PROBABILITY OF FAILURE OF BRITTLE REDUNDANT STRUCTURAL SYSTEMS IN TIME

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ABSTRACT

The probabilistic description of intermediate damage and ultimate collapse states and their evolution in time of brittle, redundant structure systems subject to Gaussian loading is studied. A time-dependent failure tree is developed. Single or multiple component failure probabilities are determined by the upcrossing approach using extensively modern FORM / SORM techniques for the necessary probability integrations. Two limiting internal load redistribution regimes, delayed load redistribution and immediate load redistribution eventually leading to progressive failures, are studied. The methodology is especially designed and suited for reliability problems connected with inspection and maintenance of structures.

1. INTRODUCTION

Brittle redundant structural systems subject to time-variant loading do not fail until all redundant elements in a set of componential (elemental) failures necessary for structural collapse have occurred. Since there are many such possible sequences to system collapse (differing in the

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number, ordering and grouping of the elements), the structure fails in one of many sequences, the union of which constitutes the structural failure event. More precisely, a structural element is said to fail when its capacity is exceeded by the demand. This event is immediately followed by a change in the mechanical properties of the failing element. In the case of ideal brittleness the element does not carry any load after failure. Depending on the properties of the system and its loading, elemental failure then causes either an immediate or a somewhat delayed redistribution of internal forces, thus changing and, in general, increasing the demand on the remaining elements. Sometimes dynamic effects need to be considered. The redundant structural elements now can fail during essentially the same loading cycle before, during or after load redistribution, or one by one in different loading cycles. This is the scenario which is the subject of this study. Certain basic tools for its logical and probabilistic analysis will be developed.

System reliability analyses so far have primarily been directed towards structural failure under extreme conditions. The shortcomings of such formulations are, first of all, the implied assumption of immediate redistribution of internal forces during the realisation of an extreme loading event. Furthermore, they do not offer the possibility to quantify intermediate structural damage states and, therefore, there is no possibility to account for results of inspections which generally are carried out during the time of use of the structure. The goal of the reliability formulation presented herein is to describe the evolution of the damage states of the system in time. The probabilistic characteristics of the times to single or multiple componential failure, the times of sequences of sets of componential failures and, finally, the time to structural collapse are determined. With such information in hand it might be possible to device appropriate inspection and repair strategies to restore, maintain or even improve the reliability of an existing structural system.

Although certain aspects of the subsequent developments have already been discussed and, in part, used elsewhere, e.g. [1–4], the basic assumptions and idealisations of the formulations are presented in detail. The formulation is, essentially, an upcrossing approach with special consideration of multiple level crossings applied to a time-variant failure tree in conjunction with an extensive use of modern FORM/SORM techniques [5] for the non-trivial numerical part.

2. BASIC MECHANICAL AND STOCHASTIC ASSUMPTIONS

Assume a linearly elastic, statically reacting, redundant truss or frame structure with M critical control points (hot spots, cross-sections, bars, etc.) denoted herein as "elements" where ideal brittle failure can occur. The structure is exposed to a stationary and ergodic Gaussian scalar load process $L(\tau)$ with mean m_L, covariance functions $c_L(|\tau_1 - \tau_2|)$ and with continuously differentiable sample paths. Furthermore, the load process is assumed to be not too broad-banded. This is, for example, an appropriate model for the dominant load due to waves on offshore structures in a given, critical sea state. The usually relatively small scale of fluctuation determines the time unit of the reliability problem. The vector $\boldsymbol{S}^l(\tau)$ of the scalar load effects in a structural state l of degradation can then be given by

$$\boldsymbol{S}^l(\tau) = \boldsymbol{B}^l L(\tau) \tag{1}$$

where $\boldsymbol{B}^l$ is a vector of transfer coefficients b_m^l. They are assumed as deterministic and time-invariant. Consequently, the dynamic effects caused by brittle componential failure are neglected. Each load effect $S_m^l(\tau)$ of $\boldsymbol{S}^l(\tau)$ has stochastic properties which can easily be determined from those of $L(\tau)$.

The failure event of the mth critical element can be associated with the first upcrossing of the resistance by the load effect. Thus, the failure domain is defined by

$$V_m^l = \{s_m^l: r_m - s_m^l \leq 0\} \tag{2}$$

where r_m denotes the capacity against load effects s_m^l. In general, the resistance r_m is the realisation of a random variable R_m which, here, is taken as a time-invariant random variable with given distribution function. These variables will be collected in the random vector $\boldsymbol{R}$ in the following. Consider a given state l. For simplification of notation the corresponding index is omitted. V_m can be transformed into the space of the standardized load process $X(\tau) = (L(\tau) - m_L)/\sigma_L$ (σ_L = standard deviation of the load process) by introducing the reduced random resistance:

$$U_m = \frac{R_m - b_m m_L}{b_m \sigma_L} \tag{3}$$

i.e.

$$V_m = \{x: u_m - x \leq 0\} \tag{4}$$

In this manner, it is possible to express in each structural state all the domains V_m by means of thresholds r_m in the same space where the sample paths of the reduced load process are defined.

Given a realisation r_m of R_m, an elemental failure event is completely determined by the distribution function of the time T_m to the first entrance of $X(\tau)$ into V_m, i.e.:

$$F_{T_m}(t) = P(T_m \leq t) \tag{5}$$

The distribution of the first-passage time of a random process into a given domain is known exactly only for a few special types of processes. If, however, the exits are rare events or, equivalently, u_m is relatively large and the exits become independent events, a well-known asymptotic result holds [6]. Provided that $X(\tau)$ starts from $\overline{V}_m$ at $\tau = 0$, T_m is approximately exponentially distributed:

$$F_{T_m}(t \mid R_m = r_m) \simeq 1 - \exp[-\nu(u_m)t] \tag{6}$$

where $\nu(u_m)$ is the so-called upcrossing rate of the reduced level u_m. It is also well-known that [7]

$$\nu(u_m) = \nu_0^+ \exp[-u_m^2/2] \tag{7}$$

with ν_0^+ the rate of (positive = upwards directed) crossings of $X(\tau)$ of the mean level $u_m = 0$.

Consider now a large "wave" of the load process causing a brittle elemental failure at the upcrossing instant of the corresponding threshold. The brittle elemental failure is followed by an abrupt change in the stiffness matrix of the structure. Even under the assumption of quasi-static loading, dynamic effects must be present. The load effects in the unfailed members will perform a more or less damped oscillation around the static load effect after redistribution (see Fig. 1). As mentioned above the possible dynamic overshooting of the static load effects will be neglected, i.e. a rather damped behavior is assumed which has a short characteristic time t_R. However, important for the considerations to come is the relation of t_R to some main period t_0 of the load process (compare Fig. 1). The case that will primarily be studied here is that of $t_R \gg t_0$, which corresponds to a delayed redistribution. It means that higher thresholds than the first upcrossed

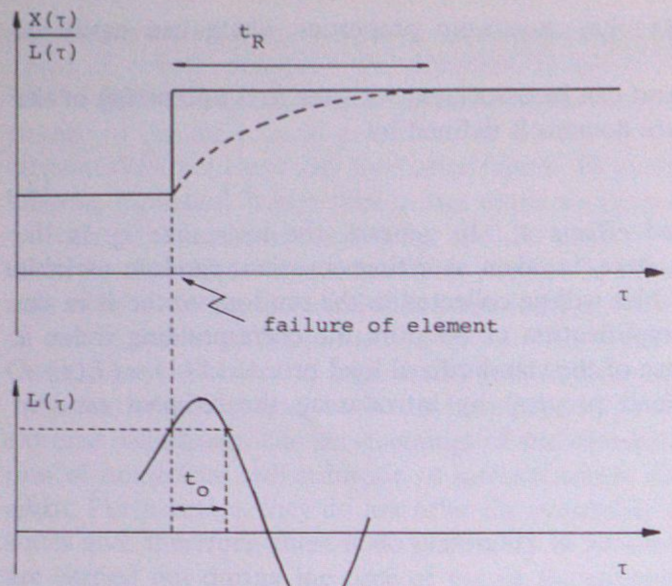

Fig. 1. Load redistribution after brittle failure.

level remain unchanged during the presence of the load wave on the structure. Load redistribution is effective at the earliest for the next loading cycle. The other limiting case, i.e. $t_R \ll t_0$ means a quasi-instantaneous redistribution after each elemental failure. In this case redistribution of internal forces must be considered during the same "extreme" wave. Then, redistribution of the load effects after failure of an element can lead to progressive failure of further intact elements during the process of redistribution. This case will briefly be discussed in the last section.

3. MULTIPLE CROSSINGS AND FAILURE TREE OVER TIME

3.1 Multiple crossings

As pointed out in the previous section, load thresholds can be defined for any arbitrary structural configuration and for every surviving control point. Conditional on $\boldsymbol{R} = \boldsymbol{r}$, this leads to an upcrossing problem as depicted in Fig. 2 for three elements. The (normalized) thresholds are ordered (for example $u_1 \leqslant u_2 \leqslant u_3$) and, therefore, have to be upcrossed in this order. It is further assumed that the excursion times of the process above the sufficiently large thresholds are relatively short as compared to the mean oscillation period around the mean value of the load process. For the long-term consideration of interest here it is then important to describe exactly the upcrossing event when the sample path upcrosses the lowest level u_1 (see Fig. 2). In a single "wave" there can occur a "single" upcrossing S of the lowest level or a "double" crossing D, i.e. the crossing of u_1 is immediately followed by a crossing of u_2 or even a "triple" crossing T. Generalizing to a structural state with n remaining control points one can define the following

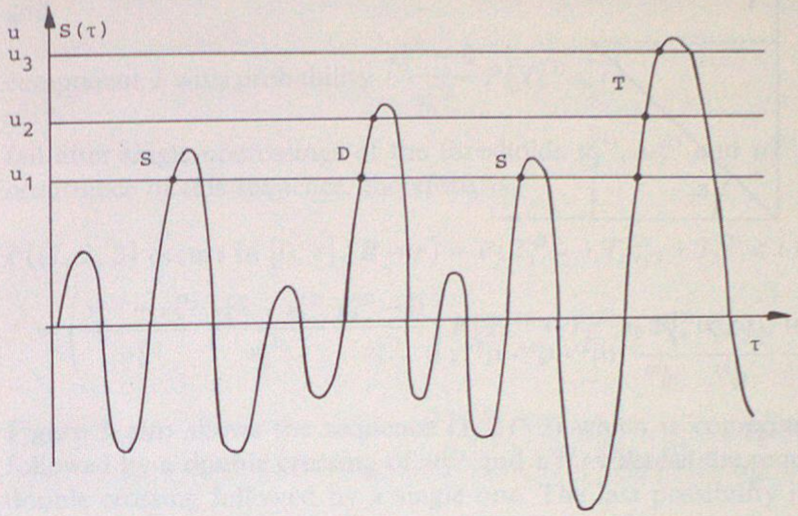

Fig. 2. Multiple crossings.

upcrossing rates for the ordered set of thresholds $u_1, u_2, \ldots, u_n$ but not the next higher one $u_{n+1} > u_n$:

single upcrossing: $\nu_{1 \cap \bar{2}} \simeq \nu_1 - \nu_2$ (8a)

double upcrossing: $\nu_{1 \cap 2 \cap \bar{3}} \simeq \nu_2 - \nu_3$ (8b)

$\vdots$

$(n-1)$ple upcrossing: $\nu_{1 \cap 2 \cap \ldots \cap (n-1) \cap \bar{n}} \simeq \nu_{n-1} - \nu_n$

$n$ple upcrossing: $\nu_{1 \cap \ldots \cap n} \simeq \nu_n - \nu_{n+1}$ (8

with $\nu_{n+1} = 0$ a dummy rate for a threshold $u_{n+1} = \infty$. In these formulae use is made of asymptotic independence of crossings of different levels [8]. When multiplied by a certain ti period those quantities are just the mean numbers of $i$ple crossings. Using similar asympt arguments one can also derive the conditional crossing rate for the ith level, given that it crossed immediately after the crossing of the lowest level. The probability of occurrence of first upcrossing of level 1 (superscript (1)) during some specified period of time $[0, t]$ that fina turns out be an upcrossing of the ith lowest threshold is given by (see also the exact derivation the Appendix):

$$P\left(T^{(1)}_{1 \cap \ldots \cap i \cap \overline{(i+1)}} \leqslant t\right) = \frac{\nu_{1 \cap \ldots \cap i \cap \overline{(i+1)}}}{\nu_1} P(T_1 \leqslant t) \simeq \frac{\nu_i - \nu_{i+1}}{\nu_1} F_{T_1}(t) \quad (9)$$

where T_1 is exponentially distributed with mean rate ν_1. This equation can be interpreted as follows: the probability distribution of the time of crossing the ith level for the first time is the probability of occurrence of a multiple crossing of the i first levels without having upcrossed u_1 before, times the probability that the first crossing of u_1 occurs during $[0, t]$. The concepts outlined before but especially the probability in eqn. (9) will now be used to derive the probability distribution of the failure times of entire failure sequences.

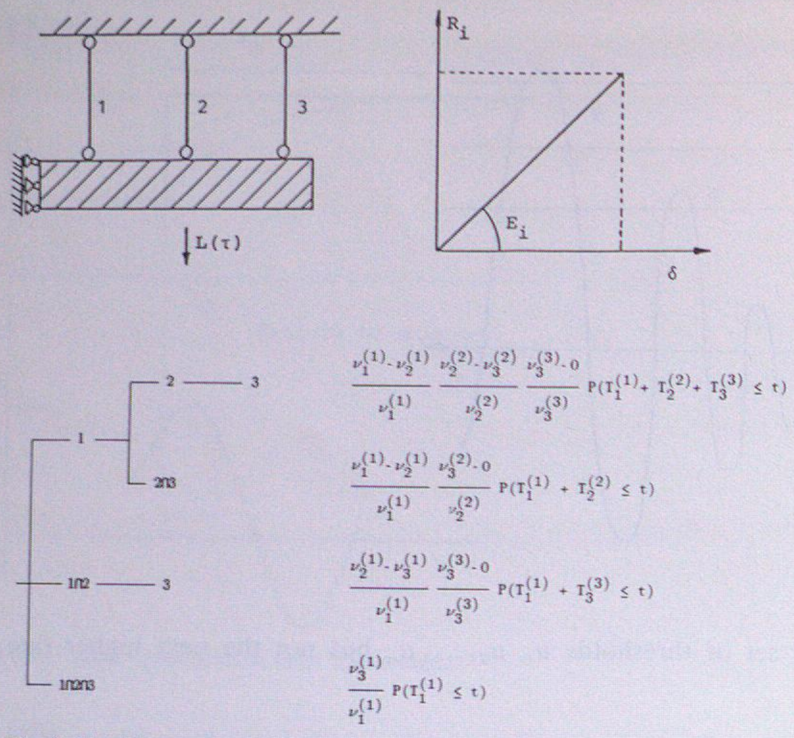

Fig. 3. Daniels system and failure tree.

3.2 Time-dependent failure tree

System degradation in time can conveniently be described by a time-dependent event tree for the elemental failures. As before, consider a system consisting of three redundant elements 1, 2 and 3 with deterministic resistances $u_1 \leqslant u_2 \leqslant u_3$. For simplicity, the elements are subject to the same loads or, equivalent, to equal load sharing in a Daniels system which has been under study repeatedly (see, for example Refs. [9,10]). A possible randomness of the capacities will be accounted for at the end of this section. Under the assumption of "delayed" load redistribution one can identify four potential failure sequences leading to system collapse (see Fig. 3). Using the property of independent crossings and, therefore, independent componential failure times, one can add the individual times in order to obtain the total time to system collapse and the corresponding probabilities of occurrence of the considered failure sequence during the interval $[0, t]$. In the following the superscripts indicate the state of the structure. The path (1, 2, 3) contains three time steps at the end of which

component 1 with probability $\dfrac{\nu_1^{(1)} - \nu_2^{(1)}}{\nu_1^{(1)}} P\left(T_1^{(1)} \leqslant t\right)$

component 2 with probability $\dfrac{\nu_2^{(2)} - \nu_3^{(2)}}{\nu_2^{(2)}} P\left(T_2^{(2)} \leqslant t\right)$

and

component 3 with probability $\dfrac{\nu_3^{(3)} - 0}{\nu_3^{(3)}} P\left(T_3^{(3)} \leqslant t\right)$

fail after single upcrossings of the thresholds $u_1^{(1)}$, $u_2^{(2)}$ and $u_3^{(3)}$, respectively. The probability of occurrence of this sequence, therefore, is:

$$P((1, 2, 3) \text{ occurs in } [0, t] \mid \boldsymbol{R} = \boldsymbol{r}) = P\left(T_{1\cap 2}^{(1)} + T_{2\cap 3}^{(2)} + T_3^{(3)} \leqslant t\right) = P_{1,2,3}(t)$$
$$= \left(\frac{\nu_1^{(1)} - \nu_2^{(1)}}{\nu_1^{(1)}} \frac{\nu_2^{(2)} - \nu_3^{(2)}}{\nu_2^{(2)}} \frac{\nu_3^{(3)} - 0}{\nu_3^{(3)}}\right) P\left(T_1^{(1)} + T_2^{(2)} + T_3^{(3)} \leqslant t\right) \tag{10a}$$

Figure 3 also shows the sequence $(1, 2 \cap 3)$ which is composed of a single upcrossing of $u_1^{(1)}$ followed by a double crossing of $u_2^{(2)}$ and $u_3^{(2)}$, whereas the sequence $(1 \cap 2, 3)$ is composed of a double crossing followed by a single one. The last possibility involves a triple crossing, i.e. the system fails under a single load wave. The corresponding probabilities of occurrence are also given below.

$$P_{1,2\cap 3}(t) = \frac{\nu_1^{(1)} - \nu_2^{(1)}}{\nu_1^{(1)}} \frac{\nu_2^{(2)} - 0}{\nu_2^{(2)}} P\left(T_1^{(1)} + T_2^{(2)} \leqslant t\right) \tag{10b}$$

$$P_{1\cap 2,3}(t) = \frac{\nu_2^{(1)} - \nu_3^{(1)}}{\nu_1^{(1)}} \frac{\nu_3^{(2)} - 0}{\nu_3^{(2)}} P\left(T_1^{(1)} + T_3^{(2)} \leqslant t\right) \tag{10c}$$

$$P_{1\cap 2\cap 3}(t) = \frac{\nu_3^{(1)}}{\nu_1^{(1)}} P\left(T_1^{(1)} \leqslant t\right) \tag{10d}$$

In the same manner the entire conditional failure tree can always be found fc complicated systems. The realisation of these failure sequences are disjoint events, a ţ which later will be used in the operation of deconditioning.

3.3 Uncertain resistances

Since the resistance properties generally must be assumed uncertain, the consideratio above are valid only for a given ordering of the random resistances. The corres conditions must, therefore, be verified when expressing the total probability of occurrence of the branch. For the sequence (1, 2, 3) considered above the total probability reads:

$$P((1, 2, 3) \text{ occurs in } [0, t]) = \int P((1, 2, 3) \text{ occurs in } [0, t] \mid \boldsymbol{R} = \boldsymbol{r}) P(r_1 \leqslant r_2 \leqslant r_3)\, \mathrm{d}F_{\boldsymbol{R}}(\boldsymbol{r}) \tag{11}$$

Similar expressions can be derived for the other failure sequences.

4. SYSTEM FAILURE PROBABILITIES

Generalization to an arbitrary system with K possible failure paths leading to system collapse is straightforward but notationally somewhat awkward. The number of time steps in the kth path is L_k. In order to completely define the sequence, an ordering of the reduced resistances must be performed for each of the time steps and the elements must be grouped into time components failing at the end of each time step. The probability of occurrence of the kth failure sequence during $[0, t]$ can be written as:

$$P(F_k(t)) = \int P_k(t \mid \boldsymbol{R} = \boldsymbol{r}) P\left(\bigcap_{l=1}^{L_k} \bigcap_{i=1}^{I_k^l} u_{n_{kl}(i)}^l \leqslant u_{n_{kl}(i+1)}^l \right) \mathrm{d}F_{\boldsymbol{R}}(\boldsymbol{r}) \tag{12}$$

where the running index $l = 1, \ldots, L_k$ represents the time steps and $i = 1, \ldots, I_k^l$ the present elements. The conditional failure probability can be derived from Section 3 as:

$$P_k(t \mid \boldsymbol{R} = \boldsymbol{r}) = P^k P\left(\sum_{l=1}^{L_k} T_k^l\left(u_{n_{kl}(1)}^l\right) \leqslant t \right) \tag{13}$$

In these equations, the following notations are used:

I_k^l denotes the number of control points surviving in the beginning and all along the nth time step.

$n_{kl}(i)$ is an integer function which assigns, in an ascending order, the number of the reduced threshold during the lth time step.

$P^k = \prod_{l=1}^{L_k} P_k^l$ is the overall weighting probability of the kth failure path where (see Section 3.1)

$$P_k^l = \frac{\nu_{n_{kl}(i_{kl})}^{(l)} - \nu_{n_{kl}(i_{kl}+1)}^{(l)}}{\nu_{n_{kl}(1)}^{(l)}}$$

is the local weighting probability for the lth time step which ends at the upcrossing of the i_{kl} thresholds after an upcrossing of the elements $n_{kl}(1)$ to $n_{kl}(i_{kl})$ but without upcrossing of the $n_{kl}(i_{kl}+1)$th level.

$T_k^l(u_{n_{kl}(1)}^l)$ is the time to the first upcrossing of the lowest relevant level for the lth time step. It also defines the length of this time step. Note that due to the expressions for multiple crossings T_k^l only depends on the lowest reduced threshold $u_{n_{kl}(1)}^l$ at the beginning of each time step.

The total system failure event then is the union of all failure sequence events $F_k(t)$, i.e. the system failure probability becomes:

$$P_f(t) = P\left(\bigcup_{k=1}^{K} F_k(t) \right) \tag{14}$$

Direct numerical evaluation of these formulae by numerical integration appears to be a formidable if not unsolveable task at least for larger systems. The same appears to be true for simulation methods unless certain non-trivial modifications of the formulation can be made. However, the following reformulation can be used with advantage for numerical analysis.

5. APPLICATION OF FORM/SORM TO NUMERICAL PROBABILITY INTEGRATIONS

5.1 Transformation into the standard normal space

The probability integrations necessary in eqns. (12) and (13) can be facilitated very much by application of modern FORM/SORM techniques. This requires a transformation of the integra-

tion domain into the so-called standard normal space. First of all, the following two probabilities are equal:

$$P(U_T \leqslant u_T) = \Phi(u_T) = P(T(\boldsymbol{r}) \leqslant t) = 1 - \exp[-\nu(\boldsymbol{r})t] \tag{15}$$

where U_T is an auxiliary standard normal variable and Φ the standard normal distribution function. Hence, each first passage time $T(\boldsymbol{r})$ in eqn. (13) conditional on $\boldsymbol{R} = \boldsymbol{r}$ can be represented as:

$$T(\boldsymbol{r}) = -\frac{1}{\nu(\boldsymbol{r})} \ln \Phi(-U_T)$$

The inequality

$$\sum_{l=1}^{L_k} T_k^l - t \leqslant 0$$

can be rewritten as

$$\sum_{l=1}^{L_k} -\frac{1}{\nu_k^l(\boldsymbol{r})} \ln \Phi(-U_{T_l}) - t \leqslant 0 \tag{16}$$

where all U_{T_l}'s are independent auxiliary variables. Thus, eqn. (13) can be evalua standard FORM/SORM techniques. In a next step, the weighting probability rewritten, conditional on $\boldsymbol{R} = \boldsymbol{r}$, as follows:

$$P^k = \prod_{l=1}^{L_k} P_k^l = P\left(U_P^k - \Phi^{-1}(P^k) \leqslant 0\right)$$

where U_P^k is another auxiliary variable. The conditioning vector $\boldsymbol{R} = \boldsymbol{r}$ can also be rep its Rosenblatt transformation [11]:

$$\boldsymbol{R} = T_{\mathrm{R}}(\boldsymbol{U_R}) \tag{18}$$

If these transformations are introduced in eqns. (12) and (13), eqn. (12) can be integrated by the FORM/SORM method [5]. The following formulation is obtained:

$$P(F_k(t)) = P\Bigg[\left(\bigcap_{l=1}^{L_k} \bigcap_{i=1}^{I_k^l} U_{n_{kl}(1)}^l - U_{n_{kl}(i+1)}^l \leqslant 0 \right) \cap \left(U_P^k - \Phi^{-1}(P^k) \leqslant 0\right) \cap \left(\sum_{l=1}^{L_k} -\frac{1}{\nu_k^l(\boldsymbol{r})} \ln \Phi(-U_{T_k^l}) - t \leqslant 0 \right) \Bigg| \left(U_P^k = \Phi^{-1}(P^k)\right) \Bigg] P^k \tag{19}$$

where the three parts on the right-hand side are due to the ordering condition, the weighting factor, and the occurrence of the time step, respectively.

It is recognised that the determination of the probability in eqn. (19) involves the evaluation of the probability of intersections of events in standard space — a classical task for FORM. The results can be improved by SORM corrections [5]. Furthermore, the evaluation of the union probability in eqn. (14) follows the well-known procedures for the union of small probability events also described in Ref. [5] and elsewhere. The most involved numerical task is the determination of the joint β-point (joint most likely failure point) in each intersection.

5.2 Application of a search algorithm for most likely sequences

There usually is an enormous number of failure sequences leading to failure, which makes a complete analysis prohibitively expensive even for smaller systems. In general, only a few of these sequences contribute substantially to the total failure probability. Fortunately, the search algorithms developed for the same problem in time-invariant system reliability can be generalised to time-variant failure trees. The main idea of such search algorithms (see Ref. [12] and the references therein) is to set out from an intact structure and to progress along incomplete failure paths by selecting at each branching point the most likely sequence among all sequences whose branch-point probabilities have already been calculated. This requires the evaluation of the corresponding intersection probabilities which can be done as described above. If a complete failure sequence is found and has been checked to be the most likely one, the corresponding sequence probability is a lower bound to the system failure probability while the probability of the union of all complete and incomplete sequences is an upper bound [13]. During that search it can be useful to simplify the considered domain, i.e. by retaining in eqn. (19) only the important conditions (those which are "active" at the joint β-point). Some limited experience indicates that only a few ordering conditions usually are necessary to be included when applying the method at larger systems.

6. NUMERICAL EXAMPLES

6.1 Four-element Daniels system

As an introductory example a Daniels system with four brittle elements is considered (compare Fig. 3). The normally distributed resistances R_i are assumed to be equicorrelated with correlation coefficient ρ. They can be represented as follows:

$$R_i = E[R] + D[R]\left(U_0\sqrt{\rho} + U_i\sqrt{1-\rho}\right)$$

U_0 and U_i's are independent standard normal variables. The use of the order statistics of the R_i's, i.e. $\hat{R}_1 < \hat{R}_2 < \hat{R}_3 < \hat{R}_4$ as described in Ref. [9] reduces the number of potential failure paths to be considered. The ordering conditions in eqn. (19) are also automatically fulfilled. For the numerical calculations it is assumed that the R_i's are distributed according to $R_i \sim N(0.9; 0.2)$. The correlation coefficient just introduced is $\rho = 0.3$. The load process is Gaussian with $X(t) \sim (N(0.5; 0.1)$ and autocorrelation parameters such that $\nu_0^+ t = 10^6$ where $\nu_0^+ t$ is the expected number of load cycles during the intended time of use t.

The complete failure tree is given in Fig. 4. For each time step the safety indices are given in parentheses. The upper value is the first-order index β_k^{I}. The lower second-order value β_k^{II} can be

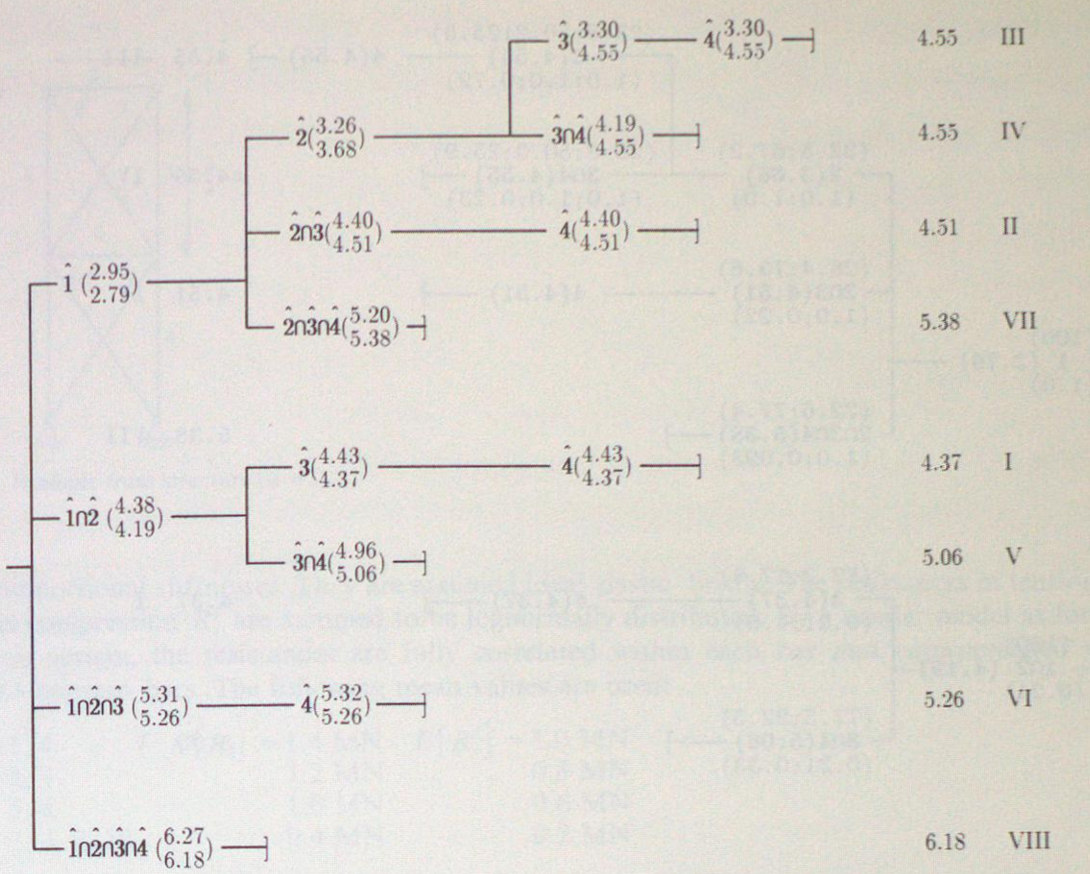

Fig. 4. Failure tree of 4-element Daniels system—first- and second-order safety indices.

related to the probability of occurrence of the corresponding complete or uncomplete failure sequence by:

$$\beta_k^{\mathrm{II}} = -\Phi^{-1}(P_k(t))$$

It is recognised that the two types of safety indices can differ. A detailed numerical analysis has shown that the second-order indices correspond very well with the exact results and should be used [14]. The safety indices increase along a path, indicating its remaining redundancy. The amount of increase is a quantitative measure of the degree of redundancy. It is seen that in some failure sequences the additional elements up to failure provide only negligible extra safety. From the numbers in Fig. 4, it is obvious that the four most likely failure sequences are $(\hat{1} \cap \hat{2}, \hat{3}, \hat{4})$, $(\hat{1}, \hat{2} \cap \hat{3}, \hat{4})$, $(\hat{1}, \hat{2}, \hat{3}, \hat{4})$ and $(\hat{1}, \hat{2}, \hat{3} \cap \hat{4})$ with three or four time steps. This is not surprising since sequences with failures in separate time steps are more likely due to the assumption of delayed load redistribution. It is, however, interesting to note that the most likely failure sequence $(\hat{1} \cap \hat{2}, \hat{3}, \hat{4})$ starts with a double crossing. The most likely values for the failure times according to FORM are given in Fig. 5. The values given just above the last failed component of the considered sequence are the time step lengths in percentage of the total time of life up to this point. When the system, for example, reaches in two time steps the state with two surviving elements, it most likely has spent 32.8% up to failure of $\hat{1}$ and 67.2% up to failure of $\hat{2}$. As the strongest element 4 alone does not contribute to redundancy anymore, the corresponding time

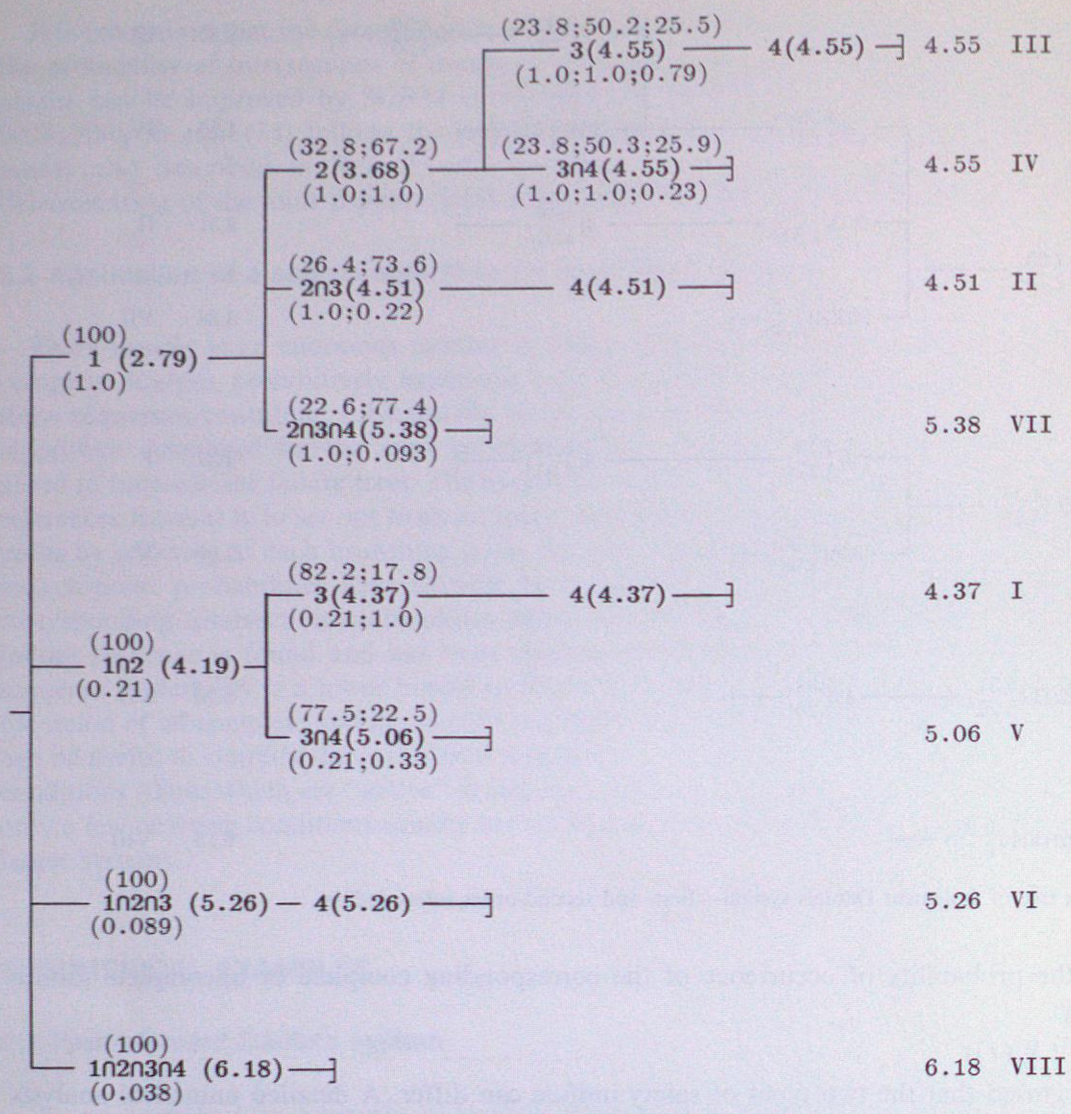

Fig. 5. Failure tree of 4-element Daniels system—failure times and branch probabilities.

step is of length zero. A correct interpretation of these relative step lengths, however, has to be made in view of the β-value of occurrence of the corresponding sequence. Hence, the time to the realisation of sequence $(\hat{1}, \hat{2})$ with $\beta^{\text{II}} = 3.68$ is smaller than for $(\hat{1}, \hat{2}, \hat{3})$ with $\beta^{\text{II}} = 4.55$. The values given just below the component correspond to the weighting probabilities P_k discussed in Section 4.

6.2 Reliability of a simple truss

As a second example, consider the simple redundant truss shown in Fig. 6. The bars have the following section areas:

$A = 1.0\ \text{m}^2$ for the elements 1 to 6
$A = 0.2\ \text{m}^2$ for the elements 7 to 10

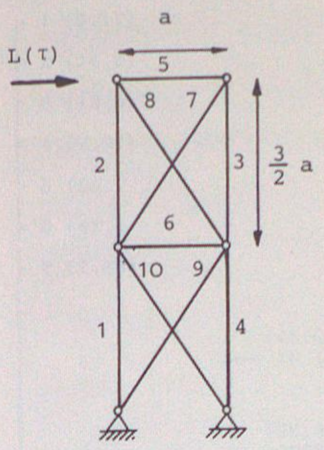

Fig. 6. Example truss structure ($a = 1$).

and proportional stiffnesses. They are assumed ideal elastic–brittle. The resistances in tension R_i and in compression R_i' are assumed to be lognormally distributed. By a similar model as for the Daniels system, the resistances are fully correlated within each bar and equicorrelated with $\rho = 0.3$ between bars. The following mean values are used:

Bars: 1, 4	$E[R_i] = 1.4$ MN	$E[R_i'] = 1.0$ MN
2, 3	1.2 MN	0.8 MN
5, 6	1.0 MN	0.6 MN
7, 8, 9, 10	0.4 MN	0.2 MN

The coefficient of variation is $V[R_i] = V[R_i'] = 0.15$. The load process is $L(\tau) \sim N(0.04; 0.02)$ having a mean number of load cycles $\nu_0^+ t = 10^6$ during the time $[0, t]$. The resulting failure tree is given in Fig. 7 with the same conventions as for Figs. 4 and 5.

Although the structure would be still small enough to carry out a complete analysis, the first dominant failure sequences are determined by the search algorithm explained in Section 5.2. It is seen that (10, 8, 9) is the dominant failure sequence, immediately followed by $(8 \cap 10, 7)$ and several other sequences with only slightly larger β values. The algorithm starts system degradation by transferring the weakest component 8 into a failure state. Then, it evaluates the failure pairs (8, 10), (8, 9) and (8, 7), the last one of which turns out to be already a sequence to structural collapse. The algorithm then restores the system and transfers component 10 into a failure state followed by component 8, 9 and 7, in that order. Next, the two sequences (8, 10, 7) and (8, 10, 9), but also (10, 8, 7) and (10, 8, 9) are investigated after the corresponding restoring operations. The last sequence is the critical one. But also the sequences starting with double crossings, i.e. $(8 \cap 10, 7)$ and $(8 \cap 10, 9)$ as well $(10 \cap 8, 7)$ and $(10 \cap 8, 9)$ are of significance. An interpretation of the times spent in different states for these sequences leads to similar conclusions as for the Daniels system. The system safety index is bounded by $3.65 \leqslant \beta \leqslant 4.09$. The true index is likely to be close to the lower bound.

The assumption of delayed load redistribution can be released at the expense of more involved ordering conditions.

In that case it is possible that the redistributed thresholds relevant immediately after the first crossing fall under the originally lowest thresholds. This can be interpreted as a double failure

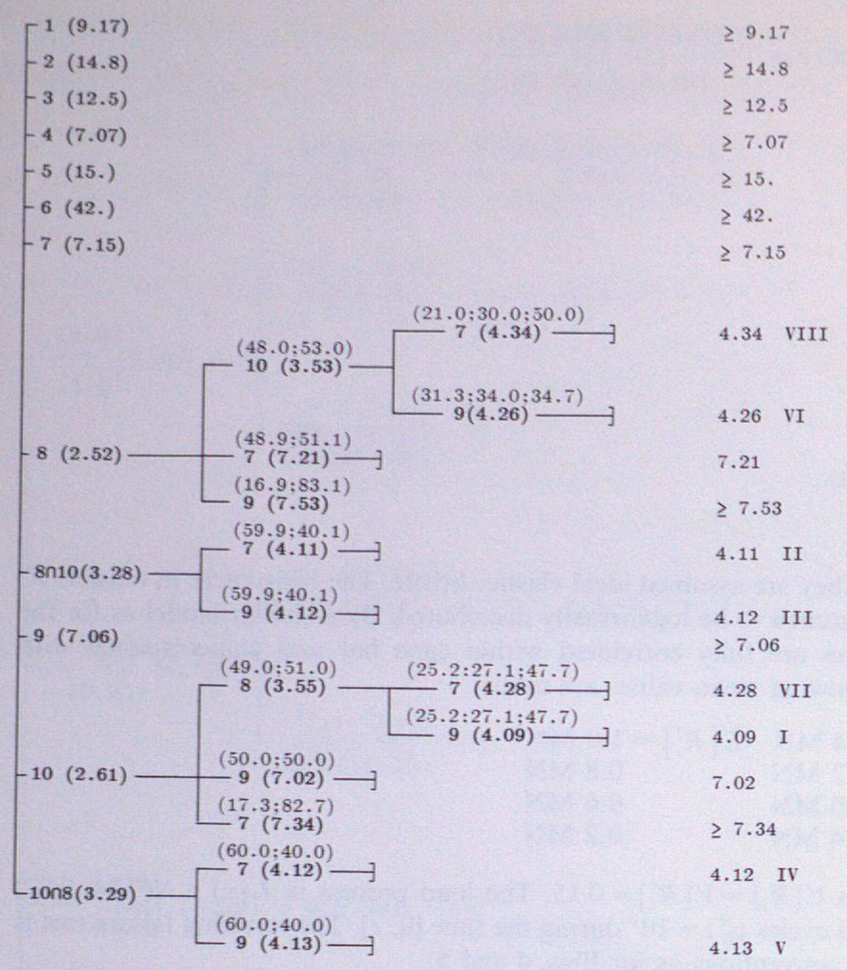

Fig. 7. Failure tree of example truss—delayed load redistribution.

caused by a single upcrossing. Therefore, it is clear that supplementary ordering conditions have to be introduced after each elemental failure. The calculations then become more laborous but do not differ in principle from the case already discussed.

In Fig. 8 the same structure is investigated under the alternative extreme assumption that immediate load redistribution occurs. Again, the proposed search algorithm is applied but consideration is given to failures during load redistribution (progressive failure). One observes that failure sequences with a greater number of time steps now become less likely. Multiple failures are more frequent. Interestingly, two progressive failure sequences dominate. Here, the system safety index is bounded by $2.98 \leqslant \beta \leqslant 3.30$.

7. DISCUSSION AND CONCLUSIONS

There are several aspects which need some discussion. First of all, eqn. (6) is only a relatively crude approximation for larger componential failure probabilities. For narrow-band load processes one can improve that equation substantially by replacing the upcrossing rate, eqn. (7),

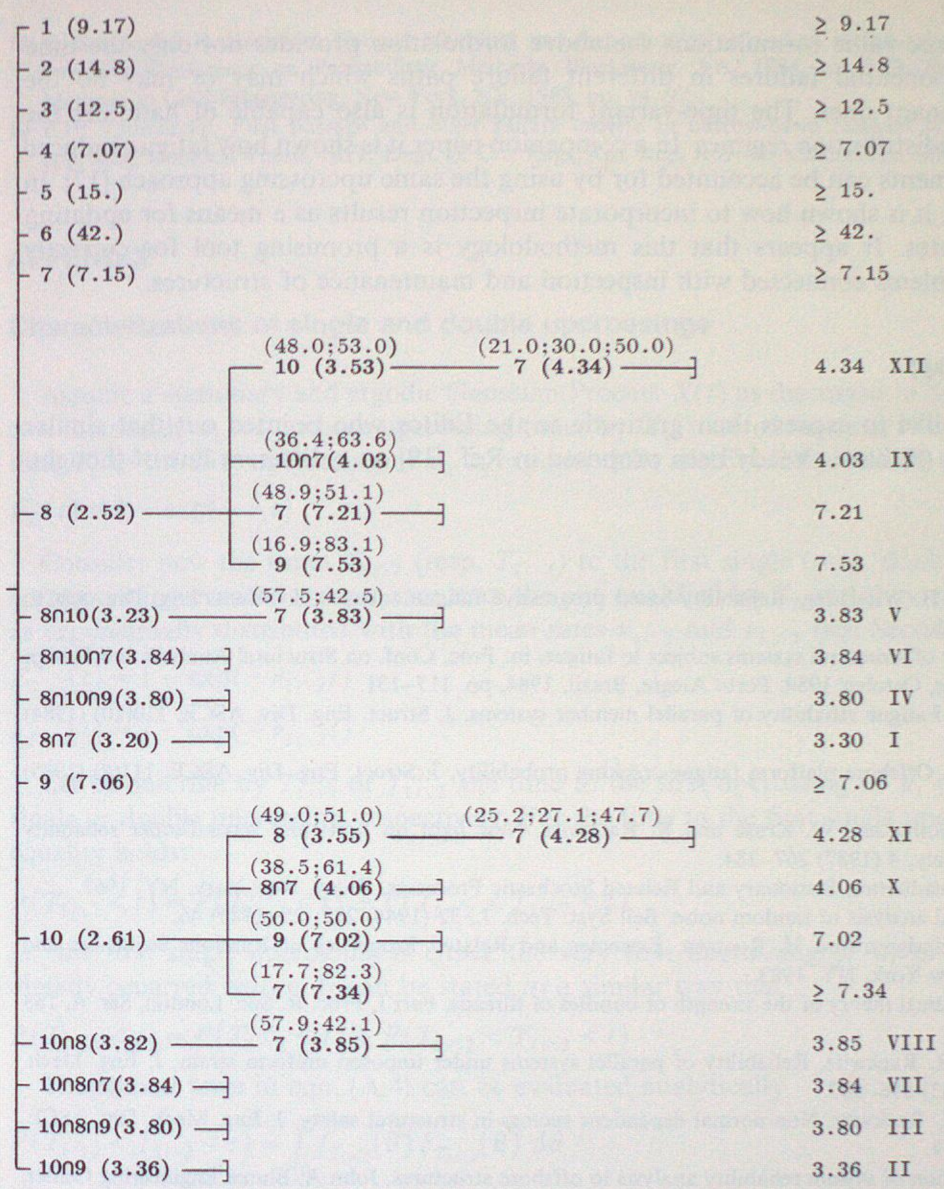

Fig. 8. Failure tree of example truss—immediate load redistribution.

by the modified upcrossing rate proposed in Ref. [15] for the envelope process.

Dynamic effects have been neglected throughout. They can be included easily by considering a limiting case, i.e. that the additional dynamic load effect in an unfailed element can be at most as large as the difference between the static load effects before and after load redistribution (see Ref. [14] for further detail). A rigorous treatment of dynamic effects appears to be rather complicated.

The numerical results for system collapse according to the above time-variant formulation have been checked with the appropriate extreme value formulations (see Refs. [14,16]). Excellent agreement was found, supporting the various asymptotic arguments in the derivations.

In contrast to extreme value formulations the above formulation provides not only the time lengths between componential failures in different failure paths which may or may not be identified as the dominant ones. The time-variant formulation is also capable of handling the various internal load redistribution regimes. In a companion paper it is shown how fatigue-induced degradation of the elements can be accounted for by using the same upcrossing approach [17]. In yet another paper [18], it is shown how to incorporate inspection results as a means for updating the probability estimates. It appears that this methodology is a promising tool for correctly treating reliability problems connected with inspection and maintenance of structures.

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REFERENCES

1 S.G. Martindale and P.H. Wirsching, Reliability-based progressive fatigue collapse, J. Struct. Eng. Div. ASCE, 109(8) (1983) 1792–1811.
2 R. Rackwitz, Reliability of structural systems subject to fatigue, in: Proc. Conf. on Structural Analysis and Design of Nuclear Power Plants, October 1984, Porto Alegre, Brazil, 1984, pp. 117–131.
3 B. Stahl and F. Geyer, Fatigue reliability of parallel member systems, J. Struct. Eng. Div. ASCE, 110(10) (1984) 2307–232.
4 E. Knapp and B. Stahl, Offshore platform fatigue cracking probability, J. Struct. Eng. Div. ASCE, 111(8) (1985) 1647–1660.
5 M. Hohenbichler, S. Gollwitzer, W. Kruse and R. Rackwitz, New light on first- and second-order reliability methods, Structural Safety, 4 (1987) 267–284.
6 H. Cramer and M.R. Leadbetter, Stationary and Related Stochastic Processes, Wiley, New York, NY, 1967.
7 S.O. Rice, Mathematical analysis of random noise, Bell Syst. Tech. J., 32 (1944) 282; 25 (1945) 46.
8 M.R. Leadbetter, G. Lindgren and H. Rootzen, Extremes and Related Properties of Random Sequences and Processes, Springer, New York, NY, 1983.
9 H.E. Daniels, The statistical theory of the strength of bundles of threads, Part I, Proc. R. Soc. London, Ser. A, 183 (1945) 405–435.
10 M. Hohenbichler and R. Rackwitz, Reliability of parallel systems under imposed uniform strain, J. Eng. Mech. Div. ASCE, 109(3) (1983) 896–907.
11 M. Hohenbichler and R. Rackwitz, Non-normal dependent vectors in structural safety, J. Eng. Mech. Div. ASCE, 107(6) (1981) 1227–1249.
12 Y.F. Guenard, Application of system reliability analysis to offshore structures, John A. Blume Engineering Center, Report No. 71, Stanford University, Stanford, CA, 1984.
13 S. Gollwitzer, Zuverlässigkeit redundanter Tragsysteme bei geometrischer und stofflicher Nichtlinearität, Dissertation, Technical University of Munich, F.R.G., 1986.
14 F. Guers, K. Dolinski and R. Rackwitz, Outcrossing formulation for redundant structural systems under fatigue, Proc. 1st IFIP WG 7.5 Working Conference, Aalborg, Denmark, May, 1987, in: P. Thoft-Christensen (Ed.), Reliability and Optimization of Structural Systems, Springer–Verlag, Berlin, 1987, pp. 199–222.
15 E.H. Vanmarcke, On the distribution of the first-passage time for normal stationary random processes, J. Appl. Mech., 42 (1975) 215—220.
16 F. Guers, Zur Zuverlässigkeit redundanter Tragsysteme bei Ermüdungsbeanspruchung durch zeitinvariante Gauß'sche Lasten, Dissertation, Technical University of Munich, F.R.G., 1986.
17 F. Guers and R. Rackwitz, Time–variant reliability of structural systems subject to fatigue, Proc. 5th International Conference on Applications of Statistics and Probability in Soils and Structural Engineering, Vancouver, May, 1987, in: N.C. Lind (Ed.), Reliability and Risk Analysis in Civil Engineering 1, Waterloo, Canada, 1987, pp. 497–505.

18 G. Schall and R. Rackwitz, Fatigue reliability and inspection in redundant structural systems, Proc. 5th ASCE Specialty Conference on Probabilistic Methods, Blacksburg, VA, 1988, in: P.D. Spanos (ed.), Probabilistic Methods in Civil Engineering, New York, NY, 1988, pp. 94–97.
19 E.H. Vanmarcke, First passage and other failure criteria in narrow-band random vibration: A discrete state approach, Doctoral Thesis, MIT, Dept. of Civ. Eng., Res. Rep. R69–68, Cambridge, MA, 1969.

APPENDIX

Characterizations of single and double upcrossings

Assume a stationary and ergodic Gaussian Process $X(t)$ as discussed in Section 3 and consider two thresholds $u_1 < u_2$ whose upcrossing times by $X(t)$ are of interest. It is well known that the time T_1 to the first upcrossing of u_1 is exponentially distributed provided that $X(0) < u_1$:

$$F_{T_1}(t) = 1 - \exp(-\nu_1 t) \qquad \text{(A.1)}$$

Consider now the times $T_{1\cap\bar{2}}$ (resp. $T_{1\cap 2}$) to the first single (resp. double) upcrossing. These corresponding events are more rare than the one qualified by T_1 and they can therefore be taken as exponentially distributed with the mean rates $\nu_{1\cap\bar{2}}$ and $\nu_{1\cap 2}$ (see Section 3).

$$F_{T_{1\cap\bar{2}}}(t) = 1 - \exp(-\nu_{1\cap\bar{2}} t) \qquad \text{(A.2)}$$

$$F_{T_{1\cap 2}}(t) = 1 - \exp(-\nu_{1\cap 2} t) \qquad \text{(A.3)}$$

Let us describe by $T^{(1)}_{1\cap\bar{2}}$ or $T^{(1)}_{1\cap 2}$ the time to the first upcrossing of u_1 (index (1)) which is a single or double upcrossing, respectively. For the time to the first single upcrossing the following equality holds:

$$P(T_{1\cap\bar{2}} < t) = P\left(T^{(1)}_{1\cap\bar{2}} < t\right) + P(T_{1\cap 2} < T_{1\cap\bar{2}} < t) \qquad \text{(A.4)}$$

i.e. the first single upcrossing is either the very first upcrossing of u_1 or a double upcrossing already occurred before. It can be stated in a similar way that

$$P(T_{1\cap 2} < t) = P\left(T^{(1)}_{1\cap 2} < t\right) + P(T_{1\cap\bar{2}} < T_{1\cap 2} < t) \qquad \text{(A.5)}$$

The second term in eqn. (A.4) can be evaluated analytically

$$P(T_{1\cap 2} < T_{1\cap\bar{2}} < t) = \int_0^t f_{T_{1\cap 2}}(\theta) F_{T_{1\cap\bar{2}}}(\theta)\, d\theta$$

$$= F_{T_{1\cap 2}}(t) \frac{\nu_{1\cap\bar{2}}}{\nu_1} F_{T_1}(t) \qquad \text{(A.6)}$$

Using eqn. (A.4), $T^{(1)}_{1\cap\bar{2}}$ finally has the probability of occurrence

$$P\left(T^{(1)}_{1\cap\bar{2}} < t\right) = \frac{\nu_{1\cap\bar{2}}}{\nu_1} F_{T_1}(t) = \frac{\nu_1 - \nu_2}{\nu_1} F_{T_1}(t) \qquad \text{(A.7)}$$

and similarly

$$P\left(T^{(1)}_{1\cap 2} < t\right) = \frac{\nu_{1\cap 2}}{\nu_1} F_{T_1}(t) = \frac{\nu_2}{\nu_1} F_{T_1}(t) \qquad \text{(A.8)}$$

These relationships can be generalized to cases with more thresholds as done in Section 3.

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