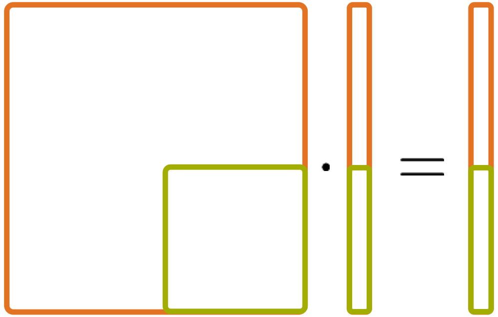
Nonlinear Moment Matching for the Simulation-Free Reduction of Structural Systems

Maria Cruz Varona, Nico Schneucker, Boris Lohmann

Chair of Automatic Control, Technical University of Munich



IFAC MECHATRONICS & NOLCOS 2019

Mathematical Modeling and Model Order Reduction

Vienna, September 4th 2019

Motivation

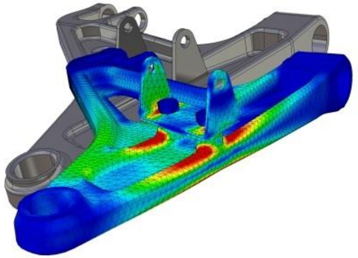
Nonlinear Moment Matching for the Simulation-Free Reduction of Structural Systems

Why nonlinear?

Geometric nonlinearities (large deformations)

Material nonlinearities

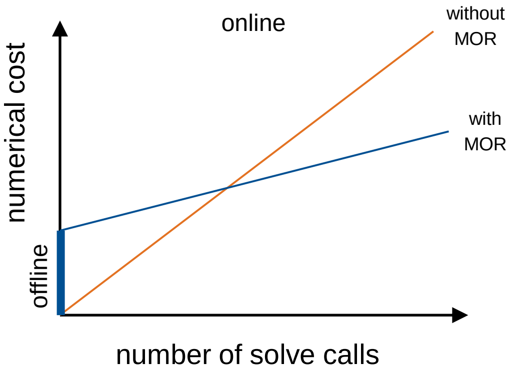
Nonlinear boundary conditions



Why simulation-free?

Avoid expensive training simulations
→ simulation-free / system-theoretic

Simulation-Free Model Order Reduction for Nonlinear Second-Order Systems

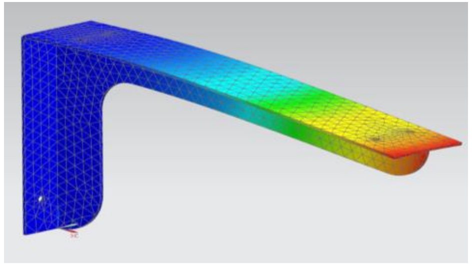


Why reduction?

Efficient numerical analysis, computer-aided design, uncertainty quantification, predictive maintenance

Why structural systems?

2nd-order systems arise in many applications: flexible structures, MEMS, vibroacoustics, biomechanics,…



Projective Model Order Reduction

Second-order nonlinear full order model (FOM)

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boxed{\boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{f}(\boldsymbol{q}(t))} = \boxed{\boldsymbol{B}\boldsymbol{F}(t)} \qquad \boldsymbol{q}(0) = \boldsymbol{q}_0,\ \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0,$$

$$\boldsymbol{y}(t) = \boxed{\boldsymbol{C}\boldsymbol{q}(t)}$$

Linear (Petrov-)Galerkin projection — MOR — $\boldsymbol{q}(t) \approx \boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}(t)\,, \quad \boldsymbol{V} \in \mathbb{R}^{n\times r} \qquad r \ll n$

Reduced order model (ROM)

$$\boldsymbol{M}_{\mathrm{r}}\ddot{\boldsymbol{q}}_{\mathrm{r}}(t) + \boldsymbol{D}_{\mathrm{r}}\dot{\boldsymbol{q}}_{\mathrm{r}}(t) + \boldsymbol{W}^{\mathsf{T}}\boldsymbol{f}\big(\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}(t)\big) = \boldsymbol{B}_{\mathrm{r}}\boldsymbol{F}(t) \qquad \{\boldsymbol{q}_{\mathrm{r}}(0),\, \dot{\boldsymbol{q}}_{\mathrm{r}}(0)\} = (\boldsymbol{W}^{\mathsf{T}}\boldsymbol{V})^{-1}\boldsymbol{W}^{\mathsf{T}}\{\boldsymbol{q}_0,\, \dot{\boldsymbol{q}}_0\}\,,$$

$$\boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{C}_{\mathrm{r}}\boldsymbol{q}_{\mathrm{r}}(t)$$

with

$$\{\boldsymbol{M}_{\mathrm{r}}, \boldsymbol{D}_{\mathrm{r}}\} = \boldsymbol{W}^{\mathsf{T}}\{\boldsymbol{M}, \boldsymbol{D}\}\,\boldsymbol{V}\,, \qquad \boldsymbol{f}_{\mathrm{r}}(\boldsymbol{q}) = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{f}(\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}})$$

Hyper reduction!

$$\boldsymbol{B}_{\mathrm{r}} = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{B},$$

$$\boldsymbol{C}_{\mathrm{r}} = \boldsymbol{C}\,\boldsymbol{V}\,,$$

In this talk: Dimensional reduction
How to choose $\boldsymbol{V}$?

Nonlinear dimensional reduction methods – Overview

First-order (state-space) nonlinear models

Simulation-based

- POD, Reduced Basis
- TPWL, Empirical Gramians
- Hyper reduction: DEIM

Simulation-free / System-theoretic

- Reduction of polynomial systems (bilinear, quadratic-bil.) using Volterra series theory (balancing and Krylov)
- Extension to 1st-order systems?
- MDs for 1st-order systems (Himpsl ’18, Meyer ’19)
- Nonlinear Balanced Truncation (Scherpen ’93)
- Nonlinear Moment Matching (Astolfi ’10, Cruz et al. ’19)

Second-order (mechanical) nonlinear models

Simulation-based

- POD, Reduced Basis
- TPWL, Empirical Gramians
- Hyper reduction: ECSW

Simulation-free / System-theoretic

- Reduction of polynomial 2nd-order systems (quadratic, cubic) using Volterra series theory
- Nonlinear Normal Modes (Rosenberg ’62)
- Basis augmentation with Modal Derivatives (MDs)
- Extension to 2nd-order systems?
- **Extension to 2nd-order systems!!**

Moment Matching for Linear Second-Order Systems

Second-order linear system

$$
\begin{aligned}
\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) &= \boldsymbol{B}\boldsymbol{F}(t), \\
\boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{q}(t).
\end{aligned}
$$

Transfer function matrix

$$\boldsymbol{G}(s) = \boldsymbol{C}(s^2\boldsymbol{M} + s\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}$$

Moments of the transfer function

The moments $\boldsymbol{m}_i(\sigma)$ at σ are the coefficients of the Taylor series of the transfer function.

Example

$$\boldsymbol{m}_0(\sigma) = \boldsymbol{G}(\sigma) = \boldsymbol{C}\,\boldsymbol{K}_\sigma^{-1}\,\boldsymbol{B}$$

$$\boldsymbol{m}_1(\sigma) = \boldsymbol{G}'(\sigma) = -\boldsymbol{C}\,\boldsymbol{K}_\sigma^{-1}\,\boldsymbol{D}_\sigma\,\boldsymbol{K}_\sigma^{-1}\,\boldsymbol{B}$$

$$\vdots$$

$$\boldsymbol{K}_\sigma = \boldsymbol{K} + \sigma\boldsymbol{D} + \sigma^2\boldsymbol{M}$$

$$\boldsymbol{D}_\sigma = \boldsymbol{D} + 2\sigma\boldsymbol{M}$$

Frequency-domain interpretation of Moment Matching

(Tangential) input Krylov subspace for proportional damping:

$$\operatorname{span}\left\{\boldsymbol{K}_{\sigma_1}^{-1}\boldsymbol{B}\,\boldsymbol{r}_1, \ldots, \boldsymbol{K}_{\sigma_r}^{-1}\boldsymbol{B}\,\boldsymbol{r}_r\right\} \subseteq \operatorname{ran}(\boldsymbol{V})$$

leads to (tangential) multipoint moment matching:

$$\boldsymbol{G}(\sigma_i)\,\boldsymbol{r}_i = \boldsymbol{G}_{\mathrm{r}}(\sigma_i)\,\boldsymbol{r}_i \iff \boldsymbol{m}_0(\sigma_i)\,\boldsymbol{r}_i = \boldsymbol{m}_{\mathrm{r},0}(\sigma_i)\,\boldsymbol{r}_i$$

Equivalence of Krylov and Sylvester equation

$$(\sigma_i^2\boldsymbol{M} + \sigma_i\boldsymbol{D} + \boldsymbol{K})\,\boldsymbol{v}_i = \boldsymbol{B}\,\boldsymbol{r}_i \iff$$

$$\boldsymbol{M}\,\boldsymbol{V}\,\boldsymbol{S}_v^2 + \boldsymbol{D}\,\boldsymbol{V}\,\boldsymbol{S}_v + \boldsymbol{K}\,\boldsymbol{V} = \boldsymbol{B}\,\boldsymbol{R}$$

Reduction parameters:

- Shifts: $\boldsymbol{S}_v = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$
- Tang. directions: $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r]$

Linear systems – Steady-state response

Notion of signal generators

Interconnecting a system with the following *linear signal generator*

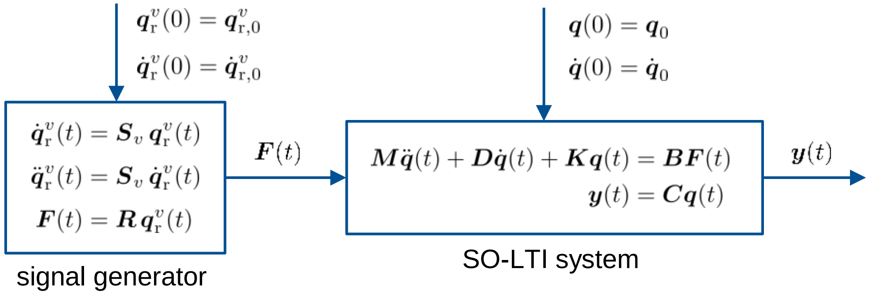
$$\left.\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t) &= \boldsymbol{S}_v\,\boldsymbol{q}_{\mathrm{r}}^{v}(t), & \boldsymbol{q}_{\mathrm{r}}^{v}(0) &= \boldsymbol{q}_{\mathrm{r},0}^{v},\\ \ddot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t) &= \boldsymbol{S}_v\,\dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t), & \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(0) &= \dot{\boldsymbol{q}}_{\mathrm{r},0}^{v},\\ \boldsymbol{F}(t) &= \boldsymbol{R}\,\boldsymbol{q}_{\mathrm{r}}^{v}(t), \end{aligned}\right\} \implies \left\{\begin{aligned} \boldsymbol{q}_{\mathrm{r}}^{v}(t) &= \mathrm{e}^{\boldsymbol{S}_v t}\,\boldsymbol{q}_{\mathrm{r},0}^{v},\\ \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t) &= \boldsymbol{S}_v\,\mathrm{e}^{\boldsymbol{S}_v t}\,\boldsymbol{q}_{\mathrm{r},0}^{v},\\ \boldsymbol{F}(t) &= \boldsymbol{R}\,\mathrm{e}^{\boldsymbol{S}_v t}\,\boldsymbol{q}_{\mathrm{r},0}^{v} = \textstyle\sum_{i=1}^{r} \boldsymbol{r}_i\,\mathrm{e}^{\sigma_i t}\,q_{\mathrm{r},0,i}^{v}, \end{aligned}\right.$$

$$\boldsymbol{S}_v = \operatorname{diag}(\sigma_1,\dots,\sigma_r)$$

$$\boldsymbol{R} = [\boldsymbol{r}_1,\dots,\boldsymbol{r}_r]$$

corresponds to exciting the system with a sum of (growing) exponentials.

Steady-state response of interconnected system



$$\boldsymbol{q}(t) = \underbrace{\boldsymbol{q}_{\mathrm{h}}(t)}_{\text{decaying homog. solution}} + \underbrace{\sum_{i=1}^{r} \overbrace{(\sigma_i^2\boldsymbol{M} + \sigma_i\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\boldsymbol{r}_i}^{\boldsymbol{v}_i}\;\overbrace{\mathrm{e}^{\sigma_i t}\,q_{\mathrm{r},0,i}^{v}}^{q_{\mathrm{r},i}^{v}(t)}}_{\text{growing part. solution}}$$

For $t \to \infty$: $\quad \boldsymbol{q}_{\mathrm{ss}}(t) = \boldsymbol{q}_{\mathrm{p}}(t) = \sum_{i=1}^{r} \boldsymbol{v}_i\, q_{\mathrm{r},i}^{v}(t)$

$$\boldsymbol{y}_{\mathrm{ss}}(t) = \sum_{i=1}^{r} \boldsymbol{C}(\sigma_i^2\boldsymbol{M} + \sigma_i\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\boldsymbol{r}_i\, q_{\mathrm{r},i}^{v}(t) = \boldsymbol{C}\,\boldsymbol{V}\,\boldsymbol{q}_{\mathrm{r}}^{v}(t)$$

Recall: $\boldsymbol{m}_0(\sigma_i)\,\boldsymbol{r}_i = \boldsymbol{C}(\sigma_i^2\boldsymbol{M} + \sigma_i\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\,\boldsymbol{r}_i = \boldsymbol{C}\,\boldsymbol{v}_i$

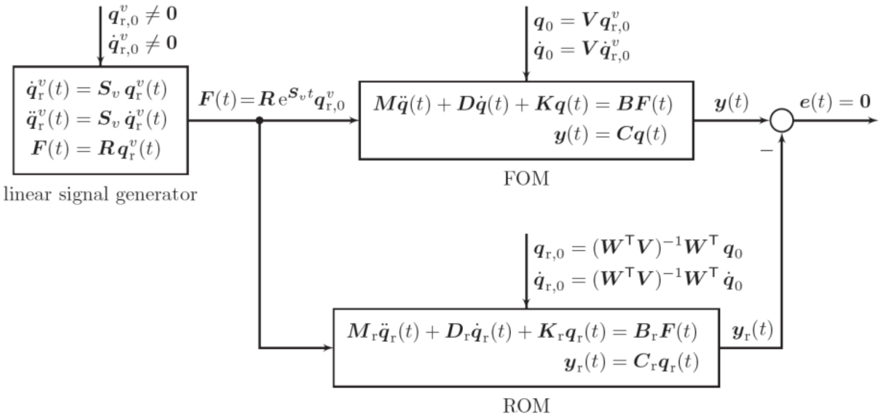
Linear Moment Matching by Interconnection

Time-domain interpretation of Moment Matching

Theorem 1: Consider the interconnection of the linear second-order system with the signal generator

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t) &= \boldsymbol{S}_v\, \boldsymbol{q}_{\mathrm{r}}^{v}(t), \quad \boldsymbol{q}_{\mathrm{r}}^{v}(0) = \boldsymbol{q}_{\mathrm{r},0}^{v} \neq \boldsymbol{0}, \\ \ddot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t) &= \boldsymbol{S}_v\, \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t), \quad \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(0) = \dot{\boldsymbol{q}}_{\mathrm{r},0}^{v} \neq \boldsymbol{0}, \\ \boldsymbol{F}(t) &= \boldsymbol{R}\, \boldsymbol{q}_{\mathrm{r}}^{v}(t). \end{aligned}$$

Let $\boldsymbol{V}$ be the solution of the Sylvester equation $\boldsymbol{M}\,\boldsymbol{V}\,\boldsymbol{S}_v^2 + \boldsymbol{D}\,\boldsymbol{V}\,\boldsymbol{S}_v + \boldsymbol{K}\,\boldsymbol{V} = \boldsymbol{B}\,\boldsymbol{R}$, and $\boldsymbol{W}$ arbitrary such that $\det(\boldsymbol{W}^{\mathsf{T}}\boldsymbol{V}) \neq 0$. Furthermore, let $\boldsymbol{q}_0 = \boldsymbol{V}\,\boldsymbol{q}_{\mathrm{r},0}^{v}$ and $\dot{\boldsymbol{q}}_0 = \boldsymbol{V}\,\dot{\boldsymbol{q}}_{\mathrm{r},0}^{v}$ with $\boldsymbol{q}_{\mathrm{r},0}^{v} \neq \boldsymbol{0},\ \dot{\boldsymbol{q}}_{\mathrm{r},0}^{v} \neq \boldsymbol{0}$ arbitrary. Then, the ROM generated by projection with $\boldsymbol{V}$ *exactly* matches the output response of the FOM, i.e. $\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{y}_{\mathrm{r}}(t) = \boldsymbol{C}\boldsymbol{q}(t) - \boldsymbol{C}\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}(t) = \boldsymbol{0} \ \ \forall\, t.$



$$\boldsymbol{y}_{\mathrm{ss}}(t) = \sum_{i=1}^{r} \boldsymbol{C}(\sigma_i^2\boldsymbol{M} + \sigma_i\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\boldsymbol{r}_i\, q_{\mathrm{r},i}^{v}(t)$$

$$\equiv \sum_{i=1}^{r} \boldsymbol{C}_{\mathrm{r}}(\sigma_i^2\boldsymbol{M}_{\mathrm{r}} + \sigma_i\boldsymbol{D}_{\mathrm{r}} + \boldsymbol{K}_{\mathrm{r}})^{-1}\boldsymbol{B}_{\mathrm{r}}\boldsymbol{r}_i\, q_{\mathrm{r},i}^{v}(t) = \boldsymbol{y}_{\mathrm{r,ss}}(t)$$

Nonlinear systems – Nonlinear projection

Second-order nonlinear system

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{D}\dot{\boldsymbol{q}}(t) + \boldsymbol{f}(\boldsymbol{q}(t)) = \boldsymbol{B}\boldsymbol{F}(t) \qquad \boldsymbol{q}(0) = \boldsymbol{q}_0,\ \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_0,$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t)$$

Nonlinear (Petrov-)Galerkin projection

Approximation ansatz $\boldsymbol{q}(t) \approx \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}(t))$ with the mapping $\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})$:

$$\dot{\boldsymbol{q}} = \frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}{\partial \boldsymbol{q}_\mathrm{r}} \dot{\boldsymbol{q}}_\mathrm{r}, \qquad \ddot{\boldsymbol{q}} = \frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}{\partial \boldsymbol{q}_\mathrm{r}} \ddot{\boldsymbol{q}}_\mathrm{r} + \frac{\partial^2 \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}{\partial \boldsymbol{q}_\mathrm{r}^2} (\dot{\boldsymbol{q}}_\mathrm{r} \otimes \dot{\boldsymbol{q}}_\mathrm{r}), \qquad \widetilde{\boldsymbol{V}}(\boldsymbol{q}_\mathrm{r}) = \frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}{\partial \boldsymbol{q}_\mathrm{r}} \in \mathbb{R}^{n \times r}$$

Inserting the ansatz and its derivatives yields an overdetermined system of equations with the residual $\boldsymbol{\varepsilon}$.

Premultiplying the equations with the Jacobian $\widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T} = \left.\frac{\partial \boldsymbol{\omega}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}$ of another mapping $\boldsymbol{\omega}(\boldsymbol{q}(t))$ and enforcing $\widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T}\boldsymbol{\varepsilon} = \boldsymbol{0}$, yields the ROM

$$\widetilde{\boldsymbol{M}}_\mathrm{r}\ddot{\boldsymbol{q}}_\mathrm{r} + \tilde{\boldsymbol{g}} + \widetilde{\boldsymbol{D}}_\mathrm{r}\dot{\boldsymbol{q}}_\mathrm{r} + \widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T}\boldsymbol{f}\big(\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})\big) = \widetilde{\boldsymbol{B}}_\mathrm{r}\boldsymbol{F},$$
$$\boldsymbol{y}_\mathrm{r} = \boldsymbol{C}\,\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}),$$

with $\left\{\widetilde{\boldsymbol{M}}_\mathrm{r}, \widetilde{\boldsymbol{D}}_\mathrm{r}\right\} = \widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T}\{\boldsymbol{M}, \boldsymbol{D}\}\,\widetilde{\boldsymbol{V}}(\boldsymbol{q}_\mathrm{r})$, $\widetilde{\boldsymbol{B}}_\mathrm{r} = \widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T}\boldsymbol{B}$, the convective term $\tilde{\boldsymbol{g}} = \widetilde{\boldsymbol{W}}(\boldsymbol{q})^\mathsf{T}\boldsymbol{M}\frac{\partial^2 \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})}{\partial \boldsymbol{q}_\mathrm{r}^2}(\dot{\boldsymbol{q}}_\mathrm{r} \otimes \dot{\boldsymbol{q}}_\mathrm{r})$ and

the initial conditions $\boldsymbol{q}_\mathrm{r}(0) = \arg\min\limits_{\boldsymbol{q}_{\mathrm{r},0}} \|\boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r},0}) - \boldsymbol{q}_0\|_2^2$, $\dot{\boldsymbol{q}}_\mathrm{r}(0) = (\widetilde{\boldsymbol{W}}_{\boldsymbol{q}_0}{}^\mathsf{T}\widetilde{\boldsymbol{V}}_{\boldsymbol{q}_{\mathrm{r},0}})^{-1}\widetilde{\boldsymbol{W}}_{\boldsymbol{q}_0}{}^\mathsf{T}\dot{\boldsymbol{q}}_0$.

How to choose $\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r})$?

Nonlinear systems – Steady-state response

Nonlinear signal generator

$$\dot{\boldsymbol{q}}_\mathrm{r}^v(t) = \boldsymbol{s}_v\big(\boldsymbol{q}_\mathrm{r}^v(t)\big), \qquad \boldsymbol{q}_\mathrm{r}^v(0) = \boldsymbol{q}_{\mathrm{r},0}^v \neq \boldsymbol{0},$$

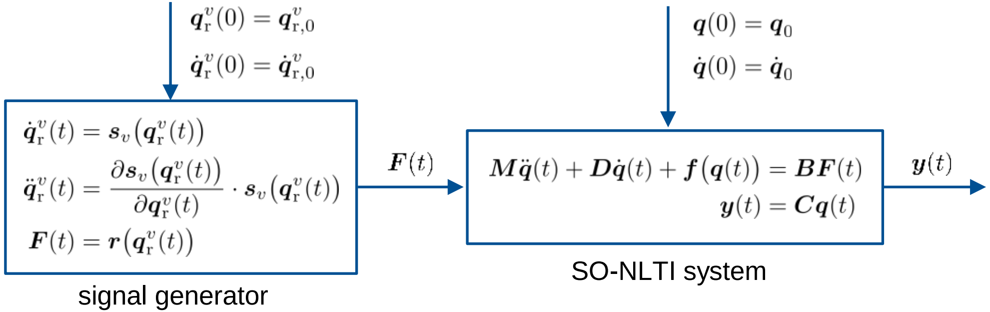
$$\ddot{\boldsymbol{q}}_\mathrm{r}^v(t) = \frac{\partial \boldsymbol{s}_v\big(\boldsymbol{q}_\mathrm{r}^v(t)\big)}{\partial \boldsymbol{q}_\mathrm{r}^v(t)} \cdot \boldsymbol{s}_v\big(\boldsymbol{q}_\mathrm{r}^v(t)\big), \qquad \dot{\boldsymbol{q}}_\mathrm{r}^v(0) = \dot{\boldsymbol{q}}_{\mathrm{r},0}^v \neq \boldsymbol{0},$$

$$\boldsymbol{F}(t) = \boldsymbol{r}\big(\boldsymbol{q}_\mathrm{r}^v(t)\big).$$

$\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v)$

$\boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v)$

Steady-state response of interconnected system



$$\boldsymbol{q}(t) = \underbrace{\boldsymbol{q}_\mathrm{h}(t)}_{\text{decaying homog. sol.}} + \underbrace{\boldsymbol{\nu}\big(\boldsymbol{q}_\mathrm{r}^v(t)\big)}_{\text{growing part. sol.}}$$

For $t \to \infty$: $\boldsymbol{y}_\mathrm{ss}(t) = \boldsymbol{C}\,\boldsymbol{q}_\mathrm{ss}(t) = \boldsymbol{C}\boldsymbol{\nu}\big(\boldsymbol{q}_\mathrm{r}^v(t)\big)$

$$:= \boldsymbol{m}_0\big(\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{q}_{\mathrm{r},0}^v\big)$$

Second-order nonlinear Sylvester-like PDE

State vector $\boldsymbol{q}_\mathrm{r}^v(t)$ **cannot be factored out**, yielding a *state-dependent*, second-order PDE

$$\boldsymbol{M}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\frac{\partial \boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) + \boldsymbol{M}\frac{\partial^2 \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^{v2}}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) \otimes \boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v)$$
$$+ \boldsymbol{D}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) + \boldsymbol{f}\big(\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)\big) = \boldsymbol{B}\,\boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v)$$

Nonlinear Moment Matching by Interconnection

Time-domain Second-Order Nonlinear Moment Matching

Theorem 2: Consider the interconnection of the nonlinear system with the nonlinear signal generator.

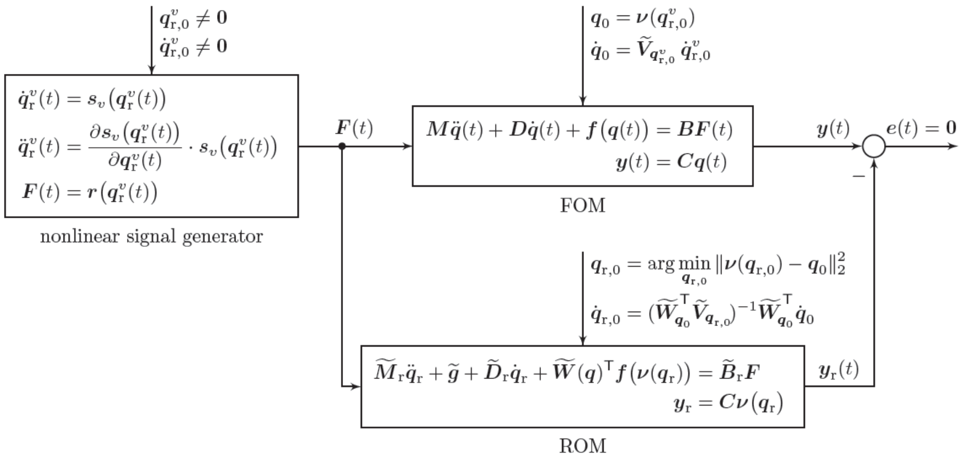
Let $\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)$ be the solution of the second-order Sylvester-PDE

$$\boldsymbol{M}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\frac{\partial \boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) + \boldsymbol{M}\frac{\partial^2 \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^{v2}}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v)\otimes \boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) + \boldsymbol{D}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)}{\partial \boldsymbol{q}_\mathrm{r}^v}\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v) + \boldsymbol{f}\big(\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)\big) = \boldsymbol{B}\,\boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v)$$

and $\boldsymbol{\omega}(\boldsymbol{q})$ arbitrary such that $\det(\widetilde{\boldsymbol{W}}^\mathsf{T}\widetilde{\boldsymbol{V}}) \neq 0$. Furthermore, let $\boldsymbol{q}_0 = \boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r},0}^v)$, $\dot{\boldsymbol{q}}_0 = \widetilde{\boldsymbol{V}}_{\boldsymbol{q}_{\mathrm{r},0}^v}\dot{\boldsymbol{q}}_{\mathrm{r},0}^v$ with $\boldsymbol{q}_{\mathrm{r},0}^v \neq \boldsymbol{0}$, $\dot{\boldsymbol{q}}_{\mathrm{r},0}^v \neq \boldsymbol{0}$ arbitrary.

Then, the ROM generated by nonlinear projection using $\boldsymbol{\nu}(\boldsymbol{q}_\mathrm{r}^v)$ *exactly* matches the output response of the FOM, i.e.

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{y}_\mathrm{r}(t) = \boldsymbol{C}\boldsymbol{q}(t) - \boldsymbol{C}\boldsymbol{\nu}\big(\boldsymbol{q}_\mathrm{r}(t)\big) = \boldsymbol{0} \quad \forall t.$$



$$\begin{aligned}\boldsymbol{y}_\mathrm{ss}(t) &= \boldsymbol{C}\,\boldsymbol{q}_\mathrm{ss}(t) := \boldsymbol{m}_0\big(\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{q}_{\mathrm{r},0}^v\big) \\ &\equiv \boldsymbol{C}\boldsymbol{\nu}\big(\boldsymbol{q}_\mathrm{r}(t)\big) := \boldsymbol{m}_{\mathrm{r},0}\big(\boldsymbol{s}_v(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{r}(\boldsymbol{q}_\mathrm{r}^v(t)), \boldsymbol{q}_{\mathrm{r},0}^v\big) \\ &= \boldsymbol{y}_{\mathrm{r,ss}}(t)\end{aligned}$$

Nonlinear Moment Matching – Simplifications

Nonlinear (state-dependent), second-order PDE is difficult to solve!

$$\boldsymbol{M}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v})}{\partial \boldsymbol{q}_{\mathrm{r}}^{v}}\frac{\partial \boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v})}{\partial \boldsymbol{q}_{\mathrm{r}}^{v}}\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v})+\boldsymbol{M}\frac{\partial^{2} \boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v})}{\partial \boldsymbol{q}_{\mathrm{r}}^{v2}}\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v})\otimes\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v})+\boldsymbol{D}\frac{\partial \boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v})}{\partial \boldsymbol{q}_{\mathrm{r}}^{v}}\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v})+\boldsymbol{f}(\boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v}))=\boldsymbol{B}\,\boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v})$$

Simplifications towards a practicable, simulation-free second-order nonlinear Moment Matching

A. ***Linear projection:*** $\boldsymbol{q}(t)=\boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))=\boldsymbol{V}\,\boldsymbol{q}_{\mathrm{r}}^{v}(t)$

→ PDE becomes an *algebraic equation*: $\boldsymbol{M}\boldsymbol{V}\frac{\partial \boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))}{\partial \boldsymbol{q}_{\mathrm{r}}^{v}(t)}\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))+\boldsymbol{D}\,\boldsymbol{V}\,\boldsymbol{s}_{v}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))+\boldsymbol{f}(\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}^{v}(t))-\boldsymbol{B}\,\boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))=\boldsymbol{0}$

B. ***Column-wise consideration:*** above equation is underdetermined → consider it column-wise for each $\boldsymbol{v}_i\in\mathbb{R}^n,\; i=1,\dots,r$

$$\boldsymbol{M}\,\boldsymbol{v}_i\frac{\partial s_{v_i}(q_{\mathrm{r},i}^{v}(t))}{\partial q_{\mathrm{r},i}^{v}(t)}s_{v_i}(q_{\mathrm{r},i}^{v}(t))+\boldsymbol{D}\,\boldsymbol{v}_i\,s_{v_i}(q_{\mathrm{r},i}^{v}(t))+\boldsymbol{f}(\boldsymbol{v}_i\,q_{\mathrm{r},i}^{v}(t))-\boldsymbol{B}\,\boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t))=\boldsymbol{0}$$

C. ***Time discretization with collocation points:*** $\{t_k^*\},\quad \boldsymbol{q}_{\mathrm{r}}^{v}(t_k^*),\quad k=1,\dots,K$

→ Equation becomes *state-independent*:

$$\boldsymbol{M}\,\boldsymbol{v}_{ik}\frac{\partial s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*))}{\partial q_{\mathrm{r},i}^{v}(t_k^*)}s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*))+\boldsymbol{D}\,\boldsymbol{v}_{ik}\,s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*))+\boldsymbol{f}(\boldsymbol{v}_{ik}\,q_{\mathrm{r},i}^{v}(t_k^*))-\boldsymbol{B}\,\boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t_k^*))=\boldsymbol{0}$$

Second-Order Nonlinear Moment Matching

Algorithm 1 Second-order NLMM (SO-NLMM)

Input: $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{f}(\boldsymbol{q})$, $\boldsymbol{B}$, $\boldsymbol{J_f}(\boldsymbol{q})$, $q^v_{\mathrm{r},i}(t^*_k)$, $\dot{q}^v_{\mathrm{r},i}(t^*_k)$, $\ddot{q}^v_{\mathrm{r},i}(t^*_k)$, $\boldsymbol{r}_i(q^v_{\mathrm{r},i}(t^*_k))$, initial guesses $\boldsymbol{v}_{0,ik}$, deflated order r_{defl}

Output: orthogonal basis $\boldsymbol{V}$

for `i = 1 : r` **do** ▷ e.g. r different shifts σ_i

    for `k = 1 : K` **do** ▷ e.g. K samples in each shift

        `fun=@(v)` $\boldsymbol{M}\,\mathrm{v}\,\ddot{q}^v_{\mathrm{r},i}(t^*_k) + \boldsymbol{D}\,\mathrm{v}\,\dot{q}^v_{\mathrm{r},i}(t^*_k) + \boldsymbol{f}(\mathrm{v}\,q^v_{\mathrm{r},ik}) - \boldsymbol{B}\,\boldsymbol{r}_i(q^v_{\mathrm{r},ik})$

        `Jfun=@(v)` $\boldsymbol{M}\,\ddot{q}^v_{\mathrm{r},i}(t^*_k) + \boldsymbol{D}\,\dot{q}^v_{\mathrm{r},i}(t^*_k) + \boldsymbol{J_f}(\mathrm{v}\,q^v_{\mathrm{r},ik})\,q^v_{\mathrm{r},ik}$

        `V(:,(i-1)*K+k)= Newton(fun,` $\boldsymbol{v}_{0,ik}$ `,Jfun)`

        `V = gramSchmidt((i-1)*K+k, V)` ▷ optional

    end for

end for

`V = svd(V,` r_{defl}`)` ▷ deflation is optional

Approximated nonlinear moments

Due to the simplications, we are *approximately* matching nonlinear moments!

$$\boldsymbol{y}_{\mathrm{ss}}(t) = \boldsymbol{C}\boldsymbol{\nu}\big(\boldsymbol{q}^v_{\mathrm{r}}(t)\big) := \boldsymbol{m}_0\big(\boldsymbol{s}_v(\boldsymbol{q}^v_{\mathrm{r}}(t)), \boldsymbol{r}(\boldsymbol{q}^v_{\mathrm{r}}(t)), \boldsymbol{q}^v_{\mathrm{r},0}\big)$$

↓ approx

$$\boldsymbol{y}_{\mathrm{ss},i}(t^*_k) = \boldsymbol{C}\,\boldsymbol{v}_{ik}\,q^v_{\mathrm{r},i}(t^*_k) := \boldsymbol{m}_0\big(s_{v_i}(q^v_{\mathrm{r},i}(t^*_k)), \boldsymbol{r}_i(q^v_{\mathrm{r},i}(t^*_k)), q^v_{\mathrm{r},0,i}, t^*_k\big)$$

Computational aspects

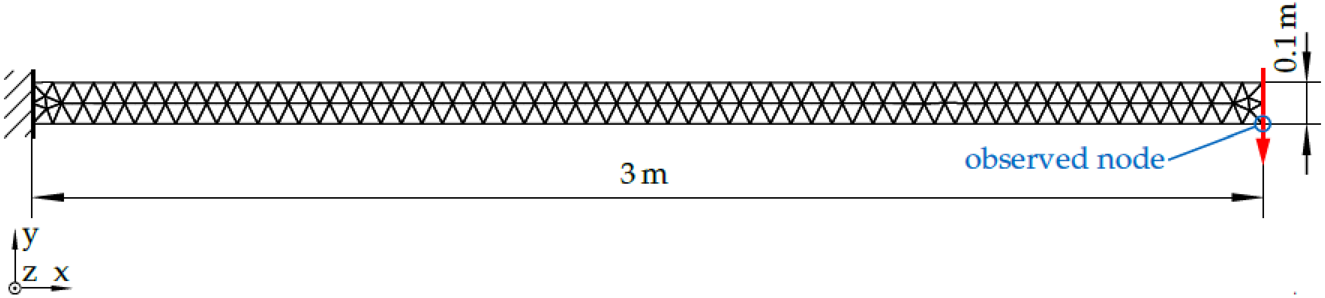
a) *Different strategies and degrees of freedom:* signal generators and collocation points $\{s_{v_i}, \boldsymbol{r}_i, q^v_{\mathrm{r},0,i}, t^*_k\}$

b) *Computational effort:* solution of nonlinear systems of equations (NLSE) with Newton

c) *Other aspects:*
   - initial guesses for Newton scheme
   - orthogonalization process and deflation

Numerical Examples – Cantilever beam

2D model of a cantilever beam



- 246 triangular Tri6 elements; 1224 dofs
- linear St. Venant-Kirchhoff material (steel)
- *geometric nonlinear* behaviour
- loading force at the tip in negative y-direction
- simulation conducted with open-source AMfe-code
- numerical integration using implicit generalized-α scheme
- comparison of SO-NLMM with POD and basis augmentation

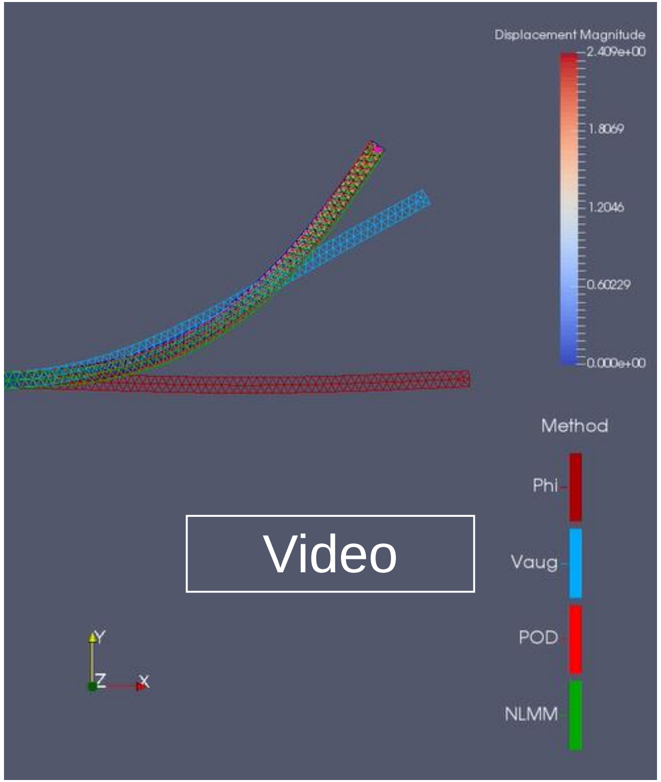
Training phase of SO-NLMM and POD:

- single signal generator with $K = 10$ or $K = 20$
- signal generator: $q_{\mathrm{r}}^{v}(t) = \sin(10t),\ \dot{q}_{\mathrm{r}}^{v}(t),\ \ddot{q}_{\mathrm{r}}^{v}(t)$
- training input: $F(t) = r(q_{\mathrm{r}}^{v}(t)) = 10^8 \cdot q_{\mathrm{r}}^{v}(t)$

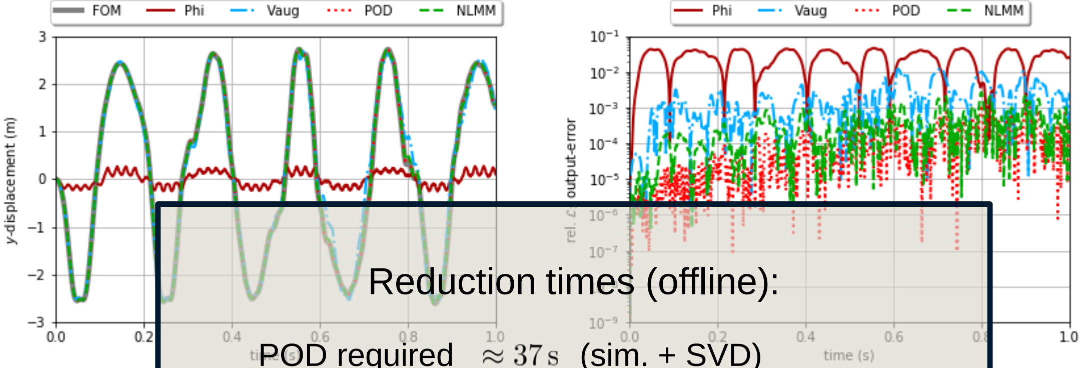
Test phase of FOM and ROMs:

- test input: $F(t) = 10^8 \cdot \sin(31t)$

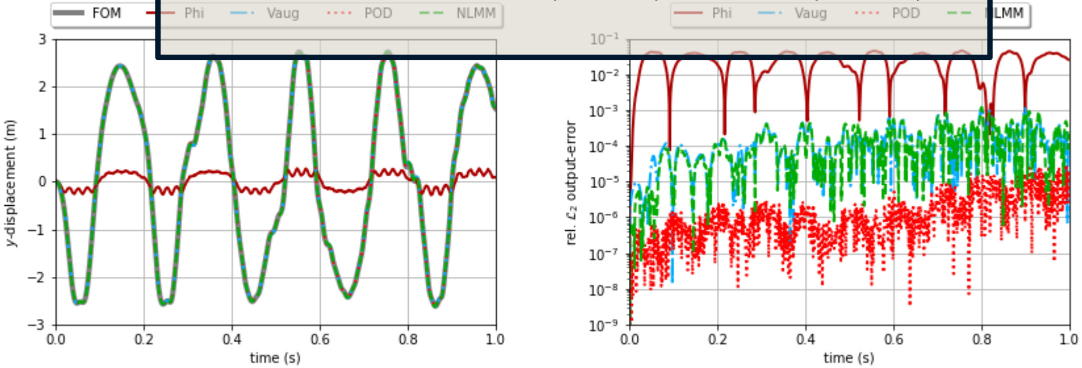
Numerical Examples – Cantilever beam



r = 10:



r = 20:



Reduction times (offline):

POD required $\approx 37\,\mathrm{s}$ (sim. + SVD)

NLMM needed $\approx 2.6\,\mathrm{s}$ $(K = 10)$ or $\approx 6\,\mathrm{s}$ $(K = 20)$

Analysis, Discussion and Limitations

1. Adequate selection of the projection ansatz

$$\boldsymbol{q} \approx \boldsymbol{V}\,\boldsymbol{q}_{\mathrm{r}} \quad \Rightarrow \quad \boldsymbol{q} \approx \sum_{k=1}^{N} \boldsymbol{V}^{(k)}\,\boldsymbol{q}_{\mathrm{r}}^{(k)} = \boldsymbol{V}^{(1)}\boldsymbol{q}_{\mathrm{r}}^{(1)} + \boldsymbol{V}^{(2)}\,(\boldsymbol{q}_{\mathrm{r}} \otimes \boldsymbol{q}_{\mathrm{r}}) + \cdots \quad \Rightarrow \quad \boldsymbol{q} \approx \boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}})$$

2. Appropriate choice of the signal generator

Signal generator determines:

- Ansatz for the dynamics $\boldsymbol{q}_{\mathrm{r}}^{v}(t),\ \dot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t),\ \ddot{\boldsymbol{q}}_{\mathrm{r}}^{v}(t)$
- Exciting input of the system $\boldsymbol{F}_{\mathrm{train},i}(t_k^*) = \boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t_k^*))$

$$\dot{\boldsymbol{q}}_{\mathrm{r}}^{v} = \sum_{k=1}^{N} \boldsymbol{S}_{v}^{(k)}\,\boldsymbol{q}_{\mathrm{r}}^{v\,(k)} = \boldsymbol{S}_{v}^{(1)}\boldsymbol{q}_{\mathrm{r}}^{v} + \boldsymbol{S}_{v}^{(2)}(\boldsymbol{q}_{\mathrm{r}}^{v} \otimes \boldsymbol{q}_{\mathrm{r}}^{v}) + \cdots,$$

$$\boldsymbol{F} = \sum_{k=1}^{N} \boldsymbol{R}^{(v)}\boldsymbol{q}_{\mathrm{r}}^{v\,(k)} = \boldsymbol{R}^{(1)}\boldsymbol{q}_{\mathrm{r}}^{v} + \boldsymbol{R}^{(2)}(\boldsymbol{q}_{\mathrm{r}}^{v} \otimes \boldsymbol{q}_{\mathrm{r}}^{v}) + \cdots,$$

3. Obtaining a state-independent matrix equation: factorization of the state vector $\boldsymbol{q}_{\mathrm{r}}^{v}(t)$ to obtain a constant matrix equation!
   → depends on both (1) the projection ansatz and (2) the signal generator (problem-dependent!)

4. Limitations of the column-wise consideration: $q_{\mathrm{r},i}^{v}(t) \in \mathbb{R},\ s_{v_i}(q_{\mathrm{r},i}^{v}(t)),\ \boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t))$ instead of $\boldsymbol{q}_{\mathrm{r}}^{v}(t) \in \mathbb{R}^{r},\ \boldsymbol{s}_v\big(\boldsymbol{q}_{\mathrm{r}}^{v}(t)\big),\ \boldsymbol{r}\big(\boldsymbol{q}_{\mathrm{r}}^{v}(t)\big)$
   → couplings in $\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}^{v}(t),\ \boldsymbol{V}\boldsymbol{s}_v(\boldsymbol{q}_{\mathrm{r}}^{v}(t))$ and $\boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v}(t))$ are not being considered.

5. Poisson stability of the signal generator

Summary & Outlook

Summary:

- Aim: simulation-free approaches for nonlinear *structural* systems
- Time-domain / Steady-state interpretation of linear moment matching with signal generators
- Astolfi‘s extension of Moment Matching to the nonlinear *second-order* case
  - ➢ Nonlinear projection ansatz yields a difficult second-order, state-dependent Sylvester-like PDE
- Simplifications proposed to achieve a practicable numerical algorithm for nonlinear MOR
  - ➢ Linear projection, column-wise consideration, time discretization yield nonlinear systems of equations (NLSE)
- Numerical examples & Discussion of different strategies, numerical aspects and choice of degrees of freedom

Ongoing / Future Work:

- Second-order NLMM for non-proportionally damped systems?
- Output Krylov subspace-based Nonlinear Moment Matching (duality!)
- Hyper-Reduction

Thank you for your attention!

Backup

References

[Astolfi ’10] *Model reduction by moment matching for linear and nonlinear systems.* IEEE TAC.

[Beattie/Gugercin ’05] *Krylov-based model reduction of second-order systems with proportional damping.* In 44th IEEE Conference on Decision and Control.

[Cruz et al. ’19] *Practicable Simulation-Free Model Order Reduction by Nonlinear Moment Matching.* https://arxiv.org/abs/1901.10750.

[Huang ’04] *Nonlinear Output Regulation: Theory and Applications.* SIAM Advances in Design & Control.

[Rutzmoser ’18] *Model Order Reduction for Nonlinear Structural Dynamics: Simulation-free Approaches*. PhD thesis, TUM

[Salimbahrami ’05] *Structure preserving order reduction of large scale second order models*. PhD thesis, TUM

[Scarciotti/Astolfi ’17] *Data-driven model reduction by moment matching for linear and nonlinear systems.* Automatica

Thank you for your attention!

Nonlinear dimensional reduction methods

Simulation-based approaches (e.g. POD)

Take snapshots of the simulated trajectory for typical (training) input force and perform SVD

$$\underset{(n,n_\mathrm{s})}{\boldsymbol{Q}} = \left[\boldsymbol{q}(t_1),\, \boldsymbol{q}(t_2),\, \cdots,\, \boldsymbol{q}(t_{n_\mathrm{s}})\right]$$

$$\boldsymbol{Q} \overset{\mathrm{SVD}}{=} \underset{(n,n)}{\boldsymbol{M}}\; \underset{(n,n_\mathrm{s})}{\boldsymbol{\Sigma}}\; \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}^\top} \approx \underset{(n,r)}{\boldsymbol{M}_\mathrm{r}}\; \underset{(r,n_\mathrm{s})}{\boldsymbol{\Sigma}_\mathrm{r}}\; \underset{(n_\mathrm{s},n_\mathrm{s})}{\boldsymbol{N}_\mathrm{r}^\top}$$

Reduction basis: $\boldsymbol{V} = \boldsymbol{M}_\mathrm{r} \in \mathbb{R}^{n\times r}$

$$\boldsymbol{q}(t) \approx \boldsymbol{V}\, \boldsymbol{q}_\mathrm{r}(t) = \sum_{i=1}^{r} \boldsymbol{v}_i\, q_{\mathrm{r},i}(t)$$

Simulation-free / System-theoretic methods

- Basis augmentation: Enrichment of a linear basis with nonlinear information

$$\boldsymbol{V}_\mathrm{aug} = \left[\boldsymbol{V}^{(1)},\, \boldsymbol{V}^{(2)}\right] \qquad \boldsymbol{q}(t) \approx \boldsymbol{V}_\mathrm{aug}\, \boldsymbol{q}_{\mathrm{r,aug}}(t)$$

\+ : Easy projection

− : Higher reduced order

- Nonlinear projection (e.g. Quadratic Manifold)

$\boldsymbol{V}^{(1)} \in \mathbb{R}^{n\times r}$

$\boldsymbol{V}^{(2)} \in \mathbb{R}^{n\times r^2}$

$$\boldsymbol{q}(t) \approx \boldsymbol{V}^{(1)} \boldsymbol{q}_\mathrm{r}(t) + \boldsymbol{V}^{(2)} \left(\boldsymbol{q}_\mathrm{r}(t) \otimes \boldsymbol{q}_\mathrm{r}(t)\right)$$

Reduced coordinates: $\boldsymbol{q}_\mathrm{r}(t) = \left[q_{\mathrm{r},1}(t), \cdots, q_{\mathrm{r},r}(t)\right]^\top$

\+ : Smaller reduced order

− : Difficult projection

Nonlinear MOR methods – Overview

Nonlinear systems

Reduction of nonlinear (parametric) systems

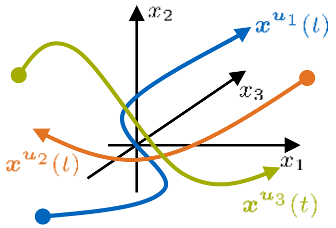
$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u) \qquad\qquad \boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\, u$$
$$y = h(\boldsymbol{x}) \qquad\qquad y = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$$

- ☑ Simulation-based:
  - POD, TPWL
  - Reduced Basis, Empirical Gramians
- ➢ **Simulation-free / System-theoretic:**
  - Nonlinear Normal Modes (Rosenberg 1962)
  - Nonlinear Balanced Truncation (Scherpen 1993)
  - Nonlinear Moment Matching (Astolfi 2010)

Proper Orthogonal Decomposition

Starting point: $\boldsymbol{E}\,\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$

$\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x})$



1. Choose suitable training input signals $\boldsymbol{u}_1(t), \boldsymbol{u}_2(t), \dots, \boldsymbol{u}_t(t)$
2. Take snapshots from simulated full order state trajectories

$$\underset{(n,n_{\mathrm{s}})}{\boldsymbol{X}} = [\boldsymbol{x}^{\boldsymbol{u}_1}(t_1), \boldsymbol{x}^{\boldsymbol{u}_1}(t_2), \cdots, \boldsymbol{x}^{\boldsymbol{u}_1}(t_N) \quad \boldsymbol{x}^{\boldsymbol{u}_2}(t_1), \boldsymbol{x}^{\boldsymbol{u}_2}(t_2), \cdots]$$

3. Perform singular value decomposition (SVD) of snapshot matrix

$$\boldsymbol{X} = \underset{(n,n)}{\boldsymbol{M}} \underset{(n,n_{\mathrm{s}})}{\boldsymbol{\Sigma}} \underset{(n_{\mathrm{s}},n_{\mathrm{s}})}{\boldsymbol{N}^{\mathsf{T}}} \approx \underset{(n,r)}{\boldsymbol{M}_{\mathrm{r}}} \underset{(r,n_{\mathrm{s}})}{\boldsymbol{\Sigma}_{\mathrm{r}}} \underset{(n_{\mathrm{s}},n_{\mathrm{s}})}{\boldsymbol{N}_{\mathrm{r}}^{\mathsf{T}}}$$

4. Reduced order basis: $\boldsymbol{V} = \boldsymbol{M}_{\mathrm{r}} \in \mathbb{R}^{n \times r}$

Advantages:

- Straightforward data-driven method
- Choice of reduced order from singular values / error bound for approx. error
- Optimal in least squares sense:

$$\min_{\mathrm{rank}(\boldsymbol{X}_{\mathrm{r}})=r} ||\boldsymbol{X} - \boldsymbol{X}_{\mathrm{r}}||_2$$

Disadvantages:

- Simulation of full order model for different input signals required
- SVD of large snapshot matrix required
- Training input dependency

Linear Systems – Sylvester equation

Equivalence of Krylov subspaces and Sylvester equations

$$\operatorname{span}\left\{(\sigma_1\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{B}\,\boldsymbol{r}_1,\ \ldots,\ (\sigma_r\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{B}\,\boldsymbol{r}_r\right\} \subseteq \operatorname{ran}(\boldsymbol{V}) \quad \Longleftrightarrow \quad \boldsymbol{E}\,\boldsymbol{V}\,\boldsymbol{S}_v - \boldsymbol{A}\,\boldsymbol{V} = \boldsymbol{B}\,\boldsymbol{R}$$

Reduction parameters: • Shifts: $\boldsymbol{S}_v = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ • Tang. directions: $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r]$

Column-wise computation via Arnoldi process

$$(\sigma_i\boldsymbol{E} - \boldsymbol{A})\,\boldsymbol{v}_i = \boldsymbol{B}\boldsymbol{r}_i \quad \overset{\boldsymbol{V}=[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r]}{\Longleftrightarrow} \quad \boldsymbol{E}\,[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix} - \boldsymbol{A}\,[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_r] = \boldsymbol{B}\,[\boldsymbol{r}_1,\ldots,\boldsymbol{r}_r]$$

Linear Sylvester equation

$$\left(\boldsymbol{E}\,\boldsymbol{V}\,\boldsymbol{S}_v - \boldsymbol{A}\,\boldsymbol{V} - \boldsymbol{B}\,\boldsymbol{R}\right)\cdot \boldsymbol{x}_{\mathrm{r}}^{v}(t) = \boldsymbol{0}, \quad \text{for} \quad \boldsymbol{x}_{\mathrm{r}}^{v}(t) = \mathrm{e}^{\boldsymbol{S}_v t}\boldsymbol{x}_{\mathrm{r},0}^{v}$$

Properties and interpretation:

- *Constant (state-independent)* linear Sylvester equation or *linear* systems of equations (LSE)
- We apply
  - a linear projection and
  - excite the system with a sum of growing exponentials (*shifts & tang. directions user-defined*)

Linear Second-Order Systems – Sylvester equation

Equivalence of Krylov subspaces and Sylvester equations

$$\operatorname{span}\left\{(\sigma_1^2 \boldsymbol{M} + \sigma_1 \boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\,\boldsymbol{r}_1, \ldots, (\sigma_r^2 \boldsymbol{M} + \sigma_r \boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{B}\,\boldsymbol{r}_r\right\} \subseteq \operatorname{ran}(\boldsymbol{V}) \quad \Longleftrightarrow \quad \boldsymbol{M}\,\boldsymbol{V}\,\boldsymbol{S}_v^2 + \boldsymbol{D}\,\boldsymbol{V}\,\boldsymbol{S}_v + \boldsymbol{K}\,\boldsymbol{V} = \boldsymbol{B}\,\boldsymbol{R}$$

Reduction parameters: • Shifts: $\boldsymbol{S}_v = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ • Tang. directions: $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r]$

Column-wise computation via Arnoldi process

$$(\sigma_i^2 \boldsymbol{M} + \sigma_i \boldsymbol{D} + \boldsymbol{K})\,\boldsymbol{v}_i = \boldsymbol{B}\boldsymbol{r}_i \quad \overset{\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r]}{\Longleftrightarrow} \quad \boldsymbol{M}\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r]\begin{bmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2\end{bmatrix} + \boldsymbol{D}\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r]\begin{bmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_r\end{bmatrix} + \boldsymbol{K}\,[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r] = \boldsymbol{B}\,[\boldsymbol{r}_1, \ldots, \boldsymbol{r}_r]$$

Linear Second-Order Sylvester equation

$$\left(\boldsymbol{M}\,\boldsymbol{V}\,\boldsymbol{S}_v^2 + \boldsymbol{D}\,\boldsymbol{V}\,\boldsymbol{S}_v + \boldsymbol{K}\,\boldsymbol{V} - \boldsymbol{B}\,\boldsymbol{R}\right) \cdot \boldsymbol{q}_{\mathrm{r}}^v(t) = \boldsymbol{0}, \quad \text{for} \quad \boldsymbol{q}_{\mathrm{r}}^v(t) = \mathrm{e}^{\boldsymbol{S}_v t}\boldsymbol{q}_{\mathrm{r},0}^v$$

Properties and interpretation:

- *Constant (state-independent)* linear Sylvester equation or *linear* systems of equations (LSE)
- We apply
  - a linear projection and
  - excite the system with a sum of growing exponentials (*shifts & tang. directions user-defined*)

Approximated Nonlinear Moments

Linear Moments

$$\boldsymbol{m}_0(\sigma_i, \boldsymbol{r}_i) = \boldsymbol{m}_0(\sigma_i)\,\boldsymbol{r}_i \qquad \{\sigma_i, \boldsymbol{r}_i\}$$

$$\boldsymbol{y}_{\mathrm{ss}}(t) = \sum_{i=1}^{r} \boldsymbol{y}_{\mathrm{ss},i}(t) = \boldsymbol{C} \sum_{i=1}^{r} (\sigma_i^2 \boldsymbol{M} + \sigma_i \boldsymbol{D} + \boldsymbol{K})^{-1} \boldsymbol{B} \boldsymbol{r}_i\, \mathrm{e}^{\sigma_i t} q_{\mathrm{r},0,i}^{v} = \sum_{i=1}^{r} \boldsymbol{m}_0(\sigma_i, \boldsymbol{r}_i)\, \mathrm{e}^{\sigma_i t} q_{\mathrm{r},0,i}^{v} = \boldsymbol{C}\boldsymbol{V}\boldsymbol{q}_{\mathrm{r}}^{v}(t)$$

Nonlinear Moments

$$\boldsymbol{m}_0\big(\boldsymbol{s}_v(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{q}_{\mathrm{r},0}^{v}\big) \qquad \big\{\boldsymbol{s}_v(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{q}_{\mathrm{r},0}^{v}\big\}$$

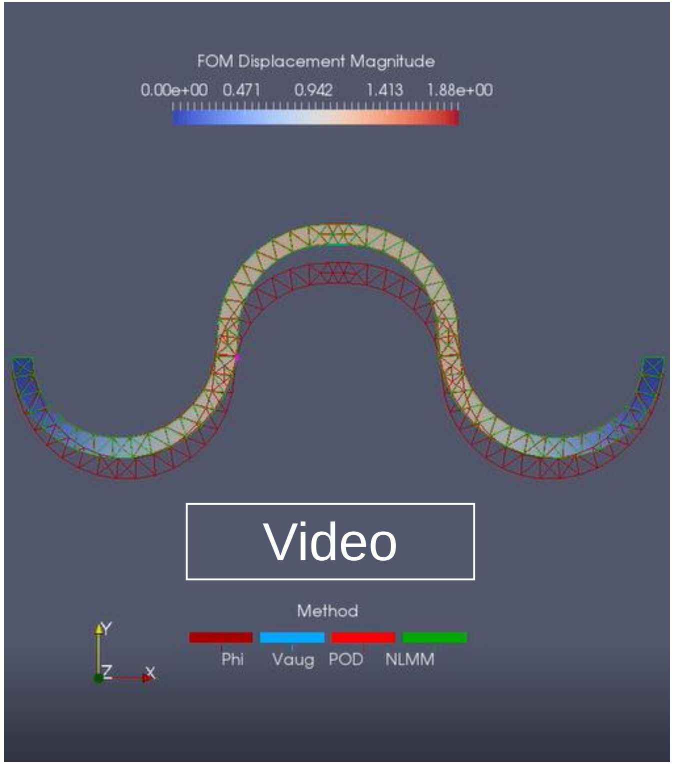
$$\begin{aligned} \boldsymbol{y}_{\mathrm{ss}}(t) &= \boldsymbol{C}\,\boldsymbol{\nu}(\boldsymbol{q}_{\mathrm{r}}^{v}(t)) \\ &:= \boldsymbol{m}_0\big(\boldsymbol{s}_v(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{r}(\boldsymbol{q}_{\mathrm{r}}^{v}(t)), \boldsymbol{q}_{\mathrm{r},0}^{v}\big) \end{aligned}$$

Approximated Nonlinear Moments

$$\boldsymbol{m}_0\big(s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*)), \boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t_k^*)), q_{\mathrm{r},0,i}^{v}, t_k^*\big) \qquad \big\{s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*)), \boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t_k^*)), q_{\mathrm{r},0,i}^{v}, t_k^*\big\}$$

$$\begin{aligned} \boldsymbol{y}_{\mathrm{ss},i}(t_k^*) &= \boldsymbol{C}\,\boldsymbol{v}_{ik}\, q_{\mathrm{r},i}^{v}(t_k^*) \\ &:= \boldsymbol{m}_0\big(s_{v_i}(q_{\mathrm{r},i}^{v}(t_k^*)), \boldsymbol{r}_i(q_{\mathrm{r},i}^{v}(t_k^*)), q_{\mathrm{r},0,i}^{v}, t_k^*\big) \end{aligned}$$

Numerical Examples – S-shape



r = 20: