Uniform Error Bounds for Gaussian Process Regression with Application to Safe Control

Motivation

Machine learning offers great promises in robotics applications.
- Policies based on machine learning unsafe in real world applications [1]
  - Safe environments to avoid damages of hardware
  - No human-robot interaction due to risk of injuries

Quantification of uncertainty in data-driven models essential for safety-critical applications
⇒ Robust control for rigorous safety certificates

How can the learning error be bounded based on the model uncertainty?
How are formal safety guarantees provided for policies based on uncertain models?

Gaussian Process Regression

- Bayesian nonparametric modeling as “distribution over functions” [2]
  \( f_0(x) \sim GP(0, k(x, x')) \)
  - Based on training data \( D = \{ x(i), y(i) = f(x(i)) + \epsilon \} \), with Gaussian noise \( \epsilon \) with variance \( \sigma_{\epsilon}^2 \), it provides mean and variance:
  \[
  \mu_y(x) = E[f_0(x)|x, D] = k(x, k + \sigma_{\epsilon}^2 I)^{-1} y
  \]

- Probabilistic uniform error bounds based on RKHS theory [3, 4] difficult to calculate in practice

Probabilistic Uniform Error Bound

- Assumption: dynamical system \( f(x, u) \) is a sample from a GP with Lipschitz constant \( L_f \)
- Lipschitz continuous posterior mean \( \mu_N (\cdot) \) and standard deviation \( \sigma_N (\cdot) \) with
  \[
  \| \mu_N (x') - \mu_N (x) \| \leq L_f \| x - x' \| \quad \| \sigma_N (x') - \sigma_N (x) \| \leq \omega_0 \| x - x' \|
  \]

Theorem

| \( P \{ | f(x) - \mu(x) | \leq \sqrt{\frac{1}{2} \log \left( \frac{1 + 1}{\delta} \right) + \log \left( \frac{1}{1 - \delta} \right)} \} \) |
| \( | f(x) - \mu(x) | \leq \sqrt{\frac{1}{2} \log \left( \frac{1 + 1}{\delta} \right) + \log \left( \frac{1}{1 - \delta} \right)} \) |

The learning error is probabilistically bounded by

| on the set \( X \) with maximal extension \( r \) for every \( \delta \in (0, 1) \), \( \tau \in \mathbb{R} \). |
| \( \left( \frac{1}{2} \log \left( \frac{1}{\delta} \right) + \log \left( \frac{1}{1 - \delta} \right) \right) \) |

Probabilistic Lipschitz Constants

- Kernel with continuous partial derivatives up to the fourth order
- Partial derivative kernels
  \[
  k^v(x, x') = \frac{\partial^v}{\partial x_v \partial x'_v} k(x, x') \quad \forall v = 1, \ldots, d.
  \]

- Lipschitz constants \( L_i \) and \( L_i^0 \)

Theorem

| \( \left( 2 \log \left( \frac{1}{\delta} \right) \right) \max_{k(x, x')} k^v(x, x') + 12 \sqrt{d} \max_{k(x, x')} k^v(x, x') \right) \) |
| \( \left( 2 \log \left( \frac{1}{\delta} \right) \right) \max_{k(x, x')} k^v(x, x') + 12 \sqrt{d} \max_{k(x, x')} k^v(x, x') \right) \) |

The constant

\[
L_f = \left\{ \begin{array}{ll}
\sqrt{2 \log \left( \frac{1}{\delta} \right) \max_{k(x, x')} k^v(x, x') + 12 \sqrt{d} \max_{k(x, x')} k^v(x, x')} \left( \frac{1}{\delta} \right) \end{array} \right.
\]

is a Lipschitz constant of \( f(\cdot) \) on \( X \) with probability of at least \( 1 - \delta \).

Safe Control of Unknown Dynamical Systems

- Nonlinear control affine dynamical system
  \[
  x_1 = x_2, \quad x_2 = f(x) + u,
  \]

- Goal: track reference \( x_d \) with \( x_1 \) such that error \( e = x - [x_2 x_1]^T \) is minimal
- Define filtered state \( \tilde{x} = \lambda x_1 + e_2, \lambda > 0 \)
- Use feedback linearizing policy
  \[
  u = \pi(x) = -f(x) + \nu
  \]

with PD-controller

| \( \nu = \tilde{x}_d - k_r \tilde{x}_1 - \lambda e_2 \) |

Theorem

The feedback linearizing controller with \( f(\cdot) = \psi_N (\cdot) \) guarantees with probability \( 1 - \delta \) that the tracking error \( e = x - x_d \) converges to

\[
B = \left\{ x \in X \left| \frac{\sqrt{D(T)}}{\max(\gamma(T))} \leq \frac{1}{\delta} \right. \right\}
\]

Numerical Evaluation on a Robotic Manipulator

References