A probabilistic view on semilinear copulas

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Abstract

This article advances the theory on multivariate upper semilinear copulas. Probabilistic features of this class are discussed and three subclasses are investigated in detail. The first one consists of upper semilinear copulas whose survival copulas are generalised Marshall–Olkin copulas. The second subclass is defined by the property of having identical multivariate diagonals. The third subclass is a family of extendible upper semilinear copulas. Stochastic models and analytical characterisation theorems are derived for each of these subclasses.

Keywords: Copula, diagonal section, semilinear copula, exogenous shock model, Marshall–Olkin copula.

1 Introduction

Sklar’s theorem, see [26], allows the decomposition of a multivariate distribution function $F$ into a copula $C$ and marginal distribution functions $F_1, \ldots, F_d$ via:

$$F(x) = C(F_1(x_1), \ldots, F_d(x_d)), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d. \quad (1)$$

The copula $C$ is itself a multivariate distribution function whose marginal distribution functions are standardised to the uniform distribution function on $[0, 1]$. The analytical decomposition in Eq. (1) is useful for stochastic modeling as well as statistical inference. Therefore, copulas have been intensively studied over the recent decades in analysis, probability theory, and statistics, see [6, 11, 16, 19, 25]. Furthermore, they are used in practical applications, e.g. in quantitative risk management, credit risk, and insurance mathematics, see [7, 9].

Copulas can be equivalently characterised by analytical properties involving the notion of $d$-monotonicity, groundedness, and the uniform margin property, see Eqs. (3a) to (3c). Consequently, in spite their probabilistic nature, they are also studied as purely analytical objects in other mathematical fields, e.g. in fuzzy set theory. Two examples are the following:

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• Bivariate copulas have the uniform margin property and are 2-increasing. In particular, this latter property implies that they are non-decreasing in each component. Hence, they belong to the classes of conjunctors and semicopulas (sometimes also called t-seminorms).\footnote{A conjunctor is a monotone non-decreasing extension of the boolean conjunction from \{0, 1\} to the interval [0, 1] and a semicopula is a conjunction with neutral element 1, see \cite{3}.}

• The same properties are also fulfilled by copulas for higher dimensions. Particularly, this implies that the distribution functions of countable infinite exchangeable sequences of random variables with uniform margins form a subclass of conjuctive aggregation operations, see \cite{5}.

As copulas can be treated and analysed as purely analytical objects, there are various families of copulas that historically have not originated from a stochastic model, but have been identified by their analytical properties from a broader class of functions. Usually, it is a difficult undertaking to link the analytical properties of a copula to a stochastic model. A stochastic representation, however, is required for applications, e.g. for simulation and model construction. Additionally, stochastic representations can yield valuable analytical insights as, for example, analytical characterisations or the proof of certain characteristics. A well-known example is the class of Archimedean copulas, see \cite{23, 24}. These originally emerged in the field of probabilistic metric spaces as a subclass of so called t-norms. A complete characterisation of this class has been presented only recently in \cite{18} by means of a unified stochastic representation. Another example is presented in Section 7, where radial symmetry follows naturally from the stochastic model whereas an analytical proof would require tedious calculations.

In this article, we discuss multivariate upper semilinear copulas (abbr. as USL\(_d\) for \(d \geq 2\)). These were introduced in \cite{2} as the solution to the following compatibility problem. Given copula diagonals \(\delta_j, j \in \{2, \ldots, d\}\), what are the necessary and sufficient conditions that these are the \(j\)-dimensional marginal diagonals of an exchangeable \(d\)-variate copula \(C\) that is linear on the segments

\[
S_{\nu,d} := \left\{ \begin{pmatrix} \mathbf{u} \\ 1 \end{pmatrix} \in [0,1]^{j+(d-j)} : \mathbf{u} = \lambda \begin{pmatrix} v_1 \\ \vdots \\ v_j \end{pmatrix} + (1 - \lambda) \begin{pmatrix} v_1 \\ \vdots \\ v_j \end{pmatrix}, \lambda \in [0,1] \right\},
\]

where \(j \in \{1, \ldots, d\}\) and \(v \in [0,1]^j\) with \(v_1 \leq \ldots \leq v_j = 1\)? These copulas are an extension of bivariate upper semilinear copulas, which were proposed in \cite{4}. \cite{2} describes the following recursive construction for \(C\). For this, denote for \(k \in \{1, \ldots, d\}\) the \(k\)-margins of \(C\) by \(C_k\) and denote for \(u = (u_1, \ldots, u_k) \in [0,1]^k\) its ordered version by \(u(1) \leq \ldots \leq u(d)\). Furthermore, represent \(u\) as the linear combination\footnote{Note, that we have made a small correction to the representation of \(u^*\) compared to the original reference \cite{2}.}

\[
u = \lambda (u(1) \cdot 1) + (1 - \lambda) u^* ,
\]
where \( \lambda = (1 - u(1)) / (u(k) - u(1)) \) and
\[
    u^*_j := u(1) + \frac{1}{1 - \lambda}(u(j) - u(1)), \quad j \in \{1, \ldots, k\}.
\]
In particular, \( u(1) \cdot 1 \) is a value on the (marginal) diagonal and \( u^* \) is a value on the (marginal) boundary. Consequently, \( C \) being linear on the segments in Eq. (2) implies for \( k \in \{2, \ldots, d\} \) the following recursion
\[
    C_k(u) = \lambda \cdot C_k(u(1) \cdot 1) + (1 - \lambda) \cdot C_k(u^*)
    = \lambda \cdot \delta_k(u(1)) + (1 - \lambda) \cdot C_{k-1}(u^*_1, \ldots, u^*_k).
\]

To the best of our knowledge, for \( d \geq 2 \), the class of multivariate upper semilinear copulas has only been investigated analytically. Furthermore, apart from the special case of comonotonicity, we are not aware of any stochastic representation for a multivariate upper semilinear copula with \( d \geq 2 \). We intend to fill this gap by investigating the following three subclasses. For all families, we provide a stochastic representation and a characterisation theorem.

(A) One subclass of upper semilinear copulas can be linked to so-called exchangeable exogenous shock models. For this, note that the survival copula \( \hat{C}_2 \) of a bivariate upper semilinear copula \( C_2 \) is called a lower semilinear copula, see [4]. A bivariate lower semilinear copula has the form
\[
    \hat{C}_2(u) = u(1) \cdot \frac{\delta^L(u(2))}{u(2)}, \quad u \in [0,1]^2.
\]
These copulas are also known as bivariate exchangeable generalised Marshall–Olkin copulas (exGMO\(_2\)), see [12, 13]. Furthermore, they have a multivariate extension with a stochastic representation which is called the exchangeable exogenous shock model, see [15]. The question that arises is: What is the intersection between the classes survival exGMO and USL for \( d > 2 \)? In Section 5, we provide an answer to this question by deriving the necessary and sufficient conditions for a copula being in the intersection of both of these classes.

(B) One example, which is presented and discussed in [2], is that of identical multivariate diagonals. We show that a realisation \( U \sim C \) of an upper semilinear copula \( C \) with identical multivariate diagonals can have at most one component that differs from the joint maximum. This implies that the ordered version of \( U \) is determined by the minimum component \( U_\wedge \), the maximum component \( U_\vee \), and the event \( \{U_\wedge \neq U_\vee\} \). In Section 6, we use this observation to derive a stochastic model, which is based on conditional sampling. In passing, we provide a novel characterisation theorem for this subclass.
In Section 7, we present a subclass which is extendible in the class of upper semilinear copulas and provide its explicit deFinetti representation. A deFinetti representation is a two-step model where first a random distribution function is sampled, from which — in the second step — an iid sample is drawn, see [1, Chp. 2 and 3]. This particular subclass extends the bivariate Dirichlet copula, see [22, Thm. 3.5.3], and is radially symmetric. Furthermore, members of this subclass are conjunctive aggregation operations.

Anticipating a few minor results, which will be proven in the following sections, the extension of the first two subclasses to higher dimensions is visualised in Fig. 1. We observe the following: In the bivariate case, both subclasses A and B coincide with the entire class of upper semilinear copulas. In the case \( d > 2 \), the only copula which is in the intersection of the subclasses A and B is the comonotonicity copula. Furthermore, the independence copula is not upper semilinear for \( d > 2 \). Figure 2 shows exemplary scatterplots for all three subclasses, A, B, and C.

The remaining paper is organised as follows: We briefly introduce the necessary key concepts from copula theory in Section 2 as well as the classes of upper semilinear copulas and exchangeable generalised Marshall–Olkin copulas. For the sake of completeness, the class of survival exchangeable Marshall–Olkin copulas is included. \( M_d \) and \( \Pi_d \) are the Fréchet–Hoeffding upper bound and the independence copula for dimension \( d \), respectively.

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3 We call a copula \( C_d \in USL_d \) extendible in the class of upper semilinear copulas if a sequence of random variables \( \{U_i\}_{i\in\mathbb{N}} \) exists such that each finite margin is upper semilinear and the \( d \)-margin is \( C_d \).

4 For the purpose of readability, we refer to these three subclasses in the remainder of this introduction as A, B, and C.

5 This statement can be extended as follows: In the case \( d > 2 \), the only copula that is in at least two of the subclasses A, B, or C is the comonotonicity copula.
Figure 2: Three scatterplots for 3-margins of 5-dimensional realisations of members of the subclasses A, B, and C (left to right). All three copulas were calibrated to the same multivariate lower tail-dependence parameter.

and Section 4 respectively. In Sections 5 to 7 we discuss the subclasses A, B, and C, while Section 8 concludes the paper.

2 Notation and mathematical background

In this section, we introduce copulas from an analytical and probabilistic point of view and recall relevant results and notation used throughout this article.

We use the following conventions for a concise notation. We use bold letters for vectors and capital letters for random variables and vectors. We apply operators component-wise, i.e. \( U \leq u \) means \( U_i \leq u_i \) for all \( i \). For a vector \( u \in [0,1]^d \), we denote its ordered version by \( u_{(1)} \leq \ldots \leq u_{(d)} \). For a (multivariate) distribution function \( F \), we write \( X \sim F \) if the random vector \( X \) has the distribution function \( F \). Furthermore, for two random vectors \( X \) and \( Y \), we write \( X \overset{d}{=} Y \) if \( X \) and \( Y \) have the same distribution function. If not stated otherwise, \( d \in \mathbb{N} \) denotes the dimension. Finally, we define \( [n] := \{1, \ldots, n\} \) for \( n \in \mathbb{N} \).

Copulas, symmetry, and diagonal functions

In the following, we summarise the most important definitions and results from the theory on copulas.

We call a function \( C : [0,1]^d \to [0,1] \) copula if it fulfills the following conditions.

\[
\begin{align*}
C(u) &= 0 & \text{for all } u \in [0,1]^d \text{ with } u_i = 0 \text{ for some } i \in [d]. \quad (3a)
C(u) &= u_i & \text{for all } u \in [0,1]^d \text{ with } u_j = 1 \forall j \neq i. \quad (3b)
V_C([a,b]) &:= \sum_{\gamma \in \times_{i=1}^d \{a_i, b_i\}} (-1)^{\sum_i 1_{i: u_i=\gamma_i}} C(\gamma) \geq 0 \quad \text{for all } [a, b] \subseteq [0,1]^d. \quad (3c)
\end{align*}
\]

Furthermore, we call \( V_C([a,b]) \) the \( C \)-volume of the rectangle \([a, b] \), \( a \leq b \), and
we call the properties in Eqs. (3a) to (3c) **groundedness, uniform margin property, and d-increasingness**.

Given a set of functions from \([0,1]^d\) to \([0,1]\), an interesting problem is to determine the copula subclass, i.e. the functions that fulfill Eqs. (3a) to (3c). To emphasise that we often discuss potential copulas, we call arbitrary functions \(C : [0,1]^d \to [0,1]\) copula candidate functions. There is a second equivalent definition for copulas that characterises copulas as probabilistic objects.

**Lemma 1** ([20 Thm. 8]). A function \(C : [0,1]^d \to [0,1]\) is a copula if and only if a random vector \(U\) on a probability space \((\Omega, A, \mathbb{P})\) exists with

\[
\mathbb{P}(U_i \leq u) = u, \quad \forall u \in [0,1], \ i \in [d],
\]

\[
\mathbb{P}(U \leq u) = C(u), \quad \forall u \in [0,1]^d.
\]

Hence, we can identify a copula with a probability measure on \([0,1]^d\).

We define the **survival copula** \(\hat{C}\) of a copula \(C\) by

\[
\hat{C}(u) := V_C([1-u,1]), \quad u \in [0,1]^d.
\]

A simple calculation shows that if \(U \sim C\), we have \(1-U \sim \hat{C}\). In particular, this implies that \(\hat{C}\) is itself a copula. Furthermore, note that \(\hat{\hat{C}} = C\). We call a copula **radially symmetric** if \(C = \hat{C}\). In the language of probability theory, this is equivalent to \(U \overset{d}{=} 1-U\) for \(U \sim C\).

We call a copula (candidate function) \(C\) **exchangeable** if

\[
C(u) = C(u_{\pi(1)}, \ldots, u_{\pi(d)}), \quad \forall u \in [0,1]^d
\]

for all permutations \(\pi\) on the index set \([d]\). For a random vector \(U \sim C\), the corresponding probabilistic interpretation is that a permutation of the components of \(U\) does not change its distribution function.

We call a copula **extendible (in a class \(C\))** if an exchangeable sequence \(\{U_i\}_{i \in \mathbb{N}}\) exists such that \((U_1, \ldots, U_d)' \sim C\) (and each finite margin is from the class \(C\)).\(^6\) Note that this implies that an extendible copula is always exchangeable. The converse, however, is not true as one can construct simple examples of exchangeable copulas that are not extendible (e.g. the bivariate counter-monotonicity copula, see [16, Rmk. 1.4]).

For an exchangeable copula (candidate function) \(C\) and \(k \in [d]\), we define the **\(k\)-margin** \(C_k\) and the **\(k\)-diagonal** \(\delta_k\) of \(C\) by

\[
C_k : [0,1]^k \to [0,1], \quad \mathbf{u} \mapsto C(u,1,\ldots,1),
\]

\[
\delta_k : [0,1] \to [0,1], \quad u \mapsto C_k(u,\ldots,u).
\]

\(^6\) Note that our definition of extendibility in a class \(C\) is stricter than the common definition of extendibility, see, e.g. [16] p. 43, where it is usually not required that each finite margin is in the prespecified class \(C\).
In a probabilistic setting, \( C_k \) is the distribution function of a \( k \)-dimensional marginal vector, e.g. \((U_1, \ldots, U_k)\), and \( \delta_k \) is the distribution of the minimum of \( k \) components, e.g. \( \min_{i \in [k]} U_i \).

If one is given the \( k \)-margin of a copula candidate function \( C \), one can use the copula definition or Lemma 1 to check if \( C_k \) is a proper copula. Similarly, there are conditions to verify whether a function \( \delta : [0,1] \to [0,1] \) is the diagonal of a \( d \)-dimensional copula. We call such a function a \( d \)-diagonal. Note that we can determine if a function \( \delta \) is a \( d \)-diagonal with the following lemma.

**Lemma 2** ([10] or [21]). A function \( \delta : [0,1] \to [0,1] \) is a \( d \)-diagonal, i.e. the diagonal of a \( d \)-dimensional copula, if and only if the following conditions are fulfilled:

\[
\begin{align*}
\delta(1) &= 1. \\
\delta(u) &\leq u, \quad \forall u \in [0,1]. \\
0 &\leq \delta(v) - \delta(u) \leq d(v - u), \quad \forall u, v \in [0,1] \text{ with } u < v.
\end{align*}
\]

### 3 Semilinear copulas

In this section, we introduce the class of upper semilinear copulas. Traditionally, this class is not defined by a stochastic model, but by an analytic, recursive construction principle. This recursive construction allows the specification of an upper semilinear copula solely by its diagonal functions. A concise overview of various approaches to specifying a copula via given diagonal functions can be found in [2, Sec. 1].

We placed a special emphasis on the characteristics and peculiarities of this class. One noteworthy property is that a realisation from such a copula has at most two distinct components. Anticipating subsequent sections, this property is crucial for the derivation of stochastic models and characterisation theorems.

**Definition 1.** We call a \( d \)-variate copula candidate function \( C : [0,1]^d \to [0,1] \) (upper) semilinear if \( C \) is exchangeable and if \( C \) is linear on the sections

\[
S_{v,d} := \left\{ \left( \begin{array}{c} u \\ 1 \end{array} \right) \in [0,1]^{j+\bar{d}-j} : \ u = \lambda \cdot \left( \begin{array}{c} v_1 \\ v_j \end{array} \right) + (1-\lambda) \cdot \left( \begin{array}{c} v_1 \\ v_j \end{array} \right), \lambda \in [0,1] \right\}, \quad \text{(rev.)}
\]

where \( j \in \{1,\ldots,d\} \) and \( v \in [0,1]^j \) with \( v_1 \leq \ldots \leq v_j = 1 \).

This property has three important implications, which can be easily derived, see [2]. For this, let \( u = (u_1, \ldots, u_k)^t \in [0,1]^k \) with ordered version \( u_{(1)} \leq \ldots \leq u_{(k)} \).

a) The \( k \)-margins \( C_k \) of \( C \) are upper semilinear copulas (resp. copula candidate functions).
b) The following recursion holds:

\[ C_k(u) = \frac{1 - u(k)}{1 - u(1)} \cdot \delta_k(u(1)) + \frac{u(k) - u(1)}{1 - u(1)} \cdot C_{k-1}(u^*), \]  
\[ \text{where } u^* \in [0,1]^{k-1} \text{ with} \]

\[ u^*_i := u(1) + \frac{(u(i) - u(1)) \cdot (1 - u(1))}{u(k) - u(1)}, \quad i \in \{1, \ldots, k-1\}. \]

\[ \text{(5a)} \]

\[ \text{c) Using the convention that } u(k+1) = 1, \text{ we can write the } k\text{-} \text{margin } C_k \text{ as} \]

\[ C_k(u) = \sum_{i=1}^{k} (u(i+1) - u(i)) \cdot \delta_i(u(1)). \]

\[ \text{(6)} \]

[2] provides the following characterisation theorem for determining the cases in which cases an upper semilinear copula function \( C \) is a proper copula.

**Theorem 1 (Characterisation, see [2, Thm. 1]).** Let \( C \) be an upper semilinear copula candidate function such that the corresponding diagonal sections \( \delta_2, \ldots, \delta_d \) are proper diagonals. \( C \) is a copula if and only if the following three conditions hold:

a) For any \( m \in \{1, \ldots, d-1\} \), the function \( \nu_d^{(m)} : [0,1) \to [0,\infty) \), defined by

\[ \nu_d^{(m)}(u) := \frac{1}{1 - u} \cdot \sum_{j=0}^{m} (-1)^j \binom{m}{j} \delta_{d-m+j}(u), \]

is non-decreasing. The \( C \)-volume of \( [0,u]^d - m \times [u,1]^m \) is \( (1-u) \cdot \nu_d^{(m)}(u) \).

b) The function \( \xi_d : [0,1] \to [0,1] \), defined by

\[ \xi_d(u) := 1 + \sum_{j=1}^{d} (-1)^j \binom{d}{j} \delta_j(u), \]

is non-increasing. The \( C \)-volume of \( [u,1]^d \) is \( \xi_d(u) \).

c) The inequality

\[ \left( \frac{\delta_d(u)}{1-u} \right)' \geq \frac{1 - \xi_d(u)}{(1-u)^2} \]

holds almost everywhere with respect to the Lebesgue measure on \((0,1)\).

The proof of this theorem, see [2, p. 293–296], shows that the three conditions from Theorem [1] are equivalent to \( C \) being \( d \)-increasing.\footnote{It follows directly from Eq. (6) that \( C \) is also grounded and has the uniform margin property.} This proof utilises some properties of semilinear copulas (resp. copula candidate functions) which we will
use in subsequent sections. Therefore, we outline the proof and highlight important intermediate results. For the complete proof, we refer the interested reader to the aforementioned reference.

In the proof, it is first observed that each \( d \)-box \([a, b] \subseteq [0, 1]^d\) can be decomposed into \( d \)-boxes, with disjoint interiors, of the form
\[
\bigtimes_{i=1}^{d} [u_i, v_i],
\]
such that for all \( i \neq j \) either \( u_j \geq v_i, u_i \geq v_j \), or \([u_i, v_i] = [u_j, v_j]\). Hence, and because of the exchangeability of \( C \), it suffices to check the \( d \)-increasingness property on \( d \)-boxes of the form
\[
\bigtimes_{j=1}^{r} [u_j, v_j]^{m_j},
\]
with \( m_1 + \ldots + m_r = d \) and
\[
0 \leq u_1 < v_1 \leq u_2 < v_2 \leq \ldots \leq u_r < v_r \leq 1.
\]

The remaining proof is split into three parts for the cases \( r = 1, r = 2, \) and \( r > 0 \) and the following is shown: The \( d \)-increasingness property is equivalent to condition \( \text{(a)} \) or conditions \( \text{(b)} \) and \( \text{(c)} \) for \( r = 1 \) or \( r = 2 \), respectively. In the case \( r > 2 \), the \( C \)-volume of the \( d \)-box is equal to zero. The last statement is summarised in the following lemma.

**Lemma 3** ([2, Proof of Thm. 1]). Let \( C \) be an upper semilinear copula candidate function and \( r \geq 2 \), \( \sum_{j=1}^{r} m_j = d \), and \( u_i > v_j \forall i > j \). Then
\[
V_C\left(\bigtimes_{j=1}^{r} [u_j, v_j]^{m_j}\right) = 0. \tag{8}
\]

Lemma 3 highlights a significant characteristic of upper semilinear copulas, as Eq. (8) implies that a realisation \( U \sim C \) must be concentrated on at most two components. Put more formally, let \( C \) be an upper semilinear copula and \( U \sim C \). Then, we can conclude with a simple probabilistic argument that
\[
\mathbb{P}(U_1 \neq U_2, U_1 \neq U_3, U_2 \neq U_3) = 0.
\]

This implies a very strong, albeit unusual, dependence structure between the components of \( U \).

Another important corollary from the proof of Theorem 1 is the following collection of closed-form expressions of \( C \)-volumes.

**Corollary 1** ([2, Proof of Thm. 1]). Let \( C \) be an upper semilinear copula. Then for \( 0 \leq u < v \leq 1 \) and \( 0 \leq u_1 < v_1 \leq u_2 < v_2 \leq 1 \), we have
\[
V_C([u, v]^d) = \delta_d(v) - \left(\frac{1-v}{1-u} \cdot \delta_d(u) + \frac{v-u}{1-u} \cdot (1 - \zeta_d(u))\right),
\]

9
\[ V_C([u_1,v_1]^{d-m} \times [u_2,v_2]^m) = (v_2 - u_2) \cdot \left( V_d^{(m)}(v_1) - V_d^{(m)}(u_1) \right), \]
\[ V_C([0,v]^{d - ([0,u]^{d} \cup [u,v]^{d})}) = \frac{v-u}{1-u} \cdot (1 - \zeta_d(u) - \delta_d(u)) \]
\[ = \frac{v-u}{1-u} \cdot V_C([0,1]^{d - ([0,u]^{d} \cup [u,1]^{d})}). \]

**Proof.** For the first identity see [2, p. 296] and for the second identity see [2, p. 294]. We obtain the third identity with the following calculation.

\[ V_C([0,v]^{d - ([0,u]^{d} \cup [u,v]^{d})}) = V_C([0,v]^{d}) - V_C([0,u]^{d}) - V_C([u,v]^{d}) \]
\[ = \delta_d(v) - \delta_d(u) - \delta_d(v) + \left( \frac{1-v}{1-u} \cdot \delta_d(u) + \frac{v-u}{1-u} \cdot (1 - \zeta_d(u)) \right) \]
\[ = \frac{v-u}{1-u} \cdot (1 - \zeta_d(u) - \delta_d(u)) \]
\[ = \frac{v-u}{1-u} \cdot V_C([0,1]^{d - ([0,u]^{d} \cup [u,1]^{d})}). \]

We conclude this section with an interesting finding: While the independence copula \( \Pi_d \) is upper semilinear for \( d = 2 \), we show in the following corollary that this is not the case for \( d > 2 \).

**Corollary 2.** Let \( d > 2 \) and \( C \) be an upper semilinear copula candidate function with \( \delta_j(u) = u^j, j \geq 2 \). Then \( C \) is not a proper copula.

**Proof.** We check this claim by contradiction. Let \( C \) be an upper semilinear copula candidate function with \( d > 2 \) and diagonal functions \( \delta_j(u) = u^j, j \geq 2 \). Assume that \( C \) is a proper copula. This implies that also the 3-margin \( C_3 \) is a proper copula. A simple calculation shows that \( \zeta_3(u) = (1 - u)^3 \). Hence, we find that the third condition of the characterisation theorem, \((\delta_3(u)/(1-u))' \geq (1 - \zeta_3(u))/(1-u)^2 \) for almost every \( u \in [0,1] \), is equivalent to
\[ 3u^2(1-u) + u^3 \geq 1 - (1-u)^3. \]
This implies for \( u = 1/2 \) that \( 4/8 \geq 1 - 1/8 \) or \( 5/8 \geq 1 \), which is a contradiction. \( \square \)

## 4 Exchangeable generalised Marshall–Olkin copulas

Upper semilinear copulas are connected to exchangeable Marshall–Olkin distributions. These have been introduced in the seminal paper [17], which also shows that they are uniquely linked to so-called exogenous shock models with independent, exponentially distributed *shocks*. The aforementioned connection emerges in the bivariate case, where the survival copula of a bivariate upper semilinear copula is an exchangeable generalised Marshall–Olkin copula and vice versa. One of the initial questions leading to this article was under which circumstance this relationship
holds in higher dimensions. To answer this question, we do a short digression on exchangeable generalised Marshall–Olkin distributions and the exogenous shock model.

**Definition 2** (Exchangeable generalised Marshall–Olkin copula). We call a $d$-variate copula candidate function $C : [0, 1]^d \to [0, 1]$ exchangeable and of generalised Marshall–Olkin type if there are functions $g_2, \ldots, g_d$ such that

$$C(u) = u(1) \cdot g_2(u(2)) \cdot \ldots \cdot g_d(u(d)), \quad u = (u_1, \ldots, u_d)'
$$

Furthermore, we call a proper copula of that form exchangeable generalised Marshall–Olkin copula (abbr. as exGMO$_d$).

An extensive monograph on these copulas is [22]; a concise article, containing all results presented in this section, is [15].

Classical Marshall–Olkin copulas arise as a special case if the functions $g_i$, $i \geq 2$, are power functions. They obtain their name from the eponymous multivariate exponential distribution which was proposed in [17] and whose survival copulas are of this form. An extensive monograph on exchangeable Marshall–Olkin copulas is [14].

[22] provides three equivalent characterising conditions for an exchangeable copula candidate function of generalised Marshall–Olkin type to be a proper copula. However, for our purposes, we only require the one presented in the following. For this, let $\mathcal{D}$ be the set of continuous distribution functions on $[0, 1]$ which are positive on $(0, 1]$, i.e.

$$\mathcal{D} := \left\{ F \in \mathcal{C}((0, 1]) : \Delta F \geq 0, \ 0 \not\in F((0, 1]), \ F(1) = 1 \right\}.$$  

**Theorem 2** (Characterisation, see [22, Thm. 3.3.1]). Let $C$ be an exchangeable copula candidate function of Marshall–Olkin type having a representation as in Eq. (9) for functions $g_2, \ldots, g_d$ with $g_i(1) = 1$, $i \geq 2$. Then $C$ is a proper copula if and only if $H_i \in \mathcal{D}$ for all $i \in \{1, \ldots, d\}$, where

$$H_i(u) := \begin{cases} \prod_{j=0}^{i-1} \left( g_{d-i+1+j}(u) \right)^{(-1)^j(\binom{i-1}{j})}, & u \in (0, 1] \\ \lim_{v \downarrow 0} H_i(v), & u = 0. \end{cases}$$

The functions $H_i$ in Eq. (10) have the interpretation that they can be used to define a stochastic representation called exchangeable exogenous shock model, see [22, p. 61 sqq.]. For this, consider a proper copula $C \in \text{exGMO}_d$ with a representation as in Eq. (9) for functions $g_2, \ldots, g_d$. Let $\{Z_I\}_{\emptyset \neq I \subseteq \{1, \ldots, d\}}$ be a family of independent random variables with

$$Z_I \sim H_{|I|}, \quad \emptyset \neq I \subseteq \{1, \ldots, d\}.$$
Then, we have that $U \sim C$, where $U$ is defined by

$$U_i = \max \{ Z_I : i \in I \}, \quad i \in \{1, \ldots, d\}. \quad (11)$$

Note that the prefix exchangeable highlights that the resulting vector $U$ is exchangeable. Furthermore, it can be shown that $U$ being exchangeable can be linked to shock distribution functions $H_I$ depending only on the cardinality of the corresponding set $I$, see [22, Prop. 3.1.2].

The following lemma shows that also every exchangeable exogenous shock model implies an exGMO copula. For this reason $C \sim \text{exGMO}_d$ is sometimes called exchangeable exogenous shock model copula.

**Lemma 4** ([22, p. 61 sq.]). Let $H_I \in \mathcal{D}$, $\emptyset \neq I \subseteq [d]$, such that

$$\prod_{j=1}^{d} H_I^{d-i} (u) = u, \quad \forall u \in [0,1].$$

Furthermore, let $\{Z_I : \emptyset \neq I \subseteq [d]\}$ be a family of independent random variables with $Z_I \sim H_I$. Define $U$ by Eq. (11). Then, the distribution function of $U$ is an exGMO copula with

$$g_i(u) = \prod_{j=1}^{d+1-i} H_I^{d-i} (u), \quad \forall u \in [0,1], \ i \in \{2, \ldots, d\}.$$

### 5 Upper semilinear and exchangeable GMO copulas

The classes of survival exGMO copulas and upper semilinear copulas are two multivariate generalisations of the same bivariate copula family. The former is attained as survival copulas in the generalisation of the bivariate exogenous shock model. In contrast, the latter is a result of a generalisation of the recursive construction principle for the copula function. Hence, both class are obtained by lifting different features of the same bivariate class to higher dimensions. However, to the best of our knowledge, the similarities and differences between these two generalisations have not been investigated in the scientific literature, yet.

A natural problem is the identification of conditions such that a copula is both, survival exGMO and upper semilinear. One must recall that a natural stochastic model for upper semilinear copulas, or at least a subclass thereof, has yet to be proposed. Hence, this problem is related to the problem of finding stochastic representations for upper semilinear copulas. In the remainder of this section, we present such conditions. These, in the anticipation of a main finding, correspond to strong restrictions on the shock model representation as well as a strong restriction on the possible choices of diagonal functions in the recursive construction principle for upper semilinear copulas.
Bivariate semilinear copulas

We start by proving for the bivariate case that the classes of survival exGMO copulas and upper semilinear copulas coincide. In order to do that, we recall results on bivariate semilinear copulas from [4]. In the bivariate case, an upper semilinear copula candidate function has the form

\[
C(u) = \frac{(1 - u_2) \cdot \delta_2(u_1) + (u_2 - u_1) \cdot u_1}{1 - u_1}, \quad u \in [0, 1]^2.
\]

A simple calculation shows that the survival counterpart, \( \hat{C} \), of \( C \) is defined by the following equations.

\[
\hat{C}(u) = V_C([1 - u, 1]) = u_1 \cdot \frac{\delta_2(u_2)}{u_2}, \quad u \in [0, 1]^2. \tag{12a}
\]
\[
\delta_2(u) = 2u - 1 + \delta_2(1 - u), \quad u \in [0, 1]. \tag{12b}
\]

A copula of this form is called (bivariate) lower semilinear copula (abbr. as LSL\(_2\)). We define \( g_2(u) := \frac{\delta_2(u)}{u}, \ u \in [0, 1] \), and we obtain the claim by comparing Eq. (12a) with Eq. (9). The linear segments of bivariate lower and upper semilinear copulas are visualised in Fig. 3a.

\[
\begin{align*}
(0, 1) & \quad (1, 1) \\
(0, 0) & \quad (1, 0)
\end{align*}
\]

(a) lower semilinear

\[
\begin{align*}
(0, 1) & \quad (1, 1) \\
(0, 0) & \quad (1, 0)
\end{align*}
\]

(b) upper semilinear

Figure 3: Visualisation of “linear segments” (dashed lines) of lower and upper semilinear copulas for \( d = 2 \), cf. [4, p. 65].

Characterisation of USL\(_d\) ∩ surv. exGMO\(_d\)

First, note that there are simple examples for copulas which are survival exGMO but not upper semilinear (e.g. the independence copula) as well as for copulas
which are survival exGMO and upper semilinear (e.g. the comonotonicity copula). Another example, which is more involved, is that of a copula which is upper semilinear but not survival exGMO (non-comonotonic with identical multivariate diagonal functions), see Section 6. Hence, we already established that neither the discussed intersection of copula classes is empty nor one of both classes is a subclass of the other.

In the following we present necessary and sufficient conditions for a copula to be both, survival exGMO and upper semilinear.

**Theorem 3.** Let \( d > 2 \) and consider the notation of Theorem 2. A \( d \)-variate exGMO copula \( C \) is the survival copula of an upper semilinear copula \( \hat{C} \), if and only if

\[
H_I \equiv 1_{[0, \infty)}, \quad \forall I : |I| < d - 1. \tag{13}
\]

**Remark.** The condition in Eq. (13) has the following intuitive interpretation. For this, consider the stochastic representation in Eq. (11), i.e. \( Z_I \sim H_I, \varnothing \neq I \subseteq [d] \), are independent random variables and \( U \sim C \) is defined by

\[
U_i := \max \left\{ Z_I : i \in I \right\}, \quad i \in \{1, \ldots, d\}. \tag{11 rev.}
\]

Hence, Eq. (13) becomes equivalent to

\[
Z_I = 0 \text{ a.s.} \quad \forall I : |I| < d - 1
\]

and the shock model simplifies considerably

\[
U_i := \max \left\{ Z_I : i \in I, |I| \in \{d - 1, d\} \right\}, \quad i \in \{1, \ldots, d\}.
\]

This imposes a strong constraint on the original exchangeable exogenous shock model definition. In this regard, these copulas constitute a rather small subclass of all survival exGMO copulas.

**Proof of Theorem 3.** First, we prove by contradiction that the exchangeable exogenous shock model of an exGMO copula whose survival copula is upper semilinear fulfills Eq. (13). Therefore, assume that \( C \) is a \( d \)-variate exGMO copula and that its survival copula \( \hat{C} \) is upper semilinear. Furthermore, assume that there exists \( i < d - 1 \) such that \( H_i \neq 1_{[0, \infty)} \). By exchangeability, we have the existence of sets \( \varnothing \neq I_1, I_2, I_3 \subseteq [d] \) with \( |I_j| = i \),

\[
I_j \cap \{1, 2, 3\} = \{j\}, \quad j \in \{1, 2, 3\}
\]

and \( 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 \leq 1 \) such that

\[
\mathbb{P}(U_1 \neq U_2, U_2 \neq U_3, U_1 \neq U_3) \geq \mathbb{P}\left( \max_{\varnothing \neq I \subseteq \{1, \ldots, d\}} \left( \max_{I \neq I_1, \{1, 2, 3\} \cap I \neq \varnothing} Z_I \right) < Z_{I_1} < Z_{I_2} < Z_{I_3} \right).
\]

14
\[
\geq \left[ \prod_{\emptyset \neq I \subseteq [d], I \neq \{1, 2, 3\}} H_{|I|}(\varepsilon_1) \right] \cdot [H_i(\varepsilon_2) - H_i(\varepsilon_1)] \\
\times [H_i(\varepsilon_3) - H_i(\varepsilon_2)] \cdot [H_i(\varepsilon_4) - H_i(\varepsilon_3)] \\
(\ast) > 0.
\]

Here, \((\ast)\) holds because the assumptions imply the existence of an interval \((a, b] \subseteq [0, 1]\) on which the functions \(H_{|I|}\) are strictly monotone. Consequently, \((\ast)\) holds for arbitrary \(\varepsilon_j \in (a, b]\) with \(\varepsilon_1 < \ldots < \varepsilon_4\), see Fig. 4. This contradicts the result of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{A stylised strictly monotone section of \(H_i\) and a possible choice for \(\varepsilon_1, \ldots, \varepsilon_4\).}
\end{figure}

Lemma 3, i.e. that all components of \(1 - U\) are concentrated on at most two distinct points.

Second, we prove that Eq. \((13)\) is a sufficient qualification such that the survival copula of an exGMO copula is upper semilinear. Therefore, assume that \(C\) is a \(d\)-variate exGMO copula such that Eq. \((13)\) holds. Lemma 4 implies that \(g_i \equiv 1_{(0, \infty)}\) for \(i > 2\) and

\[
C(u) = u_{(1)} \cdot g_2(u_{(2)}), \quad u \in [0, 1]^d, \\
g_2(u) = H_{d-1}(u), \quad u \in [0, 1].
\]

Subsequently, we obtain for \(u \in [0, 1]^d\)

\[
\dot{C}(u) = 1 + d \sum_{k=1}^d (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq d} C_k(1 - u_{(i_1)}, \ldots, 1 - u_{(i_k)}) \\
= 1 - \sum_{i=1}^d (1 - u_{(i)}) \\
+ \sum_{k=2}^d (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq d} (1 - u_{(i)}) \cdot g_2(1 - u_{(i_1)})
\]
= 1 - \sum_{i=1}^{d} (1 - u(i)) + \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} (1 - u(j)) \cdot g_2(1 - u(k)) \sum_{i=0}^{k-1} (-1)^{i+2} \binom{k-1}{i} \left(1 - u(i)\right) \nonumber
\]

Here, we used that \( \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} = 1_{\{k=1\}} \). We conclude that the \( k \)-diagonal of \( \hat{C} \) equals \( \delta_k(u) = u - (k-1)(1-u)(1-g_2(1-u)) \) and that

\[
g_2(1-u) = 1 + \frac{\delta_k(u) - \delta_{k-1}(u)}{1-u}, \quad k \in \{2, \ldots, d\}, \ u \in [0,1].
\]

Finally, we plug this identity into the last equation for \( \hat{C} \) and get with a lengthy but straightforward calculation that

\[
\hat{C}(u) = \frac{\sum_{i=1}^{d} (u(i+1) - u(i)) \cdot \delta_i(u(1))}{1-u(1)}, \quad u \in [0,1]^d.
\]

Here, we use the convention \( u(d+1) \equiv 1 \). This shows that \( \hat{C} \) is an upper semilinear copula. \hfill \Box

**Remark.** Recall that a property of upper semilinear copulas is that all components of a realisation are concentrated on at most two distinct (random) points. Theorem 3 implies that, if we additionally assume that its survival copula is exGMO, at most one component of a realisation may differ from the joint minimum. Furthermore, it follows that for \( d \geq 2 \), the only extendible copula of that subclass is the comonotonicity copula.

We can combine Theorem 3 with Theorem 2 to obtain the following analytical characterisation for an upper semilinear copula with a survival exGMO copula.

**Corollary 3.** Let \( \delta_k, k \in \{2, \ldots, d\} \), be \( k \)-diagonals and let \( C \) be the corresponding upper semilinear copula candidate function defined by Eq. 5. \( C \) is a copula and has an exGMO survival copula if and only if

\[
\delta_k(u) - \delta_{k-1}(u) = \delta_j(u) - \delta_{j-1}(u), \quad \forall k \neq j, \ u \in [0,1],
\]

and the functions \( g_2 \) and \( u/g_2(u)^{d-1} \) are non-decreasing on \([0,1]\), where \( g_2 \) is (for some \( k \geq 2 \)) defined by

\[
g_2(u) = 1 + \frac{\delta_k(1-u) - \delta_{k-1}(1-u)}{u}, \quad u \in [0,1].
\]

The second condition, namely \( g_2 \) and \( u/g_2(u)^{d-1} \) being non-decreasing on \([0,1]\), can be replaced such that we get the following alternative characterisation, cf. [4] Cor. 5.
Corollary 4. Let $\delta_k, k \in \{2, \ldots, d\}$, be $k$-diagonals and let $C$ be the corresponding upper semilinear copula candidate function defined by Eq. (6). Assume that
\[
\delta_k(u) - \delta_{k-1}(u) = \delta_j(u) - \delta_{j-1}(u), \quad \forall k \neq j, \; u \in [0, 1],
\]
and define for some $k \geq 2$
\[
g_2(u) = 1 + \frac{\delta_k(1-u) - \delta_{k-1}(1-u)}{u}, \quad u \in [0, 1].
\]
Furthermore, assume that $g_2$ is absolutely continuous and $g_2(u) > 0$, $\forall u > 0$. Then, $C$ is a copula if and only if
\[
0 \leq u \cdot (d-1) \cdot g_2'(u) \leq g_2(u),
\]
for all $u \in (0, 1)$ where $g_2'(u)$ exists.

Proof. Corollary 3 implies that $C$ is a copula if and only if $g_2(u)$ and $u/(g_2(u))^{d-1}$ are non-decreasing in $u$. The claim follows, as $g_2$ is absolutely continuous and it holds that
\[
g_2(u) \text{ and } \frac{u}{(g_2(u))^{d-1}} \text{ non-decreasing}
\]
\[
\iff g_2'(u) \geq 0 \text{ and } \left(\log \frac{u}{(g_2(u))^{d-1}}\right)' \geq 0 \text{ a.e.}
\]
\[
\iff g_2(u) \geq u \cdot (d-1) \cdot g_2'(u) \geq 0 \text{ a.e.}
\]

Remark. Adapting the notation of Theorem 1, we can show that for $C \in \text{USL}_d \cap \text{surv. exGMO}_d$
\[
\left(\frac{\delta_j(u)}{1-u}\right)' \geq \frac{1 - \zeta_d(u)}{(1-u)^2}, \quad \forall u \in [0, 1]
\]
\[
\iff u \cdot (d-1) \cdot g_2'(u) \leq g_2(u), \quad \forall u \in [0, 1].
\]
Furthermore, we can show that the non-decreasingness of $g_2$ is equivalent to the non-decreasingness of $\nu_d(m)$ or the non-increasingness of $\zeta_d$, respectively.

6 Identical multivariate diagonals

In this section, we explore the special case in which all multivariate diagonals are identical. Particularly, throughout this section we assume that $C$ is an upper semilinear copula (candidate function) with diagonals $\delta_j \equiv \delta, j \geq 2,$ for a $d$-diagonal function $\delta$. 
This assumption implies for a realisation $U \sim C$ and distinct $i, j, k$ that
\[
P\left(U_i \leq u, U_j \leq u, U_k > u\right) = 0, \quad \forall u \in [0, 1].
\]
Hence, at most one component $U_i$, $i \in [d]$, of $U$ may differ from the joint maximum $\max_{i \in [d]} U_i$. Consequently, and because $U$ is exchangeable, we can reduce sampling $U \sim C$ to sampling the first two components of the ordered version of $U$ and a random shuffling. All bivariate random vectors have a stochastic model, which is based on conditional sampling, see, e.g., [16, Algo. 1.2]. We will use this to construct a stochastic model for $U \sim C$.

The subclass in question has already been discussed and characterised in [2]. Therein, the authors establish that the copula (candidate function) simplifies to
\[
C(u) = \frac{(1-u(2)) \cdot \delta(u(1)) + (u(2)-u(1))\cdot u(1)}{1-u(1)}, \quad u \in [0, 1]^d.
\]
Furthermore, they present the following theorem, which is a refinement of their general characterisation theorem, see Theorem 1.

**Theorem 4 (Characterisation, see [2, Cor. 1]).** Let $C$ be an upper semilinear copula candidate function such that $\delta_j \equiv \delta$, $j \in \{2, \ldots, d\}$. $C$ is a copula if and only if the function $\xi$ is non-increasing and the functions $\phi$ and $\nu$ are non-decreasing, where
\[
\begin{align*}
\xi &: [0, 1] \to [0, 1], u \mapsto 1 - du + (d-1)\delta(u), \\
\phi &: [0, 1) \to \mathbb{R}, u \mapsto \frac{1-du+(d-1)\delta(u)}{(1-u)^d}, \\
\nu &: [0, 1) \to \mathbb{R}, u \mapsto \frac{u-\delta(u)}{1-u}.
\end{align*}
\]

The assumption of identical multivariate diagonals allows us to simplify several expressions. We use these simplifications, which are summarised in the following two lemmas, to derive the probability distributions for the conditional sampling of $(U(1), U(2))$.

**Lemma 5.** Let $C$ be an upper semilinear copula with equal diagonals $\delta_j \equiv \delta$, $j \in \{2, \ldots, d\}$. Then, for $u > 0$, we have
\[
\begin{align*}
\nu^{(m)}_d(u) &= \begin{cases} 
\frac{u-\delta(u)}{1-u} & m = d-1, \\
0 & \text{else}
\end{cases} \\
\zeta_d(u) &= 1 - du + (d-1)\delta(u).
\end{align*}
\]

**Proof.** First note that the binomial formula implies for $m \geq 1$ the identity $\sum_{j=0}^{m} (-1)^j \binom{m}{j} = 0$. Hence, we have
\[
\nu^{(m)}_d(u) = \frac{1}{1-u} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \delta_{d-m+j}(u)
\]
We use Corollary 1 and Lemma 5 to show the claim. Particularly, we have
\[
\delta = \frac{(u - \delta(u))/(1 - u)}{0}, \quad m = d - 1,
\]
where we use the assumption that all multivariate diagonals are identical, i.e. \(\delta_{d-m+j} = \delta, d - m + j > 1\). Similarly, we can obtain the second identity with
\[
\delta_d(u) = 1 + \sum_{j=1}^{d} (-1)^j \binom{d}{j} \delta_j(u) = 1 - du + d\delta(u) - \delta(u).
\]

\[\square\]

**Lemma 6.** Let \(C\) be an upper semilinear copula with equal diagonals \(\delta_j \equiv \delta, j \in \{2, \ldots, d\}\). Then, for \(0 \leq u < v \leq 1\) and \(0 \leq u_1 < v_1 \leq u_2 < v_2 \leq 1\), we have
\[
V_C((u,v)^d) = \delta(v) - \delta(u) - d \frac{v-u}{1-u} (u - \delta(u)) \quad \text{and} \quad V_C((u_1,v_1)\times(u_2,v_2)^{d-1}) = (v_2 - u_2) \cdot \left( \frac{v_1 - \delta(v_1)}{1-v_1} - \frac{u_1 - \delta(u_1)}{1-u_1} \right).
\]

**Proof.** We use Corollary 1 and Lemma 5 to show the claim. Particularly, we have
\[
V_C((u,v)^d) = \delta(v) - \left( \frac{1 - v}{1 - u} \cdot \delta(u) + \frac{v - u}{1 - u} \cdot (1 - \xi_d(u)) \right)
= \delta(v) - \left( \frac{1 - v}{1 - u} \cdot \delta(u) + \frac{v - u}{1 - u} \cdot (1 - 1 + du - (d - 1) \cdot \delta(u)) \right)
= \delta(v) - \delta(u) - d \frac{v - u}{1 - u} \left( u - \delta(u) \right)
\]
and
\[
V_C((u_1,v_1)\times(u_2,v_2)^{d-1}) = (v_2 - u_2) \cdot \left( V_d^{(d-1)}(v_1) - V_d^{(d-1)}(u_1) \right)
= (v_2 - u_2) \cdot \left( \frac{v_1 - \delta(v_1)}{1-v_1} - \frac{u_1 - \delta(u_1)}{1-u_1} \right).
\]

\[\square\]

**Conditional sampling approach**

The fundamental idea behind the conditional sampling approach for bivariate random vectors is to perform a separation of the joint probability distribution. In our case, we can write this separation as (for \(0 \leq u < v \leq 1\))

\[
P(U_1 \leq u, U_2 \leq v) = P(U_1 = U_2) \cdot P(U_1 = U_2) \cdot P(U_1 \leq u \mid U_1 = U_2) + P(U_1 \neq U_2) \cdot \int P(U_2 \in dv \mid U_1 = u, U_1 \neq U_2) \cdot P(U_1 \leq u \mid U_1 \neq U_2).
\]

We calculate the involved probabilities and (conditional) probability functions in the following two lemmas.
Lemma 7. Let $C$ be an upper semilinear copula with equal diagonals $\delta_j \equiv \delta$, $j \in \{2, \ldots, d\}$. Furthermore, let $U \sim C$ and define the corresponding order-statistic by $U(1) \leq \ldots \leq U(d)$. Then, for $u, v \in [0, 1]$ with $u < v$, we have

$$\mathbb{P}\left(U(1) \leq u, U(1) = U(2)\right) = \delta(u) - d \int_0^u \frac{x - \delta(x)}{1-x} \, dx,$$

$$\mathbb{P}\left(U(1) \leq u, U(2) \leq v, U(1) \neq U(2)\right) = d \left( \frac{v-u}{1-u} (u-\delta(u)) + \int_0^u \frac{x - \delta(x)}{1-x} \, dx \right).$$

Proof. We prove the two identities with results from measure theory. In particular, we use additivity and $\sigma$-continuity of the corresponding probability measure. For both identities, let $u > 0$ and let $\{Z_n\}_{n \in \mathbb{N}}$ be a refining partition of $[0, u]$ defined by

$$Z_n := \{0 = x_{1,n} < x_{2,n} < \ldots < x_{n-1,n} < x_{n,n} = u\}$$

such that $\text{Mesh}(Z_n) := \max_{k \leq n}|x_{k,n} - x_{k-1,n}| \to 0$ for $n \to \infty$.

For the first identity, consider that

$$\bigcup_{k=1}^n (x_{k-1,n}, x_{k,n}] \downarrow \left\{ x \in [0,u]^d : x_1 = \ldots = x_{d-1} \right\}.$$

Then, we use additivity and $\sigma$-continuity as well as exchangeability and the identities from Lemma 6 to establish

$$\mathbb{P}\left(U(1) \leq u, U(1) = U(2)\right) = \lim_{n \to \infty} \sum_{k=1}^n \delta(x_{k,n}) - \delta(x_{k-1,n}) - d \frac{x_{k,n} - x_{k-1,n}}{1-x_{k-1,n}} \cdot (x_{k-1,n} - \delta(x_{k-1,n}))$$

$$= \delta(u) - d \int_0^u \frac{x - \delta(x)}{1-x} \, dx.$$

For the second identity, consider that it holds for all $v \geq u$ that

$$\bigcup_{k=1}^n \bigcup_{i=1}^d (x_{k,n}, v^{d-i}] \times (x_{k-1,n}, x_{k,n}] \times (x_{k,n}, v]^{d-i} \uparrow \left\{ x \in [0,1]^d : \exists i \text{ s.t. } x_i \leq u, x_j \in (x_i, v] \ \forall j \neq i \right\}.$$

Hence, we use the same techniques as in the previous identity and integration-by-parts to show

$$\mathbb{P}\left(U(1) \leq u, U(2) \leq v, U(1) \neq U(2)\right) = d \lim_{n \to \infty} \sum_{k=1}^n \left( v - x_{k,n} \right) \left( \frac{x_{k,n} - \delta(x_{k,n})}{1-x_{k,n}} - \frac{x_{k-1,n} - \delta(x_{k-1,n})}{1-x_{k-1,n}} \right)$$

$$= d \int_0^u (v-x) \, d \left( \frac{x - \delta(x)}{1-x} \right)$$

$$= d \left( \frac{v-u}{1-u} (u-\delta(u)) + \int_0^u \frac{x - \delta(x)}{1-x} \, dx \right).$$
Note that we require \( v(u) = (u - \delta(u))/(1 - u) \) being of bounded variation, which is fulfilled if \( v \) is non-decreasing, such that this limit can be interpreted as a Riemann–Stieltjes Integral.

\[ \square \]

**Lemma 8.** Let \( C \) be an upper semilinear copula with equal diagonals \( \delta_j \equiv \delta, \ j \in \{2, \ldots, d\} \). Define \( p \equiv d\int_0^1 (x - \delta(x))/(1 - x) \, dx \). Furthermore, let \( U \sim C \) and define the corresponding order-statistic by \( U_{(1)} \leq \ldots \leq U_{(d)} \). Then, for \( u, v \in [0, 1] \) with \( u < v \), we have

\[
\mathbb{P}(U_{(1)} \neq U_{(2)}) = p = d \int_0^1 \frac{x - \delta(x)}{1 - x} \, dx.
\]

If \( p \neq 1 \), we have

\[
\mathbb{P}(U_{(1)} \leq u \mid U_{(1)} = U_{(2)}) = \frac{1}{1 - p} \left( \delta(u) - d \int_0^u \frac{x - \delta(x)}{1 - x} \, dx \right).
\]

Furthermore, if \( C \) is not equal to the comonotonicity copula, i.e. \( p \neq 0 \), we furthermore have

\[
\mathbb{P}(U_{(1)} \leq u, U_{(2)} \leq v \mid U_{(1)} \neq U_{(2)}) = \frac{d}{p} \left( \frac{v - u}{1 - u} \left( u - \delta(u) \right) + \int_0^u \frac{x - \delta(x)}{1 - x} \, dx \right),
\]

\[
\mathbb{P}(U_{(2)} \leq v \mid U_{(1)} = u, U_{(1)} \neq U_{(2)}) = \frac{v - u}{1 - u}.
\]

**Proof.** The first three identities follow directly from Lemma 7, where we use the fact that \( p \neq 1 \) in the second equation and that \( p \neq 0 \) in the third equation.

For the remaining identities, consider the following argument. Let \( U \) be defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( \Omega' := \{ \omega \in \Omega : U_{(1)}(\omega) \neq U_{(2)}(\omega) \} \), \( \mathbb{P}_{\Omega'} := \mathbb{P} \cap \mathbb{P}(\Omega') \), and consider the probability space \((\Omega, \mathcal{F}, \mathbb{P}_{\Omega'})\). Then, it suffices to show that

\[
\mathbb{P}_{\Omega'}(U_{(2)} \leq v \mid U_{(1)} = u) = \frac{v - u}{1 - u}, \quad \forall u, v \in [0, 1], \quad u < v.
\]

From the third identity of this lemma, we conclude that the push-forward measure of \( U \) with respect to \( \mathbb{P}_{\Omega'} \) is absolutely continuous with respect to the Lebesgue measure with density

\[
f_{U_{(1)}, U_{(2)}}(u, v) = \frac{d}{p} \left( 1 - \delta'(u) \right) (1 - u) + \left( u - \delta(u) \right) \left( 1 - \delta'(u) \right), \quad 1_{(u, 1)}(v), \quad u, v \in [0, 1].
\]

Here, we use that \( \delta \) is Lipschitz continuous and therefore, according to Rademacher’s theorem, \( \delta \) is differentiable with derivative \( \delta' \) almost everywhere (with respect to the Lebesgue measure). Finally, we conclude that the desired conditional density and conditional distribution function can be written for \( u, v \in [0, 1] \) as

\[
f_{U_{(2)} | U_{(1)} = u}(v) = \frac{f_{U_{(1)}, U_{(2)}}(u, v)}{f_0^1 f_{U_{(1)}, U_{(2)}}(u, v) \, dv} = \frac{1}{1 - u} 1_{(u, 1)}(v),
\]

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This establishes the claim. \(\square\)

Note that the (conditional) distribution function of \(U(2)|U(1) \neq U(2), U(1) = u\) corresponds to the uniform distribution on \([u, 1]\).

In conclusion, we obtain the desired stochastic model for arbitrary upper semilinear copulas \(C\) with equal multivariate diagonals, see Algorithm 1.

**Algorithm 1** Sampling algorithm for an upper semilinear copula \(C\) with equal multivariate diagonals \(\delta_j \equiv \delta, j \in \{2, \ldots, d\}\).

**input** An admissible \(d\)-diagonal \(\delta\).

**output** A sample from the upper semilinear copula with diagonal \(\delta\).

**function** \(\text{USLC}(\delta)\)

Draw \(I \sim \text{Bernoulli}(p)\) with \(p = d \int_0^1 \frac{x - \delta(x)}{1-x} \, dx\).

if \(I = 0\) then
  \(\text{Draw } U_\wedge \sim \text{B}(\frac{\delta(u)}{d - \int_0^u \frac{x - \delta(x)}{1-x} \, dx} / \frac{1}{1-u})\).
  \(\text{Set } U_1 = \ldots = U_d = U_\wedge\).
else
  \(\text{Draw } U_\wedge \sim d \left( (u - \delta(u)) + \int_0^u \frac{x - \delta(x)}{1-x} \, dx \right) / p\).
  \(\text{Draw } U_\lor \sim \text{Unif}(U_\wedge, 1)\).
  \(\text{Draw } K \text{ uniform from the set } \{1, \ldots, d\}\).
  \(\text{Set } U_K = U_\wedge \text{ and } U_j = U_\lor, j \neq K\).
end if

\(\text{return } U = (U_1, \ldots, U_d)'\).

end function

Besides the possibility of sampling from those copulas, we can use these results to simplify the conditions from Theorem 4.

**Corollary 5.** Let \(C\) be an upper semilinear copula candidate function with equal diagonals \(\delta_j \equiv \delta, j \in \{2, \ldots, d\}\). Then, \(C\) is a copula if and only if \(\nu\) is non-decreasing and bounded by \(\delta'(x)/d\) almost everywhere (with respect to the Lebesgue measure), where

\[
\nu: [0, 1) \rightarrow [0, \infty), x \mapsto \frac{x - \delta(x)}{1-x}.
\]

**Proof.** Note that

\[
\left( \delta(u) - d \int_0^u \frac{x - \delta(x)}{1-x} \, dx \right)' = \delta'(u) - d \frac{u - \delta(u)}{1-u} \geq 0 \text{ a.e.}
\]

is equivalent to

\[
\frac{u - \delta(u)}{1-u} \leq \frac{1}{d} \delta'(u) \text{ a.e.}
\]
For the forward direction, assume that \(C\) is a copula. The non-decreasingness of \(\nu\) follows with Theorem 4. With Lemma 7, we conclude that \(\delta(u) - d \int_0^u \nu(x) \, dx\) is non-decreasing as well, which implies \(\nu(u) \leq \delta'(u)/d\) a.e.

For the reverse direction, assume that \(\nu\) is non-decreasing and that \(\nu(u) \leq \delta'(u)/d\) holds a.e. Combined with the diagonal properties, this implies that \(u \mapsto \delta(u) - d \int_0^u \nu(x) \, dx\) is non-decreasing as well as bounded from below by zero and from above by \(\delta(u)\). Particularly, \(p\) (as defined in the previous lemmas) fulfills \(0 \leq p \leq 1\). With integration-by-parts, we obtain

\[
(u - \delta(u)) + \int_0^u \frac{x - \delta(x)}{1 - x} \, dx = \int_0^u \left( \frac{x - \delta(x)}{1 - x} \right)'(1 - x) \, dx.
\]

The non-decreasingness of \(\nu\) implies that this function is also non-decreasing. Furthermore, this implies that it is bounded from below by zero and from above by \(p\). Moreover, due to the continuity of the Riemann-integral as well as the continuity of the diagonals \(\delta\), both functions are continuous. In summary, we get that the functions

\[
u(u) = \int_0^u \frac{x - \delta(x)}{1 - x} \, dx
\]

are continuous distribution functions on \([0, 1]\) and \(p = d \int_0^1 \nu(x) \, dx\) is a probability, i.e. \(p \in [0, 1]\). Hence, we can apply Algorithm 1. Therefore, let \(U\) be a realisation of Algorithm 1 and let \(u \in [0, 1]\). As the corresponding copula is exchangeable, we can restrict our proof w.l.o.g. to the special case \(u_1 \leq \ldots \leq u_d\). We have

\[
P(U_1 \leq u_1, \ldots, U_d \leq u_d)
= P(U_1 = U_2) \cdot P(U_1 \leq u_1 | U_1 = U_2)
+ \sum_{k=2}^d P(K = k | U_1 \neq U_2) \cdot P(U_1 \leq u_1, U_k \leq u_k | U_1 \neq U_2, K = k)
= \frac{1}{1 - p} \int_0^u \frac{x - \delta(x)}{1 - x} \, dx
+ \frac{1}{d} \int_0^u \frac{x - \delta(x)}{1 - x} \, dx
+ \sum_{k=2}^d \frac{1}{d} \int_0^u \frac{x - \delta(x)}{1 - x} \, dx
\]

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\[ \begin{align*}
\delta(u_1) &= \frac{u_2 - u_1}{1 - u_1}(u_1 - \delta(u_1)) \\
&= \frac{\delta(u_1) - u_1 \delta(u_1) + (u_2 - u_1)u_1 - \delta(u_1)u_2 + \delta(u_1)u_1}{1 - u_1} \\
&= \frac{(1 - u_2) \cdot \delta(u_1) + (u_2 - u_1) \cdot u_1}{1 - u_1} = C(u).
\end{align*} \]

Examples

We conclude this section with two examples. First, in the light of Corollary 5, we call a \( d \)-diagonal which fulfills that \( \nu \) is non-decreasing and almost everywhere bounded by \( \delta'(u)/d \) \( C_d \)-admissible. It can easily be seen, that the class of \( C_d \)-admissible diagonals is closed under point-wise limits and convex combinations.

Example 1. In [2], it was shown that for fixed \( d \geq 2 \) the lowest \( C_d \)-admissible \( d \)-diagonal is

\[ \delta(u) = \frac{(1 - u)^d + du - 1}{d - 1}, \quad u \in [0, 1]. \]

Then

\[ \delta'(u) = \frac{d}{d - 1}(- (1 - u)^d + 1) \]

and simple calculations show that \( p = 1 \),

\[ \nu(u) = \frac{u - \delta(u)}{1 - u} = \frac{1}{d - 1} - \frac{(1 - u)^d - 1}{d - 1}, \]

\[ F_{U \wedge U \wedge \neq U \vee U \wedge \neq U}(u) = 1 - (1 - u)^d, \quad \text{and} \]

\[ F_{U \wedge U \wedge \neq U \vee U \wedge \neq U}(v) = \frac{1 - v}{1 - u}. \]

Note that in this case as \( \delta \) is chosen minimal, we have \( \delta'/d = \nu \).

Example 2. We can consider convex combinations between the lowest \( C_d \)-admissible diagonal and the comonotonicity diagonal (which is the highest \( C_d \)-admissible diagonal), i.e.

\[ \delta(u) = \lambda \frac{(1 - u)^d + du - 1}{d - 1} + (1 - \lambda)u, \quad u \in [0, 1]. \]

Again, simple calculations show that \( p = \lambda \),

\[ F_{U \wedge U \wedge \neq U \vee U}(u) = u, \]

\[ F_{U \wedge U \wedge \neq U \vee U}(u) = 1 - (1 - u)^d, \quad \text{and} \]

\[ F_{U \wedge U \wedge \neq U \vee U \wedge U}(v) = \frac{1 - v}{1 - u}. \]
7 An extendible subclass

In this section, we present a subclass of extendible upper semilinear copulas. Recall that an upper semilinear copula is extendible (in the class of upper semilinear copulas) if an exchangeable sequence \( \{U_i\}_{i \in \mathbb{N}} \) exists such that each finite margin is upper semilinear and \((U_1, \ldots, U_d)' \sim C\).

Extendible subclasses are interesting, because they admit a so-called de Finetti representation, see [1, Chp. 2 and 3]. That means, a realisation \( U \sim C \) can be represented as

\[
U = (F^{-1}(U_1), \ldots, F^{-1}(U_d))'
\]

for a random distribution function \( F \) with generalised inverse \( F^{-1} \) and iid uniform random variables \( U_i, i \in [d], \) independent of \( F \). Models of de Finetti kind are interesting, as they can easily be generalised in an efficient way to higher dimensions by considering an iid uniform family \( \{U_i\}_{i \in \mathbb{N}} \) instead of \( U_i, i \in [d] \).

We construct the extendible subclass directly by its de Finetti representation. For this representation, the idea is to distribute two independent uniform random variables to a vector by iid Bernoulli experiments.

**Theorem 5.** Let \( q \in [1/2, 1] \) and consider the following stochastic model: Let \( V \sim \Pi_2 \) and \( \{J_i\}_{i \in \mathbb{N}} \) an iid family of Bernoulli distributed random variables with success parameter \( q \). Define

\[
U_i := J_iV_1 + (1 - J_i)V_2 = \begin{cases} V_1, & \text{if } J_i = 1, \\ V_2, & \text{if } J_i = 0, \end{cases} \quad i \in \mathbb{N}.
\]

Then for each \( d \geq 2 \), the random vector \( U = (U_1, \ldots, U_d)' \) has the distribution function

\[
C_d(u) = u(1) \left( q^d + (1 - q)^d + \sum_{j=2}^{d} u_{(j)} \left[ q(1 - q)^{j-1} + (1 - q)q^{j-1} \right] \right), \quad u \in [0, 1]^d.
\]

Furthermore, \( C_d \) is an upper semilinear copula as a distribution function and the corresponding diagonal functions are

\[
\delta_k(u) = u \left( q^k + (1 - q)^k + \sum_{j=2}^{k} u_{(j)} \left[ q(1 - q)^{j-1} + (1 - q)q^{j-1} \right] \right), \quad u \in [0, 1].
\]

**Proof.** First, we calculate the distribution function of \( U \). Therefore, let \( d \geq 2 \) and \( u \in [0, 1]^d \) and assume w.l.o.g. that \( u_1 \leq \ldots \leq u_d \). Then, with the convention \( \min \emptyset = 1 \), we use the tower property to conclude

\[
C_d(u) = \sum_{\emptyset \subseteq L \subseteq \{1, \ldots, d\}} \Pi_2 \left( \min_{i \in L} u_i, \min_{i \notin L} u_i \right) \cdot q^{\left| L \right|} (1 - q)^{d - \left| L \right|}.
\]

**Footnote:** For a distribution function \( F \), we define the generalised inverse \( F^{-1} \) by \( F^{-1}(y) = \inf \{ x \in \mathbb{R} : F(x) \geq y \} \), where we use the convention \( \inf \emptyset = \sup \{ \text{ran} \} \), see [8].
Bernoulli distributed random variables with random success parameter \( p \). Define \( u \) as a stochastic model. Let \( p \) be a distribution function on \( \mathbb{R} \).

Corollary 6. Second, we prove that \( C \) is upper semilinear. Therefore, let \( v \in [0,1]^k \) with \( v_{(k)} = 1 \) and \( \lambda \in (0,1) \) and assume w.l.o.g. that \( v_1 \leq \ldots \leq v_k \). Then

\[
C_k (\lambda v_1 \mathbf{1} + (1-\lambda) v) = v_1 \left( q^k + (1-q)^k + \sum_{j=2}^k [\lambda v_1 + (1-\lambda) v_j] [q(1-q)^{j-1} + (1-q)^{j-1}] \right)
\]

Furthermore, this implies that the corresponding \( k \)-diagonal is

\[
\delta_k(u) = u \left( q^k + (1-q)^k + u \sum_{j=2}^k [q(1-q)^{j-1} + (1-q)^{j-1}] \right).
\]

The class of upper semilinear copulas is closed under convex combinations and point-wise limits. This implies for this example, that we can randomise the success parameter \( q \) in the previous model and stay in the class of upper semilinear copulas.

Corollary 6. Let \( F \) be a distribution function on \([1/2, 1]\) and consider the following stochastic model. Let \( Q \sim F \), \( V \sim V_2 \), and \( \{J_i\}_{i \in \mathbb{N}} \) a conditionally iid family of Bernoulli distributed random variables with random success parameter \( Q \). Define

\[
U_i := J_i V_1 + (1-J_i) V_2, \quad i \in \mathbb{N}.
\]
Then for each \( d \geq 2 \), the random vector \( \mathbf{U} = (U_1, \ldots, U_d)' \) has the distribution function (for \( u \in [0, 1]^d \))

\[
C_d(u) = u(1) \left( \mathbb{E}[Q^d] + \mathbb{E}[(1 - Q)^d] + \sum_{j=2}^{d} u(j)\mathbb{E}[Q(1 - Q)^{j-1}] + \mathbb{E}[(1 - Q)Q^{j-1}] \right).
\]

Furthermore, \( C_d \) is an upper semilinear copula and the corresponding diagonal functions are (for \( u \in [0, 1] \))

\[
\delta_k(u) = u \left( \mathbb{E}[Q^d] + \mathbb{E}[(1 - Q)^d] + u \sum_{j=2}^{d} \mathbb{E}[Q(1 - Q)^{j-1}] + \mathbb{E}[(1 - Q)Q^{j-1}] \right). \tag{14}
\]

Remark. In the previous sections, we discussed the two subclasses where either the survival copula is \( \text{exGMO} \) or all multivariate diagonals are identical. For the former subclass, at most one component of a realisation may differ from the joint minimum. In contrast, for the latter subclass, at most one component of a realisation may differ from the joint maximum. In a way these subclasses are diametrically opposed to each other in the class of upper semilinear copulas. However, both subclasses are similarly restricted, as their realisations can have at most one component differing from all other ones. Reduced on this aspect, the subclass presented in this section is very different. If \( p \neq 1 \) (resp. \( F \neq 1_{(1, \infty)} \)), all splits of the components have a positive probability.

We conclude this section by exploiting the stochastic representation to derive some of the properties of this class.

Corollary 7. The copula \( C_d \) from Corollary 6 is radially symmetric, has a lower and upper multivariate tail dependence coefficient of \( \mathbb{E}[Q^d + (1 - Q)^d] \), and is positively orthant dependent.

Proof. First, we obtain radial symmetry from the stochastic representation and \( \Pi_2 \) being radially symmetric. Second, we use Eq. (14) to show that \( \lim_{u \to 0} \delta_k(u) / u = \mathbb{E}[Q^d + (1 - Q)^d] \). Hence, the second statement follows with the radial symmetry of \( C_d \). Finally, with the following calculation, we prove that \( C_d \) is positively orthant dependent:

\[
C_d(u) = \sum_{\emptyset \subseteq L \subseteq \{1, \ldots, d\}} \Pi_L \left( \frac{\min u_i, \min u_j}{\min u_i, \min u_j} \right) \mathbb{E}[Q^d(1 - Q)^{d - |L|}]
\geq \Pi_d(u), \quad \forall u \in [0, 1]^d.
\]

8 Conclusion

We provided necessary and sufficient conditions for a copula to be in the intersection of the families of upper semilinear and survival exchangeable generalised
Marshall–Olkin copulas. More precisely, the shocks in the corresponding exchangeable exogenous shock model must be almost surely equal to zero for all index sets missing more than one component. This finding has important consequences. In particular, this implies for $d \geq 3$ that the independence copula is not upper semilinear and that the only copula which is upper semilinear, survival exGMO, and admits a deFinetti representation is the comonotonicity copula.

We derived a sampling algorithm based on conditional sampling for the subclass of upper semilinear copulas with identical multivariate diagonals. For this, a Bernoulli experiment determines whether all components are identical. Then, the minimum $U_\wedge$ and maximum $U_\vee$ are sampled conditionally, based on the outcome of the Bernoulli experiment. In the last step, the position of $U_\wedge$ is drawn with a uniform distribution on the set $\{1, \ldots, d\}$. This corresponds to shuffling the ordered components $(U_\wedge, U_\vee, \ldots, U_\vee)$ with a uniform distribution on the set of permutations.

For both aforementioned subclasses, we utilised the stochastic representations to derive simplified characterisation theorems.

Finally, we presented a novel subclass of upper semilinear copulas which are radially symmetric and admit a deFinetti representation. In particular, these copulas are conjunctive aggregation operations.

For the special case of equal multivariate diagonals, it remains an open question as to whether the sampling algorithm can be generalised for a larger subclass of upper semilinear copulas.

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References


