Number Sense Across Number Domains: An Integrated Mathematics Educational and Cognitive Psychological Perspective

Habilitation Thesis

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March 2018

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1. Introduction

Numerical abilities are indispensable for everyday life as well as for further mathematical learning. Research in cognitive psychology suggests that a key aspect of numerical learning is *number sense*, an intuitive sense for numerical magnitudes (Berch, 2005; Dehaene, 1998; Siegler, 2016). Yet, important issues related to number sense are not fully understood, including its conceptualization, its role in students' ability to transition between numbers of different domains (e.g., from natural numbers to fractions), and its psychological foundations. Accordingly, this work aims to fill gaps in number sense research—by integrating research from mathematics education and cognitive psychology. Specifically, the aim is to better understand the intuitive cognitive processes that occur when people solve natural number problems, and how these processes may affect solving number problems in another number domain, such as rational numbers or irrational numbers. A fundamental assumption of this work is that a better understanding of the cognitive mechanisms involved in numerical problem solving lays the foundation for improving mathematics teaching and learning.

From a mathematics education perspective, student understanding of numbers from different domains is a core goal of instruction from elementary education through the end of secondary education (e.g., Common Core State Standards Initiative, 2010; Kultusministerkonferenz, 2003, 2004). According to current curricula, students are supposed to learn about different types of numbers in a hierarchical manner. Children work with natural numbers from the beginning of schooling and have experience with these numbers even before they enter school. They are subsequently introduced to rational numbers—including natural numbers, negative integers, and fractions—, and then to real numbers more generally—including rational and irrational numbers.

Introducing numbers of different domains in this hierarchical manner is in line with their mathematical structure, because domains higher in the hierarchy include those below. Moreover, there are stringent ways in which one can use the numbers of a certain domain to construct the numbers of the next domain in the hierarchy. From the perspective of cognitive psychology, however, it is not obvious that learning numbers follows the same stringent logic. The transition from one domain to the next does not appear to be a smooth process but can pose great difficulties for learners. Just like learning in general, numerical learning is not a linear process. Rather, numerical learning is a complex process that is influenced by, among other factors, students' previously acquired knowledge, intuition, and the tools and symbols that are being used to represent numbers.

In view of recent advances in research on numerical processing, there is a need for research that merges perspectives from mathematics education and cognitive psychology. While research in mathematics education traditionally focusses more strongly on the inner logic of the content domain—in this case, on the inner logic of numbers—cognitive psychology has a stronger focus on the cognitive mechanisms involved in the processes of learning and problem solving. Although recent research in cognitive psychology has resulted in novel insights into numerical learning, this research has received little attention by mathematics educators. For example, theories in mathematics education have strongly been influenced by the work of Piaget (e.g., Piaget, 1952) with a key assumption being that children are able to learn certain numerical concepts only after having reached a certain level of cognitive maturation. Later work has, however, challenged this view. Research in cognitive psychology has pointed to core numerical abilities that seem to be present very early in life, including abilities to discriminate between numerosities and ratios (Dehaene, 1998; Feigenson, Dehaene, & Spelke, 2004). Cognitive neuroscience has unveiled a neural network in the brain that seems to be responsible for processing numbers, including brain regions that are able to process numerosities and ratios (Dehaene, Piazza, Pinel, & Cohen, 2003). Such new perspectives have become possible through the use of modern research methods, such as response times measures, recording of eye movements, and brain imaging. By using these methods, novel theories of numerical learning have emerged from recent research that challenge prior approaches. For example, there is now an increased focus on early intuition of numbers and quantities, and on an understanding of numerical magnitudes of all real numbers (Siegler, 2016; Siegler, Fazio, Bailey, & Zhou, 2013; Siegler, Thompson, & Schneider, 2011). Accordingly, this work focuses particularly on factors affecting students' understanding of numerical magnitudes and on intuition during their learning and processing of numbers.

Research on number sense and number magnitude processing has mostly focused on natural numbers. The cognitive mechanisms for processing rational number magnitudes—specifically in their representation as fractions—have only recently become a research focus, although student *difficulties* with rational numbers have been studied for decades. One reason for this renewed focus on rational numbers is that students' difficulties with understanding and processing them turn out to be particularly persistent and omnipresent at all grade levels, and even among adults. Moreover, rational numbers have been a research focus because their learning appears to involve core cognitive mechanisms that are relevant for numerical learning and mathematical learning more generally, such as intuition, cognitive biases, and conceptual change (Siegler et al., 2012; Vamvakoussi & Vosniadou, 2004). In view of this renewed research interest on rational numbers, this work has a main focus on the cognitive mechanisms involved in processing rational numbers. In addition, this work extends the numerical range from rational to irrational numbers, and asks whether the processing of irrational numbers can be described in ways similar to the processing of rational numbers.

The overarching questions this work poses are: What is number sense and which theoretical accounts might be fruitful in fostering number sense in children? How well is our cognitive architecture prepared for processing the magnitudes of numbers from different domains? How do intuitive processes affect the processing of numbers from a new domain? In addressing these questions, this work advances previous research in several ways. First, a novel aspect of this research project is its focus on numerical learning across several number domains. More specifically, it addresses the question of whether and how well people can activate magnitude representations of natural numbers, rational numbers, and irrational numbers. While previous studies have often focused on natural and rational numbers, this is the first study to systematically address magnitude processing of irrational numbers. Second, this work takes an interdisciplinary approach, integrating perspectives from mathematics education, cognitive psychology, and neuroscience. Many authors have discussed the challenges of integrating these perspectives, particularly the perspectives of neuroscience and education. They have suggested that while there is no direct link between neuroscience and education, such a link may be made via other disciplines such as cognitive psychology and educational psychology (Bowers, 2016; De Smedt et al., 2010; Grabner & Ansari, 2010); this view is also adopted in this work. Third, the empirical studies recorded here make use of a variety of methods, including measures of accuracy, response times, eye tracking, and verbal reports. Such a mixed methods approach seems appropriate when addressing different levels of numerical processing and may help overcoming limitations that each method has.

This work is structured into six chapters. Chapter 2 elaborates on the overarching theoretical frameworks that are fundamental to the interdisciplinary perspective on numerical learning adopted in this work. Specifically, chapter 2 first elaborates on the related notions of number sense, number magnitudes, and the mental number line, all of which have been used to describe particular ways in which the brain processes numbers. Then, the chapter elaborates on the conceptual change approach and on dual-process theories as two important theoretical frameworks for number processing and learning.

Chapters 3, 4, and 5 are the three major parts of this work. They include reports of empirical studies on the mental processing of natural numbers, rational numbers, and irrational numbers, respectively. Chapter 3 addresses early development of number sense, the effectiveness of intervention studies, and the question of how the use of specific external representations can enhance number sense. It also includes a neuroscience perspective and discusses how mathematics education may benefit from such a perspective. Chapter 4 is the most comprehensive chapter. It focusses on the mechanisms involved in learning of and working with rational numbers, predominantly with fractions. More specifically, the chapter explores the occurrence and the persistence of the natural number bias, a notion that refers to the overextension of natural number knowledge to rational number problems. The chapter explores factors that might affect the occurrence and the strength of this bias, especially the strategies that people use for solving rational number problems. Chapter 5 advances previous research on number sense for natural and rational numbers by expanding the number range from rational to irrational numbers. The chapter asks how well mathematically skilled adults have developed a sense for irrational numbers, and whether processing of irrational number symbols is affected by intuitions about natural numbers.

Chapter 6 provides a summary and a discussion of the main results of the studies recorded in this thesis. This chapter also includes a discussion of the implications of this research for classroom practices, and it suggests how multi-disciplinary perspectives may be applied to further research in numerical processing and learning.

2. Theoretical Frameworks for Number Processing

The studies recorded in this work draw on theoretical perspectives describing how our cognitive system represents numbers, and how the learning and processing of numbers occurs. This chapter introduces these overarching theoretical perspectives. The first section focuses on the interrelated concepts of number sense, number magnitudes, and the mental number line. The second section elaborates on the conceptual change approach that has traditionally been used in the context of science learning and has only more recently been used to describe the learning of numbers as well. The third section introduces dual-process accounts of number processing, which propose that solving number problems involves analytic as well as intuitive processing, which can cause bias.

2.1. Number Sense, Number Magnitudes, and the Mental Number Line

Competency with numbers encompasses an ability to use numbers flexibly and adaptively. The term "number sense" is often used to refer to this flexible and adaptive use of numbers (Berch, 2005; McIntosh, Reys, & Reys, 1992), although different conceptualizations exist (see 3.1, for an elaboration on the concept of number sense). As an example, an efficient strategy for solving the subtraction problem 701 – 698 is "adding up" the three digits from 698 to 701, rather than actually subtracting 698 from 701 (which requires more work). In order to choose the adding up strategy, it is necessary to quickly recognize that 698 and 701 are close in magnitude. Although number sense is not a novel concept (e.g., L. B. Resnick, Lesgold, & Bill, 1990), it has received greater attention in the numerical cognition literature in the past twenty years since the publication of S. Dehaene's (1998) popular book "The Number Sense".

A precondition of number sense is a quick understanding of the exact or approximate numerical magnitude represented by a number symbol. The Triple Code Model by Dehaene and colleagues (e.g., Dehaene, Bossini, & Giraux, 1993; Dehaene et al., 2003) describes this link between number symbols and their magnitude. This model proposes three formats in which numbers can be represented mentally: A symbolic format (e.g., “2”), a verbal format (e.g., “two”), and an analogue magnitude format. Because the latter format is thought to be structurally similar to a number line, it has also been referred to as a “mental number line” (Dehaene et al., 1993; Fias & Fischer, 2005; Fias & Verguts, 2004; Izard & Dehaene, 2008). An understanding of numbers includes the ability to quickly make connections between the three different formats. The link between the symbolic and the analogue magnitude format is particularly relevant to mathematics education because mathematical problems are most often represented in a symbolic format.

Empirical evidence supporting the assumption that (natural) numbers are mentally represented according to their magnitudes (i.e., on a mental number line) comes from the *distance effect*: When people have to decide which of two numbers is greater in magnitude, they are more accurate and faster when the distance between these two numbers is large rather than small. Moyer and Landauer (1967) initially documented the distance effect, and it has since been replicated in a large variety of studies including participants of different ages, ability levels, and including symbolic and non-symbolic representations of numbers (De Smedt, Verschaffel, & Ghesquière, 2009; Holloway & Ansari, 2009).

Another empirical finding supporting the existence of a mental number line is the *spatial numerical association of response code-effect (SNARC-effect)* (Daar & Pratt, 2008; Gevers & Lammertyn, 2005). This effect refers to the finding that people automatically associate relatively large numbers with the right side of external space and relatively small numbers with the left side. This left-right orientation corresponds to the conventional representation of numbers on external number lines in Western countries. As the SNARC-effect was found in problems for which the numerical magnitude was actually irrelevant to the task (e.g., deciding whether a number is odd or even, or finding the midpoint of a line that is composed of number symbols), researchers have concluded that linking number symbols to their magnitudes becomes a highly automated process, at least in educated adults (Hubbard, Piazza, Pinel, & Dehaene, 2005).

While the assumption of a mental number line is well established for natural numbers, much less research has investigated the representations of magnitudes for other numbers such as rational numbers. For these numbers, it is not obvious whether people are at all able to mentally represent their magnitudes on a mental number line. In the example of (positive) fractions, which are composed of two natural numbers (e.g., 5 and 7 in 5/7), it is possible that people would mentally represent the two natural numbers separately (i.e., 5 and 7 as two natural numbers) rather than the integrated magnitude of the fraction (i.e., the numerical value of 5/7). Even if people were able to mentally represent the numerical magnitudes of fractions, the automatized activation of the magnitudes of the natural number components could interfere with the overall fraction magnitudes. For example, such interference could occur when people compare two fractions in which the larger fraction is composed of the smaller natural number components (i.e., 4/5 versus 7/11, where the first fraction is larger in magnitude but has the smaller components). Chapter 4 further explores this potential interference between the magnitudes of natural number components and fraction magnitudes.

The ability to activate mental number magnitudes is relevant for children’s numerical development and potentially even for mathematical development more generally. De Smedt et al. (2009) found that performance in a symbolic number comparison task (a measure of the ability to activate number magnitudes) in pre-school children was predictively related to their mathematical achievement at the end of first grade. A meta-analysis by Schneider et al. (2017) showed that the ability to perform symbolic (natural) number comparison was related to mathematical achievement. There is also empirical evidence that understanding the magnitudes of rational numbers is predictive of later achievement in mathematics and particularly in algebra (Bailey, Siegler, & Geary, 2014; Siegler et al., 2012). In their integrative theory of numerical development, Siegler and colleagues (Siegler & Lortie-Forgues, 2014; Siegler et al., 2011) argue that understanding number magnitude is key to numerical development because magnitude is a common feature of all real numbers. Thus, understanding the concept of the magnitude of numbers within different domains may help learners integrate numerical concepts across domains.

Notwithstanding the importance of an *understanding* of numerical magnitudes across number domains, it is yet an open question whether people are able to quickly activate number magnitudes from number symbols of all number domains. Figure 1 illustrates the link between number symbols and their analogue magnitude representation for numbers from three domains that are considered in this work: natural numbers (Chapter 3), rational numbers (Chapter 4), and irrational numbers (Chapter 5).

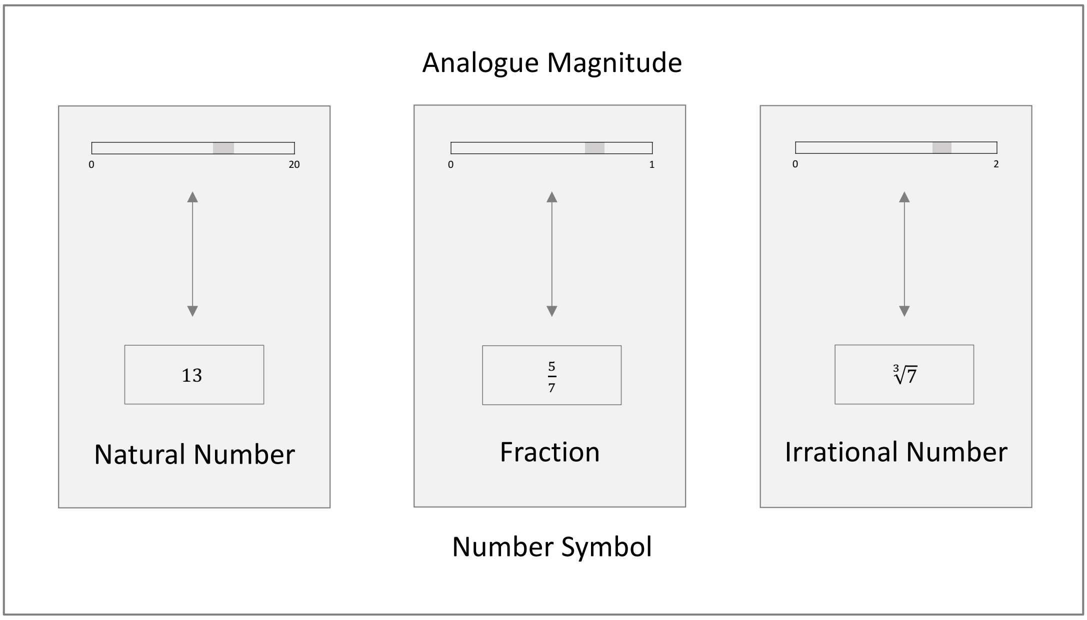

Figure 1. Overview of the proposed link between number symbols and number magnitudes for numbers from the three different domains that are the focus of Chapters 3, 4, and 5, respectively. The figure adopts elements from figures illustrating the Triple Code Model published by Kucian and von Aster (2005) and Schwank (2009). Note that the figure does not include the verbal representation format (as proposed in the Triple Code Model) because this format is not considered in this work.

The two representation formats shown in the above figure, symbolic and analogue magnitude, correspond to two of the three representation formats proposed in the Triple Code Model (Dehaene et al., 2003, see above). As already mentioned, it is well established that people are able to link number symbols to analogue magnitude representations for natural numbers (see left box in Figure 1). For rational numbers and specifically for fractions (middle box in Figure 1), research suggests that although people are in principle able to make this link, it is less automatic, more effortful, and that the magnitudes of natural number components can interfere. For irrational numbers (right box in Figure 1), there is so far no empirical evidence as to whether or not people are at all able to link a number symbol to its approximate magnitude.

Although understanding number magnitudes seems to be important for numerical processing across domains, it appears that the transition from one number domain to the next is not a smooth switch. While all real numbers share the feature of having magnitudes that allow representing them on the same number line, there are also important differences in the numerical concepts across different domains, and these differences can pose obstacles to learning. The next section focuses on this issue, and more generally on the effects that internalized knowledge about natural numbers has for learning about rational numbers.

2.2. Conceptual Change in the Transition Between Number Domains

Learning about rational numbers certainly requires previous knowledge about natural numbers. Such natural number knowledge is, for example, required to understand the meaning of representations used for rational numbers such as fractions and decimals, and also to understand the concept of operations with these numbers. Yet, rational numbers also differ from natural numbers in important ways, so that learners who make the transition from natural to rational numbers have to modify their concept of numbers (Stafylidou & Vosniadou, 2004).

A theoretical framework that has been used to describe this transition is the conceptual change approach. This framework has been established in the literature on the learning of science for a long time, and it has recently been used in studies of mathematical learning as well (Merenluoto & Lehtinen, 2002). Vamvakoussi and Vosniadou (2004) applied the conceptual framework to mathematical learning within the domain of rational numbers. While learning may in some cases refer to extending an existing concept or to acquiring a completely new concept, the conceptual change framework is particularly useful when describing instances in which learning requires the modification of an existing concept.

There are at least four important aspects in which rational numbers differ from natural numbers, and which can be challenging for students (Obersteiner, Reiss, Van Dooren, & Van Hoof, in press; Prediger, 2008; Van Hoof, Verschaffel, & Van Dooren, 2017)[1]. One important difference is the way number symbols represent numerical magnitudes. While natural numbers are represented in the base-ten system, where more digits are always associated with numbers of greater magnitude, the symbolic representation of fractions is not as easily correlated to their magnitude: neither the number of digits of the fraction components nor their individual magnitudes allow immediate inferences about a fraction's total magnitude. Rather, one needs to consider the ratio between two natural numbers, the numerator and the denominator. A second difference between natural numbers and rational numbers is that natural numbers have a unique symbolic representation, using only natural number figures (e.g., the natural symbol "3" is the only one used to represent the natural number "three"). Rational number representations are, in contrast, not unique. In fact, there are infinitely many ways to represent each rational number (e.g., 0.5, 1/2, 2/4, 4/8, etc., represent the same number). A third difference between natural numbers and rational numbers concerns density: within the natural number domain, numbers have unique predecessors (except for 1) and successors, and the number of numbers between two different natural numbers is finite. However, numbers within the rational number domain do not have predecessors or successors, and there are infinitely many numbers between any two numbers. Fourth, and finally, operations differ in the effect they have on numbers from different domains: while within natural numbers, multiplication by a number other than 1 always results in a bigger number, and division by a number other than 1 always results in a smaller number, this rule no longer applies within rational numbers. Moreover, while the multiplication of natural numbers is often explained through repeated addition, such an understanding cannot be applied to the multiplication of fractions in a meaningful way. It is difficult to understand what adding 2/3 times the fraction 3/4 should mean. Accordingly, the transition from natural numbers to rational numbers requires re-conceptualizing arithmetic operations.

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[1] Note that the differences between natural numbers and rational numbers discussed in this section also apply to natural numbers versus real numbers (including rational and irrational numbers) more generally. As there is very little research that has specifically focused on individuals' irrational number concepts, and because rational numbers are the main type of non-natural numbers considered in this work (but see Chapter 5), this section focuses on rational numbers.

There is empirical evidence that each of the four differences between natural and rational numbers mentioned above can cause difficulties for learners when they make the transition from natural to rational numbers (Vamvakoussi, Van Dooren, & Verschaffel, 2012, 2013; Van Hoof, Vamvakoussi, Van Dooren, & Verschaffel, 2017; Van Hoof, Verschaffel, et al., 2017). In addition to these conceptual challenges, intuitive processes that occur during problem solving with rational numbers may pose challenges for learners.

2.3. Dual Process Theories of Number Processing and the Natural Number Bias

While the conceptual change framework serves well in explaining the learning of concepts across different number domains, dual process theories can be used to describe the cognitive processes that occur during numerical problem solving. Dual process theories distinguish between "System 1" processes that are largely automatic and intuitive, and "System 2" processes that are more analytic and time consuming (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009). When people solve rational number problems, their strongly internalized knowledge of natural numbers might trigger intuitive processes, which interfere with the analytic processes that are particularly important when problems require reasoning about novel and less automatized features of rational numbers. The over-reliance on natural number knowledge and intuition, even in problems that require rational number reasoning, has been referred to as the "whole number bias" or "natural number bias" (Ni & Zhou, 2005; Vamvakoussi et al., 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013).

To test the natural number bias, researchers have compared performance on problems that are congruent with natural number reasoning to performance on problems that are incongruent with natural number reasoning. Problems are congruent with natural number reasoning, when this reasoning yields the correct response. Problems are incongruent when natural number reasoning yields an incorrect response. For example, in fraction comparison, a problem is congruent when comparing the denominators and numerators separately yields the correct result (e.g., $3/4 > 2/3$ with $3 > 2$ and $4 > 3$).

However, a problem is incongruent when comparing the denominators and numerators leads to an incorrect result (e.g., $1/2 > 3/7$ although $1 < 3$ and $2 < 7$). In the case of arithmetic operations with fractions, the intuition that multiplication makes numbers bigger may be accurate and lead to a correct response in problems congruent with natural number characteristics (e.g., "Is it possible that $4 \cdot x$ is larger than 4?"; where considering x a natural number will lead to a correct response). However, assuming that multiplication leads to a larger number would yield an incorrect response in problems that are incongruent (e.g., "Is it possible that $4 \cdot x$ is smaller than 4?"). Importantly, numerous studies have documented the natural number bias not only in primary and lower secondary school students but also in upper secondary students and adults (Byrnes & Wasik, 1991; Vamvakoussi et al., 2012; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013; Van Hoof et al., 2015). These findings suggest that the natural number bias in fraction problems can persist even after people have acquired a sound conceptual knowledge of fractions. This implies that solving fraction problems requires—in addition to conceptual understanding of fractions—some inhibition of intuitive knowledge about natural numbers.

Dual process theories can explain the potential interference of natural number fraction components when processing overall fraction magnitudes. Similarly, dual process theories can be used to assess the processing of irrational numbers that are represented by radical expressions (e.g., $\sqrt[3]{41}$) because these also includes two natural number components (e.g., 3 and 41). Accordingly, it is possible to explore whether people are able to process these irrational number magnitudes as a whole, or whether they are prone to processing the components separately. Moreover, as with fractions, one of the natural number components (the radicand, e.g., 41) of an irrational number is positively related to the overall magnitude (i.e., increasing the radicand makes the number bigger), while the other natural number component (the index, e.g., 3) is negatively related to the overall magnitude (i.e., increasing the index makes the number smaller). Therefore, irrational number comparison problems, just like fraction comparison problems, can be classified as being congruent with natural number characteristics (i.e., when the larger number is composed of larger components, as in

$\sqrt[8]{84} > \sqrt[5]{12}$) or incongruent with them (larger number is composed of smaller components, as in $\sqrt[9]{74} < \sqrt[6]{67}$). Accordingly, it is possible to assess whether people show a natural number bias when comparing the magnitudes of irrational numbers (see Chapter 5).

3. Number Sense for Natural Numbers

This chapter focusses on number sense for natural numbers. The four studies described in this chapter consider issues related to number sense that are important from a mathematics education perspective. The first study provides a brief overview of the use of the term number sense, of the methods that have been used to assess it, and of intervention studies aiming at enhancing number sense in children. The second study systematically reviews the theoretical approaches underlying intervention studies that aimed at enhancing number sense, especially in low-achieving children. The third study reports the results of an empirical investigation of number sense among first-grade children. This study explores the use and the potential effectiveness of a specific external representation of numbers that is widely used in first-grade mathematics classrooms in Germany and other countries to enhance children's numerical abilities. The fourth study discusses how neuroscience research, which has received increased attention over the last few years, can contribute to our understanding of teaching and learning of numbers, and how it may therefore be relevant to mathematics education. The chapter ends with a discussion and conclusion section.

3.1. Study 1: A Brief Review of Research on Number Sense

The study summarized in this section is published as: Torbeyns, J., Obersteiner, A., & Verschaffel, L. (2012). Number sense in early and elementary mathematics education. *Yearbook of the Department of Early Childhood Studies. A Journal of Research in Education and Training, 5*, 60–75.

Being competent with numbers includes a feeling for the magnitudes represented by number symbols. A term that describes this feeling is "number sense." The precise meaning of this term is a matter of discussion, and researchers have used this term in many different ways (Berch, 2005; Dehaene, 1998; McIntosh et al., 1992). Most often, number sense is used to describe the ability to work with numbers in a flexible, adaptive, and non-algorithmic way. Some researchers have distinguished between lower-order and higher-order number sense (Berch, 2005). Lower-order number sense refers to very fundamental intuitions about quantities and numerosities, and includes the abilities of comparing, estimating, and counting the numerosities of collections of objects. Higher-order number sense is more complex and includes, in addition to lower-order number sense, a deep understanding of number properties and the relationships between them, and the flexible use of numbers and strategies to solve arithmetic problems.[2]

While lower-order number sense is considered a fundamental precursor to higher-order number sense, it is not yet completely clear how children develop their lower-order number sense. However, many studies have shown, that children possess lower-order number sense already at a very young age. For example, even infants appear to be able to discriminate between two numerosities as long as the ratio between them is large enough (e.g., Xu & Arriaga, 2007; Xu, Spelke, & Goddard, 2005). It is also unclear how exactly children progress from lower-order to higher-order number sense. A commonly shared assumption is that the fundamental mental representations of numerosities required for lower-order number sense are later on linked to other numerical representations, such as verbal and symbolic representations (von Aster, 2000).

Two measures have predominantly been used in the literature to assess number sense: number line estimation and number comparison. In number line estimation tasks, one has to indicate the correct position of a number symbol or of a nonsymbolic set of objects on a number line that has no marks except for its endpoints. The relevant measure is the distance between the participant's chosen position of the number on the number line and its correct position there (e.g., Opfer & Siegler, 2007; Siegler & Booth, 2004). In number comparison tasks, participants have to pick the larger out of two given numbers. As accuracy on these problems is often extremely high, response times are the more relevant measure. In addition, the distance effect across a set of number comparison

[2] Note that while most researchers used the term "number sense" to refer to natural numbers, some researchers referred to number sense for rational numbers as "fractional number sense" (Sowder, 1988, p. 189) or "rational number sense" (Mazzocco & Devlin, 2008, p. 690).

problems is often used as an indicator of number sense. The distance effect is the empirical finding that accuracy increases and response times decrease as the numerical difference between the two numbers to be compared increases. Moyer and Landauer (1967) were the first to document this effect, and numerous studies have since replicated it in various versions (De Smedt et al., 2009; Holloway & Ansari, 2009). The occurrence of a distance effect is evidence that people actually process the numerical magnitudes of numbers as they compare their numerical values.[3] As mentioned above, the specific role of early numerical abilities in a child's further mathematical development is not completely clear yet. However, it has been shown repeatedly that performance in number comparison problems is related to, and predictive of, mathematics achievement. In a meta-analysis, Schneider et al. (2017) concluded that this relationship is stronger for symbolic number comparison than for nonsymbolic number comparison (see Fazio, Bailey, Thompson, & Siegler, 2014), suggesting that the ability to connect number symbols to their magnitudes is particularly relevant for mathematical development.

Number line tasks and number comparison tasks have also been used in intervention studies aiming at enhancing number sense in children (e.g., Obersteiner, Reiss, & Ufer, 2013; Siegler & Ramani, 2009). In these studies, children played games in which the player's task was to solve basic number problems, such as collecting the larger amount of coins, or moving on a linear board to compete in a race. Intervention studies with controlled experimental designs documented the positive effects of these kinds of number comparison tasks on children's ability to solve problems related to number sense. Yet, these positive effects were for the most part limited to children's performance in those tasks they were trained in during the intervention, with little or no transfer effects to other number sense tasks that the children were not trained in, or to broader arithmetic abilities (see 3.2, for a more detailed analysis of intervention studies).

To summarize, number sense is a term that has been used to describe a variety of basic numerical abilities. Regardless of these diverse understandings of the term number

[3] Note that it is not a priori clear that people actually use strategies based on magnitudes to compare numbers. Theoretically, one could apply other strategies such as counting up or retrieving from memory, which are as such not based on magnitudes and would not result in the distance effects described here.

sense, humans seem to possess fundamental numerical abilities that may lay the foundation for further numerical development. Number line estimation and number comparison are the tasks that have typically been used for measuring number sense. From a mathematics education point of view, an interesting question is how very early number skills can be used most effectively to develop further mathematical abilities. Intervention studies have documented relatively low intervention effects, particularly with regard to transfer effects. To better understand these mechanisms, a closer look at the theoretical models underlying intervention studies may be beneficial.

3.2. Study 2: Interventions to Foster Number Sense: A Review of Theoretical Approaches

The study summarized in this section is published as: Obersteiner, A., & Reiss, K. (2017). Interventionsstudien zur Förderung numerischer Basiskompetenzen rechenschwacher Kinder – ein Überblick über theoretische Grundlegungen und Förderansätze [Intervention studies aiming at fostering basic numerical competencies in children with low arithmetic abilities—an overview of theoretical foundations and intervention approaches]. In A. Fritz, S. Schmidt, & G. Ricken (Eds.), *Handbuch Rechenschwäche. Lernwege, Schwierigkeiten und Hilfen bei Dyskalkulie* (3rd ed., pp. 308–322). Weinheim: Beltz.

From a mathematics education point of view, a key question related to numerical learning is the question of how best to support children in developing their number sense. Theoretical psychological models of number sense might serve as important foundations for intervention programs. In fact, a large number of intervention studies have been published, and these studies differ substantially in the theoretical assumptions of how numerical abilities develop, and thus in the content of the interventions and the methods of their implementation. It is therefore worthwhile to provide an overview of the theoretical frameworks underlying intervention studies that aimed at improving number sense. By focusing on the theoretical assumptions of the reported intervention studies, this review aims at obtaining a better understanding of the assumed causal mechanisms of numerical learning.

The focus of this review by Obersteiner and Reiss (2017) is on intervention studies that included children with low numerical abilities. One reason for this focus is that these children are the ones for whom effective intervention programs are mostly needed. Moreover, most intervention studies reported in the literature do in fact focus on children with low numerical abilities. The review includes original reports of well controlled interventions on specific aspects of numerical abilities in low-achieving children. Broad evaluation programs that expanded over longer time periods (typically over several months), as well as case studies on individual children are not included in this review.

A major finding is that almost all studies included in the review referred more or less explicitly to the Triple Code Model by Dehaene (e.g., Dehaene et al., 2003). This is surprising because the Triple Code Model is not a developmental model, and because it is based on research with adults rather than students. On the other hand, the model describes fundamental cognitive modules of number processing that are necessarily included in one way or another in developing an early understanding of numbers. While most studies share their reliance on the Triple Code Model, studies differ substantially in how they operationalized number sense (see section 3.1) and how they implemented the training. Several studies operationalized number sense as the ability of placing number symbols on a number line (Käser et al., 2013; Kucian et al., 2011; Ramani & Siegler, 2008; Siegler & Ramani, 2008, 2009). Accordingly, these studies used video games or board games in which children had to place objects on number lines. Other studies operationalized number sense as the ability to quickly compare the numerical values of numbers. Accordingly, these studies used games in which children had to compare the magnitudes of number symbols or the numerosities of sets of objects (Obersteiner, Reiss, et al., 2013; Räsänen, Salminen, Wilson, Aunio, & Dehaene, 2009; Wilson, Dehaene, Dubois, & Fayol, 2009; Wilson, Revkin, Cohen, Cohen, & Dehaene, 2006).

Few studies focused their intervention on very basic abilities of enumeration (Fischer, Köngeter, & Hartnegg, 2008) or of fact retrieval (Powell, Fuchs, Fuchs, Cirino,

& Fletcher, 2009). Other studies used a more complex intervention approach that relied on a specific theoretical model of numerical development. Examples of these models are the model by Krajewski and Schneider (2009) and the model by Fritz, Ehlert, and Balzer (2013).

Regardless of the underlying theoretical model, all intervention studies reported positive effects on numerical learning. However, these effects were for the most part restricted to the specific facet of number competence in which participants were trained during the intervention. Transfer effects to other numerical abilities or arithmetic achievement more broadly were often not addressed, and if they were, these were absent or very small. These findings suggest that different cognitive processes might be involved in different types of numerical problems. The findings are in line with psychological research that assumes the existence of distinct cognitive systems for processing exact numerical information versus processing approximate numerical information (Feigenson et al., 2004).

In conclusion, the relatively small number of available studies suggests that there is still a need for further research focusing on the effects of targeted interventions as well as on the developmental trajectories of number sense. In particular, there is a need for further evidence on how the different subskills such as exact and approximate processing of numbers interact, and how they contribute to the development of broad arithmetic abilities. A better understanding of these cognitive processes would contribute to theories on numerical development, and it would also help in providing targeted support for children.

3.3. Study 3: First-Graders' Use of the Twenty-Frame as a Tool to Represent Numbers

The study summarized in this section is published as: Obersteiner, A., Reiss, K., Ufer, S., Luwel, K., & Verschaffel, L. (2014). Do first graders make efficient use of external number representations? The case of the twenty-frame. *Cognition and Instruction, 32*, 353–373.

Research in cognitive psychology and neuroscience (see 3.4) suggests that numbers are mentally represented in a way that can be described as a mental number line (e.g., Dehaene et al., 2003; Hubbard et al., 2005). There is also evidence suggesting that external representations that are used during learning can shape learners' mental representations (Chao, Stigler, & Woodward, 2000; Schnotz & Kürschner, 2008). Together, these lines of research may suggest that using number lines to represent numbers is beneficial for children learning about numbers. Mathematics education, however, often suggests the use of a variety of external representations. In early instruction on numbers, many have emphasized the role of structured representations that highlight the idea of grouping, a fundamental principle underlying the ten-based number system (McGuire, Kinzie, & Berch, 2012). Regardless of the specific external representation, there is a lack of empirical evidence for the claim that the use of such material does actually support learning, and, more fundamentally, that children who are using structured material for learning are actually aware of its specific structure.

This study by Obersteiner, Reiss, et al. (2014) focusses on the "twenty-frame," a specific representation of numbers that is widely used in first- and second-grade mathematics classrooms in Germany and many other countries (Krauthausen & Scherer, 2007). The twenty-frame is a grid with two rows of ten circles each. The ten circles of each row are grouped into two groups of five. Circles can be empty or filled, and the total number of filled circles indicates a specific number. The advantage of the five- and ten-grid structure of the twenty-frame is that these grid structures should allow the student to determine the represented number by looking at the structure groupings rather than through one-by-one counting. Instead of counting, one can use subitizing (the immediate recognition of small numbers up to 3 or 4) and conceptual subitizing (the immediate recognition of larger numbers through grouping) (Clements, 1999).

The aim of this study was to investigate whether first-grade children who have used the twenty-frame in their mathematics classroom are able to make use of the structure to determine the represented numbers. While Obersteiner, Reiss, et al. (2013) showed that using twenty-frame representations during a targeted intervention increased arithmetic performance, these authors did not address whether children actually made use of the twenty-frame, or how performance on the twenty-frame was specifically related to arithmetic achievement.

In a computerized experiment, first-graders saw twenty-frames representing the numbers one through 20 on a screen. They were asked to determine the represented number as fast as they could. In a control condition, each child completed the same task, with the only difference that this time the numbers were presented in random order rather than on a twenty-frame. Because only counting strategies should be available in the control condition, we expected that accuracy and response times would decrease and increase, respectively, as the number of dots increased. Comparing these control condition patterns to the twenty-frame condition would allow inferring whether children relied on one-by-one counting in the twenty-frame condition as well. To assess the relationship between performance on the twenty-frame task and arithmetic abilities more broadly, children also took a paper and pencil number and arithmetic test. As control variables, visuo-spatial abilities, working memory, and general reaction capacity were measured.

The results showed the expected patterns of accuracy and response times: In the control condition, there was an almost perfect linear pattern in mean accuracy and response times. The number of dots on the screen predicted 67% of the variance in accuracy and 98% of the variance in response times in the control condition, where the numbers were presented in random arrangements. In contrast, in the twenty-frame task, the number of dots on the twenty-frame predicted only 6% and 24% of the variance in accuracy and response times, respectively. These results strongly suggest that the children used other than counting strategies to determine the number of dots on the twenty-frame task. Furthermore, there was a significant correlation between response times on the twenty-frame task and performance on the number and arithmetic test, even when the control variables were included in the analysis.

The results from this study suggest that children who have used the twenty-frame in the classroom are able to make use of its structure, and that this ability is positively related to arithmetic achievement, above and beyond other cognitive measures. In further research, it would be interesting to explore whether children use a mental image of the twenty-frame when working on number problems, as theories suggest (Chao et al., 2000).

Another question for further research is how effective the use of the twenty-frame is relative to the use of other materials. As mentioned at the beginning of this section, diverse suggestions for using specific representations of numbers have been made by researchers from mathematics education and more recently also by researchers from neuroscience.

3.4. Study 4: A Neuroscience Perspective on Number Processing

The study summarized in this section is published as: Obersteiner, A. (2015). Zahlen im Kopf. Erkenntnisse neurowissenschaftlicher Forschung und Implikationen für das Lehren und Lernen von Zahlen [Numbers in your head. Insights from neuroscience research, and implications for teaching and learning of numbers]. *Mathematik differenziert, 3*, 8–10.

Research on numerical development has typically relied on data from written tests, verbal reports, and response time measures. More recently, the development of advanced brain imaging techniques has enabled researchers to "observe the brain in real time" while it is processing numbers (Kucian & von Aster, 2005). Among these techniques are functional magnetic resonance imaging (fMRI), electroencephalography (EEG), positron-emission-tomography (PET), and near-infrared spectroscopy (NIRS). These methods rely on different physiological mechanisms, so that each method has its unique advantages and limitations. For example, the methods differ in terms of their spatial and temporal resolution. All methods allow identifying brain areas that are particularly involved in specific cognitive activities.

While brain areas are strongly connected and virtually all brain areas are involved in any cognitive activity to some degree, brain imaging studies have identified a neural network that seems to be particularly active when the brain processes numerical information. Key regions within this network include the angular gyrus and the intraparietal sulcus. While the angular gyrus is responsible for representing information in a verbal format, the intraparietal sulcus is the key region for processing magnitudes of numbers. These brain areas align with the modules proposed in the Triple Code Model by Dehaene and colleagues (e.g., Dehaene et al., 2003).

While such findings are interesting in their own right, the question arises about whether these findings have any relevance for mathematics education. Researchers from various disciplines have controversially debated this question (Ansari & Coch, 2006; Bowers, 2016; Grabner & Ansari, 2010; Mason, 2009; Obersteiner et al., 2010; Schumacher, 2007). Some scholars, predominantly from the educational field, have argued that understanding brain mechanisms is either irrelevant for education because of its narrow focus, or that it is in principle relevant but that neuroscience research has not yet revealed new insights into learning mechanisms that are not already known from behavioral, educational or psychological research. Others, predominantly from the neuroscience field, have argued that understanding brain mechanisms can help us understand learning and can therefore guide teaching and learning in the classroom.

In view of this controversial debate, the aim of this paper by Obersteiner (2015) is to communicate in a brief and informative manner, neuroscience research on numerical processing to researchers and teachers in mathematics education, and to discuss the relevance of this research for teaching. Important to note is that the debate should not be about whether neuroscience findings have immediate and direct implications for classroom practice. Rather, the question is whether neuroscience findings are in any way relevant to teaching and learning and should be considered by mathematics educators. It seems that in the case of number processing, a topic that has received the most attention within educational neuroscience research, there are indeed findings that may inform mathematics education. For example, the intraparietal sulcus, an area within the parietal cortex that is known for processing spatial information, has been particularly related to approximate number processing, such as the estimation of the numerosities of sets of objects or of the outcome of arithmetic problems. The intraparietal sulcus also seems to be sensitive to the size of numerical differences between numbers and quantities. Furthermore, the gyrus angularis has been found to be active when number facts are retrieved from memory. These different functions of the intraparietal sulcus and the gyrus angularis allow for a broad distinction between strategies (estimation versus fact retrieval) at the brain level. More generally, the findings of brain imaging studies suggest that the brain is well prepared for approximate arithmetic and estimation. Early mathematics education, however, does not put much emphasis on approximation, but rather on exact calculation and exact number recognition. By emphasizing that the brain is well-prepared for approximate number processing, neuroscience research might stimulate a discussion among mathematics educators about the role of approximate number processing in early mathematics education.

Such a discussion has very concrete implications, for example when it comes to the use of specific external representations of numbers. Early learning of numbers typically focuses on exact representations of numbers. For example, the central idea of using structured material such as the twenty-frame (see 3.3) is that the exact numbers can be identified quickly. Empty number lines, in contrast, emphasize the approximate aspect of number magnitudes because it is virtually impossible to place a number on its ideal position using paper and pencil. In view of neuroscience research, some mathematics educators (Lorenz, 1998, 2010) have already proposed greater use of empty number lines in early mathematics instruction. Nevertheless, more empirical research is needed to evaluate the effectiveness of specific external representations of numbers in early mathematics instruction (Obersteiner, Reiss, et al., 2013).

In sum, neuroscience research has contributed to our better understanding of how the brain processes numbers. In that sense, this kind of research does have the potential to inform mathematics education. However, there are also a number of issues that limit the implications of previous neuroscience studies. For example, for practical and theoretical reasons, brain imaging studies have so far strongly focused on basic processing of natural numbers. Only recently a few studies have also included more sophisticated mathematics using fractions (DeWolf, Chiang, Bassok, Holyoak, & Monti, 2016) and higher-order algebra and calculus (Amalric & Dehaene, 2016). Obviously, neuroscience studies are limited in their external validity, as they do not take into account the classroom context. Considering that "educational neuroscience" (i.e., neuroscience research focusing on educationally relevant issues) is a very young research field, further research could eventually provide more insights and contribute to a more integrated view on learning (Pincham et al., 2014).

3.5. Discussion

This section has focused on number sense as a key aspect of early numerical development. The diverging meanings that have been associated with the term number sense have led to the need for precise definitions when carrying out research into number sense. One challenge in defining number sense is distinguishing it from other related concepts. This holds particularly true for number sense in its very narrow characterization, whether lower-order or higher-order: for lower-order number sense, the challenge lies in distinguishing it from purely perceptual abilities of recognizing or discriminating between visually presented stimuli, such as overall area, dot sizes, or density of object collections. As numerical and perceptual features are often confounded, separating them is also a methodological challenge (e.g., Clayton & Gilmore, 2015). For higher-order number sense, the challenge lies in differentiating number sense from arithmetic abilities in a broader sense. For example, if the term number sense is used to describe abilities including the flexible use of arithmetic strategies, the question arises of whether it is meaningful to distinguish number sense from the broader concept of arithmetic abilities (Verschaffel, Greer, & De Corte, 2007).

Future research should also address the specific link between lower-order number sense on the one hand, and higher-order number sense on the other. Although there are models that describe different levels of competence with numbers (Fritz et al., 2013; Reiss, Roppelt, Haag, Pant, & Köller, 2012), it is yet not well understood how learners can advance from one level of competence to the next (Obersteiner, Moll, Reiss, & Pant, 2015). Longitudinal research that would allow conclusions about individual learners' developmental trajectories is still scarce (but see Jordan, Glutting, & Ramineni, 2010; Reeve, Reynolds, Humberstone, & Butterworth, 2012).

Another challenge for future research concerns the validity of the measures that have been used to assess number sense. Some researchers have pointed out that the number line estimation task, which has often been used to assess mental representations of numerical magnitude, might actually not be a pure measure of magnitudes because one can also use strategies that do not require direct access to number magnitudes (Ebersbach, Luwel, & Verschaffel, 2013; Peeters, Degrande, Ebersbach, Verschaffel, &

Luwel, 2016; Peeters, Verschaffel, & Luwel, 2017). For number comparison tasks, in which the distance effect has often been considered the most interesting measure, factors other than the difference between numbers (such as absolute number sizes or congruency of digits in multi-digit number comparison problems) have been found to influence performance (Nuerk, Weger, & Willmes, 2001).

As discussed in Study 3 (see 3.3), there is very little empirical evidence for the effectiveness of specific external representations that are commonly used in mathematics classrooms. The example of the twenty-frame has shown that the use of specific representations is often grounded in sound theoretical assumptions, but there is often no or very little empirical evidence for the assumption that children use these representations in the expected way, or that using these representations actually enhances children's learning. Thus, empirical studies are needed that investigate the specific effects of the external numerical representations typically used in mathematics classrooms. Moreover, it would be important to know more about other classroom factors, such as the frequency with which certain types of number problems are used, and the extent to which these problems require number sense. Empirical education research suggests that current classroom instruction focuses more strongly on arithmetic procedures, rather than on conceptual understanding. Therefore, it is important for teachers to be aware that children can use arithmetic algorithms that rely on the manipulation of digits, but which do not require any understanding of numerical magnitudes, to solve routine arithmetic problems.

4. Number Sense for Rational Numbers and the Natural Number Bias

After children have acquired knowledge of natural numbers, they learn about a new type of numbers, namely rational numbers. Rational numbers extend the set of natural numbers in the sense that all natural numbers are also rational numbers. This means that children learn about special cases of rational numbers (i.e., natural numbers) prior to learning about rational numbers in general. Accordingly, features of natural numbers might have shaped children's ideas of what numbers are and how operating with numbers works. For that reason, when learning about other domains of rational numbers, children might struggle with those features of these other rational numbers that specifically differ from natural numbers (see 2.2). Empirical research shows that this is actually the case. Many studies have found that learners make systematic errors on rational number problems that require reasoning that differs from natural number reasoning, a phenomenon that has been referred to as the "natural number bias" (Ni & Zhou, 2005).

While students' difficulties with learning about other domains of rational numbers have been studied for decades (e.g., Behr, Wachsmuth, & Post, 1985; Behr, Wachsmuth, Post, & Lesh, 1984; Hart, 1981; Hasemann, 1981), research into the cognitive mechanisms underlying these difficulties has only increased more recently. Some researchers have studied learning about numbers from other rational number domains as an instance of conceptual change (see 1.2). It seems that learning about other rational number domains requires not only the extension but also the reorganization of existing knowledge about (natural) numbers.

Other researchers have focused more firmly on the cognitive processes that occur when people solve rational number problems. Such problem-solving processes include processes that are strongly automatic and intuitive, while other processes are more analytic and time-consuming. The theoretical framework for describing this perspective is the dual-process framework (see 1.3).

Research suggests that not only young children but also older children and adults can show a natural number bias (Van Hoof et al., 2013). Thus, it is possible that the natural number bias is not due to an insufficient conceptual understanding of rational numbers, but is rather deeply rooted in the cognitive processes involved in solving rational number problems. In that case, it might be impossible to completely overcome the bias.

This chapter aims at a better understanding of the nature and the occurrence of the natural number bias in rational number problems. The chapter includes six empirical studies. The first study investigates the persistence of the natural number bias in adults. It includes a sample of expert mathematicians to investigate whether they show a natural number bias in fraction comparison problems. The second study relies on the same data and explores further whether the occurrence of the natural number bias depends on whether people activate magnitudes of fractions. The third study focuses more closely on strategy use in fraction comparison, using eye tracking methodology in a small sample. The fourth study builds on that approach and analyses eye movements in fraction comparison more systematically, including a larger sample. The fifth study addresses the question of whether and how mathematical expertise might enable people to be completely unaffected by a natural number bias on the same problems in which people with less expertise are clearly biased. The sixth study adapts the framework of bias and intuition to investigate a less researched area: probabilistic reasoning in primary school children. The final section of this chapter discusses the main results.

4.1. Study 5: Natural Number Bias in Expert Mathematicians

The study summarized in this section is published as: Obersteiner, A., Van Hoof, J., & Verschaffel, L. (2013). Expert mathematicians' natural number bias in fraction comparison. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 393–400). Kiel, Germany: PME.

Research shows that not only young children but also older children and educated adults, including university students, can experience a natural number bias when solving rational number problems (DeWolf & Vosniadou, 2011; Vamvakoussi et al., 2012; Van Hoof et al., 2013). Accordingly, the natural number bias might not be fully explained by a deficit in conceptual understanding of fractions. Rather, subtle processing mechanisms as described by dual process theories (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009) might contribute to the bias: as natural number knowledge is strongly automatized, System 1 processing might activate knowledge about natural numbers quickly and unintentionally. This processing might initiate a response tendency that results in a correct response in problems congruent with natural number characteristics, and in an incorrect response in incongruent problems. To provide a correct response to incongruent problems, slower System 2 processes are required to override the initial response tendency. A way to evaluate such a dual process account for the natural number bias is to study individuals who have acquired a sound conceptual understanding of fractions. As these people can be expected to solve fraction problems correctly, response times are the relevant measures. If even these highly skilled people show a natural number bias in terms of response times, this would suggest that they have to inhibit intuitive System 1 processes that are based on natural number reasoning when solving rational number problems.

This study by Obersteiner, Van Hoof, and Verschaffel (2013) aimed at investigating whether even individuals with a level of mathematical expertise that is arguably among the highest possible would still show a natural number bias. To that end, the study included a sample of academic mathematicians who were staff members (PhD students, postdoctoral researchers, and professors) in a mathematics department of a university. As in earlier research, we used number comparison problems. We extended previous research by creating a set of comparison problems that concurrently controlled for several features that have not been controlled for simultaneously in previous research. Most importantly, we controlled for numerical distances between fractions by creating problems such that the mean numerical difference between the fraction pairs was the same for each problem type. Moreover, we systematically included all types of problems that can occur in fraction comparison with numbers smaller than one. Systematically varying the magnitudes of the numerators and the denominators resulted in five problem types: in the first two problem types, the fraction pairs had either the same denominators (e.g., 11/19 vs. 15/19) or the same numerators (e.g., 11/17 vs. 11/19). In the other three problem types, the fraction pairs did not have common components. These latter three problem types included fractions in which the larger fraction has both the larger numerator and the larger denominator (e.g., 11/19 vs. 24/25), both the smaller numerator and the smaller denominator (e.g., 25/36 vs. 19/24), or the larger numerator and the smaller denominator (e.g., 17/41 vs. 11/57). The study did not include any well-known fractions (such as 3/4) in order to assess fraction processing in general, rather than processing of very special cases of fractions. The participants solved fraction comparison problems on a computer that recorded accuracy and response times.

The analyses revealed that accuracy was extremely high in problems of all types. However, for problems that had common components, congruency had a significant effect on response times. Response times were longer for incongruent rather than congruent problems, which indicates a natural number bias in terms of response times within the subset of problems that had common components. For the analysis of problems that did not have common components, six items were excluded because they had extremely low response times. The analyses then showed no significant difference between congruent and incongruent problems of these types. In other words, there was no natural number bias for problems without common components.

These results may suggest that the natural number bias in fraction comparison is more likely to occur when the comparison problem can be solved by reasoning about the fraction components rather than their overall magnitudes, as it is the case for problems with common components. In comparison problems without common components, it is not sufficient to rely on direct comparisons of the fraction components only. Rather, these problems often require reasoning about the fraction magnitudes (although other component-based strategies are possible as well). In these problems, no natural number bias was found, which seems to be in conflict with research showing that magnitudes of natural number symbols are activated automatically and regardless of the specific affordances of the problem at hand (Hubbard et al., 2005). Such automatic activation of natural number magnitudes would predict a natural number bias in fraction comparison of any problem type. An explanation could be that comparing fractions without common components is a relatively complex task, so that other factors of the problem-solving process may have affected response times more strongly than the potential inhibition of initial component-based response tendencies. It seems plausible that the participants in this study used a variety of strategies to solve the comparison problems, and that these strategies vary in their complexities (Lortie-Forgues, Tian, & Siegler, 2015). Thus, it is possible that other problem features determined strategy use more strongly and masked the effect of problem congruency.

In sum, this study suggests that the occurrence of the natural number bias might depend on individual problem-solving strategies in addition to problem type. The next study explores this interaction between strategy use and the occurrence of a natural number bias in more detail.

4.2. Study 6: The Interaction Between Natural Number Bias and Magnitude Processing of Fractions

The study summarized in this section is published as: Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction, 28*, 64–72.

The results of Study 5 showed that even people with a very high level of mathematical expertise can show a natural number bias when comparing fractions. However, the occurrence of the natural number bias depended on the type of fraction comparison problem: the bias occurred only when fractions had common numerators or denominators but not when the fractions did not have common components. As discussed in the previous section, problem type might determine whether people process fractions according to their magnitudes or by their components alone, without reasoning about their magnitudes. Some research has focused on whether people process fractions by their components or holistically by their magnitudes. Initial studies have produced conflicting results (Bonato, Fabbri, Umiltà, & Zorzi, 2007; DeWolf, Grounds, Bassok,

& Holyoak, 2014; Meert, Grégoire, & Noël, 2010; Schneider & Siegler, 2010), suggesting that both types of processing are possible, and they might even occur simultaneously. However, the type of processing might also be related to the type of comparison problem, with holistic processing being more likely when comparison problems do not have common components, and component processing being more likely when comparison problems do have common numerators or denominators (because in these latter cases, the non-equal components provide enough information for a reliable decision). While several studies have addressed the question of whether people use component strategies or holistic (magnitude-based) strategies for comparison fractions, the interaction between strategy use and the occurrence of the natural number bias has not been addressed.

Accordingly, this study by Obersteiner, Van Dooren, Van Hoof, and Verschaffel (2013) analyzed in more detail the relationship between the occurrence of the natural number bias and participants' fraction comparison strategies. For addressing this issue, we used the data from our previous study on expert mathematicians (Obersteiner, Van Hoof, et al., 2013). In addition to the occurrence of the natural number bias in the five different problem types, we used multiple regression analyses to assess distance effects. We tested whether the numerical distances between the components (componential distance effect) or the numerical distance between the fraction magnitudes (holistic distance effect) predicted response times.

The results confirmed the assumption that the occurrence of a natural number bias depends on whether or not people rely on fraction magnitudes to solve the fraction comparison problems: for comparison problems with common components, for which we found a natural number bias in the response time data, there was no holistic distance effect. In contrast, for comparison problems without common components, for which we did not find a natural number bias, there was a holistic distance effect. In these problems, the distance between fractions explained 47% of the variance in response times. Although the specific strategies that the participants used were not assessed, the results suggest that the strategies used for solving fraction comparison problems without common components relied much more strongly on fraction magnitudes, whereas the strategies used for solving fraction comparison problems with common components depended far less on fraction magnitudes. It should be mentioned that we did not find component distance effects, not even for problems with common components, for which one would expect to find such an effect if participants relied on component comparisons. However, previous research has shown that the distance effect for natural number comparisons decreases with age and expertise (Sekuler & Mierkiewicz, 1977; Szücs & Goswami, 2007), so that it was presumably too small to be detected in our sample of academic mathematicians.

In conclusion, the results of this study, taken together with those of other studies, suggest that performance on fraction comparison is determined by a complex interplay between problem features (e.g., presence or absence of common components), individual expertise (e.g., low versus high fraction proficiency), and individual strategy use (e.g., components strategy versus holistic strategy) (Alibali & Sidney, 2015). The interaction between these factors might be even more complex as the broad distinction between holistic and component comparison strategies does not account for the large variety of strategies that these categories include. Further research should therefore investigate in more detail the different strategies people use for comparing fractions. Assessing these strategies is a methodological challenge, and eye tracking could be a suitable method.

4.3. Study 7: Eye Tracking as a Method for Assessing Fraction Processing

The study summarized in this section is published as: Obersteiner, A., Moll, G., Beitlich, J. T., Cui, C., Schmidt, M., Khmelivska, T., & Reiss, K. (2014). Expert mathematicians' strategies for comparing the numerical values of fractions – evidence from eye movements. In P. Liljedahl, S. Oesterle, C. Nicol, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education* (Vol. 4, pp. 338–345). Vancouver, Canada: PME.

Response times allow better insights into the strategies people use for mathematical problem solving than accuracy data alone. Yet, response times are still an indirect measure of strategy use. One reason is that different strategies can theoretically require the same amount of time. Another reason is that response time analyses are typically averaged across a set of problems or across a group of participants. A method that may allow more direct insights into participants' strategies is eye tracking (Mock, Huber, Klein, & Moeller, 2016). Several studies have used eye movements as a measure to assess mathematical problem solving, including numerical problems (Schneider et al., 2008; Sullivan, Juhasz, Slattery, & Barth, 2011). Accordingly, eye movements might also be suitable for assessing strategy use in fraction comparison.

As no study had analyzed eye movements during fraction comparison yet, the aim of this study by Obersteiner, Moll, et al. (2014) was to investigate whether eye movements would allow researchers to distinguish component strategies from holistic strategies in fraction comparison, a distinction that has previously been assessed on the basis of accuracy and response times data (Obersteiner, Van Dooren, et al., 2013). We studied adults with a high level of mathematical skill because previous research suggests that these people use comparison strategies efficiently. More precisely, they are more likely to rely on component strategies in comparison problems in which the fractions have common components, and on holistic strategies in problems in which the fractions do not have common components.

The participants in this study were eight adults who had an academic degree in mathematics, or who were university students majoring in mathematics. The participants solved fraction comparison problems that were similar to the ones used in previous studies. Particularly, there were comparison problems with fractions with or without common components. Eye movements were recorded using a remote SMI eye tracker with a sampling rate of 500 Hz.

To analyze strategy use, we defined rectangular areas of interest that surrounded the numerators or the denominators of the fraction pairs. To analyze potential differences in strategy use between problem types, we compared the eye fixation times within these areas of interest between problem types. As expected, eye fixation times were significantly larger for numerators than denominators when the comparison problems had the same denominators. Conversely, eye fixation times were significantly larger for denominators than for numerators when the comparison problems had the same numerators. For comparison problems without common components, there was no significant difference between eye fixation times on numerators and denominators.

The results from this study suggest that the distinction between holistic and component strategies in fraction comparison, which has so far been based on response times and accuracy data, can also be made on the basis of eye movements. However, this study had several limitations. First, the sample size was very small, which might limit the generalizability of the findings. Second, eye fixation times might not be the best possible way of assessing strategy use, because although they allow evaluating which parts of the fractions the participants payed particular attention to, they do not reveal which fraction components the participants compared with one another. For example, when comparing the numerators, one would expect to find two consecutive eye fixations on the two numerators, while when comparing the denominators, one would expect to find two consecutive eye fixations on the two denominators. However, this study did not provide such facile links between eye fixation and strategy use. Hence, the following study aimed at analyzing in more detail the sequences of eye fixations between fraction components.

4.4. Study 8: Fraction Comparison Strategies in Skilled Adults: A Study of Eye Movement Patterns

The study summarized in this section is published as: Obersteiner, A., & Tumpek, C. (2016). Measuring fraction comparison strategies with eye-tracking. *ZDM Mathematics Education, 48*, 255–266.

Results from Study 7 as well as studies by other researchers (Huber, Moeller, & Nuerk, 2014; Ischebeck, Weilharter, & Korner, 2016) suggest that eye tracking could in principle be suitable for assessing strategy use in fraction comparison problems. However, studies differed in the parameters that were employed to infer strategy use. Huber et al. (2014) used the number of eye fixations rather than the fixation times to analyze strategies, while Ischebeck et al. (2016) analyzed the frequencies of specific gaze path patterns in addition to eye fixations. In spite of these differences, both studies concurred in the overall conclusion that participants preferred component strategies when the fraction pairs had common components, while preferring holistic strategies when the fraction pairs did not have common components. However, there were several limitations that did not allow generalizing these findings to all cases of fraction comparison. For example, all fractions in Huber et al.'s (2014) experiment had one-digit components, so that the results might not generalize to more complex fraction comparisons with two-digit components. Moreover, Ischebeck et al. (2016) analyzed the proportions with which a specific sequence of eye movements was used for the different fraction comparison types, relative to the total number of eye movement sequences of that type. Although such an approach allows comparison of eye movement sequence use between problem types, it does not reveal the proportions of sequences within a specific problem type.

This study by Obersteiner and Tumpek (2016) aimed at identifying the different patterns of eye fixation sequences within the different problem types. We included two-digit fraction comparison problems that concurrently controlled for several features that have not been controlled for simultaneously in previous research, in order to obtain more generalizable results.

For each problem type, we analyzed, in addition to response times and eye fixation times, the number of all six types of eye saccades between the different fraction components, relative to the total number of saccades on a specific item. We interpreted saccades between numerators and saccades between denominators as indicators of componential comparison processes, and saccades between the numerator and denominator of each fraction as indicators of holistic comparison processes. Eye saccades between the numerator of one fraction and the denominator of the other fraction were considered to be indicators of diagonal comparisons that might indicate cross-multiplication strategies.

The results showed that, as expected, there were more eye saccades between numerators than between denominators when the fraction pairs had common denominators, and vice versa when fractions had common numerators. Also as expected, holistic eye saccades were most frequent when fractions did not have common components, suggesting increased holistic processing for these problems. Diagonal saccades hardly occurred in any problem type. In sum, the results confirmed previous findings that people prefer component strategies for comparing the numerical values of fractions when these have common components, while they prefer holistic strategies when the fractions do not have common components. However, this study also showed that the differences in gaze patterns were not as clear-cut as we had expected. For example, the relative number of saccades between numerators and denominators of each fraction was relatively high for all fraction comparisons, regardless of type.

With regard to the method, this study shows that eye tracking is in principle suitable for assessing strategy use in fraction comparison. Future studies could use eye tracking to assess strategy use in less skilled individuals, such as elementary students who are just learning about fractions, to see how strategy use develops over the course of systematic instruction on fractions. Moreover, to validate the method, future research could combine measures of eye tracking with verbal reports (Atagi, DeWolf, Stigler, & Johnson, 2016) to see whether it is possible to match specific eye movement patterns to the verbally reported strategies.

4.5. Study 9: The Persistence of the Natural Number Bias

The study summarized in this section is published as: Obersteiner, A., Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2016). Who can escape the natural number bias in rational number tasks? A study involving students and experts. *British Journal of Psychology, 107*, 537–555.

Studies have documented the natural number bias not only in students who are just learning about fractions, but also in older students, educated adults, and—for certain problems—even in expert mathematicians (Obersteiner, Van Dooren, et al., 2013; Vamvakoussi et al., 2013; Van Hoof et al., 2013; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). The results from these studies raise the question of whether the bias is generally unavoidable, or whether it is possible to be completely unaffected by a bias. For fraction comparison problems, the occurrence and strength of a natural number bias seems to depend on whether people process fractions according to their components or according to their holistic magnitudes (Alibali & Sidney, 2015; Obersteiner, Van Dooren, et al., 2013). The reason for the bias could be that focusing on the integer components of fractions triggers automatic activation of their magnitudes, hence encouraging a natural number bias, while processing fraction magnitudes triggers this activation less strongly. Following this argumentation, rational number problems that do not encourage strategies that rely on the activation of natural number magnitudes should be less likely to trigger a bias, or not trigger a bias at all.

As strategy use depends on individual level of mathematical expertise, the occurrence of the bias should differ between individuals with different levels of expertise. This study by Obersteiner, Van Hoof, Verschaffel, and Van Dooren (2016) tested this hypothesis using problems that require reasoning about the effect of arithmetic operations. More specifically, participants had to decide whether or not it is possible that multiplication or division, respectively, can result in a number that is either larger or smaller than the initial number. Previous studies have shown that people tend to show a natural number bias in such problems; that is, they are more reluctant to accept that multiplication can make a number smaller than to accept that multiplication can make a number larger (Siegler & Lortie-Forgues, 2015; Vamvakoussi et al., 2013; Van Hoof et al., 2015). We presented the problems in a way that allowed for at least two different strategies. While one of these strategies relied on reasoning about natural numbers, the other one relied on more abstract algebraic reasoning. Our rationale was that the occurrence of the natural number bias should depend on which of these strategies people used.

The aim of our study was to provide evidence that under certain circumstances, mathematical proficiency can allow people to be completely unaffected by a natural number bias when solving the same problems in which people with less proficiency are clearly biased. To that end, we included a sample of eighth-graders and a sample of expert mathematicians in our study. All participants solved the same problems, which required reasoning about the effects of multiplication and division. The problems were presented in algebraic format and were either congruent (e.g., "Is it possible that $4 \cdot x > 4$?") or incongruent (e.g., "Is it possible that $4 \cdot x < 4$?") with natural number reasoning.

The results confirmed earlier studies in that the eighth-grade students showed a clear natural number bias in terms of accuracy rates. The expert participants, in contrast, did not show any bias, neither in terms of accuracy nor response times. This suggests that the experts did not feel the need to suppress an initial response tendency that was based on natural number reasoning. A possible explanation is that the experts did not engage in reasoning about the effect of arithmetic operations but used other solution strategies instead. As largely confirmed by verbal reports, they often reasoned about the solvability of inequalities within the natural or the rational number set. As any inequality of the given types was solvable within the rational number domain, the correct answer ("yes") was straightforward. This abstract reasoning was not available for the eight-grade students, resulting in biased reasoning.

In conclusion, this study provided the most direct evidence that the way of reasoning influences the likelihood of exhibiting a natural number bias in rational number problems. In view of the results from fraction comparison studies, we conclude that people are more likely to show a bias when the problem representation itself requires reasoning about natural numbers to some extent (as is the case in fraction comparison) than when this is not the case (e.g., when reasoning about the solvability of inequalities). This conclusion demonstrates Alibali and Sydney's (2015) notion that strategy use, individual proficiency and features of the problem at hand might affect the probability of showing a natural number bias in rational number problems.

4.6. Study 10: Natural Number Bias in Primary School Children's Proportional Reasoning

The study summarized in this section is published as: Obersteiner, A., Bernhard, M., & Reiss, K. (2015). Primary school children's strategies in solving contingency table problems: the role of intuition and inhibition. *ZDM Mathematics Education, 47*, 825–836.

Studies on the natural number bias have focused on problems that required reasoning about rational numbers. However, the bias might also occur when people solve problems that require reasoning about multiplicative relations between natural numbers more generally. For example, multiplicative reasoning is required when the relation between two binary variables needs to be evaluated on the basis of frequency data. Contingency tables are one way to represent such data, and such tables are often used in mathematics education classrooms. Contingency table problems that present probabilistic data are particularly interesting to study from the perspective of bias and intuition, because in these problems the correct way of reasoning often requires advanced mathematical calculation, which is typically too difficult to carry out mentally. For this reason, people will have to rely on heuristic strategies based on intuitive reasoning rather than on normative strategies.

The aim of this study by Obersteiner, Bernhard, and Reiss (2015) was to investigate primary school children's reasoning on contingency table problems from the perspective of intuition and bias. Previous research suggests that children have basic abilities to solve contingency table problems, but that they also show typical error patterns (McKenzie, 1994; Reiss, Barchfeld, Lindmeier, Sodian, & Ufer, 2011; Shaklee & Paszek, 1985). We were first interested in the strategies children use to solve these problems. We also wanted to investigate whether error patterns can be explained by two types of systematic bias, the base rate bias and the natural number bias. The base rate bias means that children would base their decision only on the absolute frequencies of the variables of interest, thereby ignoring the relative frequencies of their occurrence. The natural number bias would predict that children focus more on additive relations between cell frequencies rather than on their multiplicative relations. In a second experiment, we used individual interviews to explore whether we would find indications for bias in individual students' verbal responses.

The participants in this study were second- and fourth-graders who had not received classroom instruction on how to solve contingency table problems. They saw 2x2-tables that were introduced as representing frequency data from previous random experiments. The children were asked to decide whether the two variables shown in the table were correlated. For example, the table showed the results of repeated random drawings from two bags containing an unknown number of blue and red chips. Based on the number of red and blue chips represented in the table, the children were asked to decide from which bag it would be better to draw if one wanted to get a blue chip (in other words, whether there was a correlation between bag and color). In one experiment, children solved the problems on paper and pencil, while in a second experiment, they solved them verbally and their responses were recorded. There were three different types of problems that differed in the type of reasoning required for solving them. Problems of the simplest type could be solved by comparing the data from just two of the four cells represented in the table. Additive problems required taking into account the information in all four cells, but additive rather than multiplicative reasoning yielded the correct result. The most challenging type of problems were those that required multiplicative reasoning.

To analyze response patterns in the written test, we compared accuracy on problems for which the three different types of strategies would yield the correct response. Accuracy depended on problem type and increased in the expected order. In line with previous research, the children often neglected the base rates (suggesting a base rate bias), and they relied on additive relations between numbers more often than on multiplicative relations (suggesting a natural number bias). This pattern was found in both second- and fourth-grade children.

The qualitative analyses of children's verbal responses showed that children rarely described multiplicative reasoning. This was true even for those children who had used multiplicative reasoning to solve the problems (as indicated by correct responses to multiplicative problems). Moreover, some children's responses clearly showed that their initial reasoning was not always in line with the final verbal explanations about their problem-solving strategies they gave to the interviewer. It seems that in those children, additive and multiplicative reasoning were competing strategies, and that the type of reasoning they relied on depended on the difficulty of the problem and the experimental situation (solving the problem versus verbally explaining their reasoning).

While the children's behavior was in line with the assumption of natural number bias, the bias these children showed was presumably not solely due to intuitive System 1 processes (as seems to be the case in highly skilled adults, see Study 6), but could to a larger extent be explained by their lack of conceptual knowledge of how to reason multiplicatively. Moreover, further research suggested that children struggled particularly with verbally describing their multiplicative strategies when the multiplicative relation involved a rational rather than a natural number (Obersteiner, Reiss, & Bernhard, 2016). Accordingly, the method of how strategies are assessed seems to be crucial, particularly for children during the learning phase.

4.7. Discussion

The studies reported in this chapter suggest that people are in principle able to activate magnitude representations for fractions. In fraction comparison problems without common components, response times indicate a holistic distance effect, and eye tracking reveals larger proportions of holistic saccades relative to problems with common components. These findings corroborate results from prior research. They also extend prior research in that they allow more generalization, because the fraction pairs used in the studies presented here were more complex and demanding (e.g., two-digit components, no very common fractions) than in most prior studies (Bonato et al., 2007; Huber et al., 2014; Meert et al., 2010; Schneider & Siegler, 2010; Van Hoof et al., 2013). Moreover, the use of eye tracking allowed assessing the problem-solving processes more directly than it is possible with response time measures alone (Mock et al., 2016). The eye tracking data clearly shows that although people are able to activate fractions magnitudes, they do not activate these automatically or unintentionally, as it is the case for natural numbers (Hubbard et al., 2005). Rather, people extract the magnitudes of complex fractions "online," by reasoning about the relations between numerators and denominators. Although well-known fractions were not assessed here, this conclusion is in accordance with recent research suggesting that quick and automatic activation of fraction magnitudes is only possible for some very well-known fractions such as 1/2 or 1/4 (Liu, 2017).

The studies also suggest that whether or not people activate fraction magnitudes depends on the affordances of the specific problem. When two fractions have common components, their numerical values can be compared more easily without reasoning about their magnitude. Reasoning about the magnitudes of the components is sufficient. Evidence from eye tracking research supports this conclusion. Adults switch more often between individual components of the fractions when comparing these components is sufficient to solve the problem. In fraction comparison without common components, they spend more time on switching between the numerator and denominator of each fraction, suggesting holistic strategies that are based on the overall fraction magnitudes.

Notwithstanding people's fundamental ability to process fraction magnitudes, this processing can be affected by a natural number bias. Primary school children's accuracy on contingency problems, some of which require reasoning about multiplicative relationships comparable to fraction comparison, is in line with the assumption of a natural number bias. Even academic mathematicians showed a natural number bias on fraction comparison problems, although they did so only on problems that had common components. Moreover, although the natural number bias seems to be very persistent, there is now empirical evidence that it is possible to completely overcome this bias—by using a way of reasoning that does not rely on natural numbers. In the study involving algebraic inequalities, expert mathematicians presumably relied on algebraic rules, rather than on substituting the unknown variable by a concrete number. This reasoning allowed them to be unaffected by a natural number bias. In contrast, eighth-grade students showed a typical natural number bias, presumably because they were unable to engage in algebraic reasoning.

The research presented in this chapter also suggests that people use a variety of strategies to compare fraction magnitudes. Recent research confirms that adult participants used a variety of strategies depending on the problem type (Fazio, DeWolf, & Siegler, 2016). In addition to the specific relation between numbers, a relevant distinction between problem types seems to be whether their fractions are common or uncommon. Liu (2017) suggests that people often rely on benchmarking, that is, they might reason about fraction magnitudes in terms of how close fractions are to the next common fraction, such as a half or a quarter. However, an open question remains how people can identify the closest benchmark to a given fraction in the first place, as this process already requires reasoning about the (approximate) magnitude of the given fraction. Further research is certainly needed to clarify these processes, particularly in fractions that are less common.

Taken together, the findings discussed so far suggest that there is a complex interaction between magnitude processing, problem type, strategy use, and individual abilities (Alibali & Sidney, 2015). Strategies that rely strongly on natural number components when solving rational number problems are more prone to the natural number bias, unlike strategies that rely less strongly on these components. In fraction comparison, the latter type of strategies includes those that rely on the holistic fraction magnitudes. In algebraic equation problems, these are strategies that rely on algebraic rules. Whether people choose one or the other strategy depends on the problem type, and on their individual ability to adapt the strategy to the problem type.

This work has used eye tracking with the aim of tapping into the cognitive mechanisms of number processing. While this method may in principle help overcome the shortcomings of other methods (e.g., allowing assessment in real time), the challenge of interpreting eye tracking data is an important limitation of this method. Although there are good reasons to interpret specific eye movements (such as fixations and saccades) in particular ways, it is difficult to empirically validate the accuracy of these interpretations (Schneider et al., 2008). For example, as discussed by Obersteiner and Staudinger (in press), it is difficult to distinguish the cognitive processes involved in numerical processing from those related to reading. Further studies could include a reading task as a control condition in an effort to distinguish between reading and numerical processes.

Research on numerical cognition has so far strongly focused on rational numbers, including natural numbers. However, rational numbers are not the only types of numbers children encounter at school. They also learn about real numbers more generally, which include irrational numbers. Most of the questions that have already been asked about the processing of rational numbers in these studies have not yet been empirically investigated when it comes to irrational numbers.

5. Number Sense for Irrational Numbers

Research into the cognitive mechanisms of number processing has almost exclusively focused on rational numbers (including natural numbers and integers). However, learning mathematics in K–12 includes learning about real numbers more generally, which includes learning about irrational numbers. The few studies that investigated students' understanding of real numbers showed that they have great difficulty understanding important conceptual aspects related to real numbers (such as their uncountability) (Bauer, Rolka, & Törner, 2005; Sirotic & Zazkis, 2007; Zazkis, 2005). To date, unlike for rational numbers, no study has empirically and systematically tested the question of whether people are able to mentally represent irrational numbers according to their magnitudes.

Addressing this issue is relevant to numerical cognition as it touches upon the question of whether human cognitive architecture is limited in its capacity to represent certain types of numbers (e.g., rational numbers). Moreover, it is widely accepted that a key aspect of numerical development is an understanding of numerical magnitudes, and some authors have made that argument not only about rational numbers, but also about all real numbers (Siegler et al., 2011; Torbeyns, Schneider, Xin, & Siegler, 2015). If magnitude understanding is indeed important for attaining proficiency with real numbers, then people with high mathematical expertise should be able to activate magnitudes not only for rational number but also for irrational numbers. However, mentally determining the magnitude of an irrational number might be much more difficult than it is for a rational number, since irrational numbers are rarely used in everyday life. Moreover, the algorithm for converting an irrational number that is represented as a radical (e.g., $\sqrt[7]{42}$) into its decimal representation is too complex to be carried out mentally. Therefore, it is possible that our mental capacity for determining number magnitudes is actually limited to rational numbers.

Addressing the question of whether it is possible to mentally represent irrational numbers is also relevant from a mathematics education point of view. If students are unable to assign magnitudes to irrational number symbols, this may well contribute to more general difficulties students have with understanding the concept of irrational numbers. For example, students might even be reluctant to accept irrational numbers as numbers.

The empirical study described in the next section of this chapter addresses the question of whether people are able to mentally activate the magnitudes of irrational numbers. It also investigates whether a natural number bias occurs in irrational number comparisons, similarly to how it does in fraction comparison. The second section in this chapter provides a summary and recommendations for further research.

5.1. Study 11: Number Sense of Irrational Numbers in Skilled Adults

The study summarized in this section is published as: Obersteiner, A., & Hofreiter, V. (2017). Do we have a sense for irrational numbers? *Journal of Numerical Cognition, 2*, 170–189.

Research has amply shown that people are in principle able to mentally activate the magnitudes of rational numbers (Desmet, Grégoire, & Mussolin, 2010; Van Hoof et al., 2013), including natural numbers (De Smedt et al., 2009) and integers (Varma & Schwartz, 2011). Notwithstanding people's fundamental ability for activating number magnitudes, this mental process appears to be more demanding and less automatic for larger and less common numbers (e.g., 31/73) than for smaller and frequently used numbers (e.g., 1/4). Do date, it is unclear whether people are also able to activate the magnitudes of irrational numbers represented by a radical expression (e.g., $\sqrt[7]{42}$). Addressing this question contributes to numerical cognition theories because it can clarify whether there is a limit in the human capacity for assessing magnitude information from number symbols. From a mathematics education perspective, addressing this question can provide additional explanations as to why many students struggle with understanding the concept of irrational numbers. If activating irrational number magnitudes is extremely difficult, students might consider irrational number symbols as more abstract objects, rather than as numbers.

In this study by Obersteiner and Hofreiter (2017) mathematically skilled adults were asked to solve number comparison problems with irrational numbers that were represented as radical expressions.[4] To parallel previous research on fraction comparison (e.g., Study 5 and 6), we used five problem types: problems with common components had either the same indices or the same radicands, and were thus respectively congruent or incongruent with whole number comparison strategies. Problems without common components included problems that were congruent (i.e., the larger number was composed of the larger components, incongruent (i.e., the larger number was composed of the smaller components) or neutral (i.e., the larger number was composed of the larger radicand and the smaller index) with respect to natural number reasoning.

The results of this experiment showed that the participants were highly accurate and relatively fast (compared to evidence from studies on fraction comparison) in all problems for which component strategies yielded the correct result. They also performed very high on incongruent problems without common components, for which component strategies were applicable after estimating their magnitudes. However, accuracy did not differ significantly from chance level (50%) on congruent problems without common components, which are problems for which component strategies were not applicable. There was no holistic distance effect for any type of problem. These results suggest that the participants in this study were able to compare irrational numbers only if component-based strategies were applicable, but not when problems required reasoning about the overall magnitudes of the irrational numbers. In line with research on fraction comparison, we found a natural number bias for problems that had common components, but not for problems without common components.[5]

[4] Note that this type of representation is the common way to represent the exact value of irrational numbers. Decimal representations are only approximations of these numbers, because irrational numbers have an infinite number of non-recurring digits after the decimal point.

[5] Within the set of problems without common components, we actually found a natural number bias in the opposite direction (higher accuracy for incongruent rather than for congruent comparison problems), which was, however, presumably due to the fact that incongruent, but not congruent problems, could often be solved by component strategies.

In sum, this study suggests that even mathematically skilled adults struggle with activating irrational numbers magnitudes. A limitation of this study is that we cannot determine whether our participants' inability to activate irrational number magnitudes was specifically due to the irrationality of the numbers, or to the fact that they were just unfamiliar with these numbers. It is possible that if people were trained to activate the magnitudes of specific irrational numbers, they would be able to activate magnitudes for these numbers. Another limitation is that we used number comparison problems and holistic distance effects as indicators of participants' activation of magnitudes. Although number comparison has frequently been used in the literature to assess mental representations of number magnitudes, this task is presumably not a completely reliable measure. Future studies could use other measures, such as a number line estimation task, to assess more directly people's ability to assess the magnitudes of irrational numbers. Finally, our interpretation is based on accuracy and response times data, which are only indirect measures of strategy use and cognitive processes. Further research could use additional measures such as self-reports or eye tracking to learn more about the strategies that people use to solve irrational number comparisons.

5.2. Discussion

Mathematically skilled adults are well able to solve irrational number comparison problems when comparison strategies based on the natural number components are valid. If this is not the case, and comparison problems require reasoning about the overall irrational number magnitudes, these comparison problems are extremely challenging even for the skilled adults. Thus, in general, translating the symbolic representation of an irrational number into its magnitude seems to be an extremely demanding cognitive task. In other words, people do not seem to have a sense for symbolic irrational number magnitudes. This result is not surprising because the meaning of the symbolic notation of irrational numbers is much more complex than it is for rational numbers. Assessing fraction magnitudes requires reasoning about the multiplicative relation between the numerator and the denominator, but assessing the magnitude of an irrational number symbol requires carrying out a much more complex procedure (e.g., a complex iterative algorithm). Therefore, activating irrational number magnitudes requires a high level of understanding of the concept of irrational numbers.

However, the results of this study do not prove that the human cognitive architecture is incapable of processing irrational number magnitudes per se. As some researchers have argued, the human cognitive system may be privileged when processing natural numbers (Feigenson et al., 2004). Other researchers have argued that human cognition is well prepared for processing other numbers, such as rational numbers (Huttenlocher, Duffy, & Levine, 2002; Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016). It is important to note here that this discussion of the present study on the mind’s capacity to process irrational number magnitudes has centered around studies that used *visual* representations of numbers (e.g., rectangular bars), rather than *symbolic* representations. Even though people may not be able to process the magnitudes of irrational number symbols, there is no reason to believe that people would not be able to process visual representations of irrational numbers. For example, ratio comparison problems used in the study by Matthews and Chesney (2015) could just as well include irrational ratios. After all, it should not be possible to perceptually distinguish between ratios representing rational numbers and those representing irrational numbers. Further research could empirically test this assumption. From a theoretical viewpoint, a challenge for further research is to better understand the relation between visual and symbolic number representations.

The study presented above used a number comparison task to assess participants’ magnitude processing, although prior research on natural and rational number magnitudes has also used number line estimation tasks. Accordingly, further research could use a number line task with irrational and rational numbers of approximately the same magnitude to directly compare people’s performance between these two types of tasks. Such a task would more directly assess magnitude understanding of these number symbols, because in number comparison problems it is possible to solve problems correctly without actually activating magnitude representations of the respective numbers—as the results in this study have shown.

More generally, further research is needed to better understand the conceptual knowledge of irrational numbers in K-12 students and adults. The few available studies on this topic (Bauer et al., 2005; Sirotic & Zazkis, 2007; Zazkis, 2005) have established an important basic foundation on the subject, but do not yet provide a comprehensive picture of student and adult learning or conceptual understanding of irrational numbers.

6. General Discussion

This chapter provides a general discussion of the research presented in the previous chapters. It is structured into four sections. The first section provides a brief summary of the main results. The second section reflects on the methods and the interdisciplinary approach adopted in this work. The third section discusses implications this research has for classroom teaching. Finally, the fourth section discusses directions for future research.

6.1. Brief Summary of Main Results

The notion of number sense has been used for a long time, although researchers have defined and empirically measured number sense in diverse ways (Study 1). Researchers across disciplines agree that number sense is important for children's numerical development, and that acquiring number sense is an important goal of mathematical learning at school. However, there is less agreement on the role that number sense for natural numbers plays in children's learning and processing of numbers in other domains. This work has highlighted both the importance and the nature of natural number sense (Chapter 3), as well as its potential to interfere in children's and even experienced adults' ability to solve rational number problems (Chapter 4) and irrational number problems (Chapter 5).

With regard to the importance of natural number sense, the research presented here has shown how interventions that focus explicitly on number magnitudes can enhance number sense in children (Study 2). The use of linear number representations seems to be one effective way to highlight magnitudes in instruction. Such a focus on number magnitudes and the use of linear number representations seems to be beneficial because the human brain is attuned to representing numbers in a linear way according to their magnitudes—as psychological and neuroscience research suggest (Study 4). More generally, learners may benefit from using external representations of numbers that match the existing or desired mental representations. As learners are supposed to acquire a variety of number concepts such as magnitudes and the base-ten structure, the use of various external representations (including structured representations such as the twenty-frame) in instruction may be beneficial to them in the long run (Study 3).

With regard to the potential interference of natural number magnitudes when it comes to solving rational and irrational number problems, this work has shown that the natural number bias in rational number problems can occur in a variety of tasks, including: fraction comparison (Study 5 and 6), reasoning about algebraic inequalities that represent the effect of fraction operations (Study 9), and solving contingency table problems (Study 10). Moreover, the natural number bias can be very persistent (Study 5, 6, and 9). While these results are in line with previous research, this work has provided deeper insights into two issues. First, it has shown that the occurrence of the natural number bias depends on the interplay of a number of factors, such as: the type of problem, the strategy that is used to solve that problem, and on an individual's ability to adapt their strategy (Study 6, 7, 8, and 9). This work has also specified and empirically documented the types of relevant strategies (holistic versus componential strategies in fraction comparison; algebraic reasoning versus natural number substitution in algebraic equations) that affect the occurrence of the natural number bias. Second, this work has provided first evidence that abstract mathematical reasoning can be one way to completely overcome the natural number bias on problems in which less skilled people are clearly biased (Study 9).

Researchers agree that understanding number magnitudes is important. However, empirical research has almost exclusively focused on rational numbers. Moreover, magnitude processing has often been studied using number comparison problems. This work suggests that people are well able to solve number comparison problems with irrational numbers, although they have great difficulty with activating magnitude representations for these numbers (Study 11). This finding suggests that number comparison problems may only be a valid measure of magnitude processing if they discourage strategies that do not actually require magnitude processing (e.g., by featuring unfamiliar numbers and number pairs without common components). Moreover, the current findings raise the theoretical question of whether the ability to quickly activate number magnitude representations should be considered a facet of numerical ability for all real numbers—including irrationals. Alternatively, students' understanding of how irrational number symbols can convey numerical magnitude, rather than their ability to actually activate such magnitude representations, may be a better indicator of mathematical ability.

6.2. Reflections on Methods and the Interdisciplinary Approach

This work has adopted an interdisciplinary approach that included perspectives from mathematics education, cognitive psychology, and neuroscience. Such an approach seemed necessary, because although there is increasing research in numerical cognition (Cohen Kadosh & Dowker, 2015), few attempts have been made to merge this research—and research in cognitive psychology more generally—with research in mathematics education (for recent approaches, see, however, Norton & Alibali, in press; Obersteiner, 2012; Obersteiner, Reiss, & Heinze, in press; Rau & Matthews, 2017; Zazkis & Mamolo, 2016).

Not only were the empirical studies reported above grounded on psychological theories of numerical cognition, they also used methods that tap into cognitive processing mechanisms, such as response times measures and eye tracking. Although response times may be a more distal measure of cognitive processes than eye tracking, comparison between specific problem types (e.g., of problems that are congruent versus incongruent with natural number reasoning) allow inferences about problem-solving mechanisms. A limitation of this measure is that response times were averaged across a whole sample of participants, which limits claims that can be made at the individual level. This limitation is more significant in samples with large individual variation (such as high school students) than within a more homogeneous sample (such as mathematically skilled adults), who were the participants in most of the studies reported here. An attempt to overcome this limitation was to combine response time measures with other measures, such as self-reports (Study 10) or eye movements (Study 7 and 8).

Although it is impossible to directly assess cognitive processes, eye tracking may be a relatively proximal measure. However, as discussed earlier (see 4.7), eye tracking has the drawback that interpreting eye movement data is not straightforward. Our current knowledge of eye movements during mathematical problem-solving is limited, so that it is difficult to distinguish processes that are related to numerical processing from those that are unrelated to the task, such as reading. Further research is needed to address this issue, for example by contrasting eye movement in different experimental conditions (calculation versus reading only), or by combining eye movements with other measures (e.g., self-reports) on a trial-by-trial basis to directly validate the method (as suggested by Obersteiner & Staudinger, in press).

On a more general note, the interdisciplinary perspective adopted in this work comes with challenges because researchers from different disciplines do not always speak the same language. For example, the term "magnitude" is commonly used in the numerical cognition literature. Researchers in mathematics education often associate the term number magnitude with the measurement aspect of numbers (e.g., 5 cm) (Nolte, 2015). The numerical cognition literature, however, often uses this term to refer to a variety of aspects of numbers, including the cardinality of collections of objects (e.g., Siegler, 2016).

To avoid miscommunication between researchers from different disciplines, researchers should strive to create a theoretical framework that can provide guidance for how to integrate research from different perspectives on a particular topic (e.g., numbers). Such a framework would help integrate research findings at different levels of explanations (e.g., brain level, behavioral level, or classroom level) (Nathan & Alibali, 2010).

6.3. Implications for Classroom Teaching

Although most of the empirical studies reported in this work included an adult sample, they gained insight into fundamental cognitive processing mechanisms, which may have implications for classroom teaching. This section discusses these implications, while making connections between chapters where appropriate.

As mentioned earlier, to develop number sense for natural numbers, children may benefit from an increased focus on number magnitudes in the classroom. Using number lines to represent numbers seems to be a particularly effective way to highlight numerical magnitudes. A particular advantage of the number line representation is that it is a sustainable representation. This means that a number line can be used throughout all grade levels when learning about numbers, because all real numbers can be represented on the same line (Hamdan & Gunderson, 2017; Siegler & Braithwaite, 2017; Siegler & Lortie-Forgues, 2014). In fact, empirical intervention studies show that the teaching of fractions is particularly effective when instruction focusses on fraction magnitudes and uses number lines to represent these magnitudes (Fazio, Kennedy, & Siegler, 2016; Fuchs et al., 2013; Hamdan & Gunderson, 2017). Based on such findings, the frequent use of number lines, particularly for representing fractions, is recommended in educational standards and curricula in the U.S. (Common Core State Standards Initiative, 2010; National Mathematics Advisory Panel, 2008). Importantly though, students may not spontaneously learn from using number line representations but may need specific support in using them appropriately. For example, in the study by Fazio, Kennedy, et al. (2016), children played games in which they had to find the correct positions of fraction symbols on number lines. Although one may expect that children can benefit from making links between number symbols on the number line representation, children only profited from playing the game when it was combined with feedback showing how one can divide the number line into segments to find the position of a fraction.

With regard to the sources of students' difficulties with learning fractions, it is useful to consider the studies reported in this chapter within the theoretical frameworks of dual-processes and conceptual change (see 2.2 and 2.3). In line with dual-process theories, evidence from fraction comparison studies with mathematically skilled adults suggests that in these problems, the natural number bias can be very persistent. Accordingly, students may still struggle with incongruent fraction comparison problems, even when they have developed a sound conceptual understanding of fractions. Thus, difficulties with fractions may partly be due to the representation itself (i.e., the bipartite notation with two natural numbers, and the fact that increasing a fraction's denominator decreases its numerical value) rather than to a poor understanding of the concept as such (Kallai & Tzelgov, 2012; Ni & Zhou, 2005). Instruction can thus support students in understanding this representation, but this may never be sufficient to completely overcome the natural number bias. In view of its intuitive character, a promising teaching approach could be to make explicit that people (even experts) are influenced by their intuitions while reasoning mathematically, that these intuitions can be misleading, and that conscious awareness and control are necessary to overcome this potentially misleading bias, that is, to “stop and think” (Vamvakoussi et al., 2012). This may not help to improve response times but could help to increase the accuracy of answers. Encouraging students to think about the reasonableness of their responses could also be beneficial in fraction arithmetic. Self-explanation prompts and refutation texts that prompt students to give reasons for their work could be concrete instructional ways to learn through reflection (Tippett, 2010; Van Hoof, Vamvakoussi, et al., 2017).

In line with the conceptual change framework, there was evidence that students and—for some problems—even adults struggle with two aspects of fractions that are conceptually different from those of natural numbers: the way fraction symbols represent numerical magnitudes, and the effect of arithmetic operations. Thus, an implication is that classroom teaching should pay particular attention to those aspects of rational numbers that are conceptually different from natural numbers.

On the other hand, teachers also need to highlight commonalities between natural and rational numbers, such as the fact that they both represent numerical magnitudes and can be represented on the same number line (Siegler & Lortie-Forgues, 2014). A challenge for teachers is specifying which concepts for rational numbers are similar to natural numbers concepts and which are different, to ensure that students activate their appropriate prior knowledge when learning about fractions. Empirical evidence shows that learning fraction division concepts can be more or less successful depending on whether the previous knowledge of natural numbers that is activated is helpful (conceptual similarity) or not (superficial similarity). Sidney and Alibali (2015, 2017) found that when learning about fraction division, students benefited more from practicing division of natural numbers (similar concept but different numbers), rather than from fraction problems without division (similar numbers but different concept), immediately before engaging in fraction division. This suggested sequencing of fraction problems (fraction division preceded by natural number division) differs from common mathematics textbooks, where fraction division typically follows up on fraction multiplication.

Current classroom practices seem to put more emphasis on the similarities, rather than the differences, between natural numbers and fractions (Lortie-Forgues et al., 2015; Van Hoof, Vamvakoussi, et al., 2017). Thus, students may get too little support in distinguishing between aspects of rational numbers that are in line with those of natural numbers and those that are conceptually different. Importantly, teachers need to be equipped with the necessary content knowledge about fractions. Unfortunately, research suggests that this is not always the case (Ball, 1990; Depaepe et al., 2015; Newton, 2008). Moreover, it seems that teaching materials such as textbooks do not always support student learning of fractions in the best possible way. Rather, the way problems are selected and presented in textbooks (e.g., their strong focus on procedures) may even contribute to children's difficulties (Braithwaite, Pyke, & Siegler, 2017). Together, carefully selecting problems and considering children's pre-knowledge seem to be important ways to improve teaching of fractions in the classroom.

Cognitive research suggests that the human cognitive system is well prepared to process ratios. Recently, Matthews and colleagues (Lewis, Matthews, & Hubbard, 2014, 2016; Matthews & Chesney, 2015) postulated the existence of a ratio processing system, which allows processing of ratios and proportions on a perceptual level from early on. Importantly, these perceptual abilities were associated with symbolic mathematics performance (Matthews et al., 2016). Fraction instruction could benefit from drawing more strongly on students' early ability to process visual ratios and fractions. One way to draw on such abilities could be by using visualizations of proportions and ratios before introducing symbolic fractions. As Rau, Aleven, Rummel, and Rohrbach (2012) have shown, students may benefit from gaining competence in the use of fraction visualizations in addition to receiving systematic instruction in fraction concepts. Continuous rather than discrete representations could be beneficial as they prevent students from engaging in counting (Boyer & Levine, 2015; Boyer, Levine, & Huttenlocher, 2008), which may later on support a natural number bias.

Concerning irrational numbers, the research presented in this work suggests that it is extremely difficult to activate magnitudes for these numbers. Although tentative, current research suggests that this difficulty might be due to students' lack of familiarity with irrational numbers, rather than to the type of numbers per se. Thus, as calculating the magnitudes of irrational numbers is very complex, a goal of classroom instruction may be that students learn the approximate magnitudes of particularly important irrational numbers such as $\sqrt{2}$ or π; however, it is not worthwhile to aim at teaching irrational number magnitudes in general. Concerning the learning of number concepts, many of the implications of learning about rational numbers discussed in this section may also apply to the learning of real number concepts more generally. However, this is a matter for future research, because there are not enough studies on students' understanding of irrational numbers (see, however, Bauer et al., 2005; Sirotic & Zazkis, 2007; Zazkis, 2005).

6.4. Future Directions

Several open questions that should be addressed in future research have been discussed in the respective chapters above. This section highlights three areas for future research that involve numerical learning across number domains.

The learning of rational and irrational numbers certainly requires proficiency with natural numbers. On the other hand, natural number knowledge can also bias learners and make them overgeneralize this knowledge to situations in which it is not directly applicable (Van Hoof, Verschaffel, et al., 2017). More research is certainly needed to better understand the relation between previously acquired knowledge about natural numbers and the learning of rational and irrational numbers. To that end, it is necessary to assess the same group of learners over a longer time period in a longitudinal study design. Some studies on rational number learning have used such designs (e.g., Braithwaite & Siegler, 2017; Mou et al., 2016; I. Resnick et al., 2016; Rinne, Ye, & Jordan, 2017). However, these studies focused on very specific aspects of rational number development, such as students' understanding of number magnitudes. Further research with a broader perspective could assess how a variety of variables contribute to children's numerical development. For example, studies could take into account cognitive and non-cognitive variables, but also school-related and socio-economic factors.

Recent studies suggest that how well people are able to mentally represent numbers may not only be a matter of number type, but of how familiar people are with specific numbers (Liu, 2017). For example, small natural numbers may be represented more easily than larger numbers just because they are more commonly used in everyday life. A similar argument applies to specific fractions (e.g., 1/2 or 3/4). Irrational numbers, on the other hand, are usually not used in daily life at all. Therefore, unfamiliarity with these numbers, rather than their irrational nature, could explain why people are not readily able to mentally represent magnitudes of these numbers. From a theoretical perspective, the role of familiarity is important because it concerns the general importance of magnitude understanding for numerical development (Siegler et al., 2011). More specifically, it would be interesting to understand how knowledge of the magnitudes of familiar numbers, along with the ability to assess the magnitudes of *any* number symbol—including unfamiliar ones—, contribute to numerical development.

In line with the literature, in the studies described in this work, the term "natural number bias" has been used to describe performance differences between problems that are or are not in line with natural number reasoning. The studies presented here provided evidence that even mathematically skilled adults (including expert mathematicians) show a natural number bias in certain types of fraction comparison problems (those in which fraction pairs have common components). However, they did so only in terms of response times but not accuracy, which was very high regardless of problem type. Therefore, relying on natural number reasoning in certain problem types seems to be a very efficient strategy, allowing people to be particularly quick. The term "bias," with its negative connotation, may thus not be the best term to describe performance differences in skilled adults. The term seems more suitable if the cognitive processes involved in problem solving are subtle, largely unconscious, and occur immediately after being presented with a specific problem, and less suitable if these processes stem from people's intentional choice of specific processing strategies—even if these choices are based on natural number components. To better understand the nature of the natural number bias in number comparison, future research could specifically explore the temporal dimension of the cognitive processing mechanisms involved. Mock et al. (2016) have taken a first step in that direction by presenting a general model of the temporal dynamics of eye movements in numerical cognition. Further research could refine that model and apply it in the context of number comparison. In addition to response times methods, eye tracking and brain imaging techniques may be suitable to assessing the temporal dimensions of number processing.

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