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Non-stationary and stationary coincidence probabilities for intermittent pulse load processes

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Abstract

A train of intermittent rectangular load pulses with arrival times driven by an Erlang renewal process and with durations distributed according to a truncated Erlang distribution is considered. Based on the phase approach of queueing theory the differential equations governing the probabilities of the system being in different Markov states are derived. The differential equations for the coincidence probabilities are also obtained for mutually independent loads arising from different sources. The non-stationary and stationary solution for Markov states probabilities and coincidence probabilities is formulated and these probabilities are evaluated for different models. In particular, the stationary coincidence probabilities are evaluated for the example problem of a steel column under bending and compression caused by three independent intermittent loads.

Keywords: Erlang distributed durations; Dirac delta function; Markov states

1. Introduction

Problems of combinations of intermittent random loads were the subject of research for many years. The technique of evaluating the probability of structural failure in terms of load coincidence probabilities was developed by Wen [16], Winterstein [19] and Shinozuka [10–12]. Wen and Winterstein primarily considered coincidence probabilities for sparse and short pulses, Shinozuka gave an exact solution for trains of intermittent pulse loads with Poisson distributed arrivals and both truncated exponential and truncated Erlang distributed durations. Madsen/Ditlevsen [4] developed results for simple alternating processes. More advanced cluster processes models have also been developed for intermittent load processes [9,17]. Many available techniques for load combinations are covered in a book by Wen [18]. Recently coincidence probabilities were evaluated by the authors of the present article in the case of independent intermittent loads idealized as rectangular pulses with Erlang distributed arrival times and with durations distributed according to a truncated exponential [5] and truncated Erlang distribution [6]. Further, in Ref. [7] the problem of coincidence probabilities for alternating "on/off"-phases with arbitrary distributed durations and time gaps between pulses was studied with the result that the stationary coincidence probabilities only depend on the mean times and agree with the results in Ref. [10], however with a different definition of the parameters for the "on"- and "off"-times.

In the present article the problem of coincidence probabilities of independent intermittent loads idealized as rectangular pulses with arrival times driven by an Erlang renewal process and with durations distributed according to a truncated Erlang distribution, with parameters $\varkappa k$ and μl, respectively, is considered. Based on the phase approach of queueing theory the equations governing the Markov state probabilities of a single load are derived like in Ref. [12]. A new formulation of the equations governing the coincidence probabilities is given, based on the representation of the set of coincidence probabilities as a tensor product of state probabilities of different loads. The recursive formula for constructing the matrix of coefficients of the governing equations has also been provided. Both the non-stationary and stationary solutions of the governing equations are formulated.

The solutions for the probabilities of Markov states and hence for the "off" and "on" probabilities as well as for the probabilities of coincidence of different states are given for different example models. In particular, the sensitivity of coincidence probabilities with respect to k and l is studied

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and a comparison is made with the results of Shinozuka for the case of Poisson arrivals and with previous results of the authors for the case of Erlang arrivals. The stationary coincidence probabilities are also evaluated for a practically important example problem of a steel column under bending and compression caused by all three independent intermittent loads.

2. Intermittent rectangular pulse load process with Erlang distributed arrivals and truncated Erlang distributed durations: statement of the problem

Consider a train of rectangular pulses whose arrival times are distributed according to an Erlang renewal process. The probability density of interarrival times T_a is then given by

$$g_{T_a}(t) = (\varkappa k)^k t^{k-1} \exp(-\varkappa kt)/(k-1)!, \qquad t > 0. \tag{1}$$

The mean of T_a is

$$E[T_a] = \frac{1}{\varkappa} \tag{2}$$

and the variance of T_a becomes

$$\mathrm{Var}[T_a] = \frac{1}{\varkappa^2 k}. \tag{3}$$

The pulses are assumed not to overlap, i.e. each pulse duration completes before, or is truncated at, the moment of the next pulse arrival. Hence the duration T_d of the loading process (pulse) equals either the "primary" ("original") pulse duration T'_d or the interarrival time T_a. Thus the truncated pulse duration is defined as

$$T_d = \begin{cases} T'_d & \text{if } T'_d < T_a, \\ T_a & \text{if } T'_d \geq T_a \end{cases}. \tag{4}$$

The primary pulse durations T'_d are assumed to be identically, Erlang distributed with the probability density

$$g_{T'_d}(t) = (\mu l)^l t^{l-1} \exp(-\mu lt)/(l-1)!, \qquad t > 0. \tag{5}$$

again with mean

$$E[T'_d] = \frac{1}{\mu} \tag{6}$$

and variance

$$\mathrm{Var}[T'_d] = \frac{1}{\mu^2 l}. \tag{7}$$

The probability density function $g_{T_d}(x)$ of the truncated pulse duration T_d is hence expressed as

$$\begin{aligned} g_{T_d}(x)\,dx &= \Pr\{T_d \in (x, x+dx)\} \\ &= \Pr\{T_d \in (x,x+dx) | T'_d < T_a\} \Pr\{T'_d < T_a\} \\ &\quad + \Pr\{T_d \in (x,x+dx) | T'_d \geq T_a\} \Pr\{T'_d \geq T_a\} \\ &= \Pr\{T'_d \in (x,x+dx)\} \Pr\{T_a > x\} \\ &\quad + \Pr\{T_a \in (x,x+dx)\} \Pr\{T'_d \geq x\}. \end{aligned} \tag{8}$$

with the result

$$g_{T_d}(x) = g_{T'_d}(x)\left(1 - F_{T_a}(x)\right) + g_{T_a}(x)\left(1 - F_{T'_d}(x)\right). \tag{9}$$

The probability distribution function $F_{T_d}(x)$ is obtained as

$$\begin{aligned} F_{T_d}(x) &= \int_0^x g_{T_d}(\xi)\,d\xi = \int_0^x g_{T'_d}(\xi)(1 - F_{T_a}(\xi))d\xi \\ &\quad + \int_0^x g_{T_a}(\xi)(1 - F_{T'_d}(\xi))d\xi. \end{aligned} \tag{10}$$

Integrating by parts yields

$$\begin{aligned} F_{T_d}(x) &= F_{T'_d}(\xi)(1 - F_{T_a}(\xi))|_0^x + \int_0^x F_{T'_d}(\xi) g_{T_a}(\xi)\,d\xi \\ &\quad + F_{T_a}(\xi)(1 - F_{T'_d}(\xi))|_0^x \int_0^x F_{T_a}(\xi) g_{T'_d}(\xi)d\xi \\ &= F_{T'_d}(x) - F_{T'_d}(x)F_{T_a}(x) + F_{T_a}(x) - F_{T_a}(x)F_{T'_d}(x) \\ &\quad + F_{T_a}(x)F_{T'_d}(x) \\ &= F_{T'_d}(x) + F_{T_a}(x) - F_{T'_d}(x)F_{T_a}(x), \end{aligned} \tag{11}$$

which is in agreement with the result given e.g. in Ref. [17]. Mean and variance of this distribution cannot be given analytically except for $\varkappa = \mu$.

The remaining "off" time T_r between the consecutive pulses, i.e. the time gap between them, defined as

$$T_r = T_a - T_d, \qquad T_r \geq 0, \tag{12}$$

may be expressed as

$$T_r = \begin{cases} T_a - T'_d & \text{if } T_a > T'_d, \\ 0 & \text{if } T_a \leq T'_d \end{cases}. \tag{13}$$

The probability density function $g_{T_r}(x)$ of the remaining time T_r is expressed as

$$\begin{aligned} g_{T_r}(x)\,dx &= \Pr\{T_r \in (x, x+dx)\} \\ &= \Pr\{T_a - T_d \in (x, x+dx) \wedge T_a - T_d > 0\} \\ &\quad + \Pr\{T_a - T_d \in (x, x+dx) \wedge T_a - T_d = 0\} \\ &= \Pr\{T_a - T'_d \in (x, x+dx)\} \\ &\quad + \Pr\{T_a - T_d \in (x, x+dx) \mid T_a - T_d = 0\} \Pr\{T_a - T_d = 0\} \\ &= \Pr\{T_a \in (T'_d + x, T'_d + x + dx)\} \\ &\quad + \Pr\{T_a \in (T_a + x, T_a + x + dx)\} \Pr\{T_a \leq T'_d\}, \end{aligned} \tag{14}$$

which yields

$$g_{T_r}(x) = \int_0^\infty g_{T_a}(x+\xi) g_{T'_d}(\xi)\,d\xi + \delta(x) \int_0^\infty F_{T_a}(\xi) g_{T'_d}(\xi)\,d\xi, \tag{15}$$

where $\delta(x)$ is the Dirac delta function.

The probability distribution function $F_{T_r}(x)$ is then obtained as

$$\begin{aligned} F_{T_r}(x) &= \int_0^x g_{T_r}(\xi)\,d\xi = \int_0^\infty F_{T_a}(x+\xi) g_{T'_d}(\xi)\,d\xi \\ &\quad + H(x) \int_0^\infty F_{T_a}(\xi) g_{T'_d}(\xi)\,d\xi, \end{aligned} \tag{16}$$

where $H(x)$ is the Heaviside's (unit step) function.

If the observation period is sufficiently long, the probability of the load "being on" ("being observed") attains the stationary value, given by [11]

$$P_{on} = \frac{E[T_d]}{E[T_a]} \tag{17}$$

The probability of the load being "off" is

$$P_{off} = \frac{E[T_r]}{E[T_a]} = \frac{E[T_a] - E[T_d]}{E[T_a]} = 1 - P_{on}. \tag{18}$$

It should be noted that the load model presented before is not on a Herating Erlane process due to the truncation of the durations.

3. Probabilities of Markov states for a single load process

3.1. Governing differential equations

As the arrival and loading processes are Erlang distributed with integer parameters k and l, respectively, the so-called "phase approach" will be used [2,14]. Thus the arrivals of phases are Poisson distributed with parameter $\varkappa k$. Hence a new phase of the arrival process occurs in the time interval $(t, t + \Delta t)$ with probability $\varkappa k \Delta t$. A phase of the loading process is completed in the time interval $(t, t + \Delta t)$ with probability $\mu l \Delta t$. There may be k "empty" phases of no load and each of k phases of the arrival process may coincide with each of l phases of the loading process. Thus the total number of different states is $k(l+1)$. Let us enumerate the states as $i = 1,2,\ldots,k(l+1)$. These are Markov states, because the probability $P_i(t + \Delta t)$ that at a later time $t + \Delta t$ the system is in state i, i.e., $P_i(t + \Delta t) = \Pr\{S(t + \Delta t) = i\}$ merely depends on the states at an earlier time t. These probabilities are evaluated according to the general scheme

$$P_i(t + \Delta t) = \sum_{j=1}^{k(l+1)} \Pr\{S(t + \Delta t) = i \mid S(t) = j\} P_j(t), \tag{19}$$

where the conditional probabilities are the transition probabilities of a Markov chain. In the present problem the following equations are obtained:

$$P_1(t + \Delta t) = P_1(t)(1 - \varkappa k \Delta t) + P_{k+l}(t)\mu l \Delta t, \qquad i = 1,$$

$$P_i(t + \Delta t) = P_{i-1}(t)\varkappa k \Delta t + P_i(t)(1 - \varkappa k \Delta t) + P_{k+il}(t)\mu l \Delta t,$$

$$1 < i \leq k,$$

$$\begin{aligned} P_{k+1}(t + \Delta t) &= P_k(t)\varkappa k \Delta t + P_{k+1}(t)(1 - \varkappa k \Delta t)(1 - \mu l \Delta t) \\ &\quad + \sum_{r=0}^{l-1} P_{k(l+1)-r}(t)\varkappa k \Delta t, \end{aligned}$$

$$i = 1, \ldots, \ k + 1,$$

$$P_i(t + \Delta t) = P_{i-1}(t)\mu l \Delta t + P_i(t)(1 - \varkappa k \Delta t)(1 - \mu l \Delta t),$$

$$k + 1 < i \leq k + l,$$

$$\begin{aligned} P_i(t + \Delta t) &= P_{i-l}(t)\varkappa k \Delta t + P_{i-1}(t)\mu l \Delta t \\ &\quad + P_i(t)(1 - \varkappa k \Delta t)(1 - \mu l \Delta t), \end{aligned}$$

$$i > k + l, \ \ i \neq k + sl + 1, \ \ s = 1, \ldots, k - 1,$$

$$P_i(t + \Delta t) = P_{i-l}(t)\varkappa k \Delta t + P_i(t)(1 - \varkappa k \Delta t)(1 - \mu l \Delta t),$$

$$i > k + l, \ \ i = k + sl + 1, \ \ s = 1, \ldots, k - 1. \tag{20}$$

As the matrix of coefficients of the set of equations (Eq. (20)) is the matrix of transition probabilities, it follows that the sum of elements in each column equals one. The corresponding differential equations obtained by retaining the terms of first order in Δt are

$$\dot{P}_1 = -P_l \varkappa k + P_{k+l}\mu l,$$

$$\dot{P}_i = P_{i-1}\varkappa k - P_i \varkappa k + P_{k+}\mu l, \qquad 1 < i \leq k,$$

$$\dot{P}_{k+1} = P_k \varkappa k - P_{k+1}(\varkappa k + \mu l) + \sum_{r=0}^{l-1} P_{k(l+1)-r}\varkappa k,$$

$$i = k + 1,$$

$$\dot{P} = P_{i-1}\mu l - P_i(\varkappa k + \mu l), \qquad k + 1 < i \leq k + l,$$

$$\dot{P}_i = P_{i-l}\varkappa k + P_{i-1}\mu l - P_i(\varkappa k + \mu l),$$

$$i > k + l, i \neq k + sl + 1, \qquad s = 1, \ldots, k - 1,$$

$$\dot{P}_i = P_{i-l}\varkappa k - P_i(\varkappa k + \mu l), \qquad i > k + l,$$

$$i = k + sl + 1, \ \ s = 1, \ldots, k - 1. \tag{21}$$

The governing differential equations in matrix form are

$$\frac{d}{dt}\mathbf{P}(t) = \mathbf{A}\mathbf{P}(t). \tag{22}$$

It is worth noting that the sum of elements in each column of the matrix $\mathbf{A}$ equals zero, which means that the rows are linearly dependent. Hence the matrix $\mathbf{A}$ is singular. The probabilities $P_i(t)$ must satisfy the normalization condition

$$\sum_{i=1}^{k(l+1)} P_i(t) = 1. \tag{23}$$

The probabilities of the load being "off" and "on" are given, respectively, by

$$P_{\text{off}} = \sum_{i=1}^{k} P_i, \tag{24}$$

$$P_{\text{on}} = \sum_{i=k+1}^{k(l+1)} P_i. \tag{25}$$

The differential equations must be associated with the initial conditions

$$\Pr\{S(0) = i\} = P_i(0) = P_{0,i} \tag{26}$$

i.e.

$$\mathbf{P}(0) = \mathbf{P}_0. \tag{27}$$

If the initial conditions are deterministic, i.e. the start is assumed with probability one at the phase j then

$$P_i(0) = \Pr\{S(0) = i\} = \delta_{ij}, \tag{28}$$

where δ_{ij} is a Kronecker delta.

If the start is assumed in "off" situation, the whole probability mass is only distributed over "off" states, and then

$$\sum_{i=1}^{k} P_i(0) = 1. \tag{29}$$

If the start is assumed in "on" situation, the whole probability mass is distributed only over "on" states, then

$$\sum_{i=k+1}^{k(l+1)} P_i(0) = 1. \tag{30}$$

If the start is completely random, the whole probability mass must be distributed over all possible states, then

$$\sum_{i=1}^{k(l+1)} P_i(0) = 1. \tag{31}$$

3.2. Non-stationary and stationary solution

Solving the eigenproblem for the $k(l+1)$ order matrix $\mathbf{A}$

$$\det(\mathbf{A} - \theta\mathbf{I}) = 0, \tag{32}$$

where $\mathbf{I}$ is an identity matrix, we obtain $k(l+1)$ eigenvalues $\theta_1, \theta_2, \ldots, \theta_{k(l+1)}$.

If all eigenvalues are distinct, then $k(l+1)$ linearly independent respective eigenvectors $\mathbf{w}_i, i = 1, 2, \ldots, k(l+1)$ can be found, which make up a non-singular modal matrix $\mathbf{W}$. The non-stationary solution can be then represented as

$$\mathbf{P}(t) = \mathbf{W}\{\exp(\theta_i t)\}\mathbf{C}, \tag{33}$$

where $\{\exp(\theta_i t)\}$ is the diagonal matrix of order $k(l+1)$, with $\exp(\theta_i t), i = 1, 2, \ldots, k(l+1)$.

The constants $\mathbf{C}$ are evaluated by imposing the initial condition (27) and the solution is arrived at in the form

$$\mathbf{P}(t) = \mathbf{W}\{\exp(\theta_i t)\}\mathbf{W}^{-1}\mathbf{P}_0. \tag{34}$$

If all the eigenvalues are not distinct, i.e. if some of them are multiple then the solution is formulated in a modified way. For example if there is one eigenvalue, θ_j, with multiplicity m then the solution may be represented as

$$\mathbf{P}(t) = [\mathbf{W}_d\mathbf{V}]\{\mathrm{e}\}C, \tag{35}$$

where $\mathbf{W}_d$ is the matrix made up of the eigenvectors corresponding to $k(l+1) - m$ distinct eigenvalues and $\{e\}$ is the following diagonal, $k(l+1)$ order, matrix

$$\{\mathrm{e}\} = \begin{bmatrix} \{\exp(\theta_i t)\} & 0 & 0 & 0 & 0 \\ 0 & \exp(\theta_j t) & 0 & 0 & 0 \\ 0 & 0 & t\exp(\theta_j t) & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & t^{m-1}\exp(\theta_j t) \end{bmatrix}, \tag{36}$$

where $\{\exp(\theta_i t)\}$ is the diagonal matrix of order $k(l+1) - m$ and θ_i are the distinct eigenvalues. The m linearly independent vectors $\mathbf{V}$ are found by inserting (35) into the governing equation (22) and equating the coefficients of the polynomial in t at the right and left-hand side. The constants $\mathbf{C}$ are, of course, determined from initial conditions. Stationary probabilities $\mathbf{P}_s$ may be obtained as

$$\mathbf{P}_s = \lim_{t\to\infty} \mathbf{P}(t) \tag{37}$$

or as a nontrivial solution of the set of homogeneous algebraic equations

$$\mathbf{A}\mathbf{P}_s = 0, \tag{38}$$

where $\det(\mathbf{A}) = 0$.

Hence the stationary probabilities $\mathbf{P}_s$, are found as any column of the matrix adjoint of $\mathbf{A}$, subject to the normalization condition. As the matrix $\mathbf{A}$ is singular, its first eigenvalue is equal to zero. Moreover as the rank of this matrix is only by one lower than its order, the zero eigenvalue is single, hence the corresponding eigenvector is unique and it is just the stationary solution.

Alternatively, first the normalization condition can be inserted into Eq. (38) and next the resulting set of non-homogeneous equations can be solved.

An example of non-stationary and stationary solutions for the probabilities of Markov states and on/off probabilities for Poisson/exponential distributions with $k = 1$ and $l = 1$ is as follows. The differential equations governing the probabilities of Markov states are

$$\dot{P}_1 = -\varkappa P_1 + \mu P_2,$$

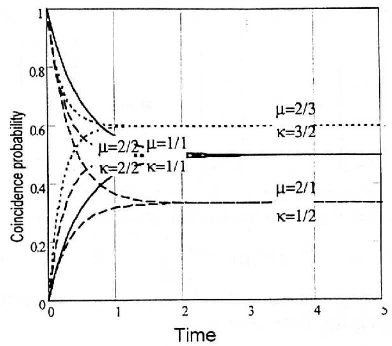

Fig. 1. Coincidence probabilities for two loads.

$$\dot{P}_2 = -\varkappa P_1 - \mu P_2,$$

where $P_1 = P_{\text{off}}$ and $P_2 = P_{\text{on}}$.

The characteristic equation becomes

$$\theta(\theta + \varkappa + \mu) = 0,$$

hence the eigenvalues are

$$\theta_1 = 0, \qquad \theta_2 = -(\varkappa + \mu).$$

The modal matrix is obtained as

$$\mathbf{W} = \begin{bmatrix} -\mu & \varkappa \\ -\varkappa & -\varkappa \end{bmatrix}.$$

Upon imposing on the first eigenvector the normalization condition the stationary solution becomes

$$\begin{bmatrix} P_{1s} \\ P_{2s} \end{bmatrix} = \begin{bmatrix} \mu/(\varkappa + \mu) \\ \varkappa/(\varkappa + \mu) \end{bmatrix} = \begin{bmatrix} 1/(1+\rho) \\ (\rho/(1+\rho) \end{bmatrix},$$

where $\rho = \varkappa/\mu$ in agreement with Ref. [11].

The non-stationary solution, in the case of the start from "off" situation, i.e. with initial conditions

$$\begin{bmatrix} P_{1,0} \\ P_{2,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

becomes (cf. Sniady and Sieniawska [13])

$$P_1(t) = \frac{\mu}{\varkappa + \mu} + \frac{\varkappa}{\varkappa + \mu}\exp(-(\varkappa + \mu)t),$$

$$P_2(t) = \frac{\varkappa}{\varkappa + \mu} - \frac{\varkappa}{\varkappa + \mu}\exp(-(\varkappa + \mu)t).$$

The start from "on" situation, i.e. with intial conditions

$$\begin{bmatrix} P_{1,0} \\ P_{2,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

yields

$$P_1(t) = \frac{\mu}{\varkappa + \mu} - \frac{\mu}{\varkappa + \mu}\exp(-(\varkappa + \mu)t),$$

$$P_2(t) = \frac{\varkappa}{\varkappa + \mu} + \frac{\mu}{\varkappa + \mu}\exp(-(\varkappa + \mu)t).$$

In the case of the random start, e.g. with initial conditions

$$\begin{bmatrix} P_{1,0} \\ P_{2,0} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix},$$

the non-stationary solution becomes

$$P_1(t) = \frac{\mu}{\varkappa + \mu} + \frac{\varkappa - \mu}{\varkappa + \mu}\exp(-(\varkappa + \mu)t),$$

$$P_2(t) = \frac{\varkappa}{\varkappa + \mu} - \frac{\varkappa - \mu}{\varkappa + \mu}\exp(-(\varkappa + \mu)t).$$

Other cases are given in Appendix A. Fig. 1 shows the convergence towards stationarity versus time for $\mu = \kappa = 1$. It is seen that convergence is reached for about two-times the mean interarrival time. It is clear that for μ and κ different stationarity is reached more slowly.

4. Probabilities of coincidence of states of different load processes

4.1. Governing differential equations

Consider an arbitrary number N of independent intermittent pulse loads each of which is characterized by parameters $\varkappa_r, k_r, \mu_r, l_r, r = 1, 2, \ldots, N$. The Markov states of each load are governed by Eq. (20).

In order to derive the differential equations governing the coincidence probabilities it is expedient to rewrite the Eq. (20) for the rth load in the form of

$$\mathbf{P}^{(r)}(t + \Delta t) = \mathbf{P}^{(r)}(t) + \mathbf{A}^{(r)}\mathbf{P}^{(r)}(t)\Delta t, \tag{39}$$

or

$$P_{i_r}^{(r)}(t + \Delta t) = P_{i_r}^{(r)}(t) + a_{i,j}^{(r)}P_j^{(r)}(t)\Delta t,$$

$$i_r = 1, 2, \ldots, k_r(l_r + 1),$$

where $\mathbf{A}^{(r)}$ is the matrix of order $k_r(l_r + 1) \times k_r(l_r + 1)$, with diagonal elements

$$a_{i,i}^r = \begin{cases} -\varkappa_r k_r, & 1 \le i < k_r + 1, \\ -\mu_r l_r, & i = k_r + 1, \quad k_r = 1, \\ -(\varkappa_r k_r + \mu_r l_r), & i = k_r + 1, \quad k_r > 1, \\ -(\varkappa_r k_r + \mu_r l_r), & i > k_r + 1 \end{cases} \tag{40}$$

and with non-zero off-diagonal elements given by

$$a_{1,k_r+l_r}^r = \mu_r l_r,$$

$$a_{i,i-1}^r = \varkappa_r k_r, \qquad 1 < i \le k_r,$$

$$a_{i,k_r+il_r}^r = \mu_r l_r, \qquad 1 < i \le k_r,$$

$$a_{k_r+1,k_r}^r = \varkappa_r k_r,$$

$$a^r_{k_r+1,j} = \varkappa_r k_r,$$

$$j = k_r(l_r + 1) - q,\quad q = 0, 1, 2, \ldots, l_r - 1,\quad q \neq k_r l_r - 1,$$

$$a^r_{i,i-1} = \mu_r l_r,\qquad k_r + 1 < i \leq k_r + l_r,$$

$$a^r_{i,i-1} = \mu_r l_r,$$

$$i > k_r + l_r,\quad i \neq k_r + s l_r + 1,\quad s = 1, \ldots, k_r - 1,$$

$$a^r_{i,i-l_r} = \varkappa_r k_r,\qquad i > k_r + l_r,$$

$$i \neq k_r + s l_r + 1,\quad s = 1, \ldots, k_r - 1,\quad l_r \neq 1,$$

$$a^r_{i,i-l_r} = \varkappa_r k_r,\qquad i > k_r + l_r,\qquad i = k_r + s l_r + 1,$$

$$s = 1, \ldots, k_r - 1,\qquad l_r \neq 1. \tag{41}$$

Let $\Pi_{i_1,\ldots,i_N}(t)$ denote the probability of coincidence of different states of N loads, i.e. the probability that at time t the rth load is in the state i_r. Each state may coincide with any other. The number of coincidence probabilities is thus $M = \Pi_j^N k_j(l_j + 1)$. As different load processes are mutually independent, the coincidence probabilities are evaluated as products of Markov states probabilities of different loads. The differential equations governing the coincidence probabilities are obtained from Eq. (39) by multiplying the left and right sides of respective equations and by retaining the terms of first order in Δt i.e.

$$\Pi_{i_1,\ldots,i_N}(t + \Delta t) = (P^{(1)}_{i_1}(t) + a^{(1)}_{i_1 j} P^{(1)}_j(t)\Delta t)(P^{(2)}_{i_2}(t) + a^{(2)}_{i_2 j} P^{(2)}_j(t)\Delta t)\cdots(P^{(N)}_{i_N}(t) + a^{(N)}_{i_N j} P^{(N)}_j(t)\Delta t). \tag{42}$$

Let us represent the set of all possible coincidence probabilities in form of a supervector $\mathbf{\Pi}^{(N)}(t)$ obtained by performing tensor products as

$$\boldsymbol{\Pi}^{(N)}(t) = \boldsymbol{P}^{(1)}(t) \otimes \boldsymbol{\Pi}^{(N-1)}(t)$$

$$\boldsymbol{\Pi}^{(N-1)}(t) = \boldsymbol{P}^{(2)}(t) \otimes \boldsymbol{\Pi}^{(N-2)}(t)$$

$$\vdots$$

$$\boldsymbol{\Pi}^{(N-i+1)}(t) = \boldsymbol{P}^{(i)}(t) \otimes \boldsymbol{\Pi}^{(N-i)}(t)$$

$$\vdots$$

$$\boldsymbol{\Pi}^{(2)}(t) = \boldsymbol{P}^{(N-1)}(t) \otimes \boldsymbol{\Pi}^{(1)}(t) = \boldsymbol{P}^{(N-1)}(t) \otimes \boldsymbol{P}^{(N)}(t), \tag{43}$$

where $\otimes$ denotes the tensor product, i.e. given vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},\qquad \mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1\mathbf{b} \\ a_2\mathbf{b} \\ \vdots \\ a_m\mathbf{b} \end{bmatrix}.$$

The differential equations governing the vector $\Pi^{(N)}(t)$ of coincidence probabilities are then given in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Pi}^{(N)}(t) = \mathbf{C}^{(N)}\boldsymbol{\Pi}^{(N)}(t), \tag{44}$$

where the coefficient matrix $\mathbf{C}^{(N)}$ is constructed by applying the following recursive procedure

$$\mathbf{C}^{(N-i+1)} = \begin{bmatrix} \mathbf{C}^{(N-i)} + \{a^{(i)}_{11}\} & \{a^{(i)}_{12}\} & \cdots & \{a^{(i)}_{1n_i}\} \\ \{a^{(i)}_{21}\} & \mathbf{C}^{(N-i)} + \{a^{(i)}_{22}\} & \cdots & \{a^{(i)}_{2n_i}\} \\ \vdots & \vdots & \ddots & \vdots \\ \{a^{(i)}_{n_i1}\} & \{a^{(i)}_{n_i2}\} & \cdots & \mathbf{C}^{(N-i)} + \{a^{(i)}_{n_in_i}\} \end{bmatrix}, \tag{45}$$

where $i = 1, 2, \ldots, N-1$ and $\{a^{(i)}_{jk}\}$ denotes the diagonal matrix formed of jkth element of the matrix $\mathbf{A}^{(i)}$ and $\mathbf{C}^{(1)} = \mathbf{A}^{(N)}$. The recursive procedure must begin with evaluation of $\mathbf{C}^{(2)}$, for $i = N - 1$, which has the form

$$\mathbf{C}^{(2)} = \begin{bmatrix} \mathbf{A}^{(N)} + \{a^{(N-1)}_{11}\} & \{a^{(N-1)}_{12}\} & \cdots & \{a^{(N-1)}_{1n_i}\} \\ \{a^{(N-1)}_{21}\} & \mathbf{A}^{(N)} + \{a^{(N-1)}_{22}\} & \cdots & \{a^{(N-1)}_{2n_i}\} \\ \vdots & \vdots & \ddots & \vdots \\ \{a^{(N-1)}_{n_i1}\} & \{a^{(N-1)}_{n_i2}\} & \cdots & \mathbf{A}^{(N)} + \{a^{(N-1)}_{n_in_i}\} \end{bmatrix}. \tag{46}$$

Next the procedure must be repeated with decreasing index i. As the matrix $\mathbf{C}^{(N-i+1)}$ is constructed by successive use of matrices $\mathbf{A}$, it can be shown that the sum of all elements in each of its column equals zero, hence the rows are linearly dependent and each matrix $\mathbf{C}^{(N-i-1)}$ and hence the matrix $\mathbf{C}^{(N)}$ is singular.

The non-stationary and stationary solutions for the coincidence probabilities can be obtained in the same way as described in the Section 3 for the Markov states probabilities. In particular the stationary solution for the coincidence probabilities $\boldsymbol{\Pi}^{(N)}_s$ can be obtained with the help of a matrix adjoint of $\mathbf{C}^{(N)}$, imposing the obvious normalization condition

$$\sum_{i=1}^{\prod_{j=1}^{N} k_j(l_j+1)} \prod_i = 1. \tag{47}$$

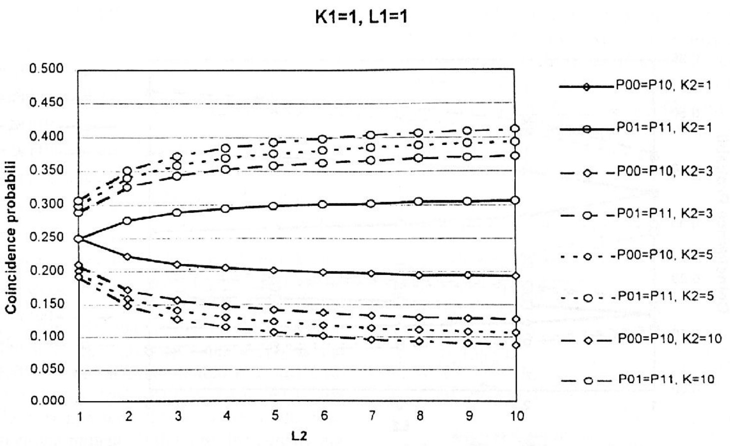

Fig. 2. Coincidence probabilities for two loads as a function of parameters k_2 and l_2 ($k_1 = 1$, $l_1 = 1$).

For an example of two loads with Poisson/exponential distributions with $k = 1$ and $l = 1$ stationary and non-stationary solution is as follows. If both loads have Poisson distributed arrivals and truncated exponential durations, the differential equations for coincidence probabilities are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{bmatrix} = \begin{bmatrix} -(\varkappa_1 + \varkappa_2) & \mu_2 & \mu_1 & 0 \\ \varkappa_2 & -(\varkappa_1 + \mu_2) & 0 & \mu_1 \\ \varkappa_1 & 0 & -(\varkappa_2 + \mu_1) & \mu_2 \\ 0 & \varkappa_1 & \varkappa_2 & -(\mu_1 + \mu_2) \end{bmatrix} \times \begin{bmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{bmatrix}.$$

The characteristic equation is obtained as

$$\theta(\theta + \varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)(\theta^2 + (\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)\theta + (\varkappa_1 + \mu_1)(\varkappa_2 + \mu_2)) = 0$$

and the eigenvalues are found to be

$$\theta_1 = 0,\ \theta_2 = -(\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2),\ \theta_3 = -(\varkappa_1 + \mu_1),\ \theta_4 = -(\varkappa_2 + \mu_2).$$

The modal matrix is obtained as

$$\mathbf{W} = \begin{bmatrix} 1/((1 + \rho_1)(1 + \rho_2)) & 1 & 1 & 1 \\ \rho_2/((1 + \rho_1)(1 + \rho_2)) & -1 & \rho_2 & -1 \\ \rho_1/((1 + \rho_1)(1 + \rho_2)) & -1 & -1 & \rho_1 \\ \rho_1\rho_2/((1 + \rho_1)(1 + \rho_2)) & 1 & -\rho_2 & -\rho_1 \end{bmatrix},$$

where $\rho_1 = \varkappa_1/\mu_1$ *and* $\rho_2 = \varkappa_2/\mu_2$ and the stationary solution is

$$\begin{bmatrix} \Pi_{1s} \\ \Pi_{2s} \\ \Pi_{3s} \\ \Pi_{4s} \end{bmatrix} = \begin{bmatrix} 1/((1 + \rho_1)(1 + \rho_2)) \\ \rho_2/((1 + \rho_1)(1 + \rho_2)) \\ \rho_1/((1 + \rho_1)(1 + \rho_2)) \\ \rho_1\rho_2/((1 + \rho_1)(1 + \rho_2)) \end{bmatrix},$$

which is the result given in Ref. [11]. The nonstationary solution, in the case when both loads start from the "off" situation, i.e. with initial conditions

$$\begin{bmatrix} \Pi_{1,0} \\ \Pi_{2,0} \\ \Pi_{3,0} \\ \Pi_{4,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

is obtained as

$$\Pi_1(t) = \frac{1}{(1 + \rho_1)(1 + \rho_2)}(1 + \rho_1\rho_2\exp(-(\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)t) + \rho_1\exp(-(\varkappa_1 + \mu_1)t) + \rho_2\exp(-(\varkappa_2 + \mu_2)t)),$$

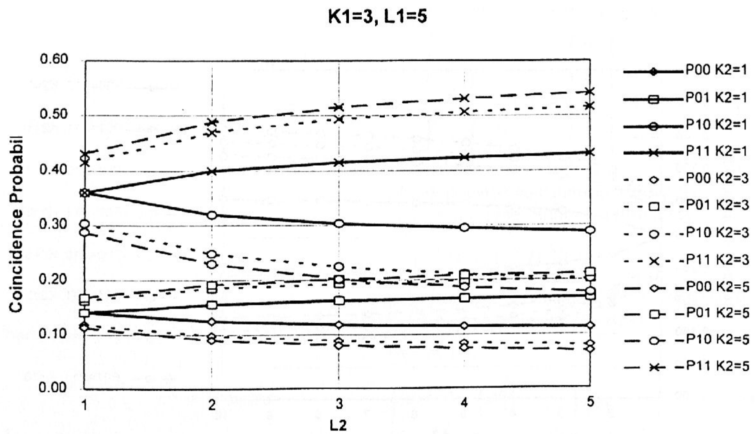

Fig. 3. Coincidence probabilities for two loads as a function of parameters k_2 and l_2 ($k_1 = 3$, $l_1 = 5$).

$$\Pi_2(t) = \frac{\rho_2}{(1+\rho_1)(1+\rho_2)}(1 - \rho_1 \times \exp(-(\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)t) + \rho_1 \exp(-(\varkappa_1 + \mu_1)t) - \exp(-(\varkappa_2 + \mu_2)t)),$$

$$\Pi_3(t) = \frac{\rho_1}{(1+\rho_1)(1+\rho_2)}(1 - \rho_2 \times \exp(-(\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)t) - \exp(-(\varkappa_1 + \mu_1)t) + \rho_2 \exp(-(\varkappa_2 + \mu_2)t)),$$

$$\Pi_4(t) = \frac{\rho_1\rho_2}{(1+\rho_1)(1+\rho_2)}(1 + \exp(-(\varkappa_1 + \varkappa_2 + \mu_1 + \mu_2)t) - \exp(-(\varkappa_1 + \mu_1)t) - \exp(-(\varkappa_2 + \mu_2)t)).$$

Figs. 2 and 3 show examples for coincidence probabilities for two loads for different k_1, l_1, k_2, l_2, and $\kappa = \mu = 1$. It is seen that the coincidence probabilities soon approach limiting values for growing k_1, l_1, k_2 and/or l_2. The effect of the mean duration parameters κ and μ is much larger.

5. Probability of structural failure

For failure probability evaluations it is convenient to distinguish between three types of variables. R-variables denote simple random variables, possibly depending on time in a deterministic manner, Q-variables denote random stationary and ergodic sequences and S-variables are random, not necessarily stationary process variables. The Q-variables may conveniently be used to model sequences of parameters for the S-variables, for example traffic states, sea states, 10 min wind regimes, etc. An upper bound for the probability of the first excursion failure in the time interval (0, T) under general loading is

$$P_f(T) \le P_f(0) + \int_0^T \sum_k P_k(t) E_r[E_Q[\nu_k^+(\tau, R, Q)]]\,d\tau \quad (48)$$

where $P_f(0)$ is the initial failure probability, $P_k(t)$ is the coincidence probability of the kth load combination and $\nu_k^+(T, R, Q)$ the mean outcrossing rate from the safe domain into the failure domain. This equation is extremely laborious to handle and will not be discussed further. Assuming the system in a stationary state with respect to the "on/off"-times but not necessarily with respect to the outcrossing rates we have

$$P_f(T) \le P_f(0) + \sum_k P_k E_R[E_Q[E[N_k^+(T, R, Q)]]]\,d\tau \quad (49)$$

with

$$E[N_k^+(T, R, Q)] = \int_0^T \nu_k^+(\tau, R, Q)\,d\tau, \quad (50)$$

where $E[N_k^+(T, R, Q)]$ is the mean number of outcrossings. Simplifying further by assuming also stationary out-crossing rates it is

$$P_f(T) \le P_f(0) + T \sum_k P_k E_R[E_Q[\nu_k^+(R, Q)]]\,d\tau. \quad (51)$$

For example, in the case of one non-intermittent, stationary load and two independent, stationary, intermittent loads of arbitrary type, Eq. (51) is given by

$$\begin{aligned} P_f(T) &\le P_f(0) + P_0 T \nu_0 + \left(P_1^1 T \nu_{0+1} + P_1^2 T \nu_{0+2} + P_2^{1+2} T \nu_{0+1+2}\right) \\ &= P_f(0) + T\left(\nu_0 P_0 + \nu_{0+1} P_1^1 + \nu_{0+2} P_1^2 + \nu_{0+1+2} P_2^{1+2}\right) \\ &= P_f(0) + T_0 \nu_0 + T_1^1 \nu_{0+1} + T_1^2 \nu_{0+2} + T_2^{1+2} \nu_{0+1+2}, \end{aligned} \quad (52)$$

where ν_0 is the outcrossing rate of the non-intermittent load only, ν_{0+1} is the outcrossing rate given the non-intermittent load and the first intermittent process are "on", ν_{0+2} is the outcrossing rate given the non-intermittent load and the second intermittent process are "on" and ν_{0+1+2} is the outcrossing rate given the non-intermittent load and both intermittent processes are "on". $P_f(0)$ is the initial failure probability, which is non-zero only if some of the load processes are non-intermittent. In this case the outcrossing rates for intermittent processes have to be computed under the condition that the non-intermittent process is always present. Making use of the ergodicity theorem the asymptotic, total duration of the various "on/off" events is $T_0 = TP_0 = TP_{\text{off}}$, $T_1^1 = TP_1^1$, $T_1^2 = TP_1^2$ and $T_2^{1+2} = TP_2^{1+2}$. This observation is useful when handling the case described by Eq. (49). There can be outcrossings of the non-intermittent processes only during the time interval [0, T$_0$] and for the intermittent processes only during their "on" periods.

In the general case of N loads it is [10]

$$P_f(T) \le P_f(0) + T\Bigg(\nu_0 P_0 + \sum_{i=1}^N \nu_{0+i} P_1^i + \sum_{i=1}^N \sum_{j=i+1}^N \nu_{0+i+j} P_2^{i+j} + \sum_{i=1}^N \sum_{j=i+1}^N \sum_{k=j+1}^N \nu_{0+i+j+k} P_3^{i+j+k} + \cdots + \nu_{0+i_1+\cdots+i_N} P_N^{i_1+\cdots+i_N}\Bigg), \quad (53)$$

where ν_{0+i} is the mean up-crossing rate when the non-intermittent load and only the ith intermittent load are "on", ν_{0+i+j} is the mean upcrossing rate when the non-intermittent load and only the two intermittent loads i and j are simultaneously "on", $\nu_{0+i+j+k}$ is the mean upcrossing rate when the non-intermittent load and only the three intermittent loads i, j and k are simultaneously "on", and so forth. Finally, $\nu_{0+i_1+\cdots+i_N}$ is the mean upcrossing rate when the non-intermittent load and all N intermittent loads are simultaneously "on". The upcrossing rates can be evaluated with the help of any techniques available in the literature, for example, according to Refs. [1,8] for rectangular wave renewal processes, Ref. [3] for processes with arbitrary pulse shape or Ref. [15] for differentiable processes. Note that the jump rates for independent components of a renewal wave process can be different from the arrival rate of its "on"-times. For differentiable vector processes all components of the vector must have the same arrival/duration rate.

In Eq. (52) the probabilities P_0 that both intermittent loads are "off", that P_1^1 that the first load is "on" while the second is "off", P_1^2 that the first load is "off" while the second is "on" and P_2^{1+2} that both loads are simultaneously "on" are given, respectively, by

$$P_0 = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \Pi_{(i-1)k_2(l_2+1)+j}, \quad (54)$$

$$P_1^1 = \sum_{i=k_1+1}^{k_1(l_1+1)} \sum_{j=1}^{k_2} \Pi_{(i-1)k_2(l_2+1)+j}, \quad (55)$$

$$P_1^2 = \sum_{i=1}^{k_1} \sum_{j=k_2+1}^{k_2(l_2+1)} \Pi_{(i-1)k_2(l_2+1)+j}, \quad (56)$$

$$P_2^{1+2} = \sum_{i=k_1+1}^{k_1(l_1+1)} \sum_{j=k_2+1}^{k_2(l_2+1)} \Pi_{(i-1)k_2(l_2+1)+j}. \quad (57)$$

In Eq. (53) the probability P_1^m that only the mth load is "on" while all other loads are "off" and the probability that the mth and nth loads are simultaneously "on" while all other loads are "off" are evaluated, respectively, as

$$P_1^m = \sum_{i_1=1}^{k_1} \cdots \sum_{i_m=k_m+1}^{k_m(l_m+1)} \cdots \sum_{i_N=1}^{k_N} \Pi_{(i_1-1)k_2(l_2+1)+\cdots k_N(l_N+1)} \times_{(i_m-1)k_{m+1}(l_{m+1}+1)\cdots k_N(l_N+1)+\cdots+(i_{N-1}-1)k_N(l_N+1)+i_N} \quad (58)$$

$$P_1^{m+n} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_m=k_m+1}^{k_m(l_m+1)} \cdots \sum_{i_n=k_n+1}^{k_n(l_n+1)} \cdots \sum_{i_N=1}^{k_N} \times \Pi_{(i_1-1)k_2(l_2+1)\cdots k_N(l_N+1)+\cdots+(i_m-1)k_{m+1}(l_{m+1}+1)\cdots k_N(l_N+1)+\cdots+(i_n-1)k_{n+1}(l_{n+1}+1)\cdots k_N(l_N+1)+\cdots+(i_{N-1}-1)k_N(l_N+1)+i_N} \quad (59)$$

and so on.

A trivial, lower probability bound can also be assessed. Under stationary conditions there is

$$P_f(T) \ge \Bigg(P_{f,0}(0)P_0 + \sum_{i=1}^N P_{f,0+i}(0)P_1^i + \sum_{i=1}^N \sum_{j=i+1}^N P_{f,0+i+j}(0)P_2^{i+j} + \sum_{i=1}^N \sum_{j=i+1}^N \sum_{k=j+1}^N P_{f,0+i+j+k}(0)P_3^{i+j+k} + \cdots + P_{f,0+i_1+i_2+\cdots+i_N}(0)P_N^{i_1+\cdots+i_N}\Bigg), \quad (60)$$

with $P_{f,\{k\}}(0)$ the failure probability if the set $\{k\}$ of loads is

Table 1
Stochastic model for illustration example

Variable	Symb.	Distr.	Type	Mean/s.d.	Unit	λ	ρ
Yield stress	F_s	LN	R	500/35	MPa	–	–
Flange breadth	B	LN	R	300/3	mm	–	–
Flange thickn.	D	LN	R	20/2	mm	–	–
Profile height	H	LN	R	300/5	mm	–	–
Initial deflect.	F_0	HN	R	m_{F_0}/σ_{F_0}	mm	–	–
Youngs Modul.	E	We	R	210 000/4200	MPa	–	–
Dead weight	G	N	R	800 000/40 000	N	–	–
Variable load	P_1	N	S	800 000/150 000	N	0.1	0.01
Variable load	P_2	Gu	S	800 000/150 000	N	10	1
Variable load	P_3	Ga	S	800 000/200 000	N	1	100

"on". Apparently, this is just the point-in-time failure probability. This lower bound is also identical with $P_f(0)$ under stationary conditions.

6. Illustration: steel column under three different stationary and intermittent loads

As a numerical example a centrally loaded steel column with initial deflection is considered with approximate limit state function in terms of the random vector **X**, the parameter (b,d,h) and auxiliary functions $\mathscr{A}_s, \mathscr{M}_s, \mathscr{M}_i, \mathscr{E}_b, \mathscr{P} = G + P_1 + P_2 + P_3$ is defined by:

$$g(\mathbf{x},\mathbf{p}) = F_s - \mathscr{P}\left(\frac{1}{\mathscr{A}_s} + \frac{F_0}{\mathscr{M}_s}\cdot\frac{\mathscr{E}_b}{\mathscr{E}_b - \mathscr{P}}\right), \qquad (61)$$

where

$\mathscr{A}_s = 2BD$, (area of section),

$\mathscr{M}_s = BDH$, (modulus of section),

$\mathscr{M}_i = \frac{1}{2}BDH^2$, (moment of inertia),

$\mathscr{E}_b = \frac{\pi^2 E \mathscr{M}_i}{s^2}$, (Euler buckling load).

The steel column has a constant length of 4000 mm. The anticipated lifetime is 50 y. The independent uncertain vector $\mathbf{X} = (F_s, P_1, P_2, P_3, B, D, H, F_0, E)$ and its stochastic characteristics are given in Table 1 where $m_{F_0} = \sqrt{(M_i/A_s)}/20 + s/500$ and $\sigma_{F_0} = 0.3m_{F_0}$.

All time-variant loads are modeled as rectangular wave processes with different jump rate λ[1/year] and different duration parameter ρ. The load P_1 may represent long duration occupancy loading, P_2 some climatic loading and P_3 short term occupancy loading. This limit state function is highly non-linear. The outcrossing rates are determined by somewhat improved SORM in standard space, i.e., by

$$\nu^+_{\{k\}}(g(\mathbf{x}_k,\mathbf{p}) \le 0) = \sum_{i=1}^{3} \lambda_i \Phi(-\beta) \prod_{j=1}^{n-1} (1 - \beta\kappa_j)^{-1/2}$$
$$\times\left[1 - \frac{\Phi(-\beta, -\beta; \rho_i)}{\Phi(-\beta)}\right], \qquad (62)$$

where β is the geometrical safety index and the last factor often is negligible because $\Phi(-\beta, -\beta; \rho_i) \ll \Phi(-\beta)$. In this formula there is $\rho_i = 1 - \alpha_i^2$, $i = 1,2,3$ with α_i the normalized mean value sensitivities (or direction cosines of the β-point) and κ_j are the main curvatures in the β-point. This yields Table 2, where for $P_{\{k\}}$ the notation $k_1/l_1, k_2/l_2, k_3/l_3$ is used. For the case 1/1, 1/1, 1/1 the second last load case clearly dominates the total failure probability because of the large coincidence probability. The initial failure probability, assuming stationary conditions, is largest for the last load case, however. The lower failure probability bound is in this case 6.37×10^{-6} and, thus, there is a factor of approximately two between the bounds. Two other combinations of parameters are computed. It is seen that the total failure probabilities differ only insignificantly in this case. Larger relative differences are observed for the dominating load case. Essentially the same findings can be demonstrated for other combinations k_1, l_1, k_2, l_2, k_3 and l_3.

Table 2
Results of illustration example

Load case	$P_{f,\{k\}}$	$P_f(0)$	$P_{\{k\}}$ 1/1, 1/1, 1/1	$P_{\{k\}}$ 10/1, 10/1, 1/1	$P_{\{k\}}$ 5/1, 2/3, 4/2
000	0		4.90×10^{-3}	3.78×10^{-3}	4.20×10^{-5}
100	1.55×10^{-21}	1.55×10^{-21}	4.90×10^{-5}	3.80×10^{-5}	4.25×10^{-7}
010	2.64×10^{-12}	7.65×10^{-13}	4.90×10^{-3}	6.02×10^{-3}	7.69×10^{-5}
001	1.28×10^{-13}	5.01×10^{-15}	4.90×10^{-1}	3.78×10^{-1}	3.42×10^{-1}
110	1.65×10^{-9}	1.61×10^{-9}	4.90×10^{-5}	6.10×10^{-5}	8.04×10^{-7}
101	8.86×10^{-10}	7.02×10^{-10}	4.90×10^{-3}	3.82×10^{-3}	3.46×10^{-3}
011	1.46×10^{-6}	5.44×10^{-9}	4.90×10^{-1}	6.02×10^{-1}	6.48×10^{-1}
111	2.09×10^{-5}	6.40×10^{-6}	4.90×10^{-3}	6.08×10^{-3}	6.54×10^{-3}
Total	–		1.00	1.00	1.00
$P_f(0) + \Sigma P_{\{k\}} P_{f,\{k\}}$			1.61×10^{-5}	1.99×10^{-5}	2.08×10^{-5}

7. Concluding remarks

Based on queueing theory a technique is devised for evaluating the probabilities of coincidence of independent intermittent loads idealized as rectangular pulses with Erlang distributed arrivals and with durations distributed according to a truncated Erlang distribution. Equations governing the coincidence of different states of load processes are derived, which constitute a set of first order ordinary differential equations. The recursive formula for constructing the coefficient matrix is provided, which allows to construct the matrix for an arbitrary number of loads. Several closed form non-stationary and stationary solutions (for small k and l) are given for the Markov state probabilities and states coincidence probabilities. It is found that the "on" probabilities and the coincidence probabilities differ only insignificantly for k and l a little larger than one, confirming the earlier results of Shinozuka [10,11] and Iwankiewicz and Rackwitz [5]. It can be found that this conclusion also holds for larger k and l and a larger number of contributing loads. However, the coincidence probabilities do vary significantly as the mean arrival time and mean durations change. It is thus concluded that the simplest model, i.e., with Poissonian arrivals and exponential durations may be recommended for practical use as a conservative one. The results discussed so far concern load processes where the "on" phases are independent. It remains to be shown that similar conclusions also hold when clustering of "on" phases occurs.

Acknowledgements

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Appendix A. Analytical results for Erlang distributions

A.1. Poisson/Erlang distributions: $k = 1$ and $l = 2$

The differential equations governing the probabilities Markov states are

$$\dot{P}_1 = -\varkappa P_1 + 2\mu\, P_3,$$

$$\dot{P}_2 = \varkappa P_1 - 2\mu\, P_2 + \varkappa P_3,$$

$$\dot{P}_3 = 2\mu\, P_2 - (\varkappa + 2\mu\,)P_3. \qquad (A.1)$$

The characteristic equation becomes

$$\theta(\theta + \varkappa + 2\mu\,)^2 = 0 \qquad (A.2)$$

hence the eigenvalues are

$$\theta_1 = 0, \qquad \theta_{2,3} = -(\varkappa + 2\mu\,). \qquad (A.3)$$

The stationary solution obtained via the matrix adjoint of **A** is

$$\begin{bmatrix} P_{1s} \\ P_{2s} \\ P_{3s} \end{bmatrix} = \begin{bmatrix} 1/(1+\rho)^2 \\ (\rho(1+\rho))/(1+\rho)^2 \\ \rho/(1+\rho)^2 \end{bmatrix}, \qquad (A.4)$$

where $\rho = \varkappa/2\mu$.

Two linearly independent vectors corresponding to the double eigenvalue are

$$\mathbf{v} = \begin{bmatrix} -1 & -(1 + (1/2\mu t)) \\ 0 & (1 + (1/2\mu t)) \\ 1 & 1 \end{bmatrix} \qquad (A.5)$$

and the non-stationary solution obtained upon imposing the initial conditions

$$\begin{bmatrix} P_{1,0} \\ P_{2,0} \\ P_{3,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad (A.6)$$

becomes

$$P_1(t) = \frac{1}{(1+\rho\,)^2}(1 + \rho((1+\rho)(2\mu t + 1) + 1)$$
$$\times \exp(-(\varkappa + 2\mu\,)t)), \qquad (A.7)$$

$$P_2(t) = \frac{\rho(1+\rho)}{(1+\rho)^2}(1 - \exp(-(\varkappa + 2\mu)t)), \qquad (A.8)$$

$$P_3(t) = \frac{\rho}{(1+\rho)^2}(1 - (2\mu t(1+\rho) + 1)\exp(-(\varkappa + 2\mu)t)). \qquad (A.9)$$

Here $P_{\text{off}} = P_1$ and $P_{\text{on}} = P_2 + P_3$.

A.2. Erlang/exponential distributions: $k = 2$ and $l = 1$

The governing differential equations are

$$\dot{P}_1 = -2\varkappa P_1 + \mu\, P_3,$$

$$\dot{P}_2 = 2\varkappa P_1 - 2\varkappa P_2 + \mu P_4,$$

$$\dot{P}_3 = 2\varkappa P_2 - (2\varkappa + \mu)P_3 + 2\varkappa P_4,$$

$$\dot{P}_4 = 2\varkappa P_3 - (2\varkappa + \mu)P_4. \qquad (A.10)$$

The characteristic equation becomes

$$\theta(\theta + 2\varkappa + \mu)(\theta^2 + (\mu + 6\varkappa)\theta + 4\varkappa\mu + 8\varkappa^2) = 0. \quad (A.11)$$

Hence the eigenvalues are

$$\theta_1 = 0,\ \ \theta_2 = -4\varkappa,\ \ \theta_{3,4} = -(2\varkappa + \mu) \tag{A.12}$$

The stationary solution obtained via the matrix adjoint of **A** is

$$P_{1s} = \frac{1+\rho}{2(1+\rho)^2}$$

$$P_{2s} = \frac{1+2\rho}{2(1+\rho)^2},$$

$$P_{3s} = \frac{\rho(1+\rho)}{2(1+\rho)^2},$$

$$P_{4s} = \frac{\rho^2}{2(1+\rho)^2}, \tag{A.13}$$

where $\rho = 2\varkappa/\mu$.

The eigenvector $\mathbf{w}_2$ corresponding to the single eigenvalue θ_2 is

$$\mathbf{w}_2 = \begin{bmatrix} 1-\rho \\ 2\rho - 1 \\ \rho(\rho-1) \\ -\rho^2 \end{bmatrix}. \tag{A.14}$$

Two linearly independent vectors corresponding to the double eigenvalue are

$$\mathbf{V} = \begin{bmatrix} 0 & -1/2\varkappa t \\ -1 & -1 \\ 0 & 1/2\varkappa t \\ 1 & 1 \end{bmatrix} \tag{A.15}$$

and the non-stationary solution obtained upon imposing the initial conditions

$$\begin{bmatrix} P_{1,0} \\ P_{2,0} \\ P_{3,0} \\ P_{4,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A.16}$$

is obtained as

$$P_1(t) = \frac{1+\rho}{2(1+\rho)^2} + \frac{1}{2(1-\rho)}\exp(-4\varkappa t) - \frac{\rho^2}{(1+\rho)(1-\rho)}\exp(-(2\varkappa+\mu)t), \tag{A.17}$$

$$P_2(t) = \frac{1+2\rho}{2(1+\rho)^2} + \frac{2\rho-1}{2(1-\rho)^2}\exp(-4\varkappa t) - \left(\frac{\rho^2}{(1+\rho)^2}\left(\frac{2\rho}{(1-\rho)^2} - \frac{1}{2}\right) + \frac{2\varkappa\rho^2 t}{(1+\rho)(1-\rho)}\right) \times \exp(-(2\varkappa+\mu)t), \tag{A.18}$$

$$P_3(t) = \frac{\rho(1+\rho)}{2(1+\rho)^2} - \frac{\rho}{2(1-\rho)}\exp(-4\varkappa t) + \frac{\rho^2}{(1+\rho)(1-\rho)}\exp(-(2\varkappa+\mu)t), \tag{A.19}$$

$$P_4(t) = \frac{\rho^2}{2(1+\rho)^2} - \frac{\rho^2}{2(1-\rho^2)}\exp(-4\varkappa t) + \left(\frac{\rho^2}{(1+\rho)^2}\left(\frac{2\rho}{(1-\rho)^2} - \frac{1}{2}\right) + \frac{2\varkappa\rho^2 t}{(1+\rho)(1-\rho)}\right) \times \exp(-(2\varkappa+\mu)t). \tag{A.20}$$

Here $P_{\text{off}} = P_1 + P_2$ and $P_{\text{on}} = P_3 + P_4$.

A.3. Erlang distributions with $k = 2$ and $l = 2$

In the case of Erlang distributions with $k = 2$ and $l = 2$ the stationary solutions are

$$P_{1s} = \frac{1+\rho}{2(1+\rho)^3},\qquad P_{2s} = \frac{1+3\rho}{2(1+\rho)^3}, \tag{A.21}$$

where $\rho = \varkappa/\mu$.

The probability of the load being "off" $P_{\text{off}} = P_1 + P_2$, and "on" are, respectively

$$P_{\text{off}} = \frac{1+2\rho}{(1+\rho)^3\}},\qquad P_{\text{on}} = \frac{\rho^3 + 3\rho^2 + \rho}{(1+\rho)^3}. \tag{A.22}$$

A.4. Erlang distributions with $k = 3$ and $l = 2$

In the case of Erlang distributions with $k = 3$ and $l = 2$ the stationary solutions are

$$P_1 = \frac{1+2\rho^2}{3(1+\rho)^4},\qquad P_2 = \frac{(1+\rho)(1+3\rho)}{3(1+\rho)^4},$$
$$P_3 = \frac{1+4\rho+6\rho^2}{3(1+\rho)^4}, \tag{A.23}$$

where $\rho = 3\varkappa/2\mu$.

The probabilities of the load being "off" $P_{\text{off}} = P_1 + P_2 + P_3$ and "on" are, respectively

$$P_{\text{off}} = \frac{3+10\rho+10\rho^2}{3(1+\rho)^4}\qquad P_{\text{on}} = \frac{2\rho+8\rho^2+12\rho^3+3\rho^4}{3(1+\rho)^4} \tag{A.24}$$

A.5. Erlang distributions with $k = 2$ and $l = 3$

In the case of Erlang distributions with $k = 2$ and $l = 3$ the stationary solutions are

$$P_1 = \frac{1+\rho}{2(1+\rho)^4},\qquad P_2 = \frac{1+4\rho}{2(1+\rho)^4}, \tag{A.25}$$

where $\rho = 2\varkappa/3\mu$.

The probabilities of the load being "off" $P_{\text{off}} = P_1 + P_2$ and "on" are respectively

$$P_{\text{off}} = \frac{2+5\rho}{2(1+\rho)^4},\qquad P_{\text{on}} = \frac{3\rho+12\rho^2+8\rho^3+2\rho^4}{2(1+\rho)^4} \tag{A.26}$$

A.6. Erlang distributions with $k = 3$ and $l = 3$

In the case of Erlang distributions with $k = 3$ and $l = 3$ the stationary solutions are

$$P_1 = \frac{(1+\rho)^2}{3(1+\rho)^5},\qquad P_2 = \frac{(1+\rho)(1+4\rho)}{3(1+\rho)^5},$$
$$P_3 = \frac{(1+5\rho+10\rho)^2}{3(1+\rho)^5}, \tag{A.27}$$

where $\rho = \varkappa/\mu$.

The probabilities of the load being "off" $P_{\text{off}} = P_1 + P_2 + P_3$ and "on" are, respectively

$$P_{\text{off}} = \frac{1+4\rho+5\rho^2}{(1+\rho)^5},$$
$$P_{\text{on}} = \frac{\rho+5\rho^2+10\rho^3+5\rho^4+\rho^5}{(1+\rho)^5}. \tag{A.28}$$

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