Information-Constrained Model Predictive Control with Application to Vehicle Platooning

V. Causevic * **Y. Fanger** * **T. Brüdigam** ** **S. Hirche** *

* *Chair of Information-oriented Control, Technical University of Munich, Munich, CO 80333 Germany (e-mails: vedad.causevic@ tum.de, yunis.fanger@tum.de, hirche@tum.de).*
** *Chair of Automatic Control Engineering, Technical University of Munich, Munich, CO 80333 Germany (e-mail: tim.bruedigam@tum.de).*

Abstract: Information constraints, induced by a delayed communication between individual decision makers (DMs) pose a significant challenge for the optimal control of physically interconnected systems, as for example vehicle platoons. In this work, we address the problem of distributed state/input power-constrained optimal control of a vehicle platoon under the assumption that neighboring vehicles communicate to each other with one step delay. In order to account for sudden changes in the environment such as different speed limits, or change of the desired relative distances between vehicles, a model-predictive control (MPC) approach is adopted. Despite the information constraints and state/input constraints we prove the optimality of linear control policy. To this end, we provide an optimal structure of control law that is imposed into the MPC optimization problem, to account for information constraints induced by one-step communication delay between neighboring vehicles. The efficacy of the approach is illustrated in simulation.

Keywords: vehicle platooning, model-predictive control, communication delays, optimal control, constrained control

1. INTRODUCTION

The developments in information technology are enabling the possibility of automated control in smart buildings (Causevic et al., 2018), transport industry etc. Of particular relevance in the transport industry is the problem of governing vehicle platoons with a large potential for improving overall traffic flow (Ioannou and Chien, 1993). The distributed nature of these systems and mobility aspect require wireless communication between individual vehicles. Consequently, the complete state information is not necessarily instantly accessible to each subsystem (vehicle). This introduces constraints on the admissible control actions for each vehicle, herein referred to as "information constraints".

In general, the design of optimal control laws under information constraints is a difficult problem. Depending on how fast the decision makers (DMs) communicate with each other, the optimal control problem might be convex or non-convex, even in settings describing linear systems with a quadratic cost function (Witsenhausen, 1968). A strong result characterizing the biggest class of information constraints under which the linear quadratic control problem can be cast as a convex problem is given in (Rotkowitz and Lall, 2006). Finally, the design of quadratically invariant information structures for distributed systems with intermittent observations is presented in (Abara et al., 2018). On the other hand, a lot of focus has been given to the design of optimal control laws for fixed information structures that have the property of being partially nested (Ho and Chu, 1972). Some of them include first explicit solutions to linear Quadratic Gaussian team problems e.g. (Lamperski and Doyle, 2012) under the assumption that information between individual decision makers is communicated at least as fast as it travels through the plant. Although such results provide insight into the structure of optimal control policy, under information constraints, such control design is not suitable for a vehicle platoon, where safety measures and limited actuation capabilities have to be considered.

From a theoretical point of view, the first result addressing linear quadratic Gaussian team problem under state/input constraints and subject to information constraints, is given in (Causevic et al., 2018). However, the latter methodology is not suitable for the vehicle platooning problem addressed here due to the following reasons. First, in (Causevic et al., 2018), the assumption is that information is communicated at the exact speed at which its propagates through the interconnected system. This is not the case for vehicle platoon control as the physical coupling is one-directional, while wireless broadcast communication between neighboring vehicles results in bidirectional information exchange. Second, the methodology is not capable to dynamically adapt to sudden changes in the environment which are highly present in vehicle platooning control. In (Feyzmahdavian et al., 2012) an aproach is proposed for optimal distributed control of platoon with delayed information sharing. The method is applied to heavy duty vehicles. However, the setting does not take into account state/input power constraints which are of importance in vehicle platoon, because of limited actuation power and/or safety measures, e.g. constraint on the deviation of relative distances between vehicles from desired values. Furthermore, the method can not deal with sudden changes in desired reference velocities/distances in a vehicle platoon. The optimal control design for a vehicle platoon with delayed information sharing, subject to state/input

⋆ This project has received funding from the German Research Foundation (DFG) within the Priority Program SPP 1914 "Cyber-Physical Networking".

power constraints and under sudden changes in the environment, e.g. different speed limits, is still a largely open problem. The contribution of this paper is a control design approach for a vehicle platoon under information constraints and sudden changes in environment such as: change in target speeds, target relative distances, change of safety constraints. The information constraints are induced by the one-step delayed information between neighboring vehicles. The framework guarantees the satisfaction of so-called safety/actuation constraints expressed as expectations of quadratic functions of state/input. To combat the issue of changing state/input constraints and target state values of the system, imposed by a dynamic environment, Model Predictive Control (MPC) is employed. Interestingly, we show the partial nestedness of the arbitrary-size vehicle platoon under one-step delayed information and further prove that linear policies are optimal even under state/input power constraints. The computation of such optimal distributed control policies is validated via simulation.

The remainder of the paper is outlined as follows. We start with problem setup in section 2. The methodology to compute distributed control law for a vehicle platoon and apply it in MPC fashion is presented in section 3. In section 4 we provide illustration of methodology via simulation. Finally conclusions are given in section 5.

Notation Vectors and matrices are denoted by bold symbols. For a time-varying vector $\boldsymbol{x}(k)$ we denote by $\boldsymbol{x}(k_1:k_2)$ stacked vector $\boldsymbol{x}(k_1:k_2)^\top = [\boldsymbol{x}^\top(k_1), \boldsymbol{x}^\top(k_1-1), \ldots, \boldsymbol{x}^\top(k_2)]$, where $k_1 < k_2$. The operator $(\cdot)^\top$ denotes the transpose. The expectation operator is denoted by $\mathbb{E}[\cdot]$ and the variance operator is denoted by $Var[\cdot]$.

2. PROBLEM SETTING

2.1 Platoon model

We consider $N > 1$ vehicles moving longitudinally in a platoon. As a simple model for the dynamics of such platoon, a double integrator model is used. Each vehicle $i \in \{1, \ldots, N\}$ measures its absolute velocity $v_i \in \mathbb{R}$. Additionally, each vehicle $i \in \{2, \ldots, N\}$ also measures its relative distance to the predecessor vehicle $i-1$ denoted by $d_i \in \mathbb{R}$. Note that since defined distances are relative, the first vehicle will only have its absolute velocity as a state variable. To this end, the state $\boldsymbol{x}_i$ of vehicle i at discrete time instants k is defined as

$$\boldsymbol{x}_i(k) := \begin{cases} v_1(k) \in \mathbb{R}, & i = 1 \\ \begin{bmatrix} d_i(k) \\ v_i(k) \end{bmatrix} \in \mathbb{R}^2, & i = 2, \ldots, N \end{cases} \tag{1}$$

By choosing the individual accelerations $u_i(k), i = 1, \ldots, N$ as control input signals, and given Δ_t as the sampling interval (time period between two discrete time instants) we next derive the dynamics equations for the introduced state variables (1). Assuming that within a sampling interval the accelerations of vehicles are uniform, velocity x_1 of the first vehicle evolves as

$$x_1(k+1) = x_1(k) + \Delta_t u_1(k) + w_1(k) \tag{2}$$

where $w_1(k) \sim \mathcal{N}(0, \Sigma_{w_1})$, i.e. $w_1(k)$ is a zero-mean Gaussian noise with finite covariance Σ_{w_1}. Similarly for vehicles $i = 2, \ldots, N$ the relative distances to respective predecessors and absolute velocities are written as

$$\begin{aligned} \boldsymbol{x}_i(k+1) &= \begin{bmatrix} d_i(k+1) \\ v_i(k+1) \end{bmatrix} = \\ &\begin{bmatrix} d_i(k) + \Delta_t(v_{i-1}(k) - v_i(k)) + \frac{1}{2}(u_{i-1}(k) - u_i(k))\Delta_t^2 + w_{i,d}(k) \\ v_i(k) + u_i(k)\Delta_t + w_{i,v}(k) \end{bmatrix} \\ &= \boldsymbol{A}_{ii}\boldsymbol{x}_i(k) + \boldsymbol{A}_{i,i-1}\boldsymbol{x}_{i-1}(k) + \boldsymbol{B}_{ii}u_i(k) + \boldsymbol{B}_{i,i-1}u_{i-1}(k) + \boldsymbol{w}_i(k) \end{aligned} \tag{3}$$

where the state space matrices are

$$\boldsymbol{A}_{ii} = \begin{bmatrix} 1 & -\Delta_t \\ 0 & 1 \end{bmatrix}, \boldsymbol{A}_{i,i-1} = \begin{cases} \begin{bmatrix} \Delta_t \\ 0 \end{bmatrix} & i = 2 \\ \begin{bmatrix} \Delta_t & 0 \\ 0 & 0 \end{bmatrix} & i = 3, \ldots, N \end{cases}$$

$$\boldsymbol{B}_{i,i-1} = \begin{bmatrix} \frac{1}{2}\Delta_t^2 \\ 0 \end{bmatrix}, \boldsymbol{B}_{ii} = \begin{bmatrix} -\frac{1}{2}\Delta_t^2 \\ \Delta_t \end{bmatrix}.$$

The process noises $\boldsymbol{w}_i(k) = [w_{i,d}(k)\, w_{i,v}(k)]^\top \in \mathbb{R}^2, i > 1$ are zero mean i.i.d. Gaussian noises with covariance matrices $\boldsymbol{\Sigma}_{w_i}$. The initial state $\boldsymbol{x}_i(0)$ is Gaussian distributed with mean $\boldsymbol{\mu}_{\boldsymbol{x}_i}$ and finite covariance $\boldsymbol{\Sigma}_{\boldsymbol{x}_i}$. Moreover, $\boldsymbol{x}_i(0)$ and $\boldsymbol{w}_i(k)$ are assumed pair-wise independent at each time instant k and for every i.

The system defined by equations (2), (3) is a dynamical system composed of N physically-coupled linear time-invariant subsystems. It can be described by a graph $\mathcal{G}^P = (\mathcal{V}, \mathcal{E})$. We will refer to it as physical interconnection graph. Each node $i \in \mathcal{V}$ corresponds to one of the subsystems (vehicles) $i \in \{1, \ldots, N\}$. An edge $(j,i) \in \mathcal{E}$ if dynamics of vehicle i is directly affected by vehicle j, either through control input u_j or state $\boldsymbol{x}_j$. According to the definition of $\mathcal{G}^P$ it can easily be verified that it is a connected chain graph, with each edge being directed. The set of physical neighbors of vehicle i is defined as $\mathcal{N}_i = \{j \,|\, (j,i) \in \mathcal{E}\}$ and is equal to $\mathcal{N}_1 = \emptyset$, $\mathcal{N}_i = \{i-1\}, i > 1$. For a more compact notation, equations (2), (3) are rewritten as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{w}(k) \tag{4}$$

where $\boldsymbol{x}(k) = (x_1^\top(k), \ldots, \boldsymbol{x}_N^\top(k))^\top \in \mathbb{R}^n$, $\boldsymbol{w}(k) = (w_1^\top(k), \ldots, \boldsymbol{w}_N^\top(k))^\top \in \mathbb{R}^n$, $\boldsymbol{u}(k) = (u_1^\top(k), \ldots, u_N^\top(k))^\top \in \mathbb{R}^m$, with $n = 1 + 2(N-1), m = N$ and the matrices $\boldsymbol{A}, \boldsymbol{B}$ are defined as

$$\begin{aligned} \boldsymbol{A} &= \left(\begin{array}{c|cccc} 1 & 0 & 0 & \ldots & 0 \\ \hline \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & 0 & \ldots & 0 \\ 0 & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & \ldots & \boldsymbol{A}_{N,N-1} & \boldsymbol{A}_{N,N} \end{array}\right), \\ \boldsymbol{B} &= \left(\begin{array}{c|cccc} \Delta_t & 0 & 0 & \ldots & 0 \\ \hline \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & 0 & \ldots & 0 \\ 0 & \boldsymbol{B}_{32} & \boldsymbol{B}_{33} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \vdots \\ 0 & 0 & \ldots & \boldsymbol{B}_{N,N-1} & \boldsymbol{B}_{N,N} \end{array}\right). \end{aligned} \tag{5}$$

Also we denote the variance of vector $\boldsymbol{w}(k)$ as $\boldsymbol{\Sigma}_{\boldsymbol{w}} = Var[\boldsymbol{w}(k)]$ and mean and variance of $\boldsymbol{x}(0)$ as $\boldsymbol{\mu}_{\boldsymbol{x}} = \mathbb{E}[\boldsymbol{x}(0)]$, $\boldsymbol{\Sigma}_{\boldsymbol{x}} = Var[\boldsymbol{x}(0)]$ respectively. An example of graph $\mathcal{G}^P$ for a platoon of four vehicles is given in Figure 1.

Remark 1. The approach developed here is applied to the vehicle platoon model described by equation (4) with values of matrices $\boldsymbol{A}_{ii}, \boldsymbol{A}_{i,i-1}, \boldsymbol{B}_{ii}, \boldsymbol{B}_{i,i-1}$ defined through equations (2),(3). However, the method also applies to more general linear time-invariant dynamics where matrices $\boldsymbol{A}_{ii}, \boldsymbol{A}_{i,i-1}, \boldsymbol{B}_{ii}, \boldsymbol{B}_{i,i-1}$ can take arbitrary values.

.

Fig. 1. Example of physical interconnection graph for a platoon consisting of four vehicles. Vehicles are denoted by $\mathcal{V}_i (i = 1,2,3,4)$. We can see that the first vehicle is not influenced by the states or control inputs of any of the other vehicles, while every other vehicle is influenced by its preceding vehicle with propagation delay of 1 (by coupling terms $\boldsymbol{A}_{i,i-1}, \boldsymbol{B}_{i,i-1}$).

2.2 Cost function and constraints

We assume that vehicle i communicates its measurements to neighboring vehicles (vehicles $i-1$, $i+1$ for $i > 2$, vehicle 2 for $i = 1$) with a one-step delay. Therefore, admissible control policies γ_k^i at time k are measurable functions of the information available to each vehicle i (also referred to as DM i)

$$u_i(k) = \gamma_k^i(\mathcal{I}_k^i) \tag{6}$$

where $\mathcal{I}_k^i$, $k = 0,\ldots,T-1$, is defined as

$$\begin{aligned} \mathcal{I}_0^i &= \{x_0^i\} \\ \mathcal{I}_k^i &= \{\mathcal{I}_{k-1}^i, x_k^i, u_{k-1}^i\} \bigcup_{j\in\mathcal{N}_i^C} \{\mathcal{I}_{k-1}^j\}, \quad k > 0, \end{aligned} \tag{7}$$

where $\mathcal{N}_i^C = \mathcal{N}_i \cup \{i+1\}$, i.e. the information set of each vehicle i is updated at time instant k by the current state and the one-step delayed information from both: the direct physical neighbor $\mathcal{N}_i$ and vehicle $i+1$. Consequently, unlike the physical interconnection, the communication is assumed bi-directional.

Remark 2. Note that (7) represents the information history that is in principle available to each vehicle i and that increases with time. For the ease of computation and memory optimization, we later introduce sufficient statistics for control policy (6).

The goal is to control the longitudinal movement of N vehicles in a way that keeps their velocity at target value v_{des} and relative distances between vehicles at d_{des}. However, since d_{des} and v_{des} might change due to sudden changes in the environment, an MPC approach is employed. By iteratively recomputing the optimal control inputs of the system with the most current information present to each DM fast reactions to changing requirements are possible (Mayne, 2014). Therefore, the objective is to iteratively minimize the following global cost

$$\begin{aligned} J_{\mathcal{C}}(k_{\text{cur}}) = \mathbb{E}\left[\sum_{k=k_{\text{cur}}}^{k_{\text{cur}}+H-1} \boldsymbol{z}(k)^\top \boldsymbol{Q} \boldsymbol{z}(k)\right] \\ + \mathbb{E}\left[(\boldsymbol{x}(k_{\text{cur}}+H) - \boldsymbol{x}_{\text{des}})^\top \boldsymbol{Q}_H (x(k_{\text{cur}}+H) - x_{\text{des}})\right] \end{aligned} \tag{8}$$

where the vector $\boldsymbol{x}_{\text{des}} = \boldsymbol{x}_{\text{des}}(k_{\text{cur}})$ represents the vector of desired distances between vehicles as well as reference absolute velocites at the current time step $k_{\text{cur}} = 0,\ldots,T-1$ and H denotes the prediction horizon. Matrix $\boldsymbol{Q}$ is partitioned according to the vector $\boldsymbol{z}^\top(k) = \left[(\boldsymbol{x}(k) - \boldsymbol{x}_{\text{des}})^\top \boldsymbol{u}(k)^\top\right]$ i.e.

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{xx} & \boldsymbol{Q}_{xu} \\ \boldsymbol{Q}_{ux} & \boldsymbol{Q}_{uu} \end{bmatrix}. \tag{9}$$

The matrix $\boldsymbol{Q}_{uu}$ is assumed to be a positive-definite matrix, while $\boldsymbol{Q}$ and $\boldsymbol{Q}_H$ are assumed to be semi-definite positive. The cost (8) is to be minimized under the following state-input power constraints

$$\mathbb{E}\left[\boldsymbol{z}(k)^\top \boldsymbol{W}_i \boldsymbol{z}(k)\right] \leq p_k^i, \quad \forall i = 1,\ldots,M \tag{10}$$

where $k = k_{\text{cur}},\ldots,k_{\text{cur}}+H-1$ and $\boldsymbol{W}_i \in \mathbb{R}^{(n+m)\times(n+m)}$, $i = 1,\ldots,M$, is a positive semi-definite weighting matrix. By an appropriate choice of $\boldsymbol{W}_i$, the set of constraints in (10) captures either constraints present in the actuation power of the platoon, or safety constraints (limited deviation of actual platoon speed and relative distances from desired values).

Example 1. An interesting example to illustrate the role of state constraints is a platoon that suddenly increases its velocity due to an increased speed limit on the road. It is crucial to limit the deviation of distances from desired ones (as the failure of one vehicle can have bigger consequences when the platoon is moving at higher velocity).

Remark 3. Constraints (10) are defined in expectation i.e. we require satisfaction of those constraints on average. This (later proven) implies the optimality of linear control policies for the problem addressed here. Furthermore, it allows to pose the control problem addressed here as a mean and covariance selection problem, and efficiently compute optimal control inputs to the vehicles using the properties of Gaussian-distributed variables.

Ultimately, the problem is formally stated as

$$\begin{aligned} \min_{\gamma_{k_{\text{cur}}:k_{\text{cur}}+H-1}} \quad & J_{\mathcal{C}} \\ \text{s.t.} \quad & (4),(6),(10) \end{aligned} \tag{11}$$

where $k \in \{k_{\text{cur}} : k_{\text{cur}}+H-1\}$ and $\gamma_k = [\gamma_k^1,\ldots,\gamma_k^N]$ is composed of all players' control policies. Before stating the main result of this section we define the notion of partial nestedness.

Definition 1. The information structure $\mathcal{I}_k = \{\mathcal{I}_k^1,\ldots,\mathcal{I}_k^N\}$ and system (4) are partially nested if, for every admissible policy (6), whenever $u_i(\tau)$ affects $\mathcal{I}_k^j$, then $\mathcal{I}_\tau^i \subset \mathcal{I}_k^j$.

Lemma 1. (Partial nestedness). The information structure defined by (7) and system (4) are partially nested.

Proof 1. Let d_{ji} be the communication delay between vehicles i and j. Since the neighboring vehicles communicate with one-step delay, it holds that $d_{ji} = |i-j|$. The proof is separated into two parts: $j < i$ and $j > i$.

Case $j < i$ The information set $\mathcal{I}_k^i$ contains $\boldsymbol{x}_i(k)$ (as decision maker i measures its state directly) as well as $\boldsymbol{x}_n(k-d_{ni}), n \neq i$. The most recent control input of vehicle j that influences the information set of vehicle i can be obtained from graph $\mathcal{G}^P$ and it is $u_j(k-d_{ji})$ due to the existence of term $\boldsymbol{B}_{i,i-1}$. Now what is left to prove is that $\mathcal{I}_{k-d_{ji}}^j \subset \mathcal{I}_k^i$. To this end, information sets of decision makers i and j are explicitly written as

$$\begin{aligned} \mathcal{I}_k^i &= \bigcup_{n=1,\ldots,N} \{\boldsymbol{x}_n(0:k-d_{ni})\}, \\ \mathcal{I}_{k-d_{ji}}^j &= \bigcup_{n=1,\ldots,N} \{\boldsymbol{x}_n(0:k-d_{nj}-d_{ji})\}, \end{aligned}$$

which reduces the partial nestedness condition to the following condition: $d_{nj}+d_{ji} \geq d_{ni}$ which holds with equality sign if $n \neq i$. If $n = i$, strict inequality holds. Therefore, the set $\mathcal{I}_{k-d_{ji}}^j$ is a proper subset of the set $\mathcal{I}_k^i$.

Case $j > i$ The difference is that in this case the dynamics of vehicle i is not influenced by u_j as $j > i$. Indeed, the information set $\mathcal{I}_k^i$ is influenced by u_j because it contains $\boldsymbol{x}_j(k-d_{ji})$. This means that unlike the case $j < i$, the most recent control input that influences $\mathcal{I}_k^i$ is $u_j(k-d_{ji}-1)$. Now one has to proof that $\mathcal{I}_{k-d_{ji}-1}^j \subset \mathcal{I}_k^i$ which is analogous to the previous case. □

Remark 4. Indeed, the partial nestedness of information structure is dictated by the communication delay between neighboring vehicles. It can easily be verified that in the case of communication delays larger than one step, the condition for partial nestedness in Definition 1 does not hold anymore.

Remark 5. Lemma 1 verifies partial nestedness of information structure (7) but it does not imply that optimal control inputs for the problem (11) are linear in the associated information, due to the presence of constraints (10). A proof for linearity of optimal control policies is given in the next corollary.

Corollary 1. Considering problem (11), the optimal control policies (6) are of the form

$$u_i(k) = \gamma_k^i(\mathscr{I}_k^i), \quad k = k_{\text{cur}}, \dots, k_{\text{cur}} + H - 1,\ i = 1, \dots, N,$$

where γ_k^i is a linear admissible map.

Proof 2. Let us define $l_i(k)$ as

$$l_i(k) = \mathbb{E}\left[\boldsymbol{z}(k)^\top \boldsymbol{W}_i \boldsymbol{z}(k)\right].$$

There exists a price $\lambda_i^\star(k)$ (Boyd and Faybusovich, 2006) such that minimizing (8) subject to (10) is equivalent to minimizing

$$J_{\mathscr{P}} = J_{\mathscr{C}} + \sum_{k=k_{\text{cur}}}^{k_{\text{cur}}+H-1} \sum_{i=1}^{M} \lambda_i^\star(k) l_i(k).$$

Since $J_{\mathscr{P}}$ is still quadratic and, from Lemma 1, the information structure (7) is partially nested, we can conclude that the optimal policies (6) to the problem (11) are linear in the associated information (Ho and Chu, 1972). □

3. STRUCTURE OF OPTIMAL CONTROL LAW

In this section we derive the structure of optimal control law for problem (11). Although the method presented here is applicable to a platoon of arbitrary size, under the assumption that each vehicle communicates its state to the neighboring vehicles with one step delay, for the sake of simplicity of derivation we demonstrate the methodology on a two-vehicle platoon. Without loss of generality, since the problem (11) has to be iteratively solved, from now on we will refer to $k_{\text{cur}} = 0$. The difference in computation of optimal control inputs for $k_{\text{cur}} > 0$ is given at the end of the section. Considering system (4) and cost function (8) it is convenient to define the transformed state

$$\tilde{\boldsymbol{x}}(k) = \boldsymbol{x}(k) - \boldsymbol{x}_{\text{des}}. \tag{12}$$

Thus the dynamics of the new state vector $\tilde{\boldsymbol{x}}$ is written as

$$\begin{aligned}\tilde{\boldsymbol{x}}(k+1) &= \boldsymbol{x}(k+1) - \boldsymbol{x}_{\text{des}} \\ &= \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}u(k) + \boldsymbol{w}(k) - \boldsymbol{x}_{\text{des}} \\ &= \boldsymbol{A}(\boldsymbol{x}(k) - \boldsymbol{x}_{\text{des}}) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{A}\boldsymbol{x}_{\text{des}} - \boldsymbol{x}_{\text{des}} + \boldsymbol{w}(k) \\ &= \boldsymbol{A}\tilde{\boldsymbol{x}}(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{A}\boldsymbol{x}_{\text{des}} - \boldsymbol{x}_{\text{des}} + \boldsymbol{w}(k)\end{aligned} \tag{13}$$

Based on the cost function (8) it is desired that sequence of $\tilde{\boldsymbol{x}}(k)$ tends to zero. Due to the linearity of optimal control policies as proven in Corollary 1, and one-step communication delay, the optimal control inputs $u_1(k), u_2(k)$ are

$$\begin{aligned}u_1(k) &= f_1(\tilde{\boldsymbol{x}}_2(0:k-1), \tilde{\boldsymbol{x}}_1(0:k-1), \tilde{x}_1(k)) \\ u_2(k) &= f_2(\tilde{\boldsymbol{x}}_1(0:k-1); \tilde{\boldsymbol{x}}_2(0:k-1); \tilde{\boldsymbol{x}}_2(k))\end{aligned}$$

where f_1, f_2 represent linear functions in respective arguments. To this end we write

$$\begin{aligned}u_1(k) &= \boldsymbol{K}_{11}\tilde{\boldsymbol{x}}_1(0:k-1) + \boldsymbol{K}_{12}\tilde{\boldsymbol{x}}_2(0:k-1) + \boldsymbol{K}_{1L}\tilde{x}_1(k) \\ u_2(k) &= \boldsymbol{K}_{21}\tilde{\boldsymbol{x}}_1(0:k-1) + \boldsymbol{K}_{22}\tilde{\boldsymbol{x}}_2(0:k-1) + \boldsymbol{K}_{2L}\tilde{\boldsymbol{x}}_2(k)\end{aligned}$$

where $\boldsymbol{K}_{11}, \boldsymbol{K}_{21} \in \mathbb{R}^{1\times k}, \boldsymbol{K}_{12}, \boldsymbol{K}_{22} \in \mathbb{R}^{1\times 2k}, K_{1L} \in \mathbb{R}, \boldsymbol{K}_{2L} \in \mathbb{R}^{1\times 2}$ represent the control gains. Note that $K_{1L}, \boldsymbol{K}_{2L}$ are gains for the states $\tilde{\boldsymbol{x}}_1(k), \tilde{\boldsymbol{x}}_2(k)$ which are available respectively to vehicles 1 and 2. Equivalently, the latter equations are written as

$$\boldsymbol{u}(k) = \boldsymbol{K}\tilde{\boldsymbol{x}}(0:k-1) + \begin{bmatrix} K_{1L}\tilde{x}_1(k) \\ \boldsymbol{K}_{2L}\tilde{\boldsymbol{x}}_2(k) \end{bmatrix} \tag{14}$$

where

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix}$$

Theorem 1. The optimal control policy for problem (11), at $k_{\text{cur}} = 0$, has the following form

$$\boldsymbol{u}(k) = \boldsymbol{\phi}(k) + \begin{bmatrix} \phi_1(k) \\ \phi_2(k) \end{bmatrix} \tag{15}$$

with the control components $\boldsymbol{\phi}(k), \phi_1(k), \phi_2(k)$ that are mutually orthogonal and given by

$$\begin{aligned}\boldsymbol{\phi}(k) &= \boldsymbol{K}^F \hat{\boldsymbol{x}}(k) \\ \phi_1(k) &= K_{1L}\omega_1(k) := K_{1L}w_1(k-1) \\ \phi_2(k) &= \boldsymbol{K}_{2L}\boldsymbol{\omega}_2(k) := \boldsymbol{K}_{2L}\boldsymbol{w}_2(k-1)\end{aligned}$$

where $\boldsymbol{K}^F, \boldsymbol{K}_{1L}, \boldsymbol{K}_{2L}$ are gains with fixed dimension and

$$\hat{\boldsymbol{x}}(k) = \boldsymbol{A}\tilde{\boldsymbol{x}}(k-1) + \boldsymbol{B}\boldsymbol{u}(k-1) + \boldsymbol{A}\boldsymbol{x}_{\text{des}} - \boldsymbol{x}_{\text{des}},\ k > 0. \tag{16}$$

The estimator $\hat{\boldsymbol{x}}(k)$ is initialized by $\hat{\boldsymbol{x}}(0) = \mathbb{E}[\tilde{\boldsymbol{x}}(0)]$.

Proof 3. Notice that in expression (14) for the optimal control law policy, the first part $\boldsymbol{\phi}(k) := \boldsymbol{K}\tilde{\boldsymbol{x}}(0:k-1)$ is proportional to the common measurement history of two vehicles, whose dimension is increasing in time. Correspondingly, this means that the dimensions of gain $\boldsymbol{K}$ would increase in time. However, as control strategy (14) is linear, and the information structure is partially nested, the computation of $\boldsymbol{\phi}(k)$ can be done by considering sufficient statistics for it (Mahajan and Nayyar, 2015)

$$\boldsymbol{\phi}(k) = \boldsymbol{K}^S \mathbb{E}[\tilde{\boldsymbol{x}}(k) | \tilde{\boldsymbol{x}}(0:k-1), \boldsymbol{u}(0:k-1)]$$

where the gain $\boldsymbol{K}^S$ has fixed dimension and estimator $\hat{\boldsymbol{x}}(k) = \mathbb{E}[\tilde{\boldsymbol{x}}(k) | \tilde{\boldsymbol{x}}(0:k-1), \boldsymbol{u}(0:k-1)]$ is based on the one-step delayed global state and input history. The computation of the estimator results from (13) and is equal to

$$\hat{\boldsymbol{x}}(k) = \boldsymbol{A}\tilde{\boldsymbol{x}}(k-1) + \boldsymbol{B}\boldsymbol{u}(k-1) + \boldsymbol{A}\boldsymbol{x}_{\text{des}} - \boldsymbol{x}_{\text{des}}$$

as the vehicles at time k know a one-step delayed information from each other and noise $w(k)$ is zero mean. Comparing the latter expression and (13) it holds that

$$\hat{\boldsymbol{x}}(k) = \tilde{\boldsymbol{x}}(k) - \boldsymbol{w}(k-1) \tag{17}$$

so we can write

$$\boldsymbol{u}(k) = \boldsymbol{K}^S \hat{\boldsymbol{x}}(k) + \begin{bmatrix} K_{1L} & 0 \\ 0 & \boldsymbol{K}_{2L} \end{bmatrix} (\hat{x}(k) + w(k-1))$$

Grouping the terms proportional to $\hat{\boldsymbol{x}}(k)$ we get (15), where

$$\boldsymbol{K}^F = \boldsymbol{K}^S + \begin{bmatrix} \boldsymbol{K}_{1L} & 0 \\ 0 & \boldsymbol{K}_{2L} \end{bmatrix}$$

The orthogonality of $\boldsymbol{\phi}(k), \phi_1(k), \phi_2(k)$ is implied by the orthogonality of $\hat{\boldsymbol{x}}(k), \omega_1(k), \boldsymbol{\omega}_2(k)$. Indeed, $\omega_1(k) = w_1(k-1), \boldsymbol{\omega}_2(k) = \boldsymbol{w}_2(k-1)$ are, by assumption, independent and zero mean so ortoghonality is given. Since $w_1(k-1)$ and $\boldsymbol{w}_2(k-1)$ are independent from $\boldsymbol{u}(k-1), \tilde{\boldsymbol{x}}(k-1)$ they are also independent of $\hat{\boldsymbol{x}}(k)$. Therefore,

$$\mathbb{E}[\hat{\boldsymbol{x}}(k)\boldsymbol{w}(k)^\top] = \mathbb{E}[\hat{\boldsymbol{x}}(k)]\mathbb{E}[\boldsymbol{w}(k)^\top] = 0 \tag{18}$$

holds which concludes the proof. □

Remark 6. The optimal control policy (15) is a superposition of two components. The component $\boldsymbol{\phi}(k)$ is proportional to the estimator $\hat{\boldsymbol{x}}(k)$ of the global state $\boldsymbol{x}(k)$, conditioned on the common information between two vehicles, and thus is computed by both decision makers locally. This common information is not the full information available in the two-vehicle platoon, as each vehicle, at time instant k, also measures only locally available state value. Thus, local corrections $\phi_i(k), i=1,2$ are applied to compensate for the discrepancy of $\hat{\boldsymbol{x}}(k)$ and actual state $\boldsymbol{x}(k)$, due to the process noise.

Remark 7. Unlike the result in Theorem 1, optimal control policies for problem (11) might be nonlinear in case of more than one-step delay.

Remark 8. Even though optimal policy (15) is derived for two-vehicle platoon, a generalization to more vehicles is straightforward based on the state decomposition method introduced in (Lamperski and Doyle, 2012) that holds for arbitrary number of DMs.

3.1 Computation of optimal control inputs

Since in problem (11) the state and input constraints are quadratic, instead of optimizing directly over $\boldsymbol{z}(k)$ it is convenient to pose the problem as a mean and covariance selection problem. To this end, the mean and variance of $\boldsymbol{z}(k)$ are

$$\boldsymbol{m}(k) = \mathbb{E}[\boldsymbol{z}(k)] = \mathbb{E}\begin{bmatrix}\tilde{\boldsymbol{x}}(k)\\ \boldsymbol{u}(k)\end{bmatrix} := \begin{bmatrix}\boldsymbol{m}_{\tilde{\boldsymbol{x}}}(k)\\ \boldsymbol{m}_{\boldsymbol{u}}(k)\end{bmatrix}$$

$$\boldsymbol{V}(k) = \mathbb{E}[(\boldsymbol{z}(k)-\boldsymbol{m}(k))(\boldsymbol{z}(k)-\boldsymbol{m}(k))^\top] := \begin{bmatrix}\boldsymbol{V}_{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}}(k) & \boldsymbol{V}_{\tilde{\boldsymbol{x}}\boldsymbol{u}}(k)\\ \boldsymbol{V}_{\boldsymbol{u}\tilde{\boldsymbol{x}}}(k) & \boldsymbol{V}_{\boldsymbol{u}\boldsymbol{u}}(k)\end{bmatrix}$$

We now translate the constraints in (11) to the corresponding constraints on $\boldsymbol{m}(k), \boldsymbol{V}(k)$. We start with (4) which was written in its equivalent form (13). Applying the mean and variance operator to the equation (13), and taking into account that $\tilde{\boldsymbol{x}}(k) = \boldsymbol{F}\boldsymbol{z}(k), \boldsymbol{F} = [\boldsymbol{I}\ \boldsymbol{0}]$ the system dynamics imposes the following constraints on the evolution of $\boldsymbol{m}(k), \boldsymbol{V}(k)$

$$\boldsymbol{F}\boldsymbol{m}(k+1) = \boldsymbol{m}_{\tilde{\boldsymbol{x}}}(k+1) = [\boldsymbol{A}\ \boldsymbol{B}]\boldsymbol{m}(k) + (\boldsymbol{A}-\boldsymbol{I})\boldsymbol{x}_{\text{des}}$$
$$\boldsymbol{F}\boldsymbol{V}(k+1)\boldsymbol{F}^\top = \boldsymbol{V}_{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}}(k+1) = [\boldsymbol{A}\ \boldsymbol{B}]\boldsymbol{V}(k)[\boldsymbol{A}\ \boldsymbol{B}]^\top + \boldsymbol{\Sigma}_{\boldsymbol{w}}. \quad (19)$$

As $\boldsymbol{z}(k)$ is a Gaussian distributed vector, constraints (10) yield

$$tr(\boldsymbol{W}_i\boldsymbol{V}(k)) + \boldsymbol{m}(k)^\top\boldsymbol{W}_i\boldsymbol{m}(k) \le p_k^i \quad (20)$$

where we have used a property that for any d-dimensional multivariate Gaussian distributed vector $\boldsymbol{q} \sim \mathcal{N}(\boldsymbol{m}_{\boldsymbol{q}}, \boldsymbol{V}_{\boldsymbol{q}})$ and any weighting matrix $\boldsymbol{W} \in \mathbb{R}^{d\times d}$ the following holds

$$\mathbb{E}[\boldsymbol{q}^\top\boldsymbol{W}\boldsymbol{q}] = tr(\boldsymbol{W}\boldsymbol{V}_{\boldsymbol{q}}) + \boldsymbol{m}_{\boldsymbol{q}}^\top\boldsymbol{W}\boldsymbol{m}_{\boldsymbol{q}} \quad (21)$$

The cost function (8) is rewritten, using the same property, as

$$J = \sum_{k=0}^{T-1}\left[tr(\boldsymbol{Q}\boldsymbol{V}(k)) + \boldsymbol{m}(k)^\top\boldsymbol{Q}\boldsymbol{m}(k)\right] + tr(\boldsymbol{F}^\top\boldsymbol{Q}_H\boldsymbol{F}V(T)) + \boldsymbol{m}(T)^\top\boldsymbol{F}^\top\boldsymbol{Q}_H F\boldsymbol{m}(T) \quad (22)$$

To account for information constaints (6), the derived optimal control structure from Theorem 1 has to be accounted for. To this end, instead of optimizing over $\boldsymbol{z}(k)$, we define

$$\hat{\boldsymbol{z}}(k) = \begin{bmatrix}\hat{\boldsymbol{x}}(k)\\ \boldsymbol{\phi}(k)\end{bmatrix} \sim \mathcal{N}\left(\hat{\boldsymbol{m}}(k), \hat{\boldsymbol{V}}(k)\right),$$

$$z_1(k) = \begin{bmatrix}\omega_1(k)\\ \phi_1(k)\end{bmatrix} \sim \mathcal{N}(\boldsymbol{m}_1(k), \boldsymbol{V}_1(k)),$$

$$z_2(k) = \begin{bmatrix}\boldsymbol{\omega}_2(k)\\ \phi_2(k)\end{bmatrix} \sim \mathcal{N}(\boldsymbol{m}_2(k), \boldsymbol{V}_2(k))$$

where

$$\hat{\boldsymbol{m}}(k) = \begin{bmatrix}\mathbb{E}(\hat{\boldsymbol{x}}(k))\\ \mathbb{E}(\hat{\boldsymbol{u}}(k))\end{bmatrix} := \begin{bmatrix}\boldsymbol{m}_{\hat{\boldsymbol{x}}}(k)\\ \boldsymbol{m}_{\hat{\boldsymbol{u}}}(k)\end{bmatrix}, \boldsymbol{m}_1(k) = \begin{bmatrix}\boldsymbol{0}\\ 0\end{bmatrix}, m_2(k) = \begin{bmatrix}0\\ 0\end{bmatrix} \quad (23)$$

and covariance matrices are

$$\hat{\boldsymbol{V}}(k) = \mathbb{E}\left[(\hat{\boldsymbol{z}}(k)-\hat{\boldsymbol{m}}(k))(\hat{\boldsymbol{z}}(k)-\hat{\boldsymbol{m}}(k))^\top\right] = \begin{bmatrix}\boldsymbol{V}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}(k) & \boldsymbol{V}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{\phi}}}(k)\\ \boldsymbol{V}_{\hat{\boldsymbol{\phi}}\hat{\boldsymbol{x}}}(k) & \boldsymbol{V}_{\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}}(k)\end{bmatrix},$$

$$\boldsymbol{V}_1(k) = \mathbb{E}\left[\begin{bmatrix}\omega_1(k)\\ \phi_1(k)\end{bmatrix}\begin{bmatrix}\omega_1(k)\\ \phi_1(k)\end{bmatrix}^\top\right] = \begin{bmatrix}V_{\omega_1\omega_1}(k) & V_{\omega_1\phi_1}(k)\\ V_{\phi_1\omega_1}(k) & V_{\phi_1\phi_1}(k)\end{bmatrix},$$

$$\boldsymbol{V}_2(k) = \mathbb{E}\left[\begin{bmatrix}\boldsymbol{\omega}_2(k)\\ \phi_2(k)\end{bmatrix}\begin{bmatrix}\boldsymbol{\omega}_2(k)\\ \phi_2(k)\end{bmatrix}^\top\right] = \begin{bmatrix}\boldsymbol{V}_{\boldsymbol{\omega}_2\boldsymbol{\omega}_2}(k) & \boldsymbol{V}_{\boldsymbol{\omega}_2\boldsymbol{\phi}_2}(k)\\ \boldsymbol{V}_{\boldsymbol{\phi}_2\boldsymbol{\omega}_2}(k) & V_{\phi_2\phi_2}(k)\end{bmatrix}.$$

Note that $z_1(k), z_2(k)$ are zero-mean since by definition variables $\omega_1(k), \boldsymbol{\omega}_2(k)$ are zero-mean Gaussian noises, and control inputs $\phi_1(k), \phi_2(k)$ are directly proportional to $\omega_1(k), \boldsymbol{\omega}_2(k)$, respectively. Finally, structural constraints on $\boldsymbol{V}_1(k)$ and $\boldsymbol{V}_2(k)$ are imposed i.e.

$$V_{\omega_1\omega_1}(k) = \Sigma_{w_1}, \boldsymbol{V}_{\boldsymbol{\omega}_2\boldsymbol{\omega}_2}(k) = \boldsymbol{\Sigma}_{\boldsymbol{w}_2}. \quad (24)$$

In order to formulate the problem in terms of $\hat{\boldsymbol{m}}(k)$, $\hat{\boldsymbol{V}}(k)$, $\boldsymbol{V}_1(k), \boldsymbol{V}_2(k)$ as decision variables, we first investigate how they are related to $\boldsymbol{m}(k), \boldsymbol{V}(k)$. From (17) and (15) it holds that

$$\boldsymbol{z}(k) = \begin{bmatrix}\tilde{\boldsymbol{x}}(k)\\ \boldsymbol{u}(k)\end{bmatrix} = \begin{bmatrix}\hat{\boldsymbol{x}}(k) + \begin{bmatrix}\omega_1(k)\\ \boldsymbol{\omega}_2(k)\end{bmatrix}\\ \boldsymbol{\phi}(k) + \begin{bmatrix}\phi_1(k)\\ \phi_2(k)\end{bmatrix}\end{bmatrix} = \hat{\boldsymbol{z}}(k) + \boldsymbol{F}_1 z_1(k) + \boldsymbol{F}_2 z_2(k) \quad (25)$$

where

$$\boldsymbol{F}_1 = \begin{bmatrix}1 & 0\\ \boldsymbol{0} & \boldsymbol{0}\\ 0 & 1\\ 0 & 0\end{bmatrix}, F_2 = \begin{bmatrix}\boldsymbol{0} & 0\\ 1 & \boldsymbol{0}\\ \boldsymbol{0} & 0\\ \boldsymbol{0} & 1\end{bmatrix}$$

Therefore, taking the expectation over (25) yields

$$\boldsymbol{m}(k) = \hat{\boldsymbol{m}}(k) + \boldsymbol{F}_1\boldsymbol{m}_1(k) + \boldsymbol{F}_2\boldsymbol{m}_2(k) = \hat{\boldsymbol{m}}(k) \quad (26)$$

due to $\boldsymbol{m}_1(k), \boldsymbol{m}_2(k)$ being zero vectors as shown in (23). Similarly, it holds

$$\begin{aligned}\boldsymbol{V}(k) &= \mathbb{E}\left[(\boldsymbol{z}(k)-\boldsymbol{m}(k))(\boldsymbol{z}(k)-\boldsymbol{m}(k))^\top\right]\\ &= \mathbb{E}\left[(\boldsymbol{z}(k)-\hat{\boldsymbol{m}}(k))(\boldsymbol{z}(k)-\hat{\boldsymbol{m}}(k))^\top\right]\\ &= \hat{\boldsymbol{V}}(k) + \boldsymbol{F}_1\boldsymbol{V}_1(k)\boldsymbol{F}_1^\top + \boldsymbol{F}_2\boldsymbol{V}_2(k)\boldsymbol{F}_2^\top \end{aligned} \quad (27)$$

Before stating the main result, we define suitable partitioning of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{Q}$. Refering to (9), for a two-vehicle system with states $x_1, \boldsymbol{x}_2$ and inputs u_1, u_2 the matrix $\boldsymbol{Q}$ is partitioned as

$$\boldsymbol{Q} = \begin{bmatrix}Q_{x_1x_1} & \boldsymbol{Q}_{x_1\boldsymbol{x}_2} & Q_{x_1u_1} & Q_{x_1u_2}\\ \boldsymbol{Q}_{\boldsymbol{x}_2x_1} & \boldsymbol{Q}_{\boldsymbol{x}_2\boldsymbol{x}_2} & \boldsymbol{Q}_{\boldsymbol{x}_2u_1} & \boldsymbol{Q}_{\boldsymbol{x}_2u_2}\\ Q_{u_1x_1} & \boldsymbol{Q}_{u_1\boldsymbol{x}_2} & Q_{u_1u_1} & Q_{u_1u_2}\\ Q_{u_2x_1} & \boldsymbol{Q}_{u_2\boldsymbol{x}_2} & Q_{u_2u_1} & Q_{u_2u_2}\end{bmatrix}$$

where we define

$$\boldsymbol{Q}^1 := \begin{bmatrix}Q_{x_1x_1} & Q_{x_1u_1}\\ Q_{u_1x_1} & Q_{u_1u_1}\end{bmatrix}, \boldsymbol{Q}^2 := \begin{bmatrix}\boldsymbol{Q}_{\boldsymbol{x}_2\boldsymbol{x}_2} & \boldsymbol{Q}_{\boldsymbol{x}_2u_2}\\ \boldsymbol{Q}_{u_2\boldsymbol{x}_2} & Q_{u_2u_2}\end{bmatrix}$$

Recalling (5), $\boldsymbol{A}$ and $\boldsymbol{B}$ are partitioned as

$$\boldsymbol{A} = [\boldsymbol{A}_1|\boldsymbol{A}_2], \boldsymbol{B} = [\boldsymbol{B}_1|\boldsymbol{B}_2],$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix}1\\ \boldsymbol{A}_{21}\end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix}0\\ \boldsymbol{A}_{22}\end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix}\Delta_t\\ \boldsymbol{B}_{21}\end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix}0\\ \boldsymbol{B}_{22}\end{bmatrix}.$$

Finally, we define the following

$$[\boldsymbol{A}^i\boldsymbol{B}^i] := \begin{cases} [\boldsymbol{A}\,\boldsymbol{B}], & i=0 \\ [\boldsymbol{A}_1\,\boldsymbol{B}_1], & i=1 \\ [\boldsymbol{A}_2\,\boldsymbol{B}_2], & i=2 \end{cases}, \boldsymbol{W}^i := \begin{cases} \boldsymbol{W}_i, & i=0 \\ \boldsymbol{F}_1^\top \boldsymbol{W}_i \boldsymbol{F}_1, & i=1 \\ \boldsymbol{F}_2^\top \boldsymbol{W}_i \boldsymbol{F}_2, & i=2 \end{cases} \quad (28)$$

$$\boldsymbol{Q}^i := \begin{cases} \boldsymbol{Q}, & i=0 \\ \boldsymbol{Q}^1, & i=1 \\ \boldsymbol{Q}^2, & i=2 \end{cases}, \boldsymbol{Q}_H^i := \begin{cases} \boldsymbol{F}^\top \boldsymbol{Q}_H \boldsymbol{F}, & i=0 \\ \boldsymbol{F}_1 \boldsymbol{Q}_H^0 \boldsymbol{F}_1, & i=1 \\ \boldsymbol{F}_2 \boldsymbol{Q}_H^0 \boldsymbol{F}_2, & i=2 \end{cases} \quad (29)$$

$$\boldsymbol{m}^0 := \hat{\boldsymbol{m}}, \quad \boldsymbol{V}^i := \begin{cases} \hat{\boldsymbol{V}}, & i=0 \\ \boldsymbol{V}_1, & i=1 \\ \boldsymbol{V}_2, & i=2 \end{cases} \quad (30)$$

Theorem 2. Problem (11), at $k_{cur}=0$ is equivalent to:

$$\min_{\boldsymbol{m}^0(k),\boldsymbol{V}^0(k),\boldsymbol{V}^1(k),\boldsymbol{V}^2(k),k=0:H-1} J_c \quad (31)$$

$$\text{s.t.} \quad \boldsymbol{F}\boldsymbol{m}^0(0) = \boldsymbol{\mu_x} - \boldsymbol{x}_{\text{des}} \quad (32)$$

$$\boldsymbol{F}\boldsymbol{V}^0(0)\boldsymbol{F}^\top = 0 \quad (33)$$

$$\boldsymbol{M}_i\boldsymbol{V}^i(0)\boldsymbol{M}_i^\top = \boldsymbol{\Sigma}_{x_i}, \forall i=1,2 \quad (34)$$

$$\boldsymbol{M}_i\boldsymbol{V}^i(k)\boldsymbol{M}_i^\top = \boldsymbol{\Sigma}_{w_i}, \forall i=1,2\ \forall k>0 \quad (35)$$

$$\boldsymbol{F}\boldsymbol{m}^0(k+1) = [\boldsymbol{A}\boldsymbol{B}]\boldsymbol{m}^0(k) + (\boldsymbol{A}-\boldsymbol{I})\boldsymbol{x}_{\text{des}}$$

$$\boldsymbol{F}\boldsymbol{V}^0(k+1)\boldsymbol{F}^\top = \sum_{j=0}^{2} [\boldsymbol{A}^i\,\boldsymbol{B}^i]\boldsymbol{V}^i(k)[\boldsymbol{A}^i\,\boldsymbol{B}^i]^\top$$

$$\sum_{j=0}^{2} tr(\boldsymbol{W}^j\boldsymbol{V}^j(k)) + \boldsymbol{m}^0(k)^\top \boldsymbol{W}^0 \boldsymbol{m}^0(k) \le p_k^i$$

where $\boldsymbol{M}_1 = [1\ 0], \boldsymbol{M}_2 = [\boldsymbol{I}^{2x2}\ 0^{2x1}]$ and

$$J_c = \boldsymbol{m}^0(H)^\top \boldsymbol{Q}_H^0 \boldsymbol{m}^0(H) + \sum_{j=0}^{2} tr(\boldsymbol{Q}_H^j \boldsymbol{V}^j(H))$$

$$+ \sum_{k=0}^{H-1} \left(\boldsymbol{m}^0(k)^\top \boldsymbol{Q}\boldsymbol{m}^0(k) + \sum_{j=0}^{2} tr(\boldsymbol{Q}^j\boldsymbol{V}^j(k) \right).$$

Proof 4. The first constraint is obtained by imposing the initial condition of the estimator in Theorem 1, i.e. $\hat{\boldsymbol{x}}(0) = \mathbb{E}[\tilde{\boldsymbol{x}}(0)] = \mathbb{E}[\boldsymbol{x}(0) - \boldsymbol{x}_{\text{des}}] = \boldsymbol{\mu_x} - \boldsymbol{x}_{\text{des}}$. Taking into account that $\hat{\boldsymbol{x}}(0)$ is deterministic it holds $\hat{\boldsymbol{x}}(0) = \mathbb{E}[\hat{\boldsymbol{x}}(0)] = \boldsymbol{F}\boldsymbol{m}^0(0)$ and its variance is zero, which yields (33). Constraint (34) is obtained starting from $Var(\boldsymbol{x}(0)) = Var(\tilde{\boldsymbol{x}}(0)) = Var(\hat{\boldsymbol{x}}(0) + \boldsymbol{\omega}(0)) = Var(\boldsymbol{\omega}(0)) = \boldsymbol{\Sigma_x}$ and by spliting the constraint into individual constraints on variance of $\omega_1(0)$ and $\boldsymbol{\omega}_2(0)$. Constraint (35) follows directly from (24). The remaining constraints and expression for J_c are obtained by imposing expressions of decomposition of $\boldsymbol{m}(k), \boldsymbol{V}(k)$, i.e. (26), (27) into (19), (20), (22), taking into account the notation defined by (28)-(30). □

Remark 9. Theorem 2 provides the computation of optimal covariances and mean values at $k_{\text{cur}} = 0$. The computation at $k_{\text{cur}} > 0$ differs only in the initialization steps, i.e. instead of constraints (32)-(34) the following constraints are used

$$\boldsymbol{F}\boldsymbol{m}^0(k_{\text{cur}}) = \boldsymbol{A}\boldsymbol{x}(k_{\text{cur}}-1) + \boldsymbol{B}\boldsymbol{u}(k_{\text{cur}}-1) + (\boldsymbol{A}-\boldsymbol{I})\boldsymbol{x}_{\text{des}}$$

$$\boldsymbol{F}\boldsymbol{V}^0(k_{\text{cur}})\boldsymbol{F}^\top = \sum_{j=0}^{2} \left[\boldsymbol{A}^i\,\boldsymbol{B}^i\right]\boldsymbol{V}^i(k_{\text{cur}}-1)\left[\boldsymbol{A}^i\,\boldsymbol{B}^i\right]^\top$$

$$\boldsymbol{M}_i\boldsymbol{V}^i(k_{\text{cur}})\boldsymbol{M}_i^\top = \boldsymbol{\Sigma}_{\boldsymbol{w}_i}\ (i=1,2)$$

Furthermore, at $k_{\text{cur}} > 0$ the values $\boldsymbol{x}_{\text{des}}$ and p_k^i can be adjusted according to the speed limits, safety constraints etc.

After computing the optimal mean values and covariances according to Theorem 2 the result is a sequence of Gaussian distributions that is optimal with regard to the cost function (8). Since the vectors $\hat{z}(k), \hat{z}_1(k), \hat{z}_2(k)$ are multivariate Gaussian distributed vectors, the optimal control inputs are found by exploiting the formula for conditional mean, i.e.

$$\hat{\boldsymbol{\phi}}(k) = \boldsymbol{m}_{\hat{\boldsymbol{u}}}(k) + \boldsymbol{V}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{\phi}}}(k)\boldsymbol{V}_{\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}}^{-1}(\hat{\boldsymbol{x}}(k) - \boldsymbol{m}_{\hat{\boldsymbol{x}}}(k))$$

$$\phi_i(k) = \boldsymbol{V}_{\boldsymbol{\omega}_i\phi_i}(k)V_{\boldsymbol{\omega}_i\boldsymbol{\omega}_i}^{-1}(k)\boldsymbol{\omega}_i(k)\ \forall i=1,2$$

Remark 10. Interestingly the Theorem 2 shows that optimal covariances and mean values to the problem (11) can be found as a solution to the simple convex program, which is important for the vehicle platooning problem as each vehicle has a local controller with (potentially) limited computing capabilities. Moreover simple structure of (11) is important for in-network (Rueth et al., 2018) implementation of derived optimal control law where control functionalities are pushed as close as possible to the controlled process exploiting the computational power of active network components - even if limited.

Remark 11. Since the developed control scheme uses the MPC framework, at time step k_{cur} only the optimal control input $\boldsymbol{u}(k_{\text{cur}})$ is applied to the system. After Δ_t has elapsed, new information is measured by/transmitted to the DMs and the optimization (31) is recomputed. By doing this iteratively power constraints and target trajectories are adapted dynamically.

4. VEHICLE PLATOON SIMULATION

The goal of this simulation is to illustrate the proposed optimal longitudinal control scheme on a platoon composed of two vehicles. The desired distance between two vehicles is set to $d_{\text{des}} = 5\,\text{m}$. As it is assumed that the vehicles encounter different speed limits on the road, over the course of the simulation the desired velocity increases from an initial $v_{\text{des},1} = 20\,\text{m/s}$ to $v_{\text{des},2} = 25\,\text{m/s}$ at $t = 7\,\text{s}$ and later decreases to $v_{\text{des},3} = 17.5\,\text{m/s}$ at $t = 27\,\text{s}$. Hereby, t denotes the time elapsed since the start of the simulation and is calculated from the discrete time variable k as $t = \Delta_t k$, where a sampling time of $\Delta_t = 0.2\,\text{s}$ is used. To make the transition between different speeds less abrupt, velocity is gradually adjusted within a short time-period.

We define a constraint according to (10) to limit the power of allowed deviation of the distance between vehicles from the desired one. The matrix $\boldsymbol{W}$ and power p_k are chosen as

$$\boldsymbol{W} = \left(\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 1 & 0 & \boldsymbol{0}^{3\times 2} \\ 0 & 0 & 0 & \\ \hline & \boldsymbol{0}^{2\times 3} & & \boldsymbol{0}^{2\times 2} \end{array} \right), \quad p_k = 0.125\,\text{m}^2.$$

This power constraint is only active during the time interval $12\,\text{s} < t \le 27\,\text{s}$ as in this interval target speed is increased. Since the power constraints are defined in expectation, we perform a Monte Carlo simulation with 100 runs to verify them. For the MPC implementation of proposed control strategy, a moving time horizon of $3\,\text{s}$ is used resulting in $H = 15$ time steps within this horizon. The initial state is sampled from a normal distribution with mean $\boldsymbol{m_x} = [20\,\text{m/s}, 5.5\,\text{m}, 20\,\text{m/s}]^\top$ and covariance $\boldsymbol{\Sigma_x} = 2 \times 10^{-2} \times \boldsymbol{I}^{3\times 3}$ and the system noise is drawn from a zero-mean normal distribution with a covariance $\boldsymbol{\Sigma_w} = 2 \times 10^{-2} \times \boldsymbol{I}^{3\times 3}$ at every time step. The cost function is defined as in (8) using $\boldsymbol{Q} = \boldsymbol{I}^{5\times 5}$ and $\boldsymbol{Q}_H = \boldsymbol{I}^{3\times 3}$.

Figure 2 shows the mean and variance of the deviation from the desired distance over time. During the time interval in which the power constraint is active, i.e. $12\,\text{s} < t \le 27\,\text{s}$, the mean remains close to the desired value and the variance is noticeably reduced. To validate the satisfaction of the power constraint,

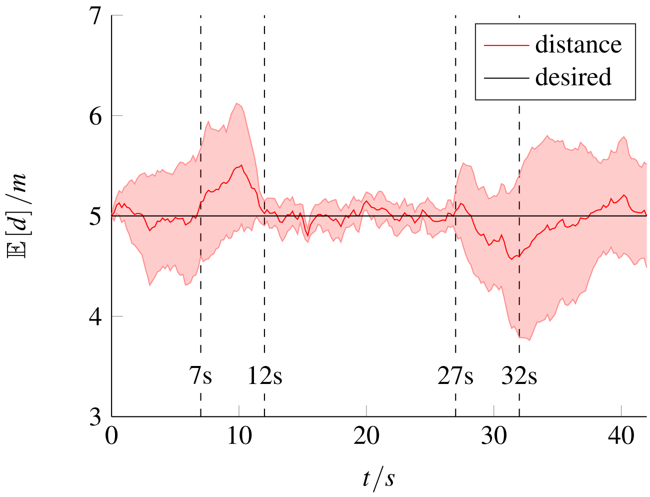

Fig. 2. Monte Carlo simulation showing mean and variance of distance trajectory with time varying power constraint.

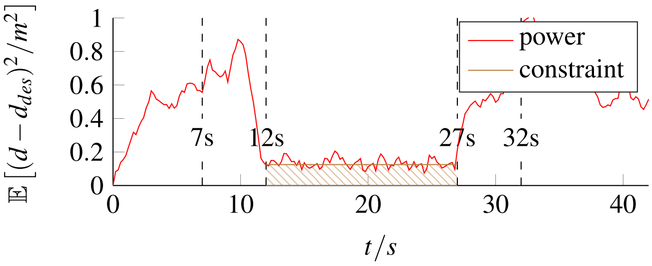

Fig. 3. Validation of enforcement of power constraint using Monte Carlo simulation.

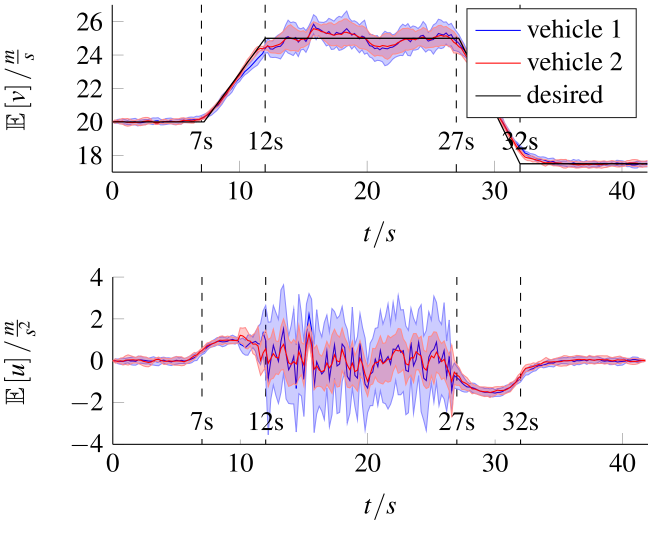

Fig. 4. Monte Carlo simulation showing mean (solid line) and variance (shaded area) of velocity/control inputs. Due to the higher importance of compensating the process noise in this scenario, the variance increases during the time interval in which the power constraint is active [12s:27s].

Figure 3 illustrates the averaged value of the squared difference between distance and desired distance over time. The power of the system stays close to the constraint boundary. It is expected that it stays below boundary when the number of Monte Carlo simulations is increased to $n \rightarrow \infty$. The controller reacts in this period by changing the acceleration rapidly and increasing its magnitude, as illustrated in Figure 4, in order to offset the process noise, which threatens to push the trajectories out of constraint boundaries. While this behavior is more expensive with regard to the cost function defined earlier, the system is able to attain a low variance on the deviation of distance.

5. CONCLUSION

We have presented an approach for vehicle platooning, under a one-step delayed information sharing pattern between neighboring vehicles, subject to state/input power constraints, that is capable of adapting to sudden changes in the environment. However, the approach can be applied to any physically interconnected system, under the assumption of one-directional coupling between neighboring subsystems and given a possibility of bi-directional communication. To this end a different (but quadratic) cost function can be considered depending on the particular application.

REFERENCES

Abara, P.U., Causevic, V., and Hirche, S. (2018). Quadratic invariance for distributed control system with intermittent observations. In *2018 IEEE Conference on Decision and Control (CDC)*, 2854–2860. doi:10.1109/CDC.2018.8618992.

Boyd, V. and Faybusovich (2006). Convex optimization. *Birches. J.*, 51(11), 123–126.

Causevic, V., Abara, P.U., and Hirche, S. (2018). Information-constrained optimal control of distributed systems with power constraints. *European Control Conference (ECC)*.

Causevic, V., Falsone, A., Ioli, D., and Prandini, M. (2018). Energy management in a multi-building set-up via distributed stochastic optimization. In *2018 Annual American Control Conference (ACC)*, 5387–5392.

Feyzmahdavian, H.R., Alam, A., and Gattami, A. (2012). Optimal distributed controller design with communication delays: Application to vehicle formations. *IEEE 51st IEEE Conference on Decision and Control (CDC)*, 34, 184–186.

Ho, Y.C. and Chu, K.C. (1972). Team decision theory and information structures in optimal control problems-part i. 17.

Ioannou, P.A. and Chien, C.C. (1993). Autonomous intelligent cruise control. *IEEE Transactions on Vehicular Technology*, 42(4), 657–672. doi:10.1109/25.260745.

Lamperski, A. and Doyle, J. (2012). Dynamic programming solutions for decentralized state-feedback lqg problems with communication delays. *American Control Conference*.

Mahajan, A. and Nayyar, A. (2015). Sufficient statistics for linear control strategies in decentralized systems with partial history sharing. *IEEE Transactions on Automatic Control*, 60(8), 2046–2056. doi:10.1109/TAC.2015.2398884.

Mayne, D.Q. (2014). Model predictive control: Recent developments and future promise. *Automatica*, 50(12), 2967–2986.

Rotkowitz, M. and Lall, S. (2006). A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2), 274–286. doi: 10.1109/TAC.2005.860365.

Rueth, J., Glebke, R., Wehrle, K., Causevic, V., and Hirche, S. (2018). Towards in-network industrial feedback control. In *SIGCOMM Workshop on In-Network Computing*, NetCompute. ACM. doi:10.1145/3229591.3229592.

Witsenhausen, H.S. (1968). A counterexample in stochastic optimum control. *SIAM J. Control*, 6, 35–47.