two components $P = P_\mathrm{p} + P_\mathrm{e}$, where P_p denotes the permanent load and P_e the environmental (snow) load. The column is assumed to have an initial deformation due to construction imperfections. The deformation has a parabolic shape with maximum amplitude δ_0 at the centre of the column.

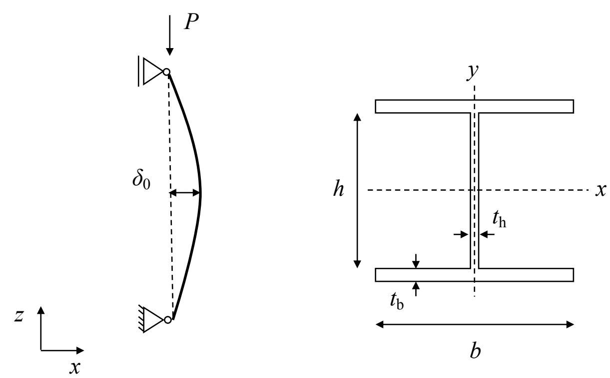

Figure 4: Wide flange steel column with initial deflection.

The column has a length of $L = 7.5$ m. The critical limit-state function that governs the performance of the column is given as follows:

$$g(\boldsymbol{X}) = 1 - \left(\frac{P}{f_{\mathrm{y}} A_{\mathrm{s}}} + \frac{P\delta_0}{f_{\mathrm{y}} W_{\mathrm{s}}} \cdot \frac{P_{\mathrm{b}}}{P_{\mathrm{b}} - P} \right) , \tag{41}$$

where $\boldsymbol{X} = [P_{\mathrm{p}}; P_{\mathrm{e}}; \delta_0; f_{\mathrm{y}}; E]$ is the vector of basic random variables. P_{b} is the Euler buckling load and is given by

$$P_{\mathrm{b}} = \frac{\pi^2 E I_{\mathrm{s}}}{L^2}. \tag{42}$$

A_{s}, W_{s} and I_{s} denote the area, section modulus and moment of inertia of the cross section around its weak axis and are given by

$$A_{\mathrm{s}} = 2bt_{\mathrm{b}} + ht_{\mathrm{h}} , \tag{43}$$

$$W_{\mathrm{s}} = \frac{ht_{\mathrm{h}}^3}{6b} + \frac{t_{\mathrm{b}}b^2}{3} , \tag{44}$$

$$I_{\mathrm{s}} = \frac{ht_{\mathrm{h}}^3}{12} + \frac{t_{\mathrm{b}}b^3}{6}. \tag{45}$$

f_{y} denotes the yield strength and E the Young's modulus of the steel material. The random vector $\boldsymbol{X}$ consists of statistically independent random variables

Table 1: Uncertain parameters of the steel column example.

Parameter	Distribution	Mean	St. Dev.
P_{p} [kN]	Normal	200	20
P_{e} [kN]	Gumbel	400	60
δ_0 [mm]	Normal	30	10
f_{y} [MPa]	Lognormal	400	32
E [MPa]	Lognormal	2.1×10^5	8.4×10^3

with marginal distributions given in Table 1. The MC reference solutions are computed with 100 independent runs with $n_s = 10^7$ samples.

The obtained MC estimate of the probability of failure is $p_F = 8.35 \times 10^{-5}$, whereas the FORM solution is $p_{F_1} = 8.19 \times 10^{-5}$. Table 2 shows the mean and coefficient of variation of the first-order and total-effect indices together with the FORM estimates and the FORM squared α-factors. The first-order and total-effect indices are also plotted in Figs. 5 and 6, respectively. We see that the only variables that have first-order index values that are not practically zero are the the environmental load P_{e} and the initial displacement δ_0. The load P_{e} is the dominant variable (with highest first-order index), which indicates that investing in reducing its uncertainty will have the highest impact on the probability of failure. The situation is different for the total-effect indices, as in this case all variables have considerable index values. The results also show that the FORM results agree well with the MC estimates, which is to be expected because of the high accuracy of the FORM probability estimate.

5.2. Elastic truss structure

Next, we consider an elastic truss that consists of 23 rods as depicted in Figure 7 [44]. Horizontal and diagonal rods have cross-sections A_1, A_2 and Young's moduli E_1, E_2, respectively. The truss sustains 6 vertical point loads P_1 - P_6. The variables $\boldsymbol{X} = [A_1; A_2; E_1; E_2; P_1; \ldots; P_6]$ are modeled by independent random variables with marginal distributions given in Table 3. The considered limit-state function restricts the maximum vertical displacement of the truss and is given by

$$g(\boldsymbol{X}) = u_{\mathrm{lim}} - u_{\mathrm{max}}(\boldsymbol{X}) \,, \tag{46}$$

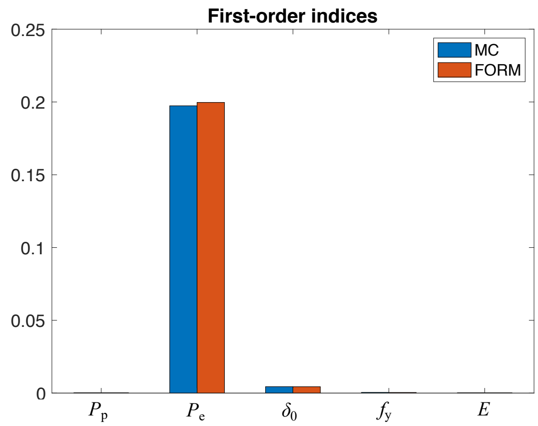

Figure 5: First-order indices for the steel column example. Comparison of MC and FORM estimates.

Table 2: Estimates of the first-order and total-effect indices for the steel column example. The MC results include the coefficient of variation of the estimate in brackets.

Variable	$S_{F,i}$		$S_{F,i}^T$		α_i^2
	MC	FORM	MC	FORM	
P_p	1.7×10^{-4} (16.1%)	1.5×10^{-4}	0.2365 (0.2%)	0.2331	0.0149
P_e	0.1974 (0.2%)	0.1997	0.9896 (< 0.1%)	0.9919	0.7524
δ_0	0.0044 (1.8%)	0.0044	0.7354 (< 0.1%)	0.7224	0.1836
f_y	4.8×10^{-4} (7.4%)	4.2×10^{-4}	0.3595 (0.1%)	0.3605	0.0368
E	1.6×10^{-4} (15.1%)	1.3×10^{-4}	0.2145 (0.2%)	0.2132	0.0124

where we set $u_{\lim} = 0.1\,\mathrm{m}$. For this example, the MC reference solutions are computed with 100 independent simulation runs with $n_s = 10^5$ samples.

The obtained MC estimate of the probability of failure is $p_F = 4.32\times10^{-2}$, whereas the FORM solution is $p_{F_1} = 2.81 \times 10^{-2}$. The first-order and total-effect indices computed by the two methods are shown in Table 4 and Figs. 8 and 9. For this example FORM significantly underestimates the probability of failure. This is reflected in a loss of accuracy in the estimates of the first-order and total-effect indices as compared to the steel column example, where the FORM probability estimate is highly accurate. However, the FORM sensitivity estimates still compare fairly well with the MC results and provide the same ranking, both for the first-order and the total-effect indices.

Table 3: Uncertain parameters of the elastic truss example.

Parameter	Distribution	Mean	St. Dev.
A_1 [m^2]	Lognormal	2×10^{-3}	2×10^{-4}
A_2 [m^2]	Lognormal	1×10^{-3}	1×10^{-4}
E_1, E_2 [MPa]	Lognormal	2.1×10^{5}	2.1×10^{4}
$P_1 - P_6$ [kN]	Gumbel	50	7.5

Table 4: Estimates of the first-order and total-effect indices for the elastic truss example for $u_{\max} = 0.1\,\text{m}$. The MC results include the coefficient of variation of the estimate in brackets

Variable	$S_{F,i}$		$S_{F,i}^T$		α_i^2
	MC	FORM	MC	FORM	
A_1	0.1202 (0.5%)	0.1044	0.7002 (0.1%)	0.7410	0.3713
A_2	0.0024 (5.3%)	0.0013	0.1172 (0.3%)	0.1229	0.0086
E_1	0.1199 (0.4%)	0.1044	0.7017 (0.1%)	0.7410	0.3713
E_2	0.0022 (9.0%)	0.0013	0.1181 (0.3%)	0.1229	0.0086
P_1	0.0010 (13.5%)	0.0006	0.0775 (0.4%)	0.0809	0.0037
P_2	0.0096 (2.4%)	0.0056	0.2309 (0.2%)	0.2457	0.0346
P_3	0.0241 (1.4%)	0.0144	0.3411 (0.2%)	0.3737	0.0818
P_4	0.0247 (1.4%)	0.0144	0.3414 (0.2%)	0.3737	0.0818
P_5	0.0101 (2.4%)	0.0056	0.2298 (0.2%)	0.2457	0.0346
P_6	0.0008 (16.9%)	0.0006	0.0779 (0.4%)	0.0809	0.0037

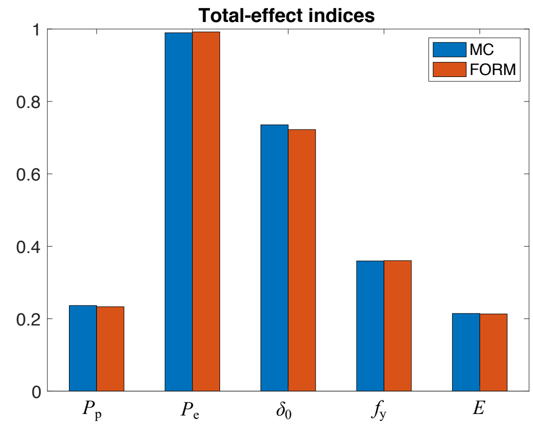

Figure 6: Total-effect indices for the steel column example. Comparison of MC and FORM estimates.

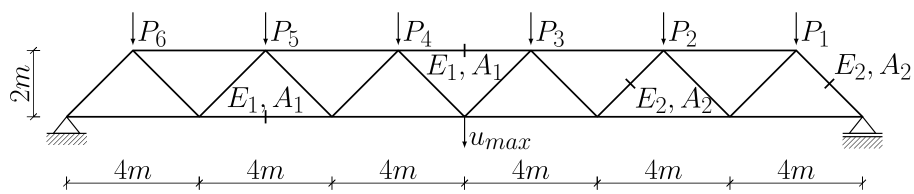

Figure 7: 2-D elastic truss structure.

6. Concluding remarks

This paper discusses the estimation of variance-based reliability sensitivities with FORM. It derives expressions for the first-order and total-effect indices of the indicator function of the linearized failure domain that depend on the FORM reliability index and the α-factors. These expressions are one-dimensional integrals and can be computed efficiently with numerical integration. A study on the dependency of the sensitivity indices on the corresponding α-factor at different probability levels showed that the first-order indices take significantly smaller values than the total-effect indices, especially at low values of the probability of failure. Nevertheless, the ranking obtained by the α-factors is the same as the one obtained by both the first-order and total-effect indices of the linearized problem. The absolute value of the α-factors gives consistent results with the the total-effect index, which supports the regular use of the α-factors for model simplification in

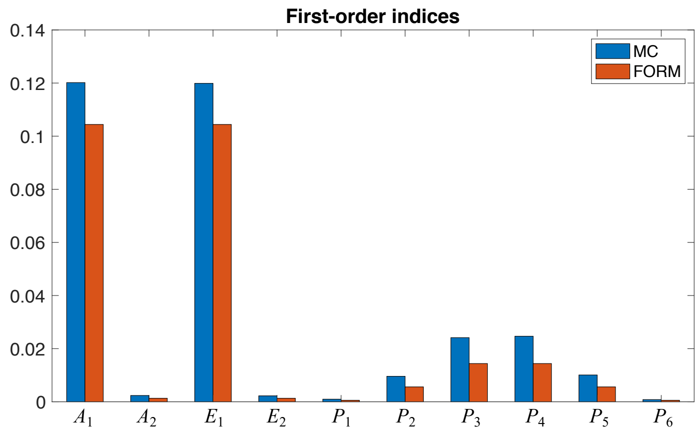

Figure 8: First-order indices for the elastic truss example. Comparison of MC and FORM estimates.

reliability analysis.

Two numerical examples illustrated that the derived expressions for the sensitivity indices of the FORM failure event provide good approximations of the variance-based sensitivities for nonlinear reliability problems for which the FORM approximation of the probability of failure is adequate. The second example showed that the FORM approximation of the sensitivities becomes worse if the approximation of the failure probability is inaccurate. We do not recommend application of the proposed approach in strongly nonlinear problems where the FORM approximation is expected to be poor.

The studied variance-based reliability sensitivities can be used in problems where the input random variables are statistically independent. A possible future research direction is to study variance-based reliability indices of dependent inputs and their relation to the FORM-indices for dependent inputs proposed in [3]. Additionally, the proposed FORM approximations can be extended to estimate the sensitivity indices of series- and parallel-system problems. These approximations could potentially be used to address nonlinear component problems with multiple design points.

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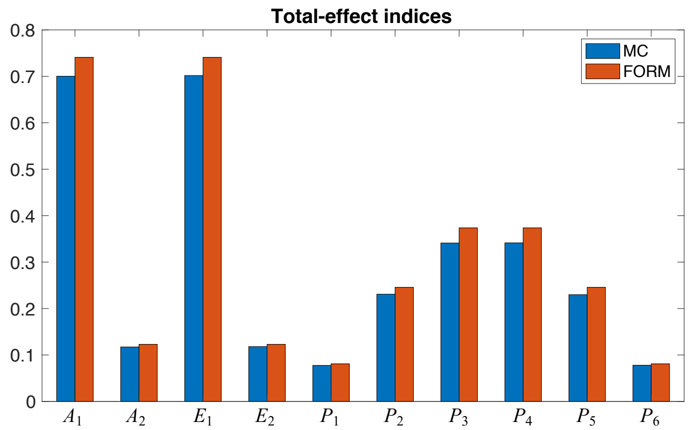

Figure 9: Total-effect indices for the elastic truss example. Comparison of MC and FORM estimates.

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