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On predictive distribution functions for the three asymptotic extreme value distributions *

S. Engelund and R. Rackwitz

Technische Universität München, Laboratorium für den Konstruktiven Ingenieurbau, Arcisstr. 21, D-8000 München 2, Germany

Abstract. This technical note is concerned with the estimation of parameters in the three extreme value distributions and the quantification of the statistical uncertainty. A Bayesian approach is used. The predictive distribution functions are determined. It is shown, that the predictive distribution functions which are determined on the basis of the noninformative priors do not lead to rational decision rules. Priors which lead to reasonable decision rules are suggested. These are, however, not noninformative.

Key words: extreme value distributions; predictive distribution; Bayesian analysis.

Extreme value distributions frequently are used when studying the variability of loading and resistance variables in structural reliability. Their parameters in general must be estimated from a limited number of observations. Therefore, it is usually necessary to consider the statistical uncertainties involved. It has been widely accepted in structural reliability, that a Bayesian approach is the most appropriate to deal with statistical uncertainties. In the following note we draw attention to a peculiarity present in the Bayesian analysis of extreme value distributions. The three extreme value distributions are given in Table 1 for easy reference.

Assume that the location parameters u, v and w in the Gumbel, Frechet and Weibull distributions, respectively, are unknown. Let n experiments be performed in order to determine the unknown parameter. The effect of statistical uncertainty can be quantified in terms of the predictive distribution function whose density is

$$f_Y(x) = \int_{\Theta} f_X(x \mid \theta) f_\theta(\theta) \, \mathrm{d}\theta \tag{1}$$

where θ is defined in the region Θ. Schrupp and Rackwitz [1] have shown that the following statistics are sufficient statistics for the parameters u, v and w, respectively:

$$r = \sum_{i=1}^{n} \exp[-\alpha x_i], \qquad s = \sum_{i=1}^{n} (x_i - \tau)^{-k}, \qquad t = \sum_{i=1}^{n} (x_i - \tau)^{h}$$

where x_i is the outcome of an experiment. When a sufficient statistic exists a conjugate prior

* Discussion is open until May 1993 (please submit your discussion paper to the Editor, Ross B. Corotis).

TABLE 1

The asymptotic extreme value distributions

Gumbel (maxima)	$f_X(x \mid u) = \alpha \exp[-\alpha(x-u) - \exp[-\alpha(x-u)]]$ $F_X(x \mid u) = \exp[-\exp[-\alpha(x-u)]]$
Frechet (maxima)	$f_X(x \mid v) = \frac{k}{v}(\frac{v}{x-\tau})^{k+1}\exp[-(\frac{v}{x-\tau})^k]$ $F_X(x \mid v) = \exp[-(\frac{v}{x-\tau})^k]$
Weibull (minima)	$f_X(x \mid w) = \frac{h}{w}(\frac{x-\tau}{w})^{h-1}\exp[-(\frac{x-\tau}{w})^h]$ $F_X(x \mid w) = 1 - \exp[-(\frac{x-\tau}{w})^h]$

exists, too (see Box and Tiao [2]). As priors for the parameters u, v and w the following functions can be proposed:

$$f'(u) \propto \exp(p\alpha u), \qquad f'(v) \propto v^{-q}, \qquad f'(w) \propto w^{-m}$$

where p, q and m are constants. The priors are improper in the sense that they integrate to infinity. They correspond to noninformative priors (see Box and Tiao [2] and Zellner [3]) when $p = 0.0$ and $q = m = 1.0$. The prior for u is noninformative in the sense that

TABLE 2

Conjugate priors and predictive functions (Γ denotes the gamma-function)

Gumbel	$f'_U(u) = \frac{\exp[(n+p)\alpha u - r\exp[\alpha u]]\alpha r^{n+p}}{\Gamma(n+p)}$ $f_Y(x) = (n+p)\frac{\alpha}{r}\exp[-\alpha x](1 + \frac{\exp[-\alpha x]}{r})^{-n-1-p}$ $F_Y(x) = (1 + \frac{\exp[-\alpha x]}{r})^{-n-p}$
Frechet	$f'_V(v) = \frac{v^{nk-q}\exp[-v^k S]ks^{n+(1-q)/k}}{\Gamma(n+(1-q)/k)}$ $f_Y(x) = (n + \frac{1-q}{k})\frac{k}{s}(\frac{(x-\tau)^{-k}}{s} + 1)^{-n-1-(1-q)/k}(x-\tau)^{-k-1}$ $F_Y(x) = (1 + \frac{(x-\tau)^{-k}}{s})^{-n-(1-q)/k}$
Weibull	$f'_w(w) = \frac{w^{-nh-m}\exp[-w^{-h}t]ht^{n+(m-1)/h}}{\Gamma(n+(m-1)/h)}$ $f_Y(x) = (n + \frac{m-1}{h})n\frac{h}{t}(\frac{(x-\tau)^h}{t} + 1)^{-n-1-(m-1)/h}(x-\tau)^{h-1}$ $F_Y(x) = 1 - (1 + \frac{(x-\tau)^h}{t})^{-n-(m-1)/h}$

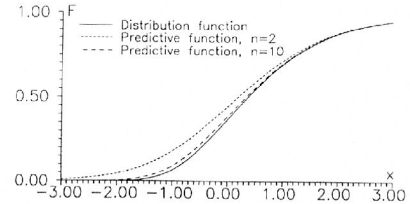

Fig. 1. Gumbel distribution ($\alpha = 1.0$, $u = 0.0$, $p = 0.0$).

$P(a < u < b)/P(c < u < d) = 0/0$ is indeterminate when a, b, c and d are finite numbers and the prior for v (or w) is noninformative in the sense that $P(v < a)/P(v > a)$ is indeterminate. Note that not even the first moment of these priors exist as one should expect. The conjugate priors and the corresponding predictive density and distribution functions are shown in Table 2.

Assume that samples are taken reproducing the initial values of the parameters, and that the predictive distribution functions are determined on basis of the noninformative priors. In Fig. 1 it is seen, that the predictive distribution function for the Gumbel distribution decreases as the number of experiments increases even for large values of $F_Y(\ )$. A similar behavior can be observed for the Frechet distribution. On the contrary for the Weibull distribution the predictive distribution function increases as the number of experiments increases. This makes no sense if the Gumbel or Frechet distribution are used to model loads or the Weibull distribution is adopted for resistances. With such choices for the priors erroneous conclusions would be drawn with regard to the effect of statistical uncertainties and the sample size required in an investigation. Similar findings have been discovered elsewhere in Bayesian analysis. In fact noninformative priors cannot always be used as a basis for rational decisions and Berger [4] concludes: "... even unanimously acclaimed noninformative priors (such as those for location parameters or scale parameters) can lead to inferior decision rules".

In the context of reliability applications one would require that $F_Y(x) > F_X(x \mid \theta)$ for small values of x and $F_Y(x) < F_X(x \mid \theta)$ for large values of x. Furthermore, one would require that an equation

$$F_X(x \mid \theta) = F_Y(x) \tag{2}$$

holds for some value $x = x^*$ independent of n. x^* may be selected as the point where $f_X(x \mid \theta)$ takes on its maximum value but other choices such as the distribution median or the

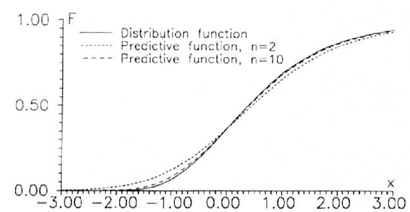

Fig. 2. Gumbel distribution ($\alpha = 1.0$, $u = 0.0$, $p = 0.46$).

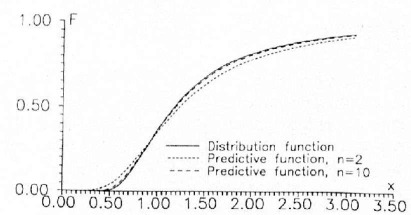

Fig. 3. Frechet distribution ($\tau = 0.0$, $k = 2.3$, $q = -0.27$).

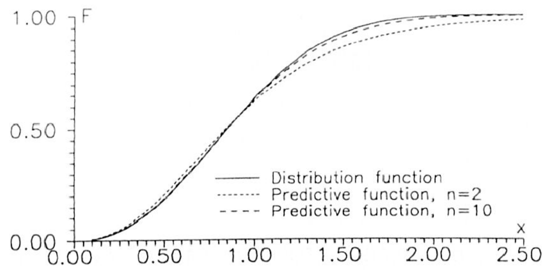

Fig. 4. Weibull distribution ($\tau = 0.0$, $h = 2.3$, $m = 1.87$).

mean are also possible. For the Frechet distribution with x^* the maximum point the following equation must then be fulfilled

$$\exp\left[-\frac{k+1}{k}\right] = \left[1 + \frac{k+1}{nk}\right]^{-n-(1-q)/k} \tag{3}$$

When k and q are constants this equation always holds for n approaching infinity but it is impossible to determine q such that the equation holds independent of n and k. It is, however, possible to find approximate solutions. The simplest approximation inferred from a systematic numerical analysis with respect to the parameters involved is

$$q = 1 - 0.55k, \quad k > 1.0 \tag{4}$$

The predictive distribution function which is determined on the basis of this prior is shown in Fig. 3. It is seen that the conjugate prior in Table 2 with eqn. (4) now leads to reasonable decision rules. For the Gumbel and Weibull distribution one analogously obtains the following approximations:

$$p = 0.46 \tag{5}$$

$$m = 1 + 0.38h, \quad h > 1.0 \tag{6}$$

The predictive distribution functions for the Gumbel and Weibull distributions are shown in Fig. 2 and 4 respectively.

It must be emphasized that these choices for the parameters in the prior distribution correspond to not noninformative priors and the values in eqns. (4) to (6) just correspond to the least informative prior. The degree of noninformativeness depends on the value of the respective nuisance parameters.

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