# Optimal LQG Control of Networked Systems under Traffic-Correlated Delay and Dropout

Dipankar Maity, *Member, IEEE*, Mohammad H. Mamduhi, Sandra Hirche, *Fellow, IEEE*, and Karl H. Johansson, *Fellow, IEEE*

***Abstract*— Transmission delay and packet dropout are inevitable network-induced phenomena that severely compromise the control performance of network control systems. The real-time network traffic is a major dynamic parameter that directly influences delay and reliability of transmission channels, and thus, acts as an unavoidable source of induced coupling among all network sharing systems. In this letter, we analyze the effects of traffic-induced delay and dropout on the finite-horizon quality-of-control of an individual stochastic linear time-invariant system, where quality-of-control is measured by an expected quadratic cost function. We model delay and dropout of the channel as generic stochastic processes that are correlated with the real-time network traffic induced by the rest of network users. This approach provides a pathway to determine the required networking capabilities to achieve a guaranteed quality-of-control for systems operating over a shared-traffic network. Numerical evaluations are performed using realistic stochastic models for delay and dropout. As a special case, we consider exponential distribution for delay with its rate parameter being traffic-correlated, and traffic-correlated Markov-based packet drop model.**

***Index Terms*- Networked control systems, latency, packet loss, network traffic, quality-of-control.**

## I. Introduction and Motivation

Transmission delay and packet dropout are two major network-induced phenomena that affect the control performance and may even lead to instability of networked control systems (NCSs) [1]. For about two decades, a significant attention has been given to analyze the effects of network-induced phenomena on stability and quality-of-control (QoC) characteristics of closed-loop network sharing systems [2]. However, most of those works either study asymptotic behavior of closed-loop systems under specific delay and packet loss scenarios, or consider stationary/independent delay and packet loss processes [3]. The consideration of more practical models of delay and packet loss wherein they are dynamically correlated with channel traffic load has received little to none attention in the context of NCSs. Moreover, from the QoC perspective, characterizing the effects of realistic delay and packet loss models on the finite horizon control performance has a notable importance especially for the state-of-the-art and time-sensitive applications of NCSs such as Industrial Internet-of-Things and Industry 4.0 [4]. There are two major reasons to study such problems: first, the novel concepts of IIoT and I4.0 require the control systems (e.g., industrial mobile robots) to be adaptable w.r.t. the changing conditions of their assigned tasks and resources [5], and secondly, the recent transform of networking technology (e.g., 5G) to a user-oriented data exchange medium has brought an extra dimension of decision-making for control systems to determine the right service characteristics needed to satisfy their expected QoC over specific time frames. This means that the systems may need to recompute their control and communication policies to satisfy new QoC requirements. This entails that the control systems predict and incorporate the effects of delay and packet loss on their performance index to decide the required levels of latency and reliability [6].

Motivated by the mentioned impacts, we focus in this letter on two major contributions, first, proposing realistic models of transmission delay and packet loss that are correlated with the real-time network traffic, and second, characterizing the effects of this traffic-correlated stochastic delay and packet loss on the finite horizon QoC of individual closed-loop control systems. We propose a generic packet loss framework and a stochastic model of delay that encompasses a wide class of distributions. Both models are correlated with the real-time network traffic such that a higher traffic load results in a higher probability of packet loss and longer delay. Furthermore, our unified approach for incorporating delay and packet loss handles the out-of-order arrival of measurements. We show that the common models of stationary delay and Markovian packet loss are special cases of our proposed traffic-dependant delay and packet loss model. We study the effects of the proposed traffic-dependent delay and packet loss processes on the control performance of an arbitrary linear time-invariant (LTI) control system that uses the shared network to close its sensor-to-controller feedback loop. Contrary to some of the existing literature where the controller is imposed to be a zero-order hold, we do not restrict ourselves to such impositions. We then discuss the maximum delay and packet loss that can be tolerated by the control system to satisfy a given QoC.

The work of D. Maity was supported, in parts, by funds provided by the University of North Carolina at Charlotte and by ARL grant ARL DCIST CRA W911NF-17-2-0181. The work of S. Hirche was supported by German Research Foundation (project SPP 1914 – Cyber-Physical Networking), and the work of K. H. Johansson was supported, in parts, by Knut and Alice Wallenberg Foundation, by Swedish Strategic Research Foundation, and by Swedish Research Council.

D. Maity is with the Department of Electrical and Computer Engineering, University of North Carolina at Charlotte, NC, 28223, USA. Email: dmaity@uncc.edu

M. H. Mamduhi is with the Automatic Control Laboratory, Department of Information Technology and Electrical Engineering, ETH Zürich, Zürich, 8092, Switzerland. Email: mmamduhi@ethz.ch

S. Hirche is with the Chair of Information-oriented Control, Department of Electrical and Computer Engineering, Technical University of Munich, Munich, 80333, Germany. Email: hirche@tum.de

K. H. Johansson is with the Division of Decision and Control Systems, School of Electrical Engineering and Computer Science, and is the Director of Digital Futures, KTH Royal Institute of Technology, Stockholm, 10044, Sweden. Email: kallej@kth.se

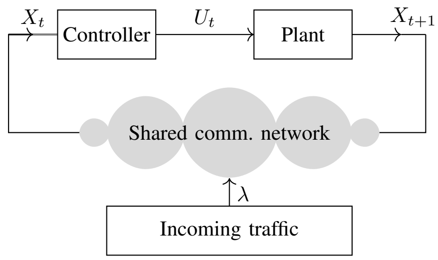


Fig. 1. Schematic of a networked control system operating over a traffic-shared network under incoming real-time traffic with rate $\lambda$.

### A. Related Works & Outline

Stability of NCSs under network-induced delay has been widely investigated for both single-user and multi-user communication channels, see [7]–[9] among many others. Stability properties of NCSs under small (less than one sampling period) and large (more than one sampling period) constant network-induced delay are analyzed using a hybrid systems technique in [7]. In [8], asymptotic and exponential stability of closed-loop NCSs under time-varying but bounded delays are studied by constructing a continuum of Lyapunov functions, and the maximum allowable delay and transmission intervals are derived to guarantee stability. In [9], stochastic stability of jump-linear systems is discussed with an output feedback controller affected by Markov-based delay. Different packet loss models ranging from i.i.d. Bernoulli-based to Markov-based scenarios are considered when addressing stability and control performance of NCSs [10], [11]. Optimal QoC in NCSs over finite time horizon and under network-induced parameters has also been addressed mostly for bounded or small delays and stationary erasure channels [12]–[14]. Similarly, over the infinite horizon, optimal control performance is discussed mainly with stationary delay processes and independent packet loss models [15], [16]. To the best of our knowledge, however, a general coupled delay and packet loss model that is in addition statistically correlated with the real-time network traffic in the context of NCSs is not studied in the literature. Quantifying the effects of such traffic-correlated stochastic delays and dropouts on the optimality of the finite horizon control performance is the subject of this letter. Additionally, a succinct discussions on the Pareto trade-off between delay and dropout parameters to achieve a desired QoC is presented.

In the rest of this letter, Section II describes the NCS model. The estimation and optimal control for general and particular models of traffic-induced delay and packet loss are discussed in Section III. Numerical results are reported in Section IV, and the letter is concluded in Section V.

## II. Networked Control System Model

### A. Control System Model

Consider an LTI NCS where the sensor-to-controller feedback loop is closed over a shared communication channel, as depicted in Fig. 1. The dynamics of the closed-loop system follows the discrete-time LTI stochastic difference equation as

$$X_{t+1} = AX_t + BU_t + W_t, \tag{1}$$

where the state $X_t \in \mathbb{R}^n$, and the control input $U_t \in \mathbb{R}^m$. The process noise $\{W_t\}_{t\in\mathbb{N}_0}$ is an independent sequence of zero-mean Gaussian random variables with $W_t \sim \mathcal{N}(0, \mathcal{W}_t)$ for some covariance matrix $\mathcal{W}_t \succeq 0$. The initial state $X_0 \sim \mathcal{N}(0, \mathcal{W}_{-1})$ is also Gaussian and independent of the noise sequence $\{W_t\}_{t\in\mathbb{N}_0}$. The QoC objective of the closed-loop system is to minimize the following finite horizon LQG cost

$$J = \mathbb{E}\Big[\sum_{t=0}^{T-1}(\|X_t\|_Q^2 + \|U_t\|_R^2) + \|X_T\|_{Q_T}^2\Big], \tag{2}$$

where $Q, Q_T \succeq 0$, $R \succ 0$, and for any two matrices $M$ and $N$ with compatible dimensions, we define $\|N\|_M^2 \triangleq N^\intercal MN$.

We assume that the controller and the actuator are collocated. The scenario where the controller-to-actuator loop is also closed over a traffic-shared channel can be studied with a suitable modification of this framework. The network traffic injected by other users affects the performance of system (1) by degrading the delay and packet dropout probabilities.

### B. Shared Channel Model

The control system (1) closes its sensor-to-controller link over a traffic-shared resource-limited network, where the system has no control over the injected traffic. Traditionally, the network provides a queuing based service to its users. Hence, whenever there is not enough transmission resources, data packets are stored and scheduled for transmission at a later time. This leads to induced delay which is directly correlated to the network traffic. Furthermore, depending on the queuing model and channel conditions and resources, transmitted packets are subject to dropouts, and such dropouts generally worsens with higher network traffic load.

Let the delay experienced by a packet transmitted at time $t$ be denoted by $d_t \in \mathbb{R}_+$. The delay is a continuous random variable with a probability measure $\mathbb{P}$, which is correlated with the network traffic. We model the NCS operations (e.g., sensing, decision-making, actuation etc.) in a discrete-time manner, and hence, a data packet belonging to time $t$ arriving at time $t + d_t$ can only be used for computing the control at the next discrete time instance. Therefore, for the purpose of controlling the discrete-time model (1), we are interested in the probabilities $\mathbb{P}(d-1 \leq d_t < d)$ for $d \in \mathbb{N}$ that characterize the *discrete-time* delays at the controller. With a slight abuse of notation, in subsequent analysis we will simply write $\mathbb{P}(d_t = d)$ to denote $\mathbb{P}(d-1 \leq d_t < d)$ for $d \in \mathcal{D}$, where $\mathcal{D} \subseteq \mathbb{N}$ denotes the sample space of the discrete delay random variables. For the simplicity of this exposition, we will assume that $d_t$ and $d_s$ are independent for all $t \neq s$. Such temporal independencies are aligned with the work [17] and discussed experimentally in [18]. The analysis can also be carried out for a more generic delay distributions in a similar manner.

Let at time $t$ the packet delivery/dropout be denoted by a binary-valued random variable $\mu_t$, and the packet dropout probability by $\gamma_t \in [0,1]$, i.e., $\mathbb{P}(\mu_t = 0) = \gamma_t$. If the packet is not dropped (i.e., $\mu_t = 1$), the probability that it experiences $d \in \mathcal{D}$ delay is denoted as $p_t(d) \triangleq \mathbb{P}(d_t = d \mid \mu_t = 1)$. When the packet is dropped, it is equivalent to an infinite delay, and hence, $\mathbb{P}(d_t = d \mid \mu_t = 0) = 0$, $\forall t \in \mathbb{N}_0$ and $\forall d \in \mathcal{D}$. Thus, we write the coupled delay and packet loss model as

$$\mathbb{P}(d_t = d) = (1-\gamma_t)p_t(d), \quad \sum_{d\in\mathcal{D}} p_t(d) = 1, \ \forall t \in \mathbb{N}_0, d \in \mathcal{D}.$$

In a compact form, one may define the set $\bar{\mathcal{D}} = \mathcal{D} \cup \{\infty\}$, where an infinite delay $d = \infty$ essentially denotes a dropout. In the subsequent analysis we will use $\bar{\mathcal{D}}$ to compactly represent both delay and dropout. To that end, for all $t \in \mathbb{N}_0$,

$$\mathbb{P}(d_t = d) = \begin{cases} q_t(d) \triangleq (1-\gamma_t)p_t(d), & d \in \mathcal{D}, \\ \gamma_t, & d = \infty. \end{cases} \tag{3}$$

Widely used models for delay and packet dropout (e.g., geometric delay and Markovian packet loss processes) are accommodated in our model setup (see Section III-C).

## III. Optimal Estimation and Control

### A. Estimation under Stochastic Delays and Dropouts

To investigate the estimation process at the controller, we construct the measurement set (information set) available to the controller. Due to the random delays and packet dropouts, not all transmitted packets prior to time $t$ will be available to the controller at time $t$. For all $s \leq t$, let $\beta_{s,t} \in \{0,1\}$ denote whether the data packet pertaining to time $s$ (i.e., $X_s$) is available to the controller at time $t$. Furthermore, at any time $t$, let the random variable $\tau_t$ denote the last time instance for which a packet has arrived at the controller, i.e., $\tau_t \triangleq \max\{s \mid \beta_{s,t} = 1\}$. We adopt the convention that $\tau_t \triangleq -1$ if and only if $\beta_{s,t} = 0$ for all $0 \leq s \leq t$. Therefore, for any $t$, the event $\{\tau_t = -1\}$ denotes that none of the transmitted packets has arrived at the controller by time $t$. The information set $\mathcal{I}_t$ at the controller contains the state information that has arrived up to time $t$, i.e., $\mathcal{I}_t = \{X_s | \ \beta_{s,t} = 1\}$. From the definition of $\tau_t$, the controller does not have any information regarding the realization of the random variables $\{W_r\}_{r\geq\tau_t}$ since $\{W_r\}_{r\geq\tau_t}$ is independent of the $\sigma$-field generated by $\{X_s\}_{s\leq\tau_t}$. Thus, we obtain $\mathbb{E}[W_r|\mathcal{I}_t] = 0$, $\forall r \geq \tau_t$.

It will be shown in Corollary 1 that the optimal controller is $U_t = -L_t \,\mathbb{E}[X_t \mid \mathcal{I}_t]$. Therefore, the optimal estimator is the conditional expectation $\mathbb{E}[X_t|\mathcal{I}_t]$, which is expressed as

$$\begin{aligned}\mathbb{E}[X_t \mid \mathcal{I}_t] =& \mathbb{E}[A^{t-\tau_t}X_{\tau_t} + \sum_{k=\tau_t}^{t-1} A^{t-k-1}(BU_k + W_k) \mid \mathcal{I}_t] \\ =& A^{t-\tau_t}X_{\tau_t} + \sum_{k=\tau_t}^{t-1} A^{t-k-1}BU_k,\end{aligned} \tag{4}$$

where we have used the fact that $X_{\tau_t}$ and $\{U_k\}_{k=0}^{t-1}$ are $\mathcal{I}_t$-measurable, and $\mathbb{E}[W_k \mid \mathcal{I}_t] = 0$ for all $k \geq \tau_t$. The estimation error, defined as $\Delta_t \triangleq X_t - \mathbb{E}[X_t|\mathcal{I}_t]$, therefore becomes

$$\Delta_t = \sum_{k=\tau_t}^{t-1} A^{t-k-1}W_k. \tag{5}$$

Note that the estimation error $\Delta_t$ in (5) depends on the random delay and dropout through the term $\tau_t$. The distribution of the random variable $\tau_t$ entirely depends on the distributions of the delay and dropout. Next, we discuss the effects of delay and dropout distributions on $\Delta_t$ (or equivalently on $\tau_t$).

From the definition of the random variable $\tau_t$, we first conclude that $\tau_t \leq t$, *almost surely*. Furthermore, the event $\{\tau_t = t\}$ denotes that the delay $d_t$ experienced by the packet transmitted at time $t$ is 0, and hence $\mathbb{P}(\tau_t = t) = \mathbb{P}(d_t = 0) = q_t(0)$. Similarly, the event $\tau_t = t-1$ denotes that the delay $d_{t-1}$ experienced by the packet sent at time $t$ is at most 1 and the delay $d_t$ experienced by the packet transmitted at time $t$ is at least 1, that is, $\{\tau_t = t-1\} = \{d_{t-1} \leq 1\} \cap \{d_t \geq 1\}$. Consequently, based on the channel model from Section II-B, we obtain that $\mathbb{P}(\tau_t = t-1) = \mathbb{P}(d_t \geq 1)\,\mathbb{P}(d_{t-1} \leq 1) = (1-q_t(0))(q_{t-1}(0) + q_{t-1}(1))$. In fact, one obtains

$$\begin{aligned}\mathbb{P}(\tau_t = t-k) &= \left[\prod_{i=0}^{k-1}\left(1 - \sum_{s=0}^{i} q_{t-i}(s)\right)\right]\sum_{\ell=0}^{k} q_{t-k}(\ell), \\ \mathbb{P}(\tau_t = -1) &= \prod_{i=0}^{t}\left(1 - \sum_{s=0}^{i} q_{t-i}(s)\right).\end{aligned} \tag{6}$$

As discussed in Section II-B, the distributions of delay and packet dropout are directly affected by the network traffic, which in turn affects the distribution of $\tau_t$ through the terms $q_{\cdot}(\cdot)$ in the expression of $\mathbb{P}(\tau_t)$ in (6). Thus, the impact of network traffic on the estimation error $\Delta_t$ appears via the random variable $\tau_t$. In the next section, we explicitly study this impact and discuss the traffic-induced performance degradation measured by the cost function (2).

### B. Optimal Control Performance with Imperfect Channel

In this section, we characterize the optimal achievable cost (2) for given distributions of delay and packet dropout. As mentioned earlier in Section II-B, several distinct models for network traffic have been discussed in the literature [19], and, based on the network infrastructure, the effects of network traffic on delay and packet loss can be different. For this section, we consider fairly general distributions for the delay and the dropouts to present our results. Later in Section III-C, we consider particular models for the network traffic and the induced delay and dropouts to discuss some special results under those modelling considerations.

Before proceeding, let us introduce the Riccati equation associated with a standard LQG problem (i.e., perfect channel with no delay or dropout)

$$P_t = Q + A^{\intercal}P_{t+1}A - N_t, \quad P_T = Q_T, \tag{7a}$$

$$N_t = L_t^{\intercal}(R + B^{\intercal}P_{t+1}B)L_t, \tag{7b}$$

$$L_t = (R + B^{\intercal}P_{t+1}B)^{-1}B^{\intercal}P_{t+1}A, \tag{7c}$$

where $P_t \succeq 0, N_t \succeq 0$ and $L_t$ are matrices with appropriate dimensions and $A, B, R, Q$ and $Q_T$ are system parameters as given in (1)-(2). Recall that the optimal LQG cost for an LTI system (1)–(2) operating over a perfect channel (i.e., without delay or packet loss) is $\sum_{t=0}^{T} \mathrm{tr}(P_t\mathcal{W}_{t-1})$, where $P_t$ follows the control Riccati equation (7), and the matrices $\mathcal{W}_{-1}$ and $\mathcal{W}_t$ are the covariances of the initial state $X_0$ and the noise $W_t$, respectively. The next theorem provides the optimal cost $J^*$ for the same system (1)–(2) using a shared network with traffic-induced stochastic delay and packet dropouts.

**Theorem 1.** Consider an LTI stochastic control system (1) that uses a traffic-shared communication network to close its feedback loop. For any given distributions of delay and dropout, the optimal achievable LQG cost, as defined in (2), over a finite horizon $[0, T]$ is

$$J^* = \sum_{t=0}^{T} \operatorname{tr}(\tilde{P}_t \mathcal{W}_{t-1}), \tag{8}$$

where

$$\tilde{P}_t = P_t + \Upsilon_{t-1}, \quad \forall t = 0, \ldots, T-1; \quad \tilde{P}_T = P_T, \tag{9a}$$

$$\Upsilon_t = \sum_{k=t+1}^{T-1} \pi_{k,t} (A^\intercal)^{k-t-1} N_k A^{k-t-1}, \tag{9b}$$

$$\pi_{t,k} = \mathbb{P}(\tau_t \leq k) = \prod_{i=k+1}^{t} \left(1 - \sum_{s=0}^{t-i} q_i(s)\right), \tag{9c}$$

and $P_t$ follows the Riccati equation given in (7).

*Proof:* Let the value function at time $t$ associated with the cost (2) be denoted as $V_t$, i.e.,

$$V_t(\mathcal{I}_t) = \min_{\{U_k\}_{k=t}^{T-1}} \mathbb{E}\left[\sum_{k=t}^{T-1} \|X_k\|_Q^2 + \|U_k\|_R^2 + \|X_T\|_{Q_T}^2\right].$$

The first step in the proof is to verify that $V_t(\mathcal{I}_t)$ is of the following decoupled form

$$V_t(\mathcal{I}_t) = \mathbb{E}[\|X_t\|_{P_t}^2] + C_t(\mathcal{I}_t), \tag{10}$$

where $C_t(\mathcal{I}_t)$ is independent of the control strategy, while it depends on the delay and packet dropout distributions. The matrix $P_t$ follows the standard Riccati equation (7). The hypothesis on the structure of $V_t$ trivially holds for time $t = T$ with $C_T(\mathcal{I}_T) = 0$. Using backward induction, we now show that (10) holds for any time $t$. To that end, we assume that the hypothesis holds for some time $k+1$ and show that this assumption leads to the fact that the hypothesis holds true for time $k$ as well. Using dynamic programming on $V_k(\mathcal{I}_k)$:

$$\begin{aligned}
V_k(\mathcal{I}_k) &= \min_{U_k} \mathbb{E}\left[\|X_k\|_Q^2 + \|U_k\|_R^2 + V_{k+1}(\mathcal{I}_{k+1})\right] \\
&\overset{(\#)}{=} \min_{U_k} \mathbb{E}\left[\|X_k\|_Q^2 + \|U_k\|_R^2 + \|X_{k+1}\|_{P_{k+1}}^2\right] + C_{k+1}(\mathcal{I}_{k+1}) \\
&= \min_{U_k} \mathbb{E}[\|U_k + L_k X_k\|_{R + B^\intercal P_{k+1} B}^2] \\
&\quad + \mathbb{E}[\|X_k\|_{P_k}^2] + \operatorname{tr}(P_{k+1}\mathcal{W}_k) + C_{k+1}(\mathcal{I}_{k+1}) \\
&\overset{(\dagger)}{=} \mathbb{E}[\|X_k\|_{P_k}^2] + C_k(\mathcal{I}_k),
\end{aligned}$$

where $(\#)$ follows from the assumption that (10) is true for time $k+1$ and $(\dagger)$ follows from the fact that the optimal $\mathcal{I}_t$-measurable controller that minimizes $\mathbb{E}[\|U_t + L_t X_t\|_{R+B^\intercal P_{t+1} B}^2]$ is $U_t = -L_t \mathbb{E}[X_t \mid \mathcal{I}_t]$, and we define

$$C_t(\mathcal{I}_t) = C_{t+1}(\mathcal{I}_{t+1}) + \operatorname{tr}(P_{t+1}\mathcal{W}_t) + \mathbb{E}[\|\Delta_t\|_{N_t}^2].$$

Based on the definition of $C_t(\mathcal{I}_t)$, we verify that $C_t(\mathcal{I}_t)$ does not depend on the control law. Furthermore, the value function $V_t(\mathcal{I}_t)$ is indeed of the form (10). Therefore, we may write

$$\begin{aligned}
J^* &= V_0(\mathcal{I}_0) = \operatorname{tr}(P_0 \mathcal{W}_{-1}) + C_0(\mathcal{I}_0) \\
&= \sum_{t=0}^{T} \operatorname{tr}(P_t \mathcal{W}_{t-1}) + \sum_{t=0}^{T-1} \mathbb{E}[\|\Delta_t\|_{N_t}^2].
\end{aligned} \tag{11}$$

Now, based on the expression of $\Delta_t$ in (5), we obtain $\mathbb{E}[\|\Delta_t\|_{N_t}^2 \mid \tau_t] = \sum_{k=\tau_t}^{t-1} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k)$, and therefore, we may write

$$\begin{aligned}
\mathbb{E}[\|\Delta_t\|_{N_t}^2] &= \sum_{s=-1}^{t-1} \sum_{k=s}^{t-1} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k)\, \mathbb{P}(\tau_t = s) \\
&= \sum_{k=-1}^{t-1} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k) \sum_{s=-1}^{k} \mathbb{P}(\tau_t = s) \\
&= \sum_{k=-1}^{t-1} \pi_{t,k} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k),
\end{aligned} \tag{12}$$

where $\pi_{t,k}$ is given in (9c). Furthermore, from (12), we obtain

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E}[\|\Delta_t\|_{N_t}^2] &= \sum_{t=0}^{T-1} \sum_{k=-1}^{t-1} \pi_{t,k} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k) \\
&= \sum_{k=-1}^{T-2} \sum_{t=k+1}^{T-1} \pi_{t,k} \operatorname{tr}(\|A^{t-k-1}\|_{N_t}^2 \mathcal{W}_k) \\
&= \sum_{k=-1}^{T-2} \operatorname{tr}(\Upsilon_k \mathcal{W}_k),
\end{aligned} \tag{13}$$

where $\Upsilon_k$ is defined in (9b). Finally, by combining (13) and (11) and using (9a), we obtain

$$J^* = \sum_{t=0}^{T} \operatorname{tr}(P_t \mathcal{W}_{t-1}) + \sum_{k=-1}^{T-2} \operatorname{tr}(\Upsilon_k \mathcal{W}_k) = \sum_{t=0}^{T} \operatorname{tr}(\tilde{P}_t \mathcal{W}_{t-1}),$$

This completes the proof. □

Theorem 1 not only provides the optimal performance but also characterizes the optimal control law for the closed-loop system. The optimal control law for this problem is of certainty-equivalence type with the feedback gain $L_t$ being the same as the gain for the standard LQG problem. The effects of the network imperfections are reflected only in the estimation process. This is formally stated in the following corollary.

**Corollary 1** (Certainty equivalence control law)**.** The optimal control that minimizes the LQG cost defined in (2) is

$$U_t = -L_t \mathbb{E}[X_t \mid \mathcal{I}_t],$$

where the optimal gain $L_t$ and the optimal estimator $\mathbb{E}[X_t \mid \mathcal{I}_t]$ are given in (7c) and (4), respectively.

**Remark 1.** The optimal achievable cost in Theorem 1 resembles the optimal cost under a perfect communication channel (i.e., no delay and dropouts). In case of perfect transmissions, the optimal cost $J^*$ is $\sum_{t=0}^{T} \operatorname{tr}(P_t \mathcal{W}_{t-1})$, which, as expected, is smaller than the cost in (8), since clearly $P_t \preceq \tilde{P}_t$, $\forall t$.

From the proof of Theorem 1, we notice that the degradation in QoC (equivalently, the increase in the value of $J^*$) due to delay and dropouts is completely characterized by the matrices $\Upsilon_t = \tilde{P}_{t+1} - P_{t+1}, \forall t$. We conclude this section with the following remark that compares the performance of the closed-loop control system with its open-loop counterpart.

**Remark 2.** The optimal cost for the NCS (1)–(2) operating in open-loop is $\sum_{t=0}^{T} \operatorname{tr}(\Psi_t \mathcal{W}_{t-1})$, with $\Psi_t = Q + A^\intercal \Psi_{t+1} A$, and $\Psi_T = Q_T$. One may verify that $\Psi_t - P_t = \sum_{k=t}^{T-1} (A^\intercal)^{k-t} N_k A^{k-t} \succeq \Upsilon_{t-1}$. Furthermore, $\Upsilon_{t-1} = \Psi_t - P_t$ if and only if $\pi_{k,t-1} = 1, \forall k \geq t$. Thus, $\Upsilon_{t-1} = \Psi_t - P_t$ for all $t$ occurs if only if $\gamma_t = 1, \forall t$. This is intuitive because $\gamma_t = 1, \forall t$ declares that all the transmitted packets will be dropped with probability one, and hence, the system operates in open-loop at all times.

### *C. Traffic-induced Quality-of-Control: A Special Case*

Now we further study the optimal cost $J^*$ by considering particular models for the stochastic delay and dropout. We consider a Poisson arrival process for the network traffic [19] with parameter $\lambda$. The discrete delay[1] induced by this network traffic is modeled by a geometric distribution with parameter $p(\lambda) \in (0,1)$, i.e., $\mathbb{P}(d_t = d \mid \mu_t = 1) = p(\lambda)(1-p(\lambda))^d$. We write $p(\lambda)$ to declare that the delay distribution correlates with real-time network traffic parameter $\lambda$. The exact relation between $p(\lambda)$ and $\lambda$ depends on the network infrastructure and available communication resource, however, $p(\cdot)$ is intuitively assumed to be non-increasing, i.e., as network traffic increases, a higher induced delay is more probable.

Packet dropouts are modelled according to the well-known non-i.i.d. (Gilbert-Elliott type) Markovian model [20]. The i.i.d packet drop model is a special case of this Markovian model. The transition probabilities for packet dropouts are $\mathbb{P}(\mu_t = 1 \mid \mu_{t-1} = 1) = \bar{\mu}(\lambda)$, $\mathbb{P}(\mu_t = 0 \mid \mu_{t-1} = 0) = \underline{\mu}(\lambda)$ and the initial probability $\mathbb{P}(\mu_0 = 0) = \gamma_0$. Therefore, for any $t$,

$$\begin{bmatrix} \gamma_t \\ 1-\gamma_t \end{bmatrix} = \begin{bmatrix} \mathbb{P}(\mu_{t+1}=0) \\ \mathbb{P}(\mu_{t+1}=1) \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\mu} & 1-\bar{\mu} \\ 1-\underline{\mu} & \bar{\mu} \end{bmatrix}^t}_{M(\lambda)^t} \begin{bmatrix} \gamma_0 \\ 1-\gamma_0 \end{bmatrix},$$

where $M(\lambda)$ is the probability transition matrix.

**Lemma 1.** Under the described Poisson network traffic, geometric delay, and Markovian packet dropout models, we have

$$\pi_{t,k} = \prod_{s=k+1}^{t} \Big( \gamma_s(\lambda) + (1-\gamma_s(\lambda))(1-p(\lambda))^{t-s+1} \Big).$$

*Proof:* The proof of this lemma follows from the construction of the probabilities $q_t(d) = \mathbb{P}(d_t = d)$ defined in (3) and using (6) to compute the final expression for $\pi_{t,k}$. A detailed derivation has been omitted due to page limitation.

[1] As discussed in Section II-B, we are interested in discrete delays due to the discrete-time nature of the dynamics (1). Following an exponential distribution with parameter $p$ for the delay (i.e., $\mathbb{P}(d_t \in \mathrm{d}x) = pe^{-px}\mathrm{d}x$ ), its discrete-time version, i.e., $\mathbb{P}(d_t = d) \triangleq \mathbb{P}(d_t \in [d\delta, (d+1)\delta))$ follows a geometric distribution with parameter $1 - e^{-p\delta}$, where $\delta$ denotes the sampling period.

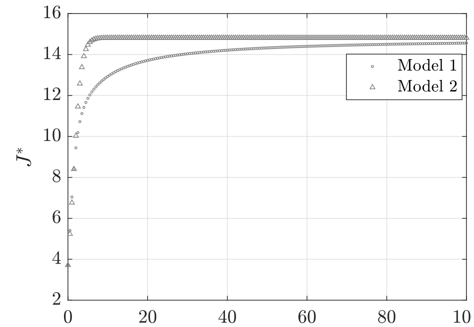


Fig. 2. Optimal cost $J^*$ vs. the traffic arrival rate $\lambda$.

## IV. SIMULATION RESULTS

In this section, we simulate the performance of a control system operating over a traffic-shared network. To obtain the results we conduct several numerical simulations. We consider $A = \begin{bmatrix} 1.01 & 0.5 \\ 0.02 & 0.7 \end{bmatrix}$, $B = I$, $Q = Q_T = R = \frac{1}{2}I$, $T = 10$ and $\mathcal{W}_t = \frac{1}{4}I$, for all $t$. We gradually increase the network traffic arrival rate and record the resulting QoC degradation.

**Model 1:** We consider the following models for $p$, $\bar{\mu}$ and $\underline{\mu}$:

$$p(\lambda) = \frac{1}{1+\lambda}, \quad \bar{\mu}(\lambda) = e^{-\lambda}, \quad \underline{\mu}(\lambda) = 0.7(1-e^{-\lambda}).$$

The selected models above ensure that, when the traffic arrival rate $\lambda$ is low, the probability of a higher delay decreases, and packet drop becomes an unlikely event. We notice that $p(\lambda) \to 0$ when $\lambda \to \infty$, which implies that packets will experience an infinite delay as traffic rate goes to infinity, and consequently, the system operates in open-loop. This is intuitive when operating over a network where *jamming* happens due to unusually high arrival of other traffic.

**Model 2:** The next models that we set for $p$, $\bar{\mu}$ and $\underline{\mu}$ are

$$p(\lambda) = e^{-\lambda}, \ \bar{\mu}(\lambda) = \frac{1}{1+\log(1+\lambda)}, \ \underline{\mu}(\lambda) = 0.5(1-e^{-\lambda}).$$

For both sets of models we consider $\gamma_0 = 0.5$. In Fig. 2, we plot $J^*$ over $\lambda$. As expected, we notice that the optimal cost increases with the traffic rate $\lambda$. Another observation from Fig. 2 is that, for the same NCS with the same traffic arrival rate $\lambda$, different performances are obtained depending on the chosen model for $p, \bar{\mu}$ and $\underline{\mu}$. Therefore, it brings out another important aspect of this problem - the delay-dropout trade-off.

### *A. Discussions: Delay-dropout Trade-off*

The effects of network traffic on the delay and dropouts are coupled and network infrastructure dependent. Based on the available resources and the network technology, it is possible to trade a higher delay with a lower dropout probability (e.g. implementing a longer queuing buffer and re-transmissions of previously dropped packets) and vice-versa for a given traffic rate. That is, for the same traffic distribution parameter $\lambda$, the network may offer a variety of choices for the parameters $p(\lambda), \bar{\mu}(\lambda)$ and $\underline{\mu}(\lambda)$ of the model discussed in Section III-C. In this numerical experiment, we investigate the allowable tolerances on delay and dropout probabilities for a given QoC. In Fig. 3 (top), we plot the optimal cost $J^*$ versus the variations in the two parameters $p(\lambda)$ and $\bar{\mu}(\lambda)$, while fixing $\underline{\mu}(\lambda) = 0.5$ and $\gamma_0 = 0.5$. We observe that for a given value of $J^*$ (i.e., QoC), several combinations of $p(\lambda)$ and $\bar{\mu}(\lambda)$ are feasible. We assume that for a given traffic arrival rate $\lambda$, the parameters $p(\lambda), \bar{\mu}(\lambda)$ and $\underline{\mu}(\lambda)$ are constrained to be $\{p(\lambda), \bar{\mu}(\lambda), \underline{\mu}(\lambda)\} \in \mathcal{C}(\lambda) \subseteq \mathbb{R}^3_+$. Such constraints may arise, for example, due to resource limitations. The network serving the control system (1) might be interested in choosing the parameters $\{p(\lambda), \bar{\mu}(\lambda), \underline{\mu}(\lambda)\} \in \mathcal{C}(\lambda)$, such that cost $J^*$ is minimized, or $J^* \leq c$, for some given $c$. To better illustrate this, we plot the feasible $(p, \bar{\mu})$ region for $J^* \leq 6$ in Fig. 3 (bottom) where we also plot a hypothetical constraint set $\mathcal{C}(\lambda)$ to illustrate how $\mathcal{C}(\lambda)$ may affect the achievable optimal cost $J^*$. The (red) square on the boundary of the set $\mathcal{C}(\lambda)$ denotes the optimal choice for $(p(\lambda), \bar{\mu}(\lambda))$ that results the minimum $J^*$. These preliminary results point toward a deeper connection between the achievable QoC and the traffic-induced network conditions for decentralized multi-agent NCSs, which is yet to be explored.

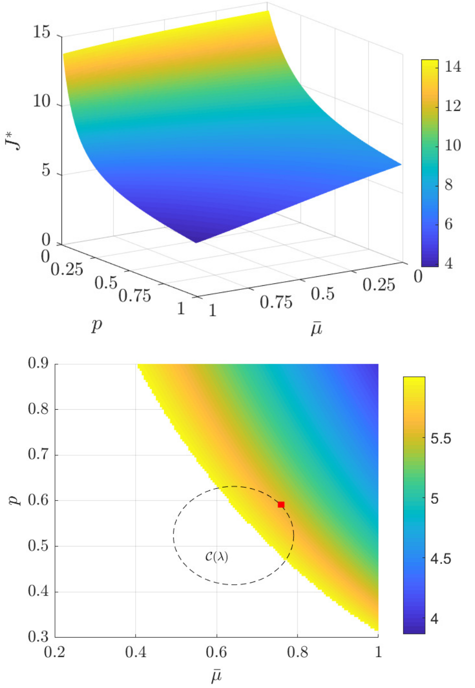


Fig. 3. Top: Cost $J^*$ versus $p$ and $\bar{\mu}$ with $\underline{\mu} = 0.5$ and $\gamma_0 = 0.5$. Bottom: The colored region on the $(p, \bar{\mu})$ plane shows the feasible choices for $p$ and $\bar{\mu}$ under the constraint $J^* \leq 6$. The interior of the dashed ellipse illustrates $\mathcal{C}(\lambda)$, i.e., the set of attainable $(p(\lambda), \bar{\mu}(\lambda))$ pairs for a given traffic parameter $\lambda$. The red square denotes the optimal $J^*$ when $(p(\lambda), \bar{\mu}(\lambda)) \in \mathcal{C}(\lambda)$.

## V. Conclusion

In this letter, we investigate the effects of network-induced latency and packet loss on the finite-horizon performance of an LTI control system operating over a shared channel. Performance of the control system is tied to the rest of network users through the network traffic that influences the real-time delay and packet loss realizations. To characterize this coupling, we consider a generic joint model of delay and packet dropout that is correlated with the network traffic. Characterizing the effects of network-induced delay and packet dropout on the LQG cost provides a crucial design contemplation for NCSs so that, depending on the traffic state, an NCS can determine its maximum tolerable combined network latency and reliability to achieve a guaranteed QoC performance. Theorem 1 captures the connection between network traffic and QoC and provides comparisons with open-loop and closed-loop (under perfect channel) operations. It also provides a formal framework to study the delay-dropout trade-off for multi-system decentralized NCSs, which is currently unexplored in the literature.

## References

[1] L. Zhang, H. Gao, and O. Kaynak, "Network-induced constraints in networked control systems—a survey," IEEE Trans. on Industrial Informatics, vol. 9, no. 1, pp. 403–416, 2013.

[2] J. P. Hespanha and A. R. Mesquita, "Networked control systems: Estimation and control over lossy networks," Encyclopedia of Systems and Control, pp. 842–849, 2015.

[3] X. Zhang, Q. Han, X. Ge, D. Ding, L. Ding, D. Yue, and C. Peng, "Networked control systems: a survey of trends and techniques," IEEE/CAA Journal of Automatica Sinica, vol. 7, no. 1, pp. 1–17, 2020.

[4] H. Xu, W. Yu, D. Griffith, and N. Golmie, "A survey on Industrial Internet of Things: A cyber-physical systems perspective," IEEE Access, vol. 6, pp. 78 238–78 259, 2018.

[5] B. Chen, J. Wan, L. Shu, P. Li, M. Mukherjee, and B. Yin, "Smart factory of Industry 4.0: Key technologies, application case, and challenges," IEEE Access, vol. 6, pp. 6505–6519, 2018.

[6] G. A. Akpakwu, B. J. Silva, G. P. Hancke, and A. M. Abu-Mahfouz, "A survey on 5G networks for the Internet of Things: Communication technologies and challenges," IEEE Access, vol. 6, pp. 3619–3647, 2018.

[7] M. S. Branicky, S. M. Phillips, and W. Zhang, "Stability of networked control systems: Explicit analysis of delay," in Proceedings of the 2000 American Control Conference. IEEE, 2000, pp. 2352–2357.

[8] W. P. M. H. Heemels, A. R. Teel, N. van de Wouw, and D. Nešić, "Networked control systems with communication constraints: Tradeoffs between transmission intervals, delays and performance," IEEE Trans. on Automatic Control, vol. 55, no. 8, pp. 1781–1796, 2010.

[9] Y. Shi and B. Yu, "Output feedback stabilization of networked control systems with random delays modeled by Markov chains," IEEE Trans. on Automatic Control, vol. 54, no. 7, pp. 1668–1674, 2009.

[10] M. H. Mamduhi, D. Tolić, A. Molin, and S. Hirche, "Event-triggered scheduling for stochastic multi-loop networked control systems with packet dropouts," in 53rd IEEE Conference on Decision and Control, Los Angeles, USA, December 15–17, 2014, pp. 2776–2782.

[11] H. Yang, Y. Xu, and J. Zhang, "Event-driven control for networked control systems with quantization and Markov packet losses," IEEE Trans. on Cybernetics, vol. 47, no. 8, pp. 2235–2243, 2017.

[12] M. Palmisano, M. Steinberger, and M. Horn, "Optimal finite-horizon control for networked control systems in the presence of random delays and packet losses," IEEE Control Systems Letters, vol. 5, no. 1, pp. 271–276, 2021.

[13] D. Maity, M. H. Mamduhi, S. Hirche, K. H. Johansson, and J. S. Baras, "Optimal LQG control under delay-dependent costly information," IEEE Control Systems Letters, vol. 3, no. 1, pp. 102–107, 2019.

[14] M. H. Mamduhi, D. Maity, S. Hirche, J. S. Baras, and K. H. Johansson, "Delay-sensitive joint optimal control and resource management in multi-loop networked control systems," IEEE Transactions on Control of Network Systems, 2021.

[15] L. Schenato, "Optimal estimation in networked control systems subject to random delay and packet drop," IEEE Trans. on Automatic Control, vol. 53, no. 5, pp. 1311–1317, 2008.

[16] Y. Ouyang, S. M. Asghari, and A. Nayyar, "Optimal infinite horizon decentralized networked controllers with unreliable communication," IEEE Trans. on Automatic Control, vol. 66, no. 4, pp. 1778–1785, 2021.

[17] J. Nilsson, B. Bernhardsson, and B. Wittenmark, "Stochastic analysis and control of real-time systems with random time delays," Automatica, vol. 34, no. 1, pp. 57–64, 1998.

[18] Y. Tsang, M. Coates, and R. D. Nowak, "Network delay tomography," IEEE Trans. on Signal Processing, vol. 51, no. 8, pp. 2125–2136, 2003.

[19] K. K. Leung, W. A. Massey, and W. Whitt, "Traffic models for wireless communication networks," IEEE Journal on Selected Areas in Communications, vol. 12, no. 8, pp. 1353–1364, 1994.

[20] G. Haßlinger and O. Hohlfeld, "The Gilbert-Elliott model for packet loss in real time services on the internet," in 14th GI/ITG Conference-Measurement, Modelling and Evalutation of Computer and Communication Systems. VDE, 2008, pp. 1–15.