Mathematik

Wahrscheinlichkeitstheorie

# Frogs and other moving particles

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Vollständiger Abdruck der von der Fakultät für Mathematik der Technischen Universität München zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften genehmigten Dissertation

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Die Dissertation wurde am 11.10.2018 bei der Technischen Universität München eingereicht und durch die Fakultät für Mathematik am 08.01.2019 angenommen.

# Acknowledgements

First of all, I am very grateful to my supervisor Nina Gantert for the excellent support during my time as a PhD student. In particular, I thank her for giving me the freedom to choose my own research problems.
I also thank Noam Berger for always helping whenever I had questions. Furthermore, I want to thank all present and former members of the probability chair for fruitful discussions and a very pleasant atmosphere. It has been a wonderful experience working with such a great group. In particular, I am very grateful to Felizitas Weidner for great collaboration and a very enjoyable time.
Finally, I would like to thank Stefan Junk and Felizitas Weidner for proof-reading this thesis.

# Contents

# Introduction

In this thesis we consider two different models of stochastic processes in discrete time. In Chapter 1 we study the frog model and in Chapter 2 and 3 we study branching random walks. Both models belong to the family of interacting random walks.
A random walk is one of the most basic objects in probability theory. In every step a particle moves randomly according to some step size distribution. The simplest case is a simple random walk on the $d$-dimensional lattice $\mathbb{Z}^d$. The particle starts at the origin and at every step it jumps to one of its $2d$ neighbours uniformly at random. Interacting random walks consist of a (possibly infinite) set of particles moving around simultaneously according to some rules. In particular, the set of particles is not necessarily fixed.
In Chapter 1 we study the frog model on $\mathbb{Z}^d$. This model was first investigated by Telcs and Wormald [57], who, however, called it egg model. The name frog model was only later suggested by Durett, see [53]. The model can be described as follows. At the beginning there is one active frog at the origin and one sleeping frog at every other site of $\mathbb{Z}^d$. The active frog performs a simple random walk. Whenever a sleeping frog is visited by an active frog, it is activated and starts a simple random walk itself, independently of all other active frogs. It can wake up other sleeping frogs as well. The frog model does not describe the behaviour of real frogs, but it can be used to model the spread of information or a disease. Active frogs have some information or an illness and pass it whenever they meet a sleeping frog. The activated frogs then start spreading the information or illness themselves.
One of the first questions for interacting random walks deals with recurrence and transience. The frog model is called recurrent, if the origin is visited infinitely often by active frogs with probability one. In particular, in the recurrent case every frog will be activated. Otherwise it is called transient. In the transient case the cloud of active frogs moves away from the origin. If the frogs perform simple random walks, i.e. they choose every direction with the same probability, Telcs and Wormald [57] showed that the frog model is recurrent on $\mathbb{Z}^d$ in any dimension $d \geq 1$. In this thesis we study the frog model with a drift in one direction. This means that there is one distinguished direction, which is chosen by the frogs with a higher probability. In Theorem 1.20-1.23 we show that for dimension $d \geq 2$ the frog model can either be recurrent or transient, depending on the drift. These results are joint work with Döbler, Gantert, Popov and Weidner and published in [22].
For dimension $d = 1$ it is shown in [31], however, that the frog model is transient for

any drift. In particular, the cloud of frogs moves away from the origin. Without loss of generality, we assume that we have a drift to the right. In Theorem 1.16 we compute the linear speed of the leftmost occupied site explicitly. Furthermore, in Theorem 1.17 and Theorem 1.18 we show that the speed of the rightmost occupied site is monotone in the drift and strictly less than 1. We also prove in Theorem 1.19 that all frogs are in some sense uniformly distributed in between the leftmost and rightmost occupied site. These results are joint work with Weidner and published in [36].

In Chapter 2 we consider a branching random walk on $\mathbb{R}$. Informally, the model can be described in the following way. At time 0 the process starts with one particle at the origin. At every time $n \in \mathbb{N}$ each particle repeats the following two steps, independently of everything else. First, it produces offspring according to some fixed offspring distribution, and then it dies. Afterwards, the offspring particles move according to some step size distribution. In contrast to the frog model on $\mathbb{Z}$, the linear speed $x^*$ of the rightmost particle is known explicitly. Even more precise asymptotics for the rightmost particle have been obtained, see Section 2.2.3 for details. In Theorem 2.19 we derive a large deviation result for the position of the rightmost particle, i.e. we determine the exponential decay rate of the probability that the rightmost particle has speed $x \neq x^*$. In the proof we dominate the branching random walk by a random number of independent random walks, for which we show a large deviation result in Theorem 2.18 as well. These results are joint work with Gantert and published in [29].

In Chapter 3 we study the same model as in Chapter 2, but we add another source of randomness. In Chapter 2 we assume that the offspring distribution is independent of time. However, for instance environmental conditions may influence the branching behaviour. Therefore, in this chapter we assume that the offspring distribution is chosen at random in every step. The sequence of (random) offspring distributions is called random environment. The probability measure conditioned on a fixed environment is called quenched measure, whereas the probability measure averaging over all possible environments is called annealed measure. For the rightmost particle of the branching random walk we show a large deviation result in Theorem 3.11 and 3.12 with respect to the annealed and quenched measure, respectively. Analogous results for independent random walks are obtained in Theorem 3.9 and 3.10. The random environment leads to additional difficulties in the proofs. Therefore, under the annealed measure we could only prove an upper large deviation result for the rightmost particle of the branching random walk.

Both the frog model as well as the branching random walk are interacting random walks with a growing set of particles. Furthermore, in both models the interaction between the particles only takes place, whenever there is a new particle added to the system, i.e. if a sleeping frog is activated in the frog model or offspring is produced in the branching random walk, respectively. Only the location and time when the particle enters the system depends on the behaviour of the other particles. However, there are two main

differences between the two models. First, in contrast to the frog model, the average number of (active) particles is known explicitly for the branching random walk. Second, the location of each particle in the branching random walk has the same distribution, namely the distribution of a random walk with the same step size distribution. At first, one might think that this is also true for the frog model, since each frog moves according to a random walk. However, since a fixed sleeping frog only gets activated by a frog that moves away from the origin fast (it has to be the first frog reaching the sleeping frog), the location of a fixed frog tends to be further away from the origin. These two properties are used extensively to show results for the branching random walk.
In particular, this explains why there are much finer results for the branching random walk. For the frog model we can only show the existence of recurrent and transient regimes, while there is a sharp criterion separating transience from recurrence for the branching random walk. Moreover, we can only prove qualitative results about the speed of the rightmost frog, while there is an explicit formula for the speed of the rightmost particle in the branching random walk.
This thesis is structured as follows. We start every chapter with a formal description of the model before we give some known results which are interesting in the context of our results. In Chapter 2 and 3 we need to introduce some rate functions before we can state the main results of the chapter. We then collect some preliminary results which are needed in the proofs of the main results.

# 1 The Frog Model

## 1.1 Description of the model

We describe the frog model in a more general setting than explained in the introduction. Let $\mathcal{G} = (V, E)$ be a connected non-oriented graph such that every vertex has finite degree. Fix a vertex $o \in V$ and call it root. Let $\eta$ be a $\mathbb{N}_0$-valued random variable with $\mathbb{P}(\eta \geq 1) > 0$. Let $\{\eta_x \colon x \in V \setminus \{o\}\}$ and $\{(S_n^x(i))_{n\in\mathbb{N}} \colon i \in \mathbb{N}, x \in V\}$ be independent families of i.i.d. random variables defined as follows: For all $x \in V \setminus \{o\}$ the random variable $\eta_x$ has the same distribution as $\eta$ and gives the initial number of sleeping frogs at $x$. If $\eta_x \geq 1$, then for all $1 \leq i \leq \eta(x)$ the sequence $(S_n^x(i))_{n\in\mathbb{N}}$ is a discrete time nearest neighbour random walk starting at $x$. It describes the trajectory of the $i$-th frog initially at vertex $x$. The transition probabilities of the random walk are given by a transition function $\pi$. If the random walk is symmetric, i.e. if in every step it chooses one of its neighbours uniformly, the transition function is denoted by $\pi_{\text{sym}}$. For $x, y \in V, x \neq y$ define the first time that a particle initially at vertex $x$ reaches vertex $y$ as

$$t(x,y) = \min_{1\leq i\leq \eta(x)} \inf\{n \in \mathbb{N} \colon S_n^x(i) = y\}.$$

Note that $t(x, y)$ might be infinite. Then, the first time a vertex $x$ is visited by an active frog is defined as

$$T_x = \inf\big\{m \in \mathbb{N}, o = x_0, x_1, \ldots, x_m = x \in V \colon \sum_{i=1}^{m} t(x_i, x_{i-1})\big\}.$$

If $T_x = \infty$, then the frogs initially at $x$ will never be activated. $T_x$ is called activation time of $x$. After time $T_x$ all frogs initially at $x$ start to follow their trajectories given by $(S_n^x(i))_{n\in\mathbb{N}}$ for $1 \leq i \leq \eta_x$. The position $Z_n^x(i)$ of the $i$-th frog initially at vertex $x \in V$ at time $n$ is defined as

$$Z_n^x(i) = \begin{cases} x & \text{for } n < T_x \\ S_{n-T_x}^x(i) & \text{for } n \geq T_x \end{cases}.$$

The frog model on the graph $\mathcal{G}$ with transition function $\pi$ and initial configuration $\eta$ is denoted by $\mathrm{FM}(\mathcal{G}, \pi, \eta)$.

## 1.2 Some known results

### 1.2.1 Symmetric frogs

In this subsection we assume that the underlying random walk of the frog model is symmetric, i.e. the transition function is given by $\pi_{\text{sym}}$. Moreover, except for Theorem 1.2, we only consider $\mathcal{G} = \mathbb{Z}^d$ for $d \geq 1$ or $\mathcal{G} = \mathbb{T}_d$ for $d \geq 2$, where $\mathbb{T}_d$ denotes the $d$-regular tree, i.e. every vertex has degree $d + 1$.
When studying interacting random walks, one often first asks about recurrence and transience, i.e. whether the origin is visited infinitely many times, or the cloud of particles moves away from the origin.

**Definition 1.1.** *The frog model* $\mathrm{FM}(\mathcal{G}, \pi, \eta)$ *is called recurrent, if*

$$\mathbb{P}(\textit{the origin is visited infinitely many times}) = 1.$$

*Otherwise it is called transient.*

If the frog model is recurrent, then every frog will be activated. Very recently, Kosygina and Zerner showed that the frog model satisfies a zero-one-law.

**Theorem 1.2** ([45, Theorem 1])**.** *In* $\mathrm{FM}(\mathcal{G}, \pi, \eta)$ *the probability that the origin is visited infinitely many times by active frogs is either* $0$ *or* $1$*.*

Kosygina and Zerner consider a more general model in [45]. For instance, their result also applies to frog models, where the trajectories of the frogs are given by a transitive and irreducible Markov chain. Note that by Theorem 1.2, to show recurrence it suffices to prove that the origin is visited infinitely often with positive probability.
If there is one sleeping frog at every vertex, i.e. if $\eta \equiv 1$, the frog model on $\mathbb{Z}^d$ is recurrent for any dimension $d \geq 1$. For $d = 1, 2$ this is obviously true, since simple random walk is recurrent in this case. Therefore, already the frog starting from the origin will return infinitely many times. For $d = 3$ simple random walk on $\mathbb{Z}^d$ is transient. However, on its way to infinity the inital frog at the origin will activate a frog with euclidean distance to the origin in $(n-1, n]$ for every $n \in \mathbb{N}$. This frog has a chance of approximately $c \cdot n^{-1}$ to ever visit the origin. Since the trajectories of these frogs are independent, the Borel-Cantelli lemma yields that infinitely many frogs return to the origin. For $d > 3$ this was first proved by Telcs and Wormald in [57] and later refined by Popov in [52]. Popov considered a random initial configuration with one sleeping frog at $x \in \mathbb{Z}^d \setminus \{0\}$ with probability $p(x)$ and zero frogs with probability $1 - p(x)$, independently of all other vertices. He found the critical decay of $p(x)$ separating transience from recurrence.

**Theorem 1.3** ([52, Theorem 1.1])**.** *For* $d \geq 3$ *let* $p\colon \mathbb{Z}^d \setminus \{0\} \to [0, 1]$ *and* $\eta = (\eta_x)_{x \in \mathbb{Z}^d \setminus \{0\}}$ *be a collection of independent random variables with* $\mathbb{P}(\eta_x = 1) = p(x) = 1 - \mathbb{P}(\eta_x = 0)$

*for all* $x \in \mathbb{Z}^d \setminus \{0\}$ *as well as* $\eta_0 = 1$. *Consider the frog model* $\mathrm{FM}(\mathbb{Z}^d, \pi_{sym}, \eta)$. *There exists* $\alpha_c = \alpha_c(d) \in (0, \infty)$ *such that*

*(i) the frog model is transient if* $\alpha < \alpha_c$ *and* $p(x) \leq \alpha \|x\|^{-2}$ *for all* $x$ *large enough,*

*(ii) the frog model is recurrent if* $\alpha > \alpha_c$ *and* $p(x) \geq \alpha \|x\|^{-2}$ *for all* $x$ *large enough.*

Very recently, the question of recurrence and transience was also solved on the $d$-regular tree for $d = 2$ and $d \geq 5$ by Hoffmann et al. in [38]. It is still open for $d = 3, 4$.

**Theorem 1.4** ([38, Theorem 1]). *Consider the frog model* $\mathrm{FM}(\mathbb{T}_d, \pi_{sym}, 1)$.

*(i) For* $d = 2$ *the frog model on* $\mathbb{T}_d$ *is recurrent.*

*(ii) For* $d \geq 5$ *the frog model on* $\mathbb{T}_d$ *is transient.*

It is conjectured in [38] that the frog model on $\mathbb{T}_d$ remains recurrent for $d = 3$, while it is transient for $d = 4$. In [37] the same authors also investigate recurrence and transience of the frog model on the $d$-regular tree with a random initial configuration given by i.i.d. Poisson($\mu$) distributed random variables. They prove that for every $d \geq 2$ there is a critical value for the parameter $\mu$ separating transience from recurrence.
If the frog model is recurrent, then every frog will be activated. On $\mathbb{Z}^d$ every frog at euclidean distance at most $n$ from the origin will be activated up to time $c \cdot n$ for all $n$ large enough, where $c$ is a positive constant independent of $n$. More precisely, for the frog model on $\mathbb{Z}^d$ with initially one frog per site Alves et al. prove in [4] that the set of vertices visited by active frogs, rescaled by time, converges to a convex set. This result is generalised to an i.i.d. initial configuration in [5].

**Theorem 1.5** ([5, Theorem 1.1]). *Consider the frog model* $\mathrm{FM}(\mathbb{Z}^d, \pi_{sym}, \eta)$ *for* $d \geq 1$ *and* $\mathbb{P}(\eta > 0) > 0$. *Let* $\xi_n(\eta)$ *be the set of all vertices visited by active frogs at time* $n$ *and define the set* $\overline{\xi}_n(\eta) := \{x + (-\frac{1}{2}, \frac{1}{2}]^d \colon x \in \xi_n(\eta)\}$. *Then, there is a non-empty convex symmetric set* $\mathcal{A} = \mathcal{A}(d, \eta) \subseteq \mathbb{R}^d$, $\mathcal{A} \neq \{0\}$, *such that for almost all initial configurations* $\eta$ *and for any* $0 < \varepsilon < 1$ *we have*

$$(1 - \varepsilon)\mathcal{A} \subseteq \frac{\overline{\xi}_n}{n} \subseteq (1 + \varepsilon)\mathcal{A}$$

*for all* $n$ *large enough almost surely.*

**Remark 1.6.** *The proof of Theorem 1.5 goes through for the "lazy" version of the frog model, where in each step a frog decides to stay where it is with probability* $q \in (0, 1)$, *independently of all other frogs.*

### 1.2.2 Frogs with death

One possible generalisation of the frog model discussed in subsection 1.2.1 is the frog model with death. For $s \in [0,1]$ it is defined just as the usual frog model, but every active frog dies at every step with probability $1-s$, independently of everything else. The parameter $s$ is called survival probability. We denote this frog model on $\mathcal{G}$ with initial configuration $\eta$ by $\mathrm{FM}^*(\mathcal{G},\pi,\eta,s)$ if the underlying random walk has transition function $\pi$. In the symmetric case, i.e. if $\pi = \pi_{\mathrm{sym}}$, the frog model with death is intensively studied in [3] and also in [27] and [47]. Note that we need some results for the frog model with death in the proofs of our main results.
The first question for this model deals with survival of the frogs.

**Definition 1.7.** *We say that the frog model survives, if there is at least one active frog at any time. Otherwise it dies out.*

Since the probability that the frog model $\mathrm{FM}^*(\mathcal{G},\pi,\eta,s)$ survives is increasing (not necessarily strictly) in $s$, we can define the critical survival probability as

$$s_c(\mathcal{G},\pi,\eta) = \inf\{s\colon \mathbb{P}(\mathrm{FM}^*(\mathcal{G},\pi,\eta,s) \text{ survives}) > 0\}. \tag{1.1}$$

**Theorem 1.8** ([3, Theorem 1.1, Theorem 1.3, Theorem 1.4])**.** *Consider the frog model* $\mathrm{FM}^*(\mathbb{Z}^d,\pi,\eta,s)$.

*(i) If* $\mathbb{E}[\log^+(\eta)] < \infty$, *then* $s_c(\mathbb{Z},\pi_{sym},\eta) = 1$.

*(ii) Let* $d \geq 2$. *If* $\mathbb{E}[(\log^+(\eta))^d] < \infty$, *then* $s_c(\mathbb{Z}^d,\pi_{sym},\eta) > 0$.

*(iii) Let* $d \geq 2$. *If* $\mathbb{P}(\eta \geq 1) > 0$, *then* $s_c(\mathbb{Z}^d,\pi_{sym},\eta) < 1$.

Under mild assumptions on $\eta$, the frog model on $\mathbb{Z}$ dies out for every $s < 1$. Furthermore, again under mild assumptions on $\eta$, the frog model on $\mathbb{Z}^d$ for $d \geq 2$ survives for all $s$ close to 1 and dies out for all $s$ close to 0.

**Theorem 1.9** ([3, Theorem 1.2, Theorem 1.6, Theorem 1.5])**.** *Consider the frog model* $\mathrm{FM}^*(\mathbb{T}_d,\pi,\eta,s)$.

*(i) Let* $d \geq 1$. *If there exists* $\delta > 0$ *such that* $\mathbb{E}[\eta^\delta] < \infty$, *then* $s_c(\mathbb{T}_d,\pi_{sym},\eta) > 0$.

*(ii) Let* $d \geq 1$. *If* $\mathbb{E}[\eta^\delta] = \infty$ *for all* $\delta > 0$, *then* $s_c(\mathbb{T}_d,\pi_{sym},\eta) = 0$.

*(iii) Let* $d \geq 2$. *If* $\mathbb{P}(\eta \geq 1) > 0$, *then* $s_c(\mathbb{T}_d,\pi_{sym},\eta) < 1$.

Under mild assumptions on $\eta$, for all $d \geq 1$ the frog model on $\mathbb{T}_d$ survives for all $s$ close to 1 and dies out for all $s$ close to 0.
Alves et al. also prove asymptotics for the critical survival probability as $d \to \infty$.

**Theorem 1.10** ([3, Theorem 1.7, Theorem 1.8]). *If $\mathbb{E}[\eta] < \infty$, then*

$$\lim_{d\to\infty} s_c(\mathbb{Z}^d, \pi_{sym}, \eta) = \lim_{d\to\infty} s_c(\mathbb{T}_d, \pi_{sym}, \eta) = \frac{1}{1+\mathbb{E}[\eta]}.$$

If the frog model survives one can again ask if the origin is visited infinitely many times.

**Theorem 1.11** ([3, Theorem 1.10, Theorem 1.12]). *For $d \geq 1$ consider the frog model $\mathrm{FM}^*(\mathbb{Z}^d, \pi_{sym}, \eta, s)$.*

*(i) If $\mathbb{E}[(\log^+(\eta))^d] < \infty$, then the probability that the origin is visited infinitely many times is 0 for all $s \in [0,1)$.*

*(ii) If there exists $\beta < d$ such that $\mathbb{P}(\eta \geq n) \geq \frac{1}{(\log n)^\beta}$ for all $n$ large enough, then the probability that the origin is visited infinitely many times is positive for all $s \in (0,1]$.*

**Theorem 1.12** ([3, Theorem 1.9, Theorem 1.11]). *For $d \geq 1$ consider the frog model $\mathrm{FM}^*(\mathbb{T}^d, \pi_{sym}, \eta, s)$.*

*(i) If $\mathbb{E}[\eta^\varepsilon] < \infty$ for all $0 < \varepsilon < 1$, then the probability that the origin 0 is visited infinitely many times is 0 for all $s \in [0,1)$.*

*(ii) If there exists $\beta < \frac{\log(d-1)}{2\log d}$ such that $\mathbb{P}(\eta \geq n) \geq \frac{1}{(\log n)^\beta}$ for all $n$ large enough, then the probability that the origin is visited infinitely many times is positive for all $s$ close to 1.*

Note that there are only finitely many returns to the origin if the frog model dies out. Therefore, the probability to have infinitely many returns is less than 1 for every $s < 1$.

### 1.2.3 Frogs with drift

Let us now return to the frog model without death. So far we considered symmetric transition probabilities. From now on we only consider $\mathcal{G} = \mathbb{Z}^d$ and transition probabilities which are balanced in all but in one direction. More precisely, let $\mathcal{E}_d = \{\pm e_j : 1 \leq j \leq d\}$, where $e_j$ denotes the $j$-th standard basis vector in $\mathbb{R}^d$, $j = 1, \dots, d$. The particles move according to the following transition probabilities, which depend on two parameters $w \in [0,1]$ and $\alpha \in [0,1]$:

$$\pi_{w,\alpha}(e) = \begin{cases} \frac{w(1+\alpha)}{2} & \text{for } e = e_1 \\ \frac{w(1-\alpha)}{2} & \text{for } e = -e_1 \\ \frac{1-w}{2(d-1)} & \text{for } e \in \{\pm e_2, \dots, \pm e_d\} \end{cases} \tag{1.2}$$

The parameter $w$ is the weight of the drift direction $e_1$, i.e. the random walk chooses to go in direction $\pm e_1$ with probability $w$. The parameter $\alpha$ describes the strength of the drift, i.e. if the random walk has chosen to move in drift direction, it takes a step in direction

$e_1$ with probability $\frac{1+\alpha}{2}$ and in direction $-e_1$ with probability $\frac{1-\alpha}{2}$. All other directions are balanced and equally probable. Sometimes we need to consider the corresponding one-dimensional model where we have to demand $w = 1$, i.e. the transition probabilities are defined by $\pi_\alpha(e_1) = 1 - \pi_\alpha(-e_1) = \frac{1+\alpha}{2}$.
In dimension $d = 1$ Gantert and Schmidt [31] found a sharp criterion on the distribution of $\eta$ separating transience from recurrence.

**Theorem 1.13** ([31, Theorem 2.2]). *Let* $\alpha \in (0,1)$. *The frog model* $\mathrm{FM}(\mathbb{Z}, \pi_{w,\alpha}, \eta)$ *is recurrent if and only if* $\mathbb{E}[\log^+ \eta] = \infty$

The recurrence part of this result was generalised to any dimension $d \geq 1$ by Döbler and Pfeifroth in [23].

**Theorem 1.14** ([23, Theorem 2.1]). *Let* $d \geq 1$ *and* $\alpha, w \in (0,1)$. *The frog model* $\mathrm{FM}(\mathbb{Z}^d, \pi_{w,\alpha}, \eta)$ *is recurrent if* $\mathbb{E}\big[(\log^+ \eta)^{(d+1)/2}\big] = \infty$.

Kosygina and Zerner also derived a recurrence criterion in [45] in a more genereal model. In our set-up their result can be stated as follows.

**Theorem 1.15** ([45, Theorem 5]). *Let* $d \geq 1$ *and* $\alpha, w \in (0,1)$. *There is a constant* $c = c(d, \alpha, w) > 0$ *such that the frog model* $\mathrm{FM}(\mathbb{Z}^d, \pi_{w,\alpha}, \eta)$ *is recurrent if*

$$\mathbb{P}(\eta \geq n) \geq \frac{c}{(\log n)^d}$$

*for all* $n$ *large enough.*

Note that these results give criteria for recurrence and transience only depending on the distribution of $\eta$. In particular, there are no assumptions on the concrete value of the drift parameters. Indeed, it was conjectured in [31] that in higher dimensions $d \geq 2$, there is also a sharp criterion, independent of the concrete value of the drift parameters, separating transience from recurrence. However, in subsection 1.3.2 we show that this is not true.

## 1.3 Main results

In this section we consider the frog model with drift on $\mathbb{Z}^d$ with initially one sleeping frog at every vertex, i.e. $\eta \equiv 1$. To abbreviate notation we write $\mathrm{FM}(d, \pi_{w,\alpha})$ instead of $\mathrm{FM}(\mathbb{Z}^d, \pi_{w,\alpha}, \eta)$ and $\mathrm{FM}^*(d, \pi_{w,\alpha}, s)$ instead of $\mathrm{FM}^*(\mathbb{Z}^d, \pi_{w,\alpha}, \eta, s)$, respectively. Furthermore, we write $(S_n^x)_{n\in\mathbb{N}}$ instead of $(S_n^x(1))_{n\in\mathbb{N}}$ for the trajectory of the frog initially at $x$. The position of this frog at time $n$ is denoted by $Z_n^x$ instead of $Z_n^x(1)$. In dimension $d = 1$ the frog model is transient for every drift parameter $\alpha > 0$. by Theorem 1.13. Therefore, the cloud of frogs moves away from the origin. In subsection 1.3.1 we investigate how fast the cloud of frogs moves to the right. We therefore consider the leftmost and rightmost active frog and get some results on their speed. In subsection 1.3.2 we consider the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ in higher dimensions $d \geq 2$. We show that there exists a recurrent and transient regime depending on the drift parameters $\alpha$ and $w$ and reveal some interesting differences between $d = 2$ and $d \geq 3$.

### 1.3.1 Frogs with drift on $\mathbb{Z}$

The results in this subsection are joint work with Weidner and published in [36]. We consider the frog model $\mathrm{FM}(1, \pi_{w,\alpha})$. Recall that $\pi(e_1) = \frac{1+\alpha}{2} = 1 - \pi(-e_1)$. To state the results we first need to introduce some more notation. Let $A_n$ denote the set of active frogs at time $n$, i.e. $A_n = \{i \in \mathbb{Z} \colon T_i \leq n\}$. Further, we define $M_n = \max_{i \in A_n} Z_n^i$ and $m_n = \min_{i \in A_n} Z_n^i$. Thus, $M_n$ describes the maximum and $m_n$ the minimum of the locations of the active frogs. We refer to $M_n$ and $m_n$ as the maximum and the minimum. One can show that there are constants $v_{\max}$ and $v_{\min}$ such that

$$v_{\max} = \lim_{n \to \infty} \frac{M_n}{n} \quad \text{a.s.} \tag{1.3}$$

$$v_{\min} = \lim_{n \to \infty} \frac{m_n}{n} \quad \text{a.s.} \tag{1.4}$$

The existence of $v_{\max}$ is well known and stated in Lemma 1.32, and the existence of $v_{\min}$ is part of Theorem 1.16 below. We call $v_{\max}$ the speed of the maximum and $v_{\min}$ the speed of the minimum. We study $v_{\max}$ and $v_{\min}$ as functions of the drift parameter $\alpha$. First, we show that the speed of the minimum equals the speed of a single frog.

**Theorem 1.16.** *For $\alpha > 0$ the speed of the minimum exists and is given by*

$$v_{\min} = \alpha.$$

In the following two theorems we discuss some properties of the speed of the maximum.

**Theorem 1.17.** *The speed of the maximum is an increasing function in $\alpha$.*

**Theorem 1.18.** *For $\alpha < 1$ it holds that $v_{\max} < 1$.*

In comparison to the last result note that for branching random walk on $\mathbb{Z}$ with binary branching the speed of the maximum equals 1 for every $\alpha \geq 0$. This follows for instance from Theorem 2.4.

In addition to studying the behaviour of the minimum and the maximum we investigate the distribution of the active frogs. In the limit they are distributed uniformly inbetween the minimum and the maximum. To make this statement precise, we rescale the positions of all active frogs at time $n$ roughly to the interval $[0, 1]$ and then consider the empirical distribution $\mu_n$, which is defined for $\alpha < 1$ by

$$\mu_n(B) = \frac{1}{|A_n|} \sum_{i \in A_n} \mathbb{1}_{\left\{\frac{Z_n^i - v_{\min} n}{(v_{\max} - v_{\min}) n} \in B\right\}}$$

for every Borel set $B \subseteq [0,1]$. Note that $\mu_n$ is a random measure.

**Theorem 1.19.** *Almost surely, the empirical distribution $\mu_n$ converges weakly to the Lebesgue measure $\lambda$ on $[0,1]$ as $n \to \infty$.*

### 1.3.2 Frogs with drift on $\mathbb{Z}^d$ for $d \geq 2$

The results in this subsection are joint work with Döbler, Gantert, Popov and Weidner and published in [22]. We consider the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ for $d \geq 2$. We show that there exist recurrent and transient regimes depending on the drift parameters $\alpha$ and $w$. First, let us discuss the extreme cases. For $w = 1$ the frog model is one-dimensional and thus transient for any $\alpha \in (0,1]$ by Theorem 1.13 and recurrent for $\alpha = 0$. For $\alpha = 1$ it is transient for any $w \in (0,1]$. More precisely, only frogs in the hyperplane $H_0 := \{x \in \mathbb{Z}^d : x_1 = 0\}$ can be awaked and return to the origin. However, the probability that a frog in $H_0$ ever visits the origin decays exponentially with its distance to the origin. Since the trajectories of the frogs are independent, a Borel-Cantelli argument shows that almost surely only finitely many frogs will ever reach the origin. If $w = 0$, then $\mathrm{FM}(d, \pi_{0,\alpha})$ is equivalent to the symmetric frog model in $d-1$ dimensions and hence recurrent. If $\alpha = 0$, we are back in the balanced case and the model is recurrent. This follows from Theorem 1.20 (i) and Theorem 1.22 below.
In dimension $d = 2$ the frog model is recurrent whenever $\alpha$ or $w$ are sufficiently small, i.e. if the underlying transition mechanism is almost balanced. It is transient for $\alpha$ or $w$ close to 1.

**Theorem 1.20.** *Let $d = 2$ and $w \in (0,1)$.*

*(i) There exists $\alpha_r = \alpha_r(w) > 0$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ is recurrent for all $0 \leq \alpha \leq \alpha_r$.*

*(ii) There exists $\alpha_t = \alpha_t(w) < 1$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ is transient for all $\alpha_t \leq \alpha \leq 1$.*

**Theorem 1.21.** *Let $d = 2$ and $\alpha \in (0,1)$.*

*(i) There exists $w_r = w_r(\alpha) > 0$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ is recurrent for all $0 \leq w \leq w_r$.*

*(ii) There exists $w_t = w_t(\alpha) < 1$ such that the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ is transient for all $w_t \leq w \leq 1$.*

In dimension $d \geq 3$ the frog model is also recurrent if the transition probabilities are almost balanced. Further, for any fixed drift parameter $\alpha \in (0,1]$ it is transient if the weight $w$ is close to 1. However, in contrast to $d = 2$, for fixed $w \in [0,1)$ there is not always a transient regime. This follows from Theorem 1.23 (i) below.

**Theorem 1.22.** *Let $d \geq 3$ and $w \in (0,1)$. There exists $\alpha_r = \alpha_r(d,w) > 0$ such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \leq \alpha \leq \alpha_r$.*

**Theorem 1.23.** *Let $d \geq 3$ and $\alpha \in (0,1)$.*

*(i) There exists $w_r > 0$, independent of $d$ and $\alpha$, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \leq w \leq w_r$.*

*(ii) There exists $w_t = w_t(\alpha) < 1$, independent of $d$, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is transient for all $w_t \leq w \leq 1$.*

These results show that, in contrast to $d = 1$, recurrence and transience do depend on the drift in every dimension $d \geq 2$. This disproves the last conjecture in [31] that some condition on the moments of $\eta$ would separate transience from recurrence as in the one-dimensional case.
The results are graphically summarised in Figure 1.1. Note that the above theorems only make statements about the existence of recurrent, respectively transient regimes. We do not describe their shapes, as might be suggested by the curves depicted in Figure 1.1. However, we believe that there is a monotone curve separating the transient from the recurrent regime.

**Conjecture 1.24.** *For every $d$ there exists a decreasing function $f_d\colon [0,1] \to [0,1]$ such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $w, \alpha \in [0,1]$ such that $w < f_d(\alpha)$ and transient for all $w, \alpha \in [0,1]$ such that $w > f_d(\alpha)$.*

Intuitively, the frog model approximates a binary branching random walk for $d \to \infty$ from below, as each frog activates a new frog in every step if there are 'infinitely' many directions to choose from. This leads to the following conjecture.

**Conjecture 1.25.** *The sequence of functions $(f_d)_{d \in \mathbb{N}}$ is increasing in $d$.*

The comparison with a binary branching random walk raises another question. Let

$$g\colon [0,1] \to [0,1],\ g(\alpha) = \min\{1, (2(1 - \sqrt{1-\alpha^2}))^{-1}\}.$$

A binary branching random walk on $\mathbb{Z}^d$ with transition probabilities as in (1.2) is recurrent if and only if $w < g(\alpha)$, see [30, Section 4].

**Question 1.26.** *Does the sequence $(f_d)_{d \in \mathbb{N}}$ converge pointwise to $g$ as $d \to \infty$?*

## 1.4 Preliminaries

In this section we collect some preliminary results which are needed in the proofs of the main results. First we fix some more notation. We refer to the frog that is initially at

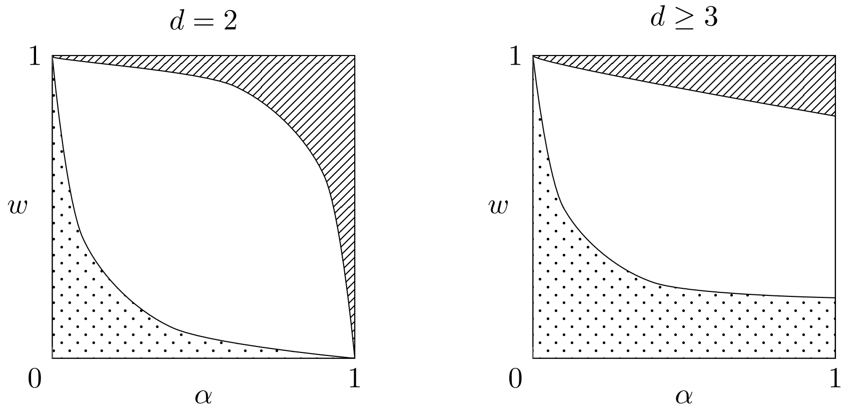


Figure 1.1: Phase diagram for the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$: the recurrent regime is marked by ⬚, the transient one by ▨.

vertex $x \in \mathbb{Z}^d$ as "frog $x$". Furthermore, for the frog model on $\mathbb{Z}$ we often need to talk about the frogs initially on negative sites. To keep the sentence structure simple we refer to them as the negative frogs. Analogously we speak of non-negative and positive frogs. Also for any $k \in \mathbb{Z}$ the frog initially on site $k$ is called frog $k$. For $x, y \in \mathbb{Z}^d$ we write $x \to y$ if frog $x$ (potentially) ever visits $y$, i.e. $y \in \{S_n^x : n \in \mathbb{N}_0\}$. For $A \subseteq \mathbb{Z}^d$ we say that there exists a frog path from $x$ to $y$ in $A$ and write $x \overset{A}{\rightsquigarrow} y$ if there exist $n \in \mathbb{N}$ and $z_1, \dots, z_n \in A$ such that $x \to z_1$, $z_i \to z_{i+1}$ for all $1 \leq i < n$ and $z_n \to y$, or if $x \to y$ directly. Note that $x, y$ are not necessarily in $A$. Also the trajectories of the frogs $z_i$, $1 \leq i \leq n$, do not need to be in $A$. For $x \in \mathbb{Z}^d$ we call the set

$$FC_x = \left\{ y \in \mathbb{Z}^d : x \overset{\mathbb{Z}^d}{\rightsquigarrow} y \right\} \tag{1.5}$$

the frog cluster of $x$. Note that, if frog $x$ ever becomes active, then every frog $y \in FC_x$ is also activated. Observe that, whenever we only deal with recurrence and transience, the exact activation times are not important, but we are only interested in whether or not a frog is activated.

Further, we often use $(d-1)$-dimensional hyperplanes $H_n$ in $\mathbb{Z}^d$ defined by

$$H_n := \{x \in \mathbb{Z}^d : x_1 = n\} \tag{1.6}$$

for $n \in \mathbb{Z}$.

### 1.4.1 Some facts about random walks

We need to deal with hitting probabilities of random walks on $\mathbb{Z}^d$. For $x, y \in \mathbb{Z}^d$ recall that $\{x \to y\}$ denotes the event that the random walk started at $x$ ever visits the vertex $y$. Analogously, for $A \subseteq \mathbb{Z}^d$ we write $\{x \to A\}$ for the event that the random walk started at $x$ ever visits a vertex in $A$.

**Lemma 1.27.** *For $d \geq 3$ and $w \in (0,1)$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,0}$. There exists a constant $c = c(d,w) > 0$ such that for all $x \in \mathbb{Z}^d$*

$$\mathbb{P}(0 \to x) \geq c\|x\|_2^{-(d-2)},$$

*where $\|x\|_2 = \big(\sum_{i=1}^d x_i^2\big)^{1/2}$ is the Euclidean norm.*

A proof of the lemma for the simple random walk, i.e. with transition function $\pi_{\text{sym}}$, can e.g. be found in [4, Theorem 2.4] and [3, Lemma 2.4]. The proof can immediately be generalised to our set-up using [46, Theorem 2.1.3].

**Lemma 1.28.** *For $d \geq 1$ and $\alpha, w \in (0,1)$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,\alpha}$. Then for each $\gamma > 0$ there is a constant $c = c(d,\gamma,w,\alpha) > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ with $x_1 = -n$ and $|x_i| \leq \gamma\sqrt{n}$, $2 \leq i \leq d$, it holds that*

$$\mathbb{P}(x \to 0) \geq cn^{-(d-1)/2}.$$

For a proof see e.g. [23, Lemma 3.1].

**Lemma 1.29.** *For $d \geq 1$ and $\alpha, w \in (0,1]$ consider a random walk on $\mathbb{Z}^d$ with transition function $\pi_{w,\alpha}$. Then for every $n \in \mathbb{N}$ and $H_{-n}$ as defined in* (1.6)

$$\mathbb{P}(0 \to H_{-n}) = \Big(\frac{1-\alpha}{1+\alpha}\Big)^n.$$

*Proof.* As $\mathbb{P}(0 \to H_{-n}) = \mathbb{P}(0 \to H_{-1})^n$ for $n \in \mathbb{N}$, it suffices to prove the lemma for $n = 1$. By the Markov property

$$\mathbb{P}(0 \to H_{-1}) = \frac{1-\alpha}{2} + \frac{1+\alpha}{2}\mathbb{P}(0 \to H_{-2}).$$

The result follows after a straightforward calculation. $\square$

### 1.4.2 Some facts about percolation

To prove recurrence we make use of the theory of independent site percolation on $\mathbb{Z}^d$ and therefore give a brief introduction here. Let $p \in [0,1]$. Every site in $\mathbb{Z}^d$ is independently of the other sites declared open with probability $p$ and closed with probability $1-p$. An open cluster is a connected component of the subgraph induced by all open sites. It is well known that for $d \geq 2$ there is a critical parameter $p_c = p_c(d) \in (0,1)$ such that for all $p > p_c$ (supercritical phase) there is a unique infinite open cluster $C$ almost surely, and for $p < p_c$ (subcritical phase) there is no infinite open cluster almost surely. Furthermore, denoting the open cluster containing the site $x \in \mathbb{Z}^d$ by $C_x$, it holds that

$\mathbb{P}(|C_x| = \infty) > 0$ for $p > p_c$, and $\mathbb{P}(|C_x| = \infty) = 0$ for $p < p_c$ and all $x \in \mathbb{Z}^d$. The following lemma states that the critical probability $p_c$ is small for $d$ large.

**Lemma 1.30.** *For independent site percolation on $\mathbb{Z}^d$,*

$$\lim_{d\to\infty} p_c(d) = 0.$$

Indeed, $p_c(d) = O\big(d^{-1}\big)$ holds. A proof of this result can e.g. be found in [13, Chapter 1, Theorem 7]. Further, in the recurrence proofs we use the fact that an infinite open cluster is "dense" in $\mathbb{Z}^d$. The following weak version of denseness suffices.

**Lemma 1.31.** *Consider supercritical independent site percolation on $\mathbb{Z}^d$. There are constants $a, b > 0$ such that*

$$\mathbb{P}\big(|A \cap C_x| \geq a|A|\big) > b$$

*for all $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$.*

*Proof.* For $a > 0$, $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ the FKG-inequality yields

$$\begin{aligned} \mathbb{P}\big(|A \cap C_x| \geq a|A|\big) &\geq \mathbb{P}\big(x \in C,\ |A \cap C| \geq a|A|\big) \\ &\geq \mathbb{P}(x \in C) \cdot \mathbb{P}\big(|A \cap C| \geq a|A|\big). \end{aligned}$$

Note that $\gamma := \mathbb{P}(x \in C) \in (0,1)$ (and $\gamma$ does not depend on $x$) since the percolation is supercritical. By the Markov inequality

$$\begin{aligned} \mathbb{P}\big(|A \cap C| \geq a|A|\big) &= 1 - \mathbb{P}\big(|A \cap C^c| \geq (1-a)|A|\big) \\ &\geq 1 - \frac{\mathbb{E}\big[|A \cap C^c|\big]}{(1-a)|A|} \\ &= 1 - \frac{1}{(1-a)|A|} \sum_{y \in A} \mathbb{P}(y \in C^c) \\ &= 1 - \frac{1-\gamma}{1-a} > 0, \end{aligned}$$

for $a$ small enough, which finishes the proof. □

### 1.4.3 Some results about frogs

The existence of the speed of the maximum defined in (1.3) is proved using Liggett's Subadditive Ergodic Theorem. Indeed, this theorem yields more information which we use in subsection 1.5.1. We summarise it in the following lemma.

**Lemma 1.32.** *Consider the frog model* $\mathrm{FM}(1, \pi_{w,\alpha})$. *For each $\alpha \in [0,1]$ there exists a positive constant $v_{\max}$ such that*

$$v_{\max} = \lim_{n\to\infty} \frac{M_n}{n} \quad a.s.$$

*Furthermore,*

$$v_{\max}^{-1} = \lim_{i\to\infty} \frac{T_i}{i} = \lim_{i\to\infty} \frac{\mathbb{E}[T_i]}{i} = \inf_{i\in\mathbb{N}} \frac{\mathbb{E}[T_i]}{i} \quad a.s. \tag{1.7}$$

*Proof.* Let $T_{i,j}$ denote the activation time of the frog at site $j$ when initially there is one active frog at site $i$ and one sleeping frog at every other site. An application of Liggett's Subadditive Ergodic Theorem (see e.g. [48]) to the times $(T_{i,j})_{i,j\in\mathbb{Z}}$ shows the existence of a positive constant $v_{\max}$ such that (1.7) holds. For $\alpha = 0$ this is proved for a more general model by Alves et al. in [4]. In our setting their argument immediately applies to $\alpha > 0$ as well.
By a standard argument it now follows that $\lim_{n\to\infty} \frac{M_n}{n}$ exists almost surely: There exists a unique random sequence $(k_n)_{n\in\mathbb{N}}$ with values in $\mathbb{N}_0$ such that $T_{k_n} \le n < T_{k_n+1}$. Note that $\lim_{n\to\infty} k_n = \infty$. Hence,

$$\lim_{n\to\infty} \frac{T_n}{n} = \lim_{n\to\infty} \frac{T_{k_n}}{k_n} = \lim_{n\to\infty} \frac{n}{k_n} \quad \text{a.s.}$$

Obviously, $k_n - (n - T_{k_n}) \le M_n \le k_n$. This implies

$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{k_n} \cdot \frac{k_n}{n}\right) \le \frac{M_n}{n} \le \frac{k_n}{n}.$$

Taking limits yields the claim. □

Further, we need a result on the frog model with death $\mathrm{FM}^*(d,\pi,s)$ for $s \in [0,1]$. We denote frog clusters in the frog model with death by $FC^*$, analogous to the notation introduced in (1.5) for the frog model without death.

**Lemma 1.33.** *For* $\mathrm{FM}(1,\pi_{1,\alpha})$ *with* $\alpha > 0$ *and* $\mathrm{FM}^*(1,\pi_{sym},s)$ *with* $s < 1$ *there is* $c > 0$ *such that* $\mathbb{P}(0 \overset{\mathbb{Z}}{\rightsquigarrow} -n) \le \mathrm{e}^{-cn}$ *for all* $n \in \mathbb{N}$.

*Proof.* Let $p$ be the probability that a frog starting from 0 ever hits the vertex $-1$. In both models we have $p < 1$. Obviously, as $s < 1$, this is true for $\mathrm{FM}^*(d,\pi_{\mathrm{sym}},s)$. For $\mathrm{FM}(1,\pi_{1,\alpha})$ it follows from Lemma 1.29.
For $n \in \mathbb{N}$ define $Y_n = |\{m > -n \colon m \to -n\}|$ if $-n \in FC_0$, respectively $-n \in FC_0^*$. Otherwise set $Y_n = 0$. If $-n$ is visited by active frogs, then $Y_n$ counts the number of frogs to the right of $-n$ that potentially ever reach $-n$. The process $(Y_n)_{n\in\mathbb{N}}$ is a Markov chain on $\mathbb{N}_0$ with

$$Y_{n+1} = \begin{cases} 0 & \text{if } Y_n = 0, \\ \mathrm{Binomial}(Y_n+1,p) & \text{if } Y_n > 0. \end{cases}$$

Note that $\mathbb{P}(0 \overset{\mathbb{Z}}{\rightsquigarrow} -n) = \mathbb{P}(Y_n > 0)$ by definition. A straightforward calculation shows that there is $k_0 \in \mathbb{N}$ such that $\mathbb{P}(Y_{n+1} < Y_n \mid Y_n = k) > \frac{2}{3}$ for all $k \ge k_0$. Hence, we can dominate the Markov chain $(Y_n)_{n\in\mathbb{N}}$ by the Markov chain $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ on $\{0, k_0, k_0+1, \ldots\}$

with transition probabilities

$$\mathbb{P}(\widetilde{Y}_{n+1} = l \mid \widetilde{Y}_n = k) = \begin{cases} \frac{1}{3} & \text{if } l = k+1,\ k > k_0, \\ \frac{2}{3} & \text{if } l = k-1,\ k > k_0, \\ (1-p)^{k_0+1} & \text{if } l = 0,\ k = k_0, \\ 1-(1-p)^{k_0+1} & \text{if } l = k+1,\ k = k_0, \\ 1 & \text{if } l = k = 0 \end{cases}$$

for all $n \in \mathbb{N}$ and starting point $\widetilde{Y}_1 = \max\{Y_1, k_0\}$. Obviously, for all $n \in \mathbb{N}$ we have $\mathbb{P}(Y_n > 0) \le \mathbb{P}(\widetilde{Y}_n > 0)$. Let $T_k = \min\{n \in \mathbb{N} \colon \widetilde{Y}_n = k\}$ and $T_{k,l} = T_l - T_k$. Note that $\mathbb{P}(\widetilde{Y}_n > 0) = \mathbb{P}(T_0 > n)$. For $t > 0$ the Markov inequality implies

$$\begin{aligned} \mathbb{P}(T_0 > n) &= \mathbb{P}\Bigg(\sum_{k=k_0}^{\widetilde{Y}_1 - 1} T_{k+1,k} + T_{k_0,0} > n\Bigg) \\ &\le \mathrm{e}^{-tn}\mathbb{E}\Bigg[\exp\Bigg(t\sum_{k=k_0}^{\widetilde{Y}_1 - 1} T_{k+1,k} + tT_{k_0,0}\Bigg)\Bigg] \\ &= \mathrm{e}^{-tn}\sum_{l=k_0}^{\infty}\prod_{k=k_0}^{l-1}\mathbb{E}\big[\exp(tT_{k+1,k})\big]\mathbb{E}\big[\exp(tT_{k_0,0})\big]\mathbb{P}(\widetilde{Y}_1 = l) \\ &= \mathrm{e}^{-tn}\sum_{l=0}^{\infty}\mathbb{E}\big[\exp(tT_{k_0+1,k_0})\big]^l\mathbb{E}\big[\exp(tT_{k_0,0})\big]\mathbb{P}(\widetilde{Y}_1 = l + k_0). \end{aligned} \tag{1.8}$$

$\widetilde{Y}_1$ can only be equal to $l + k_0$ if at least one frog to the right of $l-1$ reaches $-1$. Thus,

$$\mathbb{P}(\widetilde{Y}_1 = l + k_0) \le \sum_{i=l}^{\infty} p^{i+1} = p^l \frac{p}{1-p}. \tag{1.9}$$

Now, we choose $t > 0$ small enough such that $\mathbb{E}\big[\exp(tT_{k_0+1,k_0})\big] < p^{-1}$. Then (1.9) shows that the sum in (1.8) is finite, which yields the claim. $\square$

### 1.4.4 A lemma on Bernoulli random variables

We will repeatedly use the following simple lemma. Note that the random variables in this lemma do not have to be independent.

**Lemma 1.34.** *For $i \in \mathbb{N}$ let $X_i$ be a Bernoulli($p_i$) random variable with $\inf_{i\in\mathbb{N}} p_i =: p$. Then for every $a > 0$ and $n \in \mathbb{N}$*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \ge a\right) \ge p - a.$$

*Proof.* Since $\mathbb{E}[X_i] \geq p$ and $\frac{1}{n}\sum_{i=1}^{n} X_i \leq 1$, we have

$$p \leq \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \leq \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) + a,$$

which yields the claim. □

## 1.5 Proof of the main results

### 1.5.1 Frogs with drift on $\mathbb{Z}$

In this subsection we consider the frog model $\mathrm{FM}(1, \pi_{1,\alpha})$. In order to prove Theorem 1.16 we compare the frogs initially on non-negative sites with independent random walks. The speed of the minimum of independent random walks can be computed explicitly which is done in the first of the following lemmas. Then it remains to deal with the frogs initially on negative sites. Luckily, they can be ignored due to the transience of the frog model for all $\alpha > 0$, see Theorem 1.13.
Let $\{(\tilde{S}_n^i)_{n\in\mathbb{N}_0} \colon i \in \mathbb{Z}\}$ be a family of independent random walks starting at 0.

**Lemma 1.35.** *It holds that*

$$\lim_{n\to\infty} \frac{1}{n} \min_{i\in\{-n,\dots,n\}} \tilde{S}_n^i = \alpha \quad a.s.$$

*Proof.* We only need to prove $\liminf_{n\to\infty} \frac{1}{n} \min_{i\in\{-n,\dots,n\}} \tilde{S}_n^i \geq \alpha$. For all $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}\left(\frac{1}{n} \min_{i\in\{-n,\dots,n\}} \tilde{S}_n^i \leq \alpha - \varepsilon\right) &= \mathbb{P}\left(\bigcup_{i=-n}^{n} \left\{\frac{\tilde{S}_n^i}{n} \leq \alpha - \varepsilon\right\}\right) \\ &\leq (2n+1)\mathbb{P}\left(\frac{S_n^0}{n} \leq \alpha - \varepsilon\right). \end{aligned}$$

By Cramér's Theorem the probability in the last term of this calculation decays exponentially fast in $n$. Hence, it is summable. An application of the Borel-Cantelli Lemma and letting $\varepsilon \to 0$ completes the proof.

This result now enables us to prove a formula for the speed of the minimum of the non-negative frogs.

**Lemma 1.36.** *Let $A_n^+ = \{i \geq 0 \colon T_i \leq n\}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \min_{i\in A_n^+} Z_n^i = \alpha \quad a.s.$$

Before proving Lemma 1.36 we make another short observation. Obviously $v_{\max}$ is at least as big as the speed of a single frog, i.e. $v_{\max} \geq \alpha$. In fact, this inequality is strict for all $\alpha \geq 0$. For $\alpha = 0$ this fact is known from [4].

**Lemma 1.37.** *For $\alpha \in (0,1)$ it holds that $v_{\max} > \alpha$.*

*Proof.* Let $T_1^s = \inf\{n \in \mathbb{N} \colon S_n^0 = 1\}$ be the hitting time of the point 1 of a single simple random walk with drift parameter $\alpha$. The key point in this proof is to notice that $\mathbb{E}[T_1] < \mathbb{E}[T_1^s]$. Hence, by Lemma 1.32

$$v_{\max}^{-1} = \inf_{i \in \mathbb{N}} \frac{\mathbb{E}[T_i]}{i} \leq \mathbb{E}[T_1] < \mathbb{E}[T_1^s] = \frac{1}{\alpha}.$$

One can of course find better lower bounds for the speed of the maximum by estimating $\mathbb{E}[T_i]$ for $i \geq 1$, but this is not done in this thesis.

*Proof of Lemma 1.36.* It is enough to show $\liminf_{n\to\infty} \frac{1}{n} \min_{i \in A_n^+} Z_n^i \geq \alpha$ almost surely. In this proof we use a slightly different but equivalent way of defining the movement of the frogs. For every $i \in \mathbb{Z}$ define the position of frog $i$ at time $n$ as

$$\widetilde{Z}_n^i = \begin{cases} i & \text{for } n < \tilde{T}_i, \\ i + \tilde{S}_n^i - \tilde{S}_{\tilde{T}_i}^i & \text{for } n \geq \tilde{T}_i, \end{cases}$$

where $\tilde{T}_i$ denotes the activation time of frog $i$. Note that $(\widetilde{Z}_n^i)$ equals $(Z_n^i)$ in distribution. We now want to compare the trajectory $(\widetilde{Z}_n^i)_{n \in \mathbb{N}_0}$ of each frog with the trajectory $(\tilde{S}_n^i)_{n \in \mathbb{N}_0}$ of the corresponding random walk. From time $\tilde{T}_i$ onwards they move synchronously by the above definition. Therefore, we only need to compare their locations at time $\tilde{T}_i$. Note that $\widetilde{Z}_{\tilde{T}_i}^i = i$ and define $G = \{i \geq 0 \colon S_{\tilde{T}_i}^i \leq i\}$ to be the set of good frogs. Now $i \in A_n^+ \cap G$ implies $\tilde{S}_n^i \leq \widetilde{Z}_n^i$ for all $n \in \mathbb{N}$, i.e. all good frogs stay to the right of their corresponding random walk. Hence,

$$\begin{aligned} \min_{i \in A_n^+} \widetilde{Z}_n^i &\geq \min_{i \in A_n^+} \tilde{S}_n^i - \sum_{i \in G^c \cap A_n^+} \left(\tilde{S}_n^i - \widetilde{Z}_n^i\right) \\ &\geq \min_{i \in A_n^+} \tilde{S}_n^i - \sum_{i \in G^c} \left(\tilde{S}_{\tilde{T}_i}^i - i\right). \end{aligned} \tag{1.10}$$

We claim that the set $G^c$ is finite almost surely. For $\alpha = 1$ this is obviously true. For $\alpha < 1$ it is enough to show that

$$\lim_{i \to \infty} \frac{S_{\tilde{T}_i}^i - i}{\tilde{T}_i} = \alpha - v_{\max} \quad \text{a.s.} \tag{1.11}$$

since by Lemma 1.37 the last term is strictly negative and hence $S_{\tilde{T}_i}^i - i > 0$ can occur only for finitely many $i \geq 0$ almost surely.

Note that $(\tilde{S}^i_n)_{n\le\tilde{T}_i}$ is independent of the movement of the frogs up to time $\tilde{T}_i$. Therefore, $S^i_{\tilde{T}_i}$ equals $S^0_{\tilde{T}_i}$ in distribution. Using a standard large deviation estimate we get for every $\varepsilon > 0$

$$\mathbb{P}\Big(\frac{S^i_{\tilde{T}_i}}{\tilde{T}_i} \le \alpha - \varepsilon\Big) = \mathbb{P}\Big(\frac{S^0_{\tilde{T}_i}}{\tilde{T}_i} \le \alpha - \varepsilon\Big) \le \mathbb{E}\big[\mathrm{e}^{-c\tilde{T}_i}\big] \le \mathrm{e}^{-ci},$$

where $c = c(\varepsilon, \alpha) > 0$ is a constant. By symmetry also $\mathbb{P}\Big(\frac{S^i_{\tilde{T}_i}}{\tilde{T}_i} \ge \alpha + \varepsilon\Big)$ decays exponentially fast in $i$. An application of the Borel-Cantelli Lemma and letting $\varepsilon \to 0$ thus shows

$$\lim_{n\to\infty} \frac{S^i_{\tilde{T}_i}}{\tilde{T}_i} = \alpha \quad \text{a.s.}$$

Further, we know from Lemma 1.32 that $\lim_{i\to\infty} \frac{i}{\tilde{T}_i} = v_{\max}$ almost surely. This proves equation (1.11) which implies that $G^c$ is finite almost surely.
Therefore, the second term on the right side in inequality (1.10) is finite almost surely. Also note that it does not depend on $n$. Thus,

$$\liminf_{n\to\infty} \frac{1}{n} \min_{i\in A_n^+} \widetilde{Z}^i_n \ge \liminf_{n\to\infty} \frac{1}{n} \min_{i\in A_n^+} \tilde{S}^i_n \quad \text{a.s.}$$

As $A_n^+ \subseteq \{-n, \ldots, n\}$ an application of Lemma 1.35 finishes the proof.

*Proof of Theorem 1.16.* By Theorem 1.13 the frog model with drift $\mathrm{FM}(1, \pi_{1,\alpha})$ is transient for all $\alpha > 0$. This means that the origin is visited by only finitely many frogs almost surely. Therefore only finitely many negative frogs are ever activated. Hence, Theorem 1.16 follows from Lemma 1.36.

Next we prove that the speed of the maximum is an increasing function in the drift parameter $\alpha$. Though this statement might at first seem obvious, no direct coupling of the frog models for different drift parameters seems possible, since for smaller values of $\alpha$ more negative frogs will eventually be woken up, which might help in pushing the front forward. But we can ignore all these frogs without changing the speed of the maximum, similar to the proof of Theorem 1.16. This is shown in the next lemma. We therefore consider the frog model without negative frogs. It evolves in the same way as our usual frog model, but has another initial configuration. Here we assume that there is one sleeping frog at every positive integer, one active frog at 0 and no frogs on negative sites. We denote the activation time of the $i$-th frog in the frog model without negative frogs by $T_i^+$.

**Lemma 1.38.** *It holds that*

$$v_{\max}^{-1} = \lim_{n\to\infty} \frac{T_n^+}{n} \quad a.s.$$

*Proof.* We only need to prove $\limsup_{n\to\infty} \frac{T_n^+}{n} \leq v_{\max}^{-1}$ almost surely. First, we show that the speed of the maximum of all negative frogs in the usual frog model equals $\alpha$ almost surely, i.e. setting $A_n^- = \{i < 0 \colon T_i \leq n\}$ we prove that

$$\lim_{n\to\infty} \frac{1}{n} \max_{i\in A_n^-} Z_n^i = \alpha \quad \text{a.s.} \tag{1.12}$$

For $\alpha > 0$ only finitely many negative frogs will ever be activated almost surely as remarked in the proof of Theorem 1.16. In this case equation (1.12) is thus obvious.
If $\alpha = 0$, then by symmetry the claim follows from Lemma 1.36.
Let $E$ be the set of all positive frogs which are activated by negative frogs, meaning that at the time of their activation at least one negative frog is present. Since $v_{\max} > \alpha$ as proved in Lemma 1.37 and by equation (1.12) the set $E$ is finite almost surely.
Hence, $T = \sup_{i\in E}(T_i^+ - T_i)$ is an almost surely finite random variable. For all $i \in E$ we thus have $T_i^+ \leq T_i + T$. Actually, this inequality is true for all $i \in \mathbb{N}_0$, which immediately implies the claim of the lemma.
The inequality can e.g. be proven inductively. For $i = 0$ the inequality is obviously true as $T_0^+ = T_0 = 0$. Now assume that $i \in \mathbb{N}$ and $T_j^+ \leq T_j + T$ holds for all $0 \leq j \leq i-1$. If $i \in E$, there is nothing to show. Otherwise, let $0 \leq k \leq i-1$ be the (random) frog that activates the frog $i$ in the normal version of the model. Then we have

$$T_i^+ \leq T_k^+ + (T_i - T_k) \leq T_i + T.$$

Note here that in both models the frogs follow the same paths, they might just be activated at different times.

*Proof of Theorem 1.17.* Using a standard coupling of the random variables $(X_k^i)_{i\in\mathbb{Z},k\in\mathbb{N}}$ we can achieve that $T_i^+(\alpha)$ is monotone decreasing in $\alpha$. As $v_{\max}(\alpha) = \lim_{n\to\infty} \frac{n}{T_n^+}$ almost surely by Lemma 1.38 we conclude that $v_{\max}(\alpha)$ is increasing in $\alpha$.

In order to bound the speed of the maximum from above we prove an upper bound for the number of frogs in the maximum. We do this for a slightly modified frog model: Each time the maximum moves to the left we put a sleeping frog at the site that has just been left by the maximum. Hence, in this new model there is one sleeping frog at every site to the right of the maximum at any time. Further notice that, except at time 0, there are always at least two frogs in the maximum. We use the same notation as in the usual frog model, but add an index "mod" when referring to the modified model. Further, let $a_n$ denote the number of frogs in the maximum in the modified frog model.

**Lemma 1.39.** *For $\alpha \in (0,1)$ and all $n \in \mathbb{N}$ it holds that*

$$\mathbb{E}[a_n] \leq \frac{(3-\alpha)(1+\alpha)}{2\alpha(1-\alpha)}.$$

*Proof.* To increase the readability of this proof let $p := \frac{1+\alpha}{2}$ be the probability that a frog takes a step to the right. We prove bounds not only for the number of frogs in the maximum, but for every other site as well. Therefore, let $a_n(k)$ be the number of frogs at location $M_n^{\mathrm{mod}} - 2k$ for $k, n \in \mathbb{N}_0$. We prove by induction over $n$ that for all $n, k \in \mathbb{N}_0$

$$\mathbb{E}[a_n(k)] \leq \frac{(2-p)p}{(1-p)(2p-1)p^k}. \tag{1.13}$$

For $n = 0$ and $n = 1$ one easily checks that the claim is true. Assume that the claim holds for some integer $n \in \mathbb{N}$.
First we show inequality (1.13) for $k = 0$. Distinguishing whether all $a_n$ particles in the maximum at time $n$ move to the left or not in the next step one calculates

$$\begin{aligned}\mathbb{E}[a_{n+1}] &= \mathbb{E}\big[(1-p)^{a_n}\big(a_n + pa_n(1)\big)\big] \\ &\quad + \mathbb{E}\Big[\big(1-(1-p)^{a_n}\big)\Big(\frac{pa_n}{1-(1-p)^{a_n}} + 1\Big)\Big] \\ &= \mathbb{E}\big[(1-p)^{a_n}\big(a_n + pa_n(1) - 1\big) + pa_n + 1\big].\end{aligned}$$

Note here that the expectation of a binomial random variable with parameters $p > 0$ and $k \in \mathbb{N}$ conditioned on being at least 1 is given by $\frac{pk}{1-(1-p)^k}$. Using $a_n \geq 2$ yields

$$\mathbb{E}[a_{n+1}] \leq (1-p)^2\mathbb{E}\big[a_n + pa_n(1) - 1\big] + p\mathbb{E}[a_n] + 1. \tag{1.14}$$

Inserting the induction hypothesis (1.13) in (1.14) the claim follows after a straightforward calculation.
For $k = 1$ an analogous calculation yields

$$\begin{aligned}\mathbb{E}[a_{n+1}(1)] &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(2) + (1-p)a_n(1)\big)\big] \\ &\quad + \mathbb{E}\Big[\big(1-(1-p)^{a_n}\big)\Big(a_n - \frac{pa_n}{1-(1-p)^{a_n}} + pa_n(1)\Big)\Big] \\ &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(2) - (\alpha)a_n(1) - a_n\big)\big] \\ &\quad + \mathbb{E}\big[(1-p)a_n + pa_n(1)\big].\end{aligned} \tag{1.15}$$

For $k \geq 2$ one gets

$$\begin{aligned}\mathbb{E}[a_{n+1}(k)] &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(k+1) + (1-p)a_n(k)\big)\big] \\ &\quad + \mathbb{E}\big[\big(1-(1-p)^{a_n}\big)\big(pa_n(k) + (1-p)a_n(k-1)\big)\big] \\ &= \mathbb{E}\big[(1-p)^{a_n}\big(pa_n(k+1) - (\alpha)a_n(k) - (1-p)a_n(k-1)\big)\big] \\ &\quad + \mathbb{E}\big[(1-p)a_n(k-1) + pa_n(k)\big].\end{aligned} \tag{1.16}$$

Thus, for $k \geq 1$ equations (1.15) and (1.16) imply

$$\mathbb{E}[a_{n+1}(k)] \leq p(1-p)^2\mathbb{E}[a_n(k+1)] + p\mathbb{E}[a_n(k)] + (1-p)\mathbb{E}[a_n(k-1)]. \tag{1.17}$$

As before, inserting the induction hypothesis (1.13) into inequality (1.17) completes the proof.

*Proof of Theorem 1.18.* Consider the event that in the modified frog model at time $n$ all $a_n$ frogs sitting in the maximum move to the left. Using Jensen's inequality and Lemma 1.39, we conclude that the probability of this event is bounded from below by

$$\mathbb{E}\Big[\Big(\frac{1-\alpha}{2}\Big)^{a_n}\Big] \geq \Big(\frac{1-\alpha}{2}\Big)^{\mathbb{E}[a_n]} \geq \Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2\alpha(1-\alpha)}}.$$

Therefore, for all $n \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\big[T_{n+1}^{\mathrm{mod}} - T_n^{\mathrm{mod}}\big] &\geq 1 + 2\,\mathbb{E}\Big[\Big(\frac{1-\alpha}{2}\Big)^{a_n}\Big] \\ &\geq 1 + 2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2\alpha(1-\alpha)}}.\end{aligned}$$

Clearly, in the modified model, frogs are activated no later than in the normal version of the frog model. Thus,

$$\mathbb{E}[T_n] \geq \mathbb{E}\big[T_n^{\mathrm{mod}}\big] = \sum_{k=1}^{n} \mathbb{E}\big[T_k^{\mathrm{mod}} - T_{k-1}^{\mathrm{mod}}\big] \geq \Big(1 + 2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2\alpha(1-\alpha)}}\Big)n.$$

By Lemma 1.32 we conclude

$$v_{\max}^{-1} = \inf_{n\in\mathbb{N}} \frac{\mathbb{E}[T_n]}{n} \geq 1 + 2\Big(\frac{1-\alpha}{2}\Big)^{\frac{(3-\alpha)(1+\alpha)}{2\alpha(1-\alpha)}} > 1.$$

It remains to prove Theorem 1.19. The idea of the proof is quite simple: From the point of view of the minimum the front moves with a positive speed, but all the frogs only fluctuate around their locations with $\sqrt{n}$, so basically they stay where they are.
First, we show that for large enough times $n$ all active frogs do not deviate much from their expected locations. More precisely, let $G_n = \{i \in A_n \colon \big|Z_n^i - \mathbb{E}[Z_n^i]\big| < n^{3/4}\}$.

**Lemma 1.40.** *Almost surely, $G_n = A_n$ for all $n$ large enough.*

*Proof.* As $A_n \subseteq \{-n, \dots, n\}$ we have

$$\begin{aligned}
\mathbb{P}\big(A_n \neq G_n\big) &= \mathbb{P}\Big(\bigcup_{i \in A_n} \big\{\big|Z_n^i - \mathbb{E}[Z_n^i]\big| \geq n^{3/4}\big\}\Big) \\
&\leq \sum_{i=-n}^{n} \mathbb{P}\big(\big|Z_n^i - \mathbb{E}[Z_n^i]\big| \geq n^{3/4}\big) \\
&= \sum_{i=-n}^{n} \sum_{k=0}^{n} \mathbb{P}\big(\big|Z_n^i - \mathbb{E}[Z_n^i]\big| \geq n^{3/4} \big| T_i = k\big) \cdot \mathbb{P}(T_i = k). \qquad (1.18)
\end{aligned}$$

Further, for every $i \in \mathbb{Z}$ and $0 \leq k \leq n$ it holds that

$$\begin{aligned}
\mathbb{P}\big(\big|Z_n^i - \mathbb{E}[Z_n^i]\big| \geq n^{3/4} \big| T_i = k\big) &= \mathbb{P}\big(\big|S_{n-k}^i - \mathbb{E}[S_{n-k}^i]\big| \geq n^{3/4}\big) \\
&\leq 2 \exp\Big(-\frac{n^{3/2}}{4(n-k)}\Big) \\
&\leq 2 \exp\Big(-\frac{n^{1/2}}{4}\Big).
\end{aligned}$$

In the first inequality in the above estimate we use Höffding's inequality. Thus, (1.18) implies

$$\mathbb{P}\big(A_n \neq G_n\big) \leq 2 \exp\Big(-\frac{n^{1/2}}{4}\Big) \sum_{i=-n}^{n} \sum_{k=0}^{n} \mathbb{P}(T_i = k) \leq 2(2n+1) \exp\Big(-\frac{n^{1/2}}{4}\Big)$$

which is summable. An application of the Borel-Cantelli Lemma completes the proof.

For $\varepsilon > 0$ and $x \in [0, 1]$ define

$$L_n(x, \varepsilon) = \begin{cases} \big\{i \in \mathbb{Z}\colon -(v_{\max} - \varepsilon)n \leq i \leq \big((2x-1)v_{\max} - \varepsilon\big)n\big\} & \text{for } \alpha = 0, \\ \big\{i \in \mathbb{Z}\colon 0 \leq i \leq (xv_{\max} - \varepsilon)n\big\} & \text{for } \alpha > 0 \end{cases}$$

and

$$R_n(x, \varepsilon) = \begin{cases} \big\{i \in \mathbb{Z}\colon \big((2x-1)v_{\max} + \varepsilon\big)n \leq i \leq (v_{\max} - \varepsilon)n\big\} & \text{for } \alpha = 0, \\ \big\{i \in \mathbb{Z}\colon (xv_{\max} + \varepsilon)n \leq i \leq (v_{\max} - \varepsilon)n\big\} & \text{for } \alpha > 0. \end{cases}$$

**Lemma 1.41.** *For $n$ large enough, $i \in L_n(x, \varepsilon) \cap G_n$ implies*

$$\frac{Z_n^i - v_{\min} n}{(v_{\max} - v_{\min})n} \leq x, \qquad (1.19)$$

*whereas* $i \in R_n(x,\varepsilon) \cap G_n$ *implies*

$$\frac{Z_n^i - v_{\min} n}{(v_{\max} - v_{\min})n} \geq x. \tag{1.20}$$

*Proof.* For $\alpha = 0$ note that by symmetry $v_{\min} = -v_{\max}$. Thus, (1.19) holds if and only if $Z_n^i \leq (2x-1)v_{\max} n$. Assume $i \in L_n(x,\varepsilon) \cap G_n$. A straightforward calculation shows

$$Z_n^i \leq \mathbb{E}[Z_n^i] + n^{3/4} = i + n^{3/4} \leq (2x-1)v_{\max} n$$

for $n$ big enough. Analogously, one shows (1.20) in this case.
For $\alpha > 0$ the proof works essentially in the same way as for $\alpha = 0$, but the estimation of $\mathbb{E}[Z_n^i]$ is less trivial. We have $\mathbb{E}[Z_n^i] = i + (n - \mathbb{E}[T_i])v_{\min}$. For $i \in L_n(x,\varepsilon) \cap G_n$ we thus get

$$Z_n^i \leq \mathbb{E}[Z_n^i] + n^{3/4} = v_{\min} n + \frac{i}{v_{\max}}\Big(v_{\max} - \frac{\mathbb{E}[T_i]}{i} v_{\min} v_{\max}\Big) + n^{3/4}.$$

Lemma 1.32 yields that $\frac{\mathbb{E}[T_i]}{i} \geq \inf_{i \in \mathbb{N}} \frac{\mathbb{E}[T_i]}{i} = v_{\max}^{-1}$. Hence, for $n$ big enough

$$\begin{aligned} Z_n^i &\leq v_{\min} n + \frac{i}{v_{\max}}(v_{\max} - v_{\min}) + n^{3/4} \\ &\leq v_{\min} n + x(v_{\max} - v_{\min})n, \end{aligned}$$

as claimed in (1.19). On the other hand, $i \in R_n(x,\varepsilon) \cap G_n$ analogously implies

$$Z_n^i \geq v_{\min} n + \frac{i}{v_{\max}}\Big(v_{\max} - \frac{\mathbb{E}[T_i]}{i} v_{\min} v_{\max}\Big) - n^{3/4}.$$

Since $\lim_{i\to\infty} \frac{\mathbb{E}[T_i]}{i} = v_{\max}^{-1}$ and $i$ tends to infinity whenever $n$ does by the definition of $R_n(x,\varepsilon)$, we know that $\frac{\mathbb{E}[T_i]}{i} \leq v_{\max}^{-1} + \delta\varepsilon$ for $n$ big enough and a small constant $\delta$. Therefore,

$$Z_n^i \geq v_{\min} n + \frac{i}{v_{\max}}(v_{\max} - v_{\min} - \varepsilon\delta v_{\min} v_{\max}) - n^{3/4}.$$

Using $i \geq (x v_{\max} + \varepsilon)n$ and choosing $\delta$ small enough finishes the proof.

*Proof of Theorem 1.19.* We need to show that $\lim_{n\to\infty} \mu_n([0,x]) = \lambda([0,x])$ for every $x \in [0,1]$ almost surely.
Take a realisation of the frog model such that $A_n = G_n$ holds for sufficiently large $n$, that $\lim_{n\to\infty} \frac{M_n}{n} = v_{\max}$ and $\lim_{n\to\infty} \frac{m_n}{n} = v_{\min}$, and finally that $A_n \cap \mathbb{Z}^-$ is finite. This happens with probability 1 as we have seen in Lemma 1.40, Lemma 1.32, Theorem 1.16 and previous discussions about the transience of the frog model. Now fix $x \in [0,1]$ and

$\varepsilon > 0$ small. Lemma 1.41 yields that, for $n$ large enough,

$$\mu_n([0,x]) \geq \frac{1}{|A_n|}|G_n \cap L_n(x,\varepsilon)| = \frac{n}{|A_n|} \cdot \frac{|L_n(x,\varepsilon)|}{n}. \tag{1.21}$$

In (1.21) we used that $L_n(x,\varepsilon) \subseteq A_n$ for sufficiently large $n$ as $\lim_{n\to\infty} \frac{M_n}{n} = v_{\max}$. The definition of $L_n(x,\varepsilon)$ implies

$$|L_n(x,\varepsilon)| \geq \begin{cases} 2(xv_{\max} - \varepsilon)n & \text{for } p = \frac{1}{2}, \\ (xv_{\max} - \varepsilon)n & \text{for } p > \frac{1}{2}. \end{cases}$$

Further, $\lim_{n\to\infty} \frac{n}{|A_n|} = \frac{1}{2}v_{\max}^{-1}$ for $\alpha = 0$, respectively $\lim_{n\to\infty} \frac{n}{|A_n|} = v_{\max}^{-1}$ for $\alpha > 0$. Thus, the limit inferior of the last term in (1.21) as $n \to \infty$ is bounded from below by $x - \varepsilon v_{\max}^{-1}$. Since $\varepsilon > 0$ was chosen arbitrarily we conclude

$$\liminf_{n\to\infty} \mu_n([0,x]) \geq x.$$

On the other hand, Lemma 1.41 shows that, for $n$ large enough,

$$\mu_n([0,x]) \leq \frac{1}{|A_n|}\big|A_n \setminus \big(G_n \cap R_n(x,\varepsilon)\big)\big| = 1 - \frac{n}{|A_n|} \cdot \frac{|R_n(x,\varepsilon)|}{n} \tag{1.22}$$

since $A_n = G_n$ and $R_n(x,\varepsilon) \subseteq A_n$ for $n$ big enough. By the definition of $R_n(x,\varepsilon)$ we have

$$|R_n(x,\varepsilon)| \geq \begin{cases} 2\big((1-x)v_{\max} - \varepsilon\big)n & \text{for } p = \frac{1}{2}, \\ \big((1-x)v_{\max} - 2\varepsilon\big)n & \text{for } p > \frac{1}{2}. \end{cases}$$

Analogous to the above estimation this yields that the limit superior of the right hand side of (1.22) is bounded from above by $x + 2\varepsilon v_{\max}^{-1}$. As before we get, since $\varepsilon > 0$ is arbitrary,

$$\limsup_{n\to\infty} \mu_n([0,x]) \leq x,$$

which finishes the proof.

### 1.5.2 Frogs with drift on $\mathbb{Z}^d$ for $d \geq 2$

In this subsection we consider the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ for $d \geq 2$.

#### Recurrence for $d \geq 2$ and arbitrary weight

In this section we prove Theorem 1.20 (i) and Theorem 1.22. Throughout this section assume that $w < 1$ is fixed. To illustrate the basic idea of the proof we first sketch it for $d = 2$. We call a site $x$ in $\mathbb{Z}^2$ open if the trajectory $(S_n^x)_{n\in\mathbb{N}_0}$ of frog $x$ includes the four neighbouring vertices $x \pm e_1, x \pm e_2$ of $x$, i.e. if $x \to x \pm e_1$ and $x \to x \pm e_2$. Note that for this definition it does not matter whether frog $x$ is ever activated or not. All sites

are open independently of each other due to the independence of the trajectories of the frogs. Furthermore, the probability of a site to be open is the same for all sites. Consider the percolation cluster $C_0$ that consists of all sites that can be reached from 0 by open paths, i.e. paths containing only open sites. Note that all frogs in $C_0$ are activated as frog 0 is active in the beginning. In this sense the frog model dominates the percolation. As we are in $d = 2$, the probability of a site $x$ being open equals 1 for $\alpha = 0$ and by continuity is close to 1 if $\alpha$ is close to 0. Thus, if $\alpha$ is close enough to 0 the percolation is supercritical. Hence, with positive probability the cluster $C_0$ containing the origin is infinite. By Lemma 1.31 this infinite cluster contains many sites close to the negative $e_1$-axis. This shows that many frogs that are initially close to this axis get activated. Each of them travels in the direction of the $e_1$-axis and has a decent chance of visiting 0 on its way. Hence, this will happen infinitely many times. This argument shows that the origin is visited by infinitely many frogs with positive probability. Using the zero-one law stated in Theorem 1.2 yields the claim.
In higher dimensions the probability of a frog to visit all its neighbours is not close to 1 however small the drift may be. We can still make the argument work by using a renormalization type argument. To make this argument precise let $K$ be a non-negative integer that will be chosen later. We tessellate $\mathbb{Z}^d$ for $d \geq 2$ with cubes $(Q_x)_{x \in \mathbb{Z}^d}$ of size $(2K+1)^d$. For every $x \in \mathbb{Z}^d$ we define

$$\begin{aligned} q_x &= q_x(K) = (2K+1)x, \\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^d \colon \|y - q_x\|_\infty \leq K\}, \end{aligned} \tag{1.23}$$

where $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$ is the supremum norm. We call a site $x \in \mathbb{Z}^d$ open if for every $e \in \mathcal{E}_d$ there exists a frog path from $q_x$ to $q_{x+e}$ in $Q_x$. Otherwise, $x$ is said to be closed. The probability of a site $x$ to be open does not depend on $x$, but only on the drift parameter $\alpha$ and the cube size $K$. We denote it by $p(K, \alpha)$. This defines an independent site percolation on $\mathbb{Z}^d$, which, as mentioned before, is dominated by the frog model in the following sense: For any $x \in C_0$ the frog at $q_x$ will be activated in the frog model, i.e. $q_x \in FC_0$ with $FC_0$ as defined in (1.5).
In the next two lemmas we show that the probability $p(K, \alpha)$ of a site to be open is close to 1 if the drift parameter $\alpha$ is small and the cube size $K$ is large. We first show this claim for the symmetric case $\alpha = 0$.

**Lemma 1.42.** *For every $w < 1$ in the frog model* $\mathrm{FM}(d, \pi_{w,0})$ *we have*

$$\lim_{K \to \infty} p(K, 0) = 1.$$

*Proof.* For $d = 2$ we obviously have $p(K, 0) = 1$ for all $K \in \mathbb{N}_0$ as balanced nearest neighbour random walk on $\mathbb{Z}^2$ is recurrent. Therefore, we can assume $d \geq 3$. The proof of the lemma relies on the shape theorem (Theorem 1.5) for the frog model. This theorem assumes equal weights on all directions. As in our model the $e_1$-direction has a different weight, we need a workaround. We couple our model with a modified frog model on $\mathbb{Z}^{d-1}$ in which the frogs in every step stay where they are with probability $w$ and move according to a simple random walk otherwise. A direct coupling shows that, up to any fixed time, in the modified frog model on $\mathbb{Z}^{d-1}$ there are at most as many frogs activated as in the frog model $\mathrm{FM}(d, \pi_{w,0})$. Note that Theorem 1.5 holds true for the modified frog model on $\mathbb{Z}^{d-1}$, see Remark 1.6. Let $\xi_K$, respectively $\xi_K^{\mathrm{mod}}$, be the set of all sites visited by active frogs by time $K$ in the frog model $\mathrm{FM}(d, \pi_{w,0})$, respectively the modified frog model on $\mathbb{Z}^{d-1}$. Further, let $\overline{\xi_K^{\mathrm{mod}}} := \{x + (-\frac{1}{2}, \frac{1}{2}]^{d-1} : x \in \xi_K^{\mathrm{mod}}\}$. By Theorem 1.5 there exists a non-trivial convex symmetric set $\mathcal{A} = \mathcal{A}(d) \subseteq \mathbb{R}^{d-1}$ and an almost surely finite random variable $\mathcal{K}$ such that

$$\mathcal{A} \subseteq \frac{\overline{\xi_K^{\mathrm{mod}}}}{K}$$

for all $K \geq \mathcal{K}$. This implies that there exists a constant $c_1 = c_1(d) > 0$ such that $|\xi_K^{\mathrm{mod}}| \geq c_1 K^{d-1}$ for all $K \geq \mathcal{K}$. By the coupling the same statement holds true for $\xi_K$. As $\xi_K \subseteq Q_0(K)$ and any vertex in $\xi_K$ can be reached by a frog path from 0 in $Q_0$, this implies

$$\Big|\Big\{y \in Q_0 \colon 0 \overset{Q_0}{\rightsquigarrow} y\Big\}\Big| \geq |\xi_K| \geq c_1 K^{d-1}$$

for all $K \geq \mathcal{K}$. Thus we have at least $c_1 K^{d-1}$ vertices in the box $Q_0$ that can be reached by frog paths from 0. Each frog in $Q_0$ has a chance to reach the centre $q_e$ of a neighbouring box.More precisely, by Lemma 1.27 there is a constant $c_2 = c_2(d) > 0$ such that

$$\mathbb{P}\big(y \to q_e\big) \geq \frac{c_2}{K^{d-2}} \tag{1.24}$$

for any vertex $y \in Q_0$ and $e \in \mathcal{E}_d$. Hence, for any $e \in \mathcal{E}_d$

$$\begin{aligned}
\mathbb{P}\big((0 \overset{Q_0}{\rightsquigarrow} q_e)^c \mid K \geq \mathcal{K}\big) &= \mathbb{P}\Big(\big\{y \not\to q_e \text{ for all } y \in Q_0 \text{ with } 0 \overset{Q_0}{\rightsquigarrow} y\big\} \bigm| K \geq \mathcal{K}\Big) \\
&\leq \Big(1 - \frac{c_2}{K^{d-2}}\Big)^{c_1 K^{d-1}} \\
&\leq \mathrm{e}^{-c_1 c_2 K},
\end{aligned} \tag{1.25}$$

where we used for the first inequality the fact that a frog moves independently of all frogs in $Q_0$ once it will never return to $Q_0$ and the uniformity of the bound in (1.24). Therefore,

$$\begin{aligned}
p(K, 0) &\geq \mathbb{P}\Big(\bigcap_{e \in \mathcal{E}_d} \{0 \overset{Q_0}{\rightsquigarrow} q_e\} \Bigm| K \geq \mathcal{K}\Big)\mathbb{P}_0(K \geq \mathcal{K}) \\
&\geq \Big[1 - 2d\,\mathrm{e}^{-c_1 c_2 K}\Big]\mathbb{P}(K \geq \mathcal{K}).
\end{aligned} \tag{1.26}$$

Since $\mathcal{K}$ is almost surely finite, we have $\lim_{K\to\infty} \mathbb{P}_0(K \geq \mathcal{K}) = 1$. Thus, the right hand side of (1.26) tends to 1 in the limit $K \to \infty$.

**Lemma 1.43.** *For fixed $w < 1$, in the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *we have for all $K \in \mathbb{N}_0$*

$$\liminf_{\alpha \to 0} p(K, \alpha) \geq p(K, 0).$$

*Proof.* Let $L(a, b, c, K)$ be the number of possible realizations such that all $q_{x \pm e}$, $e \in \mathcal{E}_d$, are visited by frogs in $Q_0$ for the first time after in total (of all frogs) exactly $a$ steps in $e_1$-direction, $b$ steps in $-e_1$-direction and $c$ steps in all other directions. Note that $L(a, b, c, K)$ is independent of $\alpha$. We have

$$p(K, \alpha) = \sum_{a,b,c=1}^{\infty} L(a, b, c, K) \left( \frac{w(1+\alpha)}{2} \right)^a \left( \frac{w(1-\alpha)}{2} \right)^b \left( \frac{1-w}{2(d-1)} \right)^c .$$

The claim now follows from Fatou's Lemma.

*Proof of Theorem 1.20 (i) and Theorem 1.22.* By Lemma 1.42 and Lemma 1.43 we can assume that $K$ is big enough and $\alpha > 0$ small enough such that $p(K, \alpha) > p_c(d)$, i.e. the percolation with parameter $p(K, \alpha)$ on $\mathbb{Z}^d$ constructed at the beginning of this section is supercritical.
Consider boxes $B_n = \{-n\} \times [-\sqrt{n}, \sqrt{n}]^{d-1}$ for $n \in \mathbb{N}$. By Lemma 1.31 there are constants $a, b > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$

$$\mathbb{P}(|B_n \cap C_0| \geq a n^{(d-1)/2}) > b.$$

After rescaling, the boxes $B_n$ correspond to the boxes

$$FB_n = \{y \in \mathbb{Z}^d \colon |y_1 + (2K+1)n| \leq K,\, |y_i| \leq (2K+1)\sqrt{n} + K,\, 2 \leq i \leq d\}.$$

Recall that $FC_0$ consists of all vertices reachable by frog paths from 0 as defined in (1.5), and note that $x \in B_n \cap C_0$ implies $q_x \in FB_n \cap FC_0$. This shows

$$\mathbb{P}(|FB_n \cap FC_0| \geq a n^{(d-1)/2}) > b \tag{1.27}$$

for $n$ large enough. Analogously to (1.24), by Lemma 1.28 and (1.27) the probability that at least one frog in $FB_n$ is activated and reaches 0 is at least

$$\Big(1 - (1 - c n^{-(d-1)/2})^{a n^{(d-1)/2}}\Big) b \geq \big(1 - \mathrm{e}^{-ac}\big) b,$$

where $c = c(K, d, w) > 0$ is a constant. Altogether we get by Lemma 1.34

$$\begin{aligned}\mathbb{P}(0 \text{ visited infinitely often}) &= \lim_{n\to\infty} \mathbb{P}(0 \text{ is visited } \varepsilon n \text{ many times }) \\ &\geq \liminf_{n\to\infty} \mathbb{P}\Big(\sum_{i=1}^{n} \mathbb{1}_{\{\exists x \in FB_i \cap FC_0 \colon x \to 0\}} \geq \varepsilon n\Big) \\ &\geq \big(1 - \mathrm{e}^{-ac}\big)b - \varepsilon > 0\end{aligned}$$

for $\varepsilon$ sufficiently small. The claim now follows from Theorem 1.2.

### Recurrence for $d = 2$ and arbitrary drift

In this section we prove Theorem 1.21 (i). Throughout the section let $\alpha < 1$ be fixed. We couple the frog model with independent site percolation on $\mathbb{Z}^2$. Let $K$ be an integer that will be chosen later. We tessellate $\mathbb{Z}^2$ with segments $(Q_x)_{x\in\mathbb{Z}^2}$ of size $2K + 1$. For every $x = (x_1, x_2) \in \mathbb{Z}^2$ we define

$$\begin{aligned} q_x &= q_x(K) = \big(x_1, (2K + 1)x_2\big), \\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^2 \colon y_1 = x_1, |y_2 - (2K + 1)x_2| \leq K\}. \end{aligned}$$

We call the site $x \in \mathbb{Z}^2$ open if there are frog paths from $q_x$ to $q_{x+e}$ in $Q_x$ for all $e \in \mathcal{E}_2$. As before, we denote the probability of a site to be open by $p(K, w)$. Note that this probability does not depend on $x$.

**Lemma 1.44.** *For $\alpha < 1$, in the frog model $\mathrm{FM}(2, \pi_{w,\alpha})$ we have*

$$\lim_{K\to\infty} \liminf_{w\to 0} p(K, w) = 1.$$

*Proof.* We claim that there is a constant $c = c(\alpha) > 0$ such that for any $K \in \mathbb{N}_0$ and $x \in Q_0$

$$\liminf_{w\to 0} \mathbb{P}\Big(\bigcap_{e\in\mathcal{E}_2} \{x \to q_e\}\Big) \geq c. \tag{1.28}$$

We can estimate the probability in (1.28) by

$$\mathbb{P}\Big(\bigcap_{e\in\mathcal{E}_2} \{x \to q_e\}\Big) \geq \mathbb{P}\big(x \to q_{-e_2}\big)\mathbb{P}\big(q_{-e_2} \to q_{-e_1}\big)\mathbb{P}\big(q_{-e_1} \to q_{e_2}\big)\mathbb{P}\big(q_{e_2} \to q_{e_1}\big). \tag{1.29}$$

The probability of moving in $\pm e_2$-direction for $\lceil w^{-1}\rceil$ steps is $(1-w)^{\lceil w^{-1}\rceil}$. Conditioning on moving in this way, we just deal with a simple random walk on $\mathbb{Z}$. Therefore, there exists a constant $c_1 > 0$ such that for $w$ close to 0

$$\mathbb{P}\big(x \to q_{-e_2}\big) \geq c_1(1 - w)^{\lceil w^{-1}\rceil} \geq \frac{c_1}{4}.$$

The probability of moving exactly once in $-e_1$-direction and otherwise in $\pm e_2$-direction within $\lceil w^{-1}\rceil + 1$ steps is

$$\left(\lceil w^{-1}\rceil + 1\right)\frac{(1-\alpha)w}{2}(1-w)^{\lceil w^{-1}\rceil} \geq \frac{1-\alpha}{8}$$

for $w$ close to 0. Therefore, there exists a constant $c_2 > 0$ such that

$$\mathbb{P}\left(q_{-e_2} \to q_{-e_1}\right) \geq \frac{c_2(1-\alpha)}{8}$$

for $w$ sufficiently close to 0. The two remaining probabilities in (1.29) can be estimated analogously, which implies (1.28).

If frog 0 activates all frogs in $Q_0$ and any of these $2K$ frogs manages to visit the centres of all neighbouring segments, then 0 is open. By independence of the trajectories of the individual particles in $Q_0$ this implies

$$p(K,w) \geq \mathbb{P}\Big(\bigcap_{x\in Q_0}\{0\to x\}\Big)\bigg(1-\Big(1-\mathbb{P}\Big(\bigcap_{1\leq i\leq 4}\{x\to q_{e_i}\}\Big)\Big)^{2K}\bigg). \tag{1.30}$$

As in the proof of Lemma 1.43 one can show that for $w \to 0$ the first factor in (1.30) converges to 1. Therefore, taking limits in (1.30) and using (1.30) yields the claim.

*Proof of Theorem 1.21 (i).* By Lemma 1.44 we can choose $K$ big and $w$ small enough such that $p(K,w) > p_c(2)$, where $p_c(2)$ is the critical parameter for independent site percolation on $\mathbb{Z}^2$. As in the proof of Theorem 1.20 (i) and Theorem 1.22 the coupling with supercritical percolation now yields recurrence of the frog model. As we rescaled the lattice $\mathbb{Z}^2$ slightly different this time, the box $B_n$ defined in the proof of Theorem 1.20 (i) and Theorem 1.22 now corresponds to the box

$$FB_n = \{y \in \mathbb{Z}^2 \colon y_1 = -n,\ |y_2| \leq (2K+1)\sqrt{n} + K\}.$$

Since only asymptotics in $n$ matter for the proof, it otherwise works unchanged.

### Recurrence for arbitrary drift and $d \geq 3$

The proof of Theorem 1.23 (i) again relies on the idea of comparing the frog model with percolation. But instead of looking at the whole space $\mathbb{Z}^d$ as in the previous proofs, we consider a sequence of $(d-1)$-dimensional hyperplanes $(H_{-n})_{n\in\mathbb{N}_0}$ with $H_{-n}$ as defined in (1.6). We compare the frogs in each hyperplane with supercritical percolation, ignoring the frogs once they have left their hyperplane and all the frogs from other hyperplanes. Within a hyperplane we now deal with a frog model without drift, but allow the frogs to die in each step with probability $w$ by leaving their hyperplane, i.e. we are interested in $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$. Hence, the argument does not depend on the value of the drift

parameter $\alpha < 1$.
We start with one active particle in the hyperplane $H_0$. With positive probability this particle initiates an infinite frog cluster in $H_0$ if $w$ and therefore the probability to leave the hyperplane is sufficiently small. Every frog eventually leaves $H_0$ and has for every $n \in \mathbb{N}$ a positive chance of activating a frog in the hyperplane $H_{-n}$, which might start an infinite cluster there. This is the only time where we need $\alpha < 1$ in the proof of Theorem 1.23 (i). Using the denseness of such clusters we can then proceed as before.
We split the proof of Theorem 1.23 (i) into two parts:

**Proposition 1.45.** *There is $d_0 \in \mathbb{N}$ and $w_r > 0$, independent of $d$ and $\alpha$, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \le w \le w_r$, $0 \le \alpha < 1$ and $d \ge d_0$.*

**Proposition 1.46.** *For every $d \ge 3$ there is $w_r = w_r(d) > 0$, independent of $\alpha$, such that the frog model* $\mathrm{FM}(d, \pi_{w,\alpha})$ *is recurrent for all $0 \le w \le w_r$ and all $0 \le \alpha < 1$.*

We first prove Proposition 1.45. As indicated above we need to study the frog model with death and no drift in $\mathbb{Z}^{d-1}$. To increase the readability of the paper let us first work in dimension $d$ instead of $d-1$ and with a general survival parameter $s$, i.e. we investigate $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$ for $d \ge 2$.
We tessellate $\mathbb{Z}^d$ with cubes $(Q'_x)_{x\in\mathbb{Z}^d}$ of size $3^d$. More precisely, for $x \in \mathbb{Z}^d$ we define

$$Q'_x = \{y \in \mathbb{Z}^d \colon \|y - 3x\|_\infty \le 1\}.$$

Further, for technical reasons, for $a \in (\frac{2}{3}, 1)$ we define

$$W_x = \{y \in Q'_x \colon \|y - 3x\|_1 \le ad\},$$

where $\|z\|_1 = \sum_{i=1}^{2d} |z_i|$ is the graph distance from $z \in \mathbb{Z}^d$ to 0. Informally, $W_x$ is the set of all vertices in $Q'_x$ which are "sufficiently close" to the centre of the cube. Consider the box $Q'_x$ for some $x \in \mathbb{Z}^d$ and let $o \in W_x$. If there are frog paths in $Q'_x$ from $o$ to vertices close to the centres of all neighbouring boxes, i.e. if the event

$$\bigcap_{e\in\mathcal{E}_d} \bigcup_{y\in W_{x+e}} \{o \overset{Q'_x}{\rightsquigarrow} y\}$$

occurs, we call the vertex $o$ good. Note that this event only depends on the trajectories of all the frogs originating in the cube $Q'_x$ and the choice of $o$. If $o$ is good and is activated, then also the neighbouring cubes are visited. We show that the probability of a vertex being good is bounded from below uniformly in $d$ and this bound does not depend on the choice of $o$.

**Lemma 1.47.** *Consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$*. There are constants $\beta > 0$ and*

*$d_0 \in \mathbb{N}$ such that for all $d \geq d_0$, $s > \frac{3}{4}$, $\frac{2}{3} < a < 2 - \frac{1}{s}$, $x \in \mathbb{Z}^d$ and $o \in W_x$*

$$\mathbb{P}(o \text{ is good}) > \beta.$$

To show this we first need to prove that many frogs in the cube are activated. In the proof of Theorem 1.20 (i) and Theorem 1.22 this is done by means of Lemma 1.42 using the shape theorem. Here, we use a lemma that is analogous to Lemma 2.5 in [3].

**Lemma 1.48.** *Consider the frog model $\mathrm{FM}^*(d, \pi_{sym}, s)$. There exist constants $\gamma > 0$, $\mu > 1$ and $d_0 \in \mathbb{N}$ such that for all $d \geq d_0$, $s > \frac{3}{4}$, $\frac{2}{3} < a < 2 - \frac{1}{s}$ and $o \in W_0$ we have*

$$\mathbb{P}\Big(\big|\{y \in W_0 \colon o \overset{Q_0'}{\rightsquigarrow} y\}\big| \geq \mu^{\sqrt{d}}\Big) \geq \gamma.$$

*Proof of Lemma 1.48.* The proof consists of two parts. In the first part we show that with positive probability there are exponentially many vertices in $Q_0'$ reached from $o$ by frog paths in $Q_0'$, and in the second part we prove that many of these vertices are indeed in $W_0$. For the first part we closely follow the proof of Lemma 2.5 in [3] and rewrite the details for the convenience of the reader.

We examine the frog model with initially one active frog at $o$ and one sleeping frog at every other vertex in $Q_0'$ for $\sqrt{d}$ steps in time. Consider the sets $\mathcal{S}_0 = \{o\}$ and $\mathcal{S}_k = \{x \in Q_0' \colon \|x - o\|_1 = k, \|x - o\|_\infty = 1\}$ for $k \geq 1$ and let $\xi_k$ denote the set of active frogs which are in $\mathcal{S}_k$ at time $k$. We will show that, conditioned on an event to be defined later, the process $(\xi_k)_{k \in \mathbb{N}_0}$ dominates a process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$, which again itself dominates a supercritical branching process. The process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$ is defined as follows. Initially, there is one particle at $o$. Assume that the process has been constructed up to time $k \in \mathbb{N}_0$. In the next step each particle in $\tilde{\xi}_k$ survives with probability $s$. If it survives, it chooses one of the neighbouring vertices uniformly at random. If that vertex belongs to $\mathcal{S}_{k+1}$ and no other particle in $\tilde{\xi}_k$ intends to jump to this vertex, the particle moves there, activates the sleeping particle, and both particles enter $\tilde{\xi}_{k+1}$. Otherwise, the particle is deleted. In particular, if two or more particles attempt to jump to the same vertex, all of them will be deleted. Obviously, $\tilde{\xi}_k \subseteq \xi_k$ for all $k \in \mathbb{N}_0$.

First, we show that for $d$ large it is unlikely that two particles in $\tilde{\xi}_k$ attempt to jump to the same vertex. To make this argument precise we need to introduce some notation. For $x \in \mathcal{S}_k$ and $y \in \mathcal{S}_{k+1}$ with $\|x - y\|_1 = 1$ define

$$\begin{aligned}
\mathcal{D}_x &= \{z \in \mathcal{S}_{k+1} \colon \|x - z\|_1 = 1\}, \\
\mathcal{A}_y &= \{z \in \mathcal{S}_k \colon \|z - y\|_1 = 1\}, \\
\mathcal{E}_x &= \{z \in \mathcal{S}_k \colon \mathcal{D}_x \cap \mathcal{D}_z \neq \emptyset\}.
\end{aligned}$$

$\mathcal{D}_x$ denotes the set of possible descendants of $x$, $\mathcal{A}_y$ the set of ancestors of $y$ and $\mathcal{E}_x$ the set of enemies of $x$. Note that $\mathcal{E}_x = \bigcup_{y \in \mathcal{D}_x} (\mathcal{A}_y \setminus \{x\})$ is a disjoint union. Let

$n_x = \sum_{i=1}^{d} \mathbb{1}_{\{o_i=0,\, x_i\neq 0\}}$. Then one can check that

$$|\mathcal{D}_x| = 2(d - \|o\|_1 - n_x) + \|o\|_1 - (k - n_x) = 2d - \|o\|_1 - k - n_x, \tag{1.31}$$
$$|\mathcal{A}_y| = k + 1.$$

For $x \in \mathcal{S}_k$ let $\chi(x)$ denote the number of particles of $\tilde{\xi}_k$ in $x$. Note that $\chi(x) \in \{0, 2\}$ for any $x \in \mathcal{S}_k$ with $k \in \mathbb{N}$.
Let $\zeta_{xy}^k$ denote the indicator function of the event that there is $z \in \mathcal{E}_x$ with $\chi(z) \geq 1$ such that one of the particles at $z$ intends to jump to $y$ at time $k+1$. If $\zeta_{xy}^k = 1$, then a particle on $x$ cannot move to $y$ at time $k+1$.
Further, we introduce the event $U_x = \{\chi(z) = 2 \text{ for all } z \in \mathcal{E}_x\}$. This event describes the worst case for $x$, when it is most likely that particles at $x$ will not be able to jump. For $k \leq \sqrt{d}$ we have

$$\mathbb{P}(\zeta_{xy}^k = 1) \leq \mathbb{P}(\zeta_{xy}^k = 1 \mid U_x) \leq \sum_{z \in \mathcal{A}_y \setminus \{x\}} \frac{2s}{2d} = \frac{ks}{d} \leq \frac{1}{\sqrt{d}}.$$

Given $\sigma > 0$ we choose $d$ large such that $\mathbb{P}(\zeta_{xy}^k = 1) < \sigma$ for all $k \leq \sqrt{d}$. Now, we consider the set of all descendants $y$ of $x$ such that there is a particle at some vertex $z \in \mathcal{E}_x$ that tries to jump to $y$ at time $k+1$. This set contains $\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k$ elements. Let $\zeta_x^k$ denote the indicator function of the event $\left\{\sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > 2\sigma d\right\}$. If $\zeta_x^k = 1$, then more than $2\sigma d$ of the $2d$ neighbours of $x$ are blocked to a particle at $x$.
The random variables $\{\zeta_{xy}^k : y \in \mathcal{D}_x\}$ are independent with respect to $\mathbb{P}(\cdot \mid U_x)$ since $\mathcal{E}_x = \bigcup_{y \in \mathcal{D}_x} (\mathcal{A}_y \setminus \{x\})$ is a disjoint union. Using $2d - ad - 2k \leq |\mathcal{D}_x| \leq 2d$ and a standard large deviation estimate we get for $k \leq \sqrt{d}$

$$\begin{aligned}
\mathbb{P}(\zeta_x^k = 1) &\leq \mathbb{P}\left( \sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > 2\sigma d \,\middle|\, U_x \right) \\
&\leq \mathbb{P}\left( \frac{1}{|\mathcal{D}_x|} \sum_{y \in \mathcal{D}_x} \zeta_{xy}^k > \sigma \,\middle|\, U_x \right) \\
&\leq \mathrm{e}^{-c_1 |\mathcal{D}_x|} \\
&\leq \mathrm{e}^{-c_2 d}
\end{aligned}$$

with constants $c_1, c_2 > 0$. Next, let us consider the bad event

$$B = \bigcup_{k=1}^{\sqrt{d}} \bigcup_{x \in \tilde{\xi}_k} \{\zeta_x^k = 1\}.$$

Then with $|\tilde{\xi}_k| \le 2^k \le 2^{\sqrt{d}}$ we get

$$\mathbb{P}(B) \le \sqrt{d} \cdot 2^{\sqrt{d}} \cdot \mathrm{e}^{-c_2 d}.$$

In particular $\mathbb{P}(B)$ can be made arbitrarily small for $d$ large. Conditioned on $B^c$, in each step for every particle there are at least

$$|\mathcal{D}_x| - 2\sigma d - 1 \ge (2 - a - 2\sigma)d - 3\sqrt{d}$$

available vertices in $\mathcal{S}_{k+1}$, i.e. vertices a particle at $x$ can jump to in the next step. Thus, conditioned on $B^c$, the process $\tilde{\xi}_k$ dominates a branching process with mean offspring at least

$$\frac{\big((2 - a - 2\sigma)d - 3\sqrt{d}\big) \cdot 2 \cdot s}{2d}.$$

For $\sigma$ small and $d$ large the mean offspring is bigger than 1 as we assumed $a < 2 - \frac{1}{s}$. Since a supercritical branching process grows exponentially with positive probability, there are constants $c_3 > 1$, $q \in (0, 1)$ that do not depend on $d$ such that

$$\mathbb{P}\big(|\tilde{\xi}_{\sqrt{d}}| \ge c_3^{\sqrt{d}}\big) \ge q. \tag{1.32}$$

For the second part of the proof condition on the event $\big\{|\tilde{\xi}_{\sqrt{d}}| \ge c_3^{\sqrt{d}}\big\}$ and choose $0 < \varepsilon < a - \frac{2}{3}$. If $\|o\|_1 \le (a - \varepsilon)d$, all particles of $\tilde{\xi}_{\sqrt{d}}$ are in $W_0$ for $d$ large. This immediately implies the claim of the lemma. Otherwise, let $n = |\tilde{\xi}_{\sqrt{d}}|$, enumerate the particles in $\tilde{\xi}_{\sqrt{d}}$ and let $\tilde{S}^i$, $1 \le i \le n$, denote the position of the $i$-th particle. Further, we define for $1 \le i \le n$

$$X_i = \begin{cases} 1 & \text{if } \|\tilde{S}^i\|_1 \le \|o\|_1, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that $\mathbb{P}(X_1 = 1) > 0$. Then Lemma 1.34 applied to the random variables $X_1, \ldots, X_n$ implies that with positive probability a positive proportion of the particles in $\tilde{\xi}_{\sqrt{d}}$ indeed have $L_1$-norm smaller than $o$, and are thus in $W_0$. Together with (1.32) this finishes the proof.

For the proof of the claim let $\tilde{S}^1_k$ denote the position of the ancestor of $\tilde{S}^1$ in $\mathcal{S}_k$, where $0 \le k \le \sqrt{d}$. Note that $\tilde{S}^1_0 = o$ and $\tilde{S}^1_{\sqrt{d}} = \tilde{S}^1$.

We are interested in the process $(\|\tilde{S}^1_k\|_1)_{1 \le k \le \sqrt{d}}$. By the construction of the process $(\tilde{\xi}_k)_{k \in \mathbb{N}_0}$ it either increases or decreases by 1 in every step. The positions $\tilde{S}^1_k$ and $\tilde{S}^1_{k+1}$ differ in exactly one coordinate. If this coordinate is changed from 0 to $\pm 1$, then we have $\|\tilde{S}^1_{k+1}\|_1 = \|\tilde{S}^1_k\|_1 + 1$. If it is changed from $\pm 1$ to 0, then $\|\tilde{S}^1_{k+1}\|_1 = \|\tilde{S}^1_k\|_1 - 1$. There are at least $(a - \varepsilon)d - \sqrt{d}$ many $\pm 1$-coordinates in $\tilde{S}^1_k$ that can be changed to 0. As we also know that $\tilde{S}^1_{k+1} \in \mathcal{D}_{\tilde{S}^1_k}$, we have for all $k \leq \sqrt{d}$ by (1.31) and the choice of $\varepsilon$

$$\mathbb{P}\big(\|\tilde{S}^1_{k+1}\|_1 = \|\tilde{S}^1_k\|_1 - 1\big) \geq \frac{(a-\varepsilon)d - \sqrt{d}}{|\mathcal{D}_{\tilde{S}^1_k}|} \geq \frac{(a-\varepsilon)d - \sqrt{d}}{2d - (a-\varepsilon)d} > \frac{1}{2}$$

for $d$ large. Hence, $\|\tilde{S}^1_k\|_1$ dominates a random walk with drift on $\mathbb{Z}$ started in $\|o\|_1$. Therefore,

$$\mathbb{P}(X_1 = 1) = \mathbb{P}\big(\|\tilde{S}^1_{\sqrt{d}}\|_1 \leq \|o\|_1\big) \geq \frac{1}{2},$$

which finishes the proof.

*Proof of Lemma 1.47.* By Lemma 1.48, with probability at least $\gamma$ there are frog paths in $Q'_x$ from $o$ to at least $\mu^{\sqrt{d}}$ vertices in $W_x$ for $d$ large. We divide the frogs on these vertices into $2d$ groups of size at least $\mu^{\sqrt{d}}/2d$ and assign each group the task of visiting one of the neighbouring boxes $W_{x+e}$, $e \in \mathcal{E}_d$. Notice that this job is done if at least one of the frogs in the group visits at least one vertex in the neighbouring box. If all groups succeed, $o$ is good. Any frog in any group is just three steps away from its respective neighbouring box $W_{x+e}$, $e \in \mathcal{E}_d$, and thus has probability at least $(\frac{s}{2d})^3$ of achieving its group's goal. Hence,

$$\mathbb{P}(o \text{ is good}) \geq \Big(1 - \Big(1 - \Big(\frac{s}{2d}\Big)^3\Big)^{\mu^{\sqrt{d}}/2d}\Big)^{2d}\gamma \geq \frac{\gamma}{2}$$

for $d$ large.

In the other recurrence proofs we couple the frog model with percolation by calling a cube open if its centre is good. Here, the choice of a "starting" vertex, like the centre, is not independent of the other cubes. Therefore, we cannot directly couple the frog model with independent percolation. However, the following lemma allows us to compare the distributions of a frog cluster and a percolation cluster.

**Lemma 1.49.** *Consider the frog model* $\mathrm{FM}^*(d, \pi_{sym}, s)$*. Let* $\beta > 0$ *and assume that* $\mathbb{P}(o \text{ is good}) > \beta$ *for all* $o \in W_x$, $x \in \mathbb{Z}^d$*. Further, consider independent site percolation on* $\mathbb{Z}^d$ *with parameter* $\beta$*. Then for all sets* $A \subseteq \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ *and for all* $k \geq 0$

$$\mathbb{P}(|A \cap C_v| \geq k) \leq \mathbb{P}\Big(\Big|\bigcup_{x \in A} Q'_x \cap FC^*_{3v}\Big| \geq k\Big).$$

*Proof.* For technical reasons we introduce a family of independent Bernoulli random variables $(X_o)_{o\in\mathbb{Z}^d}$ which are also independent of the choice of all the trajectories of the frogs and satisfy $\mathbb{P}(X_o = 1) = \mathbb{P}(o \text{ is good})^{-1}\beta$. Their job will be justified soon. Further, we fix an ordering of all vertices in $\mathbb{Z}^d$.
Now we are ready to describe a process that explores a subset of the frog cluster $FC^*_{3v}$. Its distribution can be related to the cluster $C_v$ in independent site percolation with

parameter $\beta$. The process is a random sequence $(R_t, D_t, U_t)_{t\in\mathbb{N}_0}$ of tripartitions of $\mathbb{Z}^d$. As the letters indicate, $R_t$ will contain all sites reached by time $t$, $D_t$ all those declared dead by time $t$, and $U_t$ the unexplored sites. We construct the process in such a way that for all $t \in \mathbb{N}_0$, $x \in R_t$ and $e \in \mathcal{E}_d$ there is $y \in W_{x+e}$ such that there is a frog path from $3v$ to $y$ in $\bigcup_{x\in R_t} Q'_x$. We start with $R_0 = D_0 = \emptyset$ and $U_0 = \mathbb{Z}^d$. If $3v$ is good and $X_{3v} = 1$, set $U_1 = \mathbb{Z}^d \setminus \{v\}$, $R_1 = \{v\}$, and $D_1 = \emptyset$. Otherwise, stop the algorithm. If the process is stopped at time $t$, let $U_s = U_{t-1}$, $R_s = R_{t-1}$ and $D_s = D_{t-1}$ for all $s \geq t$. Assume we have constructed the process up to time $t$. Consider the set of all sites in $U_t$ that have a neighbour in $R_t$. If it is empty, stop the process. Otherwise, pick the site $x$ in this set with the smallest number in our ordering. By the choice of $x$ there is $y \in W_x$ such that there is a frog path from $3v$ to $y$ in $\bigcup_{z\in R_t} Q'_z$. Choose any vertex $y$ with this property. If $y$ is good and $X_y = 1$, set

$$R_{t+1} = R_t \cup \{x\},\ D_{t+1} = D_t,\ U_{t+1} = U_t \setminus \{x\}.$$

Otherwise, update the sets as follows:

$$R_{t+1} = R_t,\ D_{t+1} = D_t \cup \{x\},\ U_{t+1} = U_t \setminus \{x\}$$

In every step $t$ the algorithm picks an unexplored site $x$ and declares it to be reached or dead, i.e. added to the set $R_t$ or $D_t$. The probability that $x$ is added to $R_t$ equals $\beta$. This event is (stochastically) independent of everything that happened before time $t$ in the algorithm. Note that every unexplored neighbour of a reached site will eventually be explored due to the fixed ordering of all sites.
In the same way we can explore independent site percolation on $\mathbb{Z}^d$ with parameter $\beta$. Construct a sequence $(R'_t, D'_t, U'_t)_{t\in\mathbb{N}_0}$ of tripartitions of $\mathbb{Z}^d$ as above, but whenever the algorithm evaluates whether a site $x$ is declared reached or dead we toss a coin independently of everything else. Note that $\bigcup_{t\in\mathbb{N}_0} R'_t = C_v$, where $C_v$ is the cluster containing $v$. This exploration process is well known for percolation, see e.g. [13, Proof of Theorem 4, Chapter 1].
By construction, $\bigcup_{t\in\mathbb{N}_0} R_t$ equals the percolation cluster $C_v$ in distribution. The claim follows since for every $x \in \bigcup_{t\in\mathbb{N}_0} R_t$ there is a $y \in W_x$ such that there is a frog path from $3v$ to $y$, i.e. $y \in FC^*_{3v}$.

Now we can show Proposition 1.45. Note that we are again working with the frog model $\mathrm{FM}(d, \pi_{w,\alpha})$ (without death).

*Proof of Proposition 1.45.* Throughout this proof we assume that $d$ is so large that Lemma 1.47 is applicable for $d-1$ and $p_c(d-1) < \beta$, where $\beta$ is the constant introduced in the statement of Lemma 1.47. This is possible because of Lemma 1.30. These assumptions in particular imply that we can use Lemma 1.49 and that the percolation introduced there is supercritical.

Consider the sequence of hyperplanes $(H_{-n})_{n\in\mathbb{N}_0}$ defined in (1.6) and let $A$ denote the event that there is at least one frog $v_n$ activated in every hyperplane $H_{-n}$. For technical reasons we want $v_n$ of the form $v_n = (-n, 3w_n)$ for some $w_n \in \mathbb{Z}^{d-1}$. We first show that $A$ occurs with positive probability. To see this consider the first hyperplane $H_0$ and couple the frogs in this hyperplane with $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ in the following way: Whenever a frog takes a step in $\pm e_1$-direction, i.e. leaves its hyperplane, it dies instead. By [3, Theorem 1.8] (or Lemma 1.49) this process survives with positive probability if $w$ is sufficiently small (independent of the dimension $d$). This means that infinitely many frogs are activated in $H_0$. Obviously, this implies the claim.
From now on we condition on the event $A$. Note that $FC_{v_n} \subseteq FC_0$ for $n \in \mathbb{N}$. Analogously to the proofs in the last sections we introduce boxes

$$FB'_n = \{-n\} \times [-(3\sqrt{n}+1), 3\sqrt{n}+1]^{d-1}$$

for $n \in \mathbb{N}$. We claim that analogously to Lemma 1.31 there are constants $a, b > 0$ and $N \in \mathbb{N}$ such that for $n \geq N$

$$\mathbb{P}\big(|FB'_n \cap FC_0| \geq an^{(d-1)/2}\big) \geq b. \tag{1.33}$$

To prove this claim let $a, b > 0$ and $N \in \mathbb{N}$ be the constants of Lemma 1.31 for percolation with parameter $\beta$. For $n \geq N$ couple the frog model with $\mathrm{FM}^*(d-1, \pi_{\mathrm{sym}}, 1-w)$ in the hyperplane $H_n$ as above. Let $B'_n = [-\sqrt{n}, \sqrt{n}]^{d-1}$ and note that $B'_n$ corresponds to $FB'_n$ restricted to $H_n$ after rescaling. Then by Lemma 1.49 and Lemma 1.31

$$\begin{aligned}\mathbb{P}\big(|FB'_n \cap FC_{v_n}| \geq an^{(d-1)/2}|A\big) &\geq \mathbb{P}\big(|FB'_n \cap (\{-n\} \times FC^*_{3w_n})| \geq an^{(d-1)/2)}|A\big)\\ &\geq \mathbb{P}\big(|B'_n \cap C_{w_n}| \geq an^{(d-1)/2)}|A\big)\\ &\geq b.\end{aligned}$$

Here, $C_{w_n}$ is the open cluster containing $w_n$ in a percolation model with parameter $\beta$ in $\mathbb{Z}^{d-1}$, independently of the frogs. As $FC_{v_n} \subseteq FC_0$, this implies inequality (1.33).
By Lemma 1.28 and (1.33), the probability that there is at least one activated frog in $FB'_n$ that reaches 0 is at least

$$\Big(1 - (1 - c'n^{-(d-1)/2})^{an^{(d-1)/2}}\Big)b \geq \big(1 - \mathrm{e}^{-ac'}\big)b,$$

where $c' > 0$ is a constant. Altogether we get by Lemma 1.34

$$\begin{aligned}\mathbb{P}(0 \text{ visited infinitely often}) &= \lim_{n\to\infty} \mathbb{P}(0 \text{ is visited } \varepsilon n \text{ many times })\\ &\geq \lim_{n\to\infty} \mathbb{P}\Big(\sum_{i=1}^{n} \mathbb{1}_{\{\exists x\in FB'_n\cap FC_0\colon x\to 0\}} \geq \varepsilon n\Big)\\ &\geq \Big((1-\mathrm{e}^{-ac'})b-\varepsilon\Big) > 0\end{aligned}$$

for $\varepsilon$ sufficiently small. The claim now follows from Theorem 1.2. ∎

To prove Proposition 1.46 we again first study the frog model with death $\mathrm{FM}^*(d, \pi_{\mathrm{sym}}, s)$ in the hyperplanes and couple it with percolation. This time we use cubes of size $(2K+1)^d$ for some $K \in \mathbb{N}_0$. By choosing $K$ large we increase the number of frogs in the cubes. In the proof of the previous proposition this was done by increasing the dimension $d$. For $x \in \mathbb{Z}^d$ and $K \in \mathbb{N}_0$ we define

$$\begin{aligned}q_x &= q_x(K) = (2K+1)x,\\ Q_x &= Q_x(K) = \{y \in \mathbb{Z}^d\colon \|y-q_x\|_\infty \leq K\}.\end{aligned}$$

Note that this definition coincides with (1.23). In analogy to Lemma 1.49 the frog cluster dominates a percolation cluster.

**Lemma 1.50.** *For $d \geq 2$ consider the frog model $\mathrm{FM}^*(d, \pi_{sym}, s)$ and supercritical site percolation on $\mathbb{Z}^d$. There are constants $s_r(d) < 1$ and $K \in \mathbb{N}_0$ such that for any $s \geq s_r(d)$, $A \subseteq \mathbb{Z}^d$, $v \in \mathbb{Z}^d$ and for all $k \geq 0$*

$$\mathbb{P}(|A\cap C_v| \geq k) \leq \mathbb{P}\Big(\Big|\bigcup_{x\in A} Q_x \cap FC^*_{q_v}\Big| \geq k\Big).$$

*Proof.* We couple the frog model with percolation as follows: A site $x \in \mathbb{Z}^d$ is called open if for every $e \in \mathcal{E}_d$ there exists a frog path from $q_x$ to $q_{x+e}$ in $Q_x$. Note that $x \in C_v$ now implies $q_x \in FC^*_{q_v}$ for any $v \in \mathbb{Z}^d$. We denote the probability of a site $x$ to be open by $p(K,s)$. By Lemma 1.42 $p(K,1)$ is close to 1 for $K$ large. As in the proof of Lemma 1.43 one can show that $\lim_{s\to 1} p(K,s) = p(K,1)$. Thus, we can choose $K \in \mathbb{N}$ and $s_r > 0$ such that $p(K,s) > p_c(d)$ for all $s > s_r$, i.e. the percolation is supercritical. ∎

*Proof of Proposition 1.46.* Using Lemma 1.50 instead of Lemma 1.49 and boxes $Q_x$ instead of $Q'_x$, the proof is analogous to the proof of Proposition 1.45. ∎

*Proof of Theorem 1.23 (i).* Theorem 1.23 (i) follows from Proposition 1.45 and Proposition 1.46. ∎

### Transience for $d \geq 2$ and arbitrary drift

*Proof of Theorem 1.21 (ii) and Theorem 1.23 (ii).* Let the parameters $\alpha > 0$ and $d \geq 2$ be fixed throughout the proof. For $x \in \mathbb{Z}^d$ we define

$$L_x = \{y \in \mathbb{Z}^d \colon y_i = x_i \text{ for all } 2 \leq i \leq d\}. \tag{1.34}$$

$L_x$ consists of all vertices which agree in all coordinates with $x$ except the $e_1$-coordinate. The key observation used in the proof is that all particles mainly move along these lines if the weight $w$ is large.

We dominate the frog model by a branching random walk on $\mathbb{Z}^d$. At time $n = 0$ the branching random walk starts with one particle at the origin. At every step in time every particle produces offspring as follows: For every particle located at $x \in \mathbb{Z}^d$ consider an independent copy of the frog model. At any vertex $z \in \mathbb{Z}^d \setminus L_x$ the particle produces $|\{y \in L_x \colon x \overset{L_x}{\rightsquigarrow} y, y \to z\}|$ many children. Notice that this number might be 0 or infinite. The particle does not produce any offspring at a vertex in $L_x$. Further, note that the particles reproduce independently of each other as we use independent copies of the frog model to generate the offspring.

One can couple this branching random walk with the original frog model. To explain the coupling, let us briefly describe how to go from the original frog model to the branching random walk. Recall that the frog model is entirely determined by a set of trajectories $(S_n^x)_{n \in \mathbb{N}_0, x \in \mathbb{Z}^d}$ of random walks. We use this set of trajectories to produce the particles in the first generation of the branching random walk, i.e. the children of the particle initially at 0, as explained above. Now, assume that the first $n$ generations of the branching random walk have been created. Enumerate the particles in the $n$-th generation. When generating the offspring of the $i$-th particle in this generation, delete all trajectories of the frog model used for generating the offspring of a particle $j$ with $j < i$ or a particle in an earlier generation, and replace them by independent trajectories. Otherwise, use the original trajectories.

One can check that the branching random walk dominates the frog model in the following sense: For every frog in $\mathbb{Z}^d \setminus L_0$ that is activated and visits 0 there is a particle at 0 in the branching random walk. Thus, the number of visits to the origin by particles in the branching random walk is at least as big as the number of visits to 0 by frogs in the frog model, not counting those visits to 0 made by frogs initially in $L_0$. Note that, if the frog model was recurrent, then almost surely there would be infinitely many frogs in $\mathbb{Z}^d \setminus L_0$ activated that return to 0. In particular, also in the branching random walk infinitely many particles would return to 0. Therefore, to prove transience of the frog model it suffices to show that in the branching random walk only finitely many particles return to 0 almost surely.

Let $D_n$ denote the set of descendants in the $n$-th generation of the branching random walk. Further, for $i \in D_n$ let $X_n^i$ be the $e_1$-coordinate of the location of particle $i$. Define for $\theta > 0$ and $n \in \mathbb{N}_0$

$$\mu = \mathbb{E}\Big[\sum_{i \in D_1} \mathrm{e}^{-\theta X_1^i}\Big] \qquad \text{and} \qquad M_n = \frac{1}{\mu^n} \sum_{i \in D_n} \mathrm{e}^{-\theta X_n^i}. \tag{1.35}$$

We claim that $\mu < 1$ for $w$ close to 1 and $\theta$ small, which, in particular, implies that $(M_n)_{n\in\mathbb{N}_0}$ is well-defined. We show this claim in the end of the proof.
We next show that $(M_n)_{n\in\mathbb{N}_0}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ with $\mathcal{F}_n = \sigma\big(D_1, \ldots, D_n, (X_1^i)_{i\in D_1}, \ldots, (X_n^i)_{i\in D_n}\big)$. Obviously, $M_n$ is $\mathcal{F}_n$-measurable. For a particle $i \in D_n$ denote its descendants in generation $n+1$ by $D_{n+1}^i$. Since particles branch independently, we get

$$\begin{aligned} \mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\Big[\frac{1}{\mu^{n+1}} \sum_{i\in D_{n+1}} \mathrm{e}^{-\theta X_{n+1}^i} \,\Big|\, \mathcal{F}_n\Big] \\ &= \frac{1}{\mu^n} \sum_{i\in D_n} \mathrm{e}^{-\theta X_n^i} \cdot \frac{1}{\mu}\mathbb{E}\Big[\sum_{j\in D_{n+1}^i} \mathrm{e}^{-\theta\left(X_{n+1}^j - X_n^i\right)}\Big]. \end{aligned}$$

Note that the expectation on the right hand side is independent of $i$ and $n$ and therefore, by the definition of $\mu$, we conclude

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n.$$

This calculation also yields $\mathbb{E}[|M_n|] = \mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$, and therefore $M_n \in \mathcal{L}^1$. This in particular implies that $M_n$ is finite almost surely for every $n \in \mathbb{N}_0$. Thus, $X_n^i = 0$ can only occur for finitely many $i \in D_n$ almost surely for every $n \in \mathbb{N}_0$, i.e. in every generation only finitely many particles can be at 0. By the martingale convergence theorem, there exists an almost surely finite random variable $M_\infty$, such that $\lim_{n\to\infty} M_n = M_\infty$ almost surely. Combining this with $\mu < 1$, we get $\lim_{n\to\infty} \sum_{i\in D_n} \mathrm{e}^{-\theta X_n^i} = 0$ almost surely. Hence, $X_n^i = 0$ for some $i \in D_n$ occurs only for finitely many times $n$. Overall, this shows that the branching random walk is transient.
It remains to show $\mu < 1$. Note that the particles in $D_1$ are at vertices in the set $\{y \in \mathbb{Z}^d \setminus L_0 \colon 0 \overset{L_0}{\rightsquigarrow} y\}$. Therefore, for the calculation of $\mu$ we first need to consider all sites in $L_0$ that are reached from 0 by frog paths in $L_0$. The idea is to control the number of frogs activated on the negative $e_1$-axis using Lemma 1.33 and estimating the number of frogs activated on the positive $e_1$-axis by assuming the worst case scenario that all of them will be activated. Then, for every $k \in \mathbb{Z}$ we have to estimate the number of vertices with $e_1$-coordinate $k$ visited by each of these active frogs on the $e_1$-axis. Due to the definition of $\mu$, the sites visited by frogs on the positive $e_1$-axis do not contribute much to $\mu$. Recall that $H_k$ denotes the hyperplane that consists of all vertices with

$e_1$-coordinate equal to $k \in \mathbb{Z}$, see (1.6). For $k, i \in \mathbb{Z}$ define

$$N_{k,i} = |\{x \in H_k \setminus L_0 \colon (i, 0, \dots, 0) \to x\}|.$$

As $N_{k,i}$ equals $N_{k-i,0}$ in distribution for all $i, k \in \mathbb{Z}$, we get

$$\begin{aligned}
\mu &= \mathbb{E}\Big[\sum_{i \in D_1} \mathrm{e}^{-\theta X_1^i}\Big] \\
&= \sum_{i=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i, 0, \dots, 0)\big)\mathbb{E}[N_{k,i}]\mathrm{e}^{-\theta k} \\
&= \sum_{k=-\infty}^{\infty} \mathbb{E}[N_{k,0}]\mathrm{e}^{-\theta k} \sum_{i=-\infty}^{\infty} \mathrm{e}^{-\theta i}\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i, 0, \dots, 0)\big). \qquad (1.36)
\end{aligned}$$

Note that $\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i, 0, \dots, 0)\big)$ is smaller or equal than the probability of the event $\{0 \overset{\mathbb{Z}}{\rightsquigarrow} i\}$ in the frog model $\mathrm{FM}(1, 1, \alpha)$. Hence, by Lemma 1.33, there is a constant $c_1 > 0$ such that $\mathbb{P}\big(0 \overset{L_0}{\rightsquigarrow} (i, 0, \dots, 0)\big) \leq \mathrm{e}^{c_1 i}$ for all $i \leq 0$. Thus, (1.36) implies that for $\theta < c_1$ there is a constant $c_2 = c_2(\theta) < \infty$ such that

$$\mu \leq c_2 \sum_{k=-\infty}^{\infty} \mathbb{E}[N_{k,0}]\mathrm{e}^{-\theta k}. \qquad (1.37)$$

Next, we estimate $\mathbb{E}[N_{k,0}]$, the expected number of vertices in $H_k \setminus L_0$ visited by a single particle starting at 0. Recall that the trajectory of frog 0 is denoted by $(S_n^0)_{n \in \mathbb{N}_0}$. We define $T_k = \min\{n \in \mathbb{N}_0 \colon S_n^0 \in H_k\}$, the entrance time of the hyperplane $H_k$, and $T_k' = \max\{n \in \mathbb{N}_0 \colon S_n^0 \in H_k\}$, the last time frog 0 is in the hyperplane $H_k$. Obviously, $N_{k,0} = 0$ on the event $\{T_k = \infty\}$. Hence, assume we are on $\{T_k < \infty\}$. The particle can only visit a vertex in $H_k \setminus L_0$ at time $T_k$ if the random walk took at least one step in non-$e_1$-direction up to time $T_k$. This happens with probability $\mathbb{E}[1 - w^{T_k}]$. Furthermore, the number of vertices visited in $H_k$ after time $T_k$ can be estimated by the number of steps in non-$e_1$-direction taken between times $T_k$ and $T_k'$. This number is binomially distributed and, thus, its expectation equals $(1 - w)\mathbb{E}[T_k' - T_k]$. Overall, this implies

$$\mathbb{E}[N_{k,0}] \leq \mathbb{P}(T_k < \infty)\big(\mathbb{E}\big[1 - w^{T_k} \mid T_k < \infty\big] + (1 - w)\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]\big).$$

For $k < 0$ the probability $\mathbb{P}(T_k < \infty)$ decays exponentially in $k$ by Lemma 1.29. Therefore, we can choose $\theta$ small such that $\mathbb{P}(T_k < \infty)\mathrm{e}^{-\theta k} \leq \mathrm{e}^{-\theta |k|}$ for all $k \in \mathbb{Z}$. Thus, (1.37) implies

$$\mu \leq c_2 \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\theta |k|}\Big(\mathbb{E}\big[1 - w^{T_k} \mid T_k < \infty\big] + (1 - w)\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]\Big). \qquad (1.38)$$

Note that the sum in (1.38) is finite as $\mathbb{E}\big[T_k' - T_k \mid T_k < \infty\big]$ is independent of $k$. By monotone convergence $\lim_{w\to 1} \mu = 0$ and the right hand side of (1.38) is continuous in $w$. Therefore, we can choose $w$ close to 1 such that $\mu < 1$, as claimed.

### Transience for $d = 2$ and arbitrary weight

*Proof of Theorem 1.20 (ii).* Let $w > 0$ be fixed throughout the proof. As in the proof of Theorem 1.21 (ii) and Theorem 1.23 (ii) we dominate the frog model by a branching random walk. This time we use a one-dimensional branching random walk on $\mathbb{Z}$. For the construction of the process, let $\xi$ be the number of activated frogs in an independent one-dimensional frog model $\mathrm{FM}^*(1, \pi_{\mathrm{sym}}, 1-w)$ with two active frogs at 0 initially. At time $n = 0$, the branching random walk starts with one particle in the origin. At every time $n \in \mathbb{N}$, the process repeats the following two steps. First, every particle produces offspring independently of all other particles with the number of offspring being distributed as $\xi$. Then, each particle jumps to the right with probability $\frac{1+\alpha}{2}$ and to the left with probability $\frac{1-\alpha}{2}$.
As an intermediate step to understand the relation between the frog model and this branching random walk on $\mathbb{Z}$, we first couple the frog model with a branching random walk on $\mathbb{Z}^2$ with initially one particle at 0. Partition the lattice $\mathbb{Z}^2$ into hyperplanes $(H_n)_{n\in\mathbb{Z}}$ as defined in (1.6). Let the frog model $\mathrm{FM}(2, \pi_{w,\alpha})$ with initially two active frogs at $0 \in H_0$ evolve and stop every frog when it first enters $H_1$ or $H_{-1}$. Every frog leaves its hyperplane in every step with probability $w$. Thus, the number of stopped frogs is distributed according to $\xi$. A stopped frog is in $H_1$ with probability $\frac{1+\alpha}{2}$ and in $H_{-1}$ with probability $\frac{1-\alpha}{2}$. The stopped particles form the offspring of the particle at 0 in the branching random walk. We repeat this procedure to generate the offspring of an arbitrary particle in the branching random walk. Introduce an ordering of all particles in the branching random walk and let the particles branch one after another. Before generating the offspring of the $i$-th particle, refill every vertex which is no longer occupied by a sleeping frog with an extra independent sleeping frog. Unstop frog $i$ and let it continue its work as usual, ignoring all other stopped frogs. Note that there is a sleeping frog at the starting vertex of frog $i$ that is immediately activated. This explains our definition of $\xi$. Again stop every frog once it enters one of the neighbouring hyperplanes. These newly stopped frogs form the offspring of the $i$-th particle. This procedure creates a branching random walk with independent identically distributed offspring. Every vertex visited in the frog model is obviously also visited by the branching random walk.
Now, project all particles in the intermediate two-dimensional branching random walk onto the first coordinate. This creates a branching random walk on $\mathbb{Z}$ distributed as the one described above. The construction shows that transience of this one-dimensional branching random walk implies transience of the frog model.
To prove that the one-dimensional branching random walk is transient for $\alpha$ close to 1,

we proceed as in the proof of Theorem 1.21 (ii) and Theorem 1.23 (ii). The proof only differs in the calculation of the parameter $\mu$ defined by

$$\mu = \mathbb{E}\Big[\sum_{i \in D_1} \mathrm{e}^{-\theta X_1^i}\Big]$$

for $\theta > 0$ with $D_1$ denoting the set of descendants in the first generation of the branching random walk and $X_1^i$ the $e_1$-coordinate of the location of particle $i \in D_1$. Here, we immediately get

$$\mu = \frac{1}{2}\big((1-\alpha)\mathrm{e}^{\theta} + (1+\alpha)\mathrm{e}^{-\theta}\big)\mathbb{E}[\xi].$$

Lemma 1.33 implies $\mathbb{E}[\xi] < \infty$. Thus, we can choose $\theta = \log\big(2\mathbb{E}[\xi]\big)$. Then $\lim_{\alpha \to 1} \mu = \frac{1}{2}$ and by continuity $\mu < 1$ for $\alpha$ close to 1, as required.

# 2 Branching Random Walks

## 2.1 Description of the model

We study the maximum of a branching random walk on $\mathbb{R}$ in discrete time. First, we describe the branching random walk in a more formal way than explained in the introduction and fix some notation.

Let $(Z_n)_{n\in\mathbb{N}_0}$ be a Galton-Watson process with one initial particle and offspring law given by the weights $(p(k))_{k\in\mathbb{N}_0}$, where $\sum_{k=0}^{\infty} p(k) = 1$. More precisely, let $(\xi_{n,k})_{n,k\in\mathbb{N}}$ be a collection of i.i.d. random variables with $\mathbb{P}(\xi_{1,1} = k) = p(k)$ for all $k \in \mathbb{N}_0$. Let $Z_0 = 1$. The number of particles in the $n$-th generation is defined as $Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$. Let $m = \mathbb{E}[\xi_{1,1}] = \sum_{k=1}^{\infty} kp(k)$ be the reproduction mean.

To define the movement of the particles we consider the associated Galton-Watson tree denoted by $\mathcal{T} = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges. It has $Z_n$ vertices at level $n$ and for $k \leq Z_n$, the $k$-th vertex in level $n$ has $\xi_{n+1,k}$ children. For $n \in \mathbb{N}$ let $D_n$ be the set of vertices in the $n$-th level of the tree. Then, $|D_n| = Z_n$. For $v \in D_n$, the set of descendants of $v$ in the $(l+n)$-th level is denoted by $D_l^v$. Note that $|D_l^v|$ equals $|D_l|$ in distribution. The root of $\mathcal{T}$ is called $o \in V$. For $v, w \in V$ define $[v, w]$ as the set of edges along the unique path from $v$ to $w$.

We now define the locations of the particles. Let $(X_e)_{e\in E}$ be a collection of i.i.d. random variables, i.e. every edge of $\mathcal{T}$ is labelled with a random variable. For $v \in D_n$, the position of the particle $v$ at time $n$ is defined as $S_v = \sum_{w\in[o,v]} X_w$. For $n \in \mathbb{N}$ the position of the rightmost particle at time $n$ is

$$M_n = \max_{v\in D_n} S_v. \tag{2.1}$$

We set $M_n = -\infty$ if $D_n = \emptyset$. We refer to $(M_n)_{n\in\mathbb{N}}$ as the maximum of the branching random walk. For $v \in D_n$, the rightmost descendant of $v$ at time $l + n$ is defined as $M_l^v = \max_{w\in D_l^v} S_w$.

We also introduce the collection of i.i.d. random variables $(X_i^j)_{i,j\in\mathbb{N}}$, where $X_1^1$ has the same distribution as $X_e$ for some $e \in E$. Moreover, for $j, n \in \mathbb{N}$ define the random walk $S_n^j = \sum_{i=1}^{n} X_i^j$ and the maximum of independent random walks as

$$\tilde{M}_n = \max_{1\leq j\leq Z_n} S_n^j. \tag{2.2}$$

In analogy to the maximum of the branching random walk, we set $\tilde{M}_n = -\infty$ if $D_n = \emptyset$. Furthermore, for $i \in \mathbb{N}$, let $X_i$ be an independent copy of $X_i^1$ and define $S_n = \sum_{i=1}^{n} X_i$.

Note that for every time $n$ the number of particles in the branching random walk equals the number of random walks considered for $\tilde{M}_n$. However, the positions of the particles in the branching random walk are not independent. Indeed, this dependence is such that the maximum of independent random walks stochastically dominates the maximum of the branching random walk, see Lemma 2.27. We investigate the asymptotics of $M_n$ and $\tilde{M}_n$ conditioned on the event of survival of the Galton-Watson process, i.e. on the event $\{Z_n > 0 \ \forall n \in \mathbb{N}\}$. Therefore, introduce the measure

$$\mathbb{P}^*(\cdot) = \mathbb{P}(\cdot | Z_n > 0 \ \forall n \in \mathbb{N}). \tag{2.3}$$

The associated expectation is denoted by $\mathbb{E}^*$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers and let $c \in (0, \infty]$ be a constant. With a slight abuse of notation for $c = \infty$, we write $a_n = \exp(-cn + o(n))$, if $\lim_{n\to\infty} \frac{1}{n} \log a_n = -c$. Note that $a_n$ decays faster than exponentially in $n$ if $c = \infty$.

## 2.2 Some known results

In the first subsection we state some elementary results on Galton-Watson processes. In the remaining subsections we collect some results on the maximum of the branching random walk and on the maximum of independent random walks. Throughout this chapter we assume that the displacements are centred, i.e. $\mathbb{E}[X_1] = 0$. Note that every branching random walk can be restricted to this case by considering the family of displacements $(X_e - \mathbb{E}[X_1])_{e\in\mathbb{E}}$. Further note that most of the following results hold true in a more general setting, i.e. if the branching and movement of the particles are not independent.

### 2.2.1 On the Galton-Watson process

The first theorem gives the exact value of the survival probability of the Galton-Watson process, i.e. of $\mathbb{P}(Z_n > 0 \ \forall n \in \mathbb{N})$. Moreover we get a necessary and sufficient condition for positive survival probability. Let $q = \inf\{s \in [0, 1] \colon \mathbb{E}[s^{Z_1}] = s\}$.

**Assumption 1.** *The Galton-Watson process is supercritical, i.e.* $m > 1$.

**Theorem 2.1.** *The Galton-Watson process has survival probability* $1 - q$. *In particular, the survival probability is positive if and only if Assumption 1 is satisfied, or* $p(1) = 1$.

A proof can be found in [7, Chapter 1, Section 5, Theorem 1]. In the proof of Theorem 2.18 and Theorem 2.19 we also need the asymptotics of the survival probability of a critical Galton-Watson process.

**Theorem 2.2.** *Let* $m = 1$ *and* $p(1) < 1$. *Then,* $\lim_{n\to\infty} n\mathbb{P}(Z_n > 0) = \frac{2}{\mathrm{Var}(Z_1)}$.

A proof can be found in [7, Chapter 1, Section 9, Theorem 1].

We often need to estimate the number of particles at time $n$, which has expectation $m^n$. Let $W_n = \frac{Z_n}{m^n}$ and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the natural filtration of the Galton-Watson process, i.e. $\mathcal{F}_n = \sigma(Z_1, \dots, Z_n)$. The process $(W_n)_{n\in\mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Therefore, $W_n \to W$ almost surely, where $W$ is an almost surely finite random variable.

**Assumption 2.** *The Galton-Watson process satisfies* $\mathbb{E}[Z_1 \log Z_1] < \infty$.

Note that Assumption 2 implies $m < \infty$. The following well-known theorem shows that under our assumptions, the limit $W$ is non-trivial, i.e. $\mathbb{P}(W = 0) < 1$.

**Theorem 2.3** (Kesten-Stigum)**.** *If Assumption 1 and 2 are satisfied, we have*

$$\mathbb{E}[W] = 1 \quad \textit{and} \quad \mathbb{P}(W = 0) = q < 1.$$

A proof can e.g. be found in [7, Chapter 1, Section 10, Theorem 1].

### 2.2.2 First term of the maximum

In the remaining subsections we summarise some known results on the maximum of the branching random walk defined in (2.1) and on the maximum of independent random walks defined in (2.2).
For $x \in \mathbb{R}$ the rate function of the random walk $(S_n)_{n\in\mathbb{N}}$ is defined as

$$I(x) = \sup_{\lambda\in\mathbb{R}} \big(\lambda x - \log \mathbb{E}\big[e^{\lambda X_1}\big]\big). \tag{2.4}$$

**Assumption 3.** *There exists* $\varepsilon > 0$ *such that* $\mathbb{E}\big[e^{\lambda X_1}\big] < \infty$ *for all* $\lambda \in (-\varepsilon, \varepsilon)$*. Furthermore, for simplicity suppose that* $\mathbb{E}[X_1] = 0$*.*

If Assumption 1 and 3 are satisfied, then $M_n$ grows at linear speed $x^*$, where

$$x^* = \sup\big\{x \geq 0 \colon I(x) \leq \log m\big\}. \tag{2.5}$$

Note that $x^*$ is finite if $m < \infty$ and Assumption 3 are satisfied.

**Theorem 2.4.** *Suppose that Assumption 1 and 3 are satisfied. The maximum of the branching random walk has linear speed* $x^*$*, i.e.*

$$\lim_{n\to\infty} \frac{M_n}{n} = x^* \quad \mathbb{P}^*\text{-}a.s.$$

This result goes back to Biggins [11], Hammersley [35] and Kingman [44]. One can check that the speed of $(\tilde{M}_n)_n$ also equals $x^*$, see [60, Theorem 1] for deterministic branching. For general branching this is a consequence of Theorem 2.18.

**Theorem 2.5.** *Suppose that Assumption 1 and 3 are satisfied. The maximum of independent random walks has linear speed $x^*$, i.e.*

$$\lim_{n\to\infty} \frac{\tilde{M}_n}{n} = x^* \quad \mathbb{P}^*\text{-a.s.}$$

If the displacements have no exponential moments, then the maximum moves faster than linear in time. For stretched exponential tails the speed of the maximum of the branching random walk was determined by Gantert in [28].

**Theorem 2.6** ([28, Theorem 2])**.** *Assume that there exists $r \in (0,1)$ and slowly varying functions $a, L$ such that $\frac{L(t)}{t^{1-r}}$ is non-increasing and $\mathbb{P}(X_1 \geq t) = a(t)e^{-L(t)t^r}$ for all $t$ large enough. Furthermore, let $\psi$ be a positive function such that $\lim_{n\to\infty} \frac{L(\psi(n))\psi(n)^r}{n} = 1$. Then,*

$$\lim_{n\to\infty} \frac{M_n}{\psi(n)} = (\log m)^{1/r} \quad \mathbb{P}^*\text{-a.s.}$$

If for instance $L(t) \equiv b$ for some $b > 0$ we can take $\psi(n) = b^{-1/r}n^{1/r}$.
For regularly varying tails of the displacement the speed was derived by Durrett in [24]. Recall the martingale limit $W$ defined in subsection 2.2.1.

**Theorem 2.7** ([24, Theorem 1])**.** *Suppose that Assumption 2 is satisfied. Assume that there is a slowly varying function $L$ and $\alpha > 0$ such that $\mathbb{P}(X_1 \geq t) = t^{-\alpha}L(t)$ for all $t$ large enough and $\lim_{t\to\infty} \log t\mathbb{P}(X \leq -t) = 0$. Furthermore, choose $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} m^n\mathbb{P}(X_1 > a_n) = 1$. Let $r = \sum_{j=0}^{\infty} m^{-j}\mathbb{P}(Z_j > 0)$. Then, for all $x > 0$*

$$\lim_{n\to\infty} \mathbb{P}\Big(\frac{M_n}{a_n} \leq x\Big) = \int_0^\infty e^{-yx^{-\alpha}}\ \mathbb{P}(rW \in \mathrm{d}y).$$

If $L(t) \equiv b$ for some $b > 0$ we can take $a_n = b^{1/\alpha}m^{n/\alpha}$ for all $n \in \mathbb{N}$.

### 2.2.3 Second term of the maximum

In this subsection we always assume that we are in the setting of Theorem 2.4, i.e. the maximum of the branching random walk as well as the maximum of independent random walks moves with linear speed $x^*$. Throughout this subsection we make the following assumption.

**Assumption 4.** *All exponential moments of the displacements exist, i.e. $\mathbb{E}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$. Furthermore, there exists $\varepsilon > 0$ such that $\mathbb{E}[Z_1^{1+\varepsilon}] < \infty$.*

Assumption 4 implies that $I(x^*) = \log m$ and $I$ is infinitely often differentiable on $\mathbb{R}$. Note that the following results hold true under slightly weaker (more technical) assumptions and in a more general setting. We refer to the book of Shi [55] for more details. Moreover note that in most cases only critical branching random walks are considered. More precisely, let

$$\psi(t) = \log \mathbb{E}\Big[\sum_{v\in D_1} e^{tS_v}\Big] = \log m + \log \mathbb{E}\big[e^{tX_1}\big].$$

A critical branching random walk satisfies $\psi(0) > 0$ and $\psi(1) = \psi'(1) = 0$. In particular it has speed 0. However, in our setup a general branching random walk can be transformed to a critical branching random walk by considering the collection of displacements $\big(I'(x^*)(X_e - x^*)\big)_{e\in E}$.

In the setting of Assumption 4 precise asymptotics for the maximum are known. Addario-Berry and Reed [1] as well as Hu and Shi [41] obtain a logarithmic second term.

**Theorem 2.8** ([41, Theorem 1.2])**.** *Assume that Assumption 4 is satisfied. The maximum of the branching random walk has a logarithmic second term. More precisely,*

$$\begin{aligned} \liminf_{n\to\infty} \frac{M_n - x^*n}{\log n} &= -\frac{3}{2I'(x^*)} \qquad \mathbb{P}^*\text{-}a.s.\\ \limsup_{n\to\infty} \frac{M_n - x^*n}{\log n} &= -\frac{1}{2I'(x^*)} \qquad \mathbb{P}^*\text{-}a.s.\\ \lim_{n\to\infty} \frac{M_n - x^*n}{\log n} &= -\frac{3}{2I'(x^*)} \qquad \textit{in } \mathbb{P}^*\textit{-probability.} \end{aligned}$$

Addario-Berry and Reed [1] calculated $\mathbb{E}[M_n]$ to within $O(1)$ assuming that the number of offspring and displacements are bounded.

Also the maximum of independent random walks has a logarithmic second term.

**Theorem 2.9.** *Assume that Assumption 4 is satisfied. The maximum of independent random walks has a logarithmic second term. More precisely,*

$$\lim_{n\to\infty} \frac{\tilde{M}_n - x^*n}{\log n} = -\frac{1}{2I'(x^*)} \qquad \mathbb{P}^*\text{-}a.s.$$

For deterministic branching, i.e. if $\mathbb{P}(Z_1 = k) = 1$ for some $k \in \mathbb{N}$ this result follows from [60, Theorem 1]. However, the same arguments immediately apply to our setting. Note that $M_n$ and $\tilde{M}_n$ have the same speed $x^*$, but the maximum of independent random walks $\tilde{M}_n$ has a larger logarithmic term.

In [2] Aïdékon finally proves that the normalised maximum of the branching random walk converges in distribution to a random shift of a Gumbel variable.

**Theorem 2.10** ([2, Theorem 1.1])**.** *Assume that Assumption 4 is satisfied and $X_1$ is non-lattice, i.e. there exists no $a, b \in \mathbb{R}$ such that $\mathbb{P}(X_1 \in \{az + b \colon z \in \mathbb{Z}\}) = 1$. Then, for all $u \in \mathbb{R}$*

$$\lim_{n\to\infty} \mathbb{P}^*\Big(M_n - x^*n + \frac{3}{2I'(x^*)}\log n < u\Big) = \mathbb{E}^*\Big[e^{-De^{-u}}\Big],$$

*where $D$ is a $\mathbb{P}^*$-almost surely positive and finite random variable.*

Similar results were proved for branching Brownian motion by Bramson in [14]. Even the third term of $M_n$ is known.

**Theorem 2.11** ([39, Theorem 1.1])**.** *Assume that Assumption 4 is satisfied. Then,*

$$\limsup_{n\to\infty} \frac{M_n - x^* n + \frac{1}{2I'(x^*)}\log n}{\log\log n} = \frac{1}{I'(x^*)} \quad \mathbb{P}^*\text{-}a.s.$$

**Theorem 2.12** ([40, Theorem 1.1])**.** *Assume that Assumption 4 is satisfied and $X_1$ is non-lattice. Then,*

$$\liminf_{n\to\infty} \frac{M_n - x^* n + \frac{3}{2I'(x^*)}\log n}{\log\log\log n} = -\frac{1}{I'(x^*)} \quad \mathbb{P}^*\text{-}a.s.$$

### 2.2.4 Large deviations

As already explained in the introduction, we investigate the exponential decay rates of the probabilities $\mathbb{P}\big(\frac{M_n}{n} \geq x\big)$ for $x \geq x^*$ and $\mathbb{P}\big(\frac{M_n}{n} \leq x\big)$ for $x \leq x^*$. Our main result in this chapter, Theorem 2.19, characterises these exponential decay rates. We consider the same question for $\tilde{M}_n$ and determine the exponential decay rates, see Theorem 2.18. Interestingly, the rate functions coincide for $x \geq x^*$, but in general they do not coincide for $x < x^*$.

Similar questions have been studied before. Large deviation estimates for the maximum of branching Brownian motion have first been investigated by Chauvin and Rouault in [16] and very recently by Derrida and Shi in [21] and [20]. See also [19] and [56] for extensions with coalescence and selection or immigration, respectively. Note that [20] also treats continuous time branching random walks. The difference to our setup is that in the time-continuous case, the strategies can involve the exponential waiting times, while in our setup, they can involve the branching mechanism given by the offspring distribution.

Upper large deviations for the maximum of discrete time branching random walks have been investigated by Rouault in [54] in the case where the displacemets have exponential moments (i.e. we are in the setting of Theorem 2.4) and every particle has at least one offspring.

**Theorem 2.13** ([54, Theorem 2.1])**.** *Suppose that Assumption 1 and 3 is satisfied. Moreover, assume that every particle has at least one offspring, i.e. $p(0) = 0$ and there exists $\varepsilon' > 0$ such that $\mathbb{E}[Z_1(\log Z_1)^{1+\varepsilon'}] < \infty$. Then, for all $x > x^*$ and $\delta > 0$ there exists $C = C(x,\delta) > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{m^n \mathbb{P}(|S_n - xn| \leq \delta)} \mathbb{P}\big(|M_n - xn| \leq \delta\big) = C.$$

Very recently Bhattacharya proved an upper large deviation result for heavy-tailed displacements (i.e. in the setting of Theorem 2.7).

**Theorem 2.14** ([10, Corollary 2.6])**.** *Suppose that Assumption 2 is satisfied. Assume that there is a slowly varying function $L$ and $\alpha > 0$ such that $\mathbb{P}(|X_1| \geq t) = x^{-\alpha} L(t)$ for all $t$ large enough. Furthermore, there is $p, q > 0$ with $p+q = 1$ such that $\lim_{t\to\infty} \frac{\mathbb{P}(X_1 \geq t)}{\mathbb{P}(|X_1| \geq t)} = p$ and $\lim_{t\to\infty} \frac{\mathbb{P}(X_1 \leq -t)}{\mathbb{P}(|X_1| \geq t)} = q$. Choose $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} m^n \mathbb{P}(X_1 > a_n) = 1$. Let $(\gamma_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers with $\lim_{n\to\infty} \frac{a_n}{\gamma_n} = 0$.*
*For all $x > 0$*

$$\lim_{n\to\infty} \frac{1}{m^n \gamma_n^{-\alpha} L(\gamma_n)} \mathbb{P}\Big( \frac{M_n}{\gamma_n} > x \Big) = C x^{-\alpha},$$

*where $C > 0$ is a constant.*

Recently large deviation results for the empirical distribution of the branching random walk have been obtained in [17], [49], [50]. We also mention that in the case of a fixed number of offspring, much more precise results (describing not only the exponential decay rates) for first passage times were derived in [15].

## 2.3 Rate functions and assumptions

In this section we introduce the rate functions of the Galton-Watson process, which are needed to state our results in Section 2.4. Furthermore, we collect some large deviation results, which are used in the proofs of the results in Section 2.6.

### 2.3.1 Rate function of the random walk

Recall the definition of the rate function of the random walk $(S_n)_{n\in\mathbb{N}}$, see (2.4). If Assumption 3 is satisfied, Cramér's theorem implies that the probability $\mathbb{P}(S_n \geq xn)$ decays exponentially in $n$ with rate $I(x)$ for $x > 0$.

**Theorem 2.15.** *Suppose that Assumption 3 is satisfied. The random walk $(S_n)_{n\in\mathbb{N}}$ satisfies a large deviation principle, i.e.*

$$-I(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{S_n}{n} \geq x\big) & \text{for } x \geq 0, \\ \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{S_n}{n} \leq x\big) & \text{for } x \leq 0. \end{cases}$$

A proof can e.g. be found in [18, Theorem 2.2.3 and Lemma 2.2.5]. Assumption 3 ensures that $I(x) > 0$ for all $x \neq 0$ and $I(x) \to \infty$ as $|x| \to \infty$.

### 2.3.2 Rate functions of the Galton-Watson process

Due to the well-known Kesten-Stigum Theorem, Assumption 2 implies that the Galton-Watson process grows like its expectation, see Theorem 2.3. In particular, $(Z_n)_{n\in\mathbb{N}_0}$ grows exponentially with high probability.

If Assumption 1 and 2 are satisfied, there is a large deviation result for the probability that $(Z_n)_{n\in\mathbb{N}_0}$ grows at most subexponentially. A sequence $(a_n)_{n\in\mathbb{N}}$ is called subexponential, if $a_n e^{-\varepsilon n} \to 0$ as $n \to \infty$ for all $\varepsilon > 0$. Define

$$\rho := -\log \mathbb{E}[Z_1 q^{Z_1 - 1}] \in (0, \infty]. \tag{2.6}$$

Note that $\rho = -\log p(1)$ if $p(0) = 0$ (and therefore also $q = 0$).

**Assumption 5.** *Each particle has less than two offspring particles with positive probability, i.e.* $p(0) + p(1) > 0$.

Assumption 5 is often referred to as Schröder case, whereas the case $p(0) + p(1) = 0$ is called Böttcher case. We have $\rho < \infty$ if and only if Assumption 5 is satisfied. Consider the set

$$A = \big\{ l \in \mathbb{N} \colon \exists n \in \mathbb{N} \text{ such that } \mathbb{P}(Z_n = l) > 0 \big\} \tag{2.7}$$

containing all positive integers $l$ such that there are $l$ particles at some time $n$ with positive probability.

**Theorem 2.16.** *Let Assumption 1 and 2 hold. Then, for every $k \in A$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\big(Z_n = k\big) = -\rho.$$

*Moreover, for every subexponential sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \to \infty$ as $n \to \infty$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\big(Z_n \le a_n\big) = -\rho.$$

A proof of the first statement can be found in [7, Chapter 1, Section 11, Theorem 3]. The second statement is a consequence of of [9, Theorem 3.1].
For $x \in [0, \log m]$ define the rate function of the Galton-Watson process as

$$I^{\mathrm{GW}}(x) = \rho\big(1 - x(\log m)^{-1}\big). \tag{2.8}$$

Note that $I^{\mathrm{GW}}(x) > 0$ for all $x < \log m$.

**Theorem 2.17.** *Under Assumption 1 and 2 we have for $x \in [0, \log m]$*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\big(Z_n \le e^{xn}\big) = -I^{\mathrm{GW}}(x).$$

This theorem is a consequence of [9, Theorem 3.2]. Note that there is also an upper large deviation result for $\mathbb{P}^*\big(Z_n \ge e^{xn}\big)$, where $x > \log m$, see e.g. [8, Theorem 1].

## 2.4 Main results

After defining the rate functions of the random walk and the Galton-Watson process we are now able to state our main results of this chapter. The results in this section are joint work with Gantert and published in [29].
Note that $I(x^*) = \log m$ if $I(x) < \infty$ for some $x > x^*$. On the other hand, $I(x^*) < \log m$ already implies $\mathbb{P}(X_1 > x^*) = 0$. This case leads to a different shape of the rate functions, see Figure 2.1. Let

$$k^* = \inf\{k \geq 1 \colon p(k) > 0\}. \tag{2.9}$$

Note that $k = k^*$ is the smallest positive integer, such that $\mathbb{P}(Z_n = k) > 0$ for some $n \in \mathbb{N}$. Define the rate function for the maximum of independent random walks as

$$I^{\text{ind}}(x) = \begin{cases} I(x) - \log m & \text{for } x > x^*, \\ 0 & \text{for } x = x^*, \\ \rho\big(1 - \frac{I(x)}{\log m}\big) & \text{for } 0 \leq x < x^*, \\ k^* I(x) + \rho & \text{for } x \leq 0. \end{cases} \tag{2.10}$$

Note that $\rho\big(1 - \frac{I(x)}{\log m}\big) = I^{\text{GW}}(I(x))$ for $0 \leq x < x^*$. Recall the maximum $\tilde{M}_n$ of a random number of independent walks, defined in (2.2).

**Theorem 2.18.** *Suppose that Assumption 1, 2 and 3 are satisfied. Then, the laws of $\frac{\tilde{M}_n}{n}$ under $\mathbb{P}^*$ satisfy a large deviation principle with rate function $I^{\text{ind}}$. More precisely,*

$$-I^{\text{ind}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{\tilde{M}_n}{n} \geq x\big) & \textit{for } x \geq x^*, \\ \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{\tilde{M}_n}{n} \leq x\big) & \textit{for } x \leq x^*. \end{cases}$$

In the Böttcher case ($p(0) + p(1) = 0$) we have $\rho = \infty$ and therefore $I^{\text{ind}}(x) = \infty$ for all $x < x^*$. Hence, in this case the lower deviation probabilities $\mathbb{P}^*(\tilde{M}_n \leq xn)$ for $x < x^*$ decay faster than exponentially in $n$.
Let us now give some intuition for the rate function $I^{\text{ind}}$ and describe the large deviation event $\{\tilde{M}_n \geq xn\}$ for some $x > x^*$, respectively $\{\tilde{M}_n \leq xn\}$ for some $x < x^*$.
For $x > x^*$, the number of particles should be larger or equal than expected, i.e. $Z_n \geq e^{nt}$ for some $t \geq \log m$. The probability of such an event is of order $\exp(-I^{\text{GW}}(t)n + o(n))$. If there are $e^{nt}$ particles at time $n$, the probability that at least one particle reaches $xn$ is of order $\exp(-I(x)n + tn + o(n))$ for $t < I(x)$. Therefore, we need to maximize the product of these two probabilities, which amounts to minimize $I^{\text{GW}}(t) + I(x) - t$, where $t$ runs over the interval $[\log m, I(x))$. It turns out that the optimal value is $t = \log m$. This argument will go through for the maximum of the branching random walk.
If $0 \leq x < x^*$, the probability that one particle reaches $xn$ is of order $\exp(-I(x)n + o(n))$. Hence, for every $\varepsilon > 0$, if there are less than $e^{(I(x)-\varepsilon)n}$ particles, the probability that

none of these particles reaches $xn$ is close to 1. However, if there are more than $e^{(I(x)+\varepsilon)n}$ particles, this probability decays exponentially in $n$.
If $x < 0$, already the probability that a single particle is below $xn$ at time $n$ decays exponentially fast in $n$. Hence, if the number of particles $Z_n$ grows exponentially, the probability that all particles are below $xn$ at time $n$ decays faster than exponentially. Therefore, the number of particles needs to grow subexponentially. Since $\rho$ does not depend on the choice of $k$ in Theorem 2.16, there have to be only $k^*$ particles at time $n$ (provided that $\rho < \infty$).
Next, we consider the maximum of the branching random walk. For $x < x^*$ let

$$H(x) = \inf_{t\in(0,1]} \Big\{ t\rho + tI\big(t^{-1}\big(x - (1-t)x^*\big)\big) \Big\}. \tag{2.11}$$

Note that for $x > 0$ it suffices to take the infimum over $t \in (0, 1 - \frac{x}{x^*}]$. Define the rate function of the branching random walk as

$$I^{\mathrm{BRW}}(x) = \begin{cases} I(x) - \log m & \text{for } x > x^*, \\ 0 & \text{for } x = x^*, \\ H(x) & \text{for } x < x^*. \end{cases} \tag{2.12}$$

**Theorem 2.19.** *Suppose that Assumption 1, 2, 3 and 5 are satisfied. Then, the laws of $\frac{M_n}{n}$ under $\mathbb{P}^*$ satisfy a large deviation principle with rate function $I^{\mathrm{BRW}}$. More precisely,*

$$-I^{\mathrm{BRW}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{M_n}{n} \geq x\big) & \text{for } x \geq x^*, \\ \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\frac{M_n}{n} \leq x\big) & \text{for } x \leq x^*. \end{cases}$$

In contrast to the case of independent random walks we only consider the Schröder case (Assumption 5) for the branching random walk.

**Remark 2.20.** *Assumption 5 is only needed for the lower deviations ($x < x^*$) in Theorem 2.19. In the Böttcher case, i.e. if Assumption 5 is not satisfied, the strategy for lower deviations is different.*

**Proposition 2.21.** *The rate function of the maximum of the branching random walk $I^{\mathrm{BRW}}$ is convex.*

Note that the rate function of the maximum of independent random walks $I^{\mathrm{ind}}$ is concave on the interval $[0, x^*]$.
For $x > x^*$ we have $I^{\mathrm{BRW}}(x) = I^{\mathrm{ind}}(x)$. In this case the strategy is the same as for independent random walks. The strategy in the case $x < x^*$ goes as follows. At time $tn$ there are only $k^*$ particles, and the position of one of those particles is smaller than its expectation. All other $k^* - 1$ particles are killed at time $tn$. Note that by Assumption 5 either $k^* = 1$ or particles may have no offspring with positive probability. Afterwards, each particle moves and branches according to its usual behaviour.

Further notice, that in contrast to the case of independent random walks, the number of particles can also grow exponentially if $x < 0$. It suffices to have a small number of particles at time $tn$ for some $t \in (0, 1]$.

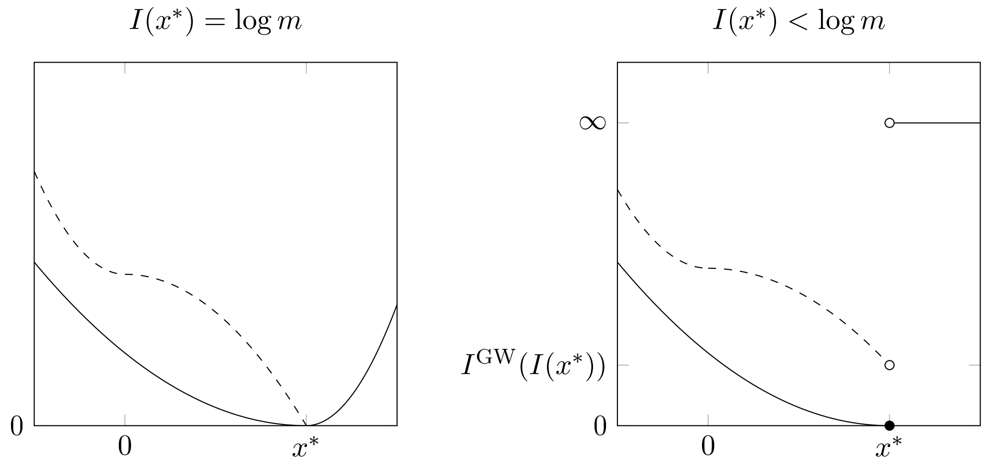


Figure 2.1: The figure shows the qualitative behaviour of the rate function of the branching random walk (——) and the rate function of independent random walks (- - -).

To compare the rate functions, note that the maximum of independent random walks stochastically dominates the maximum of the branching random walk, see Lemma 2.27. Therefore, $I^{\mathrm{ind}}(x) \leq I^{\mathrm{BRW}}(x)$ for $x > x^*$, respectively $I^{\mathrm{ind}}(x) \geq I^{\mathrm{BRW}}(x)$ for $x < x^*$. For $x < x^*$, the inequality is in general strict. For $x > x^*$, the rate functions coincide, see the argument above.

Let us now comment on the shape of the rate functions. If $I(x) = \infty$ for some $x > x^*$, also $I^{\mathrm{ind}}(x) = I^{\mathrm{BRW}}(x) = \infty$. More precisely, $I(x) = \infty$ already implies $\mathbb{P}(X_1 \geq x - \varepsilon) = 0$ for some $\varepsilon > 0$ and therefore $M_n \leq (x - \varepsilon)n$, respectively $\tilde{M}_n \leq (x - \varepsilon)n$ almost surely. If $I(x^*) = \log m$, then the rate functions $I^{\mathrm{ind}}(x)$ and $I^{\mathrm{BRW}}(x)$ are continuous from the left at $x = x^*$. However, if $I(x^*) < \log m$, the rate functions $I^{\mathrm{ind}}(x)$ and $I^{\mathrm{BRW}}(x)$ are infinite for $x > x^*$, since $I(x) = \infty$. Therefore, they are not continuous from the right at $x = x^*$. The rate function $I^{\mathrm{BRW}}(x)$ is continuous from the left at $x = x^*$, since $I^{\mathrm{BRW}}(x) \leq \rho\big(1 - \frac{x}{x^*}\big)$ for $x < x^*$. However, $I^{\mathrm{ind}}(x)$ is also not continuous from the left at $x = x^*$. In particular, $\lim_{x \nearrow x^*} I^{\mathrm{ind}}(x) \in (0, \infty)$ if $\rho < \infty$.

An intuitive explanation of this discontinuity is the following. If there are at least $\exp\big(I(x^*)n\big)$ particles at time $n$, then $\tilde{M}_n = x^*n + o(n)$ with high probability. For a smaller linear term there have to be less particles, hence for all $x < x^*$ the probability $\mathbb{P}^*(\tilde{M}_n \leq xn)$ is bounded from below by the probability to have at most $\exp\big(I(x^*)n\big)$ particles at time $n$, which decays exponentially. Note that for the branching random walk, in contrast, it suffices to have a small number of particles at the beginning.

## 2.5 Preliminaries

Before we prove the main results, we collect some preliminaries which are needed throughout the proofs.

**Lemma 2.22.** *We have the following inequalities.*

*(i) For* $x \in [0, e^{-1}]$ *it holds that* $1 - x \geq e^{-ex}$.

*(ii) For* $x \in [0, 1]$ *and* $y \geq 0$ *it holds that* $1 - (1-x)^y \geq xy(1 - xy)$.

*Proof.* Both inequalities follow after some elementary calculations.

(i) The function $f(x) = 1 - x - e^{-ex}$ is increasing on $[0, e^{-1}]$. The claim follows, as $f(0) = 0$.

(ii) We first show another inequality. The function $g(x) = 1 - e^{-x} - x(1-x)$ is increasing for $x \geq 0$. As $g(0) = 0$, we have $1 - e^{-x} \geq x(1-x)$. Using additionally the well known inequality $1 - x \leq e^{-x}$, we get

$$1 - (1-x)^y \geq 1 - e^{-xy} \geq xy(1 - xy). \qquad \square$$

Next, we need a general estimate on the sum of large deviation probabilities. For $i \in \mathbb{N}$ let $(a_n^i)_{n\in\mathbb{N}}$ be a sequence of positive numbers and $a^i = \limsup_{n\to\infty} \frac{1}{n} \log a_n^i$.

**Lemma 2.23.** *For all* $k \in \mathbb{N}$ *it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{k} a_n^i = \max_{i\in\{1,\ldots,k\}} a^i.$$

A proof can e.g. be found in [18, Lemma 1.2.15]. Theorem 2.15 gives the exponential decay rate of the probability $\mathbb{P}(S_n \geq xn)$ for $x > 0$. The following theorem gives the precise asymptotics for this probability.

**Theorem 2.24.** *Let* $x > 0$ *and* $I(x) < \infty$. *There exists an explicit constant* $c > 0$ *such that*

$$\lim_{n\to\infty} \sqrt{n} e^{I(x)n} \mathbb{P}\Big(\frac{S_n}{n} \geq x\Big) = c.$$

A proof can be found in [18, Theorem 3.7.4]. Furthermore, we need some properties of the rate function $I$.

**Lemma 2.25.** *Assume that there exists* $x \in \mathbb{R}$ *such that* $I(x) < \infty$ *and* $I(x + \varepsilon) = \infty$ *for all* $\varepsilon > 0$. *Then* $\mathbb{P}(X_1 > x) = 0$ *and* $\mathbb{P}(X_1 = x) = e^{-I(x)}$.

*Proof.* Let $x \in \mathbb{R}$ such that $I(x) < \infty$ and $I(x+\varepsilon) = \infty$ for all $\varepsilon > 0$. Assume that $\mathbb{P}(X_1 > x) > 0$. Then, there exists $\varepsilon > 0$ such that $\mathbb{P}(X_1 \geq x+\varepsilon) > 0$. However,

$$\begin{aligned} I(x+\varepsilon) &= \sup_{\lambda \in \mathbb{R}} \big(\lambda(x+\varepsilon) - \log \mathbb{E}\big[e^{\lambda X_1}\big]\big) \\ &\leq \sup_{\lambda \in \mathbb{R}} \big(\lambda(x+\varepsilon) - \log\big(e^{\lambda(x+\varepsilon)}\mathbb{P}(X_1 \geq x+\varepsilon)\big)\big) \\ &= -\log \mathbb{P}(X_1 \geq x+\varepsilon) < \infty, \end{aligned} \tag{2.13}$$

which leads to a contradiction. It remains to show that $\mathbb{P}(X_1 = x) = e^{-I(x)}$. Analogously to (2.13), we get

$$I(x) \leq \sup_{\lambda \in \mathbb{R}} \big(\lambda x - \log\big(e^{\lambda x}\mathbb{P}(X_1 = x)\big)\big) = -\log \mathbb{P}(X_1 = x).$$

Moreover, since $\mathbb{P}(X_1 > x) = 0$ we have for all $\varepsilon > 0$

$$I(x) \geq \sup_{\lambda \in \mathbb{R}} \big(\lambda x - \log\big(e^{\lambda x}\mathbb{P}(X_1 \in (x-\varepsilon, x] + e^{\lambda(x-\varepsilon)}\big).$$

Letting $\lambda \to \infty$ and $\varepsilon \to 0$ shows that

$$I(x) \geq -\log \mathbb{P}(X_1 = x),$$

which finishes the proof. □

## 2.6 Proof of the main results

The stochastic processes considered in this model are discrete time processes. However, to increase the readability, we omit integer parts if no confusion arises.

### 2.6.1 Convexity of the rate function

*Proof of Proposition 2.21.* Since the rate function $I$ is convex, it remains to show that $H$ (defined in (2.11)) is convex. Recall that $\rho < \infty$ by Assumption 5. By the definition of $H$ and the convexity of $I$, for any $x, y \in (-\infty, x^*)$ and $\varepsilon > 0$ there exists $t_x, t_y \in (0, 1]$

such that for every $\lambda \in [0,1]$

$$
\begin{aligned}
&\lambda H(x) + (1-\lambda)H(y) \\
&\geq \lambda\Big(t_x\rho + t_x I\big(t_x^{-1}(x-(1-t_x)x^*)\big)\Big) + (1-\lambda)\Big(t_y\rho + t_y I\big(t_y^{-1}(y-(1-t_y)x^*)\big)\Big) - \varepsilon \\
&= \big(\lambda t_x + (1-\lambda)t_y\big)\Big(\rho + \frac{\lambda t_x}{\lambda t_x + (1-\lambda)t_y} I\big(t_x^{-1}(x-(1-t_x)x^*)\big) \\
&\qquad\qquad + \frac{\lambda t_y}{\lambda t_x + (1-\lambda)t_y} I\big(t_y^{-1}(y-(1-t_y)x^*)\big)\Big) - \varepsilon \\
&\geq \big(\lambda t_x + (1-\lambda)t_y\big)\Big(\rho + I\Big(\frac{\lambda x + (1-\lambda)y - \big(1-\lambda t_x - (1-\lambda)t_y\big)x^*}{\lambda t_x + (1-\lambda)t_y}\Big)\Big) - \varepsilon \\
&\geq H\big(\lambda x + (1-\lambda)y\big) - \varepsilon.
\end{aligned}
$$

Letting $\varepsilon \to 0$ finishes the proof. □

### 2.6.2 Independent random walks

*Proof of Theorem 2.18.* **1. Case**: $x > x^*$
Following the strategy explained in Section 2.4, independence of the random walks and Lemma 2.22 (ii) yields

$$
\begin{aligned}
\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \geq x\Big) &= \mathbb{E}^*\Big[1-\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\Big)^{Z_n}\Big] \\
&\geq \mathbb{P}^*\Big(Z_n \geq \frac{1}{2}m^n\Big)\cdot\Big(1-\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\Big)^{\frac{1}{2}m^n}\Big) \\
&\geq \mathbb{P}^*\Big(W_n \geq \frac{1}{2}\Big)\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\frac{1}{2}m^n\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\frac{1}{2}m^n\Big). \qquad (2.14)
\end{aligned}
$$

By Theorem 2.15, $\mathbb{P}(\frac{S_n}{n} \geq x)\frac{1}{2}m^n \to 0$ as $n \to \infty$, since $\log m < I(x)$. For the first factor on the right hand side of (2.14) we have $\liminf_{n\to\infty} \mathbb{P}^*(W_n \geq \frac{1}{2}) \geq \mathbb{P}^*(W > \frac{1}{2}) > 0$, since $\mathbb{E}^*[W] \geq \mathbb{E}[W] = 1$ by Theorem 2.3. Together with Theorem 2.17 and Lemma 2.23 we conclude

$$\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \geq x\Big) \geq \exp\big(-(I(x) - \log m)n + o(n)\big),$$

which yields the lower bound. For the upper bound, the Markov inequality yields

$$\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \geq x\Big) = \mathbb{P}^*\Big(\sum_{i=1}^{Z_n} \mathbb{1}_{\{S_n^i \geq nx\}} \geq 1\Big) \leq \mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\mathbb{E}^*[Z_n] = \mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\frac{m^n}{1-q}, \qquad (2.15)$$

which immediately implies the claim.
**2. Case**: $0 < x < x^*$

Since the rate function $I$ is strictly increasing on the interval $[0, x^*]$, we can choose $\varepsilon > 0$ such that $\varepsilon < I(x) < \log m - \varepsilon$. We prove the upper bound first. Using the inequality $1 - y \leq e^{-y}$ and Theorem 2.17, we have for $n$ large enough

$$\begin{aligned}\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \leq x\Big) &= \mathbb{E}^*\Big[\Big(1 - \mathbb{P}\Big(\frac{S_n}{n} > x\Big)\Big)^{Z_n}\Big] \leq \mathbb{E}^*\Big[\exp\Big(-\mathbb{P}\Big(\frac{S_n}{n} > x\Big)Z_n\Big)\Big] \\ &\leq \mathbb{P}^*\Big(Z_n \leq e^{(I(x)+\varepsilon)n}\Big) + \exp\big(-e^{\varepsilon n + o(n)}\big) \\ &= \exp\Big(-\big(I^{\mathrm{GW}}(I(x)+\varepsilon)n + o(n)\big)\Big).\end{aligned}$$

Letting $\varepsilon \to 0$ yields the upper bound. Note that $I^{\mathrm{GW}}$ defined in (2.8) is continuous. The proof for the lower bound is similar. More precisely, since $\mathbb{P}(\frac{S_n}{n} > x) < e^{-1}$ for $n$ large enough, Lemma 2.22 (i) yields for $n$ large enough

$$\begin{aligned}\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \leq x\Big) &= \mathbb{E}^*\Big[\Big(1 - \mathbb{P}\Big(\frac{S_n}{n} > x\Big)\Big)^{Z_n}\Big] \geq \mathbb{E}^*\Big[\exp\Big(-e \cdot \mathbb{P}\Big(\frac{S_n}{n} > x\Big)Z_n\Big)\Big] \\ &\geq \mathbb{P}^*\Big(Z_n \leq e^{(I(x)-\varepsilon)n}\Big) \cdot \exp\big(-e^{-\varepsilon n + o(n)}\big) \\ &= \exp\Big(-I^{\mathrm{GW}}(I(x)-\varepsilon)n + o(n)\Big).\end{aligned}$$

Letting $\varepsilon \to 0$ yields the lower bound.

**3. Case**: $x \leq 0$

We first consider $x < 0$. For the upper bound we have for $K \in \mathbb{N}$

$$\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \leq x\Big) = \mathbb{E}^*\Big[\mathbb{P}\Big(\frac{S_n}{n} \leq x\Big)^{Z_n}\Big] \leq \sum_{k=1}^{K} \mathbb{P}\Big(\frac{S_n}{n} \leq x\Big)^{k} \mathbb{P}^*(Z_n = k) + \mathbb{P}\Big(\frac{S_n}{n} \leq x\Big)^{K}. \tag{2.16}$$

By Theorem 2.16, the probability $\mathbb{P}(Z_n = k)$ is of order $\exp(-\rho n + o(n))$ for all $k \in A$ (defined in (2.7)) and $\mathbb{P}(Z_n = k) = 0$ otherwise. For all $K \in \mathbb{N}$, Lemma 2.23 yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \leq x\Big) \leq \max\big\{-(k^* I(x) + \rho), -K I(x)\big\}.$$

Hence, letting $K \to \infty$ proves the upper bound. Note that $I(x) > 0$ for $x < 0$. As in the proof of (2.16) we have

$$\begin{aligned}\mathbb{P}^*\Big(\frac{\tilde{M}_n}{n} \leq x\Big) &= \mathbb{E}^*\Big[\mathbb{P}\Big(\frac{S_n}{n} \leq x\Big)^{Z_n}\Big] \geq \mathbb{P}\Big(\frac{S_n}{n} \leq x\Big)^{k^*} \cdot \mathbb{P}(Z_n = k^*) \\ &= \exp\big(-(k^* I(x) + \rho)n + o(n)\big),\end{aligned}$$

which shows the lower bound. For $x = 0$ the result follows from continuity of the rate function $I^{\mathrm{ind}}$ at 0.

**4. Case**: $x = x^*$
Analogously to (2.15),

$$\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \le x^*\Big) = 1 - \mathbb{P}\Big(\frac{\tilde{M}_n}{n} > x^*\Big) \ge 1 - \mathbb{P}\Big(\frac{S_n}{n} > x^*\Big)\frac{m^n}{1-q}. \tag{2.17}$$

Now we have to distinguish two cases. If $I(x^*) = \log m$, then the right hand side of (2.17) converges to 1 as $n \to \infty$ by Theorem 2.24. If $I(x^*) < \log m$, then $I(x) = \infty$ for all $x > x^*$ and therefore $\mathbb{P}(X_1 > x^*) = 0$ by Lemma 2.25. Hence, the right hand side of (2.17) equals 1. In both cases we get

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{\tilde{M}_n}{n} \le x^*\Big) = 0.$$

Since $\mathbb{P}(\tilde{M}_n \ge x^* n) \ge \mathbb{P}(M_n \ge x^* n)$, it remains to show that $\mathbb{P}(M_n \ge x^* n)$ decays slower than exponentially in $n$. If $I(x) < \infty$ for some $x > x^*$, then the rate function $I^{\mathrm{BRW}}(x)$ is continuous from the right at $x = x^*$. Since $I^{\mathrm{BRW}}(x) \to 0$ as $x \searrow x^*$ in this case, the claim follows. Therefore assume that $I(x) = \infty$ for all $x > x^*$. By Lemma 2.25 we have $\mathbb{P}(X_1 = x^*) = e^{-I(x^*)}$ in this case. Consider the following embedded process. Every particle with step size smaller than $x^*$ at any time is killed. Therefore, the reproduction mean in every step is $\mathbb{P}(X_1 = x^*)m \ge 1$. Let $q_n$ be the extinction probability of this process at time $n$. By Theorem 2.2, $q_n$ decays slower than exponentially in $n$. Since $\mathbb{P}(M_n \ge x^* n) \ge q_n$, the claim follows. □

### 2.6.3 Branching random walk

Before proving Theorem 2.19, we first show that the maximum of independent random walks stochastically dominates the maximum of the branching random walk.

**Lemma 2.26.** *Let $(X_i)_{i\in\mathbb{N}}$ and $(Y_i)_{i\in\mathbb{N}}$ be independent sequences of (not necessarily independent) random variables. Furthermore, assume that the random variables $Y_i, i \in \mathbb{N}$, have the same distribution. Then we have for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$*

$$\mathbb{P}\Big(\max_{i\in\{1,\dots,k\}}\{X_i + Y_1\} \le x\Big) \ge \mathbb{P}\Big(\max_{i\in\{1,\dots,k\}}\{X_i + Y_i\} \le x\Big).$$

*Proof.* Let $i^*$ be the smallest (random) index such that $X_{i^*} = \max_{i\in\{1,\dots,k\}} X_i$. We have

$$\mathbb{P}\Big(\max_{i\in\{1,\dots,k\}}\{X_i + Y_i\} \le x\Big) \le \mathbb{P}\big(X_{i^*} + Y_{i^*} \le x\big) = \mathbb{P}\big(X_{i^*} + Y_1 \le x\big). \qquad \square$$

As a consequence we can show that the maximum of independent random walks stochastically dominates the maximum of the branching random walk.

**Lemma 2.27.** *For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$*

$$\mathbb{P}(M_n \le x) \ge \mathbb{P}(\tilde{M}_n \le x).$$

*Proof.* We prove this lemma by induction over $n$. For $n = 1$ the inequality is obviously true. Assume that the inequality holds for some $n \in \mathbb{N}$. Let $(S_n^{i,1})_{n\in\mathbb{N}}, (S_n^{i,2})_{n\in\mathbb{N}}, \dots$ be independent copies of $(S_n^i)_{n\in\mathbb{N}}$ and define $(\tilde{M}_n^1)_{n\in\mathbb{N}}, (\tilde{M}_n^2)_{n\in\mathbb{N}}, \dots$ and $(M_n^1)_{n\in\mathbb{N}}, (M_n^2)_{n\in\mathbb{N}}, \dots$ in the same way. Furthermore, for $i \in \{1, \dots, Z_1\}$, denote by $Z_n^i$ the number of descendants of the $i$-th particle of the first generation at time $n+1$. Note that $Z_n^i$ equals $Z_n$ in distribution. Using the induction hypothesis and Lemma 2.26,

$$\begin{aligned}
\mathbb{P}(M_{n+1} \le x) &= \mathbb{P}\Big(\max_{i\in\{1,\dots,Z_1\}} \{X_1^i + M_n^i\} \le x\Big) \\
&\ge \mathbb{P}\Big(\max_{i\in\{1,\dots,Z_1\}} \{X_1^i + \tilde{M}_n^i\} \le x\Big) \\
&\ge \mathbb{P}\Big(\max_{i\in\{1,\dots,Z_1\}} \max_{j\in\{1,\dots,Z_n^i\}} \{X_1^{i,j} + S_{n+1}^{i,j} - X_1^{i,j}\} \le x\Big) \\
&= \mathbb{P}(\tilde{M}_{n+1} \le x). \qquad \square
\end{aligned}$$

The statement of Lemma 2.27 is also true with respect to $\mathbb{P}^*$.

*Proof of Theorem 2.19.* **1. Case**: $x > x^*$
Recall that $I^{\mathrm{BRW}}(x) = I^{\mathrm{ind}}(x)$ for $x \ge x^*$. Therefore, the upper bound immediately follows from Theorem 2.18 and Lemma 2.27. It remains to prove the lower bound. Let $\varepsilon > 0$ such that $(1-\varepsilon)I(x) > \log m$. Recall that for $v \in D_{\varepsilon n}$ the rightmost descendant of $v$ at time $n$ is denoted by $M_{(1-\varepsilon)n}^v$. By Lemma 2.26,

$$\begin{aligned}
\mathbb{P}^*\Big(\frac{M_n}{n} \ge x\Big) &= \mathbb{P}^*\Big(\max_{v\in D_{\varepsilon n}} \frac{M_{(1-\varepsilon)n}^v - S_v}{(1-\varepsilon)n} + \frac{S_v}{(1-\varepsilon)n} \ge \frac{x}{1-\varepsilon}\Big) \\
&\ge \mathbb{P}^*\Big(\max_{v\in D_{\varepsilon n}} \frac{M_{(1-\varepsilon)n}^v - S_v}{(1-\varepsilon)n} + \frac{S_{\varepsilon n}}{(1-\varepsilon)n} \ge \frac{x}{1-\varepsilon}\Big) \\
&\ge \mathbb{P}^*\Big(\max_{v\in D_{\varepsilon n}} \frac{M_{(1-\varepsilon)n}^v - S_v}{(1-\varepsilon)n} \ge x\Big) \cdot \mathbb{P}\Big(\frac{S_{\varepsilon n}}{\varepsilon n} \ge x\Big). \qquad (2.18)
\end{aligned}$$

It remains to estimate the first probability on the right hand side of (2.18). Therefore, let $A_k$ be the set of infinite subtrees in generation $k$, i.e. $A_k = \{v \in D_k \colon |D_l^v| > 0 \ \forall l \in \mathbb{N}\}$. Note that $(M_{(1-\varepsilon)n}^v - S_v)_{v\in D_{\varepsilon n}}$ are independent under $\mathbb{P}^*$ conditioned on $A_{\varepsilon n}$. We can now use similar estimates as in the proof of Theorem 2.18. More precisely, by independence and Lemma 2.22 (ii) we get

$$
\begin{aligned}
&\mathbb{P}^*\Big(\max_{v\in D_{\varepsilon n}} \frac{M^v_{(1-\varepsilon)n}-S_v}{(1-\varepsilon)n}\geq x\Big)\\
&=\mathbb{E}^*\Big[\mathbb{P}^*\Big(\max_{v\in D_{\varepsilon n}} \frac{M^v_{(1-\varepsilon)n}-S_v}{(1-\varepsilon)n}\geq x \mid A_{\varepsilon n}\Big)\Big]\\
&=\mathbb{E}^*\Big[1-\Big(1-\mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\Big)^{|A_{\varepsilon n}|}\Big]\\
&\geq \mathbb{P}^*\Big(Z_{\varepsilon n}\geq \frac{1}{2}m^{\varepsilon n}\Big)\cdot \mathbb{P}^*\Big(|A_{\varepsilon n}|\geq \frac{(1-q)}{2}Z_{\varepsilon n}\mid Z_{\varepsilon n}\geq \frac{1}{2}m^{\varepsilon n}\Big)\\
&\quad\cdot\Big(1-\Big(1-\mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\Big)^{\frac{1-q}{4}m^{\varepsilon n}}\Big)\\
&\geq \mathbb{P}^*\Big(W_{\varepsilon n}\geq \frac{1}{2}\Big)\cdot \mathbb{P}^*\Big(\frac{1}{Z_{\varepsilon n}}\sum_{v\in D_{\varepsilon n}}\mathbb{1}_{\{|D^v_l|>0\ \forall l\in\mathbb{N}\}}\geq \frac{(1-q)}{2}\mid Z_{\varepsilon n}\geq \frac{1}{2}m^{\varepsilon n}\Big)\\
&\quad\cdot \mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\frac{1-q}{4}m^{\varepsilon n}\cdot\Big(1-\mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\frac{1-q}{4}m^{\varepsilon n}\Big).
\end{aligned}
\tag{2.19}
$$

Analogously to (2.14), for the first probability on the right hand side of (2.19) it holds that $\liminf_{n\to\infty}\mathbb{P}^*(W_{\varepsilon n}\geq \frac{1}{2})\geq \mathbb{P}^*(W>\frac{1}{2})>0$. The second probability is at least $\frac{1-q}{2}$ by Lemma 1.34. Furthermore, analogously to (2.15), the Markov inequality and the choice of $\varepsilon$ yields

$$
\begin{aligned}
1-\mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\frac{1-q}{4}m^{\varepsilon n}&\geq 1-\mathbb{P}\Big(\frac{S_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big)\frac{m^n}{4}\\
&=1-\exp\Big(-n\Big((1-\varepsilon)I(x)-\log m\Big)+o(n)\Big)\to 1.
\end{aligned}
\tag{2.20}
$$

Combining (2.18), (2.19) and (2.20) shows

$$
\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}^*\Big(\frac{M_n}{n}\geq x\Big)\geq -\varepsilon(I(x)-\log m)+(1-\varepsilon)\liminf_{n\to\infty}\frac{1}{(1-\varepsilon)n}\log\mathbb{P}^*\Big(\frac{M_{(1-\varepsilon)n}}{(1-\varepsilon)n}\geq x\Big).
$$

This implies the lower bound.

**2. Case**: $x<x^*$

Following the strategy explained in Section 2.4, there are only $k^*$ particles at time $tn$ and the position of one particle is smaller than its expectation. Afterwards, all particles move and branch as usual. For the lower bound let $t\in(0,\min\{1-\frac{x}{x^*},1\}]$ and fix $\varepsilon>0$.

Note that $t \in (0, 1 - \frac{x}{x^*}]$ if $x > 0$ and $t \in (0, 1]$ if $x \leq 0$. We have

$$\begin{aligned}
\mathbb{P}^*\Big(\frac{M_n}{n} \leq x\Big) &\geq \mathbb{P}^*\Big(\frac{M_n}{n} \leq x \mid Z_{tn} = k^*\Big) \cdot \mathbb{P}^*(Z_{tn} = k^*) \\
&\geq q^{k^*-1}\mathbb{P}^*\Big(\frac{S_{tn} + M_{(1-t)n}}{n} \leq x\Big) \cdot \mathbb{P}^*(Z_{tn} = k^*) \\
&\geq q^{k^*-1}\mathbb{P}^*\Big(\frac{M_{(1-t)n}}{(1-t)n} \leq x^* + \varepsilon\Big) \cdot \mathbb{P}\Big(\frac{S_{tn}}{n} \leq (x - (1-t)(x^* + \varepsilon))\Big) \\
&\quad \cdot \mathbb{P}^*(Z_{tn} = k^*). \qquad (2.21)
\end{aligned}$$

Since the first probability on the right hand side of (2.21) converges to 1 almost surely as $n \to \infty$ by Theorem 2.4, we get

$$\mathbb{P}^*\Big(\frac{M_n}{n} \leq x\Big) \geq \exp\Big(-\Big[I\Big(t^{-1}(x - (1-t)(x^* + \varepsilon))\Big) + \rho\Big]tn + o(n)\Big).$$

Letting $\varepsilon \to 0$ and since this inequality holds for all $t \in (0, \min\{1 - \frac{x}{x^*}, 1\}]$, we conclude

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\Big(\frac{M_n}{n} \leq x\Big) \geq \sup_{t \in (0, \min\{1 - \frac{x}{x^*}, 1\}]} -H(x) = -\inf_{t \in (0,1]} H(x).$$

For the upper bound define

$$T_n = \inf\Big\{t \geq 0 \colon Z_{tn} \geq n^3\Big\}$$

and for $\varepsilon_1 > 0$ introduce the set

$$F = F(\varepsilon_1) = \Big\{\varepsilon_1, 2\varepsilon_1, \dots, \Big\lceil \min\Big\{\Big(1 - \frac{x}{x^*}\Big), 1\Big\}\varepsilon_1^{-1}\Big\rceil \varepsilon_1\Big\}.$$

By the definition of $T_n$ we then have

$$\begin{aligned}
&\mathbb{P}^*\Big(\frac{M_n}{n} \leq x\Big) \\
&\leq \mathbb{P}^*\Big(T_n > \min\Big\{\Big(1 - \frac{x}{x^*}\Big), 1\Big\}\Big) + \sum_{t \in F} \mathbb{P}^*\Big(\frac{M_n}{n} \leq x \mid T_n \in \big(t - \varepsilon_1, t\big]\Big)\mathbb{P}^*\big(T_n \in \big(t - \varepsilon_1, t\big]\big) \\
&\leq \mathbb{P}^*\big(Z_{(\min\{(1 - \frac{x}{x^*}), 1\})n} \leq n^3\big) + \sum_{t \in F} \mathbb{P}^*\Big(\frac{M_n}{n} \leq x \mid T_n \in \big(t - \varepsilon_1, t\big]\Big)\mathbb{P}^*(Z_{(t - \varepsilon_1)n} \leq n^3).
\end{aligned} \qquad (2.22)$$

Let $\varepsilon_2 > 0$. Recall that $A_{tn}$ is the set of infinite subtrees in generation $tn$. Using

Lemma 2.26 and the same estimate as in (2.19),

$$\begin{aligned}
&\mathbb{P}^*\Big(\frac{M_n}{n} \le x \mid T_n \in \big(t-\varepsilon_1, t\big]\Big) \\
&\le \mathbb{P}^*\left(\max_{v \in D_{tn}} \frac{S_{tn} + M^v_{(1-t)n}}{n} \le x \,\middle|\, T_n \in \big(t-\varepsilon_1, t\big]\right) \\
&\le \mathbb{P}\Big(\frac{S_{tn}}{n} \le -\big((1-t)(x^*-\varepsilon_2)-x\big)\Big) + \mathbb{P}^*\Big(\frac{M_{(1-t)n}}{(1-t)n} \le x^* - \varepsilon_2\Big)^n \\
&\qquad + \mathbb{P}^*\big(Z_{tn} \le n^2 \mid T_n \in (t-\varepsilon_1, t]\big) + \mathbb{P}^*\big(|A_{tn}| \le n \mid Z_{tn} > n^2, T_n \in (t-\varepsilon_1, t]\big). \quad (2.23)
\end{aligned}$$

The second probability on the right hand side of (2.23) converges to 0 by Theorem 2.4. Hence, the second term in (2.23) decays faster than exponentially in $n$. For the third term on the right hand side of (2.23),

$$\begin{aligned}
\mathbb{P}^*\big(Z_{tn} \le n^2 \mid T_n \in (t-\varepsilon_1, t]\big) &\le \mathbb{P}^*\big(\exists k \in \mathbb{N}\colon Z_k \le n^2 \mid Z_0 = n^3\big) \\
&\le \binom{n^3}{n^2} q^{n^3-n^2} \le \exp\big((n^3-n^2)\log q + 3n^2 \log n\big).
\end{aligned} \tag{2.24}$$

In the second inequality we used the fact that for the event we consider, at most $n^2$ of the initial $n^3$ Galton-Watson trees may survive. Note that every initial particle produces an independent Galton-Watson tree. Analogousy to (2.24), we get for the fourth term on the right hand side of (2.23)

$$\mathbb{P}^*\big(|A_{tn}| \le n \mid Z_{tn} > n^2, T_n \in (t-\varepsilon_1, t]\big) \le \binom{n^2}{n} q^{n^2-n} \le \exp\big((n^2-n)\log q + 2n\log n\big). \tag{2.25}$$

Combining (2.22), (2.23), (2.24) and (2.25) and letting $\varepsilon_1, \varepsilon_2 \to 0$, we conclude with Lemma 2.23 after a straightforward calculation

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}^*\Big(\frac{M_n}{n} \le x\Big) \le -\inf_{t \in (0, \min\{1-\frac{x}{x^*}, 1\}]} \Big\{t\rho + tI\Big(-\frac{(1-t)x^* - x}{t}\Big)\Big\} = -H(x).$$

Note that we could take the limit $\varepsilon_2 \to 0$, since $I$ is continuous from the right on $(0, \infty)$.

**3. Case**: $x = x^*$

The proof is analogous to Theorem 2.18.

□

# 3 Branching Random Walks In Random Environment

## 3.1 Description of the model

In this chapter we consider the same model as in Chapter 2, but we add another source of randomness. In Chapter 2 the branching mechanism, given by the weights $(p(k))_{k\in\mathbb{N}_0}$, was independent of time. In this chapter, at every time $n$ the weights are chosen at random, independently of everything else.
Other random environments have been considered before. In [25] Fang and Zeitouni studied branching random walks with time-dependent transition probabilities. Space-time environments (i.e. branching and movement depends on time and location of the particles) are investigated by Yoshida [58] as well as by Hu and Yoshida [42].
We keep the notation of Chapter 2 if not stated otherwise. In our model the set of possible offspring distributions is defined as

$$\mathcal{M} = \Big\{(p(k))_{k\in\mathbb{N}_0} \colon p(k) \geq 0, \sum_{k=0}^{\infty} p(k) = 1\Big\}.$$

Let $\omega_1, \omega_2, \dots$ be a sequence of i.i.d. random variables with values in $\mathcal{M}$ and define the random environment $\omega = (\omega_n)_{n\in\mathbb{N}}$, where $\omega_n = (p_n(k))_{k\in\mathbb{N}_0}$. The corresponding probability measure is denoted by P and its expectation by E. Let $(Z_n)_{n\in\mathbb{N}_0}$ be a Galton-Watson process in the random environment $\omega$ with one initial particle. More precisely, given an environment $\omega$, for all $n\in\mathbb{N}$ let $(\xi_{n,k})_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $P_\omega(\xi_{n,1} = k) = p_n(k)$ for all $k \in \mathbb{N}$ and let $Z_0 = 1$. The number of particles in the $n$-th generation is defined as $Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n,k}$. Let $m_n = E_\omega[\xi_{n,1}] = \sum_{k=1}^{\infty} k p_n(k)$ be the reproduction mean in generation $n$ and let $m$ be an independent copy of $m_1$.
For a fixed environment $\omega$, we denote by $P_\omega$ and $E_\omega$ the probability, respectively the expectation of the processes in the environment $\omega$. We refer to $P_\omega$ as the quenched probability. The annealed probability is defined as

$$\mathbb{P}(\cdot) = \int P_\omega(\cdot)\mathrm{P}(\mathrm{d}\omega).$$

We denote the associated expectation by $\mathbb{E}$. As in Chapter 2 we study large deviation probabilities of the maximum of the branching random walk and the maximum of independent random walks.

## 3.2 Some known results

### 3.2.1 On the branching process in random environment

Throughout this chapter we assume that each particle produces at least one offspring.

**Assumption 6.** *Each particle has at least one offspring, i.e. $p(0) = 0$. Furthermore,* $\mathrm{P}(p(1) = 1) < 1$.

The Galton-Watson process in random environment always survives if Assumption 6 is satisfied. Furthermore, the reproduction means satisfy $m_n \geq 1$ for all $n \in \mathbb{N}$.
In analogy to the time-homogeneous Galton-Watson process, we introduce the process $(W_n)_{n\in\mathbb{N}}$, where

$$W_n = Z_n \Big( \prod_{i=1}^{n} m_i \Big)^{-1}.$$

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be the natural filtration of the Galton-Watson process in random environment, i.e. $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. For P-a.e. $\omega$ the process $(W_n)_{n\in\mathbb{N}}$ is a $P_\omega$-martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Therefore, $W_n \to W$ $P_\omega$-a.s., where $W$ is an almost surely finite random variable.

**Assumption 7.** *For the Galton-Watson process in random environment suppose that* $\mathbb{E}[Z_1 \log Z_1] < \infty$.

The following theorem shows that under our assumptions the limit $W$ is non-trivial, i.e. $\mathbb{P}(W = 0) < 1$.

**Theorem 3.1.** *Assume that*

$$\mathrm{E}[\log m] > 0, \quad \mathrm{E}[\log(1 - p(0))] > -\infty \quad \textit{and} \quad \mathrm{E}\big[m_1^{-1} E_\omega[Z_1 \log Z_1]\big] < \infty.$$

*Then we have* $E_\omega[W] = 1$.

A proof can be found in [6, Theorem 1]. Note that the assumptions in Theorem 3.1 are satisfied if Assumption 6 and 7 are fulfilled.

### 3.2.2 First and second term of the maximum

Define

$$x^* = \sup\big\{x \geq 0 \colon I(x) \leq \mathrm{E}[\log m]\big\}. \tag{3.1}$$

In analogy to the time-homogeneous case, Huang and Liu [43] showed that $x^*$ is the speed of the maximum of the branching random walk. Note that they consider a more general model, where branching as well as movement of particles is time-dependent.

**Theorem 3.2** ([43, Theorem 3.4]). *Suppose that Assumption 6 and 7 are satisfied. Furthermore, assume that $\mathbb{E}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$. The maximum of the branching random walk has linear speed $x^*$, i.e.*

$$\lim_{n\to\infty} \frac{M_n}{n} = x^* \quad \mathbb{P}\text{-}a.s.$$

The maximum of independent random walks also has linear speed $x^*$, see (3.8). Now assume that $I(x)$ is differentiable at $x = x^*$. For $n \in \mathbb{N}$ introduce the random variables

$$K_n = \log \mathbb{E}\big[e^{I'(x^*)X_1}\big]n + \sum_{i=1}^{n} \log m_i.$$

In particular, one can check that $\frac{K_n}{n} \to I'(x^*)x^*$ P-almost surely as $n \to \infty$. Mallein and Miłoś [51] proved that the maximum of the branching random walk has a logarithmic second term.

**Theorem 3.3** ([51, Corollary 1.3]). *Suppose that Assumption 6 is satisfied. Furthermore, assume that $\mathbb{E}[e^{\lambda X_1}] < \infty$ for all $\lambda \in \mathbb{R}$, $\mathrm{E}[(\log m)^2] < \infty$ and $\mathbb{E}[Z_1^2] < \infty$. Then, for an explicit constant $c \geq 0$*

$$\lim_{n\to\infty} \frac{M_n - \frac{K_n}{I'(x^*)}}{\log n} = -\Big(\frac{3}{2I'(x^*)} + c\Big) \quad \text{in } \mathbb{P}\text{-probability.}$$

*In particular, $c = 0$ if and only if the reproduction mean $m$ is almost surely constant.*

Note that this result is in accordance with the third statement of Theorem 2.8 in the time-homogeneous case. Moreover, Theorem 3.3 shows that the logarithmic correction term is smaller for time-inhomogeneous branching random walks.
Furthermore, precise asymptotics on the empirical distribution are obtained in [32], [33] and [34].

## 3.3 Rate functions and assumptions

In this section we introduce the rate functions of the Galton-Watson process in random environment, which are needed to state our main results of this chapter. Throughout the chapter we use the convention $\infty \cdot 0 = 0$.

### 3.3.1 Rate function of the logarithmic reproduction means

For $x \geq 0$ the rate function of the logarithmic reproduction means is defined as

$$I^{\log m}(x) = \sup_{\lambda\in\mathbb{R}} (\lambda x - \log \mathrm{E}\big[e^{\lambda \log m}\big]). \tag{3.2}$$

$I^{\log m}$ is lower semicontinuous, and therefore continuous from the right for $x \le \mathrm{E}[\log m]$ and continuous from the left for $x \ge \mathrm{E}[\log m]$.

**Assumption 8.** *All moments of the reproduction mean exist, i.e.* $\mathrm{E}[m^\lambda] < \infty$ *for all* $\lambda > 0$.

Note that Assumption 8 is satisfied for constant reproduction mean. Furthermore, Assumption 8 implies $I^{\log m}(x) > 0$ for all $x \neq \mathrm{E}[\log m]$.
Since $(\log m_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables, Cramér's theorem yields a large deviation result for the logarithmic reproduction means.

**Theorem 3.4.** *Suppose that Assumption 8 is satisfied. The laws of* $\frac{\sum_{i=1}^n \log m_i}{n}$ *under* $\mathrm{P}$ *satisfy a large deviation principle with rate function* $I^{\log m}$*, i.e.*

$$-I^{\log m}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\frac{\sum_{i=1}^n \log m_i}{n} \ge x\Big) & \textit{for } x \ge \mathrm{E}[\log m], \\ \lim_{n\to\infty} \frac{1}{n} \log \mathrm{P}\Big(\frac{\sum_{i=1}^n \log m_i}{n} \le x\Big) & \textit{for } 0 \le x \le \mathrm{E}[\log m]. \end{cases}$$

### 3.3.2 Rate functions of the Galton-Watson process

We have to distinguish the behaviour of the Galton-Watson process in random environment under the annealed law and under the quenched law.
If Assumption 6, 7 and 8 are satisfied, then there is an annealed large deviation result for the lower deviation probabilities, i.e. for probabilities of the form $\mathbb{P}(Z_n \le e^{xn})$ for $x < \mathrm{E}[\log m]$, see Theorem 3.5.

**Assumption 9.** *There exists a constant* $\beta \in (1, \infty]$ *such that*

$$\lim_{k\to\infty} \frac{\log \mathbb{P}(Z_1 \ge k)}{\log k} = -\beta.$$

*Moreover, if* $\beta < \infty$*, there is* $d_1 > 0$ *such that* $\mathrm{P}\big(P_\omega(Z_1 \ge k) \le d_1 m_1 k^{-\beta}\big) = 1$. *Otherwise, if* $\beta = \infty$*, for every* $\beta' < \infty$ *there exists a constant* $d_2 > 0$ *such that* $\mathrm{P}\big(P_\omega(Z_1 \ge k) \le d_2 m_1 k^{-\beta'}\big) = 1$.

Note that Assumption 9 implies Assumption 7. If $\beta < \infty$, Assumption 9 ensures that the offspring distribution is heavy-tailed. More precisely, there are environments such that the offspring distribution has polynomial tails with coefficient $\beta$, but no environment has an offspring distribution with heavier tail.
If Assumption 8 and 9 are satisfied, then there is an annealed large deviation result for the upper deviation probabilities, i.e. for probabilities of the form $\mathbb{P}(Z_n \ge e^{xn})$ for $x > \mathrm{E}[\log m]$, see Theorem 3.5. Let

$$\rho_{\mathrm{a}} = -\log \mathrm{E}[p(1)] \in (0, \infty]. \tag{3.3}$$

Define the annealed rate function of the Galton-Watson process in random environment as

$$I_{\mathrm{a}}^{\mathrm{GW}}(x) = \begin{cases} \inf_{t\in[0,x]}\{\beta t + I^{\log m}(x-t)\} & \text{for } x \geq \mathrm{E}[\log m], \\ \inf_{t\in[0,1)}\Big\{\rho_{\mathrm{a}} t + (1-t) I^{\log m}\big((1-t)^{-1}x\big)\Big\} & \text{for } 0 \leq x \leq \mathrm{E}[\log m], \end{cases} \tag{3.4}$$

where $\beta$ is the constant of Assumption 9. Note that $I_{\mathrm{a}}^{\mathrm{GW}}(x) > 0$ for all $x \neq \mathrm{E}[\log m]$. Furthermore, also $I_{\mathrm{a}}^{\mathrm{GW}}$ is lower semicontinuous, and therefore continuous from the right for $x \leq \mathrm{E}[\log m]$ and continuous from the left for $x \geq \mathrm{E}[\log m]$. Now we can state the first annealed large deviation principle for the Galton-Watson process in random environment.

**Theorem 3.5.** *Under Assumption 6, 8 and 9 the laws of* $\frac{\log Z_n}{n}$ *under* $\mathbb{P}$ *satisfy a large deviation principle with rate function* $I_{\mathrm{a}}^{\mathrm{GW}}$*, i.e.*

$$-I_{\mathrm{a}}^{\mathrm{GW}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(Z_n \geq e^{xn}\big) & \textit{for } x \geq \mathrm{E}[\log m], \\ \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(Z_n \leq e^{xn}\big) & \textit{for } 0 \leq x \leq \mathrm{E}[\log m]. \end{cases}$$

A proof of the lower large deviation result can be found in [9, Theorem 3.1] and of the upper large deviation result in [8, Theorem 1] for $\beta \in (1,\infty)$. The result immediately follows for $\beta = \infty$ by a monotonicity argument, see also [8, Corollary 1].
Furthermore, the authors of [9] also show a large deviation result for the probability that $(Z_n)_{n\in\mathbb{N}_0}$ grows subexponentially. Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ is called subexponential, if $a_n e^{-\varepsilon n} \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.

**Theorem 3.6.** *Assume that Assumption 6, 7 and 8 is satisfied. For every subexponential sequence* $(a_n)_{n\in\mathbb{N}}$ *such that* $a_n \geq 1$ *for all* $n \in \mathbb{N}$*,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(Z_n \leq a_n\big) = -\rho_{\mathrm{a}}.$$

A proof can be found in [9, Proposition 2.1] and [9, Theorem 3.1].
Next, we consider large deviation events with respect to the quenched law. Therefore, define

$$\rho_{\mathrm{q}} = -\mathrm{E}[\log p(1)]. \tag{3.5}$$

By Jensen's inequality $\rho_{\mathrm{q}} \geq \rho_{\mathrm{a}}$. In analogy to Assumption 5 in the time-homogeneous case the following assumption ensures $\rho_{\mathrm{q}} < \infty$.

**Assumption 10.** *It holds that* $\mathrm{E}[\log p(1)] > -\infty$.

Define the quenched rate function of the Galton-Watson process in random environment as

$$I_{\mathrm{q}}^{\mathrm{GW}}(x) = \begin{cases} \beta\big(x - \mathrm{E}[\log m]\big) & \text{for } x \geq \mathrm{E}[\log m], \\ \rho_{\mathrm{q}}\big(1 - x\mathrm{E}[\log m]^{-1}\big) & \text{for } 0 \leq x \leq \mathrm{E}[\log m]. \end{cases} \tag{3.6}$$

Note that $I_{\mathrm{q}}^{\mathrm{GW}}(x) > 0$ for all $x \neq \mathrm{E}[\log m]$. We have the following quenched large deviation results.

**Theorem 3.7.** *Under Assumption 6, 8 and 9 the laws of* $\frac{\log Z_n}{n}$ *under* $P_\omega$ *for* P*-a.e.* $\omega$ *satisfy a large deviation principle with rate function* $I_{\mathrm{q}}^{\mathrm{GW}}$*, i.e.*

$$-I_{\mathrm{q}}^{\mathrm{GW}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(Z_n \geq e^{xn}\big) & \text{for } x \geq \mathrm{E}[\log m], \\ \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(Z_n \leq e^{xn}\big) & \text{for } 0 \leq x \leq \mathrm{E}[\log m]. \end{cases}$$

A proof of the upper large deviation result can be found in [12, Theorem 4.5.1]. We prove the lower large deviation result in Subsection 3.6.1. In analogy to the annealed case (Theorem 3.6), there is also a large deviation result for subexponential growth of the population size $(Z_n)_{n\in\mathbb{N}_0}$.

**Theorem 3.8.** *Suppose that Assumption 6 is satisfied. For* P*-a.e.* $\omega$ *and for every subexponential sequence* $(a_n)_{n\in\mathbb{N}}$ *such that* $a_n \geq 1$ *for all* $n \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(Z_n \leq a_n\big) = -\rho_{\mathrm{q}}.$$

This theorem follows immediately from Theorem 3.7 for $x \to 0$.

## 3.4 Main results

Recall the definition of the rate function of the random walk, see (2.4). After defining the rate functions of the Galton-Watson process in random environment we are now able to state the main results of this chapter.
Analogously to the time-homogeneous case, note that $I(x^*) = \mathrm{E}[\log m]$ if $I(x) < \infty$ for some $x > x^*$. On the other hand, $I(x^*) < \mathrm{E}[\log m]$ already implies $\mathbb{P}(X_1 > x^*) = 0$. This case leads to a different shape of the rate functions. For $x > x^*$ let

$$G(x) = \inf_{t\in[\mathrm{E}[\log m],\, I(x))} \big\{I(x) - t + I_{\mathrm{a}}^{\mathrm{GW}}(t)\big\}. \tag{3.7}$$

We exclude $t = I(x)$ in (3.7) since $I(x)$ might be infinite. Define the annealed rate function of independent random walks as

$$I_{\mathrm{a}}^{\mathrm{ind}}(x) = \begin{cases} G(x) & \text{for } x > x^*, \\ 0 & \text{for } x = x^*, \\ I_{\mathrm{a}}^{\mathrm{GW}}(I(x)) & \text{for } 0 \leq x < x^*, \\ I(x) + \rho_{\mathrm{a}} & \text{for } x \leq 0. \end{cases}$$

**Theorem 3.9.** *Suppose that Assumption 3, 6, 8 and 9 are satisfied. The laws of* $\frac{\tilde{M}_n}{n}$ *under* $\mathbb{P}$ *satisfy a large deviation principle with rate function* $I_{\mathrm{a}}^{\mathrm{ind}}$*, i.e.*

$$-I_{\mathrm{a}}^{\mathrm{ind}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\tilde{M}_n}{n} \geq x\right) & \text{for } x \geq x^*, \\ \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{\tilde{M}_n}{n} \leq x\right) & \text{for } x \leq x^*. \end{cases}$$

Theorem 3.9 implies in particular that the speed of the maximum of independent random walks equals $x^*$, i.e.

$$\lim_{n\to\infty} \frac{\tilde{M}_n}{n} = x^* \quad \mathbb{P}\text{-a.s.} \tag{3.8}$$

Let us now give some intuition for the rate function $I_{\mathrm{a}}^{\mathrm{ind}}$ and describe the large deviation event $\{\tilde{M}_n \geq xn\}$ for some $x > x^*$, respectively $\{\tilde{M}_n \leq xn\}$ for some $x < x^*$.
For $x > x^*$ the number of particles should be larger or equal than expected, i.e. $Z_n \geq e^{nt}$ for some $t \geq \mathrm{E}[\log m]$. The probability of such an event is of order $\exp(-I_{\mathrm{a}}^{\mathrm{GW}}(t)n + o(n))$. If there are $e^{nt}$ particles at time $n$, the probability that at least one particle reaches $xn$ is of order $\exp(-I(x)n + tn + o(n))$ for $t < I(x)$. Therefore, we need to maximize the product of these two probabilities.
If $0 \leq x < x^*$, the probability that one particle reaches $xn$ is of order $\exp(-I(x)n + o(n))$. Hence, for every $\varepsilon > 0$, if there are less than $e^{(I(x)-\varepsilon)n}$ particles, the probability that none of these particles reaches $xn$ is close to 1. However, if there are more than $e^{(I(x)+\varepsilon)n}$ particles, this probability decays exponentially in $n$. Note that, in contrast to the time-homogeneous case, $I_{\mathrm{a}}^{\mathrm{ind}}(x)$ can be finite in the Böttcher case ($p(0) + p(1) = 0$), since $\tilde{M}_n$ can be small due to a bad environment.
If $x < 0$, already the probability that a single particle is below $xn$ at time $n$ decays exponentially fast in $n$. Hence, if the number of particles $Z_n$ grows exponentially, the probability that all particles are below $xn$ at time $n$ decays faster than exponentially. Therefore, the number of particles needs to grow subexponentially. Since $\rho_{\mathrm{a}}$ does not depend on the subexponential sequence in Theorem 3.6, there has to be only one particle at time $n$ (provided that $\rho_{\mathrm{a}} < \infty$ and therefore $\mathrm{P}(p(1) > 0) > 0$). In particular, the lower deviation probabilities $\mathbb{P}(\tilde{M}_n \leq xn)$ decay faster than exponentially in $n$ in the Böttcher case.
Next, we state a quenched large deviation result for the independent random walks. Define the quenched rate function

$$I_{\mathrm{q}}^{\mathrm{ind}}(x) = \begin{cases} I(x) - \mathrm{E}[\log m] & \text{for } \ x > x^*, \\ 0 & \text{for } x = x^*, \\ I_{\mathrm{q}}^{\mathrm{GW}}(I(x)) & \text{for } \ 0 \leq x < x^*, \\ I(x) + \rho_{\mathrm{q}} & \text{for } \ x \leq 0. \end{cases}$$

**Theorem 3.10.** *Suppose that Assumption 3, 6, 8 and 9 are satisfied. The laws of* $\frac{\tilde{M}_n}{n}$

*under* $P_\omega$ *for* P*-a.e.* $\omega$ *satisfy a large deviation principle with rate function* $I_{\mathrm{q}}^{\mathrm{ind}}$*, i.e.*

$$-I_{\mathrm{q}}^{\mathrm{ind}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(\frac{\tilde{M}_n}{n} \geq x\big) & \text{for } x \geq x^*, \\ \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(\frac{\tilde{M}_n}{n} \leq x\big) & \text{for } x \leq x^*. \end{cases}$$

Analogously to the annealed case one can show that for $x > x^*$ we have

$$I_{\mathrm{q}}^{\mathrm{ind}}(x) = \inf_{t\in[\mathrm{E}[\log m],\, I(x))} \big\{ I(x) - t + I_{\mathrm{q}}^{\mathrm{GW}}(t) \big\}. \tag{3.9}$$

However, the infimum in (3.9) is always attained at $t = \mathrm{E}[\log m]$ by the definition of $I_{\mathrm{q}}^{\mathrm{GW}}$ and $\beta > 1$.

Next, we consider the maximum of the branching random walk. Define the annealed rate function of the branching random walk for $x > x^*$ as

$$I_{\mathrm{a}}^{\mathrm{BRW}}(x) = G(x). \tag{3.10}$$

**Theorem 3.11.** *Suppose that Assumption 3, 6, 8 and 9 are satisfied. For* $x > x^*$

$$-I_{\mathrm{a}}^{\mathrm{BRW}}(x) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{M_n}{n} \geq x\Big).$$

For $x > x^*$ we have $I_{\mathrm{a}}^{\mathrm{BRW}}(x) = I_{\mathrm{a}}^{\mathrm{ind}}(x)$. In this case the strategy is the same as for independent random walks. Let us now define the quenched rate function. For $x \leq x^*$ let

$$H_{\mathrm{q}}(x) = \inf_{t\in(0,1]} \Big\{ t\rho_{\mathrm{q}} + tI\big(t^{-1}\big(x - (1-t)x^*\big)\big) \Big\}.$$

Define the quenched rate function of the branching random walk as

$$I_{\mathrm{q}}^{\mathrm{BRW}}(x) = \begin{cases} I(x) - \mathrm{E}[\log m] & \text{for } x > x^*, \\ 0 & \text{for } x = x^*, \\ H_{\mathrm{q}}(x) & \text{for } x < x^*. \end{cases}$$

Then we have the following quenched large deviation result for the branching random walk.

**Theorem 3.12.** *Suppose that Assumption 3, 6, 8, 9 and 10 are satisfied. The laws of* $\frac{M_n}{n}$ *under* $P_\omega$ *for* P*-a.e.* $\omega$ *satisfy a large deviation principle with rate function* $I_{\mathrm{q}}^{\mathrm{BRW}}$*, i.e.*

$$-I_{\mathrm{q}}^{\mathrm{BRW}}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(\frac{M_n}{n} \geq x\big) & \text{for } x \geq x^*, \\ \lim_{n\to\infty} \frac{1}{n} \log P_\omega\big(\frac{M_n}{n} \leq x\big) & \text{for } x \leq x^*. \end{cases}$$

**Remark 3.13.** *As in the time-homogeneous case, one can check that the rate functions* *of the branching random walk are convex.*

As in the annealed case we have $I_{\mathrm{q}}^{\mathrm{BRW}}(x) = I_{\mathrm{q}}^{\mathrm{ind}}(x)$ for $x > x^*$. In this case the strategy is the same as for independent random walks. The strategy in the case $x < x^*$ is similar to the time-homogeneous case. Let $t \in (0, 1]$. At time $tn$ there is only one particle whose position is smaller than its expectation. Afterwards, each particle moves and branches according to its usual behaviour.

Further notice, that in contrast to the case of independent random walks, the number of particles can also grow exponentially if $x < 0$. It suffices to have a small number of particles at time $tn$ for some $t \in (0, 1]$.

The rate functions of the branching random walk and independent random walks have the same properties as in the time-homogeneous case discussed after Proposition 2.21.

Moreover, both annealed rate functions are smaller or equal than the quenched rate functions, i.e. $I_{\mathrm{a}}^{\mathrm{ind}} \le I_{\mathrm{q}}^{\mathrm{ind}}$ and $I_{\mathrm{a}}^{\mathrm{BRW}} \le I_{\mathrm{q}}^{\mathrm{BRW}}$. This is true in general, see Lemma 3.14.

## 3.5 Preliminaries

In the proofs of the main results we often use Lemma 2.22 and Lemma 2.23. Furthermore, the following lemma allows us to compare annealed and quenched probabilities.

**Lemma 3.14.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events. For* $\mathrm{P}$*-a.e.* $\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(A_n) \ge \limsup_{n\to\infty} \frac{1}{n} \log P_\omega(A_n),$$

*and*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(A_n) \ge \liminf_{n\to\infty} \frac{1}{n} \log P_\omega(A_n).$$

A proof can be found in [59, Lemma 2.3.8].

## 3.6 Proof of the main results

### 3.6.1 Galton-Watson process in random environment

We first need to consider the event that the population size $Z_n$ stays bounded until time $n$. Recall $\rho_{\mathrm{q}} = -\mathrm{E}[\log p(1)]$ defined in (3.5) and $p(0) = 0$ by Assumption 6.

**Lemma 3.15.** *Suppose that Assumption 6 is satisfied. For all $b \in \mathbb{N}$ and for* $\mathrm{P}$*-a.e.* $\omega$

$$\lim_{n\to\infty} \frac{1}{n} \log P_\omega(Z_n \le b) = -\rho_{\mathrm{q}}.$$

*Proof.* The lower bound follows immediately, since

$$P_\omega(Z_n \le b) \ge \prod_{i=1}^{n} p_i(1),$$

and therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega(Z_n \le b) \ge \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log p_i(1) = -\rho_{\mathrm{q}}.$$

For the upper bound first assume $\rho_{\mathrm{q}} = \infty$. For $1 \le k \le b+1$ define the sets

$$J_k = \Big\{1 + \Big\lfloor \frac{(k-1)n}{b+1} \Big\rfloor, \dots, \Big\lfloor \frac{kn}{b+1} \Big\rfloor\Big\}.$$

Note that the event $\{Z_n \le b\}$ implies that there exists $1 \le k \le b+1$, such that all particles produce exactly one offspring for every time step in $J_k$. This yields

$$P_\omega(Z_n \le b) \le \sum_{k=1}^{b+1} \prod_{j\in J_k} p_j(1).$$

Since $(p_j(1))_{j\in\mathbb{N}}$ is a sequence of i.i.d. random variables with respect to P, Lemma 2.23 yields

$$\lim_{n\to\infty} \frac{1}{n} \log P_\omega(Z_n \le b) \le \max_{1\le k\le b+1} \limsup_{n\to\infty} \frac{1}{n} \sum_{j\in J_k} \log p_j(1) = \frac{-\rho_{\mathrm{q}}}{b+1} = -\infty.$$

Now let $\rho_{\mathrm{q}} < \infty$. If $Z_n \le b$, there are at least $n-b$ generations, in which all particles can only have one offspring. Hence,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log P_\omega(Z_n \le b) &\le \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{k=0}^{b} \tbinom{n}{n-k} \big(\inf_{i\le n} p_i(1)\big)^{-k} \prod_{i=1}^{n} p_i(1)\Big)\\ &= -\rho_{\mathrm{q}} - b \liminf_{n\to\infty} \frac{1}{n} \log\big(\inf_{i\le n} p_i(1)\big).\end{aligned}$$

It remains to show that P-a.s.

$$\liminf_{n\to\infty} \frac{1}{n} \log\big(\inf_{i\le n} p_i(1)\big) = 0.$$

Since $(p_i(1))_{i\in\mathbb{N}}$, are independent, we have for all $\varepsilon > 0$ and $N \in \mathbb{N}$

$$\begin{aligned}
&\mathrm{P}\Big(\frac{1}{n}\log\inf_{i\le n} p_i(1) \ge -\varepsilon \text{ for all } n \ge N\Big)\\
&\ge \prod_{n=1}^{N} \mathrm{P}(\log p_n(1) \ge -\varepsilon N)\cdot \prod_{n=N+1}^{\infty} \mathrm{P}(\log p_n(1) \ge -\varepsilon n)\\
&= \exp\Big(N\log\big(1-\mathrm{P}(\log p_1(1) < -\varepsilon N)\big)\Big)\cdot \exp\Big(\sum_{n=N+1}^{\infty}\log\big(1-\mathrm{P}(\log p_n(1) < -\varepsilon n)\big)\Big).
\end{aligned} \tag{3.11}$$

Since $\log(1-x) \sim -x$ for $x \to 0$ and $\mathrm{E}[\log p(1)] = -\rho_{\mathrm{q}} > -\infty$ by assumption, the term in the first exponent on the right hand side of (3.11) converges to 0 as $N \to \infty$ and the sum in the second exponent is finite. Therefore, we conclude

$$\lim_{N\to\infty} \mathrm{P}\Big(\frac{1}{n}\log\inf_{i\le n} p_i(1) \ge -\varepsilon \text{ for all } n \ge N\Big) = 1,$$

which finishes the proof. □

Now we are able to prove the quenched large deviation result for the Galton-Watson process. Therefore, for $l \in \mathbb{N}$ we introduce the shift of the environment $\omega = (\omega_1, \omega_2, \dots)$ as $\theta^l\omega = (\omega_{l+1}, \omega_{l+2}, \dots)$.

*Proof of Theorem 3.7.* For $x \ge \mathrm{E}[\log m]$ a proof can be found in [12, Theorem 4.5.1]. For $x = 0$ the result follows from Lemma 3.15. Therefore, let $0 < x < \mathrm{E}[\log m]$ and choose $0 < \varepsilon < \frac{x}{\mathrm{E}[\log m]}$. Furthermore, set $t = 1 - \frac{x}{\mathrm{E}[\log m]}$. For the lower bound we have

$$\begin{aligned}
P_\omega\big(Z_n \le e^{xn}\big) &\ge P_\omega\big(Z_{(t+\varepsilon)n} = 1\big)\cdot P_{\theta^{(t+\varepsilon)n}\omega}\big(Z_{(1-t-\varepsilon)n} \le e^{xn}\big)\\
&= P_\omega\big(Z_{(t+\varepsilon)n} = 1\big)\cdot P_\omega\Big(W_{(1-t-\varepsilon)n} \le \Big(\prod_{i=(t+\varepsilon)n+1}^{n} m_i\Big)^{-1} e^{(1-t)n\mathrm{E}[\log m]}\Big).
\end{aligned} \tag{3.12}$$

The second probability on the right hand side of (3.12) converges to 1 as $n \to \infty$, since $W_n \to W < \infty$ almost surely. Using Lemma 3.15 and letting $\varepsilon \to 0$ finishes the proof of the lower bound after a straight forward calculation.
For the upper bound let $b \in \mathbb{N}$. We have

$$P_\omega\big(Z_n \le e^{xn}\big) \le P_\omega\big(Z_{(t-\varepsilon)n} \le b\big) + P_{\theta^{(t-\varepsilon)n}\omega}\big(Z_{(1-t+\varepsilon)n} \le e^{xn}\big)^b. \tag{3.13}$$

The asymptotics for the first probability on the right hand side of (3.13) are given in Lemma 3.15. The second probability in (3.13) decays exponentially by Theorem 3.5, Lemma 3.14 and the choice of $t$. Note that the proof of Lemma 3.14 also works for the shifted environment $\theta^{(t-\varepsilon)n}\omega$. Now we choose $b$ large enough. Using Lemma 2.23 and letting $\varepsilon \to 0$ finishes the proof.

□

### 3.6.2 Independent random walks

*Proof of Theorem 3.9.* **1. Case**: $x > x^*$
Let $t \in [\mathrm{E}[\log m], I(x))$ for the proof of the lower bound. Following the strategy explained in Section 3.4, independence of the random walks and Lemma 2.22 (ii) yields

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x\Big) &= \mathbb{E}\Big[1-\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\Big)^{Z_n}\Big] \\ &\geq \mathbb{P}(Z_n \geq e^{nt}) \cdot \mathbb{E}\Big[1-\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)\Big)^{e^{nt}}\Big] \\ &\geq \mathbb{P}(Z_n \geq e^{nt})\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)e^{nt}\Big(1-\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big)e^{nt}\Big).\end{aligned}$$

By Theorem 2.15, $\mathbb{P}\big(\frac{S_n}{n} \geq x\big)e^{nt} \to 0$ as $n \to \infty$, since $t < I(x)$. Together with Theorem 3.5 and Lemma 2.23 we conclude

$$\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x\Big) \geq \exp\big(-\big(I_{\mathrm{a}}^{\mathrm{GW}}(t) + I(x) - t\big)n + o(n)\big),$$

which yields the lower bound. For the upper bound, let $0 < \varepsilon < I(x) - \mathrm{E}[\log m]$ and define the set

$$F = F(\varepsilon) = \Big\{\mathrm{E}[\log m], \mathrm{E}[\log m]+\varepsilon, \dots, \mathrm{E}[\log m] + \Big\lceil \frac{I(x)-\mathrm{E}[\log m]}{\varepsilon}\Big\rceil \varepsilon\Big\}.$$

To prove the upper bound, we need to partition according to the number of particles at time $n$. This yields

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x\Big) \leq{}& \mathbb{P}\big(Z_n \geq e^{I(x)n}\big) + \mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x \;\Big|\; Z_n \leq e^{\mathrm{E}[\log m]n}\Big) \\ &+ \sum_{k\in F} \mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x \;\Big|\; Z_n \in \big(e^{kn}, e^{(k+\varepsilon)n}\big]\Big) \cdot \mathbb{P}\Big(Z_n \in \big(e^{kn}, e^{(k+\varepsilon)n}\big]\Big).\end{aligned} \tag{3.14}$$

For all $k \in F$ the Markov inequality yields

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \geq x \;\Big|\; Z_n \in \big(e^{kn}, e^{(k+\varepsilon)n}\big]\Big) &= \mathbb{P}\Big(\sum_{i=1}^{Z_n} \mathbb{1}_{\{S_n^i \geq xn\}} \geq 1 \;\Big|\; Z_n \in \big(e^{kn}, e^{(k+\varepsilon)n}\big]\Big) \\ &\leq e^{(k+\varepsilon)n}\mathbb{P}\Big(\frac{S_n}{n} \geq x\Big).\end{aligned} \tag{3.15}$$

The second summand in (3.14) can be estimated analogously. Plugging (3.15) in (3.14) and using Theorem 2.15 and Theorem 3.5, we conclude with Lemma 2.23

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n}\geq x\Big)&\leq \mathbb{P}\big(Z_n\geq e^{I(x)n}\big)+\mathbb{P}\Big(\frac{S_n}{n}\geq x\Big)\cdot e^{\mathrm{E}[\log m]n}\\&\quad+\sum_{k\in F}e^{(k+\varepsilon)n}\mathbb{P}\Big(\frac{S_n}{n}\geq x\Big)\mathbb{P}\Big(Z_n\in\big(e^{kn},e^{(k+\varepsilon)n}\big]\Big)\\&=\exp\Big(\max_{k\in F}\big(-I_{\mathrm{a}}^{\mathrm{GW}}(k)-I(x)+k+\varepsilon\big)n+o(n)\Big)\\&\leq\sup_{t\in[\mathrm{E}[\log m],I(x)+\varepsilon)}\exp\Big(-\big(I_{\mathrm{a}}^{\mathrm{GW}}(t)+I(x)-t-\varepsilon\big)n+o(n)\Big).\end{aligned}$$

Letting $\varepsilon\to 0$ yields the upper bound.

**2. Case**: $0<x<x^*$

We prove the upper bound first. Since the rate function $I$ is strictly increasing on the interval $[0,x^*]$, we can choose $\varepsilon>0$ such that $\varepsilon<I(x)<\mathrm{E}[\log m]-\varepsilon$. Using the inequality $1-y\leq e^{-y}$ and Theorem 3.5,

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n}\leq x\Big)&=\mathbb{E}\Big[\Big(1-\mathbb{P}\Big(\frac{S_n}{n}>x\Big)\Big)^{Z_n}\Big]\leq\mathbb{E}\Big[\exp\Big(-\mathbb{P}\Big(\frac{S_n}{n}>x\Big)Z_n\Big)\Big]\\&\leq\mathbb{P}\Big(Z_n\leq e^{(I(x)+\varepsilon)n}\Big)+\exp\big(-e^{\varepsilon n+o(n)}\big)\\&=\exp\Big(-I_{\mathrm{a}}^{\mathrm{GW}}(I(x)+\varepsilon)n+o(n)\Big).\end{aligned}$$

Letting $\varepsilon\to 0$ yields the upper bound. Note that $I_{\mathrm{a}}^{\mathrm{GW}}$ is continuous from the right on $(0,I(x^*))$. The proof for the lower bound works in a similar way. More precisely, since $\mathbb{P}(\frac{S_n}{n}>x)<e^{-1}$ for $n$ large enough, Lemma 2.22 (i) and Theorem 2.15 yields

$$\begin{aligned}\mathbb{P}\Big(\frac{\tilde{M}_n}{n}\leq x\Big)&=\mathbb{E}\Big[\Big(1-\mathbb{P}\Big(\frac{S_n}{n}>x\Big)\Big)^{Z_n}\Big]\geq\mathbb{E}\Big[\exp\Big(-e\cdot\mathbb{P}\Big(\frac{S_n}{n}>x\Big)Z_n\Big)\Big]\\&\geq\mathbb{P}\Big(Z_n\leq e^{I(x)n}\Big)\cdot\exp(o(n))=\exp\Big(-I_{\mathrm{a}}^{\mathrm{GW}}(I(x))n+o(n)\Big).\end{aligned}$$

This proves the lower bound.

**3. Case**: $x\leq 0$

We first consider $x<0$. For the upper bound we have for $K\in\mathbb{N}$

$$\mathbb{P}\Big(\frac{\tilde{M}_n}{n}\leq x\Big)=\mathbb{E}\Big[\mathbb{P}\Big(\frac{S_n}{n}\leq x\Big)^{Z_n}\Big]\leq\sum_{k=1}^{K}\mathbb{P}\Big(\frac{S_n}{n}\leq x\Big)^k\cdot\mathbb{P}(Z_n=k)+\mathbb{P}\Big(\frac{S_n}{n}\leq x\Big)^K. \tag{3.16}$$

By Theorem 3.6, $\mathbb{P}(Z_n=k)$ is at most of order $\exp(-\rho_{\mathrm{a}}n+o(n))$ for all $k\in\mathbb{N}$. For all

$K \in \mathbb{N}$, Lemma 2.23 yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{\tilde{M}_n}{n} \le x\Big) \le \max\big\{-(I(x)+\rho_{\mathrm{a}}), -KI(x)\big\}.$$

Hence, letting $K \to \infty$ proves the upper bound. Note that this still holds true for $\rho_{\mathrm{a}} = \infty$. As in the proof of (3.16) we have

$$\mathbb{P}\Big(\frac{\tilde{M}_n}{n} \le x\Big) = \mathbb{E}\Big[\mathbb{P}\Big(\frac{S_n}{n} \le x\Big)^{Z_n}\Big] \ge \mathbb{P}\Big(\frac{S_n}{n} \le x\Big) \cdot \mathbb{P}(Z_n = 1) = \exp\big(-(I(x)+\rho_{\mathrm{a}})n + o(n)\big),$$

which shows the lower bound. For $x = 0$ the result follows from continuity of the rate function $I$ at 0.

**4. Case**: $x = x^*$

By Lemma 3.14 it suffices to show that the quenched probabilities $P_\omega(M_n \le x^* n)$ and $P_\omega(M_n \ge x^* n)$ decay slower than exponentially in $n$ for P-a.e. $\omega$. Analogously to proof of Theorem 2.18,

$$P_\omega\Big(\frac{\tilde{M}_n}{n} \le x^*\Big) = 1 - P_\omega\Big(\frac{\tilde{M}_n}{n} > x^*\Big) \ge 1 - \mathbb{P}\Big(\frac{S_n}{n} > x^*\Big) \prod_{i=1}^{n} m_i. \tag{3.17}$$

Now we have to distinguish two cases. If $I(x^*) = \mathrm{E}[\log m]$, then the right hand side of (3.17) converges to 1 as $n \to \infty$ by Theorem 2.24. If $I(x^*) < \mathrm{E}[\log m]$, then $I(x) = \infty$ for all $x > x^*$ and therefore, $\mathbb{P}(X_1 > x) = 0$ by Lemma 2.25. Hence, the right hand side of (3.17) equals 1. In both cases we get

$$\lim_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{\tilde{M}_n}{n} \le x^*\Big) = 0.$$

Since $P_\omega(\tilde{M}_n \ge x^* n) \ge P_\omega(M_n \ge x^* n)$, it remains to show that $P_\omega(M_n \ge x^* n)$ decays slower than exponentially in $n$. This is done in the proof of Theorem 3.12. ☐

*Proof of Theorem 3.10.* **1. Case**: $x \ge x^*$

The proof of (3.9) is analogous to the proof of Theorem 3.9. The infimum on the right hand side of (3.9) is attained at $t = \mathrm{E}[\log m]$. More precisely, plugging the quenched rate function of the Galton-Watson process defined in (3.6) in equation (3.9) and using $\beta > 1$ shows that the function in the infimum is increasing in $t$. This finishes the proof.

**2. Case**: $x \le x^*$

The proof in this case works exactly as the proof of Theorem 3.9. ☐

### 3.6.3 Branching random walk

Recall that $I_{\mathrm{a}}^{\mathrm{BRW}}(x) = I_{\mathrm{a}}^{\mathrm{ind}}(x)$ and $I_{\mathrm{q}}^{\mathrm{BRW}}(x) = I_{\mathrm{q}}^{\mathrm{ind}}(x)$ for $x > x^*$. In this case, the upper bound in Theorem 3.11 and Theorem 3.12 now follows from Lemma 2.27.

In order to prove Theorem 3.11 we need another preliminary result. If $\beta < \infty$ in Assumption 9, the initial particle should produce $\exp(sn)$ offspring particles in the first step for some $s > 0$. As we want to use independence of these particles, we first derive a quenched estimate. Note that the branching is not independent under the measure $\mathbb{P}$. This quenched estimate is used in the proof of the annealed large deviation result.
For $x \in \mathbb{R}$, $K \in \mathbb{N}$ and $\varepsilon > 0$ define the process $(Y_n)_{n\in\mathbb{N}}$, where

$$Y_n = Y_n(x, K, \varepsilon) = \inf_{l\in\{0,1,\dots,\lfloor nK^{-1}\rfloor\}} \Big\{ \sum_{i=1}^{lK} \log m_i - lK(I(x) + \varepsilon) \Big\}.$$

**Proposition 3.16.** *Suppose that Assumption 3, 6, 8 and 9 are satisfied. For all $x \in \mathbb{R}$ and $\varepsilon > 0$ there exists $K_0 = K_0(x, \varepsilon) \in \mathbb{N}$ such that for* P*-a.e. $\omega$, every $n \in \mathbb{N}$ and $K \geq K_0$ there is $C = C(x, K, \varepsilon) > 0$ with*

$$P_\omega\Big(\frac{M_n}{n} \geq x\Big) \geq C\frac{e^{Y_n}}{n}.$$

Note that $Y_n \leq 0$ for all $n \in \mathbb{N}$. Furthermore, the constants $K_0$ and $C$ are independent of the environment $\omega$ (only $Y_n$ depends on $\omega$).

*Proof.* Fix $0 < \varepsilon_1 < \varepsilon$. By Theorem 2.15 we can choose $K_0$ such that

$$\mathbb{P}(S_K \geq Kx) \geq e^{-K(I(x)+\varepsilon_1)} \tag{3.18}$$

for all $K \geq K_0$. Consider the following embedded Galton-Watson process in random environment $(\hat{Z}_n)_{n\in\mathbb{N}_0}$ consisting of all particles with average displacement of at least $x$ in blocks of length $K$. More precisely, for $n \in \mathbb{N}_0$ let $\hat{D}_n$ be the set of particles in generation $n$ of the embedded process. Let $o \in \hat{D}_0$ and for $n \in \mathbb{N}_0$, $v \in \hat{D}_n$ we have $w \in \hat{D}_1^v$ if $w \in D_K^v$ and $S_w - S_v \geq Kx$. In particular, $\hat{D}_n \subseteq D_{Kn}$ for all $n \in \mathbb{N}$. The number of children of the $k$-th particle in generation $n$ of the embedded process is denoted by $\hat{\xi}_{n+1,k}$. Note that if this embedded process survives until generation $\lfloor nK^{-1}\rfloor$, there is at least one particle $v$ in generation $K\lfloor nK^{-1}\rfloor$ with $S_v \geq K\lfloor nK^{-1}\rfloor x$. Therefore,

$$\begin{aligned} P_\omega\Big(\frac{M_n}{n} \geq x\Big) &\geq P_\omega\big(\hat{Z}_{\lfloor nK^{-1}\rfloor} > 0\big) \cdot \mathbb{P}\big(S_{n-K\lfloor nK^{-1}\rfloor} \geq \big(n - K\lfloor nK^{-1}\rfloor\big)x\big) \\ &\geq P_\omega\big(\hat{Z}_{\lfloor nK^{-1}\rfloor} > 0\big) \cdot \mathbb{P}(X_1 \geq x)^K. \end{aligned} \tag{3.19}$$

Hence, it remains to estimate the survival probability up to generation $\lfloor nK^{-1}\rfloor$ of the embedded process $(\hat{Z}_n)_{n\in\mathbb{N}_0}$. For this we want to use the second moment method. Since the variance of the number of offspring $\xi_{n,k}$ might be infinite, we have to use a truncation argument. More precisely, for $B \in \mathbb{N}$ let $\xi_{n,k}^B = \max\{\xi_{n,k}, B\}$ for all $n, k \in \mathbb{N}$. We use the same notation as for the usual process, but add an index "$B$" when referring to the truncated process. Note that in particular $\xi_{n,k}^B \leq B$ and therefore, $\hat{\xi}_{n,k}^B \leq B^K$ for all

$n, k \in \mathbb{N}$. Moreover, if the truncated embedded process survives, then also the usual embedded process survives, i.e. $P_\omega\big(\hat{Z}_{\lfloor nK^{-1}\rfloor} > 0\big) \geq P_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor} > 0\big)$. By the Cauchy-Schwarz inequality,

$$E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]^2 = E_\omega\Big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\mathbb{1}_{\big\{\hat{Z}^B_{\lfloor nK^{-1}\rfloor}>0\big\}}\Big]^2 \leq E_\omega\Big[\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big)^2\Big]\cdot P_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor} > 0\big).$$

This implies

$$P_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor} > 0\big) \geq \frac{E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]^2}{E_\omega\Big[\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big)^2\Big]} = \left(\frac{\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big)}{E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]^2} + 1\right)^{-1}. \tag{3.20}$$

This term can be estimated analogously to (2.2) in [26]. More precisely, since

$$\begin{aligned}\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big) &= \mathrm{Var}_\omega\left(\sum_{k=1}^{\hat{Z}^B_{\lfloor nK^{-1}\rfloor-1}} \hat{\xi}^B_{\lfloor nK^{-1}\rfloor,k}\right)\\ &= E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor-1}\big]\,\mathrm{Var}_\omega\big(\hat{\xi}^B_{\lfloor nK^{-1}\rfloor,1}\big) + E_\omega\big[\hat{\xi}^B_{\lfloor nK^{-1}\rfloor,1}\big]^2\,\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor-1}\big),\end{aligned}$$

we get recursively

$$\begin{aligned}\frac{\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big)}{E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]^2} &= \frac{\mathrm{Var}_\omega\big(\hat{\xi}^B_{\lfloor nK^{-1}\rfloor,1}\big)}{E_\omega\big[\hat{\xi}^B_{\lfloor nK^{-1}\rfloor,1}\big]E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]} + \frac{\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor-1}\big)}{E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor-1}\big]^2}\\ &= \sum_{l=1}^{\lfloor nK^{-1}\rfloor}\frac{\mathrm{Var}_\omega\big(\hat{\xi}^B_{l,1}\big)}{E_\omega\big[\hat{\xi}^B_{l,1}\big]E_\omega\big[\hat{Z}^B_l\big]}.\end{aligned} \tag{3.21}$$

For the numerator on the right hand side of (3.21) we have for all $l \in \mathbb{N}$

$$\mathrm{Var}_\omega\big(\hat{\xi}^B_{l,1}\big) \leq E_\omega\big[\big(\hat{\xi}^B_{l,1}\big)^2\big] \leq B^{2K}. \tag{3.22}$$

With a slight abuse of notation we write $D_k(B)$ for the set of particles of the truncated process in generation $k$. For all $l \in \mathbb{N}$ and $v \in D_{(l-1)K}$

$$\begin{aligned}E_\omega\big[\hat{\xi}^B_{l,1}\big] &= E_\omega\Big[\sum_{w\in D^v_K(B)}\mathbb{1}_{\{S_w-S_v\geq Kx\}}\Big] = \mathbb{P}(S_K \geq Kx)\cdot E_\omega\big[|D^v_K(B)|\big]\\ &= \mathbb{P}(S_K \geq Kx)\cdot\prod_{i=(l-1)K+1}^{lK}E_\omega\big[\xi^B_{i,1}\big].\end{aligned} \tag{3.23}$$

In particular, since $p(0) = 0$, this implies $E_\omega\big[\hat{\xi}^B_{l,1}\big] \geq \mathbb{P}(S_K \geq Kx)$ for all $l \in \mathbb{N}$. This estimate will be used for the first term in the denominator on the right hand side of (3.21). For the second term we also have to estimate $E_\omega\big[\xi^B_{i,1}\big]$. More precisely, using

Assumption 9, we get

$$\begin{aligned}
E_\omega\big[\xi^B_{i,1}\big] &= E_\omega\big[\max\{\xi_{i,1}, B\}\big] = E_\omega\big[\xi_{i,1}\mathbb{1}_{\{\xi_{i,1}\le B\}}\big] + B\cdot P_\omega(\xi_{i,1} > B) \\
&= E_\omega\big[\xi_{i,1}\big] - E_\omega\big[\xi_{i,1}\mathbb{1}_{\{\xi_{i,1}> B\}}\big] + B\cdot P_\omega(\xi_{i,1} > B) \\
&= m_i - \sum_{k=1}^{\infty} P_\omega\big(\xi_{i,1}\mathbb{1}_{\{\xi_{i,1}> B\}} \ge k\big) + B\cdot P_\omega(\xi_{i,1} > B) \\
&= m_i - \sum_{k=B+1}^{\infty} P_\omega\big(\xi_{i,1} \ge k\big) \\
&\ge m_i\Big(1 - d_1 \sum_{k=B+1}^{\infty} k^{-\beta}\Big). \qquad (3.24)
\end{aligned}$$

In the case $\beta = \infty$ there is a similar estimate. Note that the sum on the right hand side of (3.24) converges, since $\beta > 1$. Now let $\varepsilon_2 > 0$ such that $\varepsilon_1 - \log(1-\varepsilon_2) \le \varepsilon$ and choose $B$ large enough such that $d_1\sum_{k=B+1}^{\infty} k^{-\beta} \le \varepsilon_2$. Combining (3.18), (3.23) and (3.24),

$$\begin{aligned}
E_\omega\big[\hat{Z}^B_l\big] &= \prod_{j=1}^{l} E_\omega\big[\hat{\xi}^B_{j,1}\big] = \prod_{j=1}^{l}\Big(\mathbb{P}(S_K \ge Kx)(1-\varepsilon_2)^K \prod_{i=(j-1)K+1}^{jK} m_i\Big) \\
&\ge e^{-lK(I(x)+\varepsilon_1)}(1-\varepsilon_2)^{lK}\prod_{i=1}^{lK} m_i \\
&\ge \exp\Big(\sum_{i=1}^{lK}\log m_i - lK(I(x)+\varepsilon)\Big) \\
&\ge e^{Y_n}. \qquad (3.25)
\end{aligned}$$

Altogether, using (3.21), (3.22), (3.23) and (3.25), we get

$$\frac{\mathrm{Var}_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big)}{E_\omega\big[\hat{Z}^B_{\lfloor nK^{-1}\rfloor}\big]^2} \le \frac{B^{2K}\lfloor nK^{-1}\rfloor}{\mathbb{P}(S_K \ge Kx)} e^{-Y_n}.$$

Since $Y_n \le 0$, (3.20) implies

$$P_\omega\big(\hat{Z}^B_{\lfloor nK^{-1}\rfloor} > 0\big) \ge \Big(\frac{B^{2K}\lfloor nK^{-1}\rfloor}{\mathbb{P}(S_K \ge Kx)} + 1\Big)^{-1} e^{Y_n},$$

which yields the result together with (3.19). □

To estimate the annealed probability of the event $\{M_n \ge xn\}$ it remains to investigate the asymptotics of $Y_n$ with respect to P.

**Lemma 3.17.** *Let $x > x^*$ and $0 \le y \le I(x) - \mathrm{E}[\log m]$. Then,*

$$\lim_{\varepsilon\to 0}\lim_{K\to\infty}\liminf_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big) = \lim_{\varepsilon\to 0}\lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big)$$
$$= -I^{\log m}(I(x)-y).$$

*Furthermore, for all $x\in\mathbb{R}$, $\varepsilon>0$ and $K\in\mathbb{N}$*

$$\lim_{n\to\infty}\frac{Y_n}{n} = \min\big\{0, -(I(x)+\varepsilon-\mathrm{E}[\log m])\big\} \quad \mathrm{P}\text{-}a.s.$$

*Proof.* Since we have

$$\mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big) \le \mathrm{P}\bigg(\frac{1}{\lfloor nK^{-1}\rfloor K}\sum_{i=1}^{\lfloor nK^{-1}\rfloor K}\log m_i \ge I(x)+\varepsilon - y\frac{n}{\lfloor nK^{-1}\rfloor K}\bigg),$$

taking the limsup yields

$$\limsup_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big) \le -I^{\log m}(I(x)+\varepsilon-y).$$

Letting $\varepsilon\to 0$ finishes the proof of the upper bound. Note that $I^{\log m}$ is continuous. For the lower bound let $\varepsilon_1>0$ and take $K\in\mathbb{N}$ large enough such that

$$\mathrm{P}\Big(\sum_{i=1}^{K}\log m_i \ge K(I(x)+\varepsilon-y)\Big) \ge \exp\big(-K(I^{\log m}(I(x)+\varepsilon-y)+\varepsilon_1)\big).$$

Then,

$$\begin{aligned}\mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big) &\ge \mathrm{P}\Big(\sum_{i=(l-1)K+1}^{lK}\log m_i \ge K(I(x)+\varepsilon-y)\quad \forall l\in\{1,\dots,\lfloor nK^{-1}\rfloor\}\Big)\\ &= \mathrm{P}\Big(\sum_{i=1}^{K}\log m_i \ge K(I(x)+\varepsilon-y)\Big)^{\lfloor nK^{-1}\rfloor}\\ &\ge \exp\big(-n(I^{\log m}(I(x)+\varepsilon-y)+\varepsilon_1)\big).\end{aligned}$$

Taking the liminf as $n\to\infty$ yields

$$\liminf_{n\to\infty}\frac{1}{n}\log \mathrm{P}\Big(\frac{Y_n}{n}\ge -y\Big) \ge -I^{\log m}(I(x)+\varepsilon-y)+\varepsilon_1.$$

Letting $\varepsilon_1\to 0$ and $\varepsilon\to 0$ finishes the proof of the first part of Lemma 3.17. For the second part of the lemma note that

$$\frac{Y_n}{n} \le \min\Big\{0, \frac{1}{n}\sum_{i=1}^{\lfloor nK^{-1}\rfloor K} \log m_i - \frac{\lfloor nK^{-1}\rfloor K}{n}(I(x)+\varepsilon)\Big\}.$$

Taking limits $n \to \infty$ and $\varepsilon \to 0$ yields the upper bound. For the lower bound we have to distinguish two cases. First assume that $\mathrm{E}[\log m] > I(x) + \varepsilon$. Then, however,

$$\lim_{k\to\infty} \sum_{i=1}^{k} \log m_i - k(I(x)+\varepsilon) = \infty \quad \text{P-a.s.}$$

and therefore, $\lim_{n\to\infty} Y_n > -\infty$ P-a.s., which implies the claim. For the second case let $\mathrm{E}[\log m] \le I(x) + \varepsilon$ and assume that there is $\varepsilon_1 > 0$ such that

$$\liminf_{n\to\infty} \frac{Y_n}{n} \le -(I(x) + \varepsilon - \mathrm{E}[\log m] + 2\varepsilon_1) \quad \text{P-a.s.}$$

Hence, $Y_n \le -(I(x)+\varepsilon-\mathrm{E}[\log m]+\varepsilon_1)n$ for infinitely many $n$ almost surely and therefore also $\sum_{i=1}^{n} \log m_i \le (\mathrm{E}[\log m] - \varepsilon_1)n$ for infinitely many $n$ almost surely, which leads to a contradiction. This implies the lower bound. $\square$

We are now able to prove the main results of this chapter.

*Proof of Theorem 3.11.* Let $x > x^*$. As already mentioned above, the upper bound immediately follows from Theorem 3.9 and Lemma 2.27. It remains to prove the lower bound.

First of all recall that $I_{\mathrm{a}}^{\mathrm{GW}}(t) = \inf_{s\in[0,t]}\{\beta s + I^{\log m}(t-s)\}$ for $t \ge \mathrm{E}[\log m]$, see (3.4). Now fix $t \in [\mathrm{E}[\log m], I(x))$, $s \in [0, t - \mathrm{E}[\log m]]$ and let $\varepsilon > 0$ and $K \in \mathbb{N}$ large enough such that Proposition 3.16 is applicable. Since the particles move and branch independently with respect to $P_\omega$, Lemma 2.26 yields

$$\begin{aligned}
\mathbb{P}\Big(\frac{M_n}{n} \ge x\Big) &= \mathbb{P}\Big(\max_{v\in D_1} \frac{M_{n-1}^v + S_v}{n} \ge x\Big) \ge \mathbb{P}\Big(\max_{v\in D_1} \frac{M_{n-1}^v}{n-1} \ge x\Big) \cdot \mathbb{P}(X_1 \ge x)\\
&= \mathbb{E}\Big[1 - \Big(1 - P_{\theta\omega}\Big(\frac{M_{n-1}}{n-1} \ge x\Big)\Big)^{Z_1}\Big] \cdot \mathbb{P}(X_1 \ge x)\\
&\ge \mathrm{E}\Big[1 - \Big(1 - P_{\theta\omega}\Big(\frac{M_{n-1}}{n-1} \ge x\Big)\Big)^{e^{sn}}\Big] \cdot \mathbb{P}(X_1 \ge x) \cdot \mathbb{P}\big(Z_1 \ge e^{sn}\big). \qquad (3.26)
\end{aligned}$$

Let $0 < \varepsilon_1 < I(x) - t$. By Proposition 3.16 and Lemma 2.22 (ii)

$$
\begin{aligned}
&\mathrm{E}\Big[1 - \Big(1 - P_{\theta\omega}\Big(\frac{M_{n-1}}{n-1} \geq x\Big)\Big)^{e^{sn}}\Big] \geq \mathrm{E}\Big[1 - \Big(1 - \frac{C}{n-1} e^{Y_{n-1}}\Big)^{e^{sn}}\Big] \\
&\geq \mathrm{E}\Big[\frac{C}{n-1} e^{Y_{n-1}} e^{sn}\Big(1 - \frac{C}{n-1} e^{Y_{n-1}} e^{sn}\Big) \mid 0 \leq \frac{Y_{n-1}}{n-1} + I(x) - t + s \leq \varepsilon_1\Big] \\
&\quad \cdot \mathrm{P}\Big(0 \leq \frac{Y_{n-1}}{n-1} + I(x) - t + s \leq \varepsilon_1\Big) \\
&\geq \frac{Ce^s}{n-1} e^{-(I(x)-t)(n-1)}\Big(1 - \frac{Ce^s}{n-1} e^{-(I(x)-t-\varepsilon_1)(n-1)}\Big) \\
&\quad \cdot \mathrm{P}\Big(0 \leq \frac{Y_{n-1}}{n-1} + I(x) - t + s \leq \varepsilon_1\Big).
\end{aligned} \tag{3.27}
$$

The third term on the right hand side of (3.27) converges to 1 as $n \to \infty$, since we have $I(x) - t - \varepsilon_1 > 0$. For the last term Lemma 3.17 yields

$$
\lim_{\varepsilon \to 0} \lim_{K \to \infty} \liminf_{n \to \infty} \frac{1}{n} \log \mathrm{P}\Big(0 \leq \frac{Y_{n-1}}{n-1} + I(x) - t + s \leq \varepsilon_1\Big) = -I^{\log m}(t-s). \tag{3.28}
$$

Combining (3.26), (3.27), (3.28) and Assumption 9 shows

$$
\mathbb{P}\Big(\frac{M_n}{n} \geq x\Big) \geq \exp\big(-(I(x) - t + \beta s + I^{\log m}(t-s))n + o(n)\big).
$$

Since this holds for all $s$ and $t$, the lower bound follows.

$\square$

*Proof of Theorem 3.12.* **1. Case**: $x > x^*$
The upper bound immediately follows from Theorem 3.10 and Lemma 2.27. It remains to prove the lower bound. Since $I(x) > \mathrm{E}[\log m]$, Proposition 3.16 and Lemma 3.17 yield

$$
\liminf_{n \to \infty} \frac{1}{n} \log P_\omega\Big(\frac{M_n}{n} \geq x\Big) \geq \lim_{n \to \infty} \frac{Y_n}{n} = -(I(x) + \varepsilon - \mathrm{E}[\log m]).
$$

Letting $\varepsilon \to 0$ finishes the proof of the lower bound.
**2. Case**: $x < x^*$
Analogously to the annealed case, there is only one particle at time $tn$ and the position of this particle is smaller than its expectation. Afterwards, all particles move and branch as usual. For the lower bound let $t \in \big(0, \min\{1 - \frac{x}{x^*}, 1\}\big]$ and fix $\varepsilon > 0$. Note that

$t \in (0, 1 - \frac{x}{x^*}, 1]$ if $x > 0$ and $t \in (0, 1]$ if $x \leq 0$. Then we have

$$\begin{aligned}
P_\omega\Big(\frac{M_n}{n} \leq x\Big) &\geq P_\omega\Big(\frac{M_n}{n} \leq x \mid Z_{tn} = 1\Big) \cdot P_\omega(Z_{tn} = 1) \\
&= P_{\theta^{tn}\omega}\Big(\frac{S_{tn} + M_{(1-t)n}}{n} \leq x\Big) \cdot P_\omega(Z_{tn} = 1) \\
&\geq P_{\theta^{tn}\omega}\Big(\frac{M_{(1-t)n}}{(1-t)n} \leq x^* + \varepsilon\Big) \cdot P_\omega\Big(\frac{S_{tn}}{n} \leq (x - (1-t)(x^* + \varepsilon))\Big) \cdot P_\omega(Z_{tn} = 1).
\end{aligned} \tag{3.29}$$

Since the first probability on the right hand side in (3.29) converges to 1 $P_\omega$-a.s. as $n \to \infty$ for P-a.e. $\omega$ analogously to (3.17), we get

$$P_\omega\Big(\frac{M_n}{n} \leq x\Big) \geq \exp\Big(\Big[I\Big(t^{-1}(x - (1-t)(x^* + \varepsilon))\Big) + \rho_{\mathrm{q}}\Big]tn + o(n)\Big).$$

Since this inequality holds for all $t \in \big(0, \min\{1 - \frac{x}{x^*}, 1\}\big]$, letting $\varepsilon \to 0$ yields

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{M_n}{n} \leq x\Big) \geq \sup_{t\in(0,\min\{1-\frac{x}{x^*},1\}]} -H_{\mathrm{q}}(x) = -\inf_{t\in(0,1]} H_{\mathrm{q}}(x).$$

For the upper bound, define

$$T_n = \inf\Big\{t \geq 0 \colon Z_{tn} \geq n^3\Big\}$$

and for $\varepsilon_1 > 0$ let

$$F = F(\varepsilon_1) = \Big\{\varepsilon_1, 2\varepsilon_1, \dots, \Big\lceil\Big(\min\Big\{1 - \frac{x}{x^*}, 1\Big\}\Big)\varepsilon_1^{-1}\Big\rceil\varepsilon_1\Big\}.$$

Note that $T_n \in \big(t - \varepsilon_1, t\big]$ implies $Z_{tn} \geq n^3$, since $p(0) = 0$. Therefore, we have

$$\begin{aligned}
&P_\omega\Big(\frac{M_n}{n} \leq x\Big) \\
&\leq P_\omega\Big(T_n > \min\Big\{1 - \frac{x}{x^*}, 1\Big\}\Big) + \sum_{t\in F} P_\omega\Big(\frac{M_n}{n} \leq x \mid T_n \in \big(t - \varepsilon_1, t\big]\Big) P_\omega\big(T_n \in \big(t - \varepsilon_1, t\big]\big) \\
&\leq P_\omega\big(Z_{(\min\{1-\frac{x}{x^*},1\})n} \leq n^3\big) + \sum_{t\in F} P_\omega\Big(\frac{M_n}{n} \leq x \mid T_n \in \big(t - \varepsilon_1, t\big]\Big) P_\omega(Z_{(t-\varepsilon_1)n} \leq n^3).
\end{aligned} \tag{3.30}$$

For $\varepsilon_2 > 0$, Lemma 2.26 thus shows

$$\begin{aligned} &P_\omega\Big(\frac{M_n}{n} \le x \mid T_n \in \big(t-\varepsilon_1, t\big]\Big) \\ &\le P_\omega\bigg(\max_{v\in D_{tn}} \frac{S_{tn} + M^v_{(1-t)n}}{n} \le x \,\bigg|\, T_n \in \big(t-\varepsilon_1, t\big]\bigg) \\ &\le P_\omega\Big(\frac{S_{tn}}{n} < -\big((1-t)(x^*-\varepsilon_2) - x\big)\Big) + P_{\theta^{tn}\omega}\Big(\frac{M_{(1-t)n}}{(1-t)n} < x^* - \varepsilon_2\Big)^{n^3}. \end{aligned} \tag{3.31}$$

To estimate the second probability in (3.31) we define the shifted version of $Y_n$. More precisely, for $x \in \mathbb{R}$, $K \in \mathbb{N}$ and $\varepsilon > 0$ let

$$Y'_{(1-t)n} = Y'_{(1-t)n}(x, K, \varepsilon) = \inf_{l\in\{0,1,\ldots,\lfloor(1-t)nK^{-1}\rfloor\}} \Big\{\sum_{i=tn+1}^{tn+lK} \log m_i - lK(I(x)+\varepsilon)\Big\}.$$

Analogously to Proposition 3.16 one gets

$$P_{\theta^{tn}\omega}\Big(\frac{M_{(1-t)n}}{(1-t)n} \ge x^* - \varepsilon_2\Big) \ge C\frac{e^{Y'_{(1-t)n}}}{n}.$$

Furthermore, as in the proof of Lemma 3.17 we have $\lim_{n\to\infty} Y'_{(1-t)n} > -\infty$ P-a.s. for $\varepsilon$ small enough, since $\mathrm{E}[\log m] > I(x^*-\varepsilon_2)+\varepsilon$. Therefore, the second term in (3.31) decays faster than exponentially in $n$. Combining (3.30) and (3.31) and letting $\varepsilon_1, \varepsilon_2 \to 0$, we conclude with Lemma 2.23

$$\limsup_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{M_n}{n} \le x\Big) \le -\inf_{t\in(0,1-\frac{x}{x^*}]} \Big\{t\rho_{\mathrm{q}} + tI\Big(-\frac{(1-t)x^*-x}{t}\Big)\Big\} = -H_{\mathrm{q}}(x).$$

**3. Case**: $x = x^*$
The proof for $P_\omega(M_n \le x^* n)$ is analogous to the proof of Theorem 3.9. By Proposition 3.16 and Lemma 3.17

$$\liminf_{n\to\infty} \frac{1}{n} \log P_\omega\Big(\frac{M_n}{n} \ge x^*\Big) \ge \lim_{n\to\infty} \frac{Y_n}{n} = \min\big\{0, -(I(x^*) + \varepsilon - \mathrm{E}[\log m])\big\}.$$

Since $I(x^*) \le \mathrm{E}[\log m]$, letting $\varepsilon \to$ finishes the proof.

□

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