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# Energy and Dissipation in Consensus Systems

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*The birth and death of the leaves*
*are the rapid whirls of the eddy whose wider circles*
*move slowly among stars.*

Stray Birds: 92, Rabindranath Tagore

## Abstract

The linear consensus protocol and Laplacian system paradigm form a pillar in theories that emerged with the rise of today's era of networks and massive data sets. The fragmented nature of the studies and communities working on consensus problems demonstrates its complexity and richness, despite the simple model structure. As a result, today, we face various abstract technical frameworks for the analysis and design of consensus systems, which often are hardly accessible or applicable in applications in natural sciences, technology, and other practice-oriented fields. We propose a simple, comprehensive, and integrative framework for the stability and convergence analysis and the protocol design for exponentially converging consensus networks irrespective of a linear or nonlinear appearance. Core to this framework is the transformation of a system of coupled nonlinear differential equations to an equivalent linear Laplacian form. We technically build on axiomatic properties of metrics and mean functions, combined with a general and abstract convergence result in linear consensus theory. This framework is simple, as it builds on a classical network synthesis result for general passive electric circuits, which involves a dissipative-lossless and memory-memoryless decomposition of system equations that completely defines the dissipation and energy storage behavior. This framework is integrative as the common thread of passive circuit synthesis directly yields system formulations resulting in the recent study of gradient flows and optimal transport on discrete (probability) and metric spaces. It is compatible with majorization theory and provides a majorization-passivity equivalence in a differential form; further, we show equivalence to the recent cut-balance convergence result for general linear consensus networks. This framework also includes the most known and general consensus protocol design rules and stability results, and as such, we understand it as comprehensive. We derive three main classes of consensus protocols from our framework: the comprehensive metric action consensus protocol, a novel mean-control consensus protocol class with optimality properties of the differential dynamics, rather than the known optimality of the asymptotic behavior, and an embedding protocol, which can also be obtained as a result of the existing geometric generalization of linear consensus theory to nonlinear space. Application of the mean-control protocol class to obtain geometric mean-driven consensus systems results in a consensus condition reminiscent of the Wegscheider relation and detailed balance in chemistry. Application to the analysis of the dynamics of mass-action chemical reaction networks yields a novel conductance formulation for these. This is significant as it provides for the first time a circuit formulation in which chemical potentials are directly connected through a resistor element. The new methods and analysis views we present here provide a novel direction in the study of convergence and stability in nonlinear

dynamical systems by using the equivalence to an equivalent linear Laplacian form, which shows differentially dissipative behavior.

## Zusammenfassung

Das lineare Konsensusprotokoll, einhergehend mit Laplace'schen Systemformulierungen, bildet eine Grundlage für Theorien und Methoden, die im Zuge des Aufkommens der heutigen Ära von Netzwerken und Big Data entwickelt wurden. Die Komplexität und Reichhaltigkeit des Konsensusproblems wird anhand der Fragmentiertheit und Vielzahl wissenschaftlicher Arbeiten und Communities in diesem und verbundenen Themengebieten deutlich; trotz der Einfachheit der Modellstruktur. In Konsequenz verfügen wir heute über eine Vielzahl abstrakter und anwendungsorientierter Frameworks zum Entwurf konvergierender Konsensussysteme, oder zu deren Untersuchung hinsichtlich Stabilität und Konvergenzeigenschaften. Eine Einordnung solcher Frameworks zueinander wurde bislang wenig betrachtet, und gleichzeitig sind abstrakte Entwurfs- oder Analysemethoden für Anwender in realen Problemstellungen schwer zugänglich oder kaum brauchbar. Wir führen hier ein einfaches, umfassendes und anschlussfähiges Framework ein, dass die Stabilitäts- und Konvergenzuntersuchung, sowie den Entwurf exponentiell konvergierender Konsensussysteme erlaubt, unabhängig von der linearen oder nichtlinearen Gestalt des Systems. Kern unseres Frameworks ist die Transformation nichtlinearer Systeme gekoppelter Differentialgleichungen in eine äquivalente lineare Laplace-Form, wobei wir uns technischerseits axiomatische Eigenschaften von Distanz- und Mittelfunktionen zu Nutze machen, sowie ein grundlegendes abstraktes Konvergenzergebnis der linearen Konsensustheorie. Mit der Eigenschaft der Einfachheit nehmen wir Bezug auf die konzeptuelle Basis dieses Frameworks: ein klassisches Ergebnis der Netzwerksynthese für passive Schaltungen, das die Dekomposition eines Systems in dissipative und verlustfreie, bzw. zustandslose und Zustands-behaftete Systemteile vorsieht, welches das Dissipationsverhalten und energetische Eigenschaften vollständig bestimmt. Anschlussfähigkeit bedeutet hier, dass die passive Netzwerksynthese ein roter Faden ist, der zu Systembeschreibungen führt, wie sie aktuell auch im Bereich der Analyse von Gradientenflüssen auf diskreten (Wahrscheinlichkeits-) und metrischen Räumen resultieren. Er erlaubt auch eine Äquivalenz zwischen dem Konzept der Majorisierung und Passivität jedoch in einem neuen, differentiellen Zusammenhang; darüberhinaus zeigen wir eine Äquivalenz zum kürzlich eingeführten cut-balance Kriterium für konvergente lineare Konsensusnetzwerke. Unser Framework erlaubt es zudem die bekannten Ergebnisse für den Entwurf von Konsensusprotokollen und deren Stabilitätsanalyse abzuleiten, bzw. beinhaltet diese. In diesem Sinne verstehen wir unser Framework als umfassend. Aus unserem Framework leiten wir drei fundamentale Klassen an Konsensusprotokollen ab: das umfassende Protokoll gemäß metrischen Wirkprinzip, ein neuartiges Protokoll gemäß Steuerung durch Mitteln, welches zu Optimalitätseigenschaften der Dynamik - im Gegensatz zu bekannten Asymptotischen - besitzt, sowie das Einbettungsprotokoll, welches auch als Ergebnis der bekannten geometrischen Verallgemeinerung von Konsensus auf linearen zu nichtlinearen Zustandsräumen erhalten werden kann. Anwendung unseres Frameworks zum Entwurf von Konsensusprotokollen mit dem geometrischen Mittel als treibende Kraft führt zu einer Stabilitätsbedingungen, die dem Wegscheider detailliertem

Gleichgewicht in der Chemie ähnelt. Anwendung zur Analyse von chemischen Reaktionsnetzen mit Dynamik gemäß dem Massenwirkungsgesetz führt zu einer neuen elektrischen Schaltungs- und Leitwertdarstellung, in der chemische Potenziale zum ersten Mal direkt über ein Widerstandselement verkoppelt sind und wirken. Die hier präsentierten Methoden können zu neuen Ansätzen für die Konvergenz- und Stabilitätsanalyse nichtlinearer dynamischer Systeme führen, indem die Äquivalenz zu einer linearen Laplaceform genutzt wird, die differentiell dissipatives Verhalten bewirkt.

# Notations

## Abbreviations

| | |
|---|---|
| LTI | linear time-invariant |
| w.r.t. | with respect to |
| cf. | compare |
| e.g. | for example |
| i.e. | in other words |

## Conventions

**Scalars, Vectors, and Matrices**

*Scalars* are denoted by upper and lower case letters in italic type. *Vectors* are denoted by lower case letters in bold type, as the vector $\boldsymbol{x}$ is composed of elements $x_i$. *Matrices* are denoted by upper case letters in bold type; a matrix $\mathbf{M}$ is composed of elements $[\mathbf{M}]_{ij}$, or $m_{ij}$ ($i^{\text{th}}$ row, $j^{\text{th}}$ column). For a scalar function $f : \mathbb{R} \to \mathbb{R}$, and $\boldsymbol{x} \in \mathbb{R}^n$, the element-wise notation $\boldsymbol{x} \mapsto f(\boldsymbol{x})$ is defined as the mapping whose $i$-th component is given as $(f(\boldsymbol{x}))_i = f(x_i)$.

**Subscripts and Superscripts**

| | |
|---|---|
| $(\cdot)^\top$ | Transposed |
| $(\cdot)^{-1}$ | Inverse |
| $(\cdot)^\dagger$ | Pseudoinverse |

# Contents

# 1

# Introduction

## 1.1 The emerging Laplacian paradigm

Linear consensus systems form a pillar in the emerging field of systems and control theory for networks. It is arguably the simplest setup to model nontrivial network systems dynamics; arguably, it is the simplest linear system that depicts non-trivial asymptotic behavior, with equilibrium states being generally different from the origin as in classical linear systems. Convergence and stability studies even support an argument in favor of a "perhaps not so linear nature of the underlying state space of linear consensus algorithms" [Sep11]. Linear consensus models, being governed by Laplacian matrices, are diffusion systems. And as such, they provide a connection to a variety of other fields in mathematics, physics, and applications that build on linear or nonlinear diffusion models or the heat equation. Examples range from problems in electric power grids [HC06; DB10; DB14; DH14], chemical reaction networks [EG07; vRJ13b; vRJ13a; WMv18], multi-robot stabilization and coordination problems [SPL08; BCM09], neuro-science and other synchronization phenomena [BHD10; Ace+05; SSS07; Str00], to peer-to-peer sensing and gossiping [Boy+06], image processing and data filtering [Tau95; TZG96], to opinion and social evolution systems [Jia+15], and even quantum stochastic systems [SSR10]. Noteworthy is the emergence of the term "Laplacian paradigm" [Ten10] with the rise of the era of networks and massive data sets. The significance of this paradigm is routed in the spectacular success of nearly-linear time (Laplacian) solvers for linear equation systems [ST04]; the underlying ideas originate in graph problems that connect passive electric resistor networks with random walks on graphs [DS84], i.e., Markov chains on discrete probability space.

Not only are the many different application fields largely disconnected, in which a Laplacian equation system is or might be at the core of system dynamics. Also, the systems and control literature on consensus networks itself is highly fragmented. In the young history

of consensus systems research, remarks on a relationship to electric networks and Markov chains can be found in early work such as [Wil76]. Passivity and convexity, rather than the linearity of the problem, are noted to be driving the averaging behavior [Mor04], extending exponential convergence results also to nonlinear consensus dynamics [Mor05]. General frameworks that go beyond quadratic stability, which is necessary and sufficient for linear systems stability properties, include a geometric generalization [Sep11], utterly novel system properties [HT13], or abstract consensus protocol design rules as studied in [HH08; Wei+17; BGP06]. A comprehensive study and consolidating framework in which all of these different properties driving the exponentially convergent averaging behavior come together so far is lacking. This thesis aims at providing such a framework.

## 1.2 A brief history of consensus theory and its many facets

**Origins and span-norm Lyapunov function:** The study of linear consensus systems originates in decentralized decision making in networks of economic exchange [Nor67; DeG74] and distributed computation [Tsi84; TBA86]. Tools for early convergence proofs based on Markov chain theory, i.e., properties of products of stochastic matrices and invariance of a stationary distribution. A novel twist has been introduced with the span-norm Lyapunov function $\max_i x_i(t) - \min_i x_i(t)$ by John N. Tsitsiklis [Tsi84], where $x_i(t)$ is the $i$-th state component at time $t$, with $i \in \{1, \dots, n\}$. This extremal function represents the diameter of the convex hull spanned by state components. It is a very robust contraction measure; for instance, it allows to prove convergence also for distributed consensus networks with asynchronous information exchange and additional local computational updates on the local states. Tsitsiklis span-norm Lyapunov function proves especially useful in situations where quadratic Lyapunov functions fail, see, e.g., [Ols08] for that context. This span-norm contraction measure has been applied by Luc Moreau in [Mor04][Mor05], to prove the most general convergence results for linear time-varying consensus systems: dynamics converge with exponential speed to a consensus state where all state components equal if for any time instant there is a finite time interval across which all state updates are such that the effective update law has matrix structure that exhibits a directed spanning tree. This complicated, time-dependent connectivity property is dubbed uniform connectedness. Moreau uses the span-norm function in a blend of graph and systems theoretic tools introducing a set-valued Lyapunov framework for the convergence study that applies to linear and nonlinear network problems. Aiming at applications in quantum stochastic dynamics, this Lyapunov function has been re-discovered in a study of consensus in non-commutative space by Sepulchre, Sarlette, and Rouchon [SSR10], however in logarithmic coordinates. Then, the Lyapunov function becomes invariant to scaling and is particularly suited for the structure of a projective space. This logarithmic contraction measure has been initially proposed by Garrett Birkhoff [Bir57] in the study of homogeneous monotone maps on cones. It is indeed a contraction measure, as it represents the so-called Hilbert metric, which measures the projective distance between the state of a consensus system and the set of consensus states. This discovery has led to the remark of the "not so linear nature" of linear consensus systems in [Sep11].

**Physics, coordination problems and dissipation metaphors:** Interest in the study of consensus algorithms has significantly spurred with the work [Vic+95], where the authors propose a simple discrete-time model involving averaging interactions on velocities, which leads to the emergence of self-ordered motion. This model has first been analyzed by the systems and control community in great detail by Ali Jadbabaie, Jie Lin, and Stephen Morse in [JLM03], where the authors provide a theoretical foundation for the observed dynamic behavior. Their work is based on products of stochastic matrices, algebraic properties such as null-spaces leading to invariance properties, and, in the continuous-time context, on flow maps generated by Laplacian matrices, which are contractions to spectral properties of the Laplacian. The authors also consider the case of time-varying dynamics, which they trace back to the analysis of sequenced time-invariant systems. Most importantly, the authors have put their analysis into the context of coordination problems in multi-agent systems, which lead to an explosion of practically motivated studies, see, e.g., [RBA05; OSFM07a] and references therein for early survey articles. By that, the work [JLM03] connects different approaches in interactive physical and behavioral natural systems, see also [Rey87; LF01], and multi-agent coordination problems [OSM02] with consensus schemes.

At about the same time, Alexander Fax and Richard Murray pioneer the study of coordination problems arising in vehicle formations and propose a Nyquist-like stability criterion using spectral properties of graph Laplacian matrices [FM02]. Building on that work, Reza Olfati-Saber and Murray in [SM03a; OSM04] propose convergence proofs for general linear consensus protocols, also on time-varying communication graphs. They introduced so-called (Laplacian) group disagreement and collective disagreement functions. The collective disagreement measures the disagreement in terms of sums of squared Euclidean distances of each state component to the consensus state. It serves as a Lyapunov function for strongly connected and balanced graphs. The Laplacian group disagreement is also a sum-of-squares function measuring, however, squared Euclidean distances among pairwise state components associated with nodes in a connected communication graph. Hence, the Laplacian potential is an instantaneous disagreement measure, while the collective disagreement uses asymptotic state information. The Laplacian disagreement, for symmetric and time-constant graphs, also serves as a potential in which the consensus dynamics evolve as gradient flow [OSM04; OSFM07a].

A second gradient system property has been proposed by Arjan van der Schaft in [van11] for symmetric time-constant graphs and linear consensus protocols. He argues that not the Laplacian group but the sum-of-squares collective disagreement is the appropriate potential in which consensus systems evolve as gradient flow. His argument is motivated by an electric circuit and port-Hamiltonian systems consideration: A linear consensus network with symmetric graph is a negative feedback interconnection of two open systems, one with lossless capacitor system having Hamiltonian given by the collective disagreement and one being represented by the Laplacian matrix, which models a resistor network. M. Egerstedt in [ME10] also proposes a unit-capacitor LTI electric circuit structure for LTI symmetric consensus dynamics, similar to van der Schaft. Moreau, in [Mor04], also remarked that linear Laplacian systems have close connections to passivity theory and, in particular, to electric circuits. He also referred to the work [Wil76] where it has been observed that any additive convex function may serve as a collective disagreement measure for LTI consensus systems.

**Generalizing frameworks:** Generalizations from linear to nonlinear consensus results usually follow the path of finding protocol design rules that cover maximally large interaction function classes that lead to the construction of convergent consensus systems or generalizing inherent mathematical structures of the linear consensus system extending them to nonlinear space.

Regarding the latter generalization approach, Scardovi, Sarlette, and Sepulchre in [SSS07] study consensus on the circle, respectively, on the n-torus, which is the first work to consider a non-Euclidean underlying state space instead of the typical Euclidean one in abstract consensus studies. This particular case, where the usual consensus protocol evolves on the n-torus, leads to a phase-averaging scheme, as it is known, e.g., in Kuramoto-like models. This approach is further generalized in the work of Sarlette and Sepulchre [SS09a; SS09b], see [Sep11] for a conceptual overview article: Given a state-space that is represented by a compact, homogeneous (Riemannian) manifold, a consensus system on nonlinear space evolves according to the conceptual update rule "move towards the average of your neighbors." The assumption on the underlying state space structure helps compute the necessary average of neighboring states on the nonlinear state space. The authors use the fact that an average can be defined as the solution of a sum of squared distances minimization problem. The appropriate distance is associated with the structure of the nonlinear space, as it is the natural (Riemannian), or intrinsic distance, which is given by a geodesic. To circumvent the complicated computation of geodesics, the authors apply an embedding trick: Any compact, homogeneous n-dimensional space can be embedded into at least 2n-dimensional ambient Euclidean space. Then, the average is computed in this external linear space as the arithmetic mean of state values in "embedding" coordinates and then projected back onto the nonlinear space. This additional introduction of an extrinsic geometry allows formulating an abstract protocol for averaging dynamics reaching consensus on nonlinear spaces.

While this geometric generalization requires a priori knowledge of an underlying nonlinear state-space structure with specific mathematical properties, the protocol design approach does not. The motivation here instead is the use in applications. And for that, authors following this approach usually are concerned with finding function classes for (pairwise) state interactions across a graph, such that the resulting network system dynamics still converge asymptotically to a consensus state.

Murray and Olfati-Saber introduce the essential tool for nonlinear consensus protocol design in [SM03a] in the context of so-called action graphs. In addition to a linear edge weighting, action graphs specify pairwise interactions across edges characterized by some action function. Such action functions, in most cases, take a linear state difference as input and map it to a single output value. Anti-symmetry of functions has soon been identified and used as elementary property for nonlinear consensus systems [SM03a; OSM04; BGP06; Cor08]. Additional nonlinear but positive gain functions on local interactions have been introduced and studied [BGP06; OSM04; ADJ12]. In [Wei+17] the sign-preservingness of the interaction function relative to the sign of linear state differences has been identified and rigorously analyzed for continuous and discontinuous protocol dynamics. That work also allows sign-preserving input nonlinearity, which is a generalization of typical positive gain functions. Hui and Haddad introduce another extension in [HH08], where the authors allow two independent inputs and map it to a single output for pairwise interactions. The re-

quired function property again is a function inequality, which is reminiscent of the property of anti-symmetry. Using a classical Lyapunov approach, the authors derive this inequality from network thermodynamic reasoning and prove the asymptotic stability of consensus states.

Other nonlinear consensus contexts include the study of consensus on convex metric space [MB10], on the space of probability measures [BD14], finite-time consensus problems [WX10], or mechanism design in networks of interacting agents [BGP06] with consensus on general functions being closely related [Cor08].

**Concepts for convergence analysis beyond the classical Lyapunov method:** Tracing consensus problems to the study of Markov chains and properties such as ergodicity, the behavior of products of stochastic matrices, or spectral theory for stochastic, or Laplacian matrices has been foundational in the study of consensus networks. The relationship between consensus networks and Markov chains has regained interest only recently, see, e.g., [Bol14; Bol+14; TN11], and their follow-on works. Touri and Nedic in [TN11] introduce the so-called "infinite-flow" property and relate it to ergodicity in classical Markov theory. Their framework considers consensus over random graphs. Ergodicity and the infinite-flow property of random models are equivalent when all transition matrices have a common steady state in expectation. In [Bol+14; Bol14] the authors study the so-called Sonin's decomposition-separation theorem for inhomogeneous Markov chains in the context of linear time-varying consensus systems. Sonin's result is concerned with nonhomogeneous Markov chains where no assumptions are made on the sequence of stochastic matrices [Son08]. A geometric interpretation for convergent consensus dynamics is proposed: Consider points defined by the column vectors of the transition matrix of a consensus system. These vectors are stochastic vectors with a constant 1-norm. The convex hull spanned by these points is a polygon. While consensus dynamics unfold, polygons map into each other forming sequences of nested sets until a consensus or an equilibrium is reached [Bol+14]. Geometrically familiar is the joint work of Chevalier, Hendrickx, and Jungers [CHJ14; CHJ17]. The authors study the decidability of matrix sets' stability; in particular, they consider products of stochastic matrices and ask whether or not they convergence to zero or a common set asymptotically. Convergence is measured by a polyhedral semi-norm, which for stochastic matrices is nonincreasing and forms the basis for a consensus convergence proof.

Hendrickx and Tsitsiklis introduce another matrix property that leads to convergence of time-varying linear consensus networks in [HT13]. The property is concerned with the matrix weights of differential update laws, and it is called cut-balance. Intuitively it means that dynamics are such that if any sub-group of agents affects the dynamics of the complementary subgroup, then there must also be an action back. In this sense, this matrix property bears the physical interpretation of Newton's actio-reactio law, however, not in an exact but arbitrarily scaled balance. Cut-balance is an innovation as it also allows to prove convergence for cases when consensus is not reached but some other equilibrium due to connectivity constraints. A convergence mechanism is also provided, and it is based on sums of weighted ordered state components which are non-decreasing.

The cut-balance analysis has motivated the introduction of the novel differential stability concept called consensus dichotomy by Proskurnikov and Cao in [PC17], which builds on boundedness and the contraction behavior of Laplacian flows. Similarly, consensus studies

have been influential in the recent introduction of differential Lyapunov analysis [FS14] that is adapted to systems that have non-trivial invariance properties along dynamics.

**Passivity at the core?** Despite the simple model structure, the many facets and approaches to the study of consensus systems demonstrate the complexity and richness of consensus studies. However, it also indicates the fragmented nature of the studies and communities working on consensus theory and network system applications.

The early and short work of Jan C. Willems [Wil76] on Lyapunov inequalities for diagonally dominant systems is one of only a few, if not the only work, which touches almost all approaches presented so far in the study of consensus systems. In that sense, it has an integrative character that we shall explore in detail in the course of this thesis.

In particular, Willems notes that the span norm Lyapunov function of Tsitsiklis is characteristic for diffusive systems, as both $\max_i x_i$ and $-\min_i x_i$ are non-increasing along dynamics if and only if a Laplacian generator matrix governs the dynamics. Further, based on convexity and properties of matrix weights of a Laplacian, he demonstrates that the class of additive convex functions is a class of Lyapunov functions for consensus dynamics. He connects this result to Markov chains and relative entropy, or general information divergences being non-increasing along dynamics. He remarks that Laplacian systems governed by Metzler matrices are used in modeling RC or RL electric networks and hence are elementary in passivity theory motivated by physical application. And last but not least, Willems remarks that the class of linear time-varying consensus systems may serve helpful in studying nonlinear dynamics: Transformations from a nonlinear ODE to the linear time-varying consensus system model allows to apply the available stability and convergence results of consensus theory in nonlinear contexts.

This thesis puts Willems's simple ideas on a rigorous footing. It derives a simple but fundamental passivity framework that aligns Markov chain analysis with properties such as cut-balance and nonlinear protocol or geometric generalizations. We motivate the framework with an application to chemical and polynomial network dynamics, as first described in [MXH16; Man20].

## 1.3 Summary and outline

This thesis introduces novel design tools for the synthesis and analysis of nonlinear network dynamics using linear time-varying consensus results. We demonstrate that the basis for convergence behavior is the passivity of virtual resistors across which energy stored in virtual capacitors is dissipated. We show equivalence between cut-balance, Markov chains, and passive RC networks. In that context, we introduce the novel concept called differential majorization, which is derived from the classical majorization theory put into a dynamic, differential context. Passivity and the design tool results are based on the link between convexity of stored energy with basic mathematical structures of metric and mean functions. Means can be defined via a Kolmogorov functional form or equivalently in a metric minimization problem. A metric basis for the design of consensus protocols as averaging dynamics yields a very general functional design rule class that comprises existing ones. Particular instances yield the embedding protocol as equivalent to the geometric protocol design scheme of Sepulchre

and Sarlette. Applying the presented methods to polynomial network protocols as consensus driven by the geometric mean yields novel results in chemical reaction networks.

**Chapter 2: Consensus driven by the geometric mean and chemical reaction networks** In this chapter, we consider three types of nonlinear network dynamics derived from leveraging averaging and metric structures that drive the behavior of linear consensus networks. The network protocols have polynomial, scaling-invariant, and entropic right-hand sides and are driven from geometric mean averaging considerations. The method and design principles we introduce here in a concrete example are later introduced and proven more abstractly. Hence, with its results, this chapter also serves as a motivational case study for the general framework we develop in the subsequent chapters.

We show that the introduced polynomial network and consensus protocols are closely related to mass-action chemical reaction network systems. Convergence and stability results are proven using a transformation of the nonlinear time-invariant ODE systems to one in linear time-varying consensus form. The consensus conditions we obtain are reminiscent of the Wegscheider relation and detailed balance in chemistry. We show that geometric mean-driven network dynamics are particular instances of free energy gradient flows. We further provide a novel characterization of the geometric mean of $n$ real numbers using the solution of a constrained nonlinear minimization problem, which is solved asymptotically and with exponential speed by the evolving dynamics of a geometric-mean induced consensus system. The polynomial consensus protocol asymptotically achieves a consensus value that appears to be lower bounded by the solution of an elliptic integral. Applying the results to chemical reaction networks yields a novel conductance formulation. Its significance is motivated as it directly connects to chemical potentials driving the dynamics. This conductance has the functional form of a heat exchange system in which stored Gibbs free energy of connected chemical species is dissipated.

**Chapter 3: Metrics and means in the design of dissipative consensus systems** The results of the preceding chapter are generalized, and the relationship between metrics and mean functions in the design of general consensus protocols is highlighted. In particular, we prove a connection between general averages that satisfy Kolmogorov's axioms of means and solutions of metric optimization problems that specify general means. Both metric and Kolmogorov means are used to propose design rules for consensus protocols. We summarize the main nonlinear consensus protocol design results. We show that all of these are particular instances of a more general metric consensus protocol class, for which we prove exponential convergence. Properties of metric functions are used to propose composition rules, from which more complicated consensus protocols result that are exponentially convergent by design. Consensus protocols driven by Kolmogorov means are particular instances of metric consensus protocols; they generate trajectories with an optimality characteristic across each finite time step. This optimality property is novel and differs from the known asymptotic optimality of achieved consensus states. A particular metric consensus protocol is the embedding protocol that we introduce. It combines geometric, passivity, and optimization aspects in a general gradient formulation. Its connection with the geometric consensus generalization of Sepulchre and Sarlette and links to Newton algorithms are discussed.

**Chapter 4: Dissipation mechanisms and passive circuit structure of consensus networks**
While the preceding chapters focus on general design principles and applications of nonlinear consensus networks, in this chapter, we ask what mechanisms essentially drive the exponential contraction behavior of linear or nonlinear consensus networks. We show that a passivity property is at the core by uncovering a nonlinear passive RC circuit structure in the class of nonlinear consensus protocols discussed in the preceding chapters. This passive circuit formulation is derived from structure results of dissipative systems, due to D. Hill, P. Moylan, and B.D.O. Anderson, together with a interconnected (open) passive systems approach to studying convergence and Lyapunov stability of autonomous (closed) systems. The passive circuit formulation is equivalent to a gradient system structure in which the convexity of stored energy in capacitors is equivalent to the strict passivity of nonlinear resistor elements. We prove the gradient system property not based on typical gradient representations but using a duality structure and a necessary and sufficient energy dissipation equality. We generalize the passive circuit result to a port-Hamiltonian representation, which is used in defining an equivalence between a property we call differential majorization and the existence of a passive RC circuit structure in dynamic systems. Using differential majorization, we also prove an equivalence between passivity and the cut-balance property introduced recently by Hendrickx and Tsitsiklis. The obtained results are applied to Markov chains and phase-coupled oscillator networks. We conclude with remarks on the study of interconnected stochastic systems, optimal transport and machine learning problems using the circuit formulation of Markov chains, and on recent advances in contraction studies using novel differential analysis tools.

# 2

# Consensus driven by the geometric mean and chemical reaction networks

## 2.1 Introduction

Under the umbrella of linear consensus theory, results are collected that describe the convergence and stability of a very general class of linear time-varying, arithmetic mean driven network dynamics, see, e.g., [OSFM07b; BT89; Mor04]. What suffers from this generality is the specificity needed to make immediate use of those results in applied network problems - problems that often appear as nonlinear and time-invariant dynamics that are inherently driven by non-arithmetic means. A prime example is the class of Kuramoto-type network models [Ace+05] that can be found in a wide range of important applications, e.g., in power grid studies [DB14; DH14], or in neuroscience [BHD10]. The collective averaging motion is driven by the so-called chordal mean, which is an average adapted to the circular geometry of phase angles [Sep11; SS09b; SSS07]. Significant stability results can indeed be based on linear consensus theory, see, e.g., the work [JMB04], where the authors prove stability by reverse engineering for this particular case a linear time-varying consensus structure from the nonlinear time-invariant original system model. In this chapter, the starting point is not an existing nonlinear network model, but a significant type of average, namely the geometric mean, that shall serve as the driving element in a nonlinear dynamic averaging network. In particular, we are interested in designing and studying network protocols that generate geometric mean averaging processes in the same way the arithmetic mean does in linear consensus protocols. For the novel types of geometric mean-driven network dynamics, we propose touching points to nonlinear problems in chemistry, optimization, and analog computation using networked dynamical systems.

The geometric mean plays an important role in various applications. It is the appropriate tool to evaluate averages on data that exhibits power-law relationships, as they arise in describing relative, resp., compound, or geometric growth relations [Lue98; Van10]. Examples of such relations can be found in financial and economic studies [Zen+08; Mit04; SW08], they are abundant in biology [Shi10] and chemistry [Con90; HKH11; EG07]. For instance, in gene expression networks, the geometric mean of degradation and production rates has been found to act as feedback control gain in linearized dynamics [HKH11]. Geometric mean averaging also appears in the context of algorithm design, in distributed Bayesian consensus filtering and detection schemes, see, e.g., [BC14] and [QJX15]. There, the geometric mean arises from the combination of a given network structure and a Bayesian update rule, leading to a so-called logarithmic opinion pooling as natural scheme of combining local probabilities, see [GZ86], and also [NOU15; SJ13; RJ15] for further reference.

Despite the central role of mean functions and averaging structures for stability studies in network problems, yet, there are few works on how specific means, particularly the geometric mean, drive such systems' (nonlinear) behavior. Consensus-like protocols driven by non-arithmetic means with geometric mean as particular case are for instance studied in [HK05] in the specific context of opinion dynamics in discrete time. In [BGP06], the authors introduce a nonlinear protocol design rule that leads to convergence to a consensus state given by a generalized mean that also includes the geometric mean. They also show that their protocol is the solution of individual optimizations performed locally at each network node, given local information. Works on consensus on nonlinear space [SS09b; Sep11] extend the usual arithmetic averaging in linear consensus to a nonlinear configuration space; the associated non-arithmetic mean results as a by-product of that choice of geometry. Extensions to other mathematical structures include the work on consensus on convex metric spaces [MB15], or on the Wasserstein metric space of probability measures [BD14]. None of these works puts in the center of consideration a particular type of average, from where continuous-time network dynamics shall arise by design.

In this chapter, we propose and study three novel nonlinear consensus protocols based on elementary considerations on how the arithmetic mean appears in the structure of linear consensus protocols and replacing it with the geometric mean functional relationship. The contribution is as follows:

a) We introduce novel geometric mean-driven network protocols that we call polynomial, entropic, and scaling-invariant protocol. A fourth protocol called reaction network protocol is proposed as a slight extension of the polynomial consensus protocol. This fourth protocol serves as a bridge in leveraging consensus results in the chemical reaction network application.

b) We prove convergence to consensus under appropriate connectedness conditions building on Moreau's basic stability result [Mor04; Mor05] for linear, time-varying consensus networks.

c) We show that along continuous-time dynamics of the entropic consensus network, the solution of a free energy nonlinear constrained optimization problem is computed with the (weighted) geometric mean as consensus solution.

d) We put the distinct protocols on a common footing by showing that all three protocols describe a particular type of free energy gradient descent flow.

e) Leveraging the polynomial consensus and reaction network protocol results, we introduce a novel conductance formulation for chemical reaction networks. This novel conductance is motivated by the fact that it characterizes a heat exchanger-like process the dissipation of stored Gibbs free energy in chemical species.

f) In a numerical study, we demonstrate that the solution of an elliptic integral lower bounds the consensus value obtained under the polynomial protocol on certain graph types.

The remainder of this chapter is organized as follows: In section 2.2 we give an overview of mean functions and linear consensus theory. In section 2.3 we propose the novel protocols and discuss relationships to arithmetic-mean averaging structures in linear consensus networks. In section 2.4 we prove exponential convergence and consensus value results. Next, in section 2.5.2 we provide illustrative numerical examples and conduct a numerical study on the consensus value for the polynomial consensus system. In section 2.6, we put the three novel consensus protocols in a single free energy gradient flow framework and provide a novel optimization characterization of the geometric mean on a constrained space. In section 2.7 we apply the result to chemical reaction network systems of mass-action type and derive a novel dissipation conductance formulation. We complete this chapter with a conclusion and possible further directions of research.

## 2.2 Linear consensus protocol, arithmetic and geometric mean

In this section, we present characterizations of the arithmetic and geometric mean and representation and the basic stability result of linear consensus theory.

### 2.2.1 Characterization of the arithmetic and geometric mean

Consider data points $x_1, x_2, \ldots, x_n$ taking values on the positive real line $\mathbb{R}_{>0}$, and let these elements be collected in the vector $\boldsymbol{x}$. An average or mean computed from $\boldsymbol{x}$ can be obtained as the solution of an unconstrained minimization,

$$\mathsf{mean}(\boldsymbol{x}) = \arg\min_{x\in\mathbb{R}_{>0}} \sum_{i=1}^{n} d(x_i, x)^2, \tag{2.1}$$

where $d(a, b)$ denotes a metric in $\mathbb{R}_{>0}$.

If the Euclidean distance $d_{\mathsf{E}}(a, b) := |a - b|$ is chosen in (2.1), the resulting average is the arithmetic mean,

$$\mathsf{am}(\boldsymbol{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i = \arg\min_{x\in\mathbb{R}} \sum_{i=1}^{n} |x_i - x|^2. \tag{2.2}$$

Another important metric in $\mathbb{R}_{>0}$ is the hyperbolic distance $d_{\mathsf{H}}(a,b) := |\ln a - \ln b|$, which coincides with the Euclidean metric assessed in logarithmic coordinates. It is a geodesic distance, and measures the hyperbolic length of the straight line segment joining two points in Cartesian coordinates $(x,a),(x,b)$, $x \in \mathbb{R}_{>0}$, see e.g., [Sta93] Proposition 4.3. Its significance arises from the fact that the solution of the minimization problem (2.1) using the hyperbolic metric $d_{\mathsf{H}}$ yields the geometric mean

$$\mathsf{gm}(\boldsymbol{x}) := \sqrt[n]{x_1 x_2 \cdots x_n}.$$

To see this, observe that

$$\sum_{i=1}^{n} |\ln x_i - \ln \mathsf{gm}(\boldsymbol{x})|^2 = \sum_{i=1}^{n} |\ln x_i - \mathsf{am}(\ln \boldsymbol{x})|^2, \tag{2.3}$$

which is the least-squares characterization of the arithmetic mean in logarithmic coordinates.

To complete this section, we introduce the weighted versions of the arithmetic and geometric means,

$$\mathsf{am}_w(\boldsymbol{x}) := \sum_{i=1}^{n} \omega_i x_i, \text{ and } \mathsf{gm}_w(\boldsymbol{x}) := \prod_{i=1}^{n} x_i^{\omega_i},$$

where for $i = 1,2,\ldots,n$, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

### 2.2.2 Graphs, linear consensus protocols and the arithmetic mean

Let $\mathsf{G} = (N,B,w)$ be a weighted digraph (directed graph) with set of nodes $N := \{1,2,\ldots,n\}$, set of branches $B := \{1,2,\ldots,b\} \subseteq N \times N$ having elements ordered pairs $(j,i)$ that indicate that there is a branch from node $j$ to $i$, and $w : B \to \mathbb{R}_{>0}$ is a weighting function for which we write $w((j,i)) = w_{ij}$. Define the in-neighborhood of a node $i$ as the set of connected nodes $N_i^+ := \{j \in N : (j,i) \in B\}$ and the out-neighborhood $N_i^- := \{j \in N, (i,j) \in B\}$. The (in-) degree of a node $i$ is the value $d_i := \sum_{j \in N_i^+} w_{ij}$. Set $\mathbf{D} := \mathsf{diag}\{d_1,d_2,\ldots,d_n\}$. The weighted adjacency matrix $\mathbf{W}$ is such that $[\mathbf{W}]_{ij} = w_{ij}$ for all $(j,i) \in B$; if $(j,i) \notin B$, then $[\mathbf{W}]_{ij} = 0$, and $[\mathbf{W}]_{ii} = 0$, for all $i \in N$. A graph is called balanced if $\sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji}$ and it is symmetric if $w_{ij} = w_{ji}$, $\forall (j,i) \in B$. The Laplacian matrix of a weighted digraph is defined as $\mathbf{L} := \mathbf{D} - \mathbf{W}$, and the normalized Laplacian is $\hat{\mathbf{L}} := \mathbf{I} - \hat{\mathbf{W}}$, where $\hat{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{W}$ is the matrix of normalized branch weights.

A linear consensus system evolving in continuous time is a dynamics on a family of graphs $\{\mathsf{G}(t)\}_{t \geq 0}$ governed by

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij}(t)\big(x_j - x_i\big) \iff \dot{\boldsymbol{x}} = -\mathbf{L}(t)\boldsymbol{x}, \tag{2.4}$$

where each dynamic branch weight $w_{ij}(\cdot)$ is a measurable non-negative function [HT13].

The following relationships between the arithmetic mean and consensus system representations and properties are well known in consensus theory: Using (2.2), a component-wise LTI consensus dynamics (2.4) on a normalized weighted digraph can locally be brought to

the open-loop control system form

$$\dot{x}_i = -x_i + u_i(\{x_j\}_{j\in N_i^+}), \qquad u_i = \sum_{j\in N_i^+} \hat{w}_{ij}(t)x_j \triangleq \mathsf{am}_w(\{x_j\}_{j\in N_i^+}). \tag{2.5}$$

Without the requirement of a normalized weighting, a variable time discretization can be chosen such that a local algorithmic update law (e.g., in an explicit Euler scheme) has the arithmetic mean driven form

$$x_i(t+\mathrm{d}t) = \alpha(\mathrm{d}t)x_i(t) + [1-\alpha(\mathrm{d}t)]\mathsf{am}_w(\{x_j(t)\}_{j\in N_i^+(t)},$$

where $0 \leq \alpha < 1$, cf., e.g., [SSS07].

Besides its appearance in the local dynamics at a particular instant in time, the arithmetic average also unfolds as asymptotic global system property: in the class of consensus networks being governed by Laplacians $\mathbf{L}(t)$ that are irreducible and balanced for all $t \geq 0$, the asymptotically reached uniform agreement value is given by the arithmetic mean of the initial condition [OSM04]. The problem in which the equilibrium state to be reached is uniform with consensus value $\bar{x} = \mathsf{am}(\boldsymbol{x}_0)$ is commonly known as the average consensus problem.

The goal of this work is to study the interplay between consensus protocols and the geometric mean. In that, we first seek to understand the various interaction points between the design of LTI consensus protocols and the arithmetic mean, and then leverage these observations to derive and study novel geometric mean-driven consensus protocols.

For the sake of focus and ease of understanding, we assume the underlying graph to have a constant, i.e., time-invariant weighting. In our analysis, it shall turn out elementary to transform the nonlinear time-invariant network protocols to linear time-varying consensus form so that the following convergence result becomes applicable.

**Proposition 1.** *[Adopted from [Sep11] Prop. 1 with Def. 2] A linear time-varying system evolving according to* (2.4) *in* $\mathbb{R}^n$ *converges globally and exponentially to a consensus point* $\bar{x}\mathbf{1}$*,* $\bar{x} \in \mathbb{R}$*, if the underlying digraph is uniformly connected, i.e., if for all* $t > 0$*, there exists a time horizon* $T > 0$*, such that the graph* $(N, \tilde{B}(t), \tilde{w}(t))$ *defined by*

$$\tilde{w}_{ij}(t) := \begin{cases} \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau & \text{if } \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau \geq \delta > 0 \\ 0 & \text{if } \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau < \delta \end{cases}$$

$w_{ij}(\tau)$ *a branch weight at time* $\tau$*,* $(j,i) \in B$ *if and only if* $\tilde{w}_{ij}(t) \neq 0$*, contains a node from which there is a path to every other node.*

Uniform connectivity certainly holds if at each time instant the graph $\mathsf{G}(t)$ is strongly connected and $w_{ij}(t) \geq \delta > 0$, i.e., if the graph contains a directed path from every node to every other node and the finite branch weights are positively bounded away from zero for all time.

## 2.3 Geometric mean driven network protocols

This section proposes and motivates three novel consensus protocols driven by the geometric mean in ways as the arithmetic mean does for the linear consensus protocol. Based on their functional structure, they are called the polynomial, entropic, and scaling-invariant protocol. A fourth protocol is proposed that generalizes the polynomial consensus protocol to a consensus-seeking reaction network protocol having the form of chemical equation systems.

### 2.3.1 Polynomial consensus and reaction network protocol

The polynomial protocol we consider is a dynamic on a graph where at each node $i \in N$, the differential update rule has the form

$$\dot{x}_i = -\prod_{j \in N_i^-} x_i^{w_{ji}} + \prod_{j \in N_i^+} x_j^{w_{ij}}. \tag{2.6}$$

For the polynomial protocol we assume a balanced graph weighting, i.e., $\sum_{j \in N_i^-} w_{ji} = d_i$. With that, the protocol (2.6) can be written as

$$\dot{x}_i = -x_i^{d_i} + \prod_{j \in N_i^+} x_j^{w_{ij}}.$$

Comparing this form with a linear consensus protocol, which can be stated as

$$\dot{x}_i = -d_i x_i + \sum_{j \in N_i^+} w_{ij} x_j, \quad i \in N$$

we observe an equivalence resulting upon replacing the operation of summation and multiplication with the similar[1] operations multiplication and exponentiation.

Alternatively, referring to the open-loop control representation (2.5), where weightings are normalized, replacement of the weighted arithmetic mean by the geometric average leads to the protocol

$$\dot{x}_i = -x_i + \mathrm{gm}_w(\{x_j\}_{j \in N_i^+}) = -x_i + \prod_{j \in N_i^+} x_j^{\hat{w}_{ij}},$$

from where (2.6) results again under the assumption of having a balanced weighting, that is, $\sum_{j \in N_i^-} \hat{w}_{ji} = 1$.

In its general form (2.6), the polynomial protocol has the structure of a rate equation as it occurs, for instance, in reaction networks and chemical kinetics [Con90]. We define

$$r_i^+ := \prod_{j \in N_i^+} x_j^{w_{ij}}, \tag{2.7}$$

the nonlinear rate at which some quantity "$x$" flows from in-connected nodes $j$ to node $i$, and

$$r_i^- := \prod_{j \in N_i^-} x_i^{w_{ji}}, \tag{2.8}$$

[1] These operations are similar in the sense that the addition of logarithmic variables turns the variables into products, and products result into exponentiation.

the rate at which $x$ flows along links $(i,j) \in B$ from node $i$ to the out-directed nodes $j$. The local rate of change $\dot{x}_i$ balances in- and out-flows on a graph such that

$$\dot{x}_i = r_i^+(\boldsymbol{x}) - r_i^-(\boldsymbol{x}). \tag{2.9}$$

The relation to the (weighted) geometric mean and the similarity to chemical kinetics in reaction networks is further described in the following example.

**Example 1** (Chemical kinetics)**.** In mass action chemical reaction networks, the net rate equation for a concentration of one component $i$ in one reaction involving $n$ substances indexed in $N$ is split into a difference of a forward and a backward reaction rate, each having the form

$$r_i^\pm = \gamma^\pm \prod_{j=1}^{n} x_j^{s_j^\pm} = e^{\sum_j s_j^\pm \ln x_j + \ln \gamma^\pm} \tag{2.10}$$

where $\pm$ stands either for the forward or backward rate, and $\gamma^\pm > 0$ is the associated forward/backward reaction constant. The weights $s_j^\pm > 0$ are stoichiometric coefficients. The representation (2.10) has been instrumental in the studies [vRJ13a],[vRJ13b],[Yon12] that shed light on a systems theoretic structure of chemical reaction networks: Introducing the density vector $\boldsymbol{\rho}$, with $\rho_i = \frac{x_i}{\bar{x}_i}, i \in N$, under a detailed balance assumption on the equilibrium concentrations $\bar{\boldsymbol{x}}$, it can be shown that

$$\sum_j s_j^\pm \ln x_j + \ln \gamma^\pm = \sum_j s_j^\pm \ln \rho_j.$$

Observe that

$$e^{\sum_j s_j^\pm \ln \rho_j} = e^{\ln \prod_j \rho_j^{s_j^\pm}} = \mathsf{gm}_w(\boldsymbol{\rho}),$$

i.e., the (forward or backward) reaction rate has the functional structure of a (non-normalized) weighted geometric mean.

Motivated by the application Example 1 we extend the polynomial consensus protocol (2.9) by the forward backward rate constants $\gamma^+ > 0$ and $\gamma^- > 0$, cf Example 1 to obtain the reaction rate network protocol ODE

$$\dot{x}_i = \gamma_i^+ r^+(\boldsymbol{x}) - \gamma_i^- r_i^-(\boldsymbol{x}), \quad i \in N. \tag{2.11}$$

Clearly (2.11) reduces to (2.9) when $\gamma_i^+ = \gamma_i^- = 1$ for all $i \in N$.

### 2.3.2 Entropic consensus protocol

The entropic protocol is governed by a vector field that is represented by a set of negative (weighted) divergences between local states $x_i$ and connected nodes' states $x_j$, such that

$$\dot{x}_i = -\sum_{j \in N_i^+} w_{ij} x_i \ln \frac{x_i}{x_j}. \tag{2.12}$$

The term "entropic" refers to the fact that a local vector field (2.12) is an entropic quantity. More precisely, it has the structure of negative relative entropy or information divergence between the local state $x_i$ and the adjacent states $x_j$, $j \in N_i^+$. Relative entropy as divergence from a positive vector $\boldsymbol{x}$ to another positive vector $\boldsymbol{y}$, both such that their 1-norms equal one, (i.e., these are probability mass vectors), is defined as

$$D_{\mathsf{ent}}(\boldsymbol{x}||\boldsymbol{y}) := \sum_i f_R(x_i|y_i), \quad \text{where} \quad f_R(a|b) := a \ln \frac{a}{b},$$

see for instance [CT91].

The entropic protocol can be formulated as the geometric mean version of the linear consensus protocol using a coordinate transformation, with coordinate transform taken as the scalar function that leads to the least-squares optimization characterization of the considered mean; for the geometric mean, this is the logarithm, while for the arithmetic mean no coordinate transformation is required, see (2.2) with (2.3).

Writing the consensus protocol in logarithmic coordinates leads for each $i \in N$ to the ODE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \ln x_i = \frac{1}{x_i} \dot{x}_i &= \sum_{j \in N_i^+} w_{ij} (\ln x_j - \ln x_i) \\ \Leftrightarrow \quad \dot{x}_i &= x_i \sum_{j \in N_i^+} w_{ij} (\ln x_j - \ln x_i), \end{aligned}$$

which is the entropic protocol (2.12), as

$$x_i \sum_{j \in N_i^+} w_{ij} (\ln x_j - \ln x_i) = - \sum_{j \in N_i^+} w_{ij} f_R(x_i, x_j).$$

As we shall show, the significance of the entropic protocol arises from the situation that the asymptotically reached consensus value is given by the geometric mean of the initial condition. Hence, this protocol provides an analog distributed computation routine to solve the minimization (2.1) associated with the geometric mean.

### 2.3.3 Scaling-invariant consensus protocol

The scaling-invariant protocol has the form of a LTI consensus system however following log-linear updates; it is given by the component ODE

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} (\ln x_j - \ln x_i), \quad i \in N. \tag{2.13}$$

The scaling invariant protocol (2.13) is an instance of the more general type of mean-driven network protocols given by the class

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} \mathsf{sgn}(x_j - x_i) d(x_j, x_i), \tag{2.14}$$

where the metric to be chosen is the hyperbolic metric $d_{\mathsf{H}}$ associated to the optimization characterization of the geometric mean, see Section 2.2.

The general metric driven equation (2.14) can be motivated from a system thermodynamic viewpoint; in [HH08] a network protocol is proposed with pairwise interactions of the form $f(x_i, x_j)$, where $f$ is locally Lipschitz continuous and assumed to satisfy the negativity condition $(x_i - x_j)f(x_i, x_j) \leq 0$, $f(x_i, x_j) = 0$ if $x_i = x_j$. According to the authors this assumption implies that some sort of energy or information flows from higher to lower levels thus this condition is reminiscent of a "second law"-like inequality in thermodynamics.
We observe that this negativity hypothesis is naturally fulfilled by a metric interaction form as in (2.14): for any two states the sign of the terms $(x_i - x_j)$ and $f(x_i, x_j)$ must differ. Hence, for two arguments $x_i, x_j$, $f$ has the sign $\mathsf{sign}(x_j - x_i)$. Therefore, we get the structure $f = \mathsf{sign}(x_j - x_i) f_{\text{res}}$ with residual part required to be positive definite. The choice $f_{\text{res}} = d$, i.e., the residual part is a metric function, follows naturally. However, a metric structure is not necessary, only sufficient for this condition to hold.

**Example 2.** When the metric chosen in the local ODEs is the Euclidean distance, we recover the linear consensus protocol. Olfati-Saber and Murray's nonlinear consensus protocol [SM03a],

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} \phi(x_j - x_i),$$

where $\phi$ is a continuous, increasing function that satisfies $\phi(0) = 0$, is a subclass of a network dynamics (2.14). The nonlinear interaction in phase averaging, $\phi(\cdot) = \sin(\cdot)$ on the open interval $]-\pi/2, \pi/2[$ is a famous example.

We adopt the term "scaling-invariant" as the right-hand-side of (2.13) is invariant under scaling, i.e., for any $c > 0$,

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij}(\ln c x_j - \ln c x_i) = \sum_{j \in N_i^+} w_{ij} \left( \ln \frac{c x_j}{c x_i} \right) = \sum_{j \in N_i^+} w_{ij}(\ln x_j - \ln x_i).$$

Scaling is a fundamental transitive group action besides rotation and linear translation, important in constructing homogeneous spaces. With the property of being scaling-invariant, (2.13) is a third fundamental consensus protocol added to the consensus on the circle with interactions defined by sinusoidal coupling and consensus on linear space, defined by the basic linear consensus protocol. The linear consensus algorithm is invariant to translation by a constant, i.e. $x_j - x_i = (x_j + c) - (x_i + c)$. For states being configuration on the circle, where each $x_i$ hence represents a phase angle, constant translation by an amount $c$ of each $x_i$ amounts to rotation by this angle given by $c$. Sinusoidal coupling, as interaction nonlinearity for consensus on the circle, therefore results in rotation invariance, as the coupling $\sin(x_j - x_i) = \sin((x_j + c) - (x_i + c))$. The natural configruation space for the scaling invariant dynamics under (2.13) is the set of positive rays, where a positive ray is the line $[\boldsymbol{x}] := \{c\boldsymbol{x}, c > 0, \boldsymbol{x} \in \mathbb{R}^n_{>0}\}$. Scaling-invariance is a natural property for quantities described by the components $x_i$ that are intensities, i.e., relative or specific measures, such as "power per unit area", or a concentration, such as "mol per liter".

## 2.4 Dynamic behavior of novel network protocols

In this section we show global exponential convergence to a consensus configuration of the network protocols on graphs G driven by the geometric mean, which in summary are given by

$$\dot{x}_i = -\prod_{j\in N_i^-} x_i^{w_{ji}} + \prod_{j\in N_i^+} x_i^{w_{ij}}, \tag{2.15}$$

$$\dot{x}_i = -\sum_{j\in N_i^+} w_{ij} x_i \ln\frac{x_i}{x_j}, \text{ and} \tag{2.16}$$

$$\dot{x}_i = \sum_{j\in N_i^+} w_{ij}(\ln x_j - \ln x_i). \tag{2.17}$$

For protocols (2.16) and (2.17) we characterize the reached consensus value analytically. We further give a convergence proof and derive fixed point conditions for the general reaction network protocol

$$\dot{x}_i = -\gamma_i^- \prod_{j\in N_i^-} x_i^{w_{ji}} + \gamma_i^+ \prod_{j\in N_i^+} x_i^{w_{ij}}, \quad i \in N. \tag{2.18}$$

The conditions we derive are reminiscent of the Wegscheider relation in chemical reaction networks.

### 2.4.1 Agreement and convergence under consensus protocols

To study stability of fixed-points we shall make use of the logarithmic mean, its properties and the mean value theorem: The logarithmic mean of two positive real numbers $a, b$ is defined as

$$\mathrm{lgm}(a,b) := \frac{a-b}{\ln a - \ln b}.$$

The logarithmic mean is symmetric in both arguments, i.e., $\mathrm{lgm}(a,b) = \mathrm{lgm}(b,a)$, and it is positive. The mean value theorem states that for a continuously differentiable function $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, there exists a $\xi \in [a,b]$ such that

$$\nabla f(\xi) = \frac{f(b)-f(a)}{b-a}.$$

With $f = \ln$, we get the particular identity $\mathrm{lgm}(a,b) = \xi$, where $0 < a \le \xi \le b$.

The logarithmic mean and its inverse take positive and finite values for positive and finite arguments. For approaching positive real arguments, we further have

$$\lim_{b\to a} \frac{\ln b - \ln a}{b-a} = \lim_{\epsilon\to 0^+} \frac{\ln(a+\epsilon)-\ln a}{\epsilon} \triangleq \nabla \ln \xi|_{\xi=a} = \frac{1}{a},$$

so that $\lim_{b\to a} \mathrm{lgm}(a,b) = a > 0$.

**Theorem 1** (Convergence to consensus)**.** *Consider network protocols* (2.15)-(2.17) *with initial conditions restricted to* $\mathbb{R}^n_{>0}$*. If the underlying digraph is strongly connected, then protocols* (2.16) *and* (2.17) *converge exponentially fast to a consensus configuration. If in addition the weighting is balanced, then protocol* (2.15) *converges exponentially fast to a consensus state. In all three cases the equilibrium* $\bar{x}\mathbf{1}$ *has agreement value* $\min_{i\in N} x_i(0) < \bar{x} < \max_{i\in N} x_i(0)$.

*Proof.* We start with (2.15) from where the two other cases shall follow. As $r_i^+$ and $r_i^-$, as defined in (2.7) and (2.8), are positive, we can expand the protocol (2.15) with the logarithm of these rates, so that,

$$\begin{aligned}\dot{x}_i = r_i^+ - r_i^- &= \frac{r_i^+ - r_i^-}{\ln r_i^+ - \ln r_i^-}\left(\ln r_i^+ - \ln r_i^-\right) \\ &= \mathsf{lgm}(r_i^+, r_i^-)\left(\sum_{j\in N_i^+} w_{ij}\ln x_j - \sum_{j\in N_i^-} w_{ji}\ln x_i\right) \qquad (2.19)\\ &= \mathsf{lgm}(r_i^+, r_i^-)\sum_{j\in N_i^+} w_{ij}\left(\ln x_j - \ln x_i\right). \qquad (2.20)\end{aligned}$$

In going from (2.19) to (2.20) we made use of balancedness of the weighting, so that $\sum_{j\in N_i^-} w_{ji} = \sum_{j\in N_i^+} w_{ij}$. Expanding the pairwise interactions by local pairwise state differences yields

$$\dot{x}_i = \mathsf{lgm}(r_i^+, r_i^-)\sum_{j\in N_i^+} w_{ij}\frac{\ln x_j - \ln x_i}{x_j - x_i}\left(x_j - x_i\right), \quad i\in N.$$

Define the matrix $\mathbf{L}_X(\boldsymbol{x}(t))$,

$$[\mathbf{L}_X]_{ij} := \begin{cases} -w_{ij}\mathsf{lgm}^{-1}(x_j, x_i), & \text{if } j\neq i,\\ \sum_{j\in N_i^+} w_{ij}\mathsf{lgm}^{-1}(x_j, x_i), & j = i, i\in N,\end{cases}$$

and $\mathbf{R} := \mathrm{diag}\{\mathsf{lgm}(r_1^+, r_1^-), \mathsf{lgm}(r_2^+, r_2^-), \ldots, \mathsf{lgm}(r_n^+, r_n^-)\}$.

Then, we get the vector-matrix representation for the polynomial ODE system,

$$\dot{\boldsymbol{x}} = -\mathbf{R}(\boldsymbol{x})\mathbf{L}_X(\boldsymbol{x})\boldsymbol{x}. \qquad (2.21)$$

Next, we show that for positive initial conditions, the flow generated by the ODE system (2.21) is well defined for all times: For $\boldsymbol{x}(0)\in\mathbb{R}^n_{>0}$, the matrix $-\mathbf{L}_X(\boldsymbol{x}(0))$ by definition is a Laplacian matrix with finite, non-negative and real off-diagonal elements, as the branch weights are non-negative and the logarithmic mean of positive, real and finite arguments is positive, real and finite. This follows from the mean value theorem: For $x_i, x_j\in\mathbb{R}_{>0}$, $\mathsf{lgm}^{-1}(x_i, x_j) = \frac{1}{\xi} > 0$, as $\xi$ is a value within the interval spanned by the positive real numbers $x_i$ and $x_j$. Hence for positive initial condition one can always find a threshold $\delta_X$, such that $\mathsf{lgm}^{-1}(x_i(0), x_j(0)) \geq \delta_X > 0$. The diagonal matrix $\mathbf{R}(\boldsymbol{x}(0))$ is positive definite, as for positive initial conditions $r_i^+$ and $r_i^-$ are positive, so that $\mathsf{lgm}(r_i^+, r_i^-) > 0$ as well, with value in between the two rates, again by the mean value theorem. Hence, with positive initial condition, one can always find a lower bound $\delta_R > 0$, such that $\mathsf{lgm}(r_i^+, r_i^-)|_{t=0} \geq \delta_R > 0$. Therefore, the matrix $\mathbf{R}(\boldsymbol{x}(0))\mathbf{L}_X(\boldsymbol{x}(0))$ is a Laplacian matrix characterizing a "virtual" graph with non-negative finite entries, and non-trivial "virtual" branch weights that are bounded away from zero by a threshold value $\delta$ such that $\delta \geq \min_{(j,i)\in B}\{w_{ij}\}\cdot\delta_R\cdot\delta_X > 0$. Hence, at $t = 0$ the polynomial ODE system defines a consensus network. By definition, the flow map of a consensus system is a stochastic matrix, which is a positive monotone map that leaves $\mathbb{R}^n_{>0}$ invariant, cf., e.g., [SSR10] for this monotonicity fact in consensus theory. Thus, trajectories starting in $\mathbb{R}^n_{>0}$ will remain in this set, so that $[\mathbf{RL}_X](\boldsymbol{x}(t))$ is well-defined for all

$t \geq 0$, and it characterizes a linear time-varying consensus network, where the variability of "'virtual" branch weights is endogenously determined as a function of state trajectories, which are parameterized by time as a free parameter.

As the graph G on which the protocols run is strongly connected by hypothesis, the "virtual" graph associated to the dynamic Laplacian $[\mathbf{RL}_X](\cdot)$ is uniformly connected at each time instant for all $\boldsymbol{x} \in \mathbb{R}^n_{>0}$. Therefore, the polynomial network protocol converges globally and exponentially to a consensus configuration $\bar{\boldsymbol{x}} \in \mathsf{span}\{\mathbf{1}\}$, according to Proposition 1.

Now we consider protocol (2.16) and relax the constraint of balanced to arbitrary weighting of the strongly connected graph. Define the matrix $\mathbf{X}(\boldsymbol{x}) := \mathsf{diag}\{x_1, x_2, \ldots, x_n\}$. Protocol (2.16) can be written as

$$\begin{aligned} \dot{x}_i &= x_i \sum_{j \in N_i^+} w_{ij} \mathsf{lgm}^{-1}(x_i, x_j)(x_j - x_i) \\ \Leftrightarrow \quad \dot{\boldsymbol{x}} &= -\mathbf{X}(\boldsymbol{x})\mathbf{L}_X(\boldsymbol{x})\boldsymbol{x}. \end{aligned} \tag{2.22}$$

The matrix $\mathbf{XL}_X$ is a Laplacian matrix for all parameterizations, by the same arguments as before, so that also the entropic protocol (2.16) converges to a consensus configuration with exponential speed on the positive orthant.

The last protocol (2.17) can be written as

$$\dot{\boldsymbol{x}} = -\mathbf{L}_X(\boldsymbol{x}(t))\boldsymbol{x},$$

which again is a linear time-varying consensus system with endogenously determined variability of the weighting. Hence, the system converges to consensus with exponential speed on the positive orthant, as well.

Let us turn to the last statement regarding the exponentially fast reached consensus value. All three nonlinear protocols can be brought to a linear consensus form on a dynamically weighted but strongly connected "virtual" graph. By standard linear consensus theory, the function $\max_{i \in N} x_i - \min_{i \in N} x_i$ is a (strict) Lyapunov function [Mor05]. Hence, the maximal state value is decreasing, and the minimal state value is increasing, so that the consensus value must lie in between the initial maximum and minimum state values. □

This proof technique is of interest in its own right: we make use of the Laplacian structure arising from (algebraic) interconnections on a graph in shifting nonlinearity associated with nodes to a nonlinearity in pairwise interactions across branches, leading to a "virtual" dynamic graph on which the nonlinear time-invariant network dynamics appear as linear time-varying consensus systems.

*Remark* 1 (Time-varying graphs and uniform connectedness). We note that the transformations in the proof of convergence do not rely on time-invariant weightings. This suggests that convergence to consensus should take place also under the weaker assumption of uniform graph connectivity.

In the following result, we analytically characterize the consensus value for the entropic consensus network.

**Theorem 2** (Weighted geometric mean consensus)**.** *Consider a weighted digraph that is strongly connected, with left eigenvector of the associated Laplacian* $\boldsymbol{L}$*,* $\boldsymbol{\pi} \in \mathbb{R}^n_{>0}$*, such that* $\boldsymbol{\pi}^\top \boldsymbol{L} = \mathbf{0}$*. Then, the consensus dynamics* (2.16) *starting at any* $\boldsymbol{x}(0) \in \mathbb{R}^n_{>0}$ *asymptotically reaches a fixed point* $\bar{x}\mathbf{1}$*, with*

$$\bar{x} = \mathsf{gm}_w(\boldsymbol{x}(0)) = \prod_{i=1}^{n} x_i^{\hat{\pi}_i}(0),$$

*where* $\hat{\boldsymbol{\pi}} := \boldsymbol{\pi}/|\boldsymbol{\pi}|_1$*, is the Perron vector of* $\boldsymbol{L}$*.*

*Proof.* The proof concerning convergence to consensus is analogous to the previous proof of Theorem 1, where the nonlinear time-invariant system of equations is transformed to a linear time-varying consensus form, with endogenously determined variability of the branch weights. In particular, let us start from the linear representation of the protocol in vector matrix form (2.22), which can be re-written as

$$\mathbf{X}^{-1}(\boldsymbol{x})\dot{\boldsymbol{x}} = \mathbf{L}_X(\boldsymbol{x})\boldsymbol{x} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\ln \boldsymbol{x} = \mathbf{L}\ln \boldsymbol{x},$$

as $\frac{\mathrm{d}}{\mathrm{d}t}\ln x(t) = \frac{1}{x}\dot{x}$ and the Laplacian structure allows to shift the nonlinearity from inverted logarithmic mean components in the weightings to logarithmic coordinates at nodes such that

$$\mathbf{L}_X \boldsymbol{x} = \mathbf{L}\ln \boldsymbol{x}.$$

Note that the inverse $\mathbf{X}^{-1}$ exists, as it is a diagonal matrix having positive real diagonal elements.

Next, we prove that the weighted geometric mean is the consensus value. By hypothesis, $\boldsymbol{\pi}$ is in the left kernel of $\mathbf{L}$, so that $\boldsymbol{\pi}^\top \mathbf{L}\ln \boldsymbol{x} = 0$. Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\boldsymbol{\pi}^\top \ln \boldsymbol{x}(t)] = 0 \Rightarrow \boldsymbol{\pi}^\top \ln \boldsymbol{x}(0) = \sum_{i=1}^{n} \pi_i \ln x_i(t) = \text{const.} \tag{2.23}$$

Using the fact that for $t \to \infty$ a uniform state is reached, together with basic arithmetics for the logarithm, the invariance property (2.23) implies that

$$\begin{aligned} &\sum_{i=1}^{n} \pi_i \ln \bar{x} = \sum_{i=1}^{n} \pi_i \ln x_i(0) \\ &\Leftrightarrow \quad \ln \bar{x} = \frac{1}{\sum_{i=1}^{n} \pi_i} \sum_{i=1}^{n} \pi_i \ln x_i(0) = \ln \prod_{i=1}^{n} x_i(0)^{\hat{\pi}_i}. \end{aligned}$$

Solving for the consensus value yields,

$$\bar{x} = \exp\left( \ln \prod_{i=1}^{n} x_i(0)^{\hat{\pi}_i} \right) \triangleq \mathsf{gm}_w(\boldsymbol{x}(0)),$$

which completes the proof. □

**Corollary 1.** *Consider the scaling invariant protocol* (2.17) *in the setting described in Theorem 2. The asymptotically reached consensus value is the weighted arithmetic mean of the initial condition with weights given by the components of the Perron vector, i.e.,* $\bar{x} = \sum_{i=1}^{n} \hat{\pi}_i x_i(0)$*.*

*Proof.* The proof follows from noting that $\sum_{i=1}^{n} \hat{\pi}_i x_i(t)$ remains invariant along the dynamics. □

*Remark* 2. Regarding the asymptotically reached agreement value of the polynomial consensus protocol, the maximum and minimum initial state values provide upper and lower bounds by standard linear consensus theory. However, a numerical study presented in Section 2.5.2 implies tighter results for the consensus value, with interesting application to the solution of elliptic integrals.

### 2.4.2 Convergence under reaction network protocol

Define the vector $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, $q_i = \frac{\gamma_i^+}{\gamma_i^-}$. The pseudo-inverse of $\mathbf{L}$ is denoted $\mathbf{L}^\dagger$.

**Theorem 3** (Fixed points and asymptotic behavior of reaction network protocol)**.** *Let* $\mathsf{G}$ *be a strongly connected and balanced graph. Consider the reaction network protocol* (2.18) *written with forward and backward rates* $R^+$ *and* $R^-$ *such that as*

$$\dot{x}_i = \gamma_i^+ \prod_{j \in N_i^+} x_i^{w_{ij}} - \gamma_i^- \prod_{j \in N_i^-} x_i^{w_{ji}} = R_i^+(\boldsymbol{x}) - R_i^-(\boldsymbol{x}). \tag{2.24}$$

*We assume that states are positive, i.e.,* $\boldsymbol{x} \in \mathbb{R}^n_{>0}$*. The following holds:*

*(i) Given an initial condition, the system converges to a unique* $\boldsymbol{x}^*$ *if and only if* $\prod_i q_i = 1$*. Under this condition, dynamics converge exponentially towards* $\boldsymbol{x}^*$*.*

*(ii) A fixed point* $\boldsymbol{x}^*$ *is a consensus point, i.e.,* $\boldsymbol{x}^* = \bar{x}\mathbf{1}$*, with* $\bar{x} > 0$*, if and only if* $\boldsymbol{q} = \mathbf{1}$*.*

*(iii) If* $\prod_i q_i < 1$*, then* $\lim_{t\to\infty} \boldsymbol{x}(t) \to \mathbf{0}$*, i.e., trajectories move and decelerate towards the zero state, which is never reached.*

*(iv) If* $\prod_i q_i > 1$*, then* $\boldsymbol{x}(t)$ *moves towards infinity.*

*Proof.* As $R_i^+$ and $R_i^-$ are positive, for balanced weightings, we can express the rate equation (2.24) using the expansion approach, as in the proof of Theorem 1, equivalently as

$$\dot{x}_i = \mathsf{lgm}(R_i^+, R_i^-) \left[ \sum_{j \in \mathcal{N}_i^+} w_{ij} \left( \ln x_j - \ln x_i \right) + \ln q_i \right], \tag{2.25}$$

where we recall that $\mathsf{lgm}(R_i^+, R_i^-) = \frac{R_i^+ - R_i^-}{\ln R_i^+ - \ln R_i^-}$.
Define the matrix $\mathbf{R}_\gamma(\boldsymbol{x}) := \mathsf{diag}(\{\mathsf{lgm}(R_i^+, R_i^-)\}_{i \in N})$. Then, (2.25) has the vector matrix form

$$\dot{\boldsymbol{x}} = \mathbf{R}_\gamma(\boldsymbol{x}) \left( -\mathbf{L} \ln \boldsymbol{x} + \ln \boldsymbol{q} \right). \tag{2.26}$$

Next, let us split the external input $\ln \boldsymbol{q}$ into the component orthogonal to the vector $\mathbf{1}$, denoted $\xi^\perp$, and the component parallel to the one-vector, denoted $\xi^\parallel$, such that we obtain the orthogonal vector decomposition $\ln \boldsymbol{q} = \xi^\perp + \xi^\parallel$. With that, the rejection from the one-vector is in the image of the Laplacian matrix and therefore it can be expressed as

$$\xi^\perp = \mathbf{L} \ln \bar{\boldsymbol{x}}, \quad \bar{\boldsymbol{x}} \in \mathbb{R}^n_{>0}. \tag{2.27}$$

Conversely, the projection onto $\mathbf{1}$ satisfies $\xi^{\parallel} \in \ker(\mathbf{L})$, i.e., it is proportional to the one-vector, and element of the set of consensus states $\{c\mathbf{1}, c \in \mathbb{R}_{>0}\}$.

Using this decomposition together with (2.27) we can further write (2.26) as

$$\dot{\boldsymbol{x}} = \mathbf{R}_\gamma\left(-\mathbf{L}\ln \boldsymbol{x} + \xi^{\perp} + \xi^{\parallel}\right) = \mathbf{R}_\gamma\left(-\mathbf{L}(\ln \boldsymbol{x} - \ln \bar{\boldsymbol{x}}) + \xi^{\parallel}\right) =$$

$$\Leftrightarrow \quad \dot{x}_i = \frac{\bar{x}_i}{\bar{x}_i}[\mathbf{R}_\gamma]_{ii}\left(\xi_i^{\parallel} + \sum_{j\in N_i^+} w_{ij}\left(\ln\frac{x_j}{\bar{x}_j} - \ln\frac{x_i}{\bar{x}_i}\right)\right), \quad i \in N.$$

Introducing the density notation $\boldsymbol{\rho}$, with $\rho_i = \frac{x_i}{\bar{x}_i}$ for all $i \in N$, expanding by $\rho_j - \rho_i$ and division by $\bar{x}_i$ results in the component and vector matrix consensus equation

$$\dot{\rho}_i = \frac{1}{\bar{x}_i}[\mathbf{R}_\gamma]_{ii}\left(\xi_i^{\parallel} + \sum_{j\in N_i^+} w_{ij}\mathrm{lgm}(\rho_j,\rho_i)\big(\rho_j - \rho_i\big)\right)$$

$$\Leftrightarrow \quad \dot{\boldsymbol{\rho}} = -\mathbf{L}_R(\boldsymbol{x},\boldsymbol{q})\boldsymbol{\rho} + \bar{\mathbf{X}}^{-1}\mathbf{R}_\gamma(\boldsymbol{x})\xi^{\parallel}, \tag{2.28}$$

where $\bar{\mathbf{X}} = \mathrm{diag}\{\bar{x}_1,\ldots,\bar{x}_n\}$ and $\mathbf{L}_R(\boldsymbol{x},\boldsymbol{q})$ is the irreducible Laplacian matrix function

$$[\mathbf{L}_R(\boldsymbol{q},\boldsymbol{x})]_{ij} := \begin{cases} -\frac{\mathrm{lgm}(R_i^+,R_i^-)w_{ij}}{\bar{x}_i\mathrm{lgm}(\rho_j,\rho_i)}, & \text{if } j \neq i \text{ and } (j,i) \in B, \\ \sum_{j\in N_i^+}\frac{\mathrm{lgm}(R_i^+,R_i^-)w_{ij}}{\bar{x}_i\mathrm{lgm}(\rho_j,\rho_i)}, & \text{if } i = j. \end{cases}$$

Any equilibrium point $\boldsymbol{x}^*$ must satisfy $\dot{\boldsymbol{\rho}} = \mathbf{0}$. This is the case if $\xi^{\parallel} = \mathbf{0}$, because then, (2.28) reduces to the linear consensus system on a dynamics graph with weighting $\mathbf{L}_\gamma$, which by Proposition 1 converges with exponential speed to a consensus density vector, such that $\lim_{t\to\infty}\boldsymbol{\rho}(t) = \bar{\rho}\mathbf{1}$, i.e., for all $i \in N$, $x_i^* = \bar{\rho}\bar{x}_i$. By standard consensus theory, this consensus density value satisfies $\max_{i\in N}\rho_i(t=0) \leq \bar{\rho} \leq \max_{i\in N}\rho_i(t=0)$ [Mor04].
The projection of $\ln \boldsymbol{q}$ onto $\mathbf{1}$ vanishes, i.e., $\xi^{\parallel} = \mathbf{0}$, if and only if $\ln \boldsymbol{q} \perp \mathbf{1}$, i.e.,

$$\mathbf{1}^\top \ln \boldsymbol{q} = \sum_{i=1}^n \ln q_i = \ln\left(\prod_{i=1}^n q_i\right) = 0 \Leftrightarrow \prod_{i=1}^n q_i = e^0 = 1.$$

This proves sufficiency of part (i). To see that it is also necessary, using (2.28), we observe that stationarity in an equilibrium implies

$$\mathbf{0} = -\bar{\mathbf{X}}^{-1}\mathbf{R}_\gamma\mathbf{L}\ln\boldsymbol{\rho} + \bar{\mathbf{X}}^{-1}\mathbf{R}_\gamma\xi^{\parallel} \quad \Leftrightarrow \quad \xi^{\parallel} = \mathbf{L}\ln\boldsymbol{\rho}. \tag{2.29}$$

This equivalence follows from multiplication with the inverse of $\bar{\mathbf{X}}^{-1}\mathbf{R}_\gamma$, which is well-defined, as it is a positive definite diagonal matrix. The vector $\xi^{\parallel}$ is a constant multiple of the one vector, i.e., $\xi^{\parallel} = \bar{\xi}\mathbf{1}$, so that multiplication of (2.29) with the one-vector from the left yields the contradiction

$$\mathbf{1}^\top\xi^{\parallel}\mathbf{1} = n\bar{\xi} = \mathbf{1}^\top\mathbf{L}\ln\boldsymbol{\rho} = 0, \tag{2.30}$$

because $n > 0$ and $\bar{xi} > 0$, if $\xi$ is not orthogonal to $\mathbf{1}$. Hence, $\xi$ must be orthogonal to the one-vector, in order to recover validity of the stationarity condition(2.30), i.e., $\bar{\xi} = 0$. Let us find the explicit expression for $\bar{\xi}$. The vector projection of $\xi = \ln\boldsymbol{q}$ onto $\mathbf{1}$ is given by

$$\xi^{\parallel} = \bar{\xi}\mathbf{1} = \frac{\xi^\top\mathbf{1}}{\mathbf{1}^\top\mathbf{1}}\mathbf{1} \quad \Leftrightarrow \quad \bar{\xi} = \frac{\sum_{i=1}^n \ln q_i}{n} = \ln \mathrm{gm}(\boldsymbol{q}). \tag{2.31}$$

In order for $\ln \mathrm{gm}(\boldsymbol{q}) = 0$ to be true the geometric mean of the forward-backwared reaction rate ratios must be one, i.e.,

$$\mathrm{gm}(\boldsymbol{q}) = \left(\prod_{i=1}^{n} q_i\right)^{\frac{1}{n}} = 1 \quad \Leftrightarrow \quad \prod_{i=1}^{n} q_i = 1.$$

This completes the proof of part (i).

Regarding part (ii), a consensus equilibrium is a fixed point so that $\prod_i q_i = 1$ must holds. Note that this is the case if $\boldsymbol{q} = \mathbf{1}$. As shown in the proof of part (i), fixed points are consensus densities $\bar{\rho}\mathbf{1}$, i.e., $\frac{x_i^*}{\bar{x}_i} = \bar{\rho} > 0$, for all $i \in N$. For the state $\boldsymbol{x}^*$ to be a consensus state also $\bar{\boldsymbol{x}}$ must be a constant multiple of the one vector, i.e.,

$$\bar{\boldsymbol{x}} = \bar{x}\mathbf{1} = \exp(\mathbf{L}^\dagger \ln \boldsymbol{q}). \tag{2.32}$$

This requires $\mathbf{L}^\dagger \ln \boldsymbol{q} = c\mathbf{1}$ with $c \geq 0$. That equality is true if and only be true if $c = 0$, because $\ker(\mathbf{L}^\dagger) = \ker(\mathbf{L}^\top) = \{c\mathbf{1}, c \in \mathbb{R}\}$, i.e., no constant multiple of the one-vector, with positive constant, can be obtained from the image of the pseudo-inverse of the balanced Laplacian matrix $\mathbf{L}$. Hence, (2.33) $\bar{x} = 1$, as $\exp(\mathbf{0}) = \mathbf{1}$, which is necessary and sufficient for a consensus equilibrium.

If $\boldsymbol{q} = \mathbf{1}$, $\ln \boldsymbol{q} = \mathbf{0}$ and (2.32) is true with. This completes the sufficiency part of the consensus condition proof (ii).

To proof necessity of $\boldsymbol{q} = \mathbf{1}$ for consensus, let us compute the value of $\ln \boldsymbol{q}$.
For arbitrary $\boldsymbol{q} > \mathbf{0}$ we have

$$\begin{aligned} \ln \boldsymbol{q} = \xi^{\parallel} + \xi^{\perp} \quad &\overset{(2.31)}{\Leftrightarrow} \quad \ln \boldsymbol{q} - \ln \mathrm{gm}(\boldsymbol{q})\mathbf{1} = \xi^{\perp} \\ \overset{\xi^{\perp} = \ln \boldsymbol{q}}{\Leftrightarrow} \quad \xi_i^{\perp} = \ln q_i = \ln \frac{q_i}{\mathrm{gm}(\boldsymbol{q})}, \qquad &\forall i \in N. \end{aligned} \tag{2.33}$$

The component condition (2.33) is true if and only if $\mathrm{gm}(\boldsymbol{q}) = 1$, however, also $\prod_i q_i = 1$, that is,

$$\left(\prod_{i=1}^{n} q_i\right)^{\frac{1}{n}} = \prod_{i=1}^{n} q_i \quad \Leftrightarrow \quad q_i = 1, \quad \forall i \in N.$$

This completes the necessity part of the consensus condition proof (ii).

Regarding part (iii) and (iv), we note that if $\prod_i q_i \neq 1$ no equilibrium point exists, as shown in the argument based on (2.29): Then, recall that $\xi^{\parallel} = \xi^{\parallel}\mathbf{1}$, with $\xi^{\parallel} = \ln \mathrm{gm}(\boldsymbol{q}) \neq 0$, cf. (2.31).

Regarding part (iii), note that $\mathrm{gm}(\boldsymbol{q}) < 0$ is equivalent to $\prod_{i=1}^{n} q_i < 0$. Then, $\ln \mathrm{gm}(\boldsymbol{q}) \leq \mathbf{0}$, i.e., there is a constant negative velocity input to the network system. With the Laplacian network dynamics part $\mathbf{L}_R\boldsymbol{\rho}$ being contractive, the constant negative velocity input drives the system state towards ever decreasing state values.
When $\boldsymbol{\rho}(t) \leq \mathbf{1}$, and also $\boldsymbol{x}(t) \leq \mathbf{1}$, the gains collected in $\mathbf{R}_\gamma$ become very small: The gains are the logarithmic means of rates, $\frac{R_i^+ - R_i^-}{\ln R_i^+ - \ln R_i^-} =: R_i'$, with $R_i' > 0$ a value in the interval spanned by $R_i^+$ and $R_i^-$. For a component $0 < x_i(t) << 1$, $R_i^- \to 0$, and therefore, $\ln R_i^- \to -\infty$ and $R_i' \to 0$. The closer the system is to the origin, the smaller become gains in $\mathbf{R}_\gamma$, and they

decrease faster than linearly (rates are powers). Hence, the constant negative velocity input $\xi^{\parallel}$ therefore becomes ever smaller as it acts on $\dot{\boldsymbol{\rho}}$ via the rates $\mathbf{R}_\gamma$, see (2.28), which become vanishingly small. Eventually, trajectories are therefore directed towards the origin $\mathbf{0}$, which, however, will never be reached: The logarithm of values nearing 0 from the positive real line becomes infinite in magnitude.

The proof of (iv) follows analogous to (iii): If $\mathsf{gm}(\boldsymbol{q}) > 0$, or equivalently $\prod_{i=1}^{n} q_i > 0$, Then, $\ln \mathsf{gm}(\boldsymbol{q}) \geq \mathbf{0}$. Therefore, a constant positive gain acts on the contractive network dynamics for all times, so that trajectories grow towards infinity.

□

The results and proof of this Theorem lead to the following observations: The necessary and sufficient equilibrium condition in Theorem 3 (i) has a remarkable correspondence in physical chemistry: Chemical reaction networks that converge to equilibrium concentrations are characterized by the condition on forward and backward reaction constants $\gamma_i^+, \gamma_i^-$, $i = 1, 2, \ldots, n$ indexing the reaction,

$$\frac{\gamma_1^+ \gamma_2^+ \cdots \gamma_n^+}{\gamma_1^- \gamma_2^- \cdots \gamma_n^-} = \prod_{i=1}^{n} q_i = 1. \tag{2.34}$$

In chemistry, this equilibrium condition is known under the term detailed balance or Wegscheider relation corresponding to systems that have stationary states with zero flux [Qia06]. In contrast, for chemical reaction networks with $\prod_i q_i \neq 1$ non-equilibrium stationary states may occur, indicating non-zero stationary flux, which is equivalent to the non-stationarity condition of Theorem 3 (iii) and (iv).

As the proof of this Theorem shows, Wegscheider's equilibrium condition (2.34) essentially is an orthogonality condition for the vector collecting logarithmic ratios of forward/backward reaction rates to the consensus ray $[\mathbf{1}]$. Only if there is a trivial solution to this orthogonality constraint $\mathbf{1}^\top \ln \boldsymbol{q} = 0$, which is the consensus condition, when $\boldsymbol{q} = \mathbf{1}$, i.e., it has no orthogonal component to the consensus ray, the system has a flat solution with all equilibria components equal.

The part of non-identity ratios $q_i = \frac{\gamma_i^+}{\gamma_i^-}$ that drives the equilibrium solution away from a consensus solution is the component $\xi^\perp = \ln \frac{q_i}{\mathsf{gm}(\boldsymbol{q})} = \ln q_i - \ln \mathsf{gm}(\boldsymbol{q})$. That is the logarithmic deviation of the reaction constant ratio from the geometric mean of all reaction constants.

*Remark* 3. We conjecture that in cases (iii) and (iv), the asymptotic behavior is characterized by a motion towards a constant line. The internal Laplacian network dynamics contract the diameter of a cone that moves along trajectories with diameter $\max_i \ln \rho_i - \min_i \ln \rho_i$. Hence, the maximal change of direction of the solution decreases with time so that solutions tend towards constant lines. This argument is along with the recent idea of differential Lyapunov analysis as presented by Forni and Sepulchre in [FS14]. We illustrate this behavior using numerical simulation examples in Section 2.5.3.

## 2.5 Numercial examples and agreement value study

In this section, we illustrate the derived convergence results using numerical simulations. We further demonstrate in a simulation study a relationship between the solution of an elliptic

integral and the asymptotic behavior of the polynomial network dynamics. For polynomial reaction dynamics, simulation results indicate stability properties of certain rays along which solutions converge under non-equilibrium conditions.

### 2.5.1 Consensus driven by the geometric mean

First, we compare the protocols of polynomial type (2.15), of entropic type (2.16) and the scaling-invariant one (2.17) for a digraph given in Fig. 2.1. This digraph is strongly connected and has balanced branch weights.

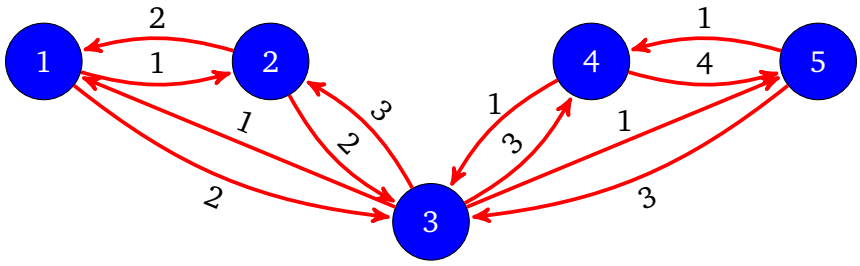



Figure 2.1: Strongly connected, balanced digraph as it is used in the numerical case study

For each of these protocols we compute trajectories starting at $\boldsymbol{x}(0) = [6.5, 0.2, 3.2, 1, 4.4]$. In accordance to Theorem 1, the novel network protocols are indeed consensus protocols that converge to a uniform equilibrium state $\bar{x}\mathbf{1}$. As the left-Perron vector for the balanced weighting is a uniform vector, the LTI consensus system must solve the average consensus problem with $\bar{x} = \mathsf{am}(\boldsymbol{x}(0)) = 3.06$, the scaling-invariant protocol, according to Corollary 1, as well, and solution curves of the entropic protocol must converge to the geometric mean of the initial state, $\mathsf{gm}(\boldsymbol{x}(0)) = 1.7886 = \bar{x}$, as shown in Theorem 2. Our observations are confirmed by Fig. 2.2.

Next, let us illustrate the results in Theorem 2 and Corollary 1 on a digraph which is strongly connected but not balanced. We consider a weighted digraph described in Fig. 2.3, which has Perron vector $\hat{\boldsymbol{\pi}} = [0.26, 0.14, 0.37, 0.09, 0.14]$.

The weighted arithmetic mean of the same initial condition using the Perron vector components as weights is $\mathsf{am}_w(\boldsymbol{x}(0)) = 3.5884$, and the weighted geometric mean becomes $\mathsf{gm}_w(\boldsymbol{x}(0)) = 2.4444$. Again, the simulation results for trajectories generated by the linear, scaling invariant and entropic protocol as depicted in Fig. 2.4 confirm our observations.

*Remark* 4. Trajectories of the polynomial consensus system are not depicted in Figure 2.4, as the underlying graph is not balanced, which, however, is a condition for convergence to a consensus equilibrium point in Theorem 1.

### 2.5.2 Agreement values of the polynomial consensus protocol

In the following, we study the consensus value of the polynomial protocol on a normalized balanced digraph using numerical simulations. We observe that the consensus value can be upper bounded by the arithmetic mean of the initial state and lower bounded by the arithmetic-geometric mean of the arithmetic mean and the geometric mean of the initial condition.

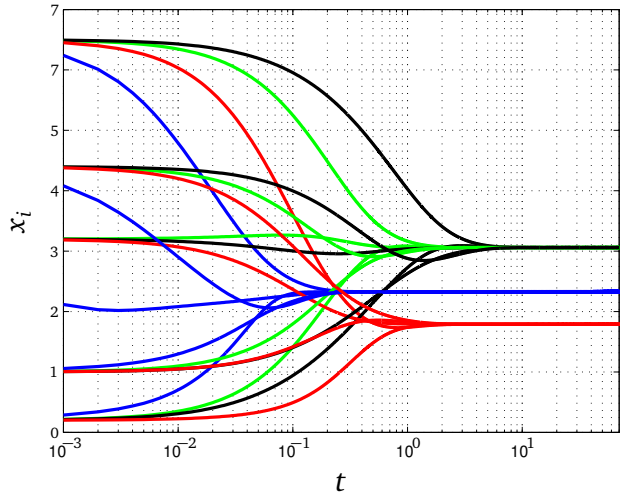



Figure 2.2: Component trajectories for polynomial (blue), entropic (red), scale invariant (black), and standard linear consensus protocol (green) on a graph as depicted in Fig. 2.1 with $\boldsymbol{x}(0) = [6.5, 0.2, 3.2, 1, 4.4]$

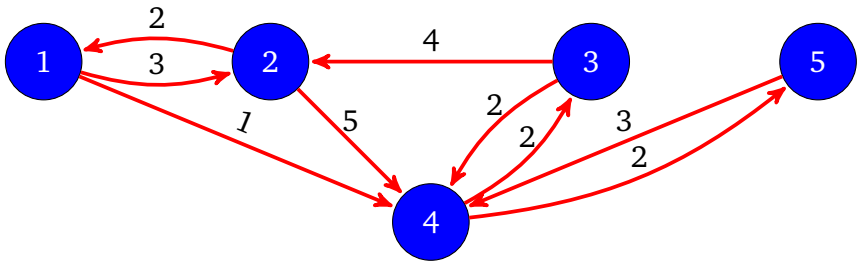



Figure 2.3: Strongly connected digraph with non-balanced weighting

The arithmetic-geometric mean $\mathsf{agm}(a, b)$ of two positive numbers $a, b$, can be defined as the limiting point of a discrete time dynamical system, $\{a_k, b_k\}_{k\geq 0}$, $k \in \mathbb{N}$ that satisfies the algorithmic update rule

$$\begin{pmatrix} a_{k+1} \\ b_{k+1} \end{pmatrix} = \begin{pmatrix} \mathsf{am}(\{a_k, b_k\}) \\ \mathsf{gm}(\{a_k, b_k\}) \end{pmatrix}. \tag{2.35}$$

It is obtained as the limit

$$\mathsf{agm}(a, b) := \lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k, \quad a_0 = a, b_0 = b;$$

The fixed-point iteration (2.35) is due to Carl Friedrich Gauss, who was concerned with computing the perimeter of ellipses, which up until today is a topic of scientific discourse [Adl12][BB87]. The arithmetic-geometric mean is related to the solution of a complete elliptic integral, as

$$\mathsf{agm}(a, b) = \frac{\pi}{2}\frac{1}{I(a, b)}, \qquad I(a, b) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}},$$

see, e.g., [Car71].

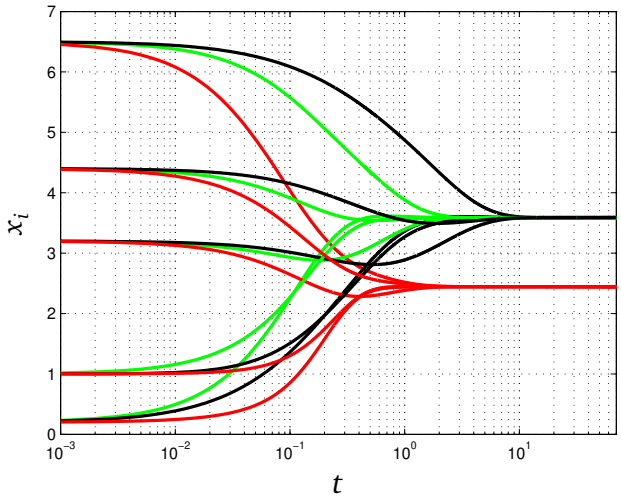



Figure 2.4: Component trajectories for the entropic (red), scaling-invariant (black), and standard linear consensus protocol (green) on a graph as depicted in Fig. 2.3 with $\boldsymbol{x}(0) = [6.5, 0.2, 3.2, 1, 4.4]$

We first consider completely connected normalized balanced graphs that differ only in the number of nodes, such that $N \in \{2, 3, \ldots, 50\}$. For each of these graphs we run the polynomial protocol for 50 random initial conditions sampled from the interval $]0, 10[$, such that $\mathsf{am}(\boldsymbol{x}(0)) = c_1$, and $\mathsf{gm}(\boldsymbol{x}(0)) = c_2$, where $c_1 > c_2 > 0$. In Fig. 2.5 the reached agreement values for this experiment are plotted as black circles. The red squares show the arithmetic mean value of the initial condition, sampled such that $c_1 = 4$ and the blue squares represent the geometric mean of the initial states, sampled such that $c_2 = 3$. We observe that for each graph, every of the reached consensus values lies above the green line, which appears to be a tight and strict lower bound. We find that the value of the green marks computes as the arithmetic-geometric mean of the arithmetic and the geometric mean of the initial state, i.e., its value corresponds to the number $\mathsf{agm}(c_1, c_2)$.

To verify that this observation is independent of the set mean value constraints $c_1$, $c_2$, we next consider the polynomial protocol on a completely connected, balanced, normalized graph with number of nodes being fixed at $N = 5$. We are interested in the values of the ratio $\frac{\mathsf{ref}}{\bar{x}}$, where $\mathsf{ref} \in \{\mathsf{am}(\boldsymbol{x}(0)), \mathsf{gm}(\boldsymbol{x}(0)), \mathsf{agm}(\mathsf{am}(\boldsymbol{x}(0)), \mathsf{gm}(\boldsymbol{x}(0)))\}$. The closer this fraction is to one, the better is "ref" suited as an estimate for the asymptotically reached consensus value, given on the basis of the initial data.

In Fig. 2.6 we plotted this ratio $\frac{\mathsf{ref}}{\bar{x}}$ for 500 random initializations sampled from the interval $]0, 10[$. The red dots mark this ratio for $\mathsf{ref} = \mathsf{am}(\boldsymbol{x}(0))$, the blue ones for $\mathsf{ref} = \mathsf{gm}(\boldsymbol{x}(0))$, and the green ones mark the ratio for reference taken as arithmetic-geometric mean of the arithmetic mean and the geometric mean of the initial state. We can confirm the previous observation that for each trajectory, the arithmetic mean of the initial condition is an upper bound for the consensus value (red dots mark above one), the geometric mean a lower bound (blue dots mark below one), and so is the arithmetic-geometric mean (green dots mark below one), whereas this value is a tighter lower bound than the geometric mean. In particular, the arithmetic-geometric mean bound appears to be, in many cases, a reasonable estimate of the

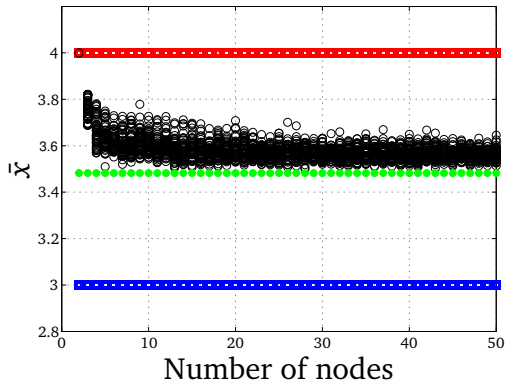



Figure 2.5: Consensus values (black) for all-to-all normalized balanced graphs for 50 random initial conditions such that the arithmetic mean of the initial condition (red square) takes value 4 and the geometric mean (blue square) has value 3. The green marks represent $\mathsf{agm}(3,4)$.

achieved consensus value as the green dots cluster very near to the black line.

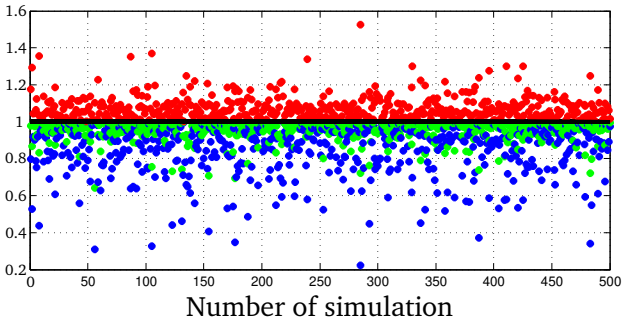



Figure 2.6: Consensus ratios $\mathsf{ref}/\bar{x}$, $\mathsf{ref} = \mathsf{am}(\boldsymbol{x}(0))$ (red), $\mathsf{ref} = \mathsf{gm}(\boldsymbol{x}(0))$ (blue), $\mathsf{ref} = \mathsf{agm}\{\mathsf{am}(\boldsymbol{x}(0)), \mathsf{gm}(\boldsymbol{x}(0)\}$ (green), and $\mathsf{ref} = \bar{x}$ (black) for 500 simulations of a normalized complete balanced graph on 5 nodes with initial conditions randomly sampled from the interval $]0, 10[$.

Eventually, we test if the agm as lower bound is independent of the normalization of the weighting and independent of the number of connected nodes that is if it is a lower bound for the consensus value for every $(N,d)$-regular graph, i.e., balanced graphs on $N$ nodes with $d \in \mathbb{N}$ nodes being connected to each node $i \in N$. In Fig. 2.7 the ratio $\mathsf{agm}(c_1,c_2)/\bar{x}$, $c_1 = \mathsf{am}(\boldsymbol{x}_0)$, $c_2 = \mathsf{gm}(\boldsymbol{x}_0)$ is plotted for $N = 30$, $d \in \{2,3,\ldots,22\}$, where the red dots mark the defined ratio for non-normalized unweighted balanced graphs and the blue dots mark this ratio for normalized ones. For each graph we computed 30 trajectories for random initial conditions sampled as before. We see that the agm lower bound holds only for the normalized case; it is independent of the degree $d$.

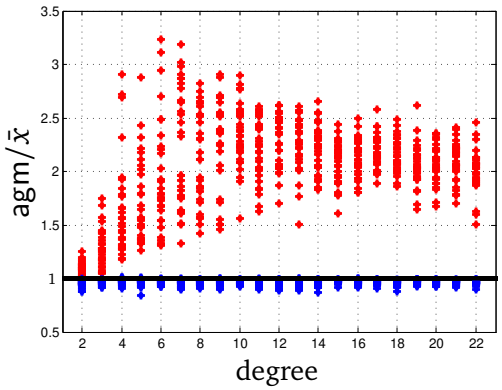



Figure 2.7: Ratio $\mathrm{agm}(c_1, c_2)/\bar{x}$, $c_1 = \mathrm{am}(\boldsymbol{x}_0)$, $c_2 = \mathrm{gm}(\boldsymbol{x}_0)$ for $(N, d)$-regular graphs, $N = 30$, $d \in \{2, 3, \ldots, 22\}$; non-normalized weighting (red) and normalized weighting (blue).

### 2.5.3 Behavior of the reaction network protocol

In the following, we illustrate the behavior of the reaction network protocol, where forward and backward reaction rates differ from one. The exponential convergence behavior to a fixed point as characterized in Theorem 3 (i) and (ii) essentially is based on the proof of convergence of a linear(time-varying) consensus system either in direct (part (ii)) or in density coordinates (part (i)). This consensus convergence behavior of polynomial interactions across a balanced digraph is covered in the discussion in section 2.5.1. Therefore, next, we consider the cases in Theorem 3 (iii) and (iv), where the asymptotic behavior is non-stationary, and we demonstrate the validity of the "contraction towards a ray" conjecture in Remark 3.

First let us consider the case where the forward and backward reactions are given as elements of the ordered sets $\gamma^+ \in \{111\}$ and $\gamma^- \in \{234\}$. Hence, $\boldsymbol{q}^\top = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, so that $\prod_i q_i = \frac{1}{12} < 1$. According to Theorem 3 (iii), trajectories asymptotically should tend towards the origin.

In Fig.2.8 we consider a complete 3-graph with unit weighting, i.e., an all-to-all connected graph with three nodes having edge weights equal one.

We can confirm that solutions tend towards the origin, and every trajectory contracts towards a single constant line, given by the green ray in Fig. 2.8. This supports the conjecture in Remark 3.

Next, we show the effect of the constant graph weighting on the line along which trajectories converge towards the origin. In Fig. 2.9 the reaction rate constants are equivalent to those used in Fig. 2.8. The all-to-all connected 3-graph has weighting $w_{12} = w_{21} = 3$, $w_{23} = w_{32} = 4$, and $w_{13} = w_{31} = 2$. Again we sample ten different initial conditions, and confirm the result in Theorem 3 (iii), as solutions tend towards zero. Comparing the envelope of 10 solutions in Fig. 2.8 with those in Fig. 2.9 we can observe that in both cases it contracts. However, the envelope over time in the latter figure forms a strongly bent cone that shrinks towards the origin. Furthermore, the angle of the constant line along which trajectories tend towards zero differs from that the case in Fig. 2.8. Despite the minor change

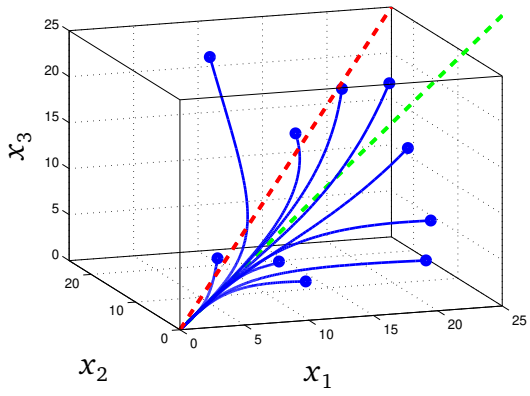



Figure 2.8: Ten trajectories generated by a reaction network protocol (2.24) on a complete 3-graph with unit weighting, $\gamma_i^+ \in \{1,1,1\}$, $\gamma_i^- \in \{2,3,4\}$. The blue dots mark the initial conditions. The red dashed line is the ray $[\mathbf{1}]$ and the green dashed ray marks the constant line along which all solutions converge towards the origin.

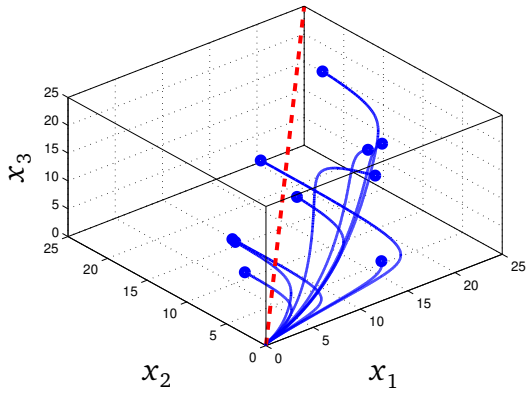



Figure 2.9: Ten trajectories generated by a reaction network protocol (2.24) on a complete 3-graph with non-unit weighting, $\gamma_i^+ \in \{1,1,1\}$, $\gamma_i^- \in \{2,3,4\}$. The blue dots mark the initial conditions. The red dashed line is the ray $[\mathbf{1}]$.

in constant graph weighting, this effect is firm, which results from the constant edge weights acting on the state in the exponent as the power.

Now, let us again consider the case of unit weighting on a complete 3-graph, but change the forward and backward reaction constants. For the 10 solutions plotted in Fig. 2.10, we used the constants $\gamma_i^+ = 1$, $\gamma_i^- = 2$, for all $i \in N$. Hence, $\prod_i q_i = \frac{1}{2^3} < 1$. According to Theorem 3 (iii) solutions should tend towards zero, which is confirmed in Fig. 2.10.

Interestingly, in this scenario, where for each component equation the reaction constant ratio is equal, so that $\boldsymbol{q} \in [\mathbf{1}]$, the ray along which trajectories move towards the origin is also the consensus ray.

Let us turn the case covered in Theorem 3 (iv), where $\prod_i q_i > 0$. In that scenario, trajectories should move towards infinity. The behavior of ten solutions generated by the reaction rate protocol on complete 3-graphs with unit weights confirms this observation in Fig. 2.11 and Fig. 2.12.

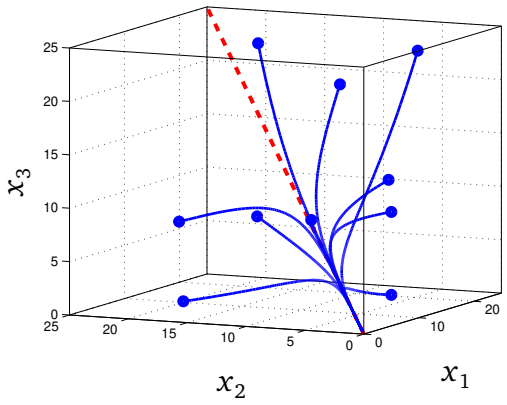



Figure 2.10: Ten trajectories generated by a reaction network protocol (2.24) on a complete 3-graph with unit weighting, $\gamma_i^+ \in \{1,1,1\}$, $\gamma_i^- \in \{2,2,2\}$. The blue dots mark the initial conditions. The red dashed line is the ray $[\mathbf{1}]$.

While in the case of Fig. 2.11, the reaction constants are $\gamma_i^+ = 1$, and $\gamma_i^- \in \{\frac{2}{10}, \frac{3}{10}, \frac{4}{10}\}$, such that $\prod_i q_i = \frac{1000}{24} > 1$, in Fig. 2.12 $\gamma_i^+ = 1$ but $\gamma_i^- = \frac{1}{2}$ for all $i \in N$, and $\prod_i q_i = 8 > 0$. In the latter case, again the vector $\boldsymbol{q} \in [\mathbf{1}]$ and we see that solutions grow the constant line represented by the consensus ray. In the former example $\boldsymbol{q} \notin [\mathbf{1}]$ and we observe a motion towards a ray different from the consensus set.

In all cases of non-stationary behaviors, we can observe a constant line that attracts solutions generated by the reaction rate protocol on balanced graphs. We also observe that in both cases of Theorem 3 (iii) and (iv), solutions are attracted by the consensus set if the vector $\boldsymbol{q}$ is an element of the consensus set (but not equal to the unit vector).

## 2.6 Gradient and optimization viewpoint

This section demonstrates that all three geometric mean-driven consensus networks can be embraced in a common setting of a projected gradient flow of free energy. On that basis, we provide a novel characterization of the geometric mean in terms of a constrained optimization problem.

### 2.6.1 Free energy gradient flow

Free energy stored in a state $\boldsymbol{x}$ w.r.t. another positive vector $\boldsymbol{y}$ can be defined as the sum-separable function [vRJ13a]

$$F(\boldsymbol{x}||\boldsymbol{y}) := \sum_{i=1}^{n} x_i \left( \ln \frac{x_i}{y_i} - 1 \right) + const.$$

For elements that are member of the set of vectors having total mass $m \in \mathbb{R}_{>0}$,

$$\mathscr{D}_m := \left\{ \boldsymbol{x} \in \mathbb{R}^n_{>0} : \sum_{i=1}^{n} x_i = m \right\},$$

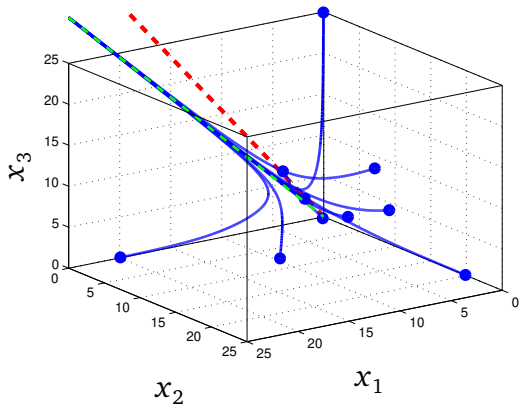



Figure 2.11: Ten trajectories generated by a reaction network protocol (2.24) on a complete 3-graph with unit weighting, $\gamma_i^+ \in \{1,1,1\}$, $\gamma_i^- \in \{0.2,0.3,0.4\}$. The blue dots mark the initial conditions. The red dashed line is the ray $[\mathbf{1}]$ and the green dashed ray marks the constant line along which all solutions converge towards the origin.

free energy is, up to an additive constant, a relative entropy; it coincides with the usual relative entropy known in information theory for vectors that are elements of the set of probability distribution vectors, $\mathscr{D}_{m=1}$, by setting $const = 1$, so that $F(\boldsymbol{x}||\boldsymbol{y}) = \sum_{i=1}^{n} x_i \ln \frac{x_i}{y_i}$.

*Remark* 5. Within the literature on network systems, relative entropy appears in the context of distributed estimation and detection algorithms, where the states represent discrete probabilities, see, e.g., [BC14; QJX15; GZ86; NOU15; SJ13; RJ15]. Free energy is used in the study on mass-action chemical reaction networks [vRJ13a].

In what follows, we show that the polynomial, entropic, and scaling-invariant consensus dynamics are all instances of a particular type of free energy gradient flow.

To start with, an ODE governing a Riemannian gradient (descent) flow in $\mathbb{R}^n$ has the generic form $\mathbf{G}(\boldsymbol{x})\dot{\boldsymbol{x}} = -\nabla E(\boldsymbol{x})$ where $E : \mathbb{R}^n \to \mathbb{R}$ is the potential and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a positive definite matrix function smoothly varying in $\boldsymbol{x}$. It defines the infinitesimal metric $\mathrm{d}\boldsymbol{x} \cdot \mathbf{G}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ in which a system is a gradient descent flow of $E$, so that $\mathbf{G}^{-1}$ defines the inverse metric, cf., e.g., [SPB14].

Let $\mathbf{L}$ be the symmetric Laplacian of an undirected connected graph. Using the eigendecomposition $\mathbf{L} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top$, where $\boldsymbol{\Lambda} := \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the diagonal matrix collecting eigenvalues of $\mathbf{L}$, and $\mathbf{V}$ collects orthogonal eigenvectors each having 2-norm one, we have

$$\mathbf{L}\boldsymbol{x} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top\boldsymbol{x} = \sum_{i=1}^{n} \boldsymbol{v}_i \lambda_i \boldsymbol{v}_i \cdot \boldsymbol{x}, \tag{2.36}$$

which is a projection of a vector $\boldsymbol{x}$ onto the set of distributions $\mathscr{D}_m, m = |\boldsymbol{x}|_1$. To see this, recall that a projection onto this set has the form

$$\mathrm{Proj}_{\mathscr{D}}\boldsymbol{x} = \sum_{i=1}^{n-1} \frac{\boldsymbol{x} \cdot \tilde{\boldsymbol{v}}_i}{\tilde{\boldsymbol{v}}_i \cdot \tilde{\boldsymbol{v}}_i} \tilde{\boldsymbol{v}}_i,$$

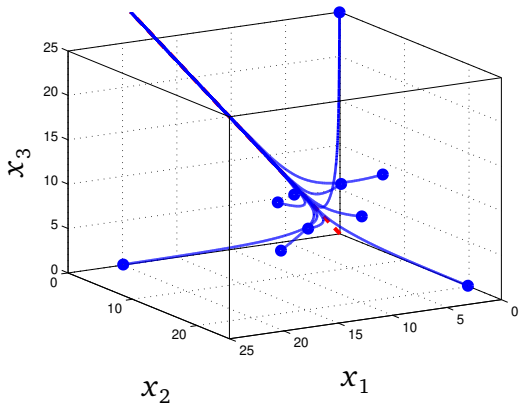



Figure 2.12: Ten trajectories generated by a reaction network protocol (2.24) on a complete 3-graph with unit weighting, $\gamma_i^+ \in \{1,1,1\}$, $\gamma_i^- \in \{0.5,0.5,0.5\}$. The blue dots mark the initial conditions.

where $\{\tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_2, \ldots, \tilde{\boldsymbol{v}}_{n-1}\}$ are linearly independent vectors that span the hyperplane $\mathscr{D}_m$. This setting is given in (2.36), as $\lambda_1 = 0$, while $\lambda_i > 0, i = 2,3,\ldots,n$, and $\boldsymbol{v}_1$ is orthogonal to any set $\mathscr{D}_m$.

Given a sum-separable convex function $\phi : \mathbb{R}^n_{>0} \to \mathbb{R}$ we introduce for the gradient $\nabla\phi$ projected onto $\mathscr{D}_m, m = |\nabla\phi|_1$, the notation $\nabla_{\mathscr{D}}\phi = \mathbf{L}\nabla\phi$. Observe that the gradient $\nabla F(\boldsymbol{x}||\mathbf{1})$ is given by the vector $\ln \boldsymbol{x}$. Using the projected gradient notation, we can write the protocols (2.15)-(2.17) in same order in vector matrix form as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= -\mathbf{R}(\boldsymbol{x})\mathbf{L}\ln \boldsymbol{x} = -\mathbf{R}(\boldsymbol{x})\nabla_{\mathscr{D}}F(\boldsymbol{x}||\mathbf{1}) \\
\dot{\boldsymbol{x}} &= -\mathbf{X}(\boldsymbol{x})\mathbf{L}\ln \boldsymbol{x} = -\mathbf{X}(\boldsymbol{x})\nabla_{\mathscr{D}}F(\boldsymbol{x}||\mathbf{1}) \\
\dot{\boldsymbol{x}} &= -\mathbf{L}\ln \boldsymbol{x} = -\nabla_{\mathscr{D}}F(\boldsymbol{x}||\mathbf{1}),
\end{aligned}$$

with $\mathbf{L}$ the constant coefficient Laplacian, and $\mathbf{R}(\boldsymbol{x}), \mathbf{X}(\boldsymbol{x})$ as in the proof of Theorem 1. As $\mathbf{R}(\boldsymbol{x})$ and $\mathbf{X}(\boldsymbol{x})$ are positive definite symmetric matrix functions for $\boldsymbol{x} \in \mathbb{R}^n_{>0}$, they define Riemannian metrics via their inverses.

The scaling-invariant protocol on an undirected graph generates a projected gradient flow in the usual Euclidean metric setting. Therefore, according to the preceding discussion, on a completely normalized graph trajectories must evolve along the steepest descent directions of free energy on the appropriate simplex of constant mass distributions.

In Fig. 2.13 this free energy gradient property is illustrated for the scaling-invariant protocol running on such a graph over three nodes. The gray outlined triangle marks the set of mass-3 distribution vectors. Color-coded are iso-level curves of $F(\boldsymbol{x}||\mathbf{1}) = \sum_i x_i(\ln x_i - 1) + 3$. This illustration also highlights the appropriateness of the $n-1$-dimensional set of mass distribution vectors within positive $n$-space in considering the free energy functional: Free energy is convex and permutation invariant on this set with minimum obtained at the consensus state. Three trajectories are plotted in black with initial conditions marked by a cross. We see that solution curves indeed follow the steepest gradient descent directions of free energy on $\mathscr{D}_{m=3}$ being directed towards the minimum of this function, which is obtained at the consensus point.

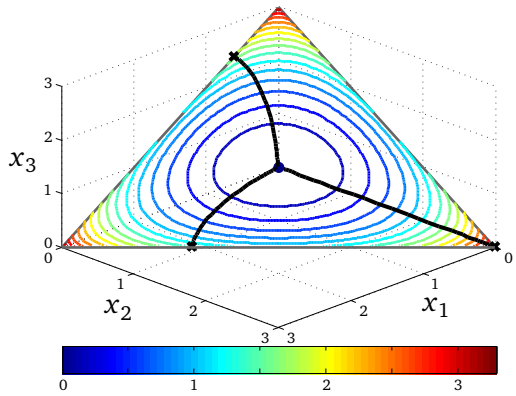



Figure 2.13: Three trajectories generated by scaling-invariant protocol converging to consensus in a free energy potential on the simplex $\mathscr{D}_{m=3}$

### 2.6.2 Constrained nonlinear optimization view

Motivated by the preceding results for the entropic protocol, we provide a novel optimization characterization of the geometric mean linking dynamic problems in consensus theory with static problems in nonlinear constrained optimization.

**Theorem 4** (Novel characterization of the geometric mean)**.** *The geometric mean of a vector* $\boldsymbol{x} \in \mathbb{R}^n_{>0}$ *is characterized as the value* $\mathsf{am}(\boldsymbol{x}^*)$*, where*

$$\boldsymbol{x}^{optimal} = \arg\min_{\boldsymbol{y}\in\mathbb{R}^n_{>0}} F(\boldsymbol{y}||\mathbf{1}), \quad \text{subject to } \prod_{i=1}^n y_i = \prod_{i=1}^n x_i. \tag{2.37}$$

*That is,* $\boldsymbol{x}^*$ *minimizes free energy on the manifold of states having a constant product of component values. In particular, this vector has the form of a consensus state with agreement value precisely the geometric mean of* $\boldsymbol{x}$*.*

*Proof.* Define the Lagrangian

$$\mathscr{L}(\mathbf{y},\lambda) = F(\mathbf{y}||\mathbf{1}) - \lambda\left(\prod_i y_i - \prod_i x_i\right).$$

The solution of the constrained free energy minimization problem satisfies the first order optimality conditions

$$\begin{aligned}\nabla_\lambda \mathscr{L} &= \prod_{j=1}^n x_j - \prod_{j=1}^n y_j = 0 \Leftrightarrow \prod_{j\neq i} y_j = \frac{\prod_{k=1}^n x_k}{y_i}, \\ \nabla_{y_i} \mathscr{L} &= \ln y_i - \lambda \prod_{j\neq i} y_j = 0, \quad i \in N\end{aligned} \tag{2.38}$$

which consequently leads to the solution characteristic

$$y_i \ln y_i = \lambda \prod_{k=1}^n x_k = constant, \quad \forall i \in N. \tag{2.39}$$

The right-hand side in (2.39) is positive (the multiplier $\lambda$ is positive and the values $x_i > 0$ by assumption), and the function $y \ln y$ is increasing on the domain where it takes positive values. Therefore, (2.39) has a unique solution and this solution is the same for all $i \in N$, i.e., a consensus state.

Next, we show that the agreement value of the consensus state is the geometric mean of the input data $\boldsymbol{x}$. Writing $\boldsymbol{y} = y\mathbf{1}$ and substituting into (2.38) yields

$$\prod_{k=1}^{n} y_k = y^n = \prod_{k=1}^{n} x_k \Leftrightarrow y = \mathsf{gm}(\boldsymbol{x}).$$

That is, if $\boldsymbol{x} \in \mathbb{R}^n_{>0}$, then the solution of (2.37) is $\boldsymbol{x}^* = \mathsf{gm}(\boldsymbol{x})\mathbf{1}$, so that $\mathsf{am}(\boldsymbol{x}^*) = \mathsf{gm}(\boldsymbol{x})$. □

Sum-separable energy functions play an axiomatic role in interconnected dissipative systems [Wil72] where they represent energy stored in local subsystems. In contrast, energy functions of the interaction type usually represent power dissipated "across", e.g., resistor elements, see for instance [MDM16] for a further discussion. Hence, the free energy minimization property seems to be the natural gradient setting for the time-continuous entropic consensus network when seen as an analog circuit device solving a minimization problem.

## 2.7 Chemical reaction networks

We apply the proposed methods and results from the study of geometric mean-driven network protocols to the application field of chemical reaction networks under mass-action kinetics and propose a novel conductance element. This element is an alternative to the existing chemical conductance formulation and motivated by the property of specifying dissipation of Gibbs free energy across resistor elements in a mechanic that has functional equivalence to heat exchange processes in classical thermodynamics.

### 2.7.1 Governing equation system and gradient form

Following [Yon12], a chemical reaction network of $n_r$ reactions and $n_s$ chemical substances, is described by a set of reaction equations

$$s'_{i1}S_1 + s'_{i2}S_2 + \ldots + s'_{in_s}S_{n_s} \underset{\gamma_i^-}{\overset{\gamma_i^+}{\rightleftharpoons}} s''_{i1}S_1 + s''_{i2}S_2 + \ldots + s''_{in_s}S_{n_s}, \tag{2.40}$$

where $i = 1, 2, \ldots, n_r$, $S_k$ is the chemical symbol of the $k$th species, $s'_{ik}/s''_{ik}$ are the non-negative stoichiometric coefficients of the $k$th species in reaction $i$, and $\gamma_i^+/\gamma_i^-$ is the positive forward/backward rate in reaction $i$.

The set of ODEs governing the concentration dynamics of a reaction system (2.40) according to the law of mass action kinetics is as follows: Let $x_k$ denote the (positive) concentration of species $k$, then,

$$\dot{x}_k = \sum_{i=1}^{n_r} \left(s''_{ik} - s'_{ik}\right) \left( \gamma_i^+ \prod_{j=1}^{n_s} x_j^{s'_{ij}} - \gamma_i^- \prod_{j=1}^{n_s} x_j^{s''_{ij}} \right). \tag{2.41}$$

*Remark* 6. Observe that for a single reaction in (2.41), the state-dependent part of the vector field governing the dynamics has the same form as the reaction network protocol (2.11) derived from the geometric mean driven consensus consideration.

Let us define the following thermodynamic quantities for the chemical network: Gibbs free energy is given by

$$G(\boldsymbol{x}) := RT \sum_{k=1}^{n_s} \left( x_k \ln \frac{x_k}{\bar{x}_k} + (\bar{x}_k - x_k) \right),$$

where $\bar{x}_k$ is a thermodynamic equilibrium concentration, $R$ is the universal gas constant and $T$ the average system temperature. Let $\rho_i = \frac{x_i}{\bar{x}_i}$, so that $\nabla G(\boldsymbol{x}) = RT \ln \boldsymbol{\rho} = \boldsymbol{\mu}(\boldsymbol{\rho})$, where $\mu_k$ is the $k$-th chemical potential. For the sake of simplicity, in the following, we set the constants $R = 1$ and $T = 1$.

It is well-known that Gibbs free energy serves as Lyapunov function for chemical reaction dynamics achieving its minimum at the configuration $\boldsymbol{x} = \bar{\boldsymbol{x}}$ [vRJ13a]. With this terminology the authors of [Yon12] show that there exists a gradient flow representation for the networked ODEs (2.41) such that

$$\dot{\boldsymbol{x}} = -\mathbf{K}(\boldsymbol{\rho}) \nabla G(\boldsymbol{x}), \tag{2.42}$$

where $\mathbf{K}(\cdot)$ is a positive semi-definite, symmetric, $n_s \times n_s$ matrix function composed as sum, such that

$$\mathbf{K}(\boldsymbol{\rho}) = -\sum_{i=1}^{n_r} \mathbf{K}_i(\boldsymbol{\rho}). \tag{2.43}$$

Next, we describe the construction of matrices $\mathbf{K}_i$, for which we shall propose an equivalent, however, explicit description using the expansion method used throughout the proofs in this chapter.

As in the polynomial network protocol, we assign

$$R_i^+ = \gamma_i^+ \prod_{j=1}^{n_s} x_j^{s'_{ij}} \quad \text{and} \quad R_i^- = \gamma_i^- \prod_{j=1}^{n_s} x_j^{s''_{ij}}.$$

Defining stoichiometric differences $\Delta s_{ik} := s''_{ik} - s'_{ij}$, we can compactly write the component ODE as $\dot{x}_k = \sum_{i=1}^{n_r} \Delta s_{ik} (R_i^+ - R_i^-)$. Using the equilibrium condition for each $i \in \{1, 2, \ldots, n_r\}$,

$$0 = R_i^+(\bar{\boldsymbol{x}}) - R_i^-(\bar{\boldsymbol{x}}) \quad \Leftrightarrow \quad \frac{\gamma_i^+}{\gamma_i^-} = \frac{\prod_{j=1}^{n_s} \bar{x}_j^{s''_{ij}}}{\prod_{j=1}^{n_s} \bar{x}_j^{s'_{ij}}},$$

we get by taking the logarithm on both sides

$$\ln \frac{\gamma_i^+}{\gamma_i^-} = \ln \gamma_i^+ - \ln \gamma_i^- = \sum_{j=1}^{n_s} s''_{ij} \ln \bar{x}_j - \sum_{j=1}^{n_s} s'_{ij} \ln \bar{x}_j = \sum_{j=1}^{n_s} \Delta s_{ij} \ln \bar{x}_j, \tag{2.44}$$

where $s''_{ij} - s'_{ij} = \Delta s_{ij}$.

With that, following [Yon12] or [vRJ13a], (2.41) can be written such that

$$\dot{x}_k = \sum_{i=1}^{n_r} \Delta s_{ik} \left( \exp\left[ \ln \gamma_i^+ + \sum_{j=1}^{n_s} s'_{ij} \ln x_j \right] - \exp\left[ \ln \gamma_i^- + \sum_{j=1}^{n_s} s''_{ij} \ln x_j \right] \right) \tag{2.45}$$

$$= \sum_{i=1}^{n_r} \Delta s_{ik} e^{\sigma_i(\boldsymbol{x})} \left( \ln \gamma_i^+ + \sum_{j=1}^{n_s} s'_{ij} \ln x_j - \ln \gamma_i^- - \sum_{j=1}^{n_s} s''_{ij} \ln x_j \right) \tag{2.46}$$

$$\overset{(2.44)}{=} -\sum_{i=1}^{n_r} \sum_{i=1}^{n_s} e^{\sigma_i(\boldsymbol{x})} \Delta s_{ik} \Delta s_{ij} \ln \rho_j. \tag{2.47}$$

In going from (2.45) to (2.46) it is assumed that $\sigma_i$ is appropriately chosen. Using the definitions

$$\alpha_i^+ := \ln \gamma_i^+ + \sum_{j=1}^{n_s} s'_{ij} \ln x_j = \sum_{j=1}^{n_s} s'_{ij} \mu_j, \quad \text{and} \quad \alpha_i^- := \ln \gamma_i^- + \sum_{j=1}^{n_s} s''_{ij} \ln x_j = \sum_{j=1}^{n_s} s''_{ij} \mu_j, \tag{2.48}$$

by the mean value theorem such a value exists and $\sigma_i \in [\alpha_i^+, \alpha_i^-]$, cf. [vRJ13a].

*Remark* 7. Observe that $\alpha_i^\pm = \ln R_i^\pm$, as used in the expansion approach in the reaction network protocol's convergence proofs.

Eventually, regarding (2.43), the matrices $\mathbf{K}_i$ are defined components-wise as

$$[\mathbf{K}_i(\boldsymbol{\rho})]_{kj} = e^{\sigma_i(\boldsymbol{\rho})} \Delta s_{ik} \Delta s_{ij},$$

see [Yon12].

The quantities $\alpha_i$ in (2.48) are so-called complex thermodynamic affinities. They play an important role in the thermodynamics of reaction networks and are related to passive resistance interpretations, see [vRJ13a], and [EG07]. Next, we apply the proposed consensus results and propose a novel resistance interpretation that also finds application in the definition of the dissipation metric tensor (2.43).

### 2.7.2 Gradient metric defined by heat exchange conductance

Going from (2.45) to (2.46) the logarithmic mean of reaction rates provides an alternative and explicit formulation for implicitly defined values $\sigma_i$.

Using the expansion method with the logarithm of forward and backward rates in each component ODE $\dot{x}_k = \sum_{i=1}^{n_r} \Delta s_{ik}(R_i^+ - R_i^-)$ leads to the equivalent explicit form of (2.46) and the result (2.47) given by

$$\dot{x}_k = \sum_{i=1}^{n_r} \Delta s_{ik} \mathrm{lgm}(R_i^+, R_i^-) \left( \ln R_i^+ - \ln R_i^- \right) \overset{Rem.7}{=} \sum_{i=1}^{n_r} \Delta s_{ik} \mathrm{lgm}(R_i^+, R_i^-) \left( \alpha_i^+ - \alpha_i^- \right)$$

$$= -\sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \mathrm{lgm}(R_i^+, R_i^-) \Delta s_{ik} \Delta s_{ij} \ln \rho_j. \tag{2.49}$$

That is, by comparing (2.49) with (2.47) we get an explicit formulation for the exponential of the "mean" complex thermodynamic affinity $\sigma_i$. Using $\mathrm{lgm}(R_i^+, R_i^-) =: R'_i$, we have

$$\mathrm{lgm}(R_i^+(\boldsymbol{x}), R_i^-(\boldsymbol{x})) = e^{\sigma_i(\boldsymbol{x})} \quad \Leftrightarrow \quad \sigma_i(\boldsymbol{x}) = \ln R'_i(\boldsymbol{x}).$$

The relationship of the exponential of the quantities $\sigma_i$, respectively of the logarithmic mean of forward and backward reaction rates, to convergence properties becomes apparent from the gradient flow formulation of Gibbs free energy. Substituting the energy gradients $\ln \rho_j = [\nabla G(\boldsymbol{x})]_j$ into (2.47) yields

$$\dot{x}_k = -\sum_{i=1}^{n_r}\sum_{j=1}^{n_s} \mathrm{lgm}(R_i^+, R_i^-)\Delta s_{ik}\Delta s_{ij}[\nabla G(\boldsymbol{x})]_j.$$

While Gibbs free energy is a Lyapunov function for the chemical network dynamics, it is not a potential for the dynamics in Euclidean space. However, the unique non-Euclidean metric in which the dynamics do evolve as gradient descent flow of $G$ is defined by the nonlinear component functions $\mathrm{lgm}(R_i^+, R_i^-)$. Hence, this unique relationship renders the logarithmic mean expression an important quantity within the context of studying chemical network dynamics using Gibbs free energy, and with that, more broadly within the study of network thermodynamics.

Let us turn to the resistance interpretation of the complex thermodynamic affinities, resp. the logarithmic mean of reaction rates. The constitutive relation for resistance, or its inverse a conductance $\kappa$, is $I = \kappa\Delta U$, where $\Delta U$ is a driving force given by a potential difference across the resistor, and $I$ is the resulting current through the resistor.
Instead of considering chemical potential differences as driving force, the authors of [EG07] take the chemical potential differences in exponential coordinates as driving force resulting in

$$\Delta U(\boldsymbol{\rho}) := \exp\left(\sum_{j=1}^{n_s} s'_{ij}\mu_j\right) - \exp\left(\sum_{j=1}^{n_s} s''_{ij}\mu_j\right) = e^{a_i^+(\boldsymbol{\rho})} - e^{a_i^-(\boldsymbol{\rho})}.$$

With that, the authors define the $i$-th reaction conductance $\kappa_i^E$, that we call Ederer conductance (according to the author's name), such that

$$I_i := R_i^+(\boldsymbol{x}) - R_i^-(\boldsymbol{x}) = \kappa_i^E \Delta U(\boldsymbol{\rho}), \quad \Leftrightarrow \quad \kappa_i^E(\tilde{\boldsymbol{x}}) := \gamma_i^+ e^{\sum_{j=1}^{n_s} s'_{ij}\ln \tilde{x}_j} = \gamma_i^- e^{\sum_{j=1}^{n_s} s''_{ij}\ln \tilde{x}_j}. \tag{2.50}$$

The Ederer conductance is a constant and essentially represents a re-formulation of the equilibrium condition (2.44). As the driving force $\Delta U$ is not the potential difference, the conductance $\kappa_i^E$ is not a classical conductance, e.g., in the Brayton-Moser gradient system sense, cf. comments in [vRJ13a] in this regard.

The logarithmic mean formulation we proposed resolves this problem by providing an alternative conductance formulation, as (2.49) shows.
To make this more explicit, observe that the complex thermodynamic affinities (2.48), as a weighted sum of chemical potentials, do indeed describe a chemical potential. Denote the complex potential difference by $\Delta U_i^\alpha := \alpha_i^+ - \alpha_i^-$. Define the alternative conductance $\kappa_i^\alpha := \mathrm{lgm}(R_i^+, R_i^-)$, so that with (2.49) we can re-write the constitutive current relation (2.50) as

$$I_i = \kappa_i^\alpha(\boldsymbol{x})(\alpha_i^+ - \alpha_i^-).$$

Here, the nonlinear conductance $\kappa_i^\alpha(\boldsymbol{x})$ serves indeed as a conductance, mapping a potential difference to a current. Other than the Ederer conductance, the alpha-conductance $\kappa_i^\alpha(\boldsymbol{x})$ is not constant but a dynamic, positive definite, and symmetric function.

Only the combination of Gibbs free energy $G$ with this new type of conductance $\kappa_i^\alpha$ specifying the appropriate dissipation metric tensor defines the gradient flow structure (2.42). The alpha-conductance has an appealing interpretation as the heat exchanger, across which heat dissipates driven by a potential difference at two terminals.

Following [KN74] Chapter 9, a heat exchanger is a device of length $\ell$,where on one terminal side a hot stream enters and it exits on the other terminal side at lower temperature, compare to Figure 2.14. We denote by $z$ the coordinate along the length of the heat exchanger, and

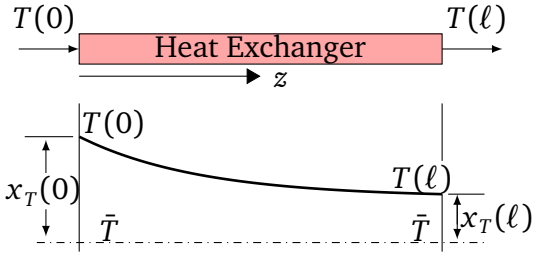



Figure 2.14: Schematic of a heat transfer process across a heat exchanger: Variables $T(\cdot)$ denote temperatures in a heat exchanger of length $\ell$, and $\bar{T}$ is a constantly held wall temperature of the heat exchange surface

we assume that the temperature of the exchange surface inside the heat exchanger is kept at a constant operating temperature $\bar{T}(z) = \bar{T} = const.$ Let us introduce the variable $x_T(z)$ denoting the temperature difference at location $z$ between the stream carrying heat and the cooler operation temperature of the heat exchanger, i.e., $x_T(z) := T(z) - \bar{T}$.

Across the heat exchanger, the stationary rate of heat transfer from the stream is the quantity

$$Q_{\text{heat}} = h \cdot \text{Area} \cdot \frac{\frac{x_T(0)}{\bar{T}} - \frac{x_T(\ell)}{\bar{T}}}{\ln \frac{x_T(0)}{\bar{T}} - \ln \frac{x_T(\ell)}{\bar{T}}} = h \cdot \text{Area} \cdot \text{lgm}\left(\frac{x_T(0)}{\bar{T}}, \frac{x_T(\ell)}{\bar{T}}\right),$$

with $h$ the heat transfer factor in units $[\frac{\text{Watt}}{m^2 \text{Kelvin}}]$, and "Area" denoting the surface area where heat exchange takes place within the heat exchanger.

Regarding (2.43), the components of the metric tensors,

$$[\mathbf{K}_i]_{kj} = \Delta s_{ik} \Delta s_{ij} \text{lgm}(R_i^+, R_i^-).$$

have the interpretation of heat rates that describe the power release driven by the current $I_i$ flowing to balance the difference in in-flux and out-flux $R_i^+$ to $R_i^-$.

## 2.8 Conclusion and outlook

In this chapter we propose, and study novel nonlinear continuous-time consensus protocols driven in three distinct ways by the geometric mean: the polynomial, the entropic, and the scaling-invariant consensus protocols. We introduce the reaction network protocol as a slight generalization of the polynomial consensus protocol, motivated by mass action kinetic chemical reaction network dynamics. The protocols are aligned in a free energy gradient property on the simplex of constant mass distribution vectors.

The entropic consensus dynamics represent a generalization of the well-known average consensus problem as the asymptotically reached agreement value corresponds to the (weighted) geometric mean of the initial state. Based on the free energy gradient property for the entropic dynamics, we provide a novel variational characterization of the geometric mean using a nonlinear constrained optimization problem that is solved with exponential speed by the continuous-time entropic consensus network dynamics.
We numerically demonstrate a relationship between the asymptotic polynomial consensus behavior and the solution of an elliptic integral. We propose a necessary and sufficient fixed-point condition for the polynomial reaction network protocol, which is equivalent to the Wegscheider characteristic of equilibrium states in chemical reactions.
We further apply the expansion techniques used in the consensus convergence proofs to chemical reaction networks and derive a novel "conductance" element for the reaction dynamics. In its original sense as resistor characteristic, this conductance also specifies the metric in which the network dynamics evolve as a gradient flow of Gibbs free energy. This dissipation metric has components that bear the interpretation of heat exchangers that connect pairwise chemical species potentials across which Gibbs free energy is released as power along the convergent dynamics.

The results of the numerical study of the polynomial protocol motivate the examination of the asymptotic characteristics. The resulting network provides a dynamical system approach to the computation of elliptic integrals, which to date still is a topic of scientific discourse.
As the case of deriving a novel chemical conductance model shows, further study of the polynomial reaction network model class may serve fruitful. In particular, the simulation examples for the reaction network protocol dynamics that violate Wegscheider's detailed balance condition show interesting non-equilibrium behavior: they converge towards a line along which dynamics seem to evolve at a constant speed. Potential directions for the analysis of non-equilibrium stationary behaviors can be along with differential dissipation concepts, as proposed, e.g., in [FS14].

3

# Metrics and means in the design of dissipative consensus systems

## 3.1 Introduction

Consensus protocols evolving on graphs have gained popularity only recently, with seminal work of Murray and Olfati-Saber [SM03b; OSM04; OSFM07a], and Moreau [Mor04; Mor05], see also [RBA05] for an early survey contribution on coordination and consensus problems. The study of consensus protocols on graphs in its original version leads to the treatment of finite-dimensional, linear time-varying diffusion processes. For this case, the contraction mechanics that drive the linear averaging behavior are well understood: linear consensus protocols can, in words, be described by the simple local update rule of "move towards the arithmetic mean of your neighbors". Such a simple idea for a generalization to nonlinear consensus protocols is lacking, with few geometry-oriented exceptions. In the large body of the nonlinear consensus networks literature, instead, a zoo of protocol types has so far been proposed, which makes it hard to keep track of novel, innovative work and relate them to another. In this chapter, we generalize the linear consensus protocol idea towards the nonlinear protocol design concept of moving towards a (possibly) nonlinear average of neighboring states and derive a novel stable-by-design framework for nonlinear consensus protocols. We formulate explicit design rules based on a fundamental relationship between averages defined as Kolmogorov means and metric functions used in defining averages in appropriate minimization problems. We propose the metric consensus framework closely related to dissipation properties that drive the exponential convergence towards consensus.

In the literature on the design of general nonlinear consensus protocol classes, authors are typically motivated by finding the maximally admissible nonlinearity in an interaction function that defines the protocol. The resulting network dynamics still converge to consensus. To give an impression of the diversity of admissible interaction function characterizations let us list only a few of them: Admissible nonlinear interaction functions may be characterized as sign-preserving [Wei+17], anti-symmetric [SM03a; BGP06; Cor08], satisfying a thermodynamically motivated inequality [HH08], be of single argument type and have non-trivial gain functions [ADJ12], accept two independent function arguments [HH08], or consider additional nodal input nonlinearity [Wei+17]. While these abstract protocol design results may be general, they often do not provide useful in nonlinear network systems applications. The specific ODE system describing a particular problem at hand is usually taken as starting point. In that sense, these nonlinear consensus protocol classes are not constructive. They do not ease understanding of a problem by helping the deconstruction or reverse-engineering of dissipation mechanisms or synthesis rules or by helping to synthesize concrete algorithms for solving specific network problems. One noteworthy exception is the work on consensus on general functions, see [Cor08], and [BGP06], with relations to mechanism design as distributed optimal control. Helpful in applications is the consideration of general (possibly nonlinear) means as driving functions of network dynamics, as shown in [MXH16] for the case of geometric mean-driven consensus with relations to chemical reaction networks, see also [vRJ13a; vRJ13b; WMv18]. The usefulness of non-arithmetic considerations for network protocol designs is also indicated in [OS+06] in the context of distributed sensing applications, where local believes are exchanged to find consensus on one believe. Here the geometric mean is relevant in the probabilistic setting.
Useful for algorithm design and profound as geometric generalization of linear consensus (on Euclidean, linear space) to consensus on nonlinear space is the framework developed by Sepulchre and co-workers in [SSS07] [SS09b] [Sar09], see [Sep11] for an overview article. Here, the starting point is the idea of a consensus update rule as locally moving towards the average of neighbors. The central idea is that averages on nonlinear spaces can be computed from minimizing a quadratic cost function defined via the intrinsic (Riemannian) metric on the underlying nonlinear space. To turn this concept into a practical algorithm, the authors assume an additional extrinsic geometry. This requires admissible nonlinear spaces of being compact and homogeneous (e.g., Riemannian manifolds) to be embeddable into ambient Euclidean linear space. Consensus on the circle (phase synchronization problems) and matrix problems (via consensus on the orthogonal group and Grassmannian manifolds) are treated. While the framework is general and profound, it requires a priori knowledge of nonlinear configuration spaces and their mathematical properties, which is usually difficult or a research task in itself in nonlinear network systems applications.
The relevance and potential use of passivity ideas in the study of consensus networks is highlighted in [Mor04]; Moreau notes that as early as 1976, Willems in [Wil76] studied consensus models as a particular sub-class of diagonally dominant systems. Willems showed that additive convex functions, known as information divergences, serve as Lyapunov functions. A sum-of-squares Lyapunov function is instrumental in the study of linear consensus systems: it defines the so-called collective disagreement [OSFM07a]. This sum-of-squared Euclidean distances function is the appropriate cost function in determining the arithmetic

mean as a unique minimizer in an optimization problem.

The importance of metrics in generalizing to nonlinear consensus design has become apparent in the geometric framework developed by Sepulchre and co-workers. The idea of using means explicitly in defining nonlinear consensus protocols is relatively new but lies at the heart of the concept of consensus protocols as averaging dynamics on graphs. Means and metrics seem to be associated via an optimization problem. The question arises on the joint context of Lyapunov functions, metrics, and means in defining linear or nonlinear consensus protocols alike, and how such a novel protocol design framework is related to existing nonlinear protocol frameworks. In this chapter, answers to these questions are derived. In particular, the contribution is as follows:

a) We work out the context of metrics and means via Lyapunov functions in linear consensus theory and prove an equivalence in the definition of a general mean via the Kolmogorov mean functional structure and a solution of a sum-of-squared-distances minimization problem.

b) We provide an overview of existing nonlinear consensus protocol classes and compare them by showing equivalences in the diverse definitions and differences.

c) A metric action consensus protocol class is introduced, for which we show that it comprises existing nonlinear consensus protocol classes as special cases. This protocol class has the advantage of not relying on difficult to define admissible nonlinearity in interactions but is purely based on the elementary structure of metric functions. We prove exponential convergence to consensus on time-varying graphs.

d) Necessary and sufficient conditions are provided to characterize functions of simple type, e.g., convex or concave functions, to define metrics. For simple functions as elementary building blocks, we propose protocol composition rules that again result in metric action protocol.

e) We introduce a mean-control consensus protocol from the direct generalization of a linear consensus contraction mechanism. This protocol generates trajectories that are optimal in an infinitesimal context. Usual consensus optimality considers the asymptotic equilibrium state given by a mean as the minimizer of an optimization problem.

f) We introduce a novel so-called embedding protocol as an instance of a metric action protocol. This protocol combines geometric, passivity, and optimization aspects in a general gradient descent formulation. It allows us to prove a novel minimization specification of Kolmogorov means in a constrained optimization problem that uses the gradient formulation potential as a cost function and associated distances in defining the constraint.

In Section 3.2 we provide the necessary background on linear consensus theory with an emphasis on the relation between the arithmetic mean, invariance, and optimality properties, Lyapunov, and potential functions in the design of the linear contraction dynamics. In Section 3.3 introduce (particular) metrics and the Kolmogorov mean, prove their equivalence and

relationship in defining general averages. In Section 3.4 major nonlinear protocol classes are surveyed and put into relation, where we also show that they are all metric action protocols. Convergence and stability results for newly introduced metric and mean-driven protocols are provided in Section 3.5. In Section 3.6 we present and discuss the embedding protocol as an instance of a metric action type before we conclude with a summary and remarks indicating possible research directions.

## 3.2 Linear consensus theory

In the following, we present the basic background of linear consensus theory. First, we introduce the general class of linear consensus protocols defined on graphs and present Moreau's most general stability and convergence result. We then show how the arithmetic mean drives the dynamic behavior for short and asymptotic times. It determines the contraction behavior at infinitesimal time steps. It is a global system invariant and appears as an optimal solution to a minimization problem involving the sum of squared distances function as a measure of collective disagreement. We comment on the context of dissipative systems and passive electric circuits.

### 3.2.1 Graphs, linear consensus protocol and general stability result

Let $\mathsf{G} = (N, B, w)$ be a weighted digraph (directed graph) with set of nodes $N := \{1, 2, \dots, n\}$, set of branches $B := \{1, 2, \dots, b\} \subseteq N \times N$ having elements ordered pairs $(j, i)$ that indicate that there is a branch from node $j$ to $i$, and $w : B \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a weighting function for which we write $w((j, i), t) = w_{ij}(t)$. We shall consider time-varying weighted digraphs $\mathsf{G}(t)$ on a set of nodes $N$, which are characterized by time-dependent sets of branches $B(t)$, where an edge $(j, i) \in B(t)$ if and only if $w_{ij}(t) > 0$.

Define the in-neighborhood of a node $i$, and the out-neighborhood, respectively, as the set of connected nodes

$$N_i^+ := \{j \in N : (j, i) \in B\} \quad \text{and} \quad N_i^- := \{j \in N, (i, j) \in B\}.$$

The (in-)degree of a node $i$ is the value $d_i := \sum_{j \in N_i^+} w_{ij}$. Set $\mathbf{D} := \mathsf{diag}\{d_1, d_2, \dots, d_n\}$. The weighted adjacency matrix $\mathbf{W}$ is such that $[\mathbf{W}]_{ij} = w_{ij}$ for all $(j, i) \in B$; if $(j, i) \notin B$, then $[\mathbf{W}]_{ij} = 0$, and $[\mathbf{W}]_{ii} = 0$, for all $i \in N$. A graph is called balanced if $\sum_{j=1}^n w_{ij} = \sum_{j=1}^n w_{ji}$ and it is symmetric if $w_{ij} = w_{ji}$, $\forall (j, i) \in B$. The Laplacian matrix of a weighted digraph is defined as $\mathbf{L} := \mathbf{D} - \mathbf{W}$, and the normalized Laplacian is $\hat{\mathbf{L}} := \mathbf{I} - \hat{\mathbf{W}}$, where $\hat{\mathbf{W}} = \mathbf{D}^{-1}\mathbf{W}$ is the matrix of normalized branch weights. We say a graph, respectively its Laplacian matrix, is irreducible if the underlying graph is strongly connected, and then, the matrix $\mathbf{L}$ has precisely one zero eigenvalue.

A linear consensus system evolving in continuous time is a dynamic on a family of time-dependent graphs $\{\mathsf{G}(t)\}_{t \geq 0}$ governed by

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij}(t)\big(x_j - x_i\big) \iff \dot{\boldsymbol{x}} = -\mathbf{L}(t)\boldsymbol{x}, \tag{3.1}$$

where each dynamic branch weight $w_{ij}(\cdot)$ is a measurable non-negative function [HT13].

The following proposition summarizes Moreau's basic stability results for general linear, time-varying consensus systems, which are pillars of linear consensus theory.

**Proposition 2.** *[Adopted from [Sep11] Prop. 1 with Def. 2] A linear time-varying system evolving according to* (3.1) *in* $\mathbb{R}^n$ *converges globally and exponentially to a consensus point* $\bar{x}\mathbf{1}$*,* $\bar{x} \in \mathbb{R}$*, if the underlying digraph is uniformly connected, i.e., if for all* $t > 0$*, there exists a time horizon* $T > 0$*, such that the graph* $(N, \tilde{B}(t), \tilde{w}(t))$ *defined by*

$$\tilde{w}_{ij}(t) := \begin{cases} \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau & \text{if } \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau \geq \delta > 0 \\ 0 & \text{if } \int_t^{t+T} w_{ij}(\tau)\mathrm{d}\tau < \delta \end{cases}$$

$w_{ij}(\tau)$ *a branch weight at time* $\tau$*,* $(j,i) \in B$ *if and only if* $\tilde{w}_{ij}(t) \neq 0$*, contains a node from which there is a path to every other node.*

Uniform connectivity certainly holds if at each time instant the graph $\mathsf{G}(t)$ is strongly connected and $w_{ij}(t) \geq \delta > 0$, i.e., if the graph contains a directed path from every node to every other node and the finite branch weights are positively bounded away from zero for all time.

### 3.2.2 Local contraction mechanics

Moreau shows in his work [Mor05] that consensus-seeking systems define mappings that move states forward in time along nested sets described by the convex hull of neighboring states. Under uniform connectivity, the convex hull of the individual states then uniformly shrinks to a point – the global consensus state of the network.

For linear consensus protocols, this contraction property has its foundation in the principal mechanism of

(i) arithmetic mean averaging of states across a neighborhood $N_i^+$, and

(ii) moving a convex combination of this local, linear average and a local state $x_i$ forward time.

More concrete, consider the weighted arithmetic mean of a set of $n$ real-valued numerical values $x_1, x_2, \ldots, x_n$, defined as

$$\mathsf{am}_w(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{n} \omega_i x_i, \tag{3.2}$$

where for $i = 1, 2, \ldots, n$, $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$.

Using (3.2), a continuous-time consensus protocol (3.1) on a normalized weighted digraph can be written as

$$\dot{x}_i = -x_i + \mathsf{am}_w(\{x_j\}_{j \in N_i^+}) \tag{3.3}$$

Discretization of this arithmetic mean driven protocol dynamics in time leads to the algorithmic update rule

$$x_i^+ = (1-\epsilon)x_i + \epsilon \mathsf{am}_w(\{x_j\}_{j \in N_i^+}) \tag{3.4}$$

with $0 < \epsilon < d_i^{-1} \leq 1$, see, e.g., [SSS07].

Convex combinations define a map into the convex hull of the input data, and mean functions, in general, satisfy a boundedness relation, such that

$$\mathsf{min}(x_1, x_2, \ldots, x_n) \leq \mathsf{mean}(x_1, x_2, \ldots, x_n) \leq \mathsf{max}(x_1, x_2, \ldots, x_n). \tag{3.5}$$

Therefore, the computation of an average (not necessarily an arithmetic one) and a convex combination of numerical values lead to an output result that can never be smaller or greater than the numerical value of any input. Intuitively, consensus protocols on uniformly connected graphs, hence, contract towards a single consensus value.

*Remark* 8. The boundedness relationship (3.5) is derived in Proposition 3.

*Remark* 9. Note that (3.4) does not depend on the particular choice of having a normalized weighting, as assumed in the definition of the arithmetic mean; a discretization step $\epsilon$ can always be chosen appropriately such that it normalizes the weighting in the algorithmic consensus update rule.

### 3.2.3 Invariance and global arithmetic mean averaging

Besides its appearance in the local dynamics at a certain instant in time, the (weighted) arithmetic mean also unfolds as asymptotic global system property: For the sake of simplicity, consider the time-invariant class of consensus networks being governed by a Laplacian matrix $\mathbf{L}$ that is irreducible. The asymptotically reached uniform agreement value $\bar{x}$ is given by the (weighted) arithmetic mean of the initial condition [OSM04]. This invariance of the arithmetic mean is a consequence of algebraic system properties:

(i) $\mathsf{ker}(\mathbf{L}) = \mathsf{span}\{\mathbf{1}\} \Leftrightarrow \mathbf{L1} = \mathbf{0}$, i.e., the Laplacian matrix has zero row sums, and an eigenvalue at zero.

(ii) $\mathsf{ker}(\mathbf{L}^\top) = \mathsf{span}\{\boldsymbol{\pi}\} \Leftrightarrow \boldsymbol{\pi}^\top \mathbf{L} = \mathbf{0}$, so that every constant multiple of $\boldsymbol{\pi}$ is a left-eigenvector associated to the trivial eigenvalue.

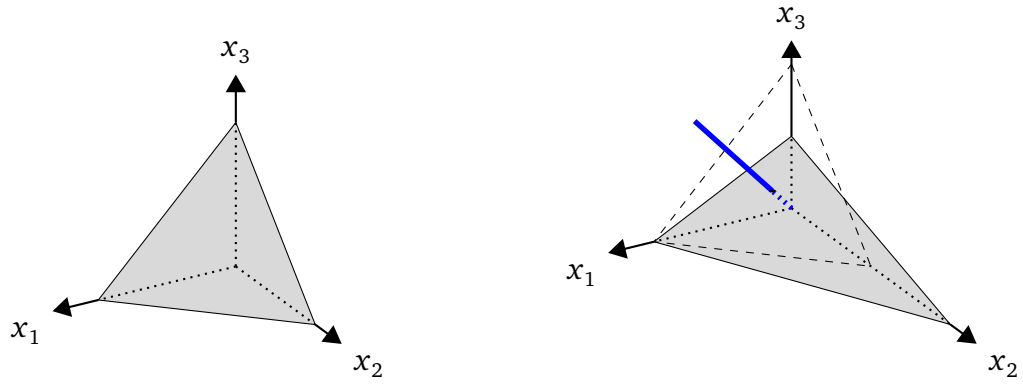
Property (i) defines stationary points of a consensus network being vectors where the individual state components are all equal. The problem in which the equilibrium state to be reached is uniform with consensus value $\bar{x} = \mathsf{am}_w(\boldsymbol{x}_0)$ is commonly known as the average consensus problem.

Property (ii) implies that the quantity $\boldsymbol{\pi}^\top \boldsymbol{x}$ remains invariant along trajectories of a consensus system, as $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\pi}^\top \boldsymbol{x}(t) = \boldsymbol{\pi}^\top \mathbf{L}\boldsymbol{x} = 0$. Hence, if a weighting for an arithmetic mean computation is chosen such that $\boldsymbol{\omega} = \boldsymbol{\pi}$, i.e., it corresponds to the principal left-eigenvector of a consensus network's Laplacian matrix, then the consensus network solves asymptotically with exponential speed the associated average consensus problem.

For a given initial condition $\boldsymbol{x}_0$ the time invariance of the quantity $\boldsymbol{\pi}^\top \boldsymbol{x}(t)$ leads to a $n-1$-dimensional configuration space of weighted integral preserving states

$$\mathscr{M}_I := \left\{ \boldsymbol{x} \in \mathbb{R}^n, \sum_{i \in N} \pi_i x_i = m \right\}, \quad \text{for some} \quad m \in \mathbb{R}. \tag{3.6}$$

The two-dimensional simplex of weighted integral preserving state configurations is illustrated in Fig. 3.1, for a fixed value of $m > 0$ and a three-dimensional irreducible consensus system: The simplex illustrated by the dashed triangle in Fig. 3.1b and Fig. 3.1 represents possible configurations of a balance consensus dynamics, where $\pi \propto \mathbf{1}$, i.e., the vector of all ones is the left kernel of the Laplacian system matrix. The gray shaded area in Fig. 3.1b depicts the configuration space of a consensus network where $\pi \not\propto \mathbf{1}$, i.e., the system matrix $\mathbf{L}$ is irreducible, but not balanced. In blue illustrated is the kernel of $\mathbf{L}$ representing the ray of consensus states.




(a) Configuration simplex for balanced graph with $\pi = (1,1,1)^\top$.

(b) Configuration simplex for non-balanced graph (grey) with $\pi = (1, 0.5, 2)^\top$, and for balanced graph (dashed) as reference.

Figure 3.1: Simplex of (weighted) integral preserving configurations for 3-state irreducible consensus dynamics.The blue ray illustrates the consensus set $\ker(\mathbf{L})$.

*Remark* 10. Note that a set of integral preserving states can still be a time-invariant configuration space for time-varying weighted digraphs. This is the case when the left eigenvector $\pi$ is the same for all Laplacians $\mathbf{L}(t)$, $t \geq 0$, or if for all times the Laplacians are balanced, i.e., the principal left-eigenvector remains a constant multiple of the vector of ones [OSM04].

### 3.2.4 Optimality, Lyapunov and potential functions

A consensus system that solves the average consensus problem is a dynamical system that computes the solution of a static optimization along its evolution in time. The trajectories converge exponentially fast to the weighted arithmetic mean of the respective initial condition; this value of the consensus equilibrium can also be characterized via a sum-of-squares minimization,

$$\mathsf{am}_w(x_1, x_2, \ldots, x_n) = \arg\min_{x \in \mathbb{R}} \sum_{i=1}^{n} \omega_i |x_i - x|^2, \tag{3.7}$$

where $\omega_i = \pi_i, i \in N$, i.e., the normalized weights correspond to the numerical values of the components of the normalized left-eigenvector of the Laplacian associated to the zero eigenvalue.

The sum-of-squares cost function in (3.7) serves as a Lyapunov function for irreducible consensus networks, as well. In fact, for that class any weighted sum-of-squares function measuring an Euclidean squared distance to an arbitrary consensus state $const\,\mathbf{1}$,

$$E_{\mathrm{SoS}}(\boldsymbol{x}) = \frac{1}{2}\sum_{i\in N} \pi_i |x_i - const)|^2 \tag{3.8}$$

is a Lyapunov function [Mor04]. This function is also referred to as collective disagreement [OSFM07a].

Following [Mor04] and [van11], this sum-of-squares Lyapunov function can also be interpreted as the stored energy in a lossless integrator system, which is dissipated across a static passive network - the two open systems (lossless and dissipative) being feedback interconnected.

In particular, considering the case of undirected, connected graphs, where $\mathbf{L}$ is symmetric and irreducible, we have $\nabla E_{\mathrm{SoS}}(\boldsymbol{x}) = \boldsymbol{x} - const\,\mathbf{1}$, and with $\mathbf{L1} = \mathbf{0}$, the consensus dynamics written as feedback interconnection of the two open systems yields the closed system consensus dynamics

$$\dot{\boldsymbol{x}} = -\mathbf{L}\nabla E_{\mathrm{SoS}}(\boldsymbol{x}) = \left\{ \begin{array}{cc} \dot{\boldsymbol{x}} = \boldsymbol{u}_1 & , \quad \boldsymbol{y}_2 = \mathbf{L}\boldsymbol{u}_2 \\ \boldsymbol{y}_1 = \nabla E_{\mathrm{SoS}}(\boldsymbol{x}) & \end{array} \right| \quad \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{y}_1 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{u}_2 \\ \boldsymbol{y}_2 \end{pmatrix}.$$

This passivity view naturally leads to a gradient system formulation, which establishes global exponential stability of the consensus state via the gradient dissipation equality

$$\dot{E}_{\mathrm{SoS}}(\boldsymbol{x}) = -\boldsymbol{x}^\top \mathbf{L}\boldsymbol{x} = -||\nabla E_{\mathrm{SoS}}(\boldsymbol{x})||^2_{\mathbf{L}}. \tag{3.9}$$

Here, the Laplacian matrix has the function of a dissipation metric, which establishes the characteristic gradient dissipation equality (3.9) showing exponential convergence to an agreement state as equilibrium.

For undirected, connected graphs a linear consensus system can also be written as gradient flow of the dissipation potential $\Psi(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^\top \mathbf{L}\boldsymbol{x}$, as

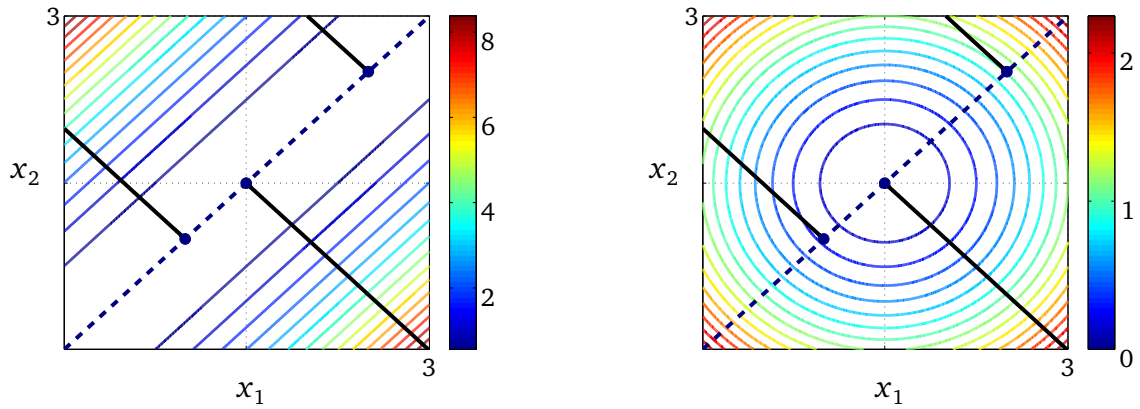
$$\dot{\boldsymbol{x}} = -\mathbf{L}\boldsymbol{x} = -\nabla\Psi(\boldsymbol{x}), \quad \dot{\Psi}(\boldsymbol{x}) = -||\nabla\Psi(\boldsymbol{x})||^2. \tag{3.10}$$

Note that, up to a constant, the dissipation potential $\Psi$ is exactly the dissipation rate of the collective disagreement, i.e., $\dot{E}_{\mathrm{SoS}} = 2\Psi$. The dissipation potential is also referred to as group disagreement [OSM04]. The two potentials in which linear symmetric consensus systems may be formulated as gradient flow are illustrated in Fig. 3.2.

The gradient formulation (3.10) is most popular in the linear consensus literature, see, e.g., [OSFM07b], [Mor04], [SM03a]. However, the less apparent gradient formulation bears a direct link to passivity theory and a direct physical network representation, cf. to the following remark (11).

*Remark* 11. In the language of electric circuits, $E_{\mathrm{SoS}}$ is an energy stored in capacitor elements, and the dissipation rate

$$\dot{E}_{\mathrm{SoS}}(\boldsymbol{x}) = -||\nabla E_{\mathrm{SoS}}(\boldsymbol{x})||^2_{\mathbf{L}} = \frac{1}{2}\sum_{(i,j)\in B} w_{ij}(x_j - x_i)^2,$$




(a) $2\Psi(\boldsymbol{x}) = \boldsymbol{x} \cdot \mathbf{L}\boldsymbol{x}$ (b) $E_{\text{SoS}}(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{x} - 1.5 \cdot \mathbf{1}||$

Figure 3.2: Trajectories of a consensus system (black) and iso-level curves of a collective and a group disagreement. The consensus dynamics is given by $\dot{\boldsymbol{x}} = -\mathbf{L}\boldsymbol{x}$, where $\mathbf{L} = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$ and the equilibrium set is indicated by the dashed dark blue line.

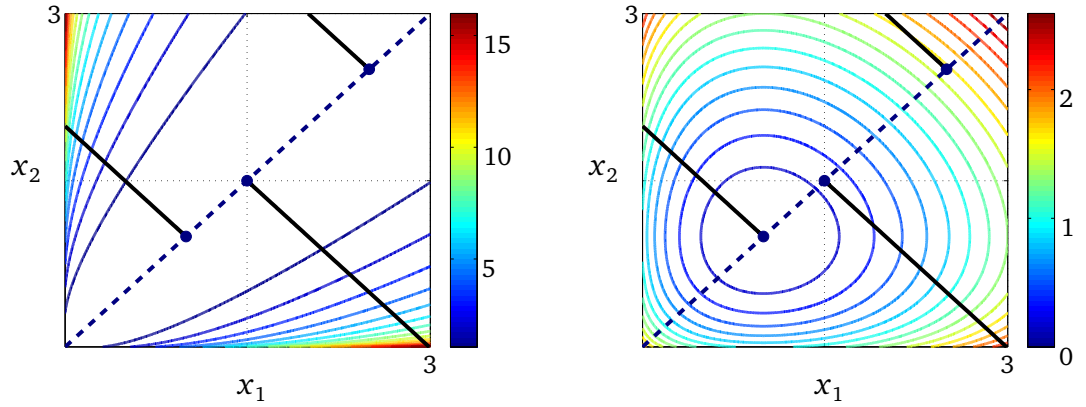
corresponds to the power dissipated across the resistor network, that connects capacitors $i \in N$, with resistor branches $(i,j) \in B$ and corresponding resistances $\frac{1}{w_{ij}}$, cf., e.g., [MDM16], [van11]. This link to a physical network system highlights the importance of considering $E_{\text{SoS}}$ as energy in which the system evolves as network gradient flow, together with the sum-of-squares minimization in defining the consensus value as minimizer of the same energy (up to a constant multiple), see (3.7).

In 1976, when linear consensus theory has not yet been invented, Willems in [Wil76] shows that any function

$$E_f(\boldsymbol{x}) = \sum_{i \in N} \pi_i f(x_i), \quad f \text{ stricly convex}, \tag{3.11}$$

serves as a Lyapunov function for the linear systems class he studies, which has a linear consensus system as special case. The sum-of-squares function (3.8) hence is just a particular case with $f(\cdot) = ||\cdot||^2$ (without loss of generality setting $const = 0$). The following example illustrates free energy as a convex function that serves as Lyapunov function and its dissipation potential, analogous to the case depicted for sum-of-squares energy in Fig. 3.2b

**Example 3** (Free energy Lyapunov dissipation in linear consensus system)**.** Consider the function $E_F = \sum_{i \in N} x_i \log x_i - x_i + 1$, which is convex and defined on the positive real line. It is an instance of the class (3.11), and it coincides with free energy studied, e.g., in network thermodynamics. According to Willems [Wil76] the function $E_F$ is a Lyapunov function for consensus states under the dynamics governed by a consensus protocol. Consider the two-state consensus system, as in Fig. 3.2. In Fig. 3.3 trajectories of this consensus system are plotted in black together with iso-level curves of free energy, see Fig. 3.3b. The dissipation potential $\dot{E} = -\log \boldsymbol{x} \cdot \mathbf{L}\boldsymbol{x} =: 2\Psi_F$ is plotted in Fig. 3.3a. Clearly, $E_F$ is a Lyapunov function for the consensus system. However, the dissipation potential $\Psi_F$ is not a potential in which trajectories evolve as steepest descent gradient curves.




(a) $2\Psi_F = \log(\boldsymbol{x}) \cdot \mathbf{L}\boldsymbol{x}$ (b) $E_F(\boldsymbol{x}) = \sum_j x_j \log x_j - x_j + 1$

Figure 3.3: Trajectories of a consensus system and iso-level curves of collective and group disagreements associated with free energy, as specified in Example 3. The dashed dark blue line indicates the consensus equilibrium set.

This raises the question of what type of (possibly nonlinear) average drives (possibly) nonlinear consensus dynamics for which a function of the class (3.11) is a natural collective disagreement measured, as the function $E_{\mathrm{SoS}}$ is for the arithmetic mean via the minimization formulation (3.7).

## 3.3 The mean-metric relationship

The concept of metrics as distance functions with quasi- and semi-metrics as special cases is introduced, together with metrics in defining general mean functions as the solution of a sum-of-squared distances minimization problem. Means have a second characterization due to Kolmogorov. We introduce Kolmogorov's axioms of means and derive on that basis a boundedness relation for Kolmogorov averages. We further show that the Kolmogorov functional structure corresponds to a mean's metric minimization characterization by choosing the same local nonlinear coordinate transformation in both the definition of the distance and the Kolmogorov characteristic.

### 3.3.1 Distance functions and metric characterization of means

Metric functions are essential elements in the mathematical toolkit. Metrics measure distances between two elements and allows the study of convergence, contraction, and stability of systems. While the definition of metric functions is well-known, the concept of semi-metric and quasi-metric functions is not. Following [Rol87] these (special) metric functions are defined as follows.

**Definition 1** ((Special) Metric functions)**.** Let $X$ be a set. The function $d : X \times X \to \mathbb{R}$ is a metric if and only if for each $x, y, z \in X$

(M1) $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$ (positive definiteness),

(M2) $d(x,y) = d(y,x)$ (symmetry),

(M3) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality or sub-additivity)

hold. The number $d(x,y)$ is called distance between $x$ and $y$ w.r.t. the metric $d$. A semi-distance denoted by $d_s$ is a function for which only (M1) and (M2) hold. A quasi-distance denoted as $d_q$ satisfies only (M1) and (M3), and for a sequence $\{x_n\}$,

(M2') $\lim_{n\to\infty} d_q(x_n, x) = 0$ if and only if $\lim_{n\to\infty} d_q(x, x_n) = 0$.

A function for which only property (M1) holds is a pre-metric $d_p$.

*Remark* 12 (Note on terminology). The terminology of generalized metrics is not unique, and sometimes quasi-metric refers to semi-metric and vice versa. We adopt the definition quasi-metrics, in particular (M2'), from [Rol87] (the author, however, refers to it as semi-metric).

The significance of quasi-metrics originates from the fact that for each quasi-metric, there exists an equivalent metric (e.g., the symmetric part of the quasi-metric), as shown, for instance by Rolewicz in [Rol87] Theorem 1.4.1. Hence, mathematical properties (convergence, continuity, etc.) of quasi-metric and metric spaces are equivalent.
The consideration of quasi-metrics is rather motivated from a modeling and design point of view regarding applications, where symmetry (M2) may be violated.
A semi-metric, where the triangle inequality is weakened or does not hold, is also said to be a non-Archimedean metric, i.e., intuitively, distances do not add up or accumulate.
A pre-metric has its significance because every pre-metric space (i.e., a set with a pre-metric defined on it) is a topological space. Therefore, elementary aspects of dynamical systems such as asymptotic properties or convergence can be studied.

**Example 4** (Transport distances: quasi- and pre-metrics)**.** Suppose $a, b$ are points in a river transport system. The cost of transport $d(a,b)$ of a cargo unit from $a$ to $b$ satisfies (M1) and (M2) in Definition 1 but not symmetry (M2), as upstream transport is more expensive than downstream transport. Hence, $d(a,b)$ is a quasi-metric.
Consider two probability distributions $p, q$ on a discrete probability space $X$. The Kullback-Leibler divergence (or relative entropy) $D_{KL}(p_1||p_2) := -\sum_{x\in X} p(x) \ln \frac{p(x)}{q(x)}$ is often referred to as "information distance" on the space of probability distributions; it also serves as Lyapunov function in the study of stochastic processes, where it can be seen as a distance between probability distribution at two points in time. However, $D_{KL}$ neither satisfies (M2) nor (M3). Hence, it is a pre-metric function.

In property (M3), the validity of the triangle inequality is equivalent to sub-additivity of $d$. It shall later be helpful in consensus protocol design. Hence we define (sub)-additivity here.

**Definition 2** ((Sub-)additive function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called sub-additive if the inequality

$$f(\boldsymbol{x} + \boldsymbol{y}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. If the inequality holds with equality, then the function is called additive.

Consider data points $x_1, x_2, \ldots, x_n$ taking values on the real line $\mathbb{R}$, and let these elements be collected in the vector $\boldsymbol{x}$. An average or mean computed from $\boldsymbol{x}$ can be obtained as the solution of an unconstrained minimization [MMO10] [Moa05],

$$M(\boldsymbol{x}) = \arg\min_{x\in\mathbb{R}} \sum_{i=1}^{n} d(x_i, x)^2, \tag{3.12}$$

Averages also carry the notion of a "centroid", "center of mass", or "(central) tendency" describing in a single numeric value a global data set property.

**Example 5** (Arithmetic and geometric mean)**.** Using the Euclidean distance

$$d_{\mathsf{E}}(a, b) := |a - b|$$

the arithmetic mean results via (3.12) as the sum-of-squares minimizer

$$\mathsf{am}(\boldsymbol{x}) := \frac{1}{n}\sum_{i=1}^{n} x_i = \arg\min_{x\in\mathbb{R}} \sum_{i=1}^{n} |x_i - x|^2.$$

An important nonlinear average is the geometric mean. It often appears in data that originates from rate equations describing growth or decay processes over time, e.g., in economics, finance, or chemistry. The geometric mean is the solution of a minimization problem (3.12), and such that

$$\mathsf{gm}(\boldsymbol{x}) := \sqrt[n]{x_1 x_2 \cdots x_n} = \arg\min_{x\in\mathbb{R}} \sum_{i=1}^{n} |\ln x_i - \ln x|^2.$$

The squared Euclidean distance of logarithmic data points represents the squared hyperbolic distance of pairwise data points. On $\mathbb{R}_{>0}$ the hyperbolic distance is defined as

$$d_{\mathsf{H}}(a, b) := |\ln a - \ln b|.$$

It is a geodesic distance measuring the hyperbolic length of the straight-line segment joining two points in Cartesian coordinates $(x, a), (x, b)$, $x \in \mathbb{R}_{>0}$, see, e.g., [Sta93] Proposition 4.3.

### 3.3.2 Kolmogorov means and sum-of-squared-distances

Besides constructing means from data sets via an optimization problem, a second characterization of mean functions via a functional structure is due to Kolmogorov. In his work [Tik91], Kolmogorov studies the structure of general mean functions, which are understood as all functions that satisfy certain natural conditions – the axioms of the mean.

**Definition 3** (Axioms of a the mean [Tik91])**.** A function $M : \mathbb{R}^n \to \mathbb{R}$ is said to be a mean function if it satisfies the following conditions:

i) $M(x_1, x_2, \cdots, x_n)$ is monotone (increasing) in each variable.

ii) $M(x_1, x_2, \cdots, x_n)$ is a symmetric function.

iii) $M(x, x, \cdots, x) = x$ , i.e., the mean of identical numbers is equal to their common value.

iv) $M(x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2}, x_{n+m}) = M(x_1, x_2, \cdots, x_n, y, y, \cdots, y)$, where $y = M(x_{n+1}, x_{n+2}, x_{n+m})$, i.e., a subset of values can be replaced with their means with no effect on the total mean.

From this definition, it also follows the boundedness property (3.5) that we now can prove.

**Proposition 3** (Boundedness of Kolmogorov means)**.** *Let $M$ be a mean function as in Definition 3. Then,*

$$\min\{x_1, x_2, ..., x_n\} < M\{x_1, x_2, ..., x_n\} < \max\{x_1, x_2, ..., x_n\},$$

*that is, the mean is bounded from below and from above by the minimal and maximal element from the set of real numbers to be averaged.*

*Proof.* To see this, note that the first inequality follows from property i) (monotonic increasing function). The second inequality can be shown by contradiction. Assume $M(x, y) > y$, for $x < y$. Then $M(y, y) > M(x, y)$ by property i). But $M(y, y) = y$ by property iii) so that $M(y, y) = y > M(x, y) > y$ is a false statement. Hence the mean cannot be greater than the maximum element. To extend this for $n$ variables one can use property iv) and replace $n-1$ variables that are not equal to the maximum element by their mean. □

Kolmogorov shows that any mean function satisfying the condition in Definition 3 is necessarily of the form

$$M(\boldsymbol{x}) = g\left(\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right), \tag{3.13}$$

where $f$ is a continuous, strictly monotone function, and $g$ is its inverse, i.e., $f \circ g = \mathrm{id}$.

Thinking of the procedure to compute a mean using a formula (3.13) shows that at the center of any such (nonlinear) average, there is the linear averaging scheme that is applied to calculate the arithmetic mean, so that we can write $M(\boldsymbol{x}) = g\left(\mathsf{am}\{f(x_1), f(x_2), \cdots, f(x_n)\}\right)$.

The following example demonstrates the Kolmogorov and minimization way of characterizing metrics.

**Example 6.** Consider the arithmetic and the geometric mean as defined in the previous Example 5. From the definition of the hyperbolic distance $d_{\mathsf{H}}(a, b) = |\ln(a) - \ln(b)|$ we get $f = \ln$. The inverse of the logarithm is the exponential function exp, as $\exp(\ln(x)) = x$. Via Kolmogorov's characterization we obtain the geometric mean as

$$\mathsf{gm}(\boldsymbol{x}) = \exp\left(\frac{\sum_{i=1}^n \ln x_i}{n}\right) = \exp\left(\frac{1}{n}\ln\prod_{i=1}^n x_i\right) = \exp\left(\ln\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}\right) = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

The geometric mean is also the solution of the minimization problem (3.12) using the hyperbolic metric $d_{\mathsf{H}}$. Inserting the geometric mean into the objective function we can write

$$\sum_{i=1}^n |\ln x_i - \ln \mathsf{gm}(\boldsymbol{x})|^2 = \sum_{i=1}^n |\ln x_i - \frac{1}{n}\sum_{j=1}^n \ln x_j|^2 = \sum_{i=1}^n |\ln x_i - \mathsf{am}(\ln \boldsymbol{x})|^2,$$

which is the least-squares characterization of the arithmetic mean in logarithmic coordinates.

As this example shows, the function $f$ can be seen as a nonlinear point-wise coordinate transformation acting on the original data points such that averaging in transformed coordinates appears as a linear sum-of-squares problem.

*Remark* 13 (Special local transform: $f = \ln$). The increasing functions ln and sin as local coordinate transformations are of particular interest. The logarithm transforms addition or subtraction of two transformed numbers into the transform of the product or quotient, e.g., $\ln(a) - \ln(b) = \ln \frac{a}{b}$. Hence, transformed differences are also scaling-invariant, as for some positive scalar $c$ we get $\ln(c \cdot a) - \ln(c \cdot b) = \ln \frac{a}{b}$.

The two alternatives to characterize a mean – the sum of squared distances optimization problem (3.12) and the structural form (3.13) – are intertwined in a manner we show next. First, we characterize metrics defined by strictly increasing scalar functions $f$.

**Lemma 1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. The function*

$$d_f(a,b) := |f(a) - f(b)|$$

*defines a metric on the real line.*

*Proof.* Observe that $|f(a) - f(b)|$ is positive definite, i.e., for $a \neq b$ it is positive, and it vanishes if and only if $a = b$. Further, $|f(a) - f(b)| = |f(b) - f(a)|$, i.e., it is symmetric. To be a metric it is by definition required to satisfy the triangle inequality. To see that this is true, let $a, b, c$ be real numbers and observe that

$$|f(a) - f(c)| = |f(a) - f(b) + f(b) - f(c)| \leq |f(a) - f(b)| + |f(b) - f(c)|,$$

where inequality follows from properties of a norm $|\cdot|$. □

**Theorem 5** (Mean - metric relationship for Kolmogorov averages)**.** *Consider a mean that has Kolmogorov structure* (3.13) *and let $f$ be differentiable. Then,*

$$g\left(\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right) = \arg\min_x \frac{1}{2} \sum_{i=1}^{n} d_f(x_i, x)^2, \tag{3.14}$$

*with $d_f(a,b) = |f(a) - f(b)|$, i.e., the coordinate transformation $f$ defines the metric appearing in the minimization characterization of a mean via the normed difference $d_f$.*

*Proof.* The function $d_f$ is indeed a metric, cf. Lemma 1.

As minimizer of the sum of squared distances the mean taken as variable satisfies the first order extremum condition,

$$0 = -2 \cdot \frac{1}{2} \sum_{i=1}^{n} |f(x_i) - f(M(\boldsymbol{x}))| f'(M(\boldsymbol{x})).$$

As $f$ is a monotone (increasing) function its derivative $f'$ is other than zero, so that

$$0 = \sum_{i=1}^{n} |f(x_i) - f(M(\boldsymbol{x}))|.$$

If $f(x_i) > f(M(\boldsymbol{x}))$ the norm brackets can be left away, and if $f(x_i) - f(M(\boldsymbol{x}) < 0$ we have $|f(x_i) - f(M(\boldsymbol{x}))| = f(M(\boldsymbol{x}) - f(x_i)$. Assume that $m$ times, $0 \leq m \leq n$, the first inequality is the case, so that (after appropriate re-indexing),

$$0 = \sum_{i=1}^{m} f(x_i) - f(M(\boldsymbol{x})) + \sum_{i=m+1}^{n} f(M(\boldsymbol{x})) - f(x_i).$$

As each difference is positive or zero, we can write the two sets of conditions

$$\sum_{i=1}^{m} f(x_i) - f(M(\boldsymbol{x})) = 0 \Leftrightarrow \sum_{i=1}^{m} f(x_i) = m f(M(\boldsymbol{x}))$$
$$\sum_{i=m+1}^{n} f(x_i) - f(M(\boldsymbol{x})) = 0 \Leftrightarrow \sum_{i=m+1}^{n} f(x_i) = (n-m) f(M(\boldsymbol{x})),$$

and equivalently, we obtain after addition of both sides

$$n f(M(\boldsymbol{x})) = \sum_{i=1}^{n} f(x_i) \Leftrightarrow M(\boldsymbol{x}) = g\left(\frac{\sum_{i=1}^{n} f(x_i)}{n}\right).$$

This completes the proof. □

This result is significant for understanding consensus systems. It generalizes optimality properties of arithmetic mean-driven consensus dynamics from linear to nonlinear consensus protocols, where nonlinearity originates from local coordinate transforms $f$.

*Remark* 14. In Theorem 5 we assume $f$ to be differentiable. This condition can be relaxed, e.g., by assuming Lipschitz continuity and using a construction for the derivative $f'$ as for instance described in [Gol77].

*Remark* 15. Without loss of generality, the factor $\frac{1}{2}$ in (3.14) can be removed or replaced by any other positive real number. It plays, however, an important role in defining appropriate convex conjugates of dissipation potentials of the Kolmogorov form (3.14), which have a constitutive character in defining gradient flows. This is a topic we discuss in Chapter 4.

## 3.4 Nonlinear consensus protocols and metric equivalents

We summarize existing classes of non-linear consensus protocols and exhibit common and different properties of the admissible nonlinearity in the respective protocol design. It turns out that the isolated consideration of admissible nonlinearity and the sign of state differences is key in characterizing a maximally large and simple class of admissible nonlinearity in consensus protocols. On that basis, we introduce the novel class of metric action protocols that subsumes the summarized existing classes. We further provide necessary and sufficient conditions for functions of simple type, such as convex or concave functions, to define a metric or its specialization.

### 3.4.1 Admissible interaction nonlinearity

Extensions from linear to nonlinear consensus protocols often focus on time-invariant networks. They are concerned with finding the largest possible class of admissible interaction nonlinearity under which consensus dynamics still converge to a uniform agreement state.

Murray and Olfati-Saber introduce the basic setup and tool in the design of nonlinear consensus protocols in [SM03a] in the context of so-called action graphs. An action graph is a weighted digraph, where in addition to a linear weighting factor, pairwise interactions across edges are characterized by an action function $\phi$.
Main elementary action function types with their associated protocol dynamics are due to Murray, and Olfati-Saber [SM03a], here referred to as class **MO**, due to Hui and Haddad, further referred to as class **HH**, and due to Wei and van der Schaft, in short, denoted class **WS**. Table 3.1 presents in summary the main properties and protocol dynamics of these nonlinear consensus protocols.

| $\phi$ | Murray/Olfati-Saber:**MO** [SM03a] | Hui/Haddad: **HH** [HH08] | Wei/v.d.Schaft:**WS** [Wei+17] |
|---|---|---|---|
| (i) | loc. Lipschitz cont. | loc. Lipschitz cont. | Leb. meas., loc. ess. bounded |
| (ii) | $\phi(x)=0 \Leftrightarrow x=0$ | $\phi(x,y)=0 \Leftrightarrow x=y$ | $\phi(x)=0 \Leftrightarrow x=0$ |
| (iii) | $(x-y)(\phi(x)-\phi(y))>0, \forall x \neq y$ | $(x-y)\phi(x,y)<0, \forall x \neq y$ | $x\phi(x)>0, x\neq 0$ |
| (iv) | $\phi(x)=-\phi(-x)$ | $\phi(x,y)=-\phi(y,x)$ | - |
| (v) | $\phi_{ij}(x)=-\phi_{ji}(-x)$ | $\phi_{ij}(x,y)=-\phi_{ji}(y,x)$ | - |
| $u_i$ | $\sum_{i\in N_i^+}\phi_{ij}(x_j-x_i)$ | $\sum_{i\in N_i^+}\phi_{ij}(x_i,x_j)$ | $\phi_i\left(\sum_{i\in N_i^+} w_{ij}\phi_{ij}(x_j-x_i)\right)$ |

Tabular 3.1: Elementary action function classes, their properties and associated consensus protocols of the form $\dot{x}_i = u_i$, $i \in N$; it is assumed that $x, y \in \mathbb{R}$, $(j,i) \in B$.

Let us discuss the nonlinear protocols summarized in Table 3.1 and in particular the underlying assumptions:

a) While in protocols MO and HH nonlinearity acts on states across edges only, WS $\phi_i$ additionally introduces an input nonlinearity at nodes $i \in N$.

b) In contrast to local Lipschitz continuity protocols of type WS can have a discontinuous right-hand side. The condition of functions $\phi$ being Lebesgue measurable and locally essentially bounded means that for almost every $x$ taken from any subset of $\mathbb{R}$, $||\phi(x)||_2 < const. < \infty$. Technically, solutions of WS are studied within the framework of Filippov.

c) Condition (iv) defines action functions as being odd or anti-symmetric.

d) Condition (iii) for protocols MO imply that action functions $\phi$ are increasing. To see this, note that for $x \neq y$,$(x-y)(\phi(x)-\phi(y))>0 \Leftrightarrow \frac{\phi(x)-\phi(y)}{x-y}>0$. Taking $x = y+\epsilon$ and $\epsilon \to 0$ shows that the slope of $\phi$ is positive; hence, $\phi$ increases at each point $y \in \mathbb{R}$.

e) Functions that satisfy WS (iii) are called "sign-preserving" as defined in [Wei+17].

f) While action functions of type MO or WS take a single scalar as an argument and map it onto another real scalar, action functions of type HH take two independent scalar variables as input and map it onto a real scalar output.

Murray and Olfati-Saber show in [SM03a] Theorem 3, and Hui with Haddad in [HH08] Theorem 3.2 with [HCN05] Theorem 3.9 that solutions of the respective protocol converge asymptotically to consensus for all initial conditions if the underlying graph is undirected and connected. Wei and van der Schaft show in [Wei+17] Theorem 19 asymptotic stability of the consensus state if the underlying graph contains a directed spanning tree.

The following result shows a hierarchy among the three leading nonlinear consensus protocols considered in the literature.

**Lemma 2** (Action function hierarchy)**.** *Consider locally Lipschitz continuous action functions $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, with arguments of the form given in protocol formulation $u_i$ in Table 3.1. For $x, y \in \mathbb{R}, x \neq y$,*

$$\{\phi : MO \ \ is\ true\} \subset \{\phi : WS \ \ is\ true\}$$
$$\{\phi : WS(iii) \ \ is\ true\} \subset \{\phi : HH(iii) \ \ is\ true\}$$

*That is, any protocol of the type MO is contained in the protocol class HH. Action functions satisfying property HH (i)-(iii) necessarily are also sign-preserving. Sign-preservingness is necessary for an action function to satisfy MO (iii), i.e., increasingness.*

*Proof.* Any action function of type MO can be obtained from the class HH by setting $\phi(a,b) = f(b) - f(a)$ with $f$ such that it satisfies properties MO (i)-(v) and with the particular choice $b = y - x$ and $a = 0$.

Next, we show that MO (iii) $\Rightarrow$ WS(iii), i.e., sign-preservingness is necessary for increasingness of the action function. With $\phi(y = 0) = 0$ we have for all $x \neq 0$

$$(x - y)(\phi(x) - \phi(y)) = x\phi(x) > 0,$$

which is the definition for sign-preservingness. Hence, condition MO (iii) is sufficient for condition WS (iii) to be true. Or, put differently, any action function satisfying MO (iii) must also be sign-preserving.

Next we show that sign-preservingness implies Hui / Haddad's inequality, i.e., WS(iii) $\Rightarrow$ HH (iii): For any $z \neq 0$, set $z = x - y$. Then,

$$z\phi(z) = (x - y)\phi(x - y) > 0 \quad \Leftrightarrow \quad (y - x)\phi(x - y) < 0.$$

For HH type action functions, we may choose $\phi(x, y) = \phi(x - y)$, so that indeed sign-preservingness implies HH (iii) restricted to the particular type of the dependency among the arguments. □

The sign-preserving action functions must not be antisymmetric or satisfy the edge symmetry condition, as for HH-type consensus protocols. These edge requirements on the nonlinearity in Hui / Haddad's approach originate in their proof technique using a quadratic Lyapunov function approach. Under the additional (mild) assumption of locally Lipschitz continuous edge actions (thus disregarding discontinuous action functions), HH's action function is most general and subsumes the other classes WS and MO.

This discussion shows that the sign of state differences and characterization of admissible action nonlinearity should be considered separate issues in designing a general and maximally large class of nonlinear consensus protocols, as we do next.

### 3.4.2 Metric action equivalents

In the following, we show that the presented classes of nonlinear action protocols can be subsumed under the metric protocol type

$$\dot{x}_i = \sum_{j \in N_i} w_{ij} \mathsf{sgn}(x_j - x_i) d(x_j, x_i)$$

where $d$ is a type of a metric function, as specified in Definition 1. Here, isolating the sign of state differences from the nonlinearity is key in characterizing a large and simple class of nonlinear consensus protocols.

Sub-homogeneity turns out as a necessary and sufficient condition in the characterization of certain action functions as metrics.

**Definition 4** ((Sub-)homogeneous function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is called subhomogeneous, if $f(\lambda \boldsymbol{x}) \leq \lambda f(\boldsymbol{x})$ for all $\lambda > 0$. It is said to be homogeneous of degree $p$, if $f(\lambda \boldsymbol{x}) = \lambda^p f(\boldsymbol{x})$, for $\lambda > 0$. We say $f$ is homogeneous if it is homogeneous of degree 1.

For scalar functions $f$ on the positive real line subhomogeneity can be tested by making use of the equivalence

$$f \text{ subhomogeneous } \iff \frac{f(x)}{x} \text{ is decreasing } \iff x f'(x) - f(x) \leq 0, \quad x \in (0, \infty),$$

see, e.g., [Lar16]. These equivalent characteristics are instrumental in proving necessary and sufficient conditions for action functions of simple type (such as convex or concave) to be metric function.

**Theorem 6** (Metric action equivalences for nonlinear consensus protocols)**.** *Consider consensus protocols MO, HH, WS as given in Table 3.1. Assume $\phi_i = \mathsf{id}$. Let $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a positive definite function. Each of the protocols is a pre-metric action protocol of the form*

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} \mathsf{sgn}(x_j - x_i) \varphi(x_i, x_j), \tag{3.15}$$

*with $\varphi = d_p$. In particular, $d_p = ||\phi||$ is a suitable choice, with $\phi$ as in Table 3.1.*

*Moreover, $\varphi = d_s$, i.e.,* (3.15) *is a semi-metric action protocol for protocols (MO) and (HH), and for protocol (WS) under the odd extension that for all $z \in \mathbb{R}^+$, $\varphi(-z) = -\varphi(z)$.*

*For HH, MO, and WS under the odd extension of the action nonlinearity, consider the cases:*

*(i) $\varphi$ convex: $\varphi$ is a metric if and only if it is sub-homogeneous.*

*Suppose $\varphi(x, y) = \varphi(r(x, y))$ with $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$. In case*

*(ii) $\frac{\varphi(r)}{r}$ is decreasing, then $\varphi$ is a metric.*

*(iii)* $\varphi$ *is concave and* $\lim_{\epsilon\to 0^+}\varphi(\epsilon)\neq-\infty$*, then* $\varphi$ *is a metric if and only if* $\lim_{\epsilon\to 0^+}\varphi(\epsilon)\geq 0$.

*If action functions of class WS are not odd, then* $\varphi = d_q$*, i.e.,* (3.15) *is a quasi-metric protocol, for the cases (i)-(iii).*

Note that (ii) neither requires convexity nor concavity.

*Proof.* Pre-and semi-metric characterization: To show the pre-metric property of all action functions, we need to show that the respective action function can be brought into the form $\mathsf{sgn}(y-x)d_p(x,y)$ with $d_p=|\phi|$.
From property (ii) in Table 3.1, all action functions satisfy the identity of indiscernibles, so that positive definiteness reduces to showing positivity of the action for non-identical arguments.

(WS protocol action): set $z=y-x$; as $z\phi(z)>0$, $\phi(z)>0\Leftrightarrow z>0$, so that in this case $\phi(z)=|\phi(z)|$ and $\phi(z)<0\Leftrightarrow z<0$ so that $\phi(z)=\mathsf{sgn}(z)|\phi(z)|$. Hence, we have $d_p(y,x)=|\phi(z)|>0$.

(HH and MO protocols are semi-metric protocols): Note that $d_s\Rightarrow d_p$, i.e, every semi-metric is also a pre-metric. We need to show $\phi(x,y)=\mathsf{sgn}(y-x)\varphi(x,y)$ for HH protocols and $\varphi$ being positive and symmetric, then also MO protocols have this property as implied by Lemma 2.
The sign of the action function $\phi(x,y)$ depends on the sign of the difference $x-y$. Hence, making additionally use of property HH(iv) in Table 3.1 (the action function is odd), we can express inequality HH (iii) in table 3.1 equivalently as

$$\mathsf{sgn}(x-y)\phi(x,y)<0\ \ \Leftrightarrow\ \ \mathsf{sgn}(x-y)\phi(y,x)>0$$

Therefore, for $x-y>0\Leftrightarrow\phi(y,x)=|\phi(y,x)|>0$ and additionally for $x-y<0\Leftrightarrow\phi(y,x)=\mathsf{sgn}(x-y)|\phi(y,x)|<0$. Hence, setting $\varphi(x,y)=|\phi(y,x)|=\mathsf{sgn}(x-y)\phi(y,x)$ we have positivity of $\varphi$, and clearly $\varphi$ is also symmetric, as $|\phi(y,x)|=|-\phi(x,y)|=|\phi(x,y)|$, Therefore, $\varphi=d_s$ so that HH and hence also MO protocols are semi-metric and by that also pre-metric protocols. By Lemma 2 WS type edge action under the additional assumption of anti-symmetry is subsumed by edge actions of HH type.

Metric property of $\varphi$ for the three protocols (and WS under odd extension): As already shown, the action function $\varphi=d_s$, i.e., positive definite and symmetric, so that the remaining property to prove for $\varphi$ to be a metric is sub-additivity of $\varphi$.

(i) Necessity ($\varphi$ sub-additive $\Rightarrow$ $\varphi$ sub-homogeneous), following [Ros50] Theorem 1.4.6: Let $\lambda>1$ and for $n\in\mathbb{N}$ choose $\lambda$ such that $n\leq\lambda\leq n+1$. For some $t\in[0,1]$ we can write $\lambda=(1-t)n+t(n+1)$. Convexity implies

$$\varphi(\lambda\boldsymbol{x})=\varphi((1-t)n\boldsymbol{x}+t(n+1)\boldsymbol{x})\leq(1-t)\varphi(n\boldsymbol{x})+t\varphi((n+1)\boldsymbol{x}).$$

As $\varphi$ is sub-additive, we obtain the inequality

$$(1-t)\varphi(n\boldsymbol{x})+t\varphi((n+1)\boldsymbol{x})\leq(1-t)n\varphi(\boldsymbol{x})+\leq t(n+1)\varphi(\boldsymbol{x})=\lambda\varphi(\boldsymbol{x}).$$

That is, $\varphi(\lambda\boldsymbol{x})\leq\lambda\varphi(\boldsymbol{x})$, which by Definition 4 is equivalent to $\varphi$ being a sub-homogeneous function.

(i) Sufficiency ($\varphi$ sub-additive $\Leftarrow$ $\varphi$ sub-homogeneous), following [Ros50] Theorem 1.4.6: If $\varphi$ is sub-homogeneous, then

$$\varphi(\boldsymbol{x}+\boldsymbol{y}) = \varphi\left(2\cdot\frac{1}{2}(\boldsymbol{x}+\boldsymbol{y})\right) \leq 2\varphi\left(\frac{1}{2}\boldsymbol{x}+\frac{1}{2}\boldsymbol{y}\right) \overset{\text{convexity}}{\leq} 2\cdot\frac{1}{2}(\varphi(\boldsymbol{x})+\varphi(\boldsymbol{y})) = \varphi(\boldsymbol{x})+\varphi(\boldsymbol{y}).$$

By Definition 2, $\varphi(\boldsymbol{x}+\boldsymbol{y}) \leq \varphi(\boldsymbol{x})+\varphi(\boldsymbol{y})$ means that $\varphi$ is sub-additive. This also completes the proof of part (i).

(ii) Following Theorem 7.2.4. in [HP57]: If $\varphi(r)/r$ is decreasing in $r$, then, for $t > 0$ we have

$$\frac{\varphi(r)}{r} \geq \frac{\varphi(r+t)}{r+t}, \tag{3.16}$$

as $r+t > r$. Using this inequality we can write

$$\begin{aligned}\varphi(r+t) &= \frac{r}{r+t}\varphi(r+t)\frac{t}{r+t}\varphi(r+t) = r\frac{\varphi(r+t)}{r+t} + t\frac{\varphi(r+t)}{r+t} \overset{(3.16)}{\leq} r\frac{\varphi(r)}{r} + t\frac{\varphi(t)}{t} \\ &= \varphi(r)+\varphi(t).\end{aligned}$$

This, by definition implies sub-additivity of $\varphi : \mathbb{R}^+ \to \mathbb{R}$ and hence $\varphi = d$, i.e., the action nonlinearity defines a metric.

(iii) Necessity ($\varphi$ is a metric $\Rightarrow \lim_{\epsilon\to 0^+}\varphi(\epsilon) \geq 0$), following Theorem 7.4.3. in [HP57]: Set $\lim_{\epsilon\to 0^+}\varphi(\epsilon) = c$. Note that by assumption $c$ is finite. Be $\{z_n\}$ a sequence such that $z_n \to 0$ and $\lim_{n\to\infty}\varphi(z_n) = c$. There is an index $n > 0$ and $\epsilon > 0$ such that $\varphi(z_n)$ is in a neighborhood characterized by $c-\epsilon \leq \varphi(2z_n) \leq 2\varphi(z_n) \leq 2(c+\epsilon)$, where the second inequality is implied by sub-additivity. As $n \to \infty$, or $\epsilon \to 0$, $\lambda \leq 2\lambda$ or equivalently $\lambda \geq 0$.

(iii) Sufficiency ($\varphi$ is a metric $\Leftarrow \lim_{\epsilon\to 0^+}\varphi(\epsilon) \geq 0$): As $\varphi$ is concave, by definition , for $t \in [0,1]$, $\varphi(tx+(1-t)y) \geq t\varphi(x)+(1-t)\varphi(y)$. Set $x = 0$, and using $\lim_{\epsilon\to 0^+}\varphi(\epsilon) \geq 0$, Jensen's inequality becomes

$$\varphi((1-t)y) \geq t\varphi(0)+(1-t)\varphi(y) \geq (1-t)\varphi(y).$$

Setting $(1-t)y = a$ and $y = b$, so that $a \leq b$, we can rewrite this inequality and obtain the equivalence

$$\varphi(a) \geq \frac{a}{b}\varphi(b) \quad\Leftrightarrow\quad \frac{\varphi(a)}{a} \geq \frac{\varphi(b)}{b}.$$

Hence, $\varphi(y)/y$ is decreasing in $y$. As shown in part (ii) of this proof, this property implies sub-additivity of the function $\varphi$, which completes the sufficiency part of (iii).

If WS-type protocols do not have an odd action function, then $\varphi$ is not symmetric but positive definite and still satisfies sub-additivity under given conditions. In those cases $\varphi = d_q$, i.e., the WS protocol is a quasi-metric action protocol by definition.

□

The existing consensus literature on admissible nonlinear action function classes, as summarized in Table 3.1, makes use of specific definitions of action function types, which are adapted and motivated by the author's chosen stability proof methods. In contrast, we proposed a single comprehensive and simple class of action functions, characterized as a specific type of metric function, that subsumes all previously discussed nonlinear action types. Moreover, we can characterize those metric action functions, which ease its construction from simple type functions, using necessary and sufficient conditions. The use of basic and generic mathematical functions and the completeness of the result highlight its significance.

## 3.5 Protocol design and convergence results

In the following, we prove stability and exponential convergence to consensus for two elementary consensus protocol classes: the previously introduced class of signed metric actions and a novel, second protocol that generalizes the linear mean-driven consensus approach using the Kolmogorov-mean-metric relationship. We propose composition rules for designing (more complex) metric consensus protocols based on functions of simple type as elementary metric building blocks. For that, we use the necessity and sufficiency relationship between metric actions and sub-homogeneous functions. We discuss optimality properties of infinitesimal solution updates generated by the second (Kolmogorov) mean-driven consensus protocol, which differs from optimality of the average consensus state, an asymptotic property.

### 3.5.1 Metric action class and protocol composition

The following result proves exponential convergence to a consensus state for the class of (signed) metric action protocols on graphs, which subsumes most of the existing nonlinear consensus protocol design and stability results and is specific enough to use it as a tool for the design of stable-by-design consensus dynamics.

**Theorem 7** (Stability for pre-metric action protocols)**.** *Let the pre-metric function $d_p$ be locally Lipschitz continuous and consider a time-varying digraph* $\mathsf{G}$*. The protocol*

$$\dot{x}_i = \sum_{j\in N_i^+} w_{ij}(t)\mathsf{sgn}(x_j - x_i)d_p(x_j, x_i), \quad i \in N, \tag{3.17}$$

*generates solutions that converge for all admissible initial conditions $\boldsymbol{x}_0 \in \mathbb{R}^n$ exponentially fast to an agreement state $\bar{x}\mathbf{1}, \bar{x} \in [\mathsf{min}(\boldsymbol{x}_0), \mathsf{max}(\boldsymbol{x}_0)]$, if the underlying digraph is uniformly connected.*

*Proof.* The basic idea is to transform the generic class to a linear time and state-dependent, but linear type, whereas the state-controlled part of the resulting "virtual" (action) graphs is such that Moreau's general stability result remains applicable.

Expand the protocol (3.17) by state differences, so that for $i \in N$ we obtain

$$\begin{aligned}\dot{x}_i &= \sum_{j\in N_i^+} w_{ij}(t)\mathsf{sgn}(x_j - x_i)d_p(x_j, x_i) = \sum_{j\in N_i^+} \underbrace{w_{ij}(t)\mathsf{sgn}(x_j - x_i)\frac{d_p(x_j, x_i)}{x_j - x_i}}_{:=\tilde{w}_{ij}(t;\boldsymbol{x})}(x_j - x_i) \\ &= \sum_{j\in N_i^+} \tilde{w}_{ij}(t;\boldsymbol{x})(x_j - x_i),\end{aligned}$$

where the new state and time-dependent weigths $\tilde{w}_{ij}(t;\boldsymbol{x}), (j,i) \in B(t)$ define a time and state-dependent "virtual" graph. The task is to show that this new weight function is well-defined and satisfies Moreau's boundedness condition, cf. Proposition 2.

By hypothesis, the function $w_{ij}(\cdot)$ is such that the time-dependent graph is uniform connected. Further, for $x_j \neq x_i, i, j \in N$,

$$\mathrm{sgn}(x_j - x_i)\frac{d_p(x_j, x_i)}{x_j - x_i} = \frac{\mathrm{sgn}(x_j - x_i)d_p(x_j, x_i)}{\mathrm{sgn}(x_j - x_i)|x_j - x_i|} > 0,$$

By assumption, the action function $d_p$ is locally Lipschitz continuous, so that for two real numbers $b, a,\ b > a$,

$$\lim_{b\to a}\frac{d_p(b,a)}{|b-a|} \leq const < \infty.$$

Hence, the elements $\tilde{w}_{ij}(t;\boldsymbol{x}), (j,i) \in B(t)$ are finite and positively bounded away from zero for all possible parametrizations. That is, functions $\tilde{w}_{ij}$ satisfy uniform connectedness as defined in Proposition 2 if time dependent functions $w_{ij}(\cdot)$ satisfy uniform connectedness, which is the case by hypothesis.

Let us turn to the statement regarding the exponentially fast reached consensus value. The metric protocol on a graph can be brought to linear consensus form on a dynamically weighted but uniformly connected "virtual" graph. By standard linear consensus theory, the function $\max_{i\in N} x_i - \min_{i\in N} x_i$ is a (strict) Lyapunov function [Mor05]. Hence, the maximal state value is decreasing, and the minimal state value is increasing, so that the consensus value must lie in between the initial maximum and minimum state values. Exponential convergence follows from Proposition 2. □

The following proposition is helpful for the composition of consensus protocols that are stable by design using simple nonlinear and locally Lipschitz continuous functions as building blocks.

**Proposition 4.** *Let $d_p$ be a pre-metric on a set $S \subset \mathbb{R}\times\mathbb{R}$, i.e., for any pair $(x,y) \in S$ of non-identical components $x, y$ the function $d_p(x,y) > 0$. Then,*

*(a) for any $\theta : \mathbb{R} \to \mathbb{R}$, such that $(\theta(x), \theta(y)) \in S$ the function $d_p(\theta(x), \theta(y)) = d'_p(x,y)$ is a pre-metric.*

*(b) for any $\alpha : \mathbb{R} \to \mathbb{R}^+$, the function $\alpha(y)d_p(x,y) = d'_p(x,y)$ is a pre-metric.*

*(c) Compositions of functions that satisfy property $(a)$ and $(b)$ result in a pre-metric function $d'_p(x,y) = \alpha(y)d_p(\theta(x),\theta(y))$.*

*(d) Let $f$ be a sign-preserving function. Then $d'_p = f(d_p)$ is a pre-metric.*

*Next, assuming $d_p = d_q$, i.e., the pre-metric is also sub-additive and hence a quasi-metric, then, the following composition rules can be used to compose more complex action functions:*

*(e) for $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$, if $f_1$ and $f_2$ are sub-additive and $c_1, c_2 > 0$, then $c_1f_1 + c_2f_2$ is sub-additive.*

*(f) if $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is sub-additive and $g : \mathbb{R}^+ \to \mathbb{R}$ increasing and sub-additive, then $g(f)$ is sub-additive in $S$.*

*Proof.* The proof of items (a)-(d) is immediate and follows from the construction of the functions. For properties (e) and (f) see [Ros50] Theorem 1.3.1. □

Using these composition rules, we can derive further special but prominent nonlinear consensus classes from the literature, as demonstrated in the three examples.

**Example 7** (Two rate-controlled consensus protocols)**.** In the paper [ADJ12] the authors study the rate-controlled consensus protocol

$$\dot{x}_i = -\gamma_i(x_i)\sum_{j\in N_i^+}\varphi_{ij}(x_i - x_j), \quad i \in N, \tag{3.18}$$

assuming that $\gamma_i(\cdot)$ is continuous and positive, $\varphi_{ij}(\cdot)$ Lipschitz continuous and sign-preserving, i.e., $x\varphi_{ij}(x) \geq 0$ and zero if and only if $x = 0$. Following the arguments of Lemma 2 or Theorem 6 we can see that $\varphi_{ij}(x_i - x_j) = \mathrm{sgn}(x_j - x_i)d_p(x_j, x_i)$ with $d_p(\cdot,\cdot) = |\varphi_{ij}(\cdot)|$, and using Proposition 4 (b) leads to the pre-metric form

$$\dot{x}_i = -\gamma_i(x_i)\sum_{j\in N_i^+}\varphi_{ij}(x_i - x_j) = \sum_{j\in N_i^+}\mathrm{sgn}(x_j - x_i)d_p'(x_j, x_i), \tag{3.19}$$

where $d_p'(x_j, x_i) = \gamma(x_i)d_p(x_j, x_i)$.

Another prominent rate-controlled consensus protocol has been introduced in [BGP06]. It is of the form

$$\dot{x}_i = -c\frac{1}{\mathrm{d}g/\mathrm{d}x_i}\sum_{j\in N_i^+}\varphi(\theta(x_j) - \theta(x_i)), \quad i \in N, \tag{3.20}$$

with $c > 0, \varphi : \mathbb{R} \to \mathbb{R}$ continuous, locally Lipschitz, odd and strictly increasing, $\theta : \mathbb{R} \to \mathbb{R}$ continuously differentiable and strictly positive and $g$ an increasing function; (this last assumption on $g$ is made implicitly in [BGP06] Assumption 2, by imposing a specific functional structure on the considered means, which turns out to be the Kolmogorov mean structure). Clearly, the rate factor $c\frac{1}{\mathrm{d}g/\mathrm{d}x_i} > 0$, and $\varphi$ satisfies properties of class MO in Table 3.1, see also Lemma 2. Hence, we can write (3.20) as pre-metric protocol as in (3.19) with

$$d_p'(x_j, x_i) = c\frac{1}{\mathrm{d}g/\mathrm{d}x_i}d_p(x_j, x_i), \text{ where } d_p(x_j, x_i) = |\varphi(\theta(x_j) - \theta(x_i))|,$$

where we make use of composition rule Proposition 4 (c).

The two cases of rate-controlled protocols discussed in Example 7 are also interesting, as they demonstrate the problem in the existing consensus literature that nonlinear protocol design is often based on very specific function assumptions which makes it complicated to relate new work to prior ones: The protocol (3.20) introduced in [BGP06] has been introduced 6 years earlier than protocol (3.18). The requirements on action functions $\varphi_{ij}$ in (3.18) are seemingly different from those of action functions in (3.20), however, as the example demonstrates (or Lemma 2), those are identical function descriptions. The gain functions describing variable rates are also both positive and $c\frac{1}{\mathrm{d}g/\mathrm{d}x}$ is just a special case for a general positive $\gamma(x)$.

In the following example, we demonstrate that composition rules as in Proposition 4 in combination with Theorem 7 can be used to prove the stability of the very general class of WS type consensus protocols that also include an input nonlinearity.

**Example 8.** Consider the consensus protocol of type WS with additional input nonlinearity,

$$\dot{x}_i = \phi_i(\sum_{j\in N_i^+} w_{ij}(t)\varphi_{ij}(x_j - x_i)), \quad i \in N. \tag{3.21}$$

As shown in Theorem 6 $\varphi_{ij}(x_j - x_i) = \mathrm{sgn}(x_j - x_i)d_p(x_j, x_i)$, i.e. the action nonlinearity is of pre-metric type, with $d_p = |\varphi_{ij}|$. Using the time-dependent in-degree function $d_i(t) = \sum_j w_{ij}(t)$ and defining $d_p' = d_i d_p$, the protocol (3.21) can be written as pre-metric action protocol on a normalized digraph

$$\dot{x}_i = \phi_i(\sum_{j\in N_i^+} \hat{w}_{ij}(t)\mathrm{sgn}(x_j - x_i)d_p'(x_j, x_i)). \tag{3.22}$$

Assume $\phi_i$ either being convex or sub-additive, then the r.h.s. of (3.22) can be bounded such that

$$\dot{x}_i = \phi_i(\sum_{j\in N_i^+} \hat{w}_{ij}(t)\mathrm{sgn}(x_j - x_i)d_p'(x_j, x_i)) \leq \sum_{j\in N_i^+} \hat{w}_{ij}(t)\mathrm{sgn}(x_j - x_i)\phi_i(d_p'(x_j, x_i))$$

As $\phi_i$ is sign-preserving $\phi_i(d_p') =: d_p''$, i.e., it leaves the pre-metric function characteristic invariant. Therefore, a pre-metric consensus protocol bounds the protocol WS with input nonlinearity from above and below. If the pre-metric version converges to consensus, the original must converge to an equilibrium state, too.

Besides the pre-metric consensus protocol class, which is motivated by a generalization of existing nonlinear protocol types, a second approach to the design of nonlinear consensus protocols arises from the study of how the arithmetic mean drives the linear consensus network via the representation (3.3). This linear approach of control by a neighborhood's average is generalized to nonlinear means in the following.

### 3.5.2 Mean-control consensus protocol and optimality of dynamics

Motivated by the Kolmogorov mean structure we introduce the function

$$M_f^w(x_1, x_2, \ldots, x_n) := g(w_1 f(x_1) + w_2 f(x_2) + \ldots + w_n f(x_n)) \tag{3.23}$$

with functions $g$ and $f$ as in the definition of the Kolmogorov mean (3.13). If $\sum_j w_j = 1$ then (3.23) is indeed a Kolmogorov mean, namely its weighted version. However, for the sake of generality in consensus system design we shall consider the general function $M_f^w$ in the following.

**Theorem 8.** *Let* G *be a time-varying digraph and consider the mean-controlled protocol*

$$\dot{x}_i = -g(d_i f(x_i)) + M_f^w(\{x_j\}_{j\in N_i^+}), \quad i \in N, \tag{3.24}$$

*where $M_f^w$ is as given in* (3.23)*, and $d_i$ is the in-degree. For any initial condition, solutions of the consensus system governed by* (3.24) *on* G *converge with exponential speed to a consensus value if the time-varying digraph is uniformly connected over time.*

*Proof.* The idea is again to transform the system to a linear consensus form on a virtual, time-varying action graph. Using the fact that $g(d_i f(x_i)) = M_f^w(\{x_i\}_{i=1,\dots,\mathsf{card}(N_i^+)})$ we can re-write the protocol dynamics (3.24) to obtain

$$\dot{x}_i = M_f^w(\{x_j\}_{j\in N_i^+}) - M_f^w(\{x_i\}_{i=1,\dots,\mathsf{card}(N_i^+)}). \tag{3.25}$$

Next, let us use the vector notation $\boldsymbol{x}_j$ for the set $\{x_j\}_{j\in N_i^+}$ and $\boldsymbol{x}_i$ for the set $\{x_i\}_{i=1,\dots,\mathsf{card}(N_i^+)}$. With that, we expand the r.h.s. of (3.25) such that

$$\frac{M_f^w(\boldsymbol{x}_j) - M_f^w(\boldsymbol{x}_i)}{f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))}\Big(f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))\Big). \tag{3.26}$$

The function $f$ is monotonously increasing by definition of the Kolmogorov mean so that the sign of the nominator and denominator of the fraction in (3.26) are identical, and hence, the fraction is positive.

If $M_f^w(\boldsymbol{x}_j) = M_f^w(\boldsymbol{x}_i) = x_i$, then the nominator is zero, the denominator, however, as well, so that the fraction is indeterminate. For that case, application of the rule de l'Hôpital shows that

$$\lim_{M_f^w(\boldsymbol{x}_j)\to x_i} \frac{M_f^w(\boldsymbol{x}_j) - M_f^w(\boldsymbol{x}_i)}{f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))} = \lim_{M_f^w(\boldsymbol{x}_j)\to x_i} \frac{1-0}{f'(M_f^w(\boldsymbol{x}_j)) - 0)}$$
$$= \frac{1}{f'(x_i)} > 0, \quad \forall x_i,$$

as $f$ is strictly increasing.
As long as the arguments are finite, the mean is finite, as well, so that for all possible parameterizations,

$$0 < \frac{M_f^w(\boldsymbol{x}_j) - M_f^w(\boldsymbol{x}_i)}{f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))} < \infty.$$

Next, let us consider the function difference with which we expanded in (3.26). Using the definition of Kolmogorov (3.13) with $g \circ f = \mathsf{id}$ we get

$$f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i)) = f(g(\sum_{j\in N_i^+} w_{ij}(t) f(x_j))) - f\,(g\,(d_i(t) f(x_i)))$$
$$= \sum_{j\in N_i^+} w_{ij}(t) f(x_j) - d_i(t) f(x_i) = \sum_{j\in N_i^+} w_{ij}(t)(f(x_j) - f(x_i))$$
$$= \sum_{j\in N_i^+} w_{ij}(t) \frac{f(x_j) - f(x_i)}{x_j - x_i}(x_j - x_i).$$

The fraction $\frac{f(x_j)-f(x_i)}{x_j-x_i} = \frac{\mathsf{sgn}(x_j-x_i)|f(x_j)-f(x_i)|}{\mathsf{sgn}(x_j-x_i)|x_j-x_i|} = \frac{|f(x_j)-f(x_i)|}{|x_j-x_i|} > 0$ and finite for all parameterizations.

We eventually can write the protocol as

$$\dot{x}_i = \frac{M_f^w(\boldsymbol{x}_j) - M_f^w(\boldsymbol{x}_i)}{f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))} \sum_{j\in N_i^+} w_{ij}(t) \frac{f(x_j) - f(x_i)}{x_j - x_i}(x_j - x_i),$$

which defines a linear consensus protocol on a state and time dependent virtual digraph with branch weight function for each $(j,i) \in B(t)$ given by

$$\tilde{w}_{ij}(\boldsymbol{x}_j, x_i) := \frac{M_f^w(\boldsymbol{x}_j) - M_f^w(\boldsymbol{x}_i)}{f(M_f^w(\boldsymbol{x}_j)) - f(M_f^w(\boldsymbol{x}_i))} w_{ij}(t) \frac{f(x_j) - f(x_i)}{x_j - x_i}.$$

Both fractions are positive, finite, and bounded positively away from zero so that this virtual weight function is positive definite and zero only if $w_{ij}(t) = 0$.

Hence, by application of Proposition 2, the system converges with exponential speed to a consensus equilibrium for all initial conditions if the weighted digraph is uniformly connected. □

For the case of having a normalized digraph, the protocol (3.24) becomes the Kolmogorov-mean-controlled version

$$\dot{x}_i = -x_i + \arg\min_{x \in \mathbb{R}} \sum_{j \in N_i^+} d_f^2(x_j, x). \tag{3.27}$$

Hence, the control input, being representative for interactions across the network, drives the stable linear dynamics $\dot{x}_i = -x_i$ using the neighborhood's Kolmogorov average being the solution of a convex optimization problem.
Suppose the time-dependent weighting is Lipschitz continuous over time. An Euler discretization of the protocol ODE (3.27) yields the equation for the updated local state

$$x_i^+ = (1 - \Delta t) x_i + \Delta t \arg\min_{x \in \mathbb{R}} \sum_{j \in N_i^+.} d_f^2(x_j, x).$$

As $\Delta t > 0$ and the values $x_i$ being constants we have the optimal characterization of an infinitesimal state update, up to the constant $const = (1 - \Delta t) x_i$,

$$x_i^+ - const = \arg\min_{x \in \mathbb{R}} \sum_{j \in N_i^+.} d_f^2(x_j, x)$$

Put differently, for a time discretization $\Delta t \to 0^+$, we have $const \approx x_i$ so that the update differential $\delta x_i(t)$ at time $t$ is optimal in the sense

$$\delta x_i(t) = \Delta t \arg\min_{x \in \mathbb{R}} \sum_{j \in N_i^+.} d_f^2(x_j(t), x), \quad i \in N.$$

The solution of the optimization problem is independent of the choice of $\Delta t$. However, the correct mapping from the optimization solution to the infinitesimal state update $\delta x_i(t)$ is obtained by scaling the optimization solution with the infinitesimal time step.

Hence, the network systems dynamics induced by the mean-control consensus protocol are optimal at each infinitesimal step in time, w.r.t. the optimality criterion defined by the metric $d_f$. This optimality of the dynamics differs from the usual optimality characteristics in the consensus literature, which is concerned about the asymptotically reached static consensus equilibrium that minimizes sum-of-squares energy as discussed in Section 3.2.

## 3.6 Application: The embedding protocol

We introduce an embedding protocol as a particular case of a metric action consensus protocol. This consensus network class combines network energy functions serving as Lyapunov and potential functions in a gradient flow schema, a geometric generalization from consensus on linear to nonlinear spaces, and extends known optimality and invariance properties of the linear to general nonlinear consensus system case.

### 3.6.1 Embedding protocol and consensus on nonlinear space

Following Kolmogorov's idea to generalize means via the arithmetic mean linear structure through coordinate transformations $f : x \mapsto f(x)$ and transforming the aggregate back via the inverse map $g$, we propose the embedding protocol as linear consensus protocol with transformed states and a transformation back to locally linear space for velocities.

Consider $\mathscr{M}_e \subset \mathbb{R}^n$ a nonlinear configuration space embedded in $n$-dimensional Euclidean space such that $\mathbb{R}^n \to \mathscr{M}_e : x_i \to e(x_i) =: \hat{x}_i, \forall i = 1, \dots, n$. The scalar embedding function $e$ is motivated from Kolmogorov's function $f$, and we assume it to be increasing, continuous and such that its first derivative $e'(x) = \frac{\mathrm{d}e}{\mathrm{d}x}$ exists.

With that, we define the embedding protocol as a metric interaction protocol with positive gain function given by the protocol differential equation

$$\dot{x}_i = (e'(x_i))^{-1} \sum_{j \in N_i^+} w_{ij}(t)(e(x_j) - e(x_i)), \quad i \in N. \tag{3.28}$$

From a geometric perspective, the embedding protocol (3.28) can formally be written as

$$\dot{x}_i = \mathsf{Proj}_{T\mathscr{M}_{e,x_i}} \left( \sum_{j \in N_i^+} w_{ij}(\hat{x}_j - \hat{x}_i) \right) \quad i \in N, \tag{3.29}$$

with $T\mathscr{M}_{e,x_i}$ the tangent space at $x_i$, where it becomes apparent that the gain function $e'^{-1}$ in (3.28) serves as a back projection to local, linear space.

**Corollary 2** (Exponential convergence for the embedding protocol)**.** *The embedding protocol* (3.28) *running on a weighted digraph* $\mathsf{G}(t)$ *converges with exponential speed to a consensus state if the underlying digraph is uniformly connected.*

*Proof.* Using Lemma 1 it follows that $|e(b) - e(a)| = d_e(a, b)$, i.e., it is a metric, and as $e$ is increasing and differentiable we have $1/e'(x) > 0$. Hence, using Theorem 7 with composition rule given in Proposition 4 (a), the embedding protocol (3.28) is an instance of a pre-metric consensus protocol and hence, when running on a uniformly connected graph, the system converges exponentially fast to a consensus state. □

To better understand the relation between the embedding protocol(3.28) and its geometric description (3.29) consider as starting point simply the linear consensus protocol ODE in embedded coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_i = \sum_{j \in N_i^+} w_{ij}(t)(\hat{x}_j - \hat{x}_i), \quad i \in N.$$

Solving for the local, linear coordinate velocity $\dot{x}_i$, the positive gain function $(e'(x_i))^{-1}$ in the embedding protocol originates from projecting the velocity of the transformed state component $\mathrm{d}\hat{x}_i/\mathrm{d}t$ back to the velocity of the local, linear coordinate velocity $\dot{x}$; using the chain rule, we get for the embedded state velocity

$$\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = \frac{\mathrm{d}e(x_i)}{\mathrm{d}x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} \quad \Leftrightarrow \quad \dot{x}_i = \left(\frac{\mathrm{d}e(x_i)}{\mathrm{d}x_i}\right)^{-1} \sum_{j\in N_i^+} w_{ij}(t)(\hat{x}_j - \hat{x}_i).$$

This approach to defining the embedding protocol is identical to the way the famous phase averaging in the Kuramoto model has been initially derived, whereas there, the embedding is from the real to the complex numbers, in which the circle as nonlinear configuration space for angles is defined. The following example demonstrates this embedding from reals to the circle in the Kuramoto case.

**Example 9** (Phase averaging on the circle)**.** Consider the circle $\mathbb{S}^1$ being embedded in Euclidean ambient space $\mathbb{R}^2$, which can equivalently be expressed as the complex plane $\mathbb{C}$ with coordinates the real and imaginary components of complex numbers. Coordinates on the circle are angles $\theta \in \mathbb{S}^1$. The embedding of angles into the complex plane leads to the embedding $\mathbb{S}^1 \to \mathbb{C} : \theta \to \hat{\theta} = e^{\imath\theta}$ and the projection back $\mathbb{C} \to \mathbb{S}^1 : \hat{\theta} \to \theta = \arg(\hat{\theta})$. Using this coordinate embedding, the consensus protocol on the circle can be derived for every $i \in N$ from

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}e^{\imath\theta_i} = e^{\imath\theta_i}\imath\dot{\theta}_i = \sum_{j\in N_i} w_{ij}(e^{\imath\theta_j} - e^{\imath\theta_i}) \\ \Leftrightarrow \quad &\imath\dot{\theta}_i = \sum_{j\in N_i} w_{ij}(e^{\imath(\theta_j-\theta_i)} - 1) = \sum_{j\in N_i} w_{ij}(\cos(\theta_j-\theta_i) - \imath\sin(\theta_j-\theta_i) - 1), \end{aligned}$$

so that solving for the angle velocity, i.e., taking the imaginary part on both sides, yields

$$\dot{\theta}_i = \sum_{j\in N_i} w_{ij}\sin(\theta_j - \theta_i), \quad i \in N.$$

This is the famous Kuramoto model for phase averaging dynamics with natural frequencies all equal (or equivalently zero).

The geometric protocol formulation (3.29) is also result of a geometric consensus framework developed by Sarlette and Sepulchre in the original work [SS09a] and [SS09b], see also Sarlette's thesis [Sar09] and Sepulchre's review [Sep11]. It aims at unifying linear and nonlinear consensus algorithm design by taking the idea of dynamics that unfolds by repeatedly computing updates of the local consensus estimate via arithmetic mean averaging in Euclidean space to computing means as an update in the direction of a local mean; however, on nonlinear spaces. A nonlinear configuration space, defined via a Riemannian metric structure, is their starting point and not a simple coordinate transformation as given here with the embedding function $e$.

For nonlinear (Riemannian) configuration spaces denoted $\mathscr{M}$ the natural generalization of computing weighted arithmetic mean via a metric minimization, cf. (3.7), is to consider the non-Euclidean, metric minimization

$$\mathrm{mean}_{\mathscr{M}}(\boldsymbol{x}) = \arg\min_{z\in\mathscr{M}} \sum_{i=1}^{n} d_{\mathscr{M}}(z, x_i)^2,$$

where $\mathrm{d}_{\mathscr{M}}$ denotes the intrinsic (Riemannian) metric, which is a geodesic distance. The fundamental problem with this idea of updating a point towards a new point by moving along the geodesic path connecting them is that the computation of a geodesic is practically not feasible [Sep11], see also [AMS08] Chapter 4, or [LY73] Chapter 12, as it involves solving a minimization problem over all paths connecting two points at each time step and in a distributed manner across the consensus network.

To overcome this computational problem and reach a practical consensus algorithm, the authors also introduce an "embedding trick": they regard the nonlinear configuration space as embedded in ambient linear, Euclidean space, and in this extrinsic linear geometry, a simple linear update is computed, which then is projected back onto the nonlinear space. Replacing the (intrinsic) geodesic with this (extrinsic) projected approximate allows to derive explicit nonlinear consensus protocol formulae according to (3.29).

In particular, under the assumption that $\mathscr{M}$ is a connected compact homogeneous manifold, it can be smoothly embedded into a Euclidean (linear) space; here, $\mathscr{M}$ be embedded in $\mathbb{R}^n$, and elements $x \in \mathscr{M}$ are again denoted $\hat{x}$ when expressed as (vector) embedding in linear space $\mathbb{R}^n$. Within the ambient Euclidean geometry the authors make use of the induced (or projected [Moa02]) weighted arithmetic mean

$$\mathsf{IAM}_w(x_1, ..., x_n) := \arg\min_{x \in \mathscr{M}} \sum_{i=1}^{n} \omega_i |\hat{x}_i - \hat{x})|^2. \tag{3.30}$$

Without the requirement of the solution being element of $\mathscr{M}$ this squared distance minimization has solution the Kolmogorov mean in non-embedded coordinates, see Theorem 5, or equivalently, the weighted arithmetic mean, $\mathsf{am}_w(\hat{x}_1, \ldots, \hat{x}_n) = \sum_{i=1}^{n} \omega_i \hat{x}_i$, in embedding coordinates. Under the restriction of $x \in \mathscr{M}$, (3.30) defines the orthogonal projection of this centroid onto the manifold $\mathscr{M}$.

Analogous to the linear consensus protocol (2.4) the authors eventually define the nonlinear generalization as differential update rule towards the weighted arithmetic mean in ambient space, which then is projected to the closest point of the manifold $\mathscr{M}$ [Sep11], leading to the protocol differential equation

$$\dot{x}_i = \mathsf{Proj}_{T\mathscr{M}_{x_i}} \left( \sum_{j \in N_i^+} w_{ij} (\hat{x}_j - \hat{x}_i) \right),$$

$T\mathscr{M}_{x_i}$ again the tangent space at $x_i$.

### 3.6.2 Coordinate embedding defines gradient descent scheme

A gradient descent scheme of some scalar energy function $F : \mathbb{R}^n \to \mathbb{R}$ in continuous time typically has the ODE form $\mathbf{G}(\boldsymbol{x})\dot{\boldsymbol{x}} = -\nabla F(\boldsymbol{x})$. The matrix function $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is positive definite and defines the infinitesimal metric $\mathrm{d}\boldsymbol{x} \cdot \mathbf{G}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, in which the system is a gradient flow of $F$.

In the following we transform the embedding protocol (3.28) on time-invariant, connected graphs with symmetric weighting into gradient systems governed by a particular projected

version of the ODE type $\dot{\boldsymbol{x}} = -\mathbf{G}^{-1}(\boldsymbol{x})\nabla\mathscr{E}(\boldsymbol{x})$. We consider the energy function $\mathscr{E} : \mathbb{R}^n \to \mathbb{R}$ to be additive and such that

$$\mathscr{E}(\boldsymbol{x}) = \sum_i E(x_i), \quad \text{with local energy components} \quad E : \mathbb{R} \to \mathbb{R}. \tag{3.31}$$

The embedding function $e$ defines scalar local energy components $E$ via its anti-derivative such that

$$E : \nabla E = e \quad \Leftrightarrow \quad E(x) = \int e(x)\mathrm{d}x, \tag{3.32}$$

where, without loss of generality, we have set the integration constant of the indefinite integral in (3.32) to zero. With this definition, note that $e = \nabla E$ and $e' = \nabla e = \nabla^2 E$.

Now, we can re-write the embedding protocol (3.28) in component ODE as

$$\dot{x}_i = (\nabla^2 E(x_i))^{-1} \sum_{j \in N_i^+} w_{ij}(\nabla E(x_j) - \nabla E(x_i)), \quad i \in N. \tag{3.33}$$

Introducing the diagonal matrix $\mathbf{H}(\boldsymbol{x}) := \mathrm{diag}\{\nabla^2 E(x_1), \ldots, \nabla^2 E(x_n)\}$, the component ODE (3.33) in vector matrix notation for the overall system becomes

$$\dot{\boldsymbol{x}} = -\mathbf{H}(\boldsymbol{x})^{-1}\mathbf{L}\nabla\mathscr{E}(\boldsymbol{x}). \tag{3.34}$$

Under the assumption of operating on a connected and symmetric graph the Laplacian matrix can be factorized such that $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$. The matrix $\mathbf{V}$ collects the orthogonal eigenvectors of the Laplacian with two-norm one and $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is the diagonal matrix of associated eigenvalues, where we remember that $\lambda_1 = 0$, and $\lambda_i > 0$ for $i = 2, 3, \ldots, n$. From the presentation in Section 3.2.3 we know that application of the system matrix $\mathbf{L}$ on an n-vector in fact projects this $n$-vector $\boldsymbol{x}$ onto the $n-1$-dimensional subspace of integral preserving states $\mathscr{M}_I$ as defined in (3.6). Further, $\boldsymbol{v}_1$ associated to $\lambda_1 = 0$ is orthogonal to $\mathscr{M}_I$, cf. Figure 3.1, so that $\{\boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ represents an orthogonal basis for $\mathscr{M}_I$.

We formally write this projection for the energy gradient vector $\nabla\mathscr{E}$ onto $\mathscr{M}_I$

$$\nabla_{\mathscr{M}_I}\mathscr{E}(\boldsymbol{x}) = \mathbf{L}\nabla\mathscr{E}(\boldsymbol{x}) = \sum_{i=2}^{n} \lambda_i \frac{\nabla\mathscr{E}(\boldsymbol{x}) \cdot \boldsymbol{v}_i}{\boldsymbol{v}_i \cdot \boldsymbol{v}_i} \boldsymbol{v}_i = \sum_{i=2}^{n} \boldsymbol{v}_i \lambda_i \boldsymbol{v}_i \cdot \nabla\mathscr{E}(\boldsymbol{x}).$$

Now, the system ODE (3.34) represents the gradient dynamics projected onto $\mathscr{M}_I$ given by

$$\dot{\boldsymbol{x}} = -\mathbf{H}(\boldsymbol{x})^{-1}\nabla_{\mathscr{M}_I}\mathscr{E}(\boldsymbol{x}). \tag{3.35}$$

We can make the following observations and comments:

a) Within the gradient systems framework $\mathbf{G}(\cdot) = \mathbf{H}(\cdot)$, i.e., the local rate functions $1/e'$ of the embedding protocol define the metric in which the system evolves as non-Euclidean gradient flow. If the matrix function $\mathbf{H}(\boldsymbol{x})$ varies smoothly in $\boldsymbol{x}$, then the gradient system ODE defines a Riemannian gradient flow.

b) While we consider gradients of $\mathscr{E}$ on the linear space given by the simplex $\mathscr{M}_I$, the consensus dynamics is constrained to evolve on the nonlinear space $\mathscr{M}_e$. Hence, along trajectories of the gradient system projected onto $\mathscr{M}_I$ the usual quadratic gradient dissipation equality

$$\dot{\mathscr{E}}(\boldsymbol{x}) = -\nabla_{\mathscr{M}_I}\mathscr{E}(\boldsymbol{x}) \cdot \mathbf{H}(\boldsymbol{x})^{-1}\nabla_{\mathscr{M}_I}\mathscr{E}(\boldsymbol{x}) = -||\nabla_{\mathscr{M}_I}\mathscr{E}(\boldsymbol{x})||^2_{\mathbf{H}^{-1}}$$

holds, that characterizes exponential convergence speed. The minimum of $\mathscr{E}$ however is achieved under the constraint of being a consensus state lying on $\mathscr{M}_e$, where trajectories actually evolve. Hence, the consensus value $\bar{x}$ can formally be uniquely defined as the intersection point of the vector $c\mathbf{1}, c > 0$ with $\mathscr{M}_e$.

c) With $\nabla^2 E(x_i)$ being the components of the metric tensor $\mathbf{H}$ and with the particular additive structure of $\mathscr{E}$, cf. (3.31), the metric tensor in fact represents the Hessian of the scalar energy field $\mathscr{E}(\boldsymbol{x})$. Hence, the non-Euclidean gradient dynamics do not follow steepest descent gradient directions on the linear space $\mathscr{M}_I$, but they also use curvature information when approaching a minimum of $\mathscr{E}$.

d) Using the Hessian of a function in scaling the gradient of the same function is reminiscent of Newton's method in optimization. Let $\epsilon \in (0,1)$; for scalar systems a Newton type update step for finding the minimizer of a function $E(x)$ is such that

$$\Delta x = -\epsilon\frac{\nabla E(x)}{\nabla^2 E(x)} \iff x^+ = x - \epsilon[\nabla^2 E(x)]^{-1}\nabla E(x), \qquad (3.36)$$
$$\text{for } \epsilon \to 0 : \dot{x} = -[\nabla^2 E(x)]^{-1}\nabla E(x).$$

Newton descent algorithms with $\epsilon = 1$ are known to converge quadratically to an equilibrium, which is faster than the exponential convergence of a usual gradient descent system. The ODE formulation in (3.36) is is reminiscent of (3.35).

e) Writing the embedding protocol system equation in component form we observe that for each $i \in N$,

$$\dot{x}_i = -d_i\frac{\nabla E(x_i)}{\nabla^2 E(x_i)} + \frac{1}{\nabla^2 E(x_i)}\sum_{j\in N_i} w_{ij}\nabla E(x_j).$$

Hence, the local, uncontrolled dynamics at each node $i$ indeed represents a Newton type descent scheme. However, convergence to a stationary local state is decelerated by the added external, neighborhood input $\frac{1}{\nabla^2 E(x_i)}\sum_{j\in N_i} w_{ij}\nabla E(x_j)$, which serves to keep trajectories confined to $\mathscr{M}_e$.

With trajectories of the embedding protocol gradient system (3.35) being constraint to evolve on some nonlinear space $\mathscr{M}_e$, the question arises of how this space looks like. The role of the Hessian $\mathbf{H}(\boldsymbol{x})$ in shaping this space becomes apparent from the geometric protocol description (3.29) cf. to (3.28). We shall discuss this topic in the next section.

### 3.6.3 Invariance and optimality properties

As presented in Section 3.2.3, for linear consensus systems the simplex $\mathscr{M}_I$ represents an invariance property of the linear ODE system. It originates from matrix properties of the Laplacian system matrix, where the components of $\boldsymbol{\pi}$ are such that $\boldsymbol{\pi} \cdot \mathbf{L} = \mathbf{0}^\top$. In the same manner, by describing an invariance property along trajectories, we can characterize the nonlinear space $\mathscr{M}_e$ as follows.

**Lemma 3.** *Consider the embedding protocol* (3.33)*, resp.* (3.35)*, on a time-invariant, irreducible graph with embedding function e, associated energy components E according to* (3.32)*, and metric* $d_e(a,b) = |e(a) - e(b)|$*. For any admissible initial condition* $\boldsymbol{z}$ *the configuration space is given as the set of vanishing signed distances*

$$\mathscr{M}_e = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^n \pi_i \mathrm{sgn}(x_i - z_i) d_e(x_i, z_i) = 0 \right\}.$$

*Along trajectories the quantity* $\sum_i \pi_i \nabla E(x_i) = \sum_i \pi_i \nabla E(z_i)$ *is invariant.*

*Proof.* Define the quantity $\sum_i \pi_i \nabla E(x_i) =: m(\boldsymbol{x})$ and consider the matrix $\mathbf{\Pi} = \mathrm{diag}\{\pi_1, \dots, \pi_n\}$; using the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}t} m(\boldsymbol{x}(t)) = \nabla m(\boldsymbol{x}(t)) \cdot \dot{\boldsymbol{x}} = \nabla^2 \mathscr{E}(\boldsymbol{x}) \cdot \mathbf{\Pi} \cdot \mathbf{H}^{-1} \mathbf{L} \boldsymbol{e}(\boldsymbol{x}) = \boldsymbol{\pi} \cdot \mathbf{L} \boldsymbol{e}(\boldsymbol{x}) = 0.$$

Here, we make use of the fact that diagonal components of $\mathbf{\Pi} \cdot \mathbf{H}^{-1}$ are elements $\pi_i / \nabla^2 E(x_i)$, so that multiplication with the vector of second derivatives of $\mathscr{E}$ yields the left-eigenvector $\boldsymbol{\pi}$ of the Laplacian matrix associated to the zero eigenvalue. Hence, the quantity $m$ is a system invariant along trajectories, so that

$$\sum_{i \in N} \pi_i \nabla E(x_i) = \sum_{i \in N} \pi_i \nabla E(z_i) = m \quad \Leftrightarrow \quad \sum_{i \in N} \pi_i (\nabla E(x_i) - \nabla E(z_i)) = 0.$$

Noting that $\mathrm{sgn}(x_i - z_i) d_e(x_i, z_i) = \nabla E(x_i) - \nabla E(z_i)$ we obtain

$$\sum_{i \in N} \pi_i (\nabla E(x_i) - \nabla E(z_i)) = \sum_{i \in N} \pi_i \mathrm{sgn}(x_i - z_i) d_e(x_i, e_i) = 0.$$

This completes the proof. □

This invariance property is formulated purely based on additive energy associated with the embedding function $e$. Associating the embedding function $e$ with the local coordinate transformation $f$ in Kolmogorov's functional requirement on means, the consensus and gradient results yield the following dynamic and constrained minimization formulation for general standards directly.

**Theorem 9** (Kolmogorov means: Dynamic and optimization characterization)**.** *Let* $\boldsymbol{x} \in \mathbb{R}^n$ *and consider positive reals* $w_1, w_2, \dots, w_n$*. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous monotonously increasing function that is either strictly convex or concave. The weighted Kolmogorov mean* $M_f^w(\boldsymbol{x})$ *is characterized as the value* $x^*$ *obtained as minimizer*

$$\begin{aligned} x^* \mathbf{1} = & \mathrm{argmin}_{\boldsymbol{y} \in \mathbb{R}^n} \mathscr{E}(\boldsymbol{y}), \qquad \mathscr{E}(\boldsymbol{y}) = \sum_{i=1}^n w_i E(y_i) \\ & \text{s.t.} \quad \sum_{i=1}^n w_i \, \mathrm{sgn}(x_i - y_i) d_f(x_i, y_i) = 0, \end{aligned} \tag{3.37}$$

*with $E$ such that $\nabla E = f$. Moreover, $x^*$ is also the consensus equilibrium obtained from the embedding protocol with $e = f$ that evolves on an irreducible, time-invariant digraph $\mathrm{G}$, with left eigenvector $\boldsymbol{\pi}$ corresponding to the zero eigenvalue of the associated Laplacian matrix $\boldsymbol{L}$, such that $\pi_i = w_i$.*

*Proof.* First, note that with $f$ being monotonously increasing, $f'$ is positive and monotonous, too, as it is assumed to be either strictly convex or concave. For the case of convexity $f'' > 0$, so that $f'$ is increasing and for concavity, $f'' < 0$ so that it $f'$ is monotonously decreasing.

Now, consider the Lagrangian function associated to the constrained optimization problem,

$$\mathscr{L}(\boldsymbol{y}, \lambda) = \mathscr{E}(\boldsymbol{y}) - \lambda \left( \sum_{i=1}^{n} w_i (f(x_i) - f(y_i)) \right).$$

Here we made use of the identity $\operatorname{sgn}(x_i - y_i) d_f(x_i, y_i) = f(x_i) - f(y_i)$, see Lemma 1. A solution of the optimization problem satisfies the first order optimality conditions

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathscr{L} = 0 \iff \sum_{i=1}^{n} w_i f(y_i) = \sum_{i=1}^{n} w_i f(x_i) \tag{3.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}y_i}\mathscr{L} = 0 \iff w_i f(y_i) + \lambda w_i f'(y_i) = 0, \quad i = 1, \ldots, n. \tag{3.39}$$

Rewriting (3.39), we get the requirement $\frac{f'(y_i)}{f(y_i)} = -\frac{1}{\lambda}$ for all $i = 1, \ldots, n$. As $f'$ is a monotonous function and so is $f$, the solution $y_i$ satisfying this equation is unique. Moreover, as the r.h.s. is identical for all indices $i$ the solution $y_1 = y_2 = \ldots = y_n$. Hence, the minimizer candidate reduces to a consensus state $y\mathbf{1}$.

Substituting this candidate state into (3.38), and defining $\bar{w} = \sum_{i=1}^{n} w_i$ yields the identity

$$\sum_{i=1}^{n} w_i f(y_i) = \bar{w} f(y) = \sum_{i=1}^{n} w_i f(x_i) \quad \Leftrightarrow \quad y = g\left( \frac{\sum_{i=1}^{n} w_i f(x_i)}{\bar{w}} \right) = M_f^w(\boldsymbol{x}).$$

This completes part one of the theorem.

To see that the weighted Kolmogorov mean is also the consensus state asymptotically reached by the embedding protocol system, note that from Corollary 2 we first know that a consensus equilibrium is achieved under the existing connectivity assumption. Further, with Laplacian matrix such that $\pi_i = w_i$, for all $i \in N$, the invariant set is exactly the constraint (3.37), according to Lemma 3. Hence, the consensus equilibrium must be the weighted Kolmogorov mean as specified. □

Remarkably, only strictly convex or concave local state transformations $f = e$ prove this result.

## 3.7 Summary and concluding remarks

In this chapter, we studied the relationship between Kolmogorov mean and metric functions and derived a framework for the design of consensus protocols that converge exponentially

fast to a consensus state. Two mean-driven classes of nonlinear consensus protocols are distinguished: a metric action protocol and a mean-control protocol. The metric action protocol is simple and generic. It subsumes well-known nonlinear consensus protocol classes by using the properties of distance functions only – a fundamental mathematical structure. The use of metrics and properties of these lets us formulate rules to build more complex consensus protocols from the composition of functions of simple type. The mean-driven protocol class is interesting, as it extends the optimality properties of the arithmetic mean as an asymptotic equilibrium state that minimizes a sum-of-squares cost function to optimality of infinitesimal state updates, hence rendering trajectories themselves optimal. The associated cost function that is minimized is a sum-of-squared-distances function that follows from the Kolmogorov-mean-metric relationship we established. Motivated by the structure of Kolmogorov means, which is a linear one up to local nonlinear coordinate transformations, we introduce the embedding protocol as a special case of the metric action protocol. We discuss geometric properties and especially the relation to the consensus on nonlinear space framework of Sepulchre and Sarlette. We show that the embedding function is naturally associated with a network energy function, in which the nonlinear embedding-consensus network dynamics evolve as gradient descent flow in a non-Euclidean metric. This gradient structure is reminiscent of a Newton algorithm. Using this dissipation view, we further prove invariance and asymptotic optimality properties that characterize Kolmogorov means also as equilibrium solution of a dynamic embedding-consensus network, or equivalently, as the solution of a constrained minimization problem with the network energy potential as cost-function and the non-Euclidean configuration space as minimization constraint.

Asymptotic optimality properties of the embedding protocol, its close relation to a distributed Newton schema, and the infinitesimal optimality of the mean-control consensus protocol motivate the further study of these systems, e.g., in the context of mechanism design and distributed optimal control as a protocol design task. The embedding protocol with its relation to consensus on nonlinear space and gradient descent schemes suggests the study of non-convex embedding functions or embeddings with complex functions, e.g., as is the case for the Kuramoto model. The gradient descent scheme we formulated for the embedding-consensus dynamics is a non-classical gradient system formulation as we use projected gradients. It would be interesting to understand the relation to the usual gradient system formulation and further exploit its characteristics and usefulness for the study, e.g., of inhomogeneous diffusion systems with state-dependent dominant left-eigenvector. This study of Laplacian systems allows the treatment of linear and non-linear systems in the same way. Moreover, it fuses different mathematical concepts in simple ways to analyze dynamical systems, e.g., gradient and Lyapunov systems, metric, and optimization schemes. It has a physical interpretation as passive electric circuits. The usefulness of Laplacian models raises the question to what extent and how linear and nonlinear ODE systems can be embedded into a Laplacian ODE system. While we usually assumed Lipschitz continuity, extensions and conditions on when a direct extension of the presented results to discontinuous settings are possible can be an interesting direction of further studies.

4

# Dissipation mechanisms and passive circuit structure of consensus networks

## 4.1 Introduction

Consensus networks, or diffusive systems on graphs, are omnipresent in network applications such as peer-to-peer sensing and gossiping [Boy+06], systems of particles and biologically motivated interaction [Vic+95], opinion and social evolution systems [Jia+15], image processing and data filtering [Tau95; TZG96], the electric power grid [DB12], or chemical reaction systems [vRJ13b], to name a few. The fundamental stability result for the prototypical linear time-varying consensus dynamics is due to Moreau in [Mor04] and [Mor05]. It shows a contraction property forward in time, making use of convexity, nested sets, and set-valued contraction measures. Since that work, a deeper understanding of contraction mechanics is sought by various authors, introducing novel tools to the study of consensus network convergence and stability, or highlighting "not-so-linear" behavior of the linear base system [Sep11] [Ols08]. In this chapter, we build from first principles a dissipative systems and passive circuit framework for the synthesis of a large class of consensus networks. We show equivalences to existing abstract, theoretical, and modeling frameworks used in the study of consensus dynamics.

The fundamental tool in studying convergence and stability of consensus systems is the span norm Lyapunov function $\max_i x_i - \min_i x_i$, where $x_i$ denotes state components of the $n$-dimensional consensus network system. It has been applied in the analysis of John Tsitsiklis in his thesis [Tsi84], see also [TBA86], where he studies consensus protocols in the context of distributed decision making and computation building on Markov chain theory. Earlier, and less known, is the work of Jan C. Willems [Wil76], where he provides a range of Lyapunov inequalities for diagonally dominant systems among which systems of diffusive type

are a particular case. He also highlights the same span norm function as Lyapunov function. Further, he works out the relationship between the convexity of Lyapunov functions, the Laplacian matrices as the generators of the transport maps and refers to connections with RC circuit systems. Moreau in [Mor04; Mor05] uses the same span-norm Lyapunov function showing convergence using set-valued inclusions and set-valued contraction measures. Moreau in [Mor04] also refers back to Willems's work [Wil76] noting that convergence in consensus systems may have close ties to passivity theory. Sepulchre, Sarlette, and Rouchon in [SSR10] recover the span norm Lyapunov function, however, in logarithmic coordinates. It represents a contracting Hilbert distance to the set of consensus states and is a result, among others, due to Birkhoff from as early as 1957 [Bir57].
The convergence and stability study focusing on spectral properties of the Laplacian matrix governing the dynamics has gained popularity with [JLM03; SM03a; OSM04]. Murray and Olfati-Saber in [OSM04] and [OSFM07a] made famous the use of quadratic Lyapunov functions with the multi-agent systems interpretation as collective and group disagreement that is being reduced along solutions to consensus. For the case of time-invariant and symmetric graphs, they exhibit a Euclidean gradient structure of the dynamics in the group disagreement serving as potential. Van der Schaft, in [van11], argues that not the group but the sum-of-squares collective disagreement is the appropriate potential in which consensus systems evolve as gradient flow. His argument is motivated by an electric circuit and port-Hamiltonian consideration. Egerstedt in [ME10] also proposes an LTI electric circuit structure for LTI symmetric consensus dynamics. For the case of non-symmetric but detailed balance weighting and in nonlinear situations, gradient and passive circuit structures are exposed in [MDM16]. A new technique in the analysis of consensus networks is proposed in [HT13] based on a property of the time-varying graph weights called cut-balance. While results do not generalize existing ones, the proof technique is novel. It indicates that sums of state components have a monotonous behavior over time, leading to convergence to equilibrium points, not necessarily consensus ones. This property highlights novel dissipation mechanics. It has led to the introduction of novel differential analysis concepts such as the so-called consensus dichotomy [PC17], building on boundedness and the contraction behavior of Laplacian flows. It also motivates the study of differential Lyapunov functions and novel incremental stability tools, see [FS14].

Since earliest studies of consensus networks, e.g., by Willems [Wil76], passivity and electric circuit ideas are every re-occurring but have never been rigorously applied to the context of consensus networks. In the following, we build a passive circuit setting suited for analyzing general nonlinear consensus problems. We comment on span-norm approaches and show equivalence to the cut-balance property by introducing a novel differential passivity concept motivated by a majorization property. The contribution of this chapter is as follows:

a) We provide a comprehensive dissipative and interconnected systems analysis of general, nonlinear consensus systems.

b) We connect classic and hardly noted structure results in network synthesis to modern problems in network systems.

c) A novel method called graph embedding is proposed to treat linear and nonlinear consensus protocols alike; we provide rigorous proofs for the resulting system properties.

d) We derive a realization of nonlinear consensus networks as an interconnection of novel nonlinear, passive capacitor, and resistor elements.

e) On that basis, we introduce a network gradient flow structure, which we prove irrespective of a particular ODE realization but purely based on duality and an energy-dissipation equality characterizing gradient system solutions.

f) We propose the novel concept of differential majorization and prove equivalences to the cut-balance property and nonlinear RC network systems realizations.

g) Application of the results yields passive circuit-based derivations of classical information inequalities and a novel dynamic electric circuit interpretation of Markov chain dynamics.

Starting with the introduction of the considered system class, we provide in Section 4.2 a background and basic definitions on dissipative and network systems, gradient systems, and introduce a tool that we use and call graph embedding. Then, in Section 4.3 we state and prove the gradient and passive circuit realization result for nonlinear consensus networks. Following that, in Section 4.4 we extend gradient and passivity results from the detailed balanced to a general irreducible dynamics case and provide equivalences to majorization and cut-balance properties. Before we end this chapter with concluding remarks, we apply the results to Markov chains.

## 4.2 Network system representation and dissipation results

We introduce a general class of nonlinear consensus protocols on weighted directed graphs that are strongly connected. We then provide background on dissipativity theory for open systems that we use to define (closed) network systems as a neutral interconnection of a passive memoryless and a passive lossless system with memory. This abstract lossless-dissipative decomposition is put into the context of structure results for the synthesis of nonlinear passive electric circuits, due to Hill, Moylan, and Anderson, which are motivated by the classical reactance extraction approach of Youla and Tissi in the LTI setting. A gradient systems definition is proposed based on trajectory data and a duality structure in which an energy-dissipation equality serves as a necessary and sufficient constitutive equation. We close this section by introducing an expansion method and its properties that we call graph embedding. It allows for shifting nonlinearity from nodes to edges and vice versa so that nonlinear consensus problems can be studied using techniques from linear consensus theory.

### 4.2.1 Consensus protocols on graphs

Let $\mathsf{G} = (N, B, w)$ be a weighted directed graph, where $N = \{1, 2, \dots, n\}$ is the set of nodes, $B = \{1, 2, \dots, b\} \subseteq N \times N$ denotes the set of branches, whose elements are ordered pairs $(j, i)$ denoting an edge from node $j$ to $i$, and $w : B \to \mathbb{R}_{>0}$ is a weighting function, such that $w((j,i)) =: w_{ij}$, if $(j,i) \in B$, else $w_{ij} = 0$. Define the in-neighborhood of a node $i$ as the set of connected nodes $N_i^+ := \{j \in N : (j,i) \in B\}$, $i \in N$. Associated to a graph is the graph Laplace matrix $\mathbf{L}$, defined component-wise as $[\mathbf{L}]_{ij} = -w_{ij}$, $[\mathbf{L}]_{ii} = \sum_{j \in N_i^+} w_{ij}$.

For strongly connected graphs, denote the dominant left-eigenvector associated to the unique zero eigenvalue of the Laplacian by $\boldsymbol{c}$, and define $\mathbf{C} := \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$. This vector has positive elements and is also called (left-) Perron vector of $\mathbf{L}$.

A graph, respectively the associated Laplacian matrix, is said to be balanced if it has zero (row and column) excess, i.e., $\sum_{j=1}^{n} w_{ij} = \sum_{j=1}^{n} w_{ji}$, for all $i \in N$, and it is symmetric if for all $(j,i) \in B$, $w_{ij} = w_{ji}$. In vector matrix notation this property is equivalent to $\mathbf{1}^\top \mathbf{L} = 0 \cdot \mathbf{1}^\top$ and $\mathbf{L1} = 0 \cdot \mathbf{1}$, i.e., the Perron vector and corresponding right-eigenvector associated to the dominant zero-eigenvalue are proportional to the vector of all ones, see, e.g., [OSM04] Theorem 6. We say a graph, respectively its Laplacian matrix is irreducible if the underlying graph is strongly connected. Then, the associated Laplacian matrix $\mathbf{L}$ has exactly one zero eigenvalue with Perron vector $\boldsymbol{c} > \mathbf{0}$, see, e.g., [OSM04] Theorem 1.

An important generalization of the symmetry condition on Laplacians that $\mathbf{L} = \mathbf{L}^\top$ is the particular type-symmetry that for some $\mathbf{C}$, and $i, j \in N$,

$$c_i w_{ij} = c_j w_{ji} \Leftrightarrow \mathbf{CL} = \mathbf{L}^\top \mathbf{C}. \tag{4.1}$$

Equation (4.1) is known in the literature on Markov chains as detailed balance, or as reversibility w.r.t. $\boldsymbol{c}$, cf., [Nor97] Chapter 2.

We consider the general class of consensus protocols on a graph G described componentwise by an ODE of the type

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij}\, \phi(x_j, x_i), \quad i \in N, \tag{4.2}$$

where $\phi(\cdot,\cdot)$ is Lipschitz continuous and sign-preserving in the sense that $(a-b)\phi(a,b) \geq 0$ for all $a, b \in \mathbb{R}$ with equality if and only if $a = b$.

*Remark* 16. The sign-preservingness property, as stated here, can also be expressed as $\phi(a,b) = \mathrm{sgn}(a-b)\varphi(a,b)$, with $\varphi$ being a pre-metric function, i.e., positive definite, with possible choice $\varphi = |\phi|$, as shown in Chapter 3,

**Example 10** (Examples for the consensus class with action function $\phi$)**.** The class (4.2) includes many known network models: The usual linear consensus system [OSFM07a] is obtained from setting $\phi(x_j, x_i) = x_j - x_i$. If $\phi(x_j, x_i) = f(x_j - x_i)(x_j - x_i)$, with the constraint $f(z) = f(-z) > 0$, then, the ODE (4.2) describes a continuous-time opinion dynamic [CFT12]. For instance, one may choose $f(z) = |\tanh(p \cdot z)|$, $p > 0$, which is a good choice for modeling saturation phenomena in the interaction. If $\phi(x_j, x_i) = \psi(x_j - x_i)$, where $\psi(-z) = -\psi(z)$, then we recover the nonlinear consensus class introduced by Olfati-Saber and Murray in [SM03a], with $\psi = \sin$ a prominent instance. Beyond the presented known interaction types, our model also includes couplings of the form $\phi(x_j, x_i) = g(x_j) - g(x_i)$, where $g$ is an increasing function, e.g., $\ln(x), e^x, x^p$, $p > 0$, on the respective domain of definition, or $\phi(x_j, x_i) = l(x_i) - l(x_j)$, where $l$ is a decreasing function, e.g., $x^p$, with $p < 0$. The latter interaction types cover a discrete version of an equation system that models the nonlinear diffusion of gas in porous media, see [EM14] (and [VÓ7] for the continuous context).

### 4.2.2 Network systems, internal passivity, and Lyapunov stability

We consider network systems as dynamical systems with constituent parts being open systems, which are interconnected according to a neutral interconnection rule, as laid out in the axiomatic framework of dissipative systems by Jan C. Willems in [Wil72], see also [PW97], and [Wil07].

To start with, an open system is a dynamical system governed in continuous time by an ODE of the type $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x})$, with state $\boldsymbol{x} \in \mathbb{R}^n$, which interacts with an environment through inputs $\boldsymbol{u} \in \mathbb{R}^r$ and outputs $\boldsymbol{y} \in \mathbb{R}^m$, such that the system embedded into an environment evolves such that

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(t=0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \end{cases} \iff \boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{f}\left(\boldsymbol{x}(\tau), \boldsymbol{u}(\tau)\right) \mathrm{d}\tau. \tag{4.3}$$

We emphasize the fact that all dynamical effects, i.e., those phenomena involving memory [Wil72], are described and attributed to the state, so that the output, or measurement, is required to be a memoryless map. Also, note that the state is an internal or latent system variable and therefore not observable from the outside; the system interacts with the outside only via the (manifest) input and output functions [PW97].

The notion of dissipativeness involves the concept of a supply (e.g., of energy) from the environment to the system, which is a function of the interaction variables over time; we denote the supply rate as a function $s : \mathbb{R}^r \times \mathbb{R}^m \to \mathbb{R}$ and assume it to be locally integrable, i.e., $\int_t^T |s(\tau)| \mathrm{d}\tau < \infty$, with $0 \leq t \leq T$. With that, the concept of dissipativity and passivity can be defined, according to [Wil72], as follows.

**Definition 5** (Dissipativity, losslessness and passivity )**.** A system $\Sigma$ is said to be dissipative with respect to the supply rate $s$ if for all times $t$ and $T$, $0 \leq t \leq T$, the dissipation inequality

$$E(\boldsymbol{x}(T)) - E(\boldsymbol{x}(t)) \leq \int_t^T s(\boldsymbol{u}(\tau), \boldsymbol{y}(\tau)) \mathrm{d}\tau \tag{4.4}$$

holds. The function $E : \mathbb{R}^n \to \mathbb{R}$ is called storage function. A system $\Sigma$ is said to be lossless if (4.4) holds with equality for all times. It is said to be passive, if the dissipation inequality (4.4) holds with supply rate given by a bilinear form.

We consider network system of type $\Sigma$ to be composed of smaller open subsystems $\Sigma_i, i \in N$ that are interconnected to each other. Hence, there is a distinction between inputs and outputs that are internal to the overall system $\Sigma$ and inputs and outputs with which $\Sigma$ interacts with the external environment. We refer to internal inputs and outputs associated to system $\Sigma_i$ by $\boldsymbol{u}^i_{\mathsf{int}}, \boldsymbol{y}^i_{\mathsf{int}}$, and external ones accordingly with subscript ext. Consequently, each subsystem $\Sigma_i$ formally is given a supply rate $s_i = s_i^{\mathsf{ext}} + s_i^{\mathsf{int}}$. Using these elements, we can define a network system as a neutral interconnection of open subsystems.

**Definition 6** (Neutral interconnection, network system, and internal passivity)**.** Consider a family of $n$ open systems $\Sigma_i$, $i \in N$. An interconnection of these $n$ systems is said to be neutral, if the internal supply rates additively nihilate, i.e., if $\sum_{i \in N} s_i^{\mathsf{int}} = 0$. A system $\Sigma$ is said to be a network system if it is composed of a family of open systems that are neutrally interconnected. It is said to be internally dissipative, resp. passive, if each subsystem is dissipative, resp. passive, w.r.t. their associated internal supply rate.

The interconnection constraint of neutral type does not introduce any new supply or dissipation to the network system. Dissipation properties of the network system hence are a consequence of dissipation properties of the constituent subsystems and might render the choice of storage functions even unique, e.g., when subsystems are lossless and supply rates given from physical considerations. In general, the additional constraint provided by the neutral interconnection structure constrains the set of possible storage functions, e.g., imposing the form of being additive. In addition, when external supply vanishes, the concept of internal dissipativity results in Lyapunov stability, as the following result shows.

**Theorem 10** (Dissipativity and Lyapunov stability for network systems)**.** *Consider a network system $\Sigma$ that is comprised of n neutrally interconnected subsystems. Assume that each subsystem $\Sigma_i, i \in N$ is dissipative with storage function $E_i$ and associated supply rate $s_i = s_i^{\text{ext}} + s_i^{\text{int}}$. Then, the network system is dissipative w.r.t. the supply rate $s(t) = \sum_{i\in N} s_i^{\text{ext}}(t)$ and with additive storage $E(t) = \sum_{i\in N} E_i(t)$. Moreover, suppose the external supply vanishes, so that $s_i^{\text{ext}} = 0$, for all $i \in N$, and let $\boldsymbol{x}^*$ be an equilibrium point of the autonomous network system. Then, the additive storage E is a Lyapunov function proving the stability of $\boldsymbol{x}^*$ in its neighborhood if E is continuous and attains a strong local minimum at this equilibrium point.*

*Proof.* Adding up local dissipation inequalities yields the network system's dissipation inequality

$$\sum_{i=1}^{n}[E_i(T)-E_i(t)] \leq \int_t^T \sum_{i=1}^{n} s_i^{\text{ext}}(\tau)+s_i^{\text{int}}(\tau)\mathrm{d}\tau$$
$$\Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} E_i(t) \leq \sum_{i=1}^{n} s_i^{\text{ext}}(t)+s_i^{\text{int}}(t) \underset{\text{interconnection}}{\overset{\text{neutral}}{=}} \sum_{i=1}^{n} s_i^{\text{ext}}(t).$$

Therefore, $E = \sum_{i=1}^{n} E_i$ is a storage function for $\Sigma$. The proof of the Lyapunov part is identical to the proof of [Wil72] Theorem 6. □

In differential form the dissipation inequality (4.4) becomes $\dot{E} \leq s$; if the system does not interact with an environment, i.e., it is isolated from it, then $\dot{E} \leq 0$ is the typical Lyapunov inequality required to prove the stability of an equilibrium state.

While a storage function and supply rate characterize dissipativeness as a systems property along trajectories, i.e., also and especially in situations away from equilibrium, Lyapunov functions are a means to characterize the stability of an equilibrium point, in the sense of convergence of a distance function from any state to the equilibrium state. The relationship in Theorem 10 between dissipativity and Lyapunov stability is interesting as it allows to identify equilibrium candidates from local minima of the storage function and provides a divide and conquer approach to stability analysis through analysis of dissipativity of simpler subsystems.

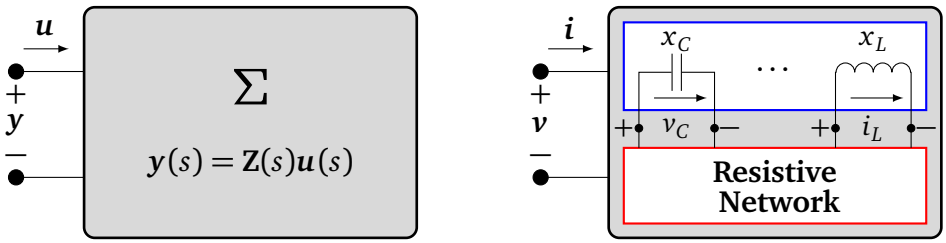
This setting, where dynamical systems are comprised of interconnected elementary subsystems, often arises in physical applications, e.g., in electric circuits or mechanical networks. There, the overall system is typically described using "local" states, and system energy is required to be additive. Various structure results on internal dissipation mechanisms are available for the case that the overall system can be decomposed into a set of lossless systems with memory that are interconnected with dissipative memoryless systems. In that case, the storage function is uniquely defined by the lossless subsystems. In contrast, the dissipative

part, being memoryless, does not contribute to stored energy, as its state space is the empty set [Wil72]. Such structure results are introduced in the following section.

### 4.2.3 Structure results of Youla-Tissi and Hill-Moylan-Anderson

Classical network synthesis is concerned with reproducing a prescribed, resp., an externally observed, manifest (linear) dynamical behavior of a black-box system, given by, say, a positive real impedance matrix function $\mathbf{Z}(s)$, (i.e., a transfer function matrix, where $s$ is the Laplace variable), in terms of a finite number of elementary, passive, linear, ideal (lumped) circuit elements and a scheme for interconnecting them, see, e.g., [AM75], [AV06]. This problem is illustrated in Figure 4.1a.

D.C. Youla and P. Tissi showed in [YT66] that a synthesized dynamic that solves this problem is structured as a (negative) feedback system, in which a dissipative and static (i.e., memoryless) network controls a lossless and passive MIMO system that is comprised of decoupled unit capacitors and inductors. The procedure to obtain this feedback representation is commonly referred to as reactance extraction, as the reactive, i.e., dynamical network elements are extracted from the composite system; this scheme is illustrated in Fig. 4.1b for the case of having only capacitor, inductor, and resistor elements at disposal that are interconnected by resistors as the only dissipative memoryless elements.




(a) Network synthesis problem

(b) Lossless-dissipative decomposition

Figure 4.1: LTI network synthesis and Youla-Tissi approach: Indexes $C, L$ denote capacitance and inductance, $x_{C/L}$ denotes capacitor charge and inductor magnetic flux, $v_C$ is capacitor voltage, $i_L$ inductor charge, and $\boldsymbol{i}, \boldsymbol{v}$ represents and input-output pairing of vector current and voltage.

**Example 11** (Linear RC system transfer function description)**.** We consider a two-port RC circuit comprising two capacitor elements having capacitance $C_1$ and $C_2$ — a standard circuit example, see, e.g., [CDK87] Chapter 7. The dynamic (output) variables are the capacitor voltages $v_{C,1}, v_{C,2}$. The external input, $\boldsymbol{u}$, is current and the output, $\boldsymbol{y}$, is voltage at the two capacitor ports. This setting corresponds to the scheme in Figure 4.1b by replacing the inductor with a second capacitor. The description of the memoryless interconnecting system for the chosen variable assignments is given by the set of algebraic network equations from

where the impedance (transfer function) behavior can be constructed, such that

$$\begin{pmatrix} y_1 \\ y_2 \\ \hline i_{C,1} \\ i_{C,2} \end{pmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & K_{11} & K_{12} \\ 0 & 1 & K_{21} & K_{22} \end{array}\right] \begin{pmatrix} u_1 \\ u_2 \\ \hline v_{C,1} \\ v_{C,2} \end{pmatrix} \Leftrightarrow \mathbf{Z}(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left( s\mathbf{I} + \begin{bmatrix} \frac{K_{11}}{C_1} & \frac{K_{12}}{C_1} \\ \frac{K_{21}}{C_2} & \frac{K_{22}}{C_2} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{C_2} \end{bmatrix}, \tag{4.5}$$

where the matrix $\mathbf{K}$ denotes the conductance matrix of the interconnecting resistor network. The equivalence in (4.5) is a consequence of Youla's and Tissi's classical result presented in [YT66], using Schur's complement formula together with the capacitor equation $sv_C = i_C$. Note that without a resistor network, i.e., $\mathbf{K} = \mathbf{0}$, the transfer function in (4.5) becomes the integrator equation $\mathbf{Z}(s) = \mathrm{diag}\{\frac{1}{sC_1}, \frac{1}{sC_2}\}$. That is, time-domain trajectories are straight lines with slope determined by $C_{1,2}$. The associated controlled ODE for voltage dynamics is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} v_{C,1}(t) \\ v_{C,2}(t) \end{pmatrix} = \begin{bmatrix} -\frac{K_{11}}{C_1} & -\frac{K_{12}}{C_1} \\ -\frac{K_{21}}{C_2} & -\frac{K_{22}}{C_2} \end{bmatrix} \begin{pmatrix} v_{C,1}(t) \\ v_{C,2}(t) \end{pmatrix} + \begin{bmatrix} -\frac{1}{C_1} & 0 \\ 0 & -\frac{1}{C_2} \end{bmatrix} \begin{pmatrix} i_{C,1}(t) \\ i_{C,2}(t) \end{pmatrix}. \tag{4.6}$$

The minus sign in the system matrices arises from noting that a response in capacitor current (the input of the dynamic part) results from exciting with a voltage source at the other port through the non-dynamic network, which is in a negative feedback connection. See [AV06] Chapter 3 for details on this sign convention.

Independent of the linearity or nonlinearity of a system, Hill, Moylan and Anderson in [AM75] [HM80], see also [Moy14] Chapter 10, proposed a structure result for the realization of a given passive system $\Sigma$ by a nonlinear circuit using a lossless-memoryless decomposition similar to the Youla-Tissi one for linear networks. In particular, in [HM80] Hill and Moylan show that a passive system governed by a set of equations (4.3) can be synthesized by the two passive systems

$$\Sigma_1 : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{u}_1(t) \\ \boldsymbol{y}_1 = \boldsymbol{h}_1(\boldsymbol{x}) \end{cases}, \quad \text{and} \quad \Sigma_2 : \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{y}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{h}(\boldsymbol{g}_2(\boldsymbol{u}_2), \boldsymbol{u}) \\ -\boldsymbol{f}(\boldsymbol{g}_2(\boldsymbol{u}_2), \boldsymbol{u}) \end{pmatrix}, \tag{4.7}$$

together with the neutral interconnection assignment of negative feedback type

$$\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \boldsymbol{u}_1^\top & \boldsymbol{u}_2^\top \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{pmatrix} = 0. \tag{4.8}$$

The authors further assume that the internal output and input maps $\boldsymbol{h}_1$ and $\boldsymbol{g}_2$ are such, that $\boldsymbol{h}_1\left[\boldsymbol{g}_2(\boldsymbol{u}_2\right] = \boldsymbol{u}_2$, and $\boldsymbol{g}_2[\boldsymbol{h}_1(\boldsymbol{x})] = \boldsymbol{x}$. With $\nabla E = \boldsymbol{h}_1$, they put invertibility of $\nabla E$ as an assumption.

*Remark* 17 (Moylan Conjecture). In [Moy14] Chapter 10 P. Moylan conjectures that convexity of $E$ is sufficient for this property of an energy gradient being an invertible function to hold.

The following result proves that the strict convexity of $E$ is necessary and sufficient for the isolated system to have a stable equilibrium point.

**Theorem 11.** *Consider a system $\Sigma$ that is composed of two passive subsystems as given in* (4.7) *with interconnection rule* (4.8)*. Then, $\Sigma$ is passive with storage function $E$, such that $\nabla E = \boldsymbol{h}_1$. In addition, assume $\Sigma$ has an equilibrium point. Then, $E$ is a Lyapunov function for vanishing inputs and outputs if and only if $E$ is strictly convex.*

*Proof.* As $E$ is a state function it can only be defined on system $\Sigma_1$. Passivity of this integrator system is equivalent to $\dot{E}(\boldsymbol{x}) = \nabla E(\boldsymbol{x}) \cdot \dot{x} \leq \boldsymbol{u}_1^\top \boldsymbol{y}_2 = \dot{\boldsymbol{x}}^\top \boldsymbol{h}_1$. This inequality can only hold with equality and with $\boldsymbol{h}_1(\boldsymbol{x}) = \nabla E(\boldsymbol{x})$. As system $\Sigma_2$ has state space the empty set, no storage can be defined on it. Hence $E$ is also the storage function of the network system $\Sigma$, which is passive according to Theorem 10, as the negative feedback interconnection is neutral and $\Sigma_2$ passive by assumption, (i.e. $0 \leq \boldsymbol{u}_2^\top \boldsymbol{y}_2$).

Next, we show that the decomposition into lossless and dissipative subsystems such that the overall system is Lyapunov stable is unique if and only if $E$ is strictly convex.
The composition is unique if $E$ is strictly convex, because then, $\nabla E$ is an increasing function, hence, it is monotonous and as such defining a one-to-one correspondence, so that for every $\boldsymbol{y}_1 = \nabla E(\boldsymbol{x})$, there is a unique inverse function $\boldsymbol{g}_2$, such that $\boldsymbol{g}_2(\boldsymbol{y}_1) = \nabla E(\boldsymbol{x})$. Therefore, the interconnection uniquely results in (4.3). The composition is unique only if $E$ is strictly convex, because $\boldsymbol{f}(\boldsymbol{g}_2(\boldsymbol{h}_1(\boldsymbol{x})), \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ requires $\boldsymbol{g}_2(\boldsymbol{h}_1(\boldsymbol{x})) = \boldsymbol{x}$ for all admissible $\boldsymbol{x}$. This implies that for output function $\boldsymbol{y}_1 = \boldsymbol{h}_1(\boldsymbol{x})$ there is a unique inverse function, and hence $\boldsymbol{h}_1$ must be a one-to-one map, i.e., bijective. This is the case if it is a monotonous function, i.e., either increasing or decreasing. With the identity $\boldsymbol{h}_1 = \nabla E$, and $\nabla E$ being either increasing or decreasing, we equivalently have a storage $E$ to be either strictly convex or concave. A strictly convex or concave function achieves a unique extremum point, either a minimum or a maximum. From Theorem 10 we know that $E$ also serves as a Lyapunov function around an equilibrium point if $E$ attains a minimum at this point. Hence, the second derivative of $E$ must be positive. This is the case if and only if $E$ is strictly convex. □

*Remark* 18. A particular construction of the inverse of energy gradients in which convexity plays a key role is given via Legendre transforms and the structure of duality in the sense of Young, as stated in Remark 19.

Together with Theorem 11, the Hill-Moylan-Anderson analogue of the Youla-Tissi LTI reactance extraction approach leads to an output feedback structure as illustrated in Fig. 4.2, for the case of nonlinear autonomous dynamical systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

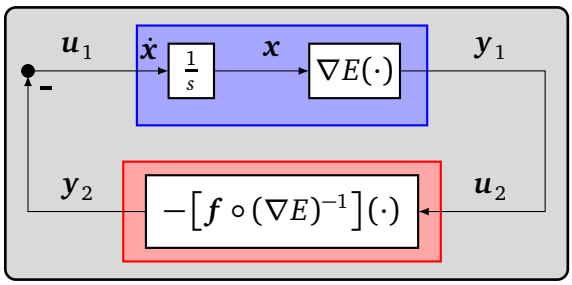



Figure 4.2: Hill-Moylan-Anderson decomposition

Stored energy is entirely defined on the lossless and dynamic integrator system part (blue), as the feedback system (red) is a memoryless mapping. While the integrator system is lossless and passive w.r.t. the supply rate $\boldsymbol{u}_1 \cdot \boldsymbol{y}_1$, dissipation of stored energy, and Lyapunov stability of an equilibrium solution of the closed-loop dynamics with Lyapunov function $E$, is entirely determined by the characteristics of $-\boldsymbol{f} \circ (\nabla E)^{-1}$.

**Example 12** (Hill-Moylan-Anderson decomposition for an RC dircuit)**.** Consider the RC circuit discussed in Example 11. The charge of a capacitor is defined as $q_i = C_i v_i$, so that $\boldsymbol{v} = \mathbf{C}^{-1}\boldsymbol{q}$. In each capacitor the electric energy stored is $E_{C,i} = \frac{1}{2}C_i v_{C,i}^2 = \frac{1}{2C_i}q_i^2$, so that $\nabla E_C(\boldsymbol{q}) = \boldsymbol{v}_C$. Choosing charge $q$ to be the state and voltage as capacitor output, the voltage dynamics (4.6) can be written as internal charge (state) dynamics

$$\dot{\boldsymbol{q}} = -\mathbf{K}\boldsymbol{u}_2 = -\mathbf{K}\nabla E_C(\mathbf{q}) = \mathbf{K}\cdot\mathsf{diag}\{C_1^{-1}, C_2^{-1}\}\boldsymbol{q} = \begin{bmatrix} -\frac{K_{11}}{C_1} & -\frac{K_{12}}{C_2} \\ -\frac{K_{21}}{G_1} & -\frac{K_{22}}{C_2} \end{bmatrix}\boldsymbol{q} \triangleq \boldsymbol{f}(\boldsymbol{q}),$$

which also confirms that the conductance matrix of the resistor network $\mathbf{K} \triangleq -\boldsymbol{f}\circ\nabla E_C^{-1}$, as $\nabla E_C^{-1}(\cdot) = \mathsf{diag}\{C_1, C_2\}$. That is, the static resistor network description is obtained from the system matrix by negation and extraction of capacitances.

### 4.2.4 Duality approach to gradient systems and their representation

In Euclidean space gradient systems have the well-known realization $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = -\nabla E(\boldsymbol{x})$, with scalar energy function $E$. Along trajectories generated by such a differential equation system, the dissipation rate is maximal, i.e., the velocity points into the direction of the steepest descent of the potential field defined by $E(\boldsymbol{x})$, and the magnitude of both the energy gradient and of the velocity need to be identical. To see this, we write for the dissipation rate

$$\frac{\mathrm{d}}{\mathrm{d}t}|E(\boldsymbol{x})| = |\nabla E(\boldsymbol{x})\cdot\dot{\boldsymbol{x}}| \leq ||\nabla E(\boldsymbol{x})||\cdot||\dot{\boldsymbol{x}}|| \leq \frac{1}{2}\left(||\nabla E(\boldsymbol{x})||^2 + ||\dot{\boldsymbol{x}}||^2\right). \tag{4.9}$$

The first inequality in (4.9) follows from the Cauchy-Schwarz inequality, saying that the scalar product of two vectors is smaller or equal to the product of the induced norms of the two vectors, with equality if and only if the two vectors are linearly dependent, i.e., they are aligned. The second inequality is a consequence of Young's inequality: For $a, b \in \mathbb{R}_{\geq 0}$, and $p, q \in \mathbb{R}_{>0}$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$a\cdot b \leq \frac{a^p}{p} + \frac{b^q}{q} \quad \text{and} \quad a\cdot b = \frac{a^p}{p} + \frac{b^q}{q} \iff a = b$$

Therefore, the assignment $a = ||\dot{\boldsymbol{x}}||$, $b = ||\nabla E(\boldsymbol{x})||$, and $p = q = 2$, yields the second inequality. Moreover, as gradient systems decrease $E$ at maximum rate, i.e., at maximum magnitude, the inequalities in (4.9) must hold with equality. This requirement is fulfilled if and only if $\nabla E(\boldsymbol{x})$ and $\dot{\boldsymbol{x}}$ are collinear (Cauchy-Schwarz) and satisfy the magnitude constraint $||\nabla E(\boldsymbol{x})|| = ||\dot{\boldsymbol{x}}||$ (Young).

To define gradient flows on general, possibly non-Euclidean spaces, using an energy dissipation equality motivated by (4.9), the concept of Legendre transformations and duality in the sense of Young are instrumental.

**Definition 7** (Legendre transform & Young duality, cf., [Arn89] Chapter 3.14)**.** Let the function $g : V \subset \mathbb{R}^n \to \mathbb{R}$ be strictly convex, with variable $\boldsymbol{p} \in V$. The Legendre transform of $g(\boldsymbol{p})$ is a new function $k$ in a variable $\boldsymbol{q}$,

$$k(\boldsymbol{q}) := \max_{\boldsymbol{p}\in V}[\boldsymbol{q}\cdot\boldsymbol{p} - g(\boldsymbol{p})].$$

Two functions are said to be dual in the sense of Young if they are the Legendre transforms of one another.

Hence, the Legendre transform of a convex function is the maximal deviation of that convex function from a linear function with a prescribed (constant) slope. A property of this transform, proven for instance in [Arn89] Chapter 3.14, is that it is involutive. The squared Legendre transform is the identity map, so that if $g$ is the Legendre transform of $k$, then $k$ is also the Legendre transform of $g$. Therefore, there is a point-wise one-to-one correspondence, i.e., a duality, between elements $k$ and $g$.

*Remark* 19 (Moylan's inversion conjecture and Legendre transforms). Moylan conjectured that convexity is sufficient for invertibility of energy gradients, see Remark 17. The Legendre transform and Young duality is the tool to identify such inverses. The variable $\boldsymbol{q}$ dual to $\boldsymbol{p}$ can be expressed in terms of the convex function $g(\boldsymbol{p})$: from the maximum condition we obtain $\nabla_p[\boldsymbol{q}\cdot\boldsymbol{p} - g(\boldsymbol{p})] = \boldsymbol{0} \Rightarrow \boldsymbol{q} = \nabla_p g(\boldsymbol{p})$. Conversely, from involutiveness we obtain $\boldsymbol{p} = \nabla_q k(\boldsymbol{q})$. Therefore, the gradient of a Legendre transform, $\nabla k(\boldsymbol{q})$, is in fact the inverse of the gradient of the original convex function $\nabla g$.

With this concept of Young duality, we can introduce a characterization of gradient flows based on trajectory data and a balance of energy with associated dissipation potentials only, which in fact may serve as a definition of gradient systems, see, e.g., [Ada+13] [Ons31].

**Proposition 5** (Energy-dissipation equality defining gradient flows)**.** *Let $\boldsymbol{x} : [0,\infty) \to \mathbb{R}^n$ be a Lipschitz continuous curve and assume that $E : \mathbb{R}^n \to \mathbb{R}$ is non-increasing along $\boldsymbol{x}(t)$ forward in time. Let $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a strictly convex function and denote by $k : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ its dual in the sense of Young, see Definition 7. If for any times $t, T \in [0,\infty)$, $t \leq T$, the energy-dissipation inequality*

$$E(\boldsymbol{x}(T)) + \int_t^T g(\dot{\boldsymbol{x}}(\tau)) + k(\nabla E(\boldsymbol{x}(\tau))\mathrm{d}\tau \leq E(\boldsymbol{x}(t)), \tag{4.10}$$

*holds with equality, then the curve $\boldsymbol{x}(t)$ defines a gradient flow of $E$, with primal and dual dissipation potentials $g$ and $k$.*

*Proof.* From the chain rule $\dot{E} = \nabla E \cdot \dot{\boldsymbol{x}}$ we have

$$E(\boldsymbol{x}(T)) + \int_t^T \nabla E(\boldsymbol{x}(\tau)) \cdot \dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau = E(\boldsymbol{x}(T)) \tag{4.11}$$

Subtraction of (4.11) from (4.10) yields

$$\int_t^T g(\dot{\boldsymbol{x}}(\tau) + k(\nabla E(\boldsymbol{x}(\tau)) - \nabla E(\boldsymbol{x}(\tau)) \cdot \dot{\boldsymbol{x}}(\tau)\mathrm{d}\tau \leq 0. \tag{4.12}$$

As both $g$ and $k$ are non-negative, and so is $-\nabla E \cdot \dot{\boldsymbol{x}}$ by hypothesis, (because $E$ is assumed to be non-increasing along the trajectory), this inequality (4.12) can only be true with equality, which is the case when

$$\dot{E}(\boldsymbol{x}(t)) = -[g(\dot{\boldsymbol{x}}(t)) + k(\nabla E(\boldsymbol{x}(t))],$$

or equivalently, if (4.10) holds. As $g$ and $k$ are dual in the sense of Young, they define a unique mapping from $\dot{\boldsymbol{x}}$ to $\nabla E(\boldsymbol{x})$ and vice versa, via the Legendre transform. Hence, from Remark 19, there is a gradient structure on the dual dissipation potential $\dot{\boldsymbol{x}} = \nabla_q k(\boldsymbol{q})$, $\boldsymbol{q} = \nabla E(\boldsymbol{x})$, that generates the state trajectory, or equivalently $\nabla_p g(\dot{\boldsymbol{x}}) = \nabla E(\boldsymbol{x})$, $\boldsymbol{p} = \dot{\boldsymbol{x}}$, with (non-Euclidean) geometry defined by the dissipation potential $g$. □

Besides the idea that this gradient system characterization according to Proposition 5 does not require a particular realization of an ODE system, but only data from a trajectory, a decreasing energy function $E$ and a dissipation function $g$, resp. its dual $k$, it also gives insights into dissipation properties: $g$ is required to be a strictly convex function on the domain of definition. It furthermore defines the geometry in which the system evolves as a gradient system. Last but not least, the definition via $E$, $g$, and $k$ defines gradient systems through a duality structure that constitutes energy and dissipation properties.

The choice of the dissipation potential and associated dual dissipation rate variables often has physical meaning; moreover, for the duality structure to work, the definition of the dissipation rate in terms of both $g$ and $k$ is essential, as the following example from passive circuits shows.

**Example 13** (Dual dissipation potentials in RC circuits)**.** Consider the dissipation function $\Psi(i,v) = \frac{1}{2}i^2R + \frac{1}{2}v^2K$, which represents the dissipated power across a resistor with resistance $R$, conductance the reciprocal value of resistance, $K = \frac{1}{R}$, and $v, i$ denoting voltage across and current through the resistor. Taking the Legendre transform of each quadratic term separately and adding them yields again the dissipation potential $\Psi(v,i) = \frac{1}{2}R\left(\frac{v}{R}\right)^2 + \frac{1}{2}K\left(\frac{i}{K}\right)^2$, and with the constitutive relation $v = Ri$, we get $\Psi(v,i) = \frac{1}{2R}v^2 + \frac{1}{2K}i^2 = \frac{1}{2}Kv^2 + \frac{1}{2}Ri^2$. Therefore, $\frac{1}{2}Ri^2$ is the Legendre transform of $\frac{1}{2}Kv^2$ and resistor voltage and current are dual variables defining dissipated power. Note that $\frac{1}{2}Ri^2 = \frac{1}{2}Kv^2$; however, if we define dissipated power as usual by the quadratic function without factor one half, $\Psi(i) = Ri^2$, then the Legendre transform is another function $\Psi'(v) = \frac{1}{4K}v^2 \neq Kv^2$, i.e., it does not correspond to dissipated power expressed in resistor voltage as the dual variable.

The classical representation of gradient systems evolving on a smooth manifold $M \subset \mathbb{R}^n$ is given by the ODE equivalence

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \iff \mathbf{G}(\boldsymbol{x})\dot{\boldsymbol{x}} = -\nabla E(\boldsymbol{x}) \iff \dot{\boldsymbol{x}} = -\mathbf{K}(\boldsymbol{x})\nabla E(\boldsymbol{x}), \tag{4.13}$$

where $\mathbf{G}(\boldsymbol{x})\mathbf{K}(\boldsymbol{x}) = \mathbf{K}(\boldsymbol{x})\mathbf{G}(\boldsymbol{x}) = \mathbf{I}$ for all $\boldsymbol{x} \in M$, and the symmetric positive semidefinite operator $\mathbf{G}(\boldsymbol{x}) : T_{\boldsymbol{x}}M \to T^*_{\boldsymbol{x}}M$ maps the tangent to the cotangent space of $M$ at $\boldsymbol{x}$. The dual matrix $\mathbf{K}(\boldsymbol{x})$ is the so-called Onsager operator, due to Onsager's work on symmetries in irreversible systems [Ons31]. We can identify $k(\nabla E) = \frac{1}{2}\nabla E \cdot \mathbf{K}\nabla E$, the dual dissipation potential, and the dissipation potential $g(\dot{\boldsymbol{x}}) = \frac{1}{2}\dot{\boldsymbol{x}} \cdot \mathbf{G}\dot{\boldsymbol{x}}$, so that the constitutive energy dissipation balance for the gradient system (4.13) becomes

$$\dot{E}(\boldsymbol{x}) = \nabla\dot{E}(\boldsymbol{x}) \cdot \dot{\boldsymbol{x}} = -\frac{1}{2}\left(\dot{\boldsymbol{x}} \cdot \mathbf{G}(\boldsymbol{x})\dot{\boldsymbol{x}} + \nabla E(\boldsymbol{x})\mathbf{K}(\boldsymbol{x})\nabla E(\boldsymbol{x})\right) = -\frac{1}{2}\left(||\dot{\boldsymbol{x}}||^2_{\mathbf{G}(\boldsymbol{x})} + ||\nabla E(\boldsymbol{x})||^2_{\mathbf{K}(\boldsymbol{x})}\right).$$

Any such triple $(M, \mathbf{K}, E)$, resp., $(M, \mathbf{G}, E)$ is said to be a classical gradient system.

### 4.2.5 Graph embedding: Slopes and equivalent linear consensus form

A key element in the derivation of a linear form from a nonlinear protocol on a graph turns out to be the concept of slopes, as (discrete) differentials on graphs, with a graph being a representation of a discrete space. In particular, given a scalar function $g$, note that we can

easily expand the local protocol ODE (4.2) such that for all elements $i \in N$,

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} \frac{\phi(x_j, x_i)}{g(x_j) - g(x_i)} \big[ g(x_j) - g(x_i) \big] = \sum_{j \in N_i^+} \tilde{w}_{ij}^g (x_j, x_i) \big[ g(x_j) - g(x_i) \big], \tag{4.14}$$

where the newly introduced nonlinear differences $g(x_j) - g(x_i)$ have been absorbed across edges in newly defined state-dependent edge weightings $\tilde{w}_{ij}^g$ and $g(x_j) - g(x_i)$ represents the new interaction function. In particular, if $g = \mathrm{id}$, the protocol equation (4.14) written in vector-matrix form has a linear consensus system representation on a dynamic, state-dependent graph. To show this, we need to assure that the newly introduced edge weights are positive and define for any given state a Laplacian matrix. As such, properties of the local slopes given by the fraction in (4.14) determine the stability and convergence properties of the consensus system.

**Lemma 4** (Positivity and symmetry of slopes)**.** *Let $I \subset \mathbb{R}$ be an interval, consider the sign-preserving interaction function $\phi : I \times I \to \mathbb{R}$, and $g : I \to \mathbb{R}$, both being Lipschitz continuous. The local slope $\frac{\phi(a,b)}{g(a)-g(b)}$ is positive definite for all $a, b \in I$ if and only if $g$ is monotonously increasing. Moreover, the local slope $\frac{\phi(a,b)}{g(a)-g(b)}$ is symmetric if and only if $\phi$ is anti-symmetric.*

*Proof.* Sign-preservingness of $\phi$ allows us to equivalently use $\phi(a,b) = \mathsf{sgn}(a-b)\varphi(a,b)$, see Remark 16, where $\varphi$ is a pre-metric function, i.e., it is positive definite on $I$. We first prove the positivity property and start with the case $a \neq b$ and then show that for $a = b$, the slopes are well-defined, in the sense that they are positive and finite.

(Sufficiency): If $g$ is increasing, then for any two distinct $a, b \in I$, the signed quotient

$$\mathsf{sgn}(a-b)\frac{\varphi(a,b)}{g(a)-g(b)} = \frac{\mathsf{sgn}(a-b)\varphi(a,b)}{\mathsf{sgn}(a-b)|g(a)-g(b)|} = \frac{\varphi(a,b)}{|g(a)-g(b)|} > 0,$$

as $\varphi$ is positive definite, and so is the norm $|g(a)-g(b)|$.
(Necessity): If $g$ were not (monotonously) increasing, but also decreasing on some subinterval on $I$, then, for $a > b$ such that $g(a) < g(b)$ we would have

$$\mathsf{sgn}(a-b)\frac{\varphi(a,b)}{g(a)-g(b)} = \frac{\varphi(a,b)}{-|g(a)-g(b)|} < 0.$$

Hence, $g$ cannot be decreasing.
Conversely, if $g$ is constant on some interval on the real line, then $g(a)-g(b) = 0$, rendering the quotient indefinite, hence not positive definite.

To see that the local slope is well-defined for approaching arguments let us expand such that

$$\lim_{a \to b} \frac{\phi(a,b)}{g(a)-g(b)} = \lim_{a \to b} \frac{\phi(a,b)}{a-b} \frac{a-b}{g(a)-g(b)} < \infty,$$

where we make use of Lipschitz continuity of both $\phi$ and $g$, so that both fractions remain finite in the limit $a \to b$.

Regarding symmetry, note that the function $d_g := |g(a)-g(b)|$ is a metric, as for instance shown in Lemma 1 in Chapter 3, i.e., $d_g$ is positive definite, subadditive and symmetric. Hence, $\mathsf{sgn}(a-b)\frac{\varphi(a,b)}{g(a)-g(b)} = \frac{\mathsf{sgn}(a-b)\varphi(a,b)}{\mathsf{sgn}(a-b)d_g(a,b)} = \frac{\varphi(a,b)}{d_g(a,b)}$ is symmetric if and only if $\varphi(a,b)$ is symmetric, which equivalently means $\phi$ is anti-symmetric. □

This process in (4.14) of defining new edge weights via embedding of local slopes that arise from the expansion with (nonlinear) differences yields a state-dependent Laplace matrix and a linear consensus system representation as the following result shows.

**Theorem 12.** *Let* G *be an irreducible weighted digraph. Consider a consensus system with interaction nonlinearity* $\phi$ *as defined for the protocol equation* (4.2)*. Let* $g : \mathbb{R} \to \mathbb{R}$ *be Lipschitz continuous and strictly increasing. Then, the matrix*

$$[\boldsymbol{L}_g^{\phi}(\boldsymbol{x})]_{ij} := \begin{cases} -\tilde{w}_{ij}^{g}(x_i, x_j) = -w_{ij}\frac{\phi(x_j,x_i)}{g(x_j)-g(x_i)}, & \text{if} \quad (j,i) \in B, \\ 0, & \text{if} \quad (j,i) \notin B, \\ \sum_{j\in N_i^+} \tilde{w}_{ij}^{g}(x_i, x_j), & \text{if} \quad j = i \in N. \end{cases} \tag{4.15}$$

*is an irreducible Laplace matrix, for all admissible* $x_i, x_j \in \mathbb{R}$*. Moreover, the nonlinear consensus system governed by* (4.2) *has the equivalent linear consensus representation*

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}_{\mathrm{id}}^{\phi}(\boldsymbol{x}(t))\boldsymbol{x}(t) \iff \dot{x}_i = \sum_{j:(j,i)\in B} \tilde{w}_{ij}^{\mathrm{id}}(t)(x_j - x_i), \quad i \in N,$$

*where the weights* $\tilde{w}_{ij}^{\mathrm{id}} = w_{ij}\frac{\phi(x_j,x_i)}{x_j-x_i}$.

*Proof.* Note that $g = \mathrm{id}$, so that $y = g(x) = x$ for all admissible $x \in \mathbb{R}$ is an increasing function and Lipschitz continuous. It remains to prove that (4.15) indeed defines an irreducible Laplace matrix, i.e., the weights $\tilde{w}_{ij}^{g}(x_i, x_j)$ need to be positive for all arguments. With the increasingness assumption on $g$ and using Lemma 4 we have $\tilde{w}_{ij}^{g}(x_i, x_j) > 0$ and the system is also well-defined when it reaches a consensus state $\boldsymbol{x} = c1, c \in \mathbb{R}$, where all state components equal. □

The graph embedding approach yields a linear form for the nonlinear protocol (4.2) on a graph, without using the process of linearization as localizing the dynamics around a particular state. This allows us to use results derived in consensus protocols as linear time-varying systems in the context of nonlinear consensus protocols. In the graph-embedding approach, the character of varying weights over time is hence endogenously determined via the local slope factors, in addition to potentially time-varying weights given by factors $w_{ij}(t)$, where the dependence on time is explicit.

For later use in proving dissipation properties, we now introduce the particular class of slopes given by divided differences of the type $\frac{f(a)-f(b)}{a-b}$. These can be seen as a discrete version of a local gradient $f'(a)$. Further, functions $f$ representing a local nonlinear state transformation play an important role for nonlinear consensus dynamics, when $f$ is strictly convex or concave and monotonously increasing, as seen in Chapter 3. In that case, they are associated with an additive convex Lyapunov function and define certain optimality properties of the network system. Strict convexity (or, resp., concavity) of $f$ is equivalent to a monotonicity property of the divided difference, as the following result shows.

**Lemma 5** (Monotonicity of slopes as divided differences)**.** *Let* $I \subset \mathbb{R}$ *be an interval and the function* $f : I \to \mathbb{R}$ *be differentiable. The slope* $\frac{f(x_1)-f(x_2)}{x_1-x_2}$ *is increasing in both arguments if and only if* $f$ *is strictly convex.*

*Proof.* Consider points from a real interval, $x_1, x, x_2 \in I$ such that $x_1 \leq x \leq x_2$. Then, $x$ can be written as the convex interpolation $x = ax_1 + (1-a)x_2$, with parameter $a \in [0,1]$. By definition of convexity via Jensen's inequality,

$$\begin{aligned} f(ax_1 + (1-a)x_2) = f(x) &\leq (1-a)f(x_2) + af(x_1) \\ \Leftrightarrow -af(x_1) &\leq (1-a)f(x_2) - f(x) \\ \Leftrightarrow a[f(x) - f(x_1)] &\leq (1-a)[f(x_2) - f(x)], \end{aligned} \tag{4.16}$$

with strict convexity if strict inequality holds. Multiplication of (4.16) with $(x_2 - x_1)$ yields

$$a(x_2 - x_1)(f(x) - f(x_1)) \leq (1-a)(x_2 - x_1)(f(x_2) - f(x)). \tag{4.17}$$

The expression for $x$ as convex interpolation can be restated as $(1-a)(x_2 - x_1) = (x - x_1)$ and equivalently $a(x_2 - x_1) = (x_2 - x)$, so that (4.17) becomes

$$(x_2 - x)(f(x) - f(x_1)) \leq (f(x_2) - f(x))(x - x_1). \tag{4.18}$$

Division of both sides in (4.18) with $(x_2 - x)(x - x_1)$ yields

$$\frac{f(x) - f(x_1)}{x - x_1} = \frac{f(x_1) - f(x)}{x_1 - x} \leq \frac{f(x_2) - f(x)}{x_2 - x}, \tag{4.19}$$

where equality follows from symmetry of the slope in both arguments, i.e., it is an even function. Inequality (4.19) shows that the slope is a monotonous function in the first argument, and it is increasing with increasing first argument, as $x_2 > x_1$, for strictly convex $f$. Due to symmetry (the slope is an even function), this monotonicity property is also true in the second argument. Conversely, if the slope has this monotonicity property in both arguments, then (4.19) together with symmetry in both arguments implies strict convexity of $f$. □

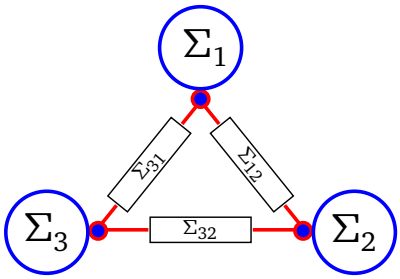
*Remark* 20. The result in Lemma 5 can easily be formulated and restated for the case of strict concavity: The divided difference is decreasing in both arguments if and only if $f$ is strictly concave.

## 4.3 Gradient and passive circuit structure

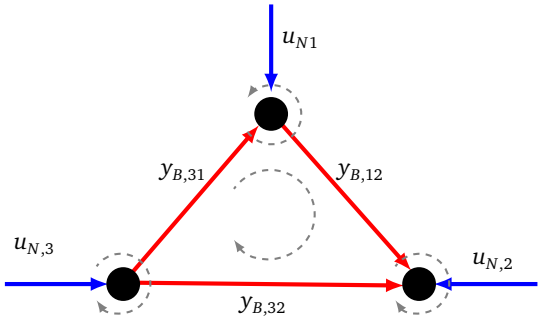
In the following, we synthesize a nonlinear consensus system governed by the protocol equation (4.2) on an irreducible graph that satisfies the detailed balance condition (4.1) using passive nonlinear resistor and capacitor elements. The resulting nonlinear RC circuit evolves as a gradient flow of energy stored in capacitor elements, which is dissipated across passive resistor elements. Internal, local passivity of the dissipative network, i.e., of each resistor, turns out equivalent to strict convexity of stored energy in capacitor elements. This is a specific realization of the Hill-Moylan-Anderson structure result for nonlinear networks. A discussion on aspects of duality follows. The results are applied in the example of phase-coupled oscillator networks with application to electric power networks.

### 4.3.1 Passive circuit synthesis of consensus gradient systems

The passive circuit realization of the nonlinear consensus network is schematically depicted in Fig. 4.3a. Node systems $\Sigma_i$, where $i \in N$, are abstract single-input-single-output, nonlinear capacitor elements. They are interconnected by branch systems $\Sigma_{ij}$, where $(j,i) \in B$, which model single-input-single-output, nonlinear resistor elements. Figure 4.3b illustrates the Kirchhoff current-balance at RC-circuit elements and in the network: The black dots are the internal network terminals, at which current in- and outflows balance. These internal terminals are represented by nodes $i \in N$ of the graph $\mathsf{G}$. The blue arrows represent current flows across capacitors; the red arrows are current flows across resistors that interconnect the internal (voltage) terminals $i \in N$.




(a) Lossless node systems (blue) and memoryless edge systems (red)




(b) Current flows to and from internal terminals (black interconnection points)

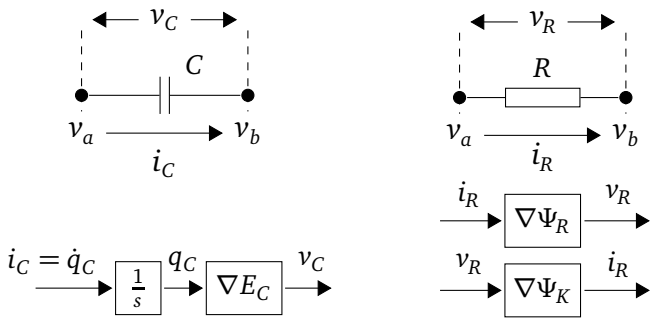
Figure 4.3: Schematic of the synthetic structure of a consensus network realized from passive circuit elements

Every circuit element is required to have two terminals. Resistor elements $(j,i) \in B$ clearly connect pairwise terminals $i, j \in N$. These graph nodes $i \in N$ also represent one terminal of each capacitor element. The second capacitor terminal (outside of the graph, connecting to an abstract environment) is assumed to be a ground terminal with voltage equal to zero. Capacitor and resistor primitives, together with their constitutive signal graph representation, are depicted in Fig. 4.4.

For each capacitor with capacitance $c_i > 0$, let $E : \mathbb{R} \to \mathbb{R}$ be a storage function with gradient denoted $h$. The node and edge open dynamic system equations for capacitors and resistors are therefore given as

$$\Sigma_{i\in N} : \begin{cases} \dot{q}_i(t) = u_i(t) \\ y_i(t) = \nabla E(c_i^{-1} q_i(t)) \end{cases} \qquad \Sigma_{(j,i)\in B} : y_{(j,i)} = k_{ij}\, u_{(j,i)}, \quad k_{ij} := c_i w_{ij} \frac{\phi(x_j, x_i)}{h(x_j) - h(x_i)}, \tag{4.20}$$

with $x_i = c_i^{-1} q_i$, for all $i \in N$. In view of Fig. 4.4, the abstract inputs $u_i$ in (4.20) at nodal capacitor elements are capacitor currents, and the associated outputs $y_i$ are the capacitor voltages across each capacitor element. With ground terminal potentials equal to zero, the capacitor outputs $y_i = v_{C,i}$ are the internal terminal voltages at nodes $i \in N$. For resistor systems, $y_{(j,i)}$ are resistor currents $i_{R,(j,i)}$ through resistor elements at branches $(j,i) \in B$, and with $u_{(j,i)} = v_{R,(j,i)}$ the resistor voltages across resistors, the parameter $k_{ij}$ in (4.20) represents




(a) Capacitor primitive (b) Resistor primitive

Figure 4.4: Functional relations between input, output, state, energy, and dissipation in RC-electric circuit elements: $E_C$ denotes electric energy, $q_c$ is the capacitor charge, i.e., the state variable, $i$ and $v$ are current, respectively voltage variables at capacitor and resistor, and $\Psi_{R,G}$ denotes a dissipation potential specified by either resistance $R$ or conductance $K$.

the resistor conductance, so that $k_{ij}^{-1}$ would be the associated resistance, mapping a current input to a voltage output at the respective resistor element.

Let us turn to the interconnection of systems $\Sigma_i$ with the set of branch systems $\Sigma_{(j,i)\in B}$. For that, first we denote with the index $e := (j,i) \in B$ a branch element, and introduce the vector of branch inputs (i.e., voltages) $\boldsymbol{u}_B^\top := (u_1, \ldots, u_b)$ and outputs (i.e., currents) $\boldsymbol{y}_B^\top := (y_1, \ldots, y_b)$ at resistors $e \in B$. In the same manner, we denote by $\boldsymbol{y}_N^\top = (y_1, \ldots, y_n)$ the vector of capacitor outputs (i.e., voltages) at, and by $\boldsymbol{u}_N^\top = (u_1, \ldots, u_n)$ the vector of capacitor currents into the terminal nodes $i \in N$, compare to Fig. 4.3b.

For the algebraic interconnection structure we define $\mathbf{B}(B) \in \mathbb{R}^{b\times n}$ the node-to-edge incidence matrix mapping nodal outputs to output differences across edges, so that, e.g., $y_e := y_i - y_j$, $e = (i,j) \in B$. This matrix is defined by the components

$$m_{ej} := \begin{cases} 1, & \text{if the edge } e \in B \text{ enters vertex } i \in N, \\ -1, & \text{if the edge } e \in B \text{ leaves node } i \in N, \\ 0, & \text{if the edge } e \in B \text{ does not touch a node } i \in N\ . \end{cases}$$

With these definitions, $B$ specifies the interconnection such that

$$\boldsymbol{u}_B = -\mathbf{B}\boldsymbol{y}_N, \qquad \text{and} \qquad \boldsymbol{u}_N = \mathbf{B}^\top \boldsymbol{y}_B. \tag{4.21}$$

The first identity in (4.21) corresponds to Kirchhoff's voltage law (KVL) $\boldsymbol{u}_B + \mathbf{B}\boldsymbol{y}_N = \mathbf{0}$, i.e., the sum of terminal voltage potential differences around any closed circle is zero. The second identity in (4.21) corresponds to Kirchhoff's current law (KCL) in a resistor network, $\boldsymbol{u}_N - \mathbf{B}^\top \boldsymbol{y}_B = \mathbf{0}$, i.e., currents flowing through resistors out of a node $i \in N$ balance at this node with the capacitor current flowing into it. The voltage and current balance laws of Kirchhoff are illustrated by the grey circles in Fig. 4.3b.

To complete the set of circuit laws, we add Ohm's law that constitutes the edges systems $\Sigma_{e\in B}$, as defined in (4.20). Define the diagonal matrix $\mathbf{W}_K \in \mathbb{R}^{b\times b}$ collecting conductances $k_e$, with $e \in B$, as $\mathbf{W}_K = \mathrm{diag}\{k_1, \ldots, k_b\}$. With that, $\boldsymbol{y}_B = \mathbf{W}_K \mathbf{u}_B$, Kirchhoff's and Ohm's laws

completely specify the memoryless edge system by the input-output relations

$$\boldsymbol{y}_N \overset{\text{(KVL)}}{\longleftrightarrow} \boldsymbol{u}_B, \qquad \boldsymbol{u}_B \overset{\text{(Ohm)}}{\longleftrightarrow} \boldsymbol{y}_B, \qquad \boldsymbol{y}_B \overset{\text{(KCL)}}{\longleftrightarrow} \boldsymbol{u}_N.$$

Together with the constitutive equations for the $n$ lossless and passive systems $\Sigma_i$, $i \in N$, see (4.20), Kirchhoff and Ohm's law can be written compactly as the system of equations

$$\begin{pmatrix} \mathbf{W}_k^{-1}(\boldsymbol{q}) & \mathbf{B}(B) \\ \mathbf{B}^\top(B) & \mathbf{0} \end{pmatrix} \begin{pmatrix} -\boldsymbol{y}_B \\ \boldsymbol{y}_N \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \boldsymbol{u}_N \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{W}_k^{-1}(\boldsymbol{q}) & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{0} \end{pmatrix} \begin{pmatrix} -\boldsymbol{y}_B \\ \nabla E(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \dot{\boldsymbol{x}} \end{pmatrix}. \tag{4.22}$$

In (4.22), the coefficient matrix containing the graph's incidence structure and the inverse of the diagonal edge weight matrix is a KKT (Karush-Kuhn-Tucker) matrix, so that the Kirchhoff-Ohm network synthesis equations form a saddle-point or KKT system, for reference, see, e.g., [BV09] Chapter 10, or [Str10] Chapter 2. From this observation, we can derive the gradient structure result as follows.

**Theorem 13** (Gradient structure and internal passivity equivalence)**.** *Let* $\mathsf{G} = (N, B, w)$ *be a strongly connected, weighted digraph with edge weights satisfying detailed balance* (4.1) *and consider the nonlinear consensus protocol*

$$\dot{x}_i = \sum_{j \in N_i^+} w_{ij} \phi(x_j, x_i), \quad i \in N, \tag{4.23}$$

*with action function* $\phi$ *as given in* (4.2)*. We further assume* $\phi$ *to be anti-symmetric, introduce the variable* $q_i = c_i x_i$*,* $c_i$ *the Perron vector components for all* $i \in N$ *and* $r_e = \frac{1}{k_e}$ *for all* $e \in B$*. Consider the sum-separable functions*

$$E(\boldsymbol{q}) = \sum_{i \in N} c_i H(c_i^{-1} q_i), \qquad \Psi(\boldsymbol{y}_B) = \frac{1}{2} \sum_{e \in B} r_e y_e^2 \tag{4.24}$$

*where* $H : \mathbb{R} \to \mathbb{R}$ *is any differentiable function with Lipschitz continuous derivative. If H, or equivalently E, is strictly convex, then the consensus network evolves as gradient flow of E such that for all times* $t \leq T$*,*

$$E(T) - E(t) = \frac{1}{2} \int_t^T \Psi(\tau) + \Psi^\star(\tau) \mathrm{d}\tau, \tag{4.25}$$

*with dissipation potential* $\Psi^\star$ *being dual in the sense of Young to* $\Psi$*, according to Definition 7. Moreover, the system defined by* (4.23) *is internally, locally passive, i.e., each resistance* $r_e > 0$ *defining the dissipation potential* $\Psi$*, if and only if E is strictly convex. Then, the consensus network has the representation*

$$\dot{\boldsymbol{q}} = -\boldsymbol{K}(\boldsymbol{q}) \nabla E(\boldsymbol{q}), \tag{4.26}$$

*with* $\boldsymbol{K} = \boldsymbol{C}\boldsymbol{L}_h^\phi$*, and* $\boldsymbol{L}_h^\phi$ *as defined in Theorem 12, being an irreducible, symmetric graph Laplace matrix.*

*Proof.* According to Proposition 5 gradient flows are characterized as ODE systems with trajectories satisfying the energy-dissipation equality involving dual dissipation potentials.

Note that the dissipation potential given in (4.24) can be written with internal input and output vectors as

$$\Psi(\boldsymbol{y}_B) = \frac{1}{2}\sum_{e\in B} r_e y_e^2 = \frac{1}{2}\boldsymbol{y}_B^\top \mathbf{W}_k^{-1}(\boldsymbol{\rho})\boldsymbol{y}_B.$$

Now let us show that the dual dissipation potential $\Psi^\star$ satisfies $\Psi(\boldsymbol{y}_B) = \Psi^\star(\boldsymbol{u}_B)$ and $\boldsymbol{u}_B$ being the dual variable associated with $\boldsymbol{y}_B$. According to Definition 7 the identity

$$\Psi^\star(\boldsymbol{u}_B) = \max_{\boldsymbol{y}_B}\left(\boldsymbol{u}_B^\top \boldsymbol{y}_B - \Psi(\boldsymbol{y}_B)\right)$$

must hold. Equivalently, $\frac{\partial}{\partial y_e}\left(\boldsymbol{u}_B^\top \boldsymbol{y}_B - \Psi(\boldsymbol{y}_B)\right) = u_e - r_e y_e = 0$ must hold for each $e \in B$. Hence, we have for the dual variable components $u_e = r_e y_e$. Substituting the correspondence $y_e = \frac{u_e}{r_e}$ into the equation of $\Psi$, we obtain the dual as

$$\Psi(\boldsymbol{y}_B) = \frac{1}{2}\sum_{e\in B} r_e y_e^2 = \frac{1}{2}\sum_{e\in B} r_e\left(\frac{u_e}{r_2}\right)^2 = \frac{1}{2}\sum_{e\in B} k_e u_e^2 = \Psi^\star(\boldsymbol{u}_B).$$

Now, let us show that the dual dissipation potentials are strictly convex, as required in Proposition 5. For that to be true, $r_e > 0$, respectively $k_e > 0$, must be true. We observe that by the chain rule with $c_i^{-1} q_i = x_i$ we get for energy gradients

$$\frac{\partial}{\partial q_i} E(\boldsymbol{q}) = c_i \frac{\partial H(x_i)}{\partial x_i}\frac{\partial x_i}{\partial q_i} = c_i h(x_i) c_i^{-1} = h(x_i).$$

The components $r_{e=(j,i)} = \frac{1}{c_i w_e}\frac{h(x_j)-h(x_i)}{\phi(x_j,x_i)}$ are positive if and only if $E$ is strictly convex, as shown in Lemma 4 together with the fact that both $c_i$ and $w_{ij}$ are positive. That is, each resistor at $e \in B$ with resistance $r_e$, or conductance $k_e$, is passive, i.e., $r_e > 0$, if and only if $E$ is strictly convex. This shows strict convexity and proves the local, internal passivity property.

It remains to show that $\dot{E}(t) = \frac{1}{2}(\Psi(t) + \Psi^\star(t))$ and that $\Psi$ is a function of the consensus system velocity $\dot{\boldsymbol{x}}$ and its dual a function of energy gradients $\nabla E(\boldsymbol{q})$.
Kirchhoff's voltage law (KVL) specifies $\boldsymbol{u}_B = \mathbf{B}\boldsymbol{y}_N$ and $\boldsymbol{y}_N = \nabla E(\boldsymbol{q})$ according to the capacitor equation (4.20). As $\mathbf{B}$ is a matrix with components $1, -1, 0$ only, $\Psi^\star(\boldsymbol{u}_B) = \Psi^\star(\nabla E(\boldsymbol{q}))$. The same argument applies with Krichhoff's current law (KCL), so that $\Psi(\boldsymbol{y}_B) = \Psi(\dot{\boldsymbol{x}})$.

With that, we can write Young's inequality, cf. [BV09] Chapter 2,

$$\Psi(\dot{\boldsymbol{x}})) + \Psi^\star(\nabla E(\boldsymbol{q})) \geq \dot{\boldsymbol{x}} \cdot \nabla E(\boldsymbol{q})$$

which holds with equality if and only if $\Psi$ and $\Psi^\star$ are dual in the sense of Young. This, we have already shown and hence, we have proven the gradient flow property for the consensus system.

Finally, let us show that the representation (4.26) does indeed hold. The protocol dynamics (4.23) can be written equivalently for every $i \in N$ as

$$\frac{1}{c_i}c_i\dot{x}_i = \sum_{j:(j,i)\in B} w_{ij}\phi(x_j,x_i) \iff \dot{q}_i = \sum_{j:(j,i)\in B} c_i w_{ij}\phi(x_j,x_i).$$

Expanding this equation by $h(x_j) - h(x_i)$ yields the $i$-th component ODE and vector-matrix system equation

$$\dot{q}_i = \sum_{j:(j,i)\in B} c_i w_{ij}\frac{\phi(x_j,x_i)}{h(x_j)-h(x_i)}\left(\frac{\partial}{\partial q_j}E(\boldsymbol{q}) - \frac{\partial}{\partial q_i}E(\boldsymbol{q})\right) \iff \dot{\boldsymbol{q}} = -\mathbf{C}\mathbf{L}_h^\phi(\boldsymbol{x})\nabla E(\boldsymbol{q}),$$

with Laplacian $\mathbf{L}_h^\phi$ as in Theorem 12. The off-diagonal components of $\mathbf{CL}_h^\phi$ are exactly the values $k_e$. Conversely, taking the Schur complement of the KKT matrix in (4.22), the saddle-node equation system becomes

$$\dot{\boldsymbol{q}} = -\mathbf{B}^\top \mathbf{W}_k(\boldsymbol{q})\,\mathbf{B}\,\nabla E(\boldsymbol{q}),$$

so that we eventually obtain the identity

$$\mathbf{B}^\top \mathbf{W}_k(\boldsymbol{q})\,\mathbf{B} = \mathbf{CL}_h^\phi(\boldsymbol{q}) = \mathbf{K}(\boldsymbol{q}).$$

This completes the proof. □

The proof of this theorem entirely builds on the local passivity of circuit elements and the duality approach to the energy-dissipation equality (4.25) as defining equation for gradient systems. Usual proof techniques to exhibit gradient system structures instead build on vector-matrix representations of the Onsager type (4.26), as for instance done in [MDM16] for the same result as in Theorem 13. Without the network synthesis pathway using the introduced passive circuit primitives of capacitors and resistors, it is not possible to explicitly derive the dissipation potential $\Psi(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}} \cdot \mathbf{G}(\boldsymbol{q})\dot{\boldsymbol{q}}$ respecting the connectivity structure of the network. The matrix function $\mathbf{K}(\cdot)$ however, does mirror the network structure, but it is singular, so that it is not straightforward to compute a structure-preserving $\mathbf{G}$ as pseudo-inverse of the Onsager matrix $\mathbf{K}$.

It is remarkable that this gradient system property can be established for the $q$-variable system, with the charge being the state $\boldsymbol{q}$, instead of the original consensus system state $\boldsymbol{x}$. In view of J. Willems's dissipative systems framework, the state must be internal, i.e., it cannot be measured at terminals. In this context, the charges $q_i$ occur as appropriate state definition. For the case that $E$ is quadratic defining electric energy stored in capacitors, $x_i = c_i^{-1} q_i$ are voltages at internal network nodes $i \in N$, and hence are manifest (external) variables.

To complete this section, we prove convergence to consensus using the introduced dual dissipation potential

$$\Psi^\star(\boldsymbol{u}_B) = \frac{1}{2}\sum_{e\in B} k_e u_e^2 = \frac{1}{2}\sum_{e=(j,i)\in B} k_e \big(h(x_j) - h(x_i)\big)^2 = \frac{1}{2}\nabla E(\boldsymbol{q}) \cdot \mathbf{K}\nabla E(\boldsymbol{q}),$$

as introduced in the proof of Theorem 13.

**Corollary 3** (Exponential convergence and arithmetic mean consensus state)**.** *Consider the system as in Theorem 13. The energy function $E$ is a Lyapunov function establishing exponential stability of the consensus state $\bar{x}\mathbf{1}$, with $\bar{x} = \frac{\sum_i c_i x_i(0)}{\sum_i c_i}$ the weighted arithmetic mean of the initial condition.*

*Proof.* Let us show that the consensus state is the only equilibrium state. In equilibrium the energy $E$ reaches a minimum and the dissipation rate equals zero, i.e.,

$$\dot{E}(\boldsymbol{q}) = -\nabla E(\boldsymbol{q}) \cdot \mathbf{K}(\boldsymbol{q})\nabla E(\boldsymbol{q}) = \sum_{e=(j,i)\in B} k_e \big(h(x_j) - h(x_i)\big)^2 = 0. \tag{4.27}$$

As $k_e > 0$ for $x_j \to x_i$, see proof of Lemma 4, the identity (4.27) holds if and only if we have $h(x_j) = h(x_i)$, which is equivalent to $x_j = x_i$, as $h$ is strictly increasing, i.e., bijective. Therefore, the only equilibrium state is a consensus state and furthermore, with $\mathbf{1}^\top \mathbf{K}(q)\nabla E(q) = 0$,

$$\sum_{i\in N} q_i(t) = \sum_{i\in N} c_i x_i(t) = \bar{x} \sum_{i\in N} c_i = const \quad \Leftrightarrow \quad \bar{x} = \frac{\sum_{i\in N} c_i x_i(0)}{\sum_{i\in N} c_i},$$

which is the weighted arithmetic mean of the initial condition. The function $E$ hence establishes asymptotic stability of this weighted arithmetic mean consensus state. The quadratic dissipation rate equation (4.27) further shows that the system evolves with exponential speed to this consensus equilibrium state. □

### 4.3.2 Hill-Moylan-Anderson structure and duality

Let us discuss this gradient system representation and the internal, local passivity property in the context of the Hill-Moylan-Anderson structure result. The consensus network synthesized via Kirchhoff-Ohm laws and the constitutive equation for the capacitor bank results in a lossless-dissipative decomposition of the consensus system such that

$$\dot{x} = -\mathbf{L}_{\mathrm{id}}^{\phi}(x)x \quad \Leftrightarrow \quad \begin{cases} \dot{q} = \mathbf{u}_N \\ y_N = \nabla E(q) \\ u_N = -\mathbf{B}^\top \mathbf{K}(q)\,\mathbf{B}\,y_N \end{cases} . \tag{4.28}$$

This negative feedback interconnection of the two passive systems is illustrated in Fig. 4.5. The result in Theorem 13 therefore is the application of the abstract Hill-Moylan-Anderson structure result to nonlinear consensus protocols on graphs.

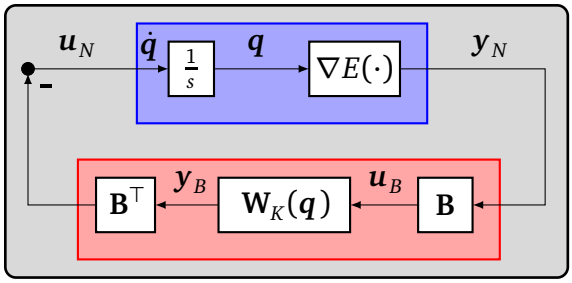



Figure 4.5: Output feedback representation of network dynamics (4.28) with controller system (red) in Kirchhoff-Ohm factorized form and lossless nonlinear integrator system (blue): Variables $y_{B,N}$, $u_{B,N}$ denote internal vector outputs and inputs of the edge (branch) system (index $B$) and of the node system (index $N$).

The expansion method we introduced with our graph-embedding approach in Section4.2.5 is a natural construction method for the abstract Hill-Moylan-Anderson dissipative system module $f \circ (\nabla E)^{-1}$. For one, we showed that strict convexity of $E$ is necessary and sufficient for the uniqueness of the inversion, but also for local passivity of the resulting conductances that define dissipation from the stored energy in the capacitor bank.

Karush-Kuhn-Tucker equation systems occur in convex optimization problems involving equality constraints. The optimization problem associated to the Kirchhoff-Ohm KKT equation system (4.22) is:

$$\begin{aligned} \text{minimize} \quad & \Psi(\boldsymbol{y}_B) = \tfrac{1}{2}\boldsymbol{y}_B^\top \mathbf{W}_k^{-1}(\boldsymbol{q})\boldsymbol{y}_B, \\ \text{subject to} \quad & \boldsymbol{u}_N = \mathbf{B}^\top \boldsymbol{y}_B, \end{aligned}$$

where $\boldsymbol{y}_B$ is the optimization variable, and $\boldsymbol{q}$ serves as parameter that is fixed. Note that in Section 4.2.4 gradient systems are defined as systems that maximize the magnitude of the dissipation rate, while here, it is minimized.

To resolve this, consider the dual problem associated to the constraint minimization involving the Lagrangian function

$$\mathscr{L}_\Psi(\boldsymbol{y}_B, \boldsymbol{y}_N) := \frac{1}{2}\boldsymbol{y}_B^\top \mathbf{W}_k^{-1}\boldsymbol{y}_B + \boldsymbol{y}_N^\top \big(\boldsymbol{u}_N - \mathbf{B}^\top \boldsymbol{y}_B\big),$$

where the vector $\boldsymbol{y}_N = \nabla E(\boldsymbol{q})$ represents the vector of Lagrange multipliers, or sensitivities. The dual problem then is given as

$$\max_{\boldsymbol{y}_N} \min_{\boldsymbol{y}_B} \mathscr{L}_\Psi(\boldsymbol{y}_B, \boldsymbol{y}_N).$$

A solution to this min-max problem describes a saddle-point. It balances minimization of dissipation in the edge system across branches indexed in the set $B$ and maximization of the extraction of Energy $E$ from the capacitor bank with each capacitor indexed in $N$. The result is the gradient system structure as proposed.

*Remark* 21. We observe that the constitutive network system equations in KKT form (4.22) are recovered from the Lagrangian via the necessary optimality conditions

$$\begin{aligned} \frac{\partial \mathscr{L}_\Psi}{\partial \boldsymbol{y}_B} &= \mathbf{W}_k^{-1}(\boldsymbol{q})\boldsymbol{y}_B - \mathbf{B}\boldsymbol{y}_N = \mathbf{0} && \overset{(Ohm)}{\Leftrightarrow} (KVL) : \boldsymbol{u}_B = \mathbf{B}\boldsymbol{y}_N \\ \frac{\partial \mathscr{L}_\Psi}{\partial \boldsymbol{y}_N} &= -\mathbf{B}^\top \boldsymbol{y}_B + \boldsymbol{u}_N = \mathbf{0} && \Leftrightarrow (KCL). \end{aligned}$$

For strictly convex $\Psi$, as is the case here, these conditions are also sufficient for the solution's optimality.

In the following, we discuss two examples of nonlinear consensus protocols on graphs that satisfy detailed balance and discuss their circuit gradient realization and local passivity.

### 4.3.3 Passive circuit view on coupled oscillator models

A generic model in the study of phase-coupled oscillator networks is given by the ODE system on a graph G,

$$\dot{\theta}_i = \omega_i + \sum_{j:(j,i)\in B} w_{ij} \sin(\theta_j - \theta_i), \quad i \in N, \tag{4.29}$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$ is the vector of natural (driving) frequencies, and the state $\boldsymbol{\theta} \in \mathbb{T}^n$ is an $n$-vector of angles as elements of the $n$-Torus.

If we set $w_{ij} = \frac{K}{n}$, $K > 0$ for all $(j,i) \in B$, then (4.29) represents Kuramoto's oscillator model [Ace+05]. If $w_{ij} = \frac{|v_i||v_j|\Im(y_{ij})}{D_i}$, $|v_i|$ a voltage magnitude, $y_{ij}$ the complex admittance of a line $(j,i)$, and $D_i > 0$ a damping parameter, then (4.29) describes a so-called droop control setup for frequency stabilization of generators in an electric power grid whose diffusively coupled voltage angles $\theta_i$ are driven by nominal power inputs $\omega_i$, see, e.g., [DB14].

Observe that the detailed balance condition (4.1) naturally applies in this setting, because $c_i w_{ij} = c_j w_{ji}$ implies that weights $w_{ij}$ have the form $w_{ij} = \frac{1}{r_{ij} c_i}$, $r_{ij} = r_{ji}$, with $r$ a resistance. Following this RC circuits view, we may take $r_{ij}^{-1} = \Im(y_{ij})$ having unit Siemens (one over Ohm), and $c_i^{-1} = |v_i||v_j|/D_i$. Capacitance has unit Farad $F = \frac{\text{Ampere·sec}}{\text{Volt}}$, so that $D_i$ should carry the unit Volt-Ampere-Seconds (an energy), which is similar to a measure of a power deviation per base frequency. This indeed matches the meaning of the factor $D_i$ in the droop control setting.

Using our reactance extraction approach we can write (4.29) as (driven) gradient system, with $\boldsymbol{q} = \mathbf{C}\boldsymbol{\theta}$, $\boldsymbol{\omega}_C := \mathbf{C}\boldsymbol{\omega}$, and the gradient $\boldsymbol{h}(\boldsymbol{\theta}) = \nabla E(\boldsymbol{q})$,

$$\dot{\boldsymbol{q}} = \boldsymbol{\omega}_C - \mathbf{K}(\boldsymbol{\theta})\nabla E(\boldsymbol{q}), \quad [\mathbf{K}]_{ij} = c_j w_{ij} \frac{\sin(\theta_j - \theta_i)}{h(\theta_j) - h(\theta_i)}, \tag{4.30}$$

For instance, if we choose $E = \sum_i c_i \theta_i^2 = \sum_i \frac{1}{c_i} q_i^2$, (electric energy), then, the graph embedding methods yields conductance factors $\frac{\sin(\theta_j - \theta_i)}{\theta_j - \theta_i} = \text{sinc}(\theta_j - \theta_i)$, the sine cardinalis, or sampling function. We denote the corresponding inverse metric by $\mathbf{K}_{\text{sinc}}$. Let $\boldsymbol{\omega}_C = c\mathbf{1}$, $c \in \mathbb{R}$, so that without loss of generality we can study the dynamics in a rotating frame at speed $c$ and set $\boldsymbol{\omega} = \mathbf{0}$ in (4.30) [DB14] [Mor05]. Phase synchronization takes place if $\max_{i,j \in N} |\theta_j - \theta_i| < \pi$, because in that case $\text{sinc}(\cdot) > 0$, i.e., all nonlinear resistances $[\mathbf{K}_{\text{sinc}}]_{ij}^{-1}$ are passive, so that $\lim_{t\to\infty} \boldsymbol{\theta}(t) \to \theta_\infty \mathbf{1}$, according to Theorem 13. By that we recover a known phase synchronization result, see, e.g., [DB14], but via a passive circuits approach. The sine cardinalis together with the interval in which the corresponding resistances are passive is illustrated in Fig. 4.6.

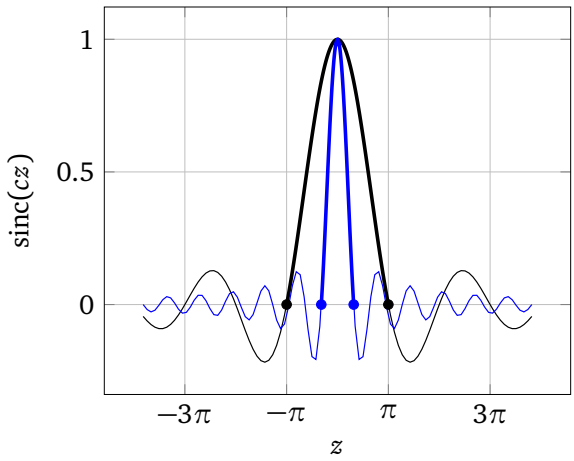



Figure 4.6: Sine cardinalis function with usual normalization $c = \pi$ (blue), and $c = 1$ (black).

We can bring the coupled oscillator model (4.29) (in uniform rotating frame) into $q$-variable form, when it is mass preserving and hence defining an inhomogeneous Markov chain dynamic, see also Section 4.5. The fact that the class of information divergences, see Table 4.1 for examples, are Lyapunov functions, allows to explore mixing time bounds in Markov chains for convergence bounds in dynamics on a graph, which often are tighter than usual bounds based on the second largest eigenvalue of a Laplace matrix $\mathbf{K}$, see [MT05] for an overview.

| Divergence name | $f(\rho)$ | Collective disagreement form |
|---|---|---|
| Kullback-Leibler | $\rho\log\rho$ | $\sum_{i\in N} p_i \log\frac{p_i}{q_i}$ |
| Jeffrey | $(\rho-1)\log\rho$ | $\sum_{i\in N}(p_i-q_i)\log\frac{p_i}{q_i}$ |
| Pearson $\chi^2$ | $\rho^2-1$ | $\sum_{i\in N}\frac{(p_i-q_i)^2}{q_i}$ |
| (Kolmogorov) total variation distance | $\lvert\rho-1\rvert$ | $\sum_{i\in N}\lvert p_i-q_i\rvert$ |
| squared Hellinger | $(\sqrt{\rho}-1)^2$ | $\sum_{i\in N}\sqrt{p_i}-\sqrt{q_i}$ |
| $\alpha$-Hellinger ($\alpha>1$) | $\rho^\alpha$ | $\sum_{i\in N} q_i^{1-\alpha}p_i^\alpha$ |

Tabular 4.1: Examples of $f$-divergences that are collective disagreement measures. Components $p_i$, $\pi_i$ are probabilities and $\rho_i=\frac{p_i}{q_i}$ is the component of a relative measure, i.e., a density.

Let us turn to the case when $\boldsymbol{\omega}_C \notin \mathsf{span}\{\mathbf{1}\}$. An open problem for driven systems (4.29) is to find (closed form) conditions under which frequency-synchronized solutions $\bar{\boldsymbol{\theta}}$ exist, i.e., conditions under which the system of equations

$$\omega_i = \sum_{j:(j,i)\in B} w_{ij}\sin(\bar{\theta}_j-\bar{\theta}_i),\quad i\in N, \tag{4.31}$$

can be solved uniquely, also for situations in which angles are further apart than $\pi$ [DH14]. The problem of existence of solutions is of importance for all power systems studies as it is directly concerned with the feasibility of the power flow equations [DH14].

We can re-write the equation system (4.31) in the form

$$\boldsymbol{\omega}_C = -\mathbf{K}_{\mathsf{sinc}}(\bar{\boldsymbol{\theta}})\bar{\boldsymbol{\theta}}. \tag{4.32}$$

The problem then is to characterize the vectors $\bar{\boldsymbol{\theta}}$, for which there exists a $\mathbf{G}(\bar{\boldsymbol{\theta}})$, such that $\mathbf{G}(\bar{\boldsymbol{\theta}})\boldsymbol{\omega}_C = -\bar{\boldsymbol{\theta}}$. Observe the similarity to the problem of associating the metric $\mathbf{G}$ to a non-invertible inverse metric $\mathbf{K}$. A necessary and sufficient condition is that $\frac{1}{2}\bar{\boldsymbol{\theta}}^\top\mathbf{K}_{\mathsf{sinc}}\bar{\boldsymbol{\theta}}$ is a strictly convex function on the simplex of mass preserving vectors. For this to be true, certain angles may be further apart than $\pi$, so that at certain edges $\mathsf{sinc}(\theta_j-\theta_i)<0$. The associated $\mathbf{G}$-matrix is characterized via Young duality, i.e., $\frac{1}{2}\boldsymbol{\omega}_C^\top\mathbf{G}(\bar{\boldsymbol{\theta}})\boldsymbol{\omega}_C$ is a Legendre transform of the convex potential $\frac{1}{2}\bar{\boldsymbol{\theta}}^\top\mathbf{K}_{\mathsf{sinc}}\bar{\boldsymbol{\theta}}$.

*Remark* 22. An interesting direction of research in this context would be the study of definiteness properties of $\mathbf{K}$ by means of passivity of worst-case effective resistance/conductance [Spi12] [ZB14]. In fact, with $\mathbf{K}$ representing a conductance matrix, (4.32) represents a (nonlinear) version of Ohm's law where $\boldsymbol{\omega}_C$ is a current vector and $\bar{\boldsymbol{\theta}}$ a vector of voltages.

## 4.4 Passivity equivalence to majorization and cut-balance

We have seen that the convexity of an additive Lyapunov function for detailed-balance consensus networks is equivalent to the local passivity of virtual resistor elements across which dissipation of energy stored in a capacitor bank is wholly defined. In the following, we extend the Lyapunov part of the result to general irreducible consensus networks that are weighted

(arithmetic) average preserving. The proof technique bases on a result by Willems from 1976. We show that this result is equivalent to the passive circuit dissipation principle via a formulation of the network system via a generalized Hamiltonian realization. We then show that this Lyapunov property and local passivity is equivalent to the concept of majorization, or more specifically, to a variant that we introduce and define as differential majorization. Further, an equivalence to the cut-balance property in linear consensus systems, as proposed by Hendrickx and Tsitsiklis, is derived based on differential majorization and local passivity.

### 4.4.1 Invariance and dissipation

According to Theorem 12 the consensus network has linear form on a state-dependent graph. The associated state-dependent graph Laplacian is the generator of a transfer operator, such that given an initial state $\boldsymbol{x}(0)$, for any time $t > 0$,

$$\dot{\boldsymbol{x}}(t) = -\mathbf{L}_{\mathsf{id}}^{\phi}(\boldsymbol{x})\boldsymbol{x}(t) \quad \Rightarrow \quad \boldsymbol{x}(t) = e^{-\mathbf{L}_{\mathsf{id}}^{\phi}(\boldsymbol{x})t}\boldsymbol{x}(0).$$

We write for the transfer operator given by the matrix exponential $e^{-\mathbf{L}_{\mathsf{id}}^{\phi}(\boldsymbol{x})t} =: \mathbf{A}(\boldsymbol{x}, t)$. It has the following property.

**Proposition 6** (Stochastic transfer operator and its generator, [Wil76] Proposition 4)**.** *Given any admissible $\boldsymbol{x} \in \mathbb{R}^n$, and $t > 0$, the transfer operator matrix $\boldsymbol{A}(\boldsymbol{x}, t)$ is a stochastic matrix, i.e., it has row sums equal one, if and only if the generator $\boldsymbol{L}_{\mathsf{id}}^{\phi}$ is a matrix with zero row-sums, i.e., a Laplacian matrix. Matrices $\boldsymbol{A}$ are doubly stochastic, i.e., both row and column sums equal one, if and only if $\boldsymbol{L}_{\mathsf{id}}^{\phi}(\boldsymbol{x})$ is a balanced generator matrix so that both column and row sums equal zero.*

Via the transfer matrix $\mathbf{A}(\cdot)$ an equivalence between average preservingness as global system invariance property and non-increasingness of network energy functions can be established, which together establish Lyapunov stability of the weighted arithmetic mean.

**Lemma 6** (Network Lyapunov functions)**.** *Let $\mathsf{G}$ be an irreducible weighted digraph and consider a consensus protocol according to* (4.2) *evolving on $\mathsf{G}$. The additive function*

$$E(\boldsymbol{q}) = \sum_{i \in N} c_i H(c_i^{-1} q_i)$$

*with convex $H : \mathbb{R} \to \mathbb{R}$, cf. Theorem 13, is a Lyapunov function for the weighted arithmetic mean of the initial condition as given in Corollary 3 if and only if*

$$\sum_{i=1}^{n} c_i a_{ij} = c_j, \quad j \in N \qquad \Leftrightarrow \qquad \boldsymbol{c}^{\top}\boldsymbol{A}(\boldsymbol{x}, t) = \boldsymbol{c}^{\top}. \tag{4.33}$$

*That is, the invariance of the weighted arithmetic mean along trajectories is necessary and sufficient for the class $E$ to be non-increasing along solutions and also serve as Lyapunov function class establishing stability of $\boldsymbol{c}^{\top}\boldsymbol{x}(0)\mathbf{1}$, which is the unique consensus equilibrium state.*

*Proof.* Sufficiency, i.e., (4.33) $\Rightarrow E$ is non-increasing: Define the output $\boldsymbol{y} := \boldsymbol{A}\boldsymbol{x}$. Then,

$$\sum_{i=1}^{n} c_i H(y_i) = \sum_{i=1}^{n} c_i H\left(\sum_{j=1}^{n} a_{ij} x_j\right) \leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_i a_{ij} H(x_j) \overset{(4.33)}{=} \sum_{j=1}^{n} c_j H(x_j). \tag{4.34}$$

Here we used Jensen's inequality for convex functions together with the fact that row-sums of the transfer matrix given by **A** add up to one, see Proposition 6.
Necessity, i.e., $E$ is non-increasing $\Rightarrow$ (4.33): Follows from taking any norm as extreme case of a convex function. Then, e.g., for $\boldsymbol{x} \in \mathbb{R}^n_{>0}$, and $f(\cdot) = ||\cdot||_2$, so that $||x_i||_2 = x_i$, the quantity $\sum_i c_i x_i = const$, so that equality must hold in (4.34) and this implies (4.33). □

This result is remarkable due to its simplicity. It demonstrates an equivalence between the network energy function class representing Lyapunov functions and an invariance property of the system, making use only of Jensen's inequality for convex functions.

*Remark* 23 (Admissible action functions for arithmetic mean invariance). For the synthesis of consensus protocols, it may be useful to know which nonlinearity on $\phi$ is admissible to have transfer operators satisfying (4.33). Anti-symmetry of the action function $\phi$ together with detailed balance is sufficient, as Theorem 13 demonstrates. In that case, the slopes $\frac{\phi(x_i,x_j)}{h(x_i)-h(x_j)}$ are symmetric functions, see Lemma 4. However, for $\phi = \mathrm{id}$, detailed balance is not required. Hence, it is an open problem to characterize the class of nonlinearity of pairwise actions, under which the weighted arithmetic mean remains a constant along trajectories of the nonlinear network dynamics, and as such, for which $E$ is a Lyapunov function class.

*Remark* 24 (Span-norm Lyapunov functions). Jan C. Willems in [Wil76] also shows that both $\max_i x_i$ and $-\min_i x_i$ are Lyapunov functions if and only if the generator of the transfer matrix has zero row excess, i.e., it is a Laplacian matrix associated to a strongly connected graph. Willems further comments that this non-quadratic Lyapunov function is characteristic of diffusive systems. It therefore is a predecessor of consensus analysis using the non-quadratic span-norm Lyapunov function $\max_i x_i - \min_i x_i$.

In the following, we demonstrate that this Lyapunov property of being non-increasing along trajectories forward in time is equivalent to local passivity of "virtual" resistor elements, across which stored energy of a lossless system is dissipated such that there holds quadratic energy-dissipation equality. Here, this circuit-related local passivity property is proposed for the class of general irreducible consensus system networks, which is larger than the detailed balance class considered in the gradient circuit synthesis approach in Section 4.3.

**Theorem 14** (Generalized Hamiltonian realization and local passivity)**.** *Consider a consensus network* (4.2) *that is weighted average preserving, i.e., the condition* (4.33) *in Lemma 6 holds and we assume action functions $\phi$ to be anti-symmetric. The system has the realization*

$$\dot{\boldsymbol{q}} = [\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{K}(\boldsymbol{x})]\nabla E(\boldsymbol{q})$$

*where $\boldsymbol{K}$ is an $n \times n$ conductance matrix defined as*

$$\boldsymbol{K} := \boldsymbol{B}^\top \boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{B}, \quad \boldsymbol{R}^{-1} = \mathrm{diag}\left\{\frac{\phi(x_j,x_i)}{h(x_j)-h(x_i)}\frac{c_i w_{ij} + c_j w_{ji}}{2}\right\}_{(j,i)\in B},$$

*and $\boldsymbol{J}$ is an $n \times n$ skew-symmetric matrix function defined as $\boldsymbol{C}\boldsymbol{L}_{\mathsf{id}}^{\phi} - \boldsymbol{K}$, with $\boldsymbol{L}_{\mathsf{id}}^{\phi}$ as in Theorem 12. That is, invariance and validity of the network Lyapunov inequality are equivalent to the existence of a virtual, locally passive resistor network across which energy stored in a lossless system is dissipated.*

*Proof.* First, recall that the $i$-th component of the gradient vector $\nabla E(\boldsymbol{q}) = h(x_i)$. Using the graph embedding approach introduced in Section 4.2.5 and the result Theorem 12, the consensus system dynamics can be written as $\dot{\boldsymbol{x}} = -\mathbf{L}_h^{\phi}(\boldsymbol{x})\nabla E(\boldsymbol{q})$. By assumption, the weighted arithmetic mean is a system invariant, i.e., $\boldsymbol{c}$ is a left-eigenvector associated to the zero eigenvalue of the state dependent Laplacian $\mathbf{L}_h^{\phi}(\cdot)$. This eigenvalue equation can be written as

$$\boldsymbol{c}^{\top}\mathbf{L}_h^{\phi}(\boldsymbol{x}) = \mathbf{1}^{\top}\mathbf{C}\mathbf{L}_h^{\phi}(\boldsymbol{x}) = 0\boldsymbol{c}^{\top}, \tag{4.35}$$

where we recall that $\mathbf{C}$ is the diagonal matrix with non-zero entries the values $c_i$.

Define the matrix $\mathbf{L}_b(\boldsymbol{x}) := \mathbf{C}\mathbf{L}_h^{\phi}(\boldsymbol{x})$. As $\mathbf{L}_h^{\phi}$ is an irreducible Laplacian matrix, $\mathbf{L}_b$ is an irreducible Laplacian matrix, too. Hence, also $\mathbf{L}_b\mathbf{1} = 0$, i.e., $\mathbf{1}$ is a right-eigenvector associated to the dominant zero eigenvalue. Moreover, according (4.35), $\mathbf{1}$ is also left-eigenvector of the newly defined Laplacian matrix. Therefore, for any admissible $\boldsymbol{x} \in \mathbb{R}^n$, the matrix $\mathbf{L}_b(\boldsymbol{x})$ by definition is a balanced graph Laplace matrix, as the eigenvalue equation shows that it has zero row and column sums. Balanced graph Laplace matrices have a symmetric / skew-symmetric matrix decomposition such that

$$\mathbf{L}_b(\boldsymbol{x}) = \mathbf{L}_b^{\mathsf{sym}}(\boldsymbol{x}) + \mathbf{L}_b^{\mathsf{skew}}(\boldsymbol{x}) = \frac{1}{2}\big(\mathbf{L}_b(\boldsymbol{x}) + \mathbf{L}_b^{\top}(\boldsymbol{x})\big) + \frac{1}{2}\big(\mathbf{L}_b(\boldsymbol{x}) - \mathbf{L}_b^{\top}(\boldsymbol{x})\big).$$

The matrix $\mathbf{L}_b^{\mathsf{sym}}(\boldsymbol{x})$ is a symmetric graph Laplace matrix for all admissible arguments, i.e., it is a conductance matrix. The matrix $\mathbf{L}_b^{\mathsf{skew}}$ is skew symmetric. Both matrices have left and right kernel given by $c\mathbf{1}$, $c > 0$.

Now, let us rewrite the consensus network using the symmetric / skew-symmetric decomposition to get

$$\begin{aligned} &\dot{\boldsymbol{x}} = -\mathbf{L}_h^{\phi}(\boldsymbol{x})\nabla E(\boldsymbol{q}) = -\mathbf{C}^{-1}\mathbf{C}\mathbf{L}_h^{\phi}(\boldsymbol{x})\nabla E(\boldsymbol{q}) \Leftrightarrow \dot{\boldsymbol{q}} = -\mathbf{L}_b(\boldsymbol{x})\nabla E(\boldsymbol{q}) \\ \Leftrightarrow\ &\dot{\boldsymbol{q}} = -\big[\mathbf{L}_b^{\mathsf{sym}}(\boldsymbol{x}) + \mathbf{L}_b^{\mathsf{skew}}(\boldsymbol{x})\big]\nabla E(\boldsymbol{q}) \end{aligned}$$

With the assignment $\mathbf{K} = \mathbf{L}_b^{\mathsf{sym}}$ and $\mathbf{J}^{\top} = \mathbf{L}_b^{\mathsf{skew}}$ the representation result follows, as for any skew-symmetric matrix the identity $\mathbf{J}^{\top} = -\mathbf{J}$ holds. The conductance matrix $\mathbf{K}$ can be factorized as $\mathbf{B}^{\top}\mathbf{R}^{-1}\mathbf{B}$ where the diagonal matrix $\mathbf{R}^{-1}$ collects positive conductances as defined, cf. Theorem 12 together with $\phi$ being anti-symmetric, so that the slopes are symmetric functions. This completes the proof. □

Generalized Hamiltonian realizations are defined and studied in, e.g., [LC03], [HHG08]. The network system representation associated to such a realization of an autonomous ODE system is illustrated in Fig. 4.7. The red loop represents a negative feedback system, where the memoryless resistor network defined via the conductance matrix $\mathbf{K}$ extracts stored energy from the blue system loop. The blue loop represents a positive feedback system, where skew-symmetry of $\mathbf{J}$ implies that stored energy is conserved. At the same time, it may circulate within the lossless system being re-allocated among states.

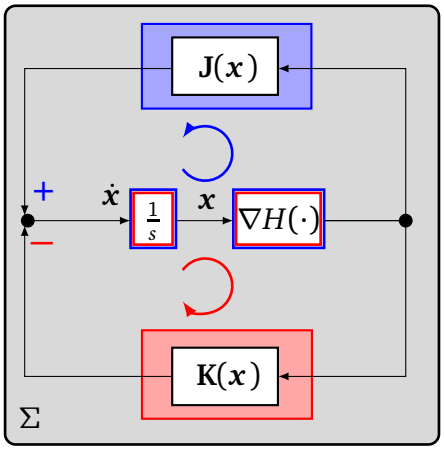



Figure 4.7: Generalized Hamiltonian realization of a dynamical system $\Sigma$ defined by a dynamics $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$: The lossless system part (blue) is controlled by a skew symmetric map in positive feedback connection, while the dissipative system part (red) is controlled by a symmetric non-negative map in negative feedback connection. The two loops define internal flows of supply.

A generalized Hamiltonian realization results in a quadratic energy dissipation equality along solutions given by

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\boldsymbol{x}) = -\nabla E(\boldsymbol{x}) \cdot \mathbf{K}(\boldsymbol{x})\nabla E(\boldsymbol{x}) + \nabla E(\boldsymbol{x}) \cdot \mathbf{J}(\boldsymbol{x})\nabla E(\boldsymbol{x}) = -||\nabla E(\boldsymbol{x})||^2_{\mathbf{K}}. \tag{4.36}$$

Moreover, the dissipation potential $||\nabla E(\boldsymbol{x})||^2_{\mathbf{K}}$ is strictly convex, so that one may define its dual in the sense of Young and with that a valid energy dissipation equality as in the defining equation for gradient systems (4.25), see also Proposition 5. However, from such a dual dissipation potential, and for non-vanishing $\mathbf{J}$, one cannot reconstruct the original ODE system that generates the gradient flow solutions. Hence, (4.36) does not serve as gradient system characterization.

*Remark* 25 (Generalized Hamiltonian realization and port-Hamiltonian framework). The prototypical example of a dynamics in generalized Hamiltonian form results from second order models describing damped oscillator systems, e.g., classical mechanical and/or electrical systems as they appear in electric power grid dynamics, cf., e.g., [vJ14] and [CT14], but also in switched systems, e.g., power electronics, see [EvO99]. A generalized Hamiltonian realization is a particular instance of a negative feedback connection of two port-controlled Hamiltonian systems, see [van00] Chapter 4. Port-Hamiltonian systems with dissipation cover systems that admit a generalized Hamiltonian realization and in addition include inputs and outputs, see, e.g., [vJ14] for a recent overview.

### 4.4.2 Majorization dynamics and local passivity

Majorization is a powerful technique in deriving functional inequalities, which by the authors of [MOA11] is even put at the basis of a "theory of inequalities". We denote with square bracket index the non-increasingly ordered components of a vector $\boldsymbol{x}$, such that

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}. \tag{4.37}$$

**Definition 8** (Majorization)**.** Let $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ and define the relation

$$\boldsymbol{y} \preceq \boldsymbol{z} \text{ if } \begin{cases} \sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} z_{[i]}, & \text{for } 1 \leq k \leq n-1, \\ \sum_{i=1}^{n} y_{[i]} = \sum_{i=1}^{n} z_{[i]}. \end{cases}$$

If $\boldsymbol{y} \preceq \boldsymbol{z}$, then $\boldsymbol{y}$ is said to be majorized by $\boldsymbol{z}$.

Loosely speaking, if a vector $\boldsymbol{y}$ is majorized by a vector $\boldsymbol{z}$, then $\boldsymbol{y}$ is more "spread out" than the vector $\boldsymbol{z}$, i.e., the components are more mixed.

**Example 14.** Consider $n$ numbers $p_i \in (0,1)$, $1 \leq i \leq n$, such that $\sum_i p_i = 1$. Then,

$$(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \preceq (p_1, p_2, \ldots, p_n) \preceq (1, 0, \ldots, 0).$$

In this example, $\frac{1}{n}\mathbf{1}^\top$ can be thought of as the invariant probability distribution of a Markov chain with doubly stochastic update law, $\boldsymbol{p}^\top$ represents a probability distribution at time $t > 0$, and $(1, 0, \ldots, 0)$ for instance an initial condition. The initial condition is less spread out, or put differently, probability mass is more concentrated on discrete probability space than in the two other vectors being majorized by $(1, 0, \ldots, 0)$.

The following majorization equivalence result is due to, among others, Weyl, Birkhoff, Karamata, and Hardy-Littlewood-Polya, see [HLP34], [Kar32], [Mal04], or [MOA11] Chapter 4 for general reference.

**Proposition 7** (Majorization equivalences)**.** *Let $I$ be a real interval and $\boldsymbol{y}, \boldsymbol{z} \in I^n \subset \mathbb{R}^n$. The following statements are equivalent:*

*(i)* $\boldsymbol{y} \preceq \boldsymbol{z}$*;*

*(ii)* $\boldsymbol{y} \in \mathsf{conv}(\{\boldsymbol{\Pi z} : \boldsymbol{\Pi}$ *a permuation matrix*$\})$*;*

*(iii)* $\sum_{i=1}^{n} f(y_i) \leq \sum_{i=1}^{n} f(z_i)$ *for all continuous convex $f$ on $I$.*

*Remark* 26 (Birkhoff result and majorization). While a relationship between convexity and majorization is apparent, the connection to doubly-stochastic matrices, which are transfer operators in a dynamical system generated from balanced Laplacian matrices, is due to a result of Birkhoff: Every doubly stochastic matrix is a convex combination of permutation matrices.

The equivalence in Proposition 7 allows visualizing the concept of majorization via nested sets, as Fig. 4.8 illustrates: The points defining the polygons of different gray-to-red colors each have coordinates given by the permutation of one single vector with a constant sum of components (i.e., constant mass vector). The nested sets define collections of vectors that are majorized by the elements of sets lying outward and being of identical mass. In blue depicted is the ray of consensus states. Any convex energy defined as $E = \sum_i H(x_i)$ on the simplex of constant mass vectors decreases along elements taken from nested sets into the direction of the consensus ray, achieving a minimum at the vector where all components are equal.

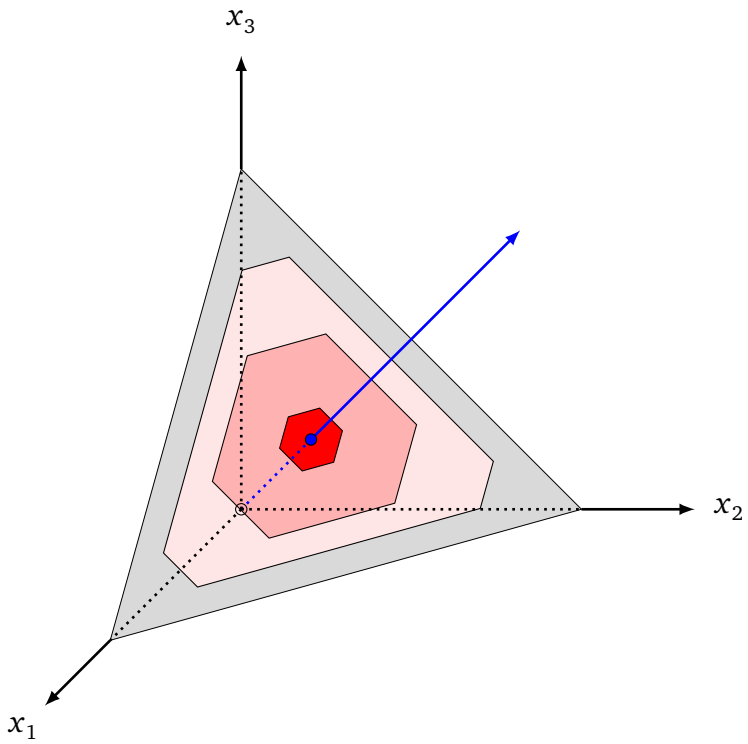



Figure 4.8: Illustration of the concept of majorization as nested sets

In Lemma 6 and Theorem 14 we have linked the more general class of storage functions $E$ and the dissipation of $E$ to passivity of (virtual) resistor elements, across which dissipation is completely defined. Next, we show that the validity of the Lyapunov inequality for $E$, due to internal passive resistors, is equivalent to the majorization of state vectors forward in time, i.e., the past is majorized by the present, which is majorized by the future.

First, assume $\boldsymbol{x}(t)$ to be a differentiable curve and define the $k$-th sum for $k \in \{0, 1, \ldots, n\}$ of sorted state vector components given by (4.37) and its rate

$$S_k(t) := \sum_{i=1}^{k} c_i x_{[i]}(t), \quad \dot{S}_k(t) := \sum_{i=1}^{k} c_i \dot{x}_{[i]}(t), \tag{4.38}$$

where $S_0(t) = 0$; further $S_n(t) = const$, resp., $\dot{S}_n(t) = 0$, if and only if $c_i$ are components of the Perron vector.

**Theorem 15** (Majorization – local passivity equivalence)**.** *Consider a consensus system* (4.2) *on a graph* $\mathsf{G} = (N, B, w)$ *that is constant, and strongly connected for all time so that* $\mathbf{c}$ *is the left Perron vector associated to the graph Laplacian of* $\mathsf{G}$*. The following two statements are equivalent:*

*(i) For all* $t < T$*, the majorization inequality* $S_k(T) \leq S_k(t)$ *holds, with* $S_n(t) = const$*.*

*(ii) The consensus network admits a generalized Hamiltonian realization according to Theorem 14, where dissipation is entirely defined by passive resistors.*

*That is, majorization of two vectors is equivalent to the existence of a possibly nonlinear, passive resistor network which, together with constant, but not necessarily identical capacitances collected in the vector* $\mathbf{c}$*, connects the two vectors with a trajectory generated by the RC circuit.*

*Proof.* First note that from Theorem 14 the characterization of $E(\boldsymbol{x}) := \sum_{i=1}^{n} c_i H(x_i)$ as Lyapunov function with $E$ decreasing over time is equivalent to the existence of a locally passive resistor network, across which dissipation dynamics of $E$ is completely characterized. Hence, we need to show that the decreasingness of $E$ is equivalent to the majorization property along dynamics as stated in (i).

Consider the $k$-th sum as defined in (4.38). Note that we can write

$$c_i x_{[i]}(t) = S^t_{[i]} - S^t_{[i-1]}, \quad \text{for } i > 1. \tag{4.39}$$

Define the slope of sorted components for $0 \leq t < T$,

$$R_{H,i} = \frac{H(x_{[i]}(T)) - H(x_{[i]}(t))}{x_{[i]}(T) - x_{[i]}(t)}. \tag{4.40}$$

For the function $E(\cdot)$ to be a Lyapunov function it is required that

$$\begin{aligned} &\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( E(\boldsymbol{x}(t+\epsilon)) - E(\boldsymbol{x}(t)) \right) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \sum_{i=1}^{n} c_i(t) H(x_i(t+\epsilon)) - \sum_{i=1}^{n} c_i(t) H(x_i(t)) \right) \leq 0 \end{aligned}$$

holds for all $t \geq 0$ everywhere except the set of consensus states $\operatorname{span}\{\mathbf{1}\}$.

Rewriting the strict Lyapunov inequality with sorted components we obtain

$$\begin{aligned} 0 &\geq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n} c_i \left( H(x_{[i]}(t+\epsilon)) - H(x_{[i]}(t)) \right) \\ &\overset{(4.40)}{=} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n} c_i R_{H,i} \left( x_{[i]}(t+\epsilon) - x_{[i]}(t) \right) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n} R_{H,i} \left( c_i x_{[i]}(t+\epsilon) - c_i x_{[i]}(t) \right) \\ &\overset{(4.39)}{=} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n} R_{H,i} S_{[i]}(t+\epsilon) - S_{[i-1]}(t+\epsilon) - S_{[i]}(t) + S_{[i-1]}(t) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n} R_{H,i} (S_{[i]}(t+\epsilon) - S_{[i]}(t)) - \sum_{i=1}^{n} R_{H,i} (S_{[i-1]}(t+\epsilon) - S_{[i-1]}(t)). \end{aligned} \tag{4.41}$$

Using the fact that $\dot{S}_n = 0$, i.e., $S_n(t+\epsilon) - S_n(t) = 0$, and shifting the indexes in the second sum by one counter, we can write with (4.41) the Lyapunov inequality

$$0 \geq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n-1} R_{H,i} (S_{[i]}(t+\epsilon) - S_{[i]}(t)) - \sum_{i=0}^{n-1} R_{H,i+1} (S_{[i]}(t+\epsilon) - S_{[i]}(t)). \tag{4.42}$$

Using $S_0(t) = 0$ leads to

$$\begin{aligned} 0 &\geq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n-1} \left( R_{H,i} - R_{H,i+1} \right) \left( (S_{[i]}(t+\epsilon) - S_{[i]}(t) \right) \\ &= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \sum_{i=1}^{n-1} \left( R_{H,i} - R_{H,i+1} \right) \left( \sum_{j=1}^{i} c_j \left( x_{[j]}(t+\epsilon) - x_{[j]}(t) \right) \right). \end{aligned} \tag{4.43}$$

By definition of the ordering of the sorted vector $\boldsymbol{x}$ the components satisfy $x_{[i+1]} \leq x_{[i]}$. By Lemma 5, strict convexity of $H$ is equivalent to increasing $R_{H,[i]}(\cdot,\cdot)$ for increasing arguments. Therefore,

$$\begin{aligned} & H \text{ is strictly convex} \\ \Leftrightarrow\ & R_{H,i+1}\left(x_{[i+1]}(t), x_{[i+1]}(0)\right) < R_{H,i}\left(x_{[i]}(t), x_{[i]}(0)\right) \\ \Leftrightarrow\ & R_{H,i} - R_{H,i+1} > 0. \end{aligned} \tag{4.44}$$

As $H$ is strictly convex by assumption, so that (4.44) holds, the inequality (4.43) is true if and only if each $c_j x_{[j]}(t+\epsilon) - c_j x_{[j]}(t) < 0$ for $j \in \{1,\dots,n-1\}$. Together with the property $S_n = const$ this completes the proof. □

While the concept of majorization (in the finite-dimensional case) is a tool to compare any two vectors in a particular property, here we provide a simple and profound necessary and sufficient dynamical perspective and passive circuits context to majorization. This "physics of majorization" view links the passivity of resistors to the Lyapunov inequality via Theorem 13, which shows the equivalence of convexity of the network Lyapunov function to local passivity of virtual resistor elements. In the context of the evolution of continuous distributions, such a dynamical context to majorization has been given, e.g., in [GCS19].

Instrumental in the proof is (i) the majorization relation according to Definition 8, respectively the weighted vector component sum, and (ii) the definition of the slope as divided difference forward in time (4.40). We can make the following observations:

(i) While we show majorization for vectors with weighted components $c_i x_{[i]}$, this scaling of the state is only required in the step from (4.41) to (4.42), where $S_n = const$ is used. That is, the weighting is needed in its role as defining the system invariant. It is not needed in proving the majorization inequality for indexes $j = 1,\dots,n-1$, as (4.43) shows: here, $\dot{S}_j \leq 0$, due $x_{[j]}(t+\epsilon) < x_{[j]}(t)$, irrespective of the weightings $c_j$. Note that, while state components $x_{[j]}$ are ordered, the components $c_j$ are not necessarily in the same non-increasing order. In fact the ordering in the vector $\boldsymbol{c}$ may be chosen independent from the ordering of $\boldsymbol{x}$, as the action of a stochastic matrix is invariant to permutation [Sep11].

(ii) As (4.43) shows, the interplay with convexity of $E$ and the Lyapunov property arises from the divided differences $R_{H,i}$ as defined in (4.40). These have the functional form of the nonlinear resistor class that we define, e.g., in Theorem 13. Here, the functions $H$ are not local energy gradients but local energy itself; however, put into relation forward in time. The interplay of convexity with decreasingness of $E$ follows, however, from a spatial context, where at fixed time $t$ sorted energy resistances are compared (where convexity yields $R_{H,i} - R_{H,i+1} > 0$).

The result of Theorem 15 allows us to define the concept of differential majorization, which may serve helpful in stability studies of dynamical systems, in particular those which converge to a one-dimensional attractor set, as in consensus networks.

**Definition 9** (Differential majorization)**.** Let $I$ be a real interval and $\boldsymbol{x} : \mathbb{R} \to I^n \subset \mathbb{R}^n$ a Lipschitz continuous one-parameter curve $\boldsymbol{x}(t)$. We say $\boldsymbol{x}(t)$ is differentially majorized, if for each $t \geq 0$, there is a graph $\mathsf{G} = (N, B, w)$, with left-Perron vector $\boldsymbol{c}$ associated to the corresponding Laplacian matrix such that

$$\dot{S}_k = \sum_{i=1}^{k} c_i \dot{x}_{[i]}(t) = \sum_{i=1}^{k} c_i \left( \sum_{j=1, j\neq i}^{n} w_{ij}(x_{[j]}(t) - x_{[i]}(t)) \right) \leq 0, \quad \text{for } k = 1, \ldots, n-1, \quad \text{and}$$
$$\dot{S}_n = 0,$$

holds.

Intuitively, differential majorization indicates that at time $t$ the velocity vector along a curve always points into the interior of the $n-1$-dimensional convex set spanned by permutations of $\boldsymbol{x}(t)$ on the manifold, where invariance $\dot{S}_n = 0$ holds.

Note that differential majorization does not require differentiability of trajectories. It may as well be used for switching systems governed by Laplacians $\mathbf{L}(t)$.

*Remark* 27 (Contracting polyhedron and differential majorization). In [CHJ14] and [CHJ17] the authors use a polyhedral norm to study convergence to consensus. They show that consensus systems contract the polyhedron defined by lines parallel to the $\mathbf{1}$-vector, each having a different permutation of the state $\boldsymbol{x}$ an element of the line. Within the definition of differential majorization, the velocity vectors point inwards this polyhedron. The constraint $\dot{S}_n = 0$, however, further specifies the manifold on which trajectories evolve.

*Remark* 28. As in Definition 9 the graph and associated Perron vector are assumed constant over time, equivalently, differential majorization holds if for every convex $E$ as in (4.24),

$$\dot{E}(\boldsymbol{x}) = -||\nabla E(\boldsymbol{x})||^2_{\mathbf{K}} \tag{4.45}$$

holds, with Onsager matrix $\mathbf{K}$ as given in Theorem 14. However, in applications the Laplacian graph may change over time, e.g., when $\boldsymbol{c}$ is a state-dependent function, as for instance is the case in the embedding protocol introduced in Chapter 3, or in the case of the reaction network protocol introduced in Chapter 2. While differential majorization then still holds, and so does convergence to consensus, the quadratic energy inequality (4.45) may not hold, or at least turn out to be more challenging to establish, as differentiation of $\boldsymbol{c}(\boldsymbol{x})$ needs to be considered in the proof. Hence, the appropriate stored energy function weighting needs to be derived.

### 4.4.3 Cut-balance and differential majorization

In [HT13] J.M. Hendrickx and J.N. Tsitsiklis study the classical linear consensus system on a time-varying graph $\mathsf{G}(t) = (N, B(t), w)$,

$$\dot{x}(t) = \sum_{j \in N_i^+} w_{ij}(t)(x_j(t) - x_i(t)), \tag{4.46}$$

and introduce the cut-balance property.

**Definition 10** (Cut-balance, acc. to [HT13])**.** Let $K \geq 1$. A consensus system governed by (4.46) is said to satisfy the cut-balance condition if for any nonempty proper subset $M$ of $N = \{1, 2, \ldots, n\}$ we have

$$\frac{1}{K} \sum_{i \in M, j \notin M} w_{ji}(t) \leq \sum_{i \in M, j \notin M} w_{ij}(t) \leq K \sum_{i \in M, j \notin M} w_{ji}(t). \tag{4.47}$$

This condition bears the physics-oriented notion of proportional back-actions, i.e., if a group of agents influences the remaining ones, the former group is affected by the remaining ones by at least a proportional amount [HT13].

*Remark* 29 (Cut-balance and leader-follower networks). Leader-follower networks do not satisfy cut-balance, as a leader typically does not receive a back-action from the followers. For instance, consider a two-node leader-follower graph, so that the Laplacian matrix $\mathbf{L}$ has a lower triangular structure. In that case, the graph is not strongly connected, and it does not satisfy cut-balance, e.g., as $w_{12} = 0$, but $w_{21} > 0$, so that for any $K \geq 1$ the second inequality $Kw_{21} \geq w_{12}$, but the first one $w_{12} \geq \frac{1}{K} w_{21}$ can never be true. The leader-follower system nevertheless converges to a consensus state, as the graph has a directed spanning tree, which guarantees convergence to the leader state, cf., e.g., [Mor04].

While this cut-balance condition is hard to verify and does not yield a broader result for the convergence to consensus; its study is, however, motivated by the use of a different proof technique as is typically used in convergence proofs, which rely on quadratic or span-norm contraction properties.

The authors show that consensus protocols (4.46) that satisfy the cut-balance condition converge to a consensus state if the time-varying graph is uniformly connected (over time). The dissipation mechanism underlying the cut-balance property uses sorted state vectors and bases on the rate of change of weighted sums of state components. The following result is instrumental for the cut-balance convergence proof.

**Lemma 7** (Cut-balance property, adapted from [HT13])**.** *Let $b_{ij}$ be non-negative coefficients that satisfy the cut-balance condition* (4.47) *as specified in Definition 10. Then,*

$$\sum_{i=1}^{m} K^{-i} \left( \sum_{j=1, j \neq i}^{n} b_{ij} (x_{[j]} - x_{[i]}) \right) \leq 0, \tag{4.48}$$

*for every sorted vector $\boldsymbol{x} \in \mathbb{R}^n$, and every $m \leq n$.*

*Proof.* The proof follows the exact steps as the one of [HT13] Lemma 2, with the following difference: here, the vector $\boldsymbol{x}$ is sorted non-increasinlgy, such that (4.37) holds, whereas the authors of [HT13] consider a non-decreasing order, i.e., such that $x_1 \leq x_2 \leq \cdots \leq x_n$. While in our case, this leads to the inequality (4.48) of being smaller or equal zero, the authors of [HT13], with their respective vector sorting, obtain inequality (4.48) with the left-hand-side being greater or equal zero. □

*Remark* 30. In [HT13] the authors proof the inequality (4.48) for the case where weighting factors $K^{-i}$ are replaced by weights $w_i > 0$, such that $w_1 > w_2 > \cdots > w_n$, i.e., for weights that are in non-increasing order, as is the case for the vector $\boldsymbol{x}$ in our setting.

Comparison of the cut-balance condition in Lemma 7 with our differential majorization condition in Definition 9 shows that the two defining inequalities are identical upon association of the weights $b_{ij}$ with $w_{ij}$ and the factors $K^{-i}$, resp. factors $w_i$ see Remark 30, with the components $c_i$ in non-increasing order.

**Theorem 16** (Cut-balance – differential majorization equivalence)**.** *A consensus system that converges to a consensus state satisfies cut-balance if and only if differential majorization holds at each $t \geq 0$, as specified in Definition 9. That is, cut-balance is equivalent to the existence of an RC network with locally passive resistor elements.*

*Proof.* To prove (4.48), in [HT13] the authors show that

$$\sum_{i=1}^{n} w_i \left( \sum_{j=1}^{n} b_{ij}(x_{[j]}) - x_{[i]} \right) \leq 0, \tag{4.49}$$

with $w_i$ as in Remark 30. For that they rewrite (4.49) as

$$\sum_{i=1}^{n} x_{[i]} \left( \sum_{j=1}^{n} w_j b_{ji} - \sum_{j=1}^{n} w_i b_{ij} \right) = \sum_{i=1}^{n} x_{[i]} e_i$$

where the last equality serves as defining equation for $e_i$. The proof works by showing that $\sum_{i=1}^{n} x_{[i]} e_i \leq 0$ if and only if (a) $\sum_{i=1}^{n} e_i = 0$ and (b) $\sum_{i=1}^{k} e_i \leq 0$.

Here, using property (a), we show that differential majorization holds, which then establishes (4.49) directly.

Let $b_{ij}$ be the weights $w_{ij}$ of the Laplacian matrix that generates a consensus flow (which is by assumption the case) at time $t$ . Assign $w_i = c_{[i]}$, with $\mathbf{c}$ the left Perron vector of $\mathbf{L}(t)$, where bracket index notation indicates a non-increasing order, i.e., $c_1 > c_2 > \cdots > c_n > 0$. This re-ordering does not alter the dynamic properties of the consensus system, as consensus dynamics are invariant to permutations [Sep11]. Hence, without loss of generality one may permute the rows of $\mathbf{L}(t)$ such that the left Perron vector is appropriately ordered.

Observe that condition (a) essentially requires the matrix $\mathbf{CL}$ to be balanced. As $\mathbf{L}$ is a Laplacian, it has zero row-excess, i.e., $\sum_{j=1}^{n} c_i w_{ij} = 0$, or in vector matrix form, $\mathbf{CL1} = \mathbf{0}$. Therefore,

$$\sum_{j=1}^{n} c_j w_{ji} = \sum_{j=1}^{n} c_i w_{ij} = 0 \quad \Leftrightarrow \quad \mathbf{1}^\top \mathbf{L}_b(t) = \mathbf{0}^\top,$$

where $\mathbf{L}_b = \mathbf{CL}$ is a balanced Laplacian matrix, by definition, as it has zero row - and column-excess.

With that, and using our re-naming of coefficients, the inequality (4.49) is identical to

$$\sum_{i=1}^{n} c_{[i]} \left( \sum_{j=1}^{n} w_{ij}(x_{[j]}) - x_{[i]} \right) = \sum_{i=1}^{n} c_{[i]} x_{[i]}) \triangleq \dot{S}_n \leq 0,$$

where $\dot{S}_n$ is as defined in Definition 9, so that $\dot{S}_k \leq 0$, for indexes $k < n$. This establishes sufficiency of differential majorization for (4.49) to hold.

To see the necessity, we need to show that (4.49) implies differential majorization. Following the authors of [HT13], if (4.49) holds and the graph is uniformly connected over time, then a consensus state is reached. Convergence to consensus requires differential majorization to hold over uniformly connected graphs, as there must be a stochastic matrix that maps any state to the consensus equilibrium state; the Laplacian that acts as a generator of this transfer operator, see Proposition 6, then establishes differential majorization.

The equivalence cut-balance to local passivity follows from Theorem 15, which shows the local-passivity – differential majorization equivalence. □

This majorization context to the cut-balance property links the cut-balance relation on weights of the generator matrix for the network dynamics, with the concept of passivity and energy functions serving as Lyapunov functions for the case of convergence to consensus, see Remark 28. The differential majorization property is also a contraction property, which originates in the nested set equivalence for majorizing states, see Fig. 4.8. Hence, beyond the context of Lyapunov functions, the non-increasing character of an energy function, which is in equivalence with a majorization property, leads to convergence of dynamics, or any two trajectories resulting from the action of the same transfer operator matrix, independent of the knowledge of a particular equilibrium state. This relationship may serve as a starting point for novel differential or incremental stability studies in dynamical systems that do not rely on linearization approaches but rather a vector field transformation to a Laplacian form.

## 4.5 Application to Markov chains

The passive circuit formulation of nonlinear consensus networks is applied to Markov chains. This provides a dynamic RC circuit formulation for Markov dynamics and a passivity-based proof for information divergences serving as Lyapunov functions. We close this section by discussing a discrete version of the continuous De Bruijn identity in information theory and stochastic processes. Our framework also provides the context of passive circuits.

### 4.5.1 Markov chains in passive circuit formulation

A pillar of the circuit concept is the conservation of total charge, see for instance [Val74] Chapter 1. In the following, we relate the nonlinear charge dynamics associated with the consensus class (4.2) to the dynamics of a (spatially) inhomogeneous Markov chain.

With $\mathbf{K}(\cdot)$ being a symmetric, irreducible Laplace matrix, $\ker(\mathbf{K}(\cdot)) = \mathsf{span}\{\mathbf{1}\}$, so that $\sum_{i\in N} q_i(t) = const.$ for all times $t \geq 0$, since $\mathbf{1}^\top \dot{\boldsymbol{q}} = 0$. This recovers the conservation principle for charge in our circuit interpretation of Theorem 13. Without loss of generality, we can choose $\boldsymbol{q}(0) \in \mathbb{R}^n_{>0}$ such that $\sum_i q_i(0) = 1$. The normalized $\boldsymbol{q}$-vector then also has the interpretation of a probability mass distribution on a discrete probability space: Each node $i \in N$ is a possible state, and $q_i(t)$ is the probability of a random walker on a graph $\mathsf{G}$ of being in state $i$ at time $t$, see, e.g., [Lov93].

The equation system describing the probability transport associated to the nonlinear averaging dynamics (4.2) in $\boldsymbol{x}$-variables follows from the gradient formulation in Theorem 13 with the admissible choice $H(x) = \frac{1}{2}x^2$, so that $\nabla E(\boldsymbol{q}) = \boldsymbol{x}$:

Define $\mathbf{F}^\top(\cdot) := \mathbf{K}(\cdot)\mathbf{C}^{-1}$ and observe that $\mathbf{F} = \mathbf{C}^{-1}\mathbf{K}$ is an irreducible (non-symmetric) Laplace matrix satisfying detailed balance. Then,

$$\dot{\boldsymbol{q}} = -\mathbf{K}(\boldsymbol{q})\boldsymbol{x} = -\mathbf{K}(\boldsymbol{q})\mathbf{C}^{-1}\boldsymbol{q} = -\mathbf{F}^\top(\boldsymbol{q})\boldsymbol{q}.$$

where $\mathbf{F}^\top$ is the infinitesimal generator of a (spatially inhomogeneous) Markov chain. This Markov chain asymptotically reaches the invariant probability measure given by the normalized capacitances $\boldsymbol{c}$ with $\sum_{i\in N} c_i = 1$.

This equivalence between passive $RC$-circuits and Markov chains as dynamical systems bears the following novelties:

(i) The usual relation between electric circuits and Markov chains in the applied mathematics literature restricts to a static equivalence between a resistor network and the probability transition kernel of the Markov chain, cf., the seminal work [DS84] and references therein. In the engineering literature, a dynamical relationship is known only for LTI symmetric consensus systems, which are equivalent to homogeneous, symmetric Markov chains and unit-capacitance RC-circuits, see, e.g., [ME10] Chapter 3. We extend those results to dynamical, nonlinear $RC$-circuits, where we show the relationship between detailed balance and non-unit capacitances.

(ii) With capacitances $\boldsymbol{c}$ such that $||\boldsymbol{c}||_1 = 1$, stored energy $E(\boldsymbol{q}) = \sum_i c_i H(c_i^{-1} q_i)$, $H$ strictly convex, corresponds to the class of information-divergences of a probability distribution $\boldsymbol{q}$ to the equilibrium distribution $\boldsymbol{c}$, introduced by Ali and Silvey [AS66] and Csiszár, cf., [CS04], and see Table 4.1 for examples. The usual technique to prove decreasingness of $E$ is based on Jensen's inequality [LV06], see also [Wil76]. Theorem 13 and Theorem 14 establish this dissipation inequality for the class of Csiszár's information divergences in a novel way, namely by exhibiting a passive RC circuit structure.

Next, we interpret a well-known information equality for stochastic processes in the discrete context of passive electric circuits realized by interconnecting the proposed nonlinear capacitor and resistor elements.

### 4.5.2 Discrete De Bruijn's identity

Two elementary quantities in information theory are differential entropy, a measure of the descriptive complexity of a random variable, and Fisher information, a measure of the minimum error in estimating a parameter from a distribution. Let $S \subseteq \mathbb{R}$ be the support set of a random variable $X$ of finite variance, and let $x(\xi) > 0$, $\forall \xi \in S$, be a probability density distribution[1] for $X$. The differential entropy then is defined as given in [CT91] Chapter 9,

$$\mathrm{Ent}(X) := -\int_S x(\xi) \ln x(\xi) \mathrm{d}\xi.$$

The Fisher information of the distribution of $X$ can be defined as in [CT91] Chapter 16.6,

$$\mathscr{J}(X) := \int_S x(\xi) \left( \frac{\nabla x(\xi)}{x(\xi)} \right)^2 \mathrm{d}\xi = \int_S |\nabla \ln x(\xi)|^2 x(\xi) \mathrm{d}\xi. \tag{4.50}$$

[1] i.e., the gradient of the cumulative probability distribution function on $S$

This is a special form of the Fisher information, taken with respect to a translation parameter on the continuous support $S$, which does not involve an explicit parameter as in its most general definition [CT91] Chapter 12, see also [PV06].

De Bruijn's identity provides a relationship between these two quantities: For a process $Y = X + \sqrt{t}Z$, where $Z$ is a normally distributed random variable,

$$\frac{\partial}{\partial t}\mathrm{Ent}(Y) = \frac{1}{2}\mathscr{J}(Y),$$

see [CT91] Theorem 16.6.2.

In the context of porous medium equations, Erbar and Maas in [EM14] propose the discrete version of Fisher information

$$J(\boldsymbol{x}) := \frac{1}{2}\sum_{i,j\in N} c_i w_{ij}\phi(x_j, x_i)\big(h(x_j) - h(x_i)\big),$$

where $\phi(x_j, x_i) = g(x_j) - g(x_i)$, $g$ an increasing function. Note that the gradient of a function on a discrete space $(N, B, w)$ is given by the (weighted) difference of the function values at connected nodes.

For instance, if we choose $g(x_j) - g(x_i) = x_j - x_i$, and relative entropy $E(\boldsymbol{q}) = \sum_i c_i q_i \ln \frac{q_i}{c_i}$ as energy, so that $h = \ln$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\boldsymbol{q}) = -J(\boldsymbol{x}) = \frac{1}{2}\sum_{i,j\in N} c_i w_{ij}\mathsf{lgm}(x_j, x_i)\left|\ln x_j - \ln x_i\right|^2, \tag{4.51}$$

where $\mathsf{lgm}(x_j, x_i) := \frac{x_j - x_i}{\ln x_j - \ln x_i}$ is the logarithmic mean of two positive variables, and the squared difference $|\ln x_j - \ln x_i|^2$ is the discrete equivalent to $|\nabla \ln x|^2$ in (4.50).

In the definition of Fisher information this gradient is integrated w.r.t. $x\mathrm{d}\xi$. On a discrete space, gradients live on the set of edges $B$, while the density vector $\boldsymbol{x}$ is defined for elements indexed in the set $N$. The logarithmic mean accounts for this lack of support in the discrete case: By the mean value theorem, there exists a value $x_{ij} \in [x_i, x_j]$, (where we suppose that density components satisfy $x_i < x_j$), such that

$$\nabla \ln x_{ij} = \frac{\ln x_j - \ln x_i}{x_j - x_i} \iff x_{ij} = \mathsf{lgm}(x_j, x_i).$$

Further, $\mathsf{lgm}^{-1}(x_i, x_j) \equiv \int_0^1 \frac{\mathrm{d}\xi}{x_i\xi + (1-\xi)x_j}$ [Car72], so that an "edge density" $x_{ij}$ can be seen as a (convex) interpolation of the density across edges $e = (j, i) \in B$ based on knowledge of density components $x_i, x_j$ defined on nodes $i, j \in N$.

Let us consider the discrete De Bruijn inequality for a system 4.2 with coupling defined by the action function $\phi(x_j, x_i = \sin(x_j - x_i)$. Then,

$$J(\boldsymbol{x}) = \frac{1}{2}\sum_{i,j\in N} c_i w_{ij}\frac{\sin(x_j - x_i)}{\ln x_j - \ln x_i}\left|\ln x_j - \ln x_i\right|^2.$$

Using a discrete chain rule, $x_{ij} = \frac{\sin(x_j - x_i)}{x_j - x_i} \frac{x_j - x_i}{\ln x_j - \ln x_i} \triangleq \mathrm{sinc}(x_j - x_i)\mathrm{lgm}(x_j, x_i)$. This is the logarithmic mean modulated by a sampling function kernel that takes values between zero and one (on sets where each edge corresponds to a strictly passive resistor).

As far as we know, this connection between De Bruijn's identity in information theory and the dissipation equality (4.51) as a discrete version of it is novel. A discrete version on domains characterized by graphs is natural in applications, where high-dimensional data reside on nodes of graphs, and (nonlinear) weightings may describe an application's peculiarity in terms of the irregularity of the domain. It would be interesting to further understand the role of discrete instead of discretized continuous information inequalities, as the presented one of De Bruijn, within the field of signal processing on graphs [Shu+13].

## 4.6 Summary and concluding remarks

In this chapter, we provided various equivalences of known dissipation mechanisms in consensus theory to the passivity of resistors. When interconnected neutrally with a lossless capacitor bank, we entirely determine the consensus dynamics' dissipation behavior. The study of a passive circuit formulation of nonlinear consensus networks is motivated by the general structure result for passive nonlinear systems due to Hill, Moylan, and Anderson. It results in the synthesis of a gradient structure for nonlinear consensus networks using nonlinear passive resistor and capacitor elements only. The passivity of resistors turns out equivalent to the strict convexity of stored energy. The gradient and passive circuit formulation of consensus networks lead us to study majorization and cut-balance techniques in the context of dissipation mechanisms for consensus networks. It motivates the introduction of the concept of differential majorization, which lays out a physical and passive electric circuit framework for majorization theory. Cut-balance, too, turns out equivalent to local passivity via differential majorization. The connection between the cut-balance proof for convergence in consensus dynamics and majorization is highlighted. We apply the obtained results to Markov chains and discuss the example of phase-coupled oscillators models in electric power grid dynamics and control.

The concepts developed in this chapter can be further applied in the context of Markov chains, e.g., in the study of interconnections of stochastic systems. The difficulty lies in finding the appropriate state definition for the resulting interconnected system, based on probability measures serving as state functions for the interconnectants. The discrete setting of probability spaces represented as graphs in combination with the passive circuit formulation we developed here eases the study, as interconnecting electric circuits is a process that is well-defined via the concept of terminals and ports. Further, the nonlinear resistor elements given by the logarithmic mean and entropy as storage function allow studying the limit where the number of nodes in the graph goes to infinity, as these functions are smooth.
Another fruitful direction is the study of Monge-Kantorovich optimal transport problems via gradient flows through the setting of de Giorgi's minimizing movement scheme, see [San15] for general reference, using the setting of passive electric circuits and information divergences as potential functions. Optimal transport has recently gained strong interest also in the machine-learning community. It provides rigorous proof and expresses mechanisms for

evolving and redistributing information from one state to another, as opposed to Bayes theorem, with only an axiomatic foundation, relates prior and posterior distributions to another without an evolution prescription. Deep learning processes have recently been represented in a differential equation system framework using a time-discretization scheme, see [Che+18], which also is reminiscent of de Giorgi's minimizing movement scheme.
The further development of the concept of differential majorization and application to systems with state-dependent Perron vectors may be a fruitful next step. It provides a means of proofing convergence in a dynamical system using a dissipation inequality without the need to specify an equilibrium point. This situation reminds of dissipativity studies for open systems, however, in the context of autonomous dynamical systems. Our use of the class of network systems seems to be a good fit. It further allows studying the stability of 1-dimensional sets that may be dynamic due to state-dependent or exogenously driven Perron vector, which specifies asymptotic and invariance properties. As such, the concept of differential majorization seems to be related to the recent differential dissipativity studies in [FS14], without, however, requiring a localization, i.e., linearizing dynamics around states along trajectories.

5

# Summary and outlook

We introduce a novel framework for designing and analyzing linear and nonlinear dynamical systems alike that form consensus networks. The significance of this framework results i) from its capability to comprehend existing application-oriented and more abstract methods for the design of convergent consensus protocols; ii) it leverages Moreau's most general stability and convergence result for the linear, time-varying consensus case by using axiomatic function properties of metrics and means as tools to transform large classes of nonlinear consensus equation systems to linear form with dynamic Laplacian system matrix, such that it is compatible with Moreau's result. And iii) we demonstrate that the mathematical tools and transformations we propose result in a system description based on passive electric network elements that interact in a lossless-dissipative negative feedback, i.e., RC loop. The framework we introduce has passivity and network theory at its core, which we use in deriving a passivity-equivalence to the famous majorization property and to the recent cut-balance property for convergent consensus systems. The abstract mean-metric consensus design framework results in three distinct classes of consensus protocols: the metric action consensus protocol, the mean-driven consensus protocol, and the embedding protocol. The metric action protocol comprises most large existing consensus design methods, i.e., those proposed by Hui and Haddad, Wei and van der Schaft, and Murray and Olfati-Saber. The mean-driven design rule has novel optimality properties characterizing the dynamics of the consensus network infinitesimally rather than the asymptotically reached equilibrium state. The embedding protocol derived from a mean-metric generalization of the linear consensus dynamics is an instance of the more complicated geometric generalization of Sarlette and Sepulchre on consensus dynamics from linear to nonlinear spaces. We show connections to gradient flow and algorithmic (Newton-type gradient descent) properties. Application of the mean-driven design rule to geometric mean-driven consensus networks yields consensus conditions reminiscent of Wegscheider's detailed balance condition in chemistry. Based on

this observation, we study the equation system of mass-action chemical reaction networks and transform it to a linear Laplacian form, which represents a novel passive circuit and conductance formulation of this chemical network system. The significance of this formulation stems from the fact that here, other than in other circuit representations, the chemical potentials directly drive the species dynamics and do so by acting across a resistor element with conductance formulation that has the form of a classical heat exchanger element across which chemical potential differences lead to dissipation.

Further research directions can follow application-oriented and more abstract, methods-oriented pathways. In view of the geometric mean-driven consensus dynamics, it seems interesting to further analyze the relationship between solutions of elliptic integrals and the consensus value of the polynomial network protocol and its connection to arithmetic-geometric mean nested algorithms. The non-equilibrium convergence behavior of the polynomial network protocol serves as an interesting case study for analyzing Laplacian systems with state-dependent dominant left-eigenvector and as an instance to work on differential dissipation concepts. The novel passive circuit representation of mass-action chemical reaction systems seems worth further exploring, e.g., in view of non-equilibrium stationary states in practice. In general, the highlighted optimality properties of the differential dynamics seem to be worth exploring within the gradient formulation using duality and the KKT formulation with the idea to solve relevant computational problems through dynamic systems synthesized by nonlinear passive RC circuits. This RC circuits view on computational problems may serve as a physical and engineering basis to the mathematical theory of optimal transport, which recently has gained popularity in the machine learning community. It serves as a mathematical foundation to the discipline rather than the axiomatic Bayes theorem. Further, the relationship to Newton's gradient descent scheme in the embedding protocol in connection with state-dependent dominant left-eigenvectors might help study accelerated, distributed gradient descent algorithms. A fruitful conceptual direction is the study of interconnected stochastic (Markov) systems by using the electric circuit equivalence for which interconnection is well-defined. From there, limits from discrete to continuous state (i.e., probability) spaces and the effect of external (control) inputs seem interesting. In view of methods for studying stability and convergence of dynamical systems, the transformation methods to linear Laplacian form with the passivity view serve as a starting point to derive novel convergence and differential stability methods using Lyapunov-like energy functions. A key problem is to consider the flow map of the dynamics and find transformations that lead to an equivalent form given by a possibly state- and time-dependent stochastic differential update matrix.

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