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Edited by

G.I.SCHUËLLER
University of Innsbruck, Austria

M.SHINOZUKA
Princeton University, New Jersey, USA

J.T.P.YAO
Texas A & M University, College Station, Texas, USA

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Combination of non-stationary rectangular wave renewal processes

R. Rackwitz
Technical University of Munich, Germany

ABSTRACT:
Suitable, widely asymptotic formulae for the failure probability under combined non-stationary rectangular wave renewal processes are derived via the outcrossing approach. Non-stationarity can exist either in the limit state function or the parameters of the stochastic models. An importance sampling scheme for the treatment of non-ergodic variables is proposed. An example illustrates some theoretical findings.

1 INTRODUCTION

Rectangular wave processes are frequently used to model the time variations of occupancy loading. Such processes can also approximate other loading phenomena and, in particular, may be used to model the main characteristics in so called missions, e.g. the journey of a ship between two places, the different sea states it experiences during the journey or the loading environment of a processing plant between the shut down periods. In some cases another more rapidly fluctuating loading process then is superimposed upon the simple rectangular wave.

Stationary rectangular wave processes have been studied repeatedly. The special case of constant durations has been proposed first and has found the earliest solutions (Ferry Borges/Castanheta, 1971; Rackwitz/Fießler, 1978). The more general model of multivariate rectangular wave renewal process and its combination has been studied by Breitung/Rackwitz (1982) and (Rackwitz, 1985) via the outcrossing approach. Interesting and practically useful generalizations of the simple rectangular wave model have been proposed by Wen (1990), Shinozuka (1981) and Schrupp/Rackwitz (1988) and others. Considerable improvement and simplification was achieved by applying asymptotic concepts, i.e. when failure probabilities tend to zero (Breitung, 1984). Breitung did not only show that computation of outcrossing rates essentially reduces to simple volume integral evaluations but also indicated that under quite general conditions the optimal asymptotic expansion point of the limit state surface is, in fact, the same as in time-invariant analysis. The asymptotic Poissonian nature of the crossings into the failure domain has already been shown earlier.

In the following an approximate computation scheme for the non-stationary crossing rates and first passage times is proposed together with some indications how to deal consistently with non-ergodic variables. Those variables distort the asymptotic nature of crossings. Therefore integration with respect to these variables has to be performed separately. Special emphasis is given to a number of non-trivial numerical problems.

2 GENERAL CONCEPTS FOR TIME VARIANT RELIABILITY

Consider the general task of estimating the probability $P_f(t)$ that a realization $z(\tau)$ of a random state vector $Z(\tau)$ enters the failure domain $V = \{z(\tau) | g(z(\tau),\tau) \leq 0, 0 \leq \tau \leq t\}$ for the first time given that $Z(\tau)$ is in the safe domain at $\tau = 0$. $g(\cdot)$ is the state function. The limit state is defined for $g(.) = 0$. $Z(\tau)$ may conveniently be separated into three components as

$$Z(\tau)^T = (R^T, Q(\tau)^T, S(\tau)^T) \quad (1)$$

where $\mathbf{R}$ is a vector of random variables independent of time, $\mathbf{Q}(\tau)$ is a slowly varying stationary and ergodic random vector sequence and $\mathbf{S}(\tau)$ is a vector of not necessarily stationary, but sufficiently mixing random process variables having fast fluctuations as compared to $\mathbf{Q}(t)$.

Consider first the case where only $\mathbf{S}(\tau)$ is present. If it can be assumed that the stream of crossings of the vector $\mathbf{S}(\tau)$ into the failure domain V is Poissonian it is well known that the failure probability $P_f(t)$ can be estimated from

$$P_f(t) \approx 1 - \exp(-\mathbb{E}[N_S^+(t)]) \leq \mathbb{E}[N_S^+(t)] \quad (2)$$

with

$$\mathbb{E}[N_{\mathbf{S}}^{+}(t)] = \int_0^t \nu_{\mathbf{S}}^{+}(\tau)\, d\tau \tag{3}$$

for high reliability problems. $\mathbb{E}[N_S^+(t)]$ is the expected number of crossings of $\mathbf{S}(\tau)$ into the failure domain V in the considered time interval [0,t] and $\nu_S^+(\tau)$ the outcrossing rate. It is assumed that there is negligible probability of failure at $\tau = 0$ and $\tau = t$, respectively. The upper bound in eq. (2) is a strict upper bound but close to the exact result only for rather small $P_f(t)$. The approximation in eq. (2) has found many applications in the past not only because of its relative simplicity but also because there has been no real practical alternative except in some special cases. It is already worth noting that for the upper bound solution there is no particular problem of integration because one needs not to distinguish between the different types of variables as introduced above (see below).

When both process variables $\mathbf{S}(\tau)$ and time invariant random variables **R** are present the Poissonian nature of outcrossings is lost. Eq. (2) can furnish only conditional probabilities. The total failure probability must be obtained by integration over the probabilities of all possible realizations of **R**. Then the equivalent to eq. (2) is

$$\begin{aligned} P_f(t) &\approx \mathbb{E}_{\mathbf{R}}[1 - \exp(-\mathbb{E}[N_{\mathbf{S}}^{+}(t|\mathbf{R})])] \\ &= 1 - \mathbb{E}_{\mathbf{R}}[\exp(-\mathbb{E}[N_{\mathbf{S}}^{+}(t|\mathbf{R})])] \\ &\leq \mathbb{E}_{\mathbf{R}}[\mathbb{E}[N_{\mathbf{S}}^{+}(t|\mathbf{R})]] \end{aligned} \tag{4}$$

In the general case where all the different types of random variables **R**, $\mathbf{Q}(\tau)$ and $\mathbf{S}(\tau)$ are present the failure probability $P_f(t)$ not only must be integrated up over the time in-variant variables **R** but an expectation operation must be performed over the slowly varying variables $\mathbf{Q}(\tau)$. In Schall et al. (1991) the following formula has been established in part by making use of the ergodicity theorem

$$\begin{aligned} P_f(t) &\approx 1 - \mathbb{E}_{\mathbf{R}}[\exp(-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N_{\mathbf{S}}^{+}(t|\mathbf{R},\mathbf{Q})]])] \\ &\leq \mathbb{E}_{\mathbf{R}}[\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N_{\mathbf{S}}^{+}(t|\mathbf{R},\mathbf{Q})]]] \end{aligned} \tag{5}$$

Eq. (5) is a rather good approximation for the stationary case but must be considered as a first approximation whenever $\mathbf{S}(\tau)$ is non-stationary or the limit state function exhibits strong dependence on τ as shown in the mentioned reference. The approximation concerns the expectation operation with respect to **Q** in the non-stationary case. The bounds given in eqs. (2, 4 and 5) again are strict but close to the exact result only for even smaller failure probabilities. In fact, while the approximation with respect to the expectation operation inside the exponent in eq. (5) may be accepted also for the non-stationary case in most practical applications the expectation with respect to **R** must be taken outside the exponent because, depending on the relative magnitude of the variabilities of the **R**- and the **S**, **Q**-variables, errors up to several orders of magnitude can occur (see Schall et al. 1990). An exact evaluation of eq. (5) may also be necessary if the failure event in eq. (5) has to be conditioned on some other event, e.g. in inspection planning. Therefore, it is of particular interest to design effective computations schemes especially in view of the fact that the **R**-vector can be high-dimensional (e.g. in stochastic finite elements with hundreds of variables describing random system properties). To be complete it ought to be mentioned that consideration of the initial and final conditions of the processes, i.e. at $\tau = 0$ and $\tau = t$, respectively, sometimes can result in noticeable improvements of the results (Plantec/Rackwitz, 1988). In the following those effects are not considered.

3 CONDITIONAL OUTCROSSING RATES FOR NON-STATIONARY RECTANGULAR WAVE RENEWAL PROCESSES

If the components of a stationary rectangular wave renewal process are independent with marks $\mathbf{S}_k$ with distribution function $F_S(\mathbf{s};\mathbf{q},\mathbf{r})$ and renewal rates λ_i it has been shown that the mean number of exits into the failure domain is (Breitung/Rackwitz, 1982)

$$\mathbb{E}[N^+(t_1,t_2;\mathbf{q},\mathbf{r})] = (t_2 - t_1) \sum_{i=1}^{n} \lambda_i\, \mathbb{P}(\{\mathbf{S}_i^- \in \bar{V};\mathbf{q},\mathbf{r}\} \cap \{\mathbf{S}_i^+ \in V;\mathbf{q},\mathbf{r}\}) \tag{6}$$

where $\bar{V}$ and V are the safe and failure domain, respectively. $\mathbf{S}_i^+$ is the total load vector when the i-th component of the renewal process had a renewal. $\mathbf{S}_i^-$ denotes the total load vector just before the renewal. Therefore, $\mathbf{S}_i^-$ and $\mathbf{S}_i^+$ differ by the vector $\mathbf{S}_i$ which is to be introduced as an independent vector in the second set. Applying asymptotic concepts and using eq. (6) it can further be shown that asymptotically (Breitung 1984)

$$E[N^+(t_1,t_2;\mathbf{r},\mathbf{q})] \sim (t_2 - t_1) \sum_{i=1}^{n} \lambda_i\, \mathbb{P}(\{\mathbf{S} \in V;\mathbf{q},\mathbf{r}\}) \tag{7}$$

with $\mathbb{P}(\{\mathbf{S} \in V;\mathbf{r},\mathbf{q}\})$ computed as a volume integral in the usual manner by SORM. Very rarely this formula is noticeably improved for not small probabilities $\mathbb{P}(\{\mathbf{S} \in V;\mathbf{r},\mathbf{q}\})$ by replacing the term $\mathbb{P}(\{\mathbf{S} \in V;\mathbf{r},\mathbf{q}\})$ by

$$\begin{aligned} &\mathbb{P}(\{\mathbf{S}_i^- \in \bar{V};\mathbf{q},\mathbf{r}\} \cap \{\mathbf{S}_i^+ \in V;\mathbf{q},\mathbf{r}\}) = \\ &= \mathbb{P}(\{\mathbf{S} \in V;\mathbf{r},\mathbf{q}\}) - \mathbb{P}(\{\mathbf{S}_i^- \in V;\mathbf{r},\mathbf{q}\} \cap \{\mathbf{S}_i^+ \in V;\mathbf{q},\mathbf{r}\}) \end{aligned}$$

as in eq. (6). Note that integration with respect to **q** is performed simultaneously with the integration with respect to **s**. If unconditional mean numbers of exits need to be computed as in eq. (5) integration is also over **r** (see below). If there is complete dependence of jump events for subsets of wave processes summation in eq. (6) or (7) is only over the independent components.

The non-stationary case is not substantially more difficult. The renewal rates $\lambda_k(\tau)$, k = 1,2,..., are assumed to vary slowly in time. The distribution function of **S** may contain distribution parameters $\mathbf{r}(\tau)$ varying in time and the failure domain can be a function of time, i.e. $V = \{g(\mathbf{s},\mathbf{q},\mathbf{r},\tau) \leq 0\}$. Then, eq. (6) needs to be modified as

$$\begin{aligned} \mathbb{E}[N^+(t_1,t_2|\mathbf{r})] &\sim \int_{t_1}^{t_2} \sum_{i=1}^{m} \lambda_i(\tau)\, \mathbb{P}(\{\mathbf{S} \in V)|\mathbf{q},\mathbf{r},\tau\})\, d\tau \\ &= \int_{t_1}^{t_2} \int_V \sum_{i=1}^{n} \lambda_i(\tau)\, f_{\mathbf{S},\mathbf{Q}}(\mathbf{s},\mathbf{q},\tau|\mathbf{r})\, d\mathbf{s}\, d\mathbf{q}\, d\tau \end{aligned} \tag{8}$$

The time-volume integral (8) can be approximated using FORM/SORM concepts. For small $\mathbb{P}(\{\mathbf{S} \in V|\mathbf{q},\mathbf{r},\tau\})$ the integrand is dominated by the probability term in the neighborhood of the most likely failure point $(\mathbf{s}^*,\mathbf{q}^*,\tau^*)$ in $\{\mathbf{S} \in V|\mathbf{q},\mathbf{r},\tau\}$ to be determined by an appropriate algorithm. One such algorithm has been proposed by Abdo/Rackwitz (1991). Here and in the following integrations are best performed in the standard space after applying a suitable probability distribution transformation. Because $\lambda_k(\tau)$ is slowly varying it is drawn in front of the integral with value $\lambda_k(\tau^*)$ by virtue of the mean value theorem of integral theory. Classical FORM/SORM algorithms can then be applied according to (Hagen/Tvedt, 1991) after transforming the integral in eq. (8) into a simple probability integral by introducing an additional uniform density $f_T(\tau) = (t_2 - t_1)^{-1}$ into eq.(8) such that

$$\begin{aligned} \mathbb{E}[N^+(t_1,t_2|\mathbf{r})] = (t_2 - t_1) \sum_{k=1}^{m} \Big[\lambda_i(\tau^*) \times \\ \times \int_{\mathbb{R}^1} \int_V f_{\mathbf{S},\mathbf{Q}}(\mathbf{s},\mathbf{q},\tau|\mathbf{r})\, f_T(\tau)\, d\mathbf{s}\, d\mathbf{q}\, d\tau \Big] \end{aligned} \tag{9}$$

With the transformation $\tau = (t_2 - t_1)\,\Phi(u_\tau)$ the probability integral can be determined in the usual manner. $\Phi(.)$ is the standard normal integral. The results turn out to be quite accurate whenever τ^* lies within a large interval $[t_1,t_2]$ and the associated norm $\|\mathbf{u}\|$ of the standard uncertainty vector is large and thus the asymptotic conditions are met. For smaller intervals $[t_1,t_2]$ the results become less accurate. They are no more acceptable in general when τ^* is a boundary point.

Therefore, a slightly different scheme is advantageous in all cases. In the critical point $(\mathbf{s}^*,\mathbf{q}^*,\tau^*)$ the probability $\mathbb{P}(\{\mathbf{S} \in V|\mathbf{q},\mathbf{r},\tau\})$ is estimated as

$$\mathbb{P}(\{\mathbf{S} \in V|\mathbf{q},\mathbf{r},\tau\}) = \Phi(-\beta(\tau^*)) \times C(\mathbf{s}^*,\mathbf{q}^*,\tau^*|\mathbf{r}) \tag{10}$$

with $C(\mathbf{s}^*,\mathbf{q}^*,\tau^*)$ the well known curvature correction term (in the **s**-**q**-space) in SORM. Then, one can write

$$\begin{aligned} &\mathbb{E}[N^+(t_1,t_2|\mathbf{r})] \\ &\sim \int_{t_1}^{t_2} \sum_{i=1}^{n} \lambda_i(\tau|\mathbf{r})\, \Phi(-\beta(\tau|\mathbf{r})) \times C(\mathbf{s}^*,\mathbf{q}^*,\tau|\mathbf{r})\, d\tau \\ &\approx C(\mathbf{s}^*,\mathbf{q}^*,\tau^*|\mathbf{r}) \sum_{i=1}^{n} \lambda_i(\tau^*|\mathbf{r}) \int_{t_1}^{t_2} \Phi(-\beta(\tau|\mathbf{r}))\, d\tau \\ &= C(\mathbf{s}^*,\mathbf{q}^*,\tau^*|\mathbf{r}) \times \\ &\quad \times \sum_{i=1}^{n} \lambda_i(\tau^*|\mathbf{r}) \int_{t_1}^{t_2} \exp[\ln[\Phi(-\beta(\tau|\mathbf{r}))]]\, d\tau \\ &= C(\mathbf{s}^*,\mathbf{q}^*,\tau^*|\mathbf{r}) \sum_{i=1}^{n} \lambda_i(\tau^*|\mathbf{r}) \int_{t_1}^{t_2} \exp[f(\tau)]\, d\tau \end{aligned} \tag{11}$$

where $f(\tau) = \ln[\Phi(-\beta(\tau))]$. The time integral in eq. (11) is perfectly suited for application of Laplace's integral approximation. Expanding $f(\tau)$ to first and second order with derivatives

$$f'(\tau) = -\frac{\varphi(-\beta(\tau))}{\Phi(-\beta(\tau))} \frac{\partial\beta(\tau)}{\partial\tau} \approx -\beta(\tau) \frac{\partial\beta(\tau)}{\partial\tau}$$

$$\begin{aligned} f''(\tau) = -\frac{\varphi(-\beta(\tau))}{\Phi(-\beta(\tau))} \Bigg[\left[\frac{\partial\beta(\tau)}{\partial\tau}\right]^2 \times \\ \times \left[\beta(\tau) + \frac{\varphi(-\beta(\tau))}{\Phi(-\beta(\tau))}\right] + \frac{\partial^2\beta(\tau)}{\partial\tau^2} \Bigg] \end{aligned}$$

yields integrals which have analytical solutions. While $\partial\beta/\partial\tau$ is directly obtained as a parametric sensitivity the second derivative $\partial^2\beta(\tau)/\partial\tau^2$ must be determined numerically by a simple difference scheme. The results are

- $\tau^* = t_1$ or $\dfrac{\partial\beta(\tau)}{\partial\tau} > 0$:

$$\begin{aligned} \mathbb{E}[N^+(t_1,t_2|\mathbf{r})] \approx C(\mathbf{s}^*,\mathbf{q}^*,t_1|\mathbf{r}) \sum_{i=1}^{n} \lambda_i(t_1|\mathbf{r})\, \Phi(-\beta(t_1|\mathbf{r})) \\ \times \left\{ \frac{\exp[f'(t_1)\, t_2] - \exp[f'(t_1)\, t_1]}{\exp[f'(t_1)\, t_1]\; f'(t_1)} \right\} \end{aligned} \tag{12a}$$

- $\tau^* = t_2$ or $\frac{\partial\beta(\tau)}{\partial\tau} < 0$:

$$\mathbb{E}[N^+(t_1,t_2|\mathbf{r})]$$
$$\approx C(\mathbf{s}^*,\mathbf{q}^*,t_2|\mathbf{r}) \sum_{i=1}^{n} \lambda_i(t_2|\mathbf{r})\, \Phi(-\beta(t_2|\mathbf{r})) \times$$
$$\times \left\{\frac{\exp[f'(t_2)\,t_2] - \exp[f'(t_2)\;t_1]}{\exp[f'(t_2)\,t_2]\;f'(t_2)}\right\} \qquad (12b)$$

- $t_1 < \tau^* < t_2$ or $\frac{\partial\beta(\tau)}{\partial\tau} = 0$ and $\frac{\partial^2\beta(\tau)}{\partial\tau^2} > 0$:

$$\mathbb{E}[N^+(t_1,t_2|\mathbf{r})]$$
$$\approx C(\mathbf{q}^*\mathbf{s}^*,\tau^*|\mathbf{r}) \sum_{i=1}^{n} \lambda_i(\tau^*|\mathbf{r})\, \Phi(-\beta(t_1|\mathbf{r}))$$
$$\times \left[\frac{2\pi}{|f''(\tau^*)|}\right]^{1/2} \times$$
$$\times \left\{\Phi(|f''(\tau^*)|^{1/2}(t_2 - \tau^*)) - \Phi(|f''(\tau^*)|^{1/2}(t_1 - \tau^*))\right\} \qquad (12c)$$

If the conditions for eq. (12c) are met but the critical point is at one of the boundaries the mean number of outcrossings is just one half of the value in eq. (12c) (see Bleistein/Handelsman, 1986). The modifications in eqs. (12) result in more accurate exit means than eq. (9) especially for smaller time distances $t_2 - t_1$ although the interaction between τ and the other variables is neglected. A genuine first order result does not exist because time integration always is an approximation in the second order sense. In many applications the computation of the correction factor $C(\mathbf{s}^*,\mathbf{q}^*,\tau^*|\mathbf{r})$ involving the second order derivatives in the **s**, **q**-space will yield only small improvements, however. Of course, some conditions must be met for the validity of the approach. In particular, there should be $\beta(\tau^*) > 1$ and $\mathbb{E}[N^+(t_1,t_2|\mathbf{r})] \ll 1$.

4 INTEGRATION WITH RESPECT TO TIME-INVARIANT VARIABLES R

If there are time-invariant random vectors **R** several possibilities exist the most straightforward being numerical integration. However, even for small dimensions of **R** the computational effort can be considerable. So called nested FORM/SORM has been proposed as an alternative. Unfortunately, it turned out to be rather time consuming and not reliable in the non-stationary case. Also a simple first order Taylor expansion of the exponent in eq. (1) or application of a standard (inner point) result of Laplace's integral approximation method has been found to be not sufficiently accurate at least if the expansion point $\mathbf{r}^*$ is not exactly the critical point. A first approximation for this point can be obtained from one of the upper bound solutions as in eq. (5). The exact location would require some iteration which now is quite involved as the eqs. (12) must be solved in each iteration step. Even then Laplace' s solution still contains an error which has been found to become quite large in extreme cases.

Alternatively and arbitrarily exact at increasing numerical effort, the expectation operation in eq. (5) can be performed either by crude Monte Carlo integration or, more efficiently, by importance sampling. For convenience it is assumed that R is an uncorrelated standardized Gaussian vector which can always be achieved by a suitable probability distribution transformation. There is

$$\mathbb{E}_{\mathbf{R}}[1 - \exp\{-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{R})]]\}]$$
$$= \int_{\mathbb{R}^{n_r}} \left[1 - \exp\{-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{r})]]\}\right] \frac{\varphi_{\mathbf{R}}(\mathbf{r})}{h_{\mathbf{R}}(\mathbf{r})}\, h_{\mathbf{R}}(\mathbf{r})\, d\mathbf{r} \qquad (13)$$

where $h_{\mathbf{R}}(\mathbf{r})$ is the sampling density. Then,

$$\mathbb{E}_{\mathbf{R}}[1 - \exp(-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{R})]])]$$
$$\approx \frac{1}{N}\sum_{i=1}^{N} \left[1 - \exp\{-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{r}_i)]]\}\right] \frac{\varphi_{\mathbf{R}}(\mathbf{r}_i)}{h_{\mathbf{R}}(\mathbf{r}_i)} \qquad (14)$$

The sampling density (standard space) can conveniently be chosen as the standard normal density with mean $\mathbf{r}^*$ from the upper bound solution eq. (5) and covariance matrix **I**. A crude conservative estimate is already obtained for $\mathbf{r}_i = \mathbf{r}^*$. From extensive testing of examples it is concluded that the scheme according to eq. (14) is rather robust provided that an efficient and reliable algorithm is available to locate the critical point $(\mathbf{s}^*,\mathbf{q}^*,\tau^*)$ for every simulated **r**.

5 HAZARD RATES

The hazard function as an additional useful reliability characteristic for time-variant reliability problems can also be computed. By FORM/SORM the hazard function is computed simply from parametric sensitivities. It is

$$h(t) = \frac{f_T(t)}{1 - P_f(t)} = \frac{1}{1 - P_f(t)}\frac{\partial P_f(t)}{\partial t}$$
$$= \frac{\varphi(\beta(t))}{\Phi(\beta(t))}\frac{\partial\beta(t)}{\partial t} \qquad (15)$$

by assuming that the considered time interval is $[0,t]$. $\beta(t)$ is to be interpreted as the equivalent reliability index $\beta(t) = \Phi^{-1}(P_f(t))$. The foregoing theory allows to compute these hazard rates rigorously if the parametric sensitivities $\partial\beta(t)/\partial t$ are available. Unfortunately, this can be done only at the expense of some additional numerical effort.

However, good approximations are possible if the computation is for the critical point τ^* only. For a critical inner point the hazard rate then remains essentially constant even if there are parameter variations. Rewriting eq. (5) as

$$P_f(t) = \mathbb{E}_{\mathbf{R}}[1 - \exp(-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{R})]])]$$
$$= \int_{\mathbb{R}^{n_r}} \left[\varphi_{\mathbf{R}}(\mathbf{r}) - \frac{1}{(2\pi)^{n_r/2}}\right.$$
$$\left.\times \exp(-\mathbb{E}_{\mathbf{Q}}[\mathbb{E}[N^{\pm}_{\mathbf{S}}(t|\mathbf{Q},\mathbf{r})]] - \frac{1}{2}\mathbf{r}^T\mathbf{r})\right] d\mathbf{r} \qquad (16)$$

and formal application eq. (15) yields

$$h(t) \approx \exp\left[-\frac{1}{2}\mathbf{r}^{*T}\mathbf{r}^*\right] \frac{\exp\left[-E_Q[E[N^+(t)]]\right]}{1 - P_f(t)} \times$$
$$\times E_Q[\nu^+(\tau^*)] \qquad (17)$$

An estimate of the (critical) conditional outcrossing rate, of course, is

$$\nu^+(\tau^*|\mathbf{r}^*) = E_Q[\nu^+(\tau^*|\mathbf{r}^*)] \qquad (18)$$

and the unconditional outcrossing rate can be obtained from

$$\nu^+(\tau^*) = \exp\left[-\frac{1}{2}\mathbf{r}^{*T}\mathbf{r}^*\right] E_Q[\nu^+(\tau^*)] \qquad (19)$$

Eqs. (17) to (19) rest on the assumption that the r-variables have only small variability as compared with the q- and s-variables. Therefore, the hazard rate as computed by eq. (17) or (19) must be considered as a crude approximation whenever the r-variables dominate. According to Krzykacs/Kersken-Bradley (1976) the hazard rate should decrease in the stationary case.

6 DISCUSSION

It should be noted first that the rectangular wave renewal process model always yields slightly larger failure probabilities than the simpler model of rectangular waves with constant durations and integer ratios between the so called repetition numbers. This is due to the partial or complete removal of jump dependencies in the latter case.

The above developments concentrated on various bounds and then on certain approximations. The motivation for the approximations was to keep the numerical effort as small as possible especially in high dimensional spaces. All approximations are based on some information from an upper bound solution. Its accuracy is practically always sufficient if the resulting failure probability is small, say, smaller than 10^{-3} the reason simply being the fact that then $1 - \exp[-x] \approx x$ in very good approximation. This is true even when the non-ergodic variables dominate. It should be clear that any upper bound solution is an upper bound solution only under the condition that the mean number of upcrossings can be computed exactly. This is, in principle, possible by adding to FORM or even SORM a suitable importance sampling scheme in the usual manner, i.e. the scheme proposed by Hohenbichler/Rackwitz (1988). For small failure probabilities very accurate results are thus obtained. For larger failure probabilities an "exact" computation of the upper bound makes hardly sense. If the mean number of upcrossings is found only by FORM (and sometimes also by SORM) the upper bound is no more strict. Remember that the time integration is always a SORM solution according to the approximate scheme followed for the derivation of eqs. (12). Whenever the dimension of the **r**-vector is large a complete SORM solution may be out of reach due to the considerable effort required for determination of the full Hessian matrix. Therefore, it must be recommended to compute the "upper bound solution" with the simplest method possible in this case.

Eqs. (12) can be evaluated strictly by FORM or SORM in the **q**,**s**-space but it should be noted that asymptotic concepts are applied in both cases when time-variant jump rates are involved (see transition from eq. (8) to (9)).

In applications with large non-ergodic uncertainties the integration over **r** is most critical for the accuracy of the final result. Surprisingly, the point $\mathbf{r}^*$ obtained for the upper bound solution according to eq. (5) can be remarkably away from the correct one in extreme cases (i.e. for large failure probabilities). It may, therefore, be advisable to use some adaptive importance sampling scheme in those cases. Experience with such schemes showed, however, that the gain in required sampling points is rather modest in the cases investigated. The reason appears to be a rather large domain in the **r**-space where considerable contributions to the integral are obtained. In this case it is clear that a sampling distribution with relatively large spread can also account for the inaccuracy in the central point.

7 ILLUSTRATION

For the purpose of illustration we take the following simple function of structural states.

$$g(t,p,R,Q,S) = p_1(1 + p_2 t + p_3 t^2)\frac{\pi}{4}R_1 R_2^2$$
$$- (p_4 S_1 + p_6 S_2 + p_7 R_3)$$

with the p_i certain deterministic constants and a fixed time window of [0, 2]. The stochastic model and the parameters are summarized in table 1.

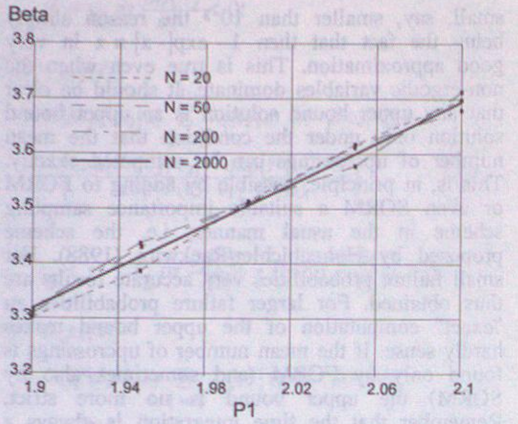

Figure 1: Convergence of equivalent reliability index with number of sampling points versus parameter p_1

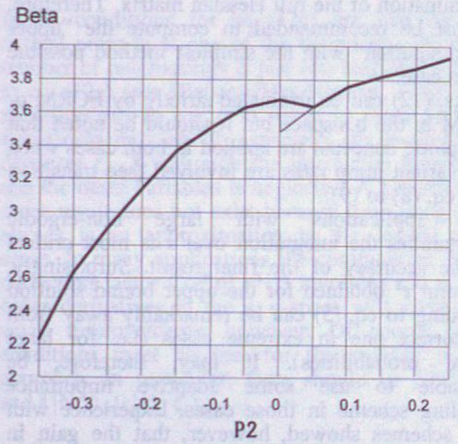

Figure 2: Transition from interior to boundary point solution with increasing parameter p_1

It is seen that this example includes almost all types of non-stationarities and dependencies which one can imagine to occur in practice. By variation of the parameters any kind degree of non-stationarity can be produced. Letting p_2 running between - 0.35 and + 0.25 will change the nature of the solution at $p_2 \approx 0.0$ from an interior point solution to a boundary point solution. It should be clear from the theoretical developments that there can be correlations between **r**-variables and **q**-variables, respectively, but no cross correlations. Also, there may be dependencies of each of the **s**-variable via its distribution parameters on **q**- and possibly **r**-variables (e.g. for taking account of statistical uncertainties) but all **q**-variables must be conditionally independent of each other. Figure 1 gives an impression of the convergence of the simulations for the **r**-variables towards the correct solution for various p_1. It is seen that roughly 50 simulations are already sufficient in this case. Figure 2 with only N = 20 illustrates what happens when the interior point solution switches over to a boundary point solution for running parameter p_2. Typically, the failure probability jumps by roughly a factor of two near $p_2 \approx 0.0$. Around this point both solutions are not very reliable. Figure 3 finally compares the first order and second order upper bound solution with the second order solution in the **q**,**s**-space and with number of samples for the **r**-integration of N = 500, which brings the coefficient of variation for the resulting failure probability down to less than 3 %. It is seen that the differences are quite small. The first order upper bound is not a strict upper bound over the whole parameter domain.

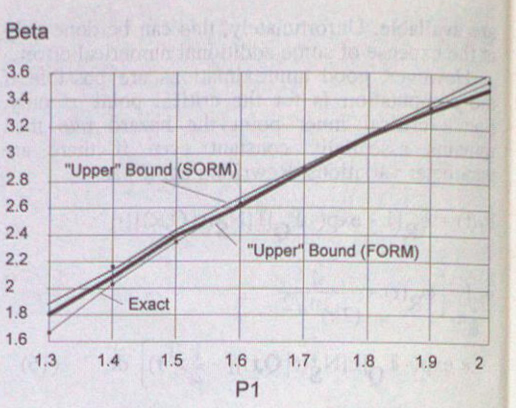

Figure 3: First order upper bound solution and simulation versus parameter p_1

8 SUMMARY AND CONCLUSIONS

Conditional non-stationary crossing rates for rectangular renewal wave processes are computed in part making use of asymptotic concepts. Hereby, the parameters of the wave processes may depend on slowly fluctuating random and ergodic sequences. Numerical analysis is performed by applying classical FORM/SORM concepts. For time integration a separate computational step is proposed. Some effort is spent to remove the conditioning by simple random variables by appropriate numerical schemes. It is found that a importance sampling scheme is the by far most satisfying numerical method in the stationary as well as the non-stationary case whenever failure probabilities are not small. For small failure probabilities the upper bound solution always is sufficiently accurate for practical purposes. It, nevertheless, must be admitted that time-variant reliabilities can be considerably more difficult and laborious to determine by FORM/SORM than time-invariant reliabilities.

Table 1: Stochastic Model for Example

Name	Distribution	Mean Value	Std. Deviation	Jump Rates
R_1	Lognormal	2.0	0.2	-
R_2	Lognormal	3.0	0.3	-
R_3	Gumbel	6.0	0.6	-
Q_1	Lognormal	5.0	0.5	-
S_1	Normal	10.0	$Q_1 (1 + p_5 t)$	10.0
S_2	Gumbel	8.0	3.0	$p_8 (1 + p_9 t)$
p_1	2.000	Ultimate resistance multiplier		
p_2	-0.100	Time multiplier for linear term		
p_3	0.100	Time multiplier for quadratic term		
p_4	0.330	Load multiplier for first life load		
p_5	0.010	Time multiplier for standard deviation function $\sigma = Q_1 (1 + p_5 t)$ for Q_1		
p_6	0.330	Load multiplier for second life load		
p_7	0.330	Load multiplier for dead load		
p_8	0.100	Constant parameter of jump rate function $\lambda_1 = p_8 (1 + p_9 t)$ for Q_2		
p_9	1.000	Linear parameter of jump rate function $\lambda = p_8 (1 + p_9 t)$ for Q_2		

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