SONDERFORSCHUNGSBEREICH 96

LABORATORIUM FÜR DEN KONSTRUKTIVEN INGENIEURBAU (LKI)
TECHNISCHE UNIVERSITÄT MÜNCHEN

# BERICHTE
# zur
# ZUVERLÄSSIGKEITSTHEORIE DER BAUWERKE

## AN ASYMPTOTIC FORMULA FOR THE CROSSING RATE OF NORMAL PROCESSES INTO INTERSECTIONS

M. Hohenbichler

Heft 75/1984

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## SONDERFORSCHUNGSBEREICH 96 (SFB 96)

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## PREFACE

The concepts of asymptotic analysis have proved to be very useful for the theoretical justification of essential fundamentals in structural reliability methods. Here, another verification of approximations in the so-called Beta-point for time-variant problems is given. The work is part of the projects A5 and A11, respectively.

Munich, September 1984 The author

## VORWORT

Die Konzepte der asymptotischen Analysis haben sich als außerordentlich fruchtbar für die theoretische Rechtfertigung der wesentlichen Grundlagen der Methoden der Zuverlässsigkeitstheorie der Bauwerke erwiesen. In diesem Beitrag erfolgt eine weitere Begründung von Näherungen im sogenannten Beta-Punkt bei zeitvarianten Problemen. Die Arbeit entstand im Rahmen der Teilprojekte A5 bzw. A11.

München, September 1984 Der Autor

Let, for $n \geq 2$, $\underline{U} = \underline{U}(t) = (U_1(t), \ldots, U_n(t))^T$ be a stationary normal process with continuously differentiable sample paths (see [1]), whose autocorrelation functions $r_i(t)$ of $U_i(t)$ are twice differentiable at $t=0$. The time derivative of $\underline{U}(t)$ is denoted by $\underline{\dot{U}} = \underline{\dot{U}}(t) = (\dot{U}_1(t), \ldots, \dot{U}_n(t))^T$. Without loss of generality it is assumed that, for each fixed value of t, the variables $U_i = U_i(t)$ are stochastically independent with zero mean and unit variance, which implies that also the velocities $\dot{U}_i(t)$ have zero mean.

Let further $k \geq 2$ and $g_1 = g_1(\underline{u}), \ldots, g_k = g_k(\underline{u})$ be piecewise continuously differentiable functions such that for the probability density $\varphi(\underline{u})$ of $\underline{U}$ (omitting the argument t in $\underline{U}(t)$ or $\underline{\dot{U}}(t)$ means, that t is fixed, but arbitrary) the surface integrals

(A1) $$S_i = \int_{\{g_i=0\}} \|\underline{u}\| \, \varphi(\underline{u}) \, ds(\underline{u}) < \infty$$

over $\{g_i = 0\}$ exist, where $\{g_i = 0\}$ is the boundary of $\{g_i < 0\}$, $\|\underline{u}\| = (\underline{u}^T\underline{u})^{1/2}$ the Euklidean norm of $\underline{x}$ and $\varphi(\underline{u}) = \varphi_n(\underline{u}; \underline{\underline{I}})$ the multinormal density function. The symbol $ds(\underline{u})$ is the scalar infinitesimal surface element at the point $\underline{u}$ on the surface.

We are now going to investigate the outcrossing rate $\nu_F$ of the process $\underline{U}(t)$ from a socalled "safe domain" $\mathbb{R}^n \backslash F$ into the "failure domain" defined by

(1) $$F := \bigcap_{i=1}^{k} \{g_i < 0\}$$

which is given by the generalized Rice formula [4]

(2) $$\nu_F = \int_{\partial F} E[\{-\underline{\alpha}(\underline{u}) \cdot \underline{\dot{U}}\}^+ | \underline{U} = \underline{u}] \, \varphi(\underline{u}) \, ds(\underline{u}).$$

Here, $\alpha(\underline{u})$ is the outwards directed unit normal vector at a point $\underline{u}$ on the surface $\partial F$ of F, $E[.|.]$ is the conditional mean and $\{x\}^+ := \max\{0, x\}$. If, for example, there is $g_i(\underline{u})=0$ but $g_j(\underline{u})<0$ for all $j \in \{1, \ldots, k\} \backslash \{i\}$, then $\underline{u}$ lies in $\partial F$ and

(3) $\underline{\alpha}(\underline{u})$ = grad $g_i(\underline{u})$ / $\|$grad $g_i(\underline{u})\|$

In the sequel an asymptotic formula for the integral (2) is derived under conditions, which are quite similar to those given in reference [2].

**Further Assumptions (A) and Notations (N)**

(A2) The failure domain F has a unique Beta-point $\underline{u}^*$ (i.e. a point $\underline{u}^*$ in F with minimal distance to the origin). The origin is not contained in F (which implies that $\underline{u}^* \in \partial F$).

(A3) In an environment $\mathcal{U}$ of $\underline{u}^*$, the functions $g_i$ $(1 \leqslant i \leqslant k)$ are twice continuously differentiable, and it is $g_i(\underline{u}^*)=0$ for $1 \leqslant i \leqslant k$.

(A4) The gradients $\underline{a}_i$ := grad $g_i(\underline{u}^*)$ $(1 \leqslant i \leqslant k)$ are linearly independent, and it is

$$\|a_i\| = 1 \quad \text{for } 1 \leqslant i \leqslant k.$$

If originally $0 \neq \|\underline{a}_i\| \neq 1$, one obtains $\|\underline{a}_i\| = 1$ by multiplying $g_i$ with the constant factor $1/\|\underline{a}_i\|$.

It can be shown that $\underline{u}^*$ is always a linear combination

$$\underline{u}^* = \sum_{i=1}^{k} \gamma_i \, \underline{a}_i \quad \text{with } \gamma_i \leqslant 0 \text{ for } 1 \leqslant i \leqslant k$$

of the $\underline{a}_i$'s, where due to (A4) the $\gamma_i$'s are uniquely determined. In addition, we assume here that

(A5) $\underline{u}^* = \sum_{i=1}^{k} \gamma_i \, \underline{a}_i$ with $\gamma_i < 0$ for $1 \leqslant i \leqslant k$.

Introducing now the cross-covariance matrix $\underline{\underline{C}}$ between $\underline{U}$ and $\underline{\dot{U}}$ (in contrast to $\underline{x}^T\underline{y}$, which is a scalar value, $\underline{x}\underline{y}^T$ is a matrix)

(N3) The (n-k)-dimensional matrix $\underline{\underline{D}} = (d_{ij}: k+1 \leq i,j \leq n)$ is defined by

$$d_{ij} = \sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j} \qquad \text{for } k+1 \leq i,j \leq n$$

(N4) $\underline{\underline{I}} = (\delta_{ij}: k+1 \leq i,j \leq n)$ is the (n-k)-dimensional unit matrix ($\delta_{ii}=1$, $\delta_{ij}=0$ for $i \neq j$) (caution: $\underline{\underline{I}}$ is a matrix, but I a set of indices)

For n=k, $\underline{\underline{D}}$ and $\underline{\underline{I}}$ are empty; in this case we define the determinant as

$$d := \det(\underline{\underline{I}} - \underline{\underline{D}}) := 1 \quad (\text{for } k=n)$$

In general we assume that

(A8) $d := \det(\underline{\underline{I}} - \underline{\underline{D}}) > 0$

Note that there is always $d \geq 0$ [2].

(N5) For $i \in \{1,...,k\}$ there is

$\beta_i := \underline{a}_i \cdot \underline{u}^*$

$\underline{c}_i := (c_{is} : 1 \leq s \leq k,\ s \neq i) \in \mathbb{R}^{k-1}$ with

$c_{is} = \beta_s - (\underline{u}^* \cdot \underline{a}_i)(\underline{a}_s \cdot \underline{a}_i)$ ,

and

$\underline{\underline{R}}_i := (r_{ist} : 1 \leq s,t \leq k,\ s \neq i,\ t \neq i) \in \mathbb{R}^{k-1,k-1}$

$r_{ist} := \underline{a}_s \cdot \underline{a}_t - (\underline{a}_s \cdot \underline{a}_i)(\underline{a}_t \cdot \underline{a}_i)$

We are now able to describe the asymptotic behaviour of the outcrossing rate

$$v_F(b) := b^n \int_{\partial F} E[\{-\underline{\alpha}(\underline{u}) \cdot \frac{1}{b}\,\dot{\underline{U}}\}^+ \,|\, \frac{1}{b}\,\underline{U} = \underline{u}]\ \varphi(b\underline{u})\ ds(\underline{u})$$

of the process $\frac{1}{b}\,\underline{U}(t)$ (with derivative $\frac{1}{b}\,\dot{\underline{U}}(t)$) from $\mathbb{R}^n \backslash F$ into F, in the limit $b \to \infty$.

(N1) $\underline{\underline{C}} := E[\underline{\dot{U}}\ \underline{U}^T] = (E[\dot{U}_i U_j]:\ 1 \leq i,j \leq n)$,

the conditional mean value of $\underline{\dot{U}}(t)$ given $\underline{U}(t)$ is (see first part of the proof in the appendix)

(4) $E[\underline{\dot{U}}|\underline{U}] = \underline{\underline{C}}\ \underline{U}$.

In particular it is assumed here that

(A6) $\underline{a}_i \cdot (\underline{\underline{C}}\ \underline{u}^*) := \underline{a}_i^T \underline{\underline{C}}\ \underline{u}^* \neq 0$ for $1 \leq i \leq k$

(the dot between vectors "$\underline{x} \cdot \underline{y}$" indicates the scalar product $\underline{x} \cdot \underline{y} = \Sigma x_i y_i = \underline{x}^T \underline{y}$). Since there is always

(5) $\underline{u} \cdot (\underline{\underline{C}}\ \underline{u}) = \underline{u}^T \underline{\underline{C}}\ \underline{u} = 0$

(see appendix), while for k=1 the gradient $\underline{a}_1$ is a multiple of $\underline{u}^*$, the condition (A6) cannot be satisfied for k=1. On the other hand, (A6) usually holds for k≥2. As a simple consequence of (A5) and (A6) let us note that even

(6) $\underline{a}_i \cdot (\underline{\underline{C}}\ \underline{u}^*) < 0$ for at least one i $(1 \leq i \leq k)$,

since due to (5)

$$0 = (\underline{u}^*)^T \underline{\underline{C}}\ \underline{u}^* = \sum_{i=1}^{k} \gamma_i (\underline{a}_i^T \underline{\underline{C}}\ \underline{u}^*).$$

Without loss of generality, it is also assumed that the last n-k components of the vectors $\underline{a}_i = (a_{i1}, \dots, a_{in})$ are zero:

(A7) $a_{ij} = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$.

In addition to (N1), the following notations are used:

(N2) I is the set of all indices i $(1 \leq i \leq k)$, for which $\underline{a}_i \cdot (\underline{\underline{C}}\ \underline{u}) < 0$.
Due to eq. (6), I is non-empty.

To simplify the notations, the abbreviation

$$f_1(b) \backsim f_2(b)$$

instead of

$$\lim_{b\to\infty} \frac{f_1(b)}{f_2(b)} = 1$$

is used. $\varphi(.)$ is the univariate standard normal density, $\Phi(.)$ the corresponding normal integral and $\Phi_k(\underline{c};\underline{\underline{R}})$ the distribution function with argument $\underline{c}\in\mathbb{R}^k$ of a k-dimensional normal random vector with covariance-matrix $\underline{\underline{R}}$ and zero mean.

Theorem

Under the described assumptions, there is

$$\psi(\underline{a},b,\underline{u}) := E[\{-\underline{a}\ \frac{1}{b}\ \underline{\dot{U}}\}^+ | \frac{1}{b}\ \underline{U} = \underline{u}] =$$

$$= -\underline{a}^T\underline{\underline{C}}\ \underline{u}\ \Phi(b\ \frac{-\underline{a}^T\underline{\underline{C}}\ \underline{u}}{(\underline{a}^T\underline{\underline{S}}\ \underline{a})^{1/2}}) + \frac{1}{b}\ (\underline{a}^T\underline{\underline{S}}\ \underline{a})^{1/2}\ \varphi(b\ \frac{-\underline{a}^T\underline{\underline{C}}\ \underline{u}}{(\underline{a}^T\underline{\underline{S}}\ \underline{a})^{1/2}})$$

where

$$\underline{\underline{S}} := E[\underline{\dot{U}}\ \underline{\dot{U}}^T] - \underline{\underline{C}}\ \underline{\underline{C}}^T .$$

With $\partial F_j := \partial F \cap \{g_j = 0\}$, the outcrossing rate $\nu_F(b)$ becomes asymptotically

$$\nu_F(b) = b^n \int_{\partial F} \psi(\underline{\alpha}(\underline{u}),b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u}) \backsim$$

$$\backsim b^n \sum_{j=1}^{k} [\psi(\underline{a}_j,b,\underline{u}^*) \int_{\partial F_j} \varphi(b\underline{u})\ ds(\underline{u})] \backsim$$

$$\sim \frac{b}{\sqrt{d}} \sum_{j=1}^{k} [\psi(\underline{a}_j, b, \underline{u}^*)\ \varphi(b\beta_j)\ \phi_{k-1}(b\underline{c}_j ; \underline{\underline{R}}_j)] \sim$$

$$\sim \frac{b}{\sqrt{d}} \sum_{i \in I} [-\underline{a}_i{}^T \underline{\underline{C}}\ \underline{u}^*\ \varphi(b\beta_i)\ \phi_{k-1}(b\underline{c}_i ; \underline{\underline{R}}_i)] .$$

## Proof of the theorem

First the conditional mean

$$\psi(\underline{\alpha}, b, \underline{u}) := E[\{-\underline{\alpha}^T \frac{1}{b} \dot{\underline{U}}\}^+ | \frac{1}{b} \underline{U} = \underline{u}] = \frac{1}{b} \psi(\underline{\alpha}, 1, b\underline{u})$$

is evaluated. Since $(\underline{U}, \dot{\underline{U}})$ is normally distributed, $\dot{\underline{U}}$ can be represented as

$$\dot{\underline{U}} = \underline{\underline{C}}\ \underline{U} + \underline{V}$$

where $\underline{V}$ is normal and independent of $\underline{U}$ with zero mean, and

$$\underline{\underline{C}} = \underline{\underline{C}}\ E[\underline{U}\ \underline{U}^T] + E[\underline{V}\ \underline{U}^T] = E[(\underline{\underline{C}}\ \underline{U} + \underline{V})\underline{U}^T] = E[\dot{\underline{U}}\ \underline{U}^T] .$$

Denoting by

$$\underline{\underline{B}} = E[\dot{\underline{U}}\ \dot{\underline{U}}^T]$$

the covariance matrix of $\dot{\underline{U}}$, we obtain further for the covariance matrix $\underline{\underline{S}}$ of $\underline{V}$

$$\underline{\underline{S}} = E[\underline{V}\ \underline{V}^T] = E[(\dot{\underline{U}} - \underline{\underline{C}}\ \underline{U})(\dot{\underline{U}} - \underline{\underline{C}}\ \underline{U})^T] = \underline{\underline{B}} - \underline{\underline{C}}\ \underline{\underline{C}}^T .$$

It is now easily seen that

$$E[\dot{\underline{U}} | \underline{U}] = \underline{\underline{C}}\ \underline{U}$$

$$E[-\underline{\alpha}^T \dot{\underline{U}} | \underline{U}] = -\underline{\alpha}^T E[\dot{\underline{U}} | \underline{U}] = -\underline{\alpha}^T \underline{\underline{C}}\ \underline{U}$$

$$\text{var}\,[-\underline{\alpha}^T\dot{\underline{U}}|\underline{U}] = \text{var}\,[-\underline{\alpha}^T\underline{V}] = E[\underline{\alpha}^T\underline{V}\,\underline{V}^T\alpha] = \underline{\alpha}^T\underline{\underline{S}}\,\underline{\alpha}$$

and, consequently, because $E[X^+] = \mu\Phi(\frac{\mu}{\sigma}) + \sigma\varphi(\frac{\mu}{\sigma})$

$$\psi(\underline{\alpha},b,\underline{u}) = -\underline{\alpha}^T\underline{\underline{C}}\,\underline{u}\,\Phi\left(b\frac{-\underline{\alpha}^T\underline{\underline{C}}\,\underline{u}}{(\underline{\alpha}^T\underline{\underline{S}}\,\underline{\alpha})^{1/2}}\right) + \frac{1}{b}(\alpha^T\underline{\underline{S}}\,\alpha)^{1/2}\;\varphi\left(b\frac{-\underline{\alpha}^T\underline{\underline{C}}\,\underline{u}}{(\underline{\alpha}^T\underline{\underline{S}}\,\underline{\alpha})^{1/2}}\right).$$

Using the abbreviation

$$\psi(b,\underline{u}) = \psi(\underline{\alpha}(\underline{u}),b,\underline{u})$$

the outcrossing rate of the process $\frac{1}{b}\dot{\underline{U}}$ becomes

$$\nu_F(b) = b^n \int_{\partial F} \psi(b,\underline{u})\;\varphi(b\underline{u})\;ds(\underline{u})$$

Since $\underline{\alpha}^T(\underline{u})\,\underline{\underline{S}}\,\underline{\alpha}(\underline{u}) \leq c < \infty$ for some constant c depending only on $\underline{\underline{S}}$ (note that $\underline{\alpha}^T(\underline{u})\underline{\alpha}(\underline{u}) = 1$), we have for $b\to\infty$

(B1) $b\psi(b,\underline{u}) \to 0$ uniformly, if $\sup\{-\underline{\alpha}(\underline{u})^T\underline{\underline{C}}\,\underline{u}\} < 0$

(B2) $\psi(b,\underline{u}) \to -\underline{\alpha}(\underline{u})^T\underline{\underline{C}}\,\underline{u}$ uniformly, if $\inf\{-\underline{\alpha}(\underline{u})^T\underline{\underline{C}}\,\underline{u}\} > 0$.

In the sequel, the following notations are used:

$$F(b) \sim g(b) \quad :\Longleftrightarrow \quad \lim_{b\to\infty}\frac{F(b)}{g(b)} = 1$$

$$F(b) < g(b) \quad :\Longleftrightarrow \quad \lim_{b\to\infty}\frac{F(b)}{g(b)} = 0$$

$$\partial F_i := \partial F \cap \{g_i = 0\}$$

The theorem is an immediate consequence of the following lemmas 2 and 3, in connection with eqs. (B1) and (B2). Lemma 1 contains technical details:

**Lemma 1**

Let $\mho \subset \mathbb{R}^n$ be an environment of $\underline{u}^*$, and $i \in I$ (i.e. $\underline{a}_i^T \underline{\underline{C}}\ \underline{u} < 0$) and $j \in \{1,\dots,k\}$. Then

a) $\int_{\partial F_j \setminus \mho} \psi(b,\underline{u})\ \varphi(b,\underline{u})\ ds(\underline{u}) \ll \int_{\partial F_i \cap \mho} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u})$

b) $\int_{\partial F_j \cap \mho} \varphi(b\underline{u})\ ds(\underline{u}) \sim \int_{\partial F_j} \varphi(b\underline{u})\ ds(\underline{u})$

**Lemma 2**

If $d = \det(\underline{\underline{I}} - \underline{\underline{D}}) > 0$, then for $j \in \{1,\dots,k\}$

$$b^{n-1} \int_{\partial F_j} \varphi(b\underline{u})\ ds(\underline{u}) \sim \frac{1}{\sqrt{d}}\ \varphi(b\beta_j)\ \phi_{k-1}\ (b\underline{c}_j ; \underline{\underline{R}}_j)$$

**Lemma 3**

For $i \in I$ and $j \notin I$ there is

a) $\int_{\partial F_j} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u}) \ll \int_{\partial F_i} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u})$, while

$\int_{\partial F_j} \varphi(b\underline{u})\ ds(\underline{u})$ and $\int_{\partial F_i} \varphi(b\underline{u})\ ds(\underline{u})$ are, for $b \to \infty$, of the same order of magnitude.

b) $\int_{\partial F_i} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u}) \sim \psi(\underline{a}_i, b, \underline{u}^*) \int_{\partial F_i} \varphi(b\underline{u})\ ds(\underline{u})$

**Proof of lemma 1**

a) Due to the continuity of $\underline{\alpha}(\underline{u})$ for $\underline{u} \in \partial F_i$, $\mho$ can, without loss of generality, be chosen so small that

$$\delta := \frac{1}{2} \inf \{-\underline{\alpha}(\underline{u})^T \underline{\underline{C}}\ \underline{u} : \underline{u} \in \partial F_i \cap \mho\} > 0$$

Eq.(B2) above implies now, that for some $0 < b_0 < \infty$ and $\underline{u} \in \partial F_i \cap \mho$

there is

$\psi(b,\underline{u}) > \delta > 0$ for $b>b_0$.

Therefore we have for $b>b_0$

$$\int_{\partial F_i \cap \mathcal{U}} \psi(b,\underline{u})\, \varphi(b\underline{u})\, ds(\underline{u}) \geq \delta \cdot \int_{\partial F_i \cap \mathcal{U}} \varphi(b\underline{u})\, ds(\underline{u}) .$$

On the other hand,

$|\psi(b,\underline{u})| \leq c_1 \|\underline{u}\| + \frac{1}{b} c_2$ for some $c_1, c_2 < \infty$ and all $\underline{u} \in \partial F$.

Let now $\varepsilon := \frac{1}{2} (\inf \{\|\underline{u}\| : \underline{u} \in F \setminus \mathcal{U}\} - \beta)$ $(\beta := \|\underline{u}^*\|)$.

In [3] it is proven that $\varepsilon>0$. Since

$$\varphi(b\underline{u}) = \varphi(\underline{u}) \exp \{-\tfrac{1}{2}(b^2-1)\, \|\underline{u}\|^2\}$$

we obtain for $b\geq 1$

$$(1) \quad \int_{\partial F_j \setminus \mathcal{U}} \psi(b,\underline{u})\, \varphi(b\underline{u})\, ds(\underline{u}) \leq$$

$$\leq \exp \{-\tfrac{1}{2} (b^2-1)(\beta + 2\varepsilon)^2\} \int_{\partial F_j \setminus \mathcal{U}} (c_1 \|\underline{u}\| + \tfrac{1}{b} c_2)\, \varphi(\underline{u})\, ds(\underline{u}) .$$

$< \infty$ (see assumption (A1))

while for $\mathcal{W} := \{\underline{u} : \|\underline{u}\| < \beta + \varepsilon\}$ and $b\geq 1$ we obtain (note that $F \cap \mathcal{W} \subset F \cap \mathcal{U}$)

$$(2) \quad \int_{\partial F_i \cap \mathcal{U}} \psi(b,\underline{u})\, \varphi(b\underline{u})\, ds(\underline{u}) \geq \delta \int_{\partial F_i \cap \mathcal{W}} \varphi(b\underline{u})\, ds(\underline{u}) \geq$$

$$\geq \delta \exp \{-\tfrac{1}{2} (b^2-1)(\beta + \varepsilon)^2\} \int_{\partial F_i \cap \mathcal{W}} \varphi(\underline{u})\, ds(\underline{u})$$

$> 0$ (compare proof of lemma 2)

Lemma 1a is now a simple consequence of (1) and (2).

b) Quite similarly it is shown that

$$\int_{\partial F_i \setminus \mathcal{U}} \varphi(b\underline{u})\, ds(\underline{u}) \ll \int_{\partial F_i \cap \mathcal{U}} \varphi(b\underline{u})\, ds(\underline{u}) .$$

This implies again part b of the lemma.

**Proof of lemma 2**

We show that for sufficiently small environments $\mathcal{U}$ of $\underline{u}^*$ there is

$$(*) \quad b^{n-1} \int_{\partial F_j \cap \mathcal{U}} \varphi(b\underline{u})\, ds(\underline{u}) \sim \frac{1}{\sqrt{d}} \varphi(b\beta_j)\, \phi_{k-1}(b\underline{c}_j;\underline{\underline{R}}_j)$$

Lemma 2 is just a consequence of (*) and lemma 1b.

For a proof of (*), we may without loss of generality assume that $j=1$ and

(1) $\underline{a}_1 = \underline{e}_1$ ($\underline{e}_j$ the j-th unit vector).

Due to (1) and assumptions (A4) and (A5), $\underline{u}^*$ and $\underline{e}_1$ are linearly independent. Therefore, $\underline{u}^* = \Sigma\alpha_i\underline{e}_i$, where $\alpha_i \neq 0$ for some $i_0 \geq 2$; without loss of generality $i_0=2$. Then $\underline{u}^*$ cannot be represented as a linear combination of $\underline{e}_1, \underline{e}_3, \ldots, \underline{e}_n$, and thus

(2) $(\underline{e}_1, \underline{u}^*, \underline{e}_3, \ldots, \underline{e}_n)$ are linearly independent.

In order to parametrize $\partial F_i$ appropriately, we define the mappings

$$(3a) \quad \underline{S}_1(\underline{u}) = (u_1 - u_1^*, \frac{1}{2} \sum_{i=1}^{n} u_i^2, u_3, \ldots, u_n)^T$$

$$(3b) \quad \underline{S}_2(\underline{u}) = (g_1(\underline{u}), \frac{1}{2} \sum_{i=1}^{n} u_i^2, u_3, \ldots, u_n)^T .$$

According to (2), the Jacobian $\underline{\underline{D}}[\underline{S}_2](\underline{u}^*)$ of $\underline{S}_2$ at $\underline{u}^*$ is non-zero.

From the assumptions on the $g_i$'s it now follows easily, that

(4) $\underline{T}(\underline{u}) := \underline{S}_2^{-1} \circ \underline{S}_1(\underline{u})$

is defined in an environment $\mathfrak{V}$ of $\underline{u}^*$ and is twice continuously differentiable, with

(5a) $\underline{T}(\underline{u}^*) = \underline{u}^*$

(5b) $\underline{\underline{D}}[\underline{T}](\underline{u}^*) = \{\underline{\underline{D}}[\underline{S}_2](\underline{T}(\underline{u}^*))\}^{-1} \circ \underline{\underline{D}}[\underline{S}_1](\underline{u}^*) =$

$= \{(\underline{e}_1, \underline{u}^*, \underline{e}_3, \dots, \underline{e}_n)^T\}^{-1} \circ (\underline{e}_1, \underline{u}^*, \underline{e}_3, \dots, \underline{e}_n)^T = \underline{\underline{E}}$

($\underline{\underline{E}}$ is the n-dimensional unit-matrix).

Since for $\underline{u} \in \mathfrak{V}$

$$(g_1(\underline{T}(\underline{u})), \frac{1}{2} \sum_{i=1}^{n} T_i(\underline{u})^2, T_3(\underline{u}), \dots, \underline{T}(\underline{u}))^T = \underline{S}_2(\underline{T}(\underline{u})) =$$

$$= \underline{S}_1(\underline{u}) = (u_1 - u_1^*, \frac{1}{2} \sum_{i=1}^{n} u_i^2, u_3, \dots, u_n)^T$$

there is for $\underline{u} \in \mathfrak{V}$

(6a) $T_j(\underline{u}) = u_j$ for $j \geq 3$

(6b) $u_1 = u_1^* \Longleftrightarrow g_1(\underline{T}(\underline{u})) = 0$

(6c) $\sum_{i=1}^{n} T_i(\underline{u})^2 = \sum_{i=1}^{n} u_i^2$ .

Let now

$$H_1 = \{\underline{u} : u_1 = u_1^*\} \cap \bigcap_{j=2}^{k} \{g_j(\underline{T}(\underline{u})) \leq 0\} .$$

Due to (5) and (6b), for $\mathfrak{V}$ small enough $\mathfrak{W} := \underline{T}(\mathfrak{V})$ is an open environment of $\underline{u}^*$, and the restriction $\hat{\underline{T}}$ of $\underline{T}$ to $H_1 \cap \mathfrak{V}$ is a proper parametrization of $\{g_1=0\}\cap\mathfrak{W}$. Therefore, using eq. (6c)

$$(7)\quad \int_{\partial F_1\cap\mathfrak{W}} \varphi(b\underline{u})\, ds(\underline{u}) = \int_{H_1\cap\mathfrak{V}} \varphi(b\underline{T}(\underline{u}))\, r(\underline{u})\, ds(\underline{u}) =$$

$$= \int_{H_1\cap\mathfrak{V}} \varphi(b\underline{u})\, r(\underline{u})\, ds(\underline{u})$$

where

$$(8)\quad r(\underline{u}) = \sqrt{\det[\underline{\underline{D}}(\underline{u})^T\, \underline{\underline{D}}(\underline{u})]}\ ,$$

$$\underline{\underline{D}}(\underline{u}) := \underline{\underline{D}}[\underline{T}](\underline{u})\ ,$$

or finally

$$(9)\quad \int_{\partial F_1\cap\mathfrak{W}} \varphi(b\underline{u})\, ds(\underline{u}) = \varphi(bu_1^*) \int_{\bigcap_{s=2}^{k}\{f_s<0\}\cap\tilde{\mathfrak{V}}} \varphi(b\underline{v})\, r(u_1^*,\underline{v})\, d\underline{v}$$

where $\tilde{\mathfrak{V}} = \{\underline{v} : (u_1^*,\underline{v}) \in \mathfrak{V}\}$ and

$\underline{v} = (v_2,\ldots,v_n) = (u_2,\ldots,u_n)$

$f_s(\underline{v}) = g_s(\underline{T}(u_1^*,\underline{v}))$ for $\underline{v}\in\tilde{\mathfrak{V}}$ .

We also define

$\underline{v}^* = (v_2^*,\ldots,v_n^*) := (u_2^*,\ldots,u_n^*)$

and introduce the abbreviations

$\underline{b} = (b_1,\ldots,b_n) = \underline{b}(\underline{u}) := (\text{grad } g_1)(\underline{T}(\underline{u}))$

$\underline{T} := (T_1,\ldots,T_n) := \underline{T}(\underline{u})$ .

<u>Evaluation of $r(\underline{u}^*)$</u>:

Since $\underline{\underline{D}}[\underline{T}](\underline{u}^*) = \underline{\underline{E}} = (\underline{e}_1,\ldots,\underline{e}_n)$, there is $\underline{\underline{D}}(\underline{u}^*) = \underline{\underline{D}}[\hat{\underline{T}}](\underline{u}^*) = (\underline{e}_2,\ldots,\underline{e}_n)$ and therefore

$$r(\underline{u}^*) = \{\det(\underline{\underline{D}}(\underline{u}^*)^T\, \underline{\underline{D}}(\underline{u}^*))\}^{\frac{1}{2}} = 1$$

## Evaluation of the first and second derivatives of $\underline{T}$:

$$\underline{\underline{D}}[\underline{T}](\underline{u}) = \{\underline{\underline{D}}[\underline{S}_2](\underline{T}(\underline{u}))\}^{-1} \; \underline{\underline{D}}[S_1](\underline{u}) =$$

$$= \begin{pmatrix} \underline{\underline{B}}_1 & \underline{\underline{B}}_2 \\ \underline{\underline{0}} & \underline{\underline{E}} \end{pmatrix}^{-1} (\underline{e}_1, \underline{u}, \underline{e}_3, \ldots, \underline{e}_n)^T$$

with

$$\underline{\underline{B}}_1 := \begin{pmatrix} b_1 & b_2 \\ T_1 & T_2 \end{pmatrix}, \qquad \underline{\underline{B}}_2 := \begin{pmatrix} b_3, \ldots, b_n \\ T_3, \ldots, T_n \end{pmatrix}$$

$\underline{\underline{0}}$ the zero-matrix in $\mathbb{R}^{n-2,2}$

$\underline{\underline{E}}$ the unit matrix in $\mathbb{R}^{n-2,n-2}$ .

It follows that

$$\underline{\underline{D}}[\underline{T}](\underline{u}) = \begin{pmatrix} \underline{\underline{B}}_1^{-1} & -\underline{\underline{B}}_1^{-1}\underline{\underline{B}}_2 \\ \underline{\underline{0}} & \underline{\underline{E}} \end{pmatrix} (\underline{e}_1, \underline{u}, \underline{e}_3, \ldots, \underline{e}_n)^T ,$$

and since

$$\underline{\underline{B}}_1^{-1} = \frac{1}{b_1T_2 - b_2T_1} \begin{pmatrix} T_2 & -b_2 \\ -T_1 & b_1 \end{pmatrix}$$

one obtains, using the abbreviations

$$d = b_1T_2 - b_2T_1$$

$$x_j = b_2T_j - b_jT_2$$

$$y_j = b_jT_1 - b_1T_j ,$$

the result

$$\underline{\underline{D}}[\underline{T}](\underline{u}) = \frac{1}{d}\begin{pmatrix} T_2 & -b_2 & x_3 & \cdots & x_n \\ -T_1 & b_1 & y_3 & \cdots & y_n \\ & \underline{\underline{0}} & & d\underline{\underline{E}} & \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ u_1 & u_2 & u_3 & \cdots & u_n \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} .$$

Observing that $T_j = u_j$ for $j \geq 3$, this implies

$$\frac{\partial T_1}{\partial u_1}(\underline{u}) = \frac{1}{d}(T_2 - b_2 u_1)$$

$$\frac{\partial T_1}{\partial u_2}(\underline{u}) = \frac{1}{d}(-u_2 b_2)$$

$$\frac{\partial T_1}{\partial u_j}(\underline{u}) = \frac{1}{d}(-b_2 u_j + x_j) = \frac{1}{d}(-b_j T_2) \quad \text{for } j \geq 3$$

$$\frac{\partial T_2}{\partial u_1}(\underline{u}) = \frac{1}{d}(-T_1 + b_1 u_1)$$

$$\frac{\partial T_2}{\partial u_2}(\underline{u}) = \frac{1}{d}(b_1 u_2)$$

$$\frac{\partial T_2}{\partial u_j}(\underline{u}) = \frac{1}{d}(b_1 u_j + y_j) = \frac{1}{d}(b_j T_1) \quad \text{for } j \geq 3$$

$$\frac{\partial T_s}{\partial u_j}(\underline{u}) = \delta_{sj} \quad \text{for } s \geq 3 \text{ , } j \geq 1 \text{ .}$$

As already noted above, this reduces in the special case $\underline{u} = \underline{u}^*$ to

$$\frac{\partial T_s}{\partial u_j}(\underline{u}^*) = \delta_{sj} .$$

The second derivatives are only evaluated at $\underline{u}=\underline{u}^*$, since for $\underline{u}=\underline{u}^*$ calculations simplify by using the relations

$$d = u_2^* ,$$
$$b_1 = 1 \quad \text{and} \quad b_2 = \ldots = b_n = 0$$
$$T_i = u_i^*$$

$$\frac{\partial b_i}{\partial u_j} = \frac{\partial b_j}{\partial u_i} = \gamma_{ij} := \frac{\partial^2 g_1(\underline{u}^*)}{\partial u_i \partial u_j}$$

$$\left[ = \sum_{s=1}^{n} \frac{\partial^2 g_1}{\partial u_s \partial u_i}(\underline{T}(\underline{u}^*)) \frac{\partial T_s(\underline{u}^*)}{\partial u_j} \right]$$

$$\frac{\partial T_i}{\partial u_j} = \frac{\partial u_i}{\partial u_j} = \delta_{ij}$$

$$\frac{\partial d}{\partial u_j} = \delta_{2j} + \gamma_{1j}u_2^* - \gamma_{2j}u_1^* .$$

This leads, for $1 \leq i \leq n$, to

$$\frac{\partial T_1(\underline{u}^*)}{\partial u_i \partial u_1} = \frac{1}{d^2} \{d[\delta_{2i} - \gamma_{2i}u_1^*] - [\delta_{2i} + \gamma_{1i}u_2^* - \gamma_{2i}u_1^*]\ u_2^*\} =$$

$$= -\frac{1}{d}\gamma_{1i}u_2^* = -\gamma_{i1}$$

$$\frac{\partial T_1(\underline{u}^*)}{\partial u_i \partial u_j} = \frac{1}{d^2} \{d[-\gamma_{ij}u_2^*] + 0\} = -\gamma_{ij} \quad \text{for } j \geq 2$$

$$\frac{\partial T_2(\underline{u}^*)}{\partial u_i \partial u_1} = \frac{1}{d^2}\{d[-\delta_{i1} + \gamma_{i1}u_1^* + \delta_{i1}] - 0\} = \frac{u_1^*}{u_2^*}\gamma_{i1}$$

$$\frac{\partial T_2(\underline{u}^*)}{\partial u_i \partial u_2} = \frac{1}{d^2}\{d[\gamma_{1i}u_2^* + \delta_{2i}] - [\delta_{2i} + \gamma_{1i}u_2^* - \gamma_{2i}u_1^*]\ u_2^*\} =$$

$$= \frac{u_1^*}{u_2^*}\gamma_{i2}$$

or

$$= \frac{1}{d^2}\{d[\gamma_{ij}u_1^*] - 0\} = \frac{u_1^*}{u_2^*}\gamma_{ij} \quad \text{for } j \geq 3$$

or

$$\frac{\partial T_2(\underline{u}^*)}{\partial u_i \partial u_j} = \frac{u_1^*}{u_2^*}\gamma_{ij} \quad \text{for all } 1 \leq i,\ j \leq n \ ;$$

$$\frac{\partial T_s(\underline{u}^*)}{\partial u_i \partial u_j} = 0 \quad \text{for } s \geq 3 \text{ , and all } 1 \leq i,\ j \leq n$$

Evaluation of the first and second derivatives of $f_s$:

$(s \geq 2,\ i \geq 2,\ j \geq 2)$

$$\frac{\partial f_s}{\partial v_j}(\underline{v}) = \sum_{t=1}^{n} \frac{\partial g_s}{\partial u_t}(\underline{T}(u_1^*,\underline{v}))\ \frac{\partial T_t(u_1^*,\underline{v})}{\partial v_j}$$

$$\frac{\partial f_s}{\partial v_j}(\underline{v}^*) = \frac{\partial g_s}{\partial u_j}(\underline{u}^*)$$

$$\frac{\partial f_s(\underline{v}^*)}{\partial v_i \partial v_j} = \sum_{t=1}^{n} \frac{\partial}{\partial v_i}\left[\frac{\partial g_s}{\partial u_t}(\underline{T}(u_1^*,\underline{v}))\ \frac{\partial T_t(u_1^*,\underline{v})}{\partial v_j}\right]_{\underline{v}=\underline{v}^*} =$$

$$= \sum_{t=1}^{n} \{[\sum_{p=1}^{n} \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_p \partial u_t} \delta_{ip}] \delta_{tj} + \frac{\partial g_s}{\partial u_t} (\underline{u}^*) \frac{\partial^2 T_t(\underline{u}^*)}{\partial v_i \partial v_j}\} =$$

$$= \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j} + \sum_{t=1}^{2} \frac{\partial g_s}{\partial u_t} (\underline{u}^*) \frac{\partial^2 T_t(\underline{u}^*)}{\partial v_i \partial v_j}\} =$$

$$= \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j} - \frac{\partial g_s(\underline{u}^*)}{\partial u_1} \frac{\partial^2 g_1(\underline{u}^*)}{\partial u_i \partial u_j} + \frac{\partial g_s(\underline{u}^*)}{\partial u_2} \frac{u_1^*}{u_2^*} \frac{\partial^2 g_1(\underline{u}^*)}{\partial u_i \partial u_j} ,$$

$$\frac{\partial^2 f_s(\underline{v}^*)}{\partial v_i \partial v_j} = \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j} - \frac{\partial^2 g_1(\underline{u}^*)}{\partial u_i \partial u_j} [\frac{\partial g_s(\underline{u}^*)}{\partial u_1} - \frac{u_1^*}{u_2^*} \frac{\partial g_s(\underline{u}^*)}{\partial u_2}]$$

Now the results in [2] can be applied to find an asymptotic solution of the integral (9), or the somewhat simpler integral

$$(10) \quad \int_{G \cap \tilde{U}} \varphi(b\underline{v})\, d\underline{v} \quad \text{with } G = \bigcap_{s=2}^{k} \{f_s < 0\} \cap \tilde{U} \subset \mathbb{R}^{n-1}$$

(where the points of $\mathbb{R}^{n-1}$ are written as $\underline{v} = (v_2, \dots, v_n)$)

Noting that $\underline{u}^*$ is the unique Beta-point of $\partial F_1$, eq. (6c) implies that $\underline{v}^*$ is the unique Beta-point of $G \cap \tilde{U}$. Due to eq. (1) and assumption (A5) there is

$$(11) \quad \underline{v}^* = \sum_{i=2}^{k} \gamma_i \text{ grad } f_i(\underline{v}^*) \quad (\gamma_i < 0) .$$

Further, for $k+1 \leq i, j \leq n$, due to

$$\sum_{s=2}^{k} \gamma_s \frac{\partial g_s(\underline{u}^*)}{\partial u_2} = \sum_{s=1}^{k} \gamma_s \frac{\partial g_s(\underline{u}^*)}{\partial u_2} = u_2^*$$

$$\sum^{k} \gamma_s \frac{\partial g_s(\underline{u}^*)}{} = u_1^* - \gamma_1 \frac{\partial g_1(\underline{u}^*)}{} = u_1^* - \gamma_1 .$$

we have

$$\sum_{s=2}^{k} \gamma_s \frac{\partial^2 f_s(\underline{v}^*)}{\partial v_i \partial v_j} = \sum_{s=2}^{k} [\gamma_s \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j}] - \frac{\partial^2 g_1(\underline{u}^*)}{\partial u_i \partial u_j} [(u_1^* - \gamma_1) - u_1^*] =$$

$$= \sum_{s=1}^{k} \gamma_s \frac{\partial^2 g_s(\underline{u}^*)}{\partial u_i \partial u_j} = d_{ij} \quad \text{(compare (N3))}.$$

The theorem in [2] implies now

$$(12) \quad b^{n-1} \int_{G \cap \tilde{\mho}} \varphi(b\underline{v})\, d\underline{v} \backsim \frac{\phi_{k-1}(b\underline{c};\underline{\underline{R}})}{\sqrt{\det(\underline{\underline{I}}-\underline{\underline{D}})}}$$

where

$$(13a) \quad \underline{c} = (c_2, \ldots, c_k)$$

$$c_s = \underline{v}^* \cdot \text{grad } f_s(\underline{v}^*) = \underline{u}^* \cdot \text{grad } g_s(\underline{u}^*) - u_1^* \cdot \frac{\partial g_s(\underline{u}^*)}{\partial u_1} =$$

$$= \underline{u}^* \cdot \underline{a}_s - (\underline{u}^* \cdot \underline{a}_1)(\underline{a}_s \cdot \underline{a}_1)$$

$$(13b) \quad \underline{\underline{R}} = (r_{st} : 2 \leqslant s, t \leqslant k),$$

$$r_{st} = \text{grad } f_s(\underline{v}^*) \cdot \text{grad } f_t(\underline{v}^*) =$$

$$= \text{grad } g_s(\underline{u}^*) \cdot \text{grad } g_t(\underline{u}^*) - \frac{\partial g_s(\underline{u}^*)}{\partial u_1} \frac{\partial g_t(\underline{u}^*)}{\partial u_1} =$$

$$= \underline{a}_s \cdot \underline{a}_t - (\underline{a}_s \cdot \underline{a}_1)(\underline{a}_t \cdot \underline{a}_1)$$

On the other hand, $r(u_1^*, \underline{v})$ is a continuous function of $\underline{v}$ with $r(u_1^*, \underline{v}^*) = 1$. Therefore (see [3] or proof of lemma 1)

$$\int_{G \cap \tilde{\mho}} \varphi(b\underline{v})\, d\underline{v} \backsim \int_{G \cap \tilde{\mho}} r(u_1^*, \underline{v})\, \varphi(b\underline{v})\, d\underline{v}).$$

Together with eqs. (12) and (9) this implies

$$b^{n-1} \int_{\partial F_1} \varphi(b\underline{u})\, ds(\underline{u}) \backsim \varphi(bu_1^*) \frac{\Phi_{k-1}(b\underline{c};\underline{\underline{R}})}{\sqrt{\det(\underline{\underline{I}}-\underline{\underline{D}})}} ,$$

and in connection with lemma 1b, using $u_1^* = \underline{a}_1 \cdot \underline{u}^*$, we finally obtain

$$b^{n-1} \int_{\partial F_1} \varphi(b\underline{u})\, ds(\underline{u}) \backsim \varphi(b(\underline{a}_1 \cdot u^*)) \frac{\Phi_{k-1}(b\underline{c};\underline{\underline{R}})}{\sqrt{\det(\underline{\underline{I}}-\underline{\underline{D}})}} .$$

Since $\underline{a}_1 \cdot \underline{u}^*$ and the last expressions, given in eqs. (13) for $\underline{c}$ and $\underline{\underline{R}}$, are invariant under orthogonal transformations, and also $\det(\underline{\underline{I}} - \underline{\underline{D}})$ with $\underline{\underline{D}}$ defined as in (N3) is invariant under orthogonal transformations preserving (A7) (since the result in [2] must exhibit the same invariance), this proves lemma 2 also in the general case where $\underline{a}_1 \neq \underline{e}_1$.

**Proof of lemma 3**

Using the continuity of $\underline{\alpha}(\underline{u})$ for $\underline{u} \in \partial F_i$ at $\underline{u} = \underline{u}^*$, eq. (B2) and lemmas 1a (with i=j) and 2, we find

$$(1) \quad \int_{\partial F_i} \psi(b,\underline{u})\, \varphi(b\underline{u})\, ds(\underline{u}) \backsim \int_{\partial F_i} [-\underline{\alpha}(\underline{u})^T \underline{\underline{C}}\, \underline{u}]\, \varphi(b\underline{u})\, ds(\underline{u}) \backsim$$

$$\backsim (-\underline{a}_i^T \underline{\underline{C}}\, \underline{u}^*) \int_{\partial F_i} \varphi(b\underline{u})\, ds(\underline{u}) \backsim \psi(\underline{a}_i, b, \underline{u}^*) \int_{\partial F_i} \varphi(b\underline{u})\, ds(\underline{u}) \backsim$$

$$\backsim (-\underline{a}_i^T \underline{\underline{C}}\, \underline{u}^*) b^{1-n}\, \varphi(b\beta_i)\, \Phi_{k-1}(b\underline{c}_i;\underline{\underline{R}}_i)$$

$$\backsim -\underline{a}_i^T \underline{\underline{C}}\, \underline{u}^* \, b^{1-n} \frac{1}{\sqrt{d}}\, \varphi(b\beta_i)\, \Phi_{k-1}(b\underline{c}_i;\underline{\underline{R}}_i)$$

This proves part b of lemma 3. Quite similarly, using eq. (B1) instead of (B2) it follows for some environment $\mho$ of $\underline{u}^*$

$$(2) \quad \int_{\partial F_j \cap \mho} \psi(b,\underline{u})\, \varphi(b\underline{u})\, ds(\underline{u}) \ll \int_{\partial F_j \cap \mho} \varphi(b\underline{u})\, ds(\underline{u}) \backsim$$

$$\backsim\ b^{1-n} \frac{1}{\sqrt{d}} \varphi(b\beta_j)\, \Phi_{k-1}(b\underline{c}_j; \underline{\underline{R}}) \ .$$

It is now easily verified, that

$$\Phi_{k-1}(b\underline{c}_j; \underline{\underline{R}}_j) = P[\bigcap_{\substack{s=1\\ s\neq j}}^{k} \{\underline{\hat{a}}_s(\underline{u} - \underline{\hat{u}}^*) < 0\}]$$

where

$$\underline{\hat{a}}_s := \underline{a}_s - (\underline{a}_j \cdot \underline{a}_s)\cdot \underline{a}_j$$

$$\underline{\hat{u}}^* = \underline{u}^* - (\underline{a}_j \cdot \underline{u}^*)\cdot \underline{a}_j \ .$$

Furthermore, due to assumption (A5), the relation

$$\sum_{\substack{s=1\\ s\neq j}}^{n} \gamma_s\, \underline{\hat{a}}_s = \underline{\hat{u}}^*$$

holds. The theorem and remark 2 in [2] imply therefore

$$\Phi_{k-1}(b\underline{c}_j; \underline{\underline{R}}_j) \backsim K_j \frac{(-1)^{k-1}}{b^{k-1}} \left(\prod_{\substack{s=1\\ s\neq j}}^{k} \frac{1}{\gamma_s}\right) \varphi(b\underline{\hat{u}}^*) \ ,$$

where $0<K_j<\infty$ is a constant not depending on b. Since further

$$(\underline{\hat{u}}^* \cdot \underline{\hat{u}}^*) + (\underline{a}_j \cdot \underline{u}^*)^2 = \underline{u}^* \cdot \underline{u}^* \ ,$$

there is

$$\varphi(b\beta_j)\, \varphi(b\underline{\hat{u}}^*) = \varphi(b(\underline{a}_j \cdot \underline{u}^*)\, \varphi(b\underline{\hat{u}}^*) = \frac{1}{\sqrt{2\pi}}\, \varphi(b\underline{u}^*)$$

(not depending on j).

This implies that both, $\varphi(b\beta_i)\ \Phi_{k-1}(b\underline{c}_i;\underline{\underline{R}}_i)$ and $\varphi(b\beta_j)\ \Phi_{k-1}(b\underline{c}_j;\underline{\underline{R}}_j)$ are asymptotically (for $b\to\infty$) of the same order of magnitude, and remembering eqs. (1) and (2) it is observed that for some environment $\mho$ of $\underline{u}^*$

$$\int_{\partial F_j \cap \mho} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u}) \ll \int_{\partial F_i} \psi(b,\underline{u})\ \varphi(b\underline{u})\ ds(\underline{u}) .$$

This, in connection with lemma 1a, proves lemma 3a.

## Appendix

Since $0 = \mathrm{cov}(U_i, \dot{U}_i) = \mathrm{cov}(U_j, \dot{U}_j) = \mathrm{cov}(U_i + U_j, \dot{U}_i + \dot{U}_j)$, we have

$$0 = \mathrm{cov}(U_i + U_j, \dot{U}_i + \dot{U}_j) = \mathrm{cov}(U_i, \dot{U}_j) + \mathrm{cov}(U_j, \dot{U}_i)$$

or

$$\underline{\underline{C}}^T = -\underline{\underline{C}} \; .$$

This implies

$$\underline{u}^T \underline{\underline{C}} \; \underline{u} = (\underline{u}^T \underline{\underline{C}} \; \underline{u})^T = \underline{u}^T \underline{\underline{C}}^T \; \underline{u} = -\underline{u}^T \underline{\underline{C}} \; \underline{u}$$

whence

$$\underline{u}^T \underline{\underline{C}} \; \underline{u} = 0 \; .$$

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