Some Results on Parametric Reduction of Port-Hamiltonian Systems

within the DFG-ANR-Project INFIDHEM: Interconnected Infinite-Dimensional Systems for Heterogeneous Media

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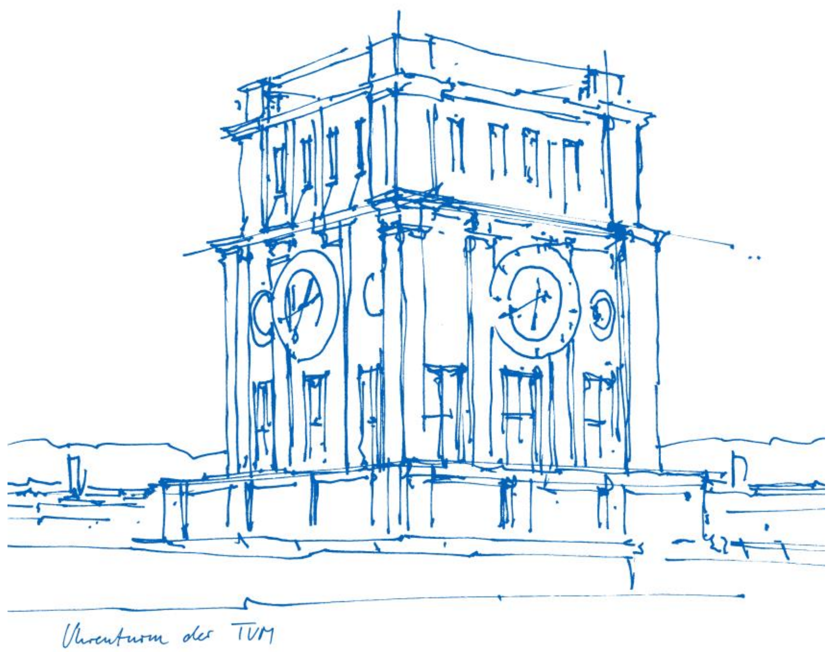
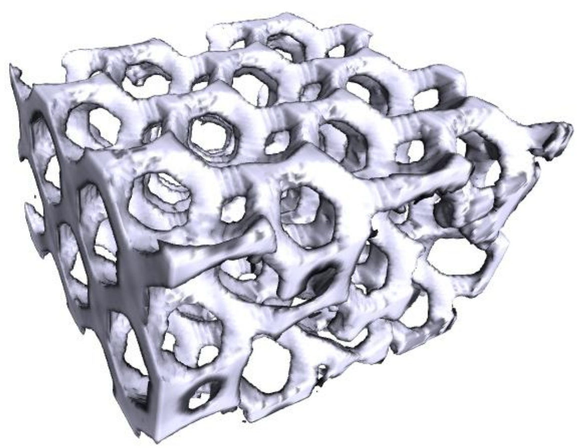


INFIDHEM
(Interconnected INfinite-Dimensional systems for Heterogeneous Media)

- French-German project with Universities of
  Besançon (Prof. Le Gorrec), Toulouse (Prof. Matignon), Lyon (Prof. Maschke), Wuppertal (Prof. Jacob), Kiel (Prof. Meurer), Munich (Prof. Lohmann)
- Period: 2017-2020, DFG-ANR funded

- Fluid-thermo-structure interaction on heterogeneous media
- Active material (e.g. Piezo)
- Modelling as port-Hamiltonian system



Metallic foam as an example for heterogeneous media[1]

[1] Tomography data from LAGEP, Université Claude Bernard Lyon 1 processed with iMorph (http://www.imorph.fr/)

Port-Hamiltonian-System:

$$\dot{\boldsymbol{x}} = (\boldsymbol{J} - \boldsymbol{R})\nabla H(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \boldsymbol{B}^T \nabla H(\boldsymbol{x})$$

$\boldsymbol{J}$: Interconnection Matrix, skew symmetric

$\boldsymbol{R}$: Dissipation Matrix, pos. semidef.

is passive, since $\dot{H} \leq \boldsymbol{y}^T \boldsymbol{u}$ with

pos. def. Energy Function $H(\boldsymbol{x})$.

Linear PH-System (with $H = \tfrac{1}{2}\boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x}$)

a) in *standard form*:

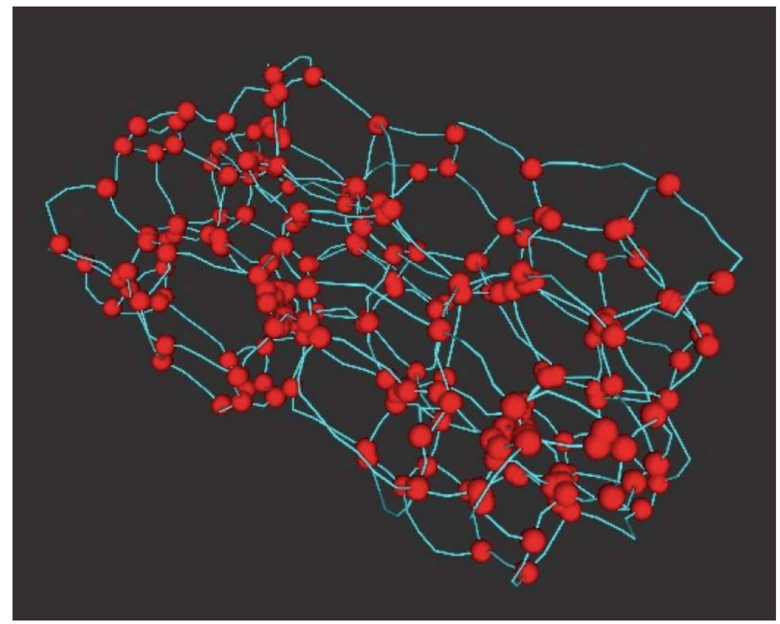
$$\dot{\boldsymbol{x}} = \overbrace{(\boldsymbol{J} - \boldsymbol{R})\boldsymbol{Q}}^{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T \boldsymbol{Q}}_{C}\boldsymbol{x}$$

PH-Model

b) in *co-energie form* (by Trf. $\boldsymbol{Q}\boldsymbol{x} = \boldsymbol{e}$):

$$\overbrace{\boldsymbol{Q}^{-1}}^{E}\dot{\boldsymbol{e}} = \overbrace{(\boldsymbol{J} - \boldsymbol{R})}^{A}\boldsymbol{e} + \boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T}_{C}\boldsymbol{e}$$

CoPH-Model



Topology of a metallic foam, extracted by image processing from tomography data (figure from DFG-ANR-application)

Starting Point: Projective reduction of linear state-space models:

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \Rightarrow \quad \begin{aligned} \overbrace{W^T E V}^{E_r}\dot{x}_r &= \overbrace{W^T A V}^{A_r} x_r + \overbrace{W^T B}^{B_r} u \\ y &= \underbrace{CV}_{C_r} x_r \end{aligned}$$

$$\begin{aligned} \dot{x} &= \overbrace{(J - R)Q}^{A} x + Bu \\ y &= \underbrace{B^T Q}_{C} x \end{aligned} \quad \overset{\text{modal}:\ x = Vz}{\Rightarrow} \quad \begin{aligned} \dot{z} &= V^{-1}(J - R)\overbrace{V^{-T}V^T}^{I} QVz + V^{-1}Bu \\ y &= B^T QVz \end{aligned}$$

Outline:

How to choose V, W so that:

- *Rational Interpolation / Moment Matching is achieved*,
- *PH structure is preserved*

How to perform parametric reduction *by Matrix Interpolation*

Application to a discretized model of the two-dimensional wave-equations

Notation

	Matrix	State vector
High dimensional parametric system:	$\boldsymbol{A}(\boldsymbol{p})$	$\boldsymbol{x}$
High dimensional sample system (for specific value of parameter):	$\boldsymbol{A}_i = \boldsymbol{A}(\boldsymbol{p}_i)$	$\boldsymbol{x}$
Locally reduced model:	$\boldsymbol{A}_{r,i}$	$\boldsymbol{x}_{r,i}$
Reduced model, transformed in joint subspace:	$\boldsymbol{A}^*_{r,i}$	$\boldsymbol{x}^*_{r,i} = \boldsymbol{x}^*_r$
Reduced parametric model (from Interpolation):	$\boldsymbol{A}^*_r(\boldsymbol{p})$	$\boldsymbol{x}^*_{r,i} = \boldsymbol{x}^*_r$

Structure preserving reduction of the co-energy form [1]

A projection matrix $\boldsymbol{V}$ is calculated by some known method (Moment Matching, POD). Then, with the choice $\boldsymbol{W} = \boldsymbol{V}$ we find the reduced model in co-energy form (descriptor form)

$$\overbrace{\boldsymbol{V}^T\boldsymbol{Q}^{-1}\boldsymbol{V}}^{\boldsymbol{Q}_r^{-1}}\dot{\boldsymbol{e}}_r = \overbrace{\boldsymbol{V}^T(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{V}}^{\boldsymbol{J}_r-\boldsymbol{R}_r}\boldsymbol{e}_r + \overbrace{\boldsymbol{V}^T\boldsymbol{B}}^{\boldsymbol{B}_r}\boldsymbol{u}$$

$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T\boldsymbol{V}}_{\boldsymbol{B}_r^T}\boldsymbol{e}_r$$

CoPH-MOR

where $\boldsymbol{Q}_r$ is pos. def., $\boldsymbol{J}_r$ skew symmetric, $\boldsymbol{R}_r$ pos. semidef.

Structure preserving reduction of the standard PH form [2, 3] (DesPH)

A projection matrix $\boldsymbol{V}$ is calculated by some known method (Moment Matching, POD). Then, with the choice $\boldsymbol{W} = \boldsymbol{Q}\boldsymbol{V}$ we first find

$$\overbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{Q}_r^{-1}}\dot{\boldsymbol{x}}_r = \overbrace{\boldsymbol{V}^T\boldsymbol{Q}(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{J}_r-\boldsymbol{R}_r}\boldsymbol{x}_r + \overbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{B}}^{\boldsymbol{B}_r}\boldsymbol{u}$$

$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{V}}_{\boldsymbol{B}_r^T}\boldsymbol{x}_r$$

PH-MOR

Applying the *state transformation* $\boldsymbol{z}_r = \boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}\boldsymbol{x}_r$ leads to the reduced model in standard PH form

$$\dot{\boldsymbol{z}}_r = \overbrace{\boldsymbol{V}^T\boldsymbol{Q}(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{J}_r-\boldsymbol{R}_r}\overbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}}^{\boldsymbol{Q}_r}\boldsymbol{z}_r + \overbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{B}}^{\boldsymbol{B}_r}\boldsymbol{u}$$

$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{V}}_{\boldsymbol{B}_r^T}\underbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}}_{\boldsymbol{Q}_r}\boldsymbol{z}_r$$

PH-MOR

However, because of the state transformation, the vector $\boldsymbol{z}_r$ has the physical meaning of a co-state vector $\boldsymbol{e}_r = \boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}\boldsymbol{x}_r$.

Structure preserving reduction of the standard form (consistent state vector) [4]

A projection matrix $\boldsymbol{V}$ is calculated by some known method (Moment Matching, POD). Then, with the choice $\boldsymbol{W}=\boldsymbol{Q}\boldsymbol{V}$ we first find

$$\overbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{Q}_r}\dot{\boldsymbol{x}}_r=\boldsymbol{V}^T\boldsymbol{Q}(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{Q}\boldsymbol{V}\boldsymbol{x}_r+\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{B}\boldsymbol{u}$$
$$\boldsymbol{y}=\boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{V}\boldsymbol{x}_r \qquad (*)$$

PH-MOR

By pre-multiplication with $(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}$ and by inserting the identity matrix $(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})$, this becomes a reduced model in standard form

$$\dot{\boldsymbol{x}}_r=\overbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}\boldsymbol{V}^T\boldsymbol{Q}(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{Q}\boldsymbol{V}(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}}^{\boldsymbol{J}_r-\boldsymbol{R}_r}\overbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})}^{\boldsymbol{Q}_r}\boldsymbol{x}_r+\overbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{B}}^{\boldsymbol{B}_r}\boldsymbol{u}$$
$$\boldsymbol{y}=\underbrace{\boldsymbol{B}^T\boldsymbol{Q}\boldsymbol{V}(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}}_{\boldsymbol{B}_r^T}\underbrace{(\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})}_{\boldsymbol{Q}_r}\boldsymbol{x}_r \qquad (**)$$

PH-MOR

This model is in *standard form and physically consistent* (By a different derivation, it was first suggested in [4], another version can be found in [7]).

Structure-preserving reduction of the standard PH-form using Cholesky-factorization [7]

The linear system is reduced with so far unknown matrices $\boldsymbol{V}$ and $\boldsymbol{W}^T$. Additionally we perform the approximation $\boldsymbol{Q} \approx \boldsymbol{W}\boldsymbol{V}^T\boldsymbol{Q}$ and demand $\boldsymbol{V}$ and $\boldsymbol{W}^T$ to be biorthogonal ($\boldsymbol{W}^T\boldsymbol{V} = \boldsymbol{I}$).

$$\overbrace{\boldsymbol{W}^T\boldsymbol{V}}^{\boldsymbol{I}}\dot{\boldsymbol{x}}_r = \overbrace{\boldsymbol{W}^T(\boldsymbol{J}-\boldsymbol{R})\boldsymbol{W}}^{\boldsymbol{J}_r-\boldsymbol{R}_r}\overbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}}^{\boldsymbol{Q}_r}x_r + \overbrace{\boldsymbol{W}^T\boldsymbol{B}}^{\boldsymbol{B}_r}\boldsymbol{u}$$

$$\boldsymbol{y} = \underbrace{\boldsymbol{B}^T\boldsymbol{W}}_{\boldsymbol{B}_r^T}\underbrace{\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V}}_{\boldsymbol{Q}_r}\boldsymbol{x}_r$$

PH-MOR

To achieve biorthogonality, we use Cholesky-factorization to generate $\boldsymbol{V}$ and $\boldsymbol{W}^T$ from a $\tilde{\mathbf{V}}$ resulting from an arbitrary reduction method:

$$\tilde{\boldsymbol{V}}^T\boldsymbol{Q}\tilde{\boldsymbol{V}} = \boldsymbol{R}^T\boldsymbol{R}$$

The reduction matrices are finally computed as:

$$\boldsymbol{V} = \tilde{\boldsymbol{V}}\boldsymbol{R}^{-1} \qquad \boldsymbol{W} = \boldsymbol{Q}\boldsymbol{V}$$

Parametric Reduction of the co-energy form [1]

The matrices $\boldsymbol{J}, \boldsymbol{R}, \boldsymbol{Q}^{-1}, \boldsymbol{B}$ depend on a parameter p. We now specify some values p_i and reduce the corresponding models with corresponding individual projection matrices $\boldsymbol{V}_i$:

$$\boldsymbol{V}_i^T \boldsymbol{Q}_i^{-1} \boldsymbol{V}_i \dot{\boldsymbol{e}}_{r,i} = \boldsymbol{V}_i^T (\boldsymbol{J}_i - \boldsymbol{R}_i) \boldsymbol{V}_i \boldsymbol{e}_{r,i} + \boldsymbol{V}_i^T \boldsymbol{B}_i \boldsymbol{u} \quad , \quad i = 1, \ldots, k$$
$$\boldsymbol{y}_i = \boldsymbol{B}_i^T \boldsymbol{V}_i \boldsymbol{e}_{r,i}$$

CoPH-pMOR

Matrix Interpolation: for some value p of interest, we like to find the matrices $\boldsymbol{J}_r(p), \boldsymbol{R}_r(p), \boldsymbol{Q}_r^{-1}(p), \boldsymbol{B}_r(p)$ by *Interpolation* between the matrices $\boldsymbol{J}_{r,i}, \boldsymbol{R}_{r,i}, \boldsymbol{Q}_{r,i}^{-1}, \boldsymbol{B}_{r,i}$. In order to make this interpolation physically meaningful, we first have to adapt/adjust the state spaces of the local models to each others. This is done by

- *State transformation* of each local model,

  $$\boldsymbol{x}_{r,i}^* = \boldsymbol{T}_i \boldsymbol{x}_{r,i}$$

  with $\boldsymbol{T}_i = \boldsymbol{U}^T \boldsymbol{V}_i$ (where $\boldsymbol{U}$ is from an $SVD\{[\boldsymbol{V}_1, \ldots, \boldsymbol{V}_k]\}$, [5]) and
- *Pre-multiplication* of each local model by a matrix

  $\boldsymbol{M}_i = (\boldsymbol{V}_i^T \boldsymbol{U})^{-1}$ (see [5, 1]. Alternatives in [6]) :

$$\overbrace{(\boldsymbol{V}_i^T\boldsymbol{U})^{-1}\boldsymbol{V}_i^T\boldsymbol{Q}_i^{-1}\boldsymbol{V}_i(\boldsymbol{U}^T\boldsymbol{V}_i)^{-1}}^{\boldsymbol{Q}_{r,i}^{-1}}\dot{\boldsymbol{e}}_{r,i}^* = \overbrace{(\boldsymbol{V}_i^T\boldsymbol{U})^{-1}\boldsymbol{V}_i^T(\boldsymbol{J}_i-\boldsymbol{R}_i)\boldsymbol{V}_i(\boldsymbol{U}^T\boldsymbol{V}_i)^{-1}}^{\boldsymbol{J}_{r,i}-\boldsymbol{R}_{r,i}}\boldsymbol{e}_{r,i}^* + \overbrace{(\boldsymbol{V}_i^T\boldsymbol{U})^{-1}\boldsymbol{V}_i^T\boldsymbol{B}_i}^{\boldsymbol{B}_{r,i}}\boldsymbol{u}$$

$$\boldsymbol{y}_i = \underbrace{\boldsymbol{B}_i^T\boldsymbol{V}_i(\boldsymbol{U}^T\boldsymbol{V}_i)^{-1}}_{\boldsymbol{B}_{r,i}^T}\boldsymbol{e}_{r,i}^*$$

Matrix Interpolation:

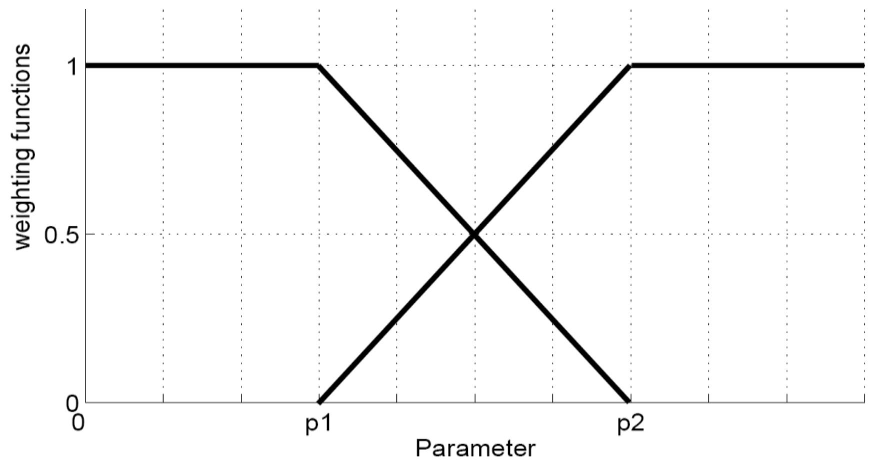
$$\boldsymbol{J}_r(p) = \sum_i \omega_i(p)\boldsymbol{J}_{r,i} \ ,$$

$$\boldsymbol{R}_r(p) = \sum_i \omega_i(p)\boldsymbol{R}_{r,i} \, ,$$

$$\boldsymbol{Q}_r^{-1}(p) = \sum_i \omega_i(p)\boldsymbol{Q}_{r,i}^{-1} \ ,$$

$$\boldsymbol{B}_r(p) = \sum_i \omega_i(p)\boldsymbol{B}_{r,i}$$

with $\omega_i \geq 0, \quad \sum_i \omega_i = 1$



Example of two weighting functions $\omega_1(p), \omega_2(p)$

Reduced parametric model in co-energy form:

$$\boldsymbol{Q}_r^{-1}(p)\dot{\boldsymbol{e}}_r = \left(\boldsymbol{J}_r(p) - \boldsymbol{R}_p(p)\right)\boldsymbol{e}_r + \boldsymbol{B}_r(p)\boldsymbol{u}$$

$$\boldsymbol{y} = \boldsymbol{B}_r^T(p)\boldsymbol{e}_r$$

CoPH-pMOR

Parametric Reduction of the standard PH form (new)

a) Interpolate descriptor form, *then* convert to state-space form (Int/PH)

PH-pMOR

The locally reduced models (*) in descriptor form are

$$\boldsymbol{V}_i^T \boldsymbol{Q}_i \boldsymbol{V}_i \dot{\boldsymbol{x}}_{r,i} = \boldsymbol{V}_i^T \boldsymbol{Q}_i (\boldsymbol{J}_i - \boldsymbol{R}_i) \boldsymbol{Q}_i \boldsymbol{V}_i \boldsymbol{x}_{r,i} + \boldsymbol{V}_i^T \boldsymbol{Q}_i \boldsymbol{B}_i \boldsymbol{u}$$

$$\boldsymbol{y}_i = \boldsymbol{B}_i^T \boldsymbol{Q}_i \boldsymbol{V}_i \boldsymbol{x}_{r,i}$$

Preparation of Interpolation by state transformation $\boldsymbol{T}_i = \boldsymbol{U}^T \boldsymbol{V}_i$ and pre-multiplier $\boldsymbol{M}_i = (\boldsymbol{V}_i^T \boldsymbol{U})^{-1}$:

$$\overbrace{(\boldsymbol{V}_i^T \boldsymbol{U})^{-1} \boldsymbol{V}_i^T \boldsymbol{Q}_i \boldsymbol{V}_i (\boldsymbol{U}^T \boldsymbol{V}_i)^{-1}}^{\boldsymbol{Q}_{r,i}} \dot{\boldsymbol{x}}_{r,i}^* = \overbrace{(\boldsymbol{V}_i^T \boldsymbol{U})^{-1} \boldsymbol{V}_i^T \boldsymbol{Q}_i (\boldsymbol{J}_i - \boldsymbol{R}_i) \boldsymbol{Q}_i \boldsymbol{V}_i (\boldsymbol{U}^T \boldsymbol{V}_i)^{-1}}^{\boldsymbol{J}_{r,i} - \boldsymbol{R}_{r,i}} \boldsymbol{x}_{r,i}^* + \overbrace{(\boldsymbol{V}_i^T \boldsymbol{U})^{-1} \boldsymbol{V}_i^T \boldsymbol{Q}_i \boldsymbol{B}_i}^{\boldsymbol{B}_{r,i}} \boldsymbol{u}$$

$$\boldsymbol{y}_i = \underbrace{\boldsymbol{B}_i^T \boldsymbol{Q}_i \boldsymbol{V}_i (\boldsymbol{U}^T \boldsymbol{V}_i)^{-1}}_{\boldsymbol{B}_{r,i}^T} \boldsymbol{x}_{r,i}^*$$

Matrix interpolation leads to:

$$\tilde{\boldsymbol{J}}_r(p) = \sum_i \omega_i(p) \boldsymbol{J}_{r,i} \ , \quad \tilde{\boldsymbol{R}}_r(p) = \sum_i \omega_i(p) \boldsymbol{R}_{r,i} \ ,$$

$$\boldsymbol{Q}_r(p) = \sum_i \omega_i(p) \boldsymbol{Q}_{r,i} \ , \quad \tilde{\boldsymbol{B}}_r(p) = \sum_i \omega_i(p) \boldsymbol{B}_{r,i}$$

and the reduced model in descriptor form is

$$\boldsymbol{Q}_r(p)\dot{\boldsymbol{x}}_r^* = \left(\tilde{\boldsymbol{J}}_r(p) - \tilde{\boldsymbol{R}}_r(p)\right)\boldsymbol{x}_r^* + \tilde{\boldsymbol{B}}_r(p)\boldsymbol{u}$$
$$\boldsymbol{y} = \tilde{\boldsymbol{B}}_r^T(p)\boldsymbol{x}_r^*$$

By pre-multiplication with $\boldsymbol{Q}_r^{-1}$ and by inserting the identity matrix $\boldsymbol{Q}_r^{-1}\boldsymbol{Q}_r$ we get

parametric model in standard form:

$$\dot{\boldsymbol{x}}_r^* = \overbrace{\boldsymbol{Q}_r^{-1}(p)\left(\tilde{\boldsymbol{J}}_r(p) - \tilde{\boldsymbol{R}}_r(p)\right)\boldsymbol{Q}_r^{-1}(p)}^{\boldsymbol{J}_r(p)-\boldsymbol{R}_r(p)}\boldsymbol{Q}_r(p)\boldsymbol{x}_r^* + \overbrace{\boldsymbol{Q}_r^{-1}(p)\tilde{\boldsymbol{B}}_r(p)}^{\boldsymbol{B}_r(p)}\boldsymbol{u}$$
$$\boldsymbol{y} = \underbrace{\tilde{\boldsymbol{B}}_r^T(p)\boldsymbol{Q}_r^{-1}(p)}_{\boldsymbol{B}_r^T(p)}\boldsymbol{Q}_r(p)\boldsymbol{x}_r^*$$

PH-pMOR

Parametric reduction of the standard PH form (new)

b) Convert to state-space form, *then* interpolate (PH/Int)

PH-pMOR

The locally reduced models (**) in standard form are

$$\begin{aligned}\dot{\boldsymbol{x}}_{r,i} &= (\boldsymbol{J}_{r,i} - \boldsymbol{R}_{r,i})\boldsymbol{Q}_{r,i}\boldsymbol{x}_{r,i} + \boldsymbol{B}_{r,i}\boldsymbol{u} \\ \boldsymbol{y}_{r,i} &= \boldsymbol{B}^T_{r,i}\boldsymbol{Q}_{r,i}\boldsymbol{x}_{r,i}\end{aligned}$$

Preparation of the interpolation by state transformation $\boldsymbol{T}_i = \boldsymbol{U}^T\boldsymbol{V}_i$ and by inserting an identity matrix:

$$\dot{\boldsymbol{x}}^*_{r,i} = \overbrace{(\boldsymbol{U}^T\boldsymbol{V}_i)(\boldsymbol{J}_{r,i} - \boldsymbol{R}_{r,i})(\boldsymbol{U}^T\boldsymbol{V}_i)^T}^{\boldsymbol{J}^*_{r,i}-\boldsymbol{R}^*_{r,i}}\overbrace{(\boldsymbol{U}^T\boldsymbol{V}_i)^{-T}\boldsymbol{Q}_i(\boldsymbol{U}^T\boldsymbol{V}_i)^{-1}}^{\boldsymbol{Q}^*_{r,i}}\boldsymbol{x}^*_{r,i} + \overbrace{(\boldsymbol{U}^T\boldsymbol{V}_i)\boldsymbol{B}_{r,i}}^{\boldsymbol{B}^*_{r,i}}\boldsymbol{u}$$

$$\boldsymbol{y}_i = \underbrace{\boldsymbol{B}^T_{r,i}(\boldsymbol{U}^T\boldsymbol{V}_i)^T}_{\boldsymbol{B}^{*T}_{r,i}}\underbrace{(\boldsymbol{U}^T\boldsymbol{V}_i)^{-T}\boldsymbol{Q}_i(\boldsymbol{U}^T\boldsymbol{V}_i)^{-1}}_{\boldsymbol{Q}^*_{r,i}}\boldsymbol{x}^*_{r,i}$$

Matrix Interpolation determines the matrices of the reduced model in standard form:

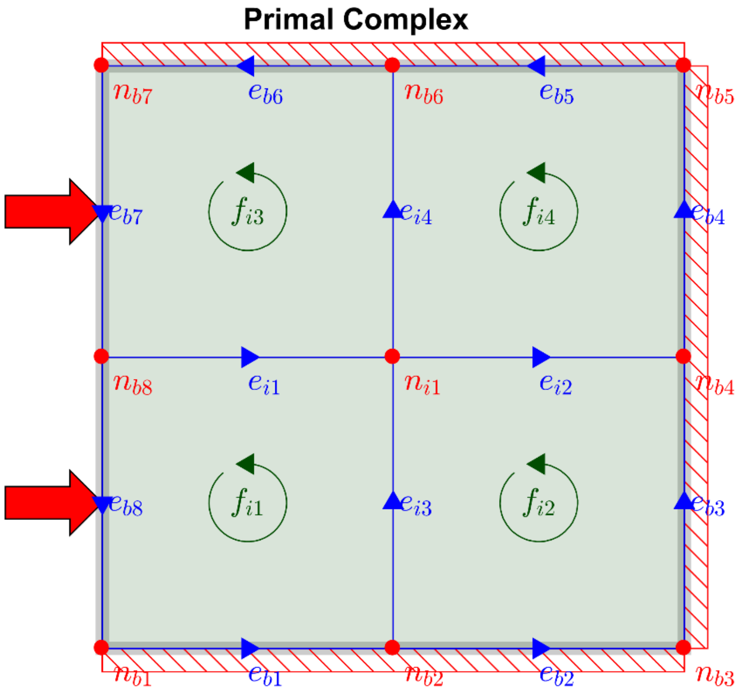
$$\boldsymbol{J}^*_r(p) = \sum_i \omega_i(p)\boldsymbol{J}^*_{r,i}\ , \quad \boldsymbol{R}^*_r(p) = \sum_i \omega_i(p)\boldsymbol{R}^*_{r,i}\ ,$$

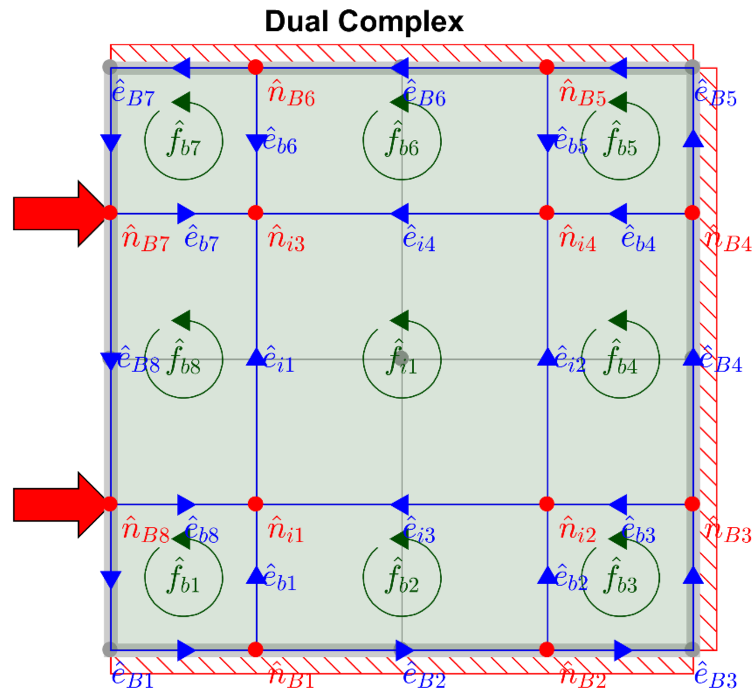
$$\boldsymbol{Q}^*_r(p) = \sum_i \omega_i(p)\boldsymbol{Q}^*_{r,i}\ , \quad \boldsymbol{B}^*_r(p) = \sum_i \omega_i(p)\boldsymbol{B}^*_{r,i}$$

Example: discretized linear wave equation [8]

$$\frac{\partial u}{\partial t} = -div\ \mathbf{v} \qquad\qquad \frac{\partial \mathbf{v}}{\partial t} = -\text{grad}\ u - c\mathbf{v}$$

- Discretizing PDE on dual complexes leads to a Port-Hamiltonian system.
- Physical values are connected to their geometrical domain.
- The discretized equations are an exact representation of the PDE (in each geometric element)





Reduction

Full Order: **682**

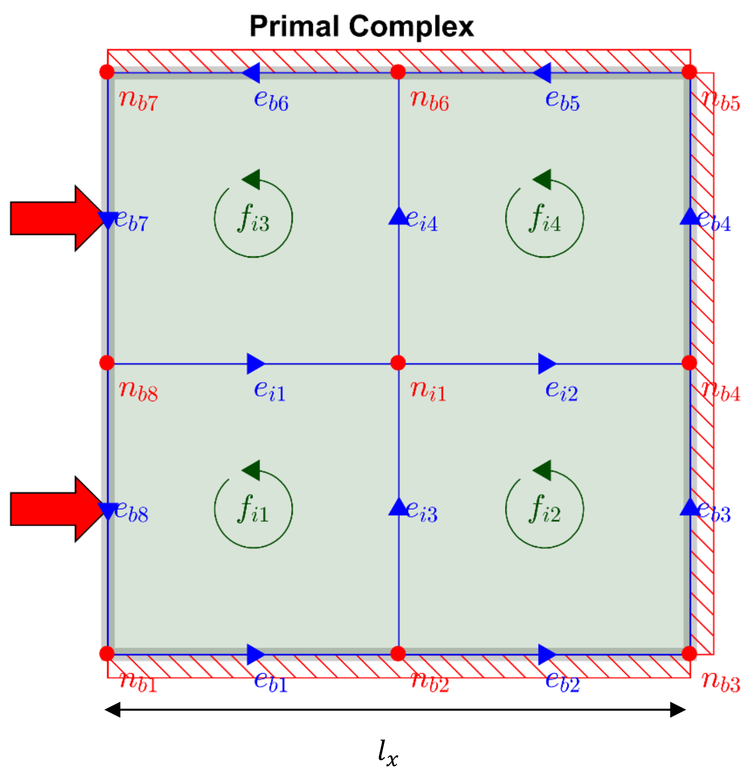
Reduced Order: **50**

Parameter: l_x

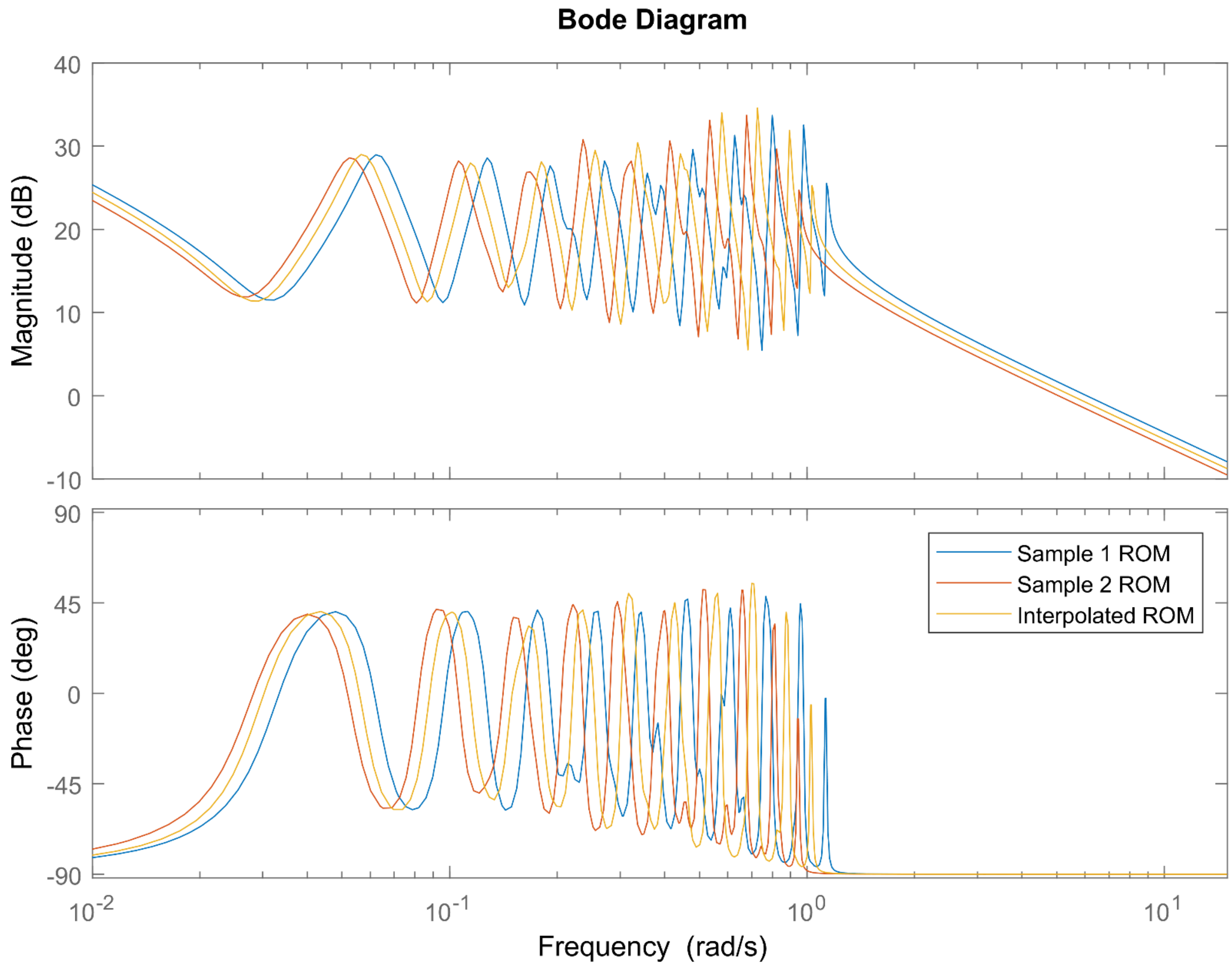
$l_{x_1} = 50$

$l_{x_2} = 60$

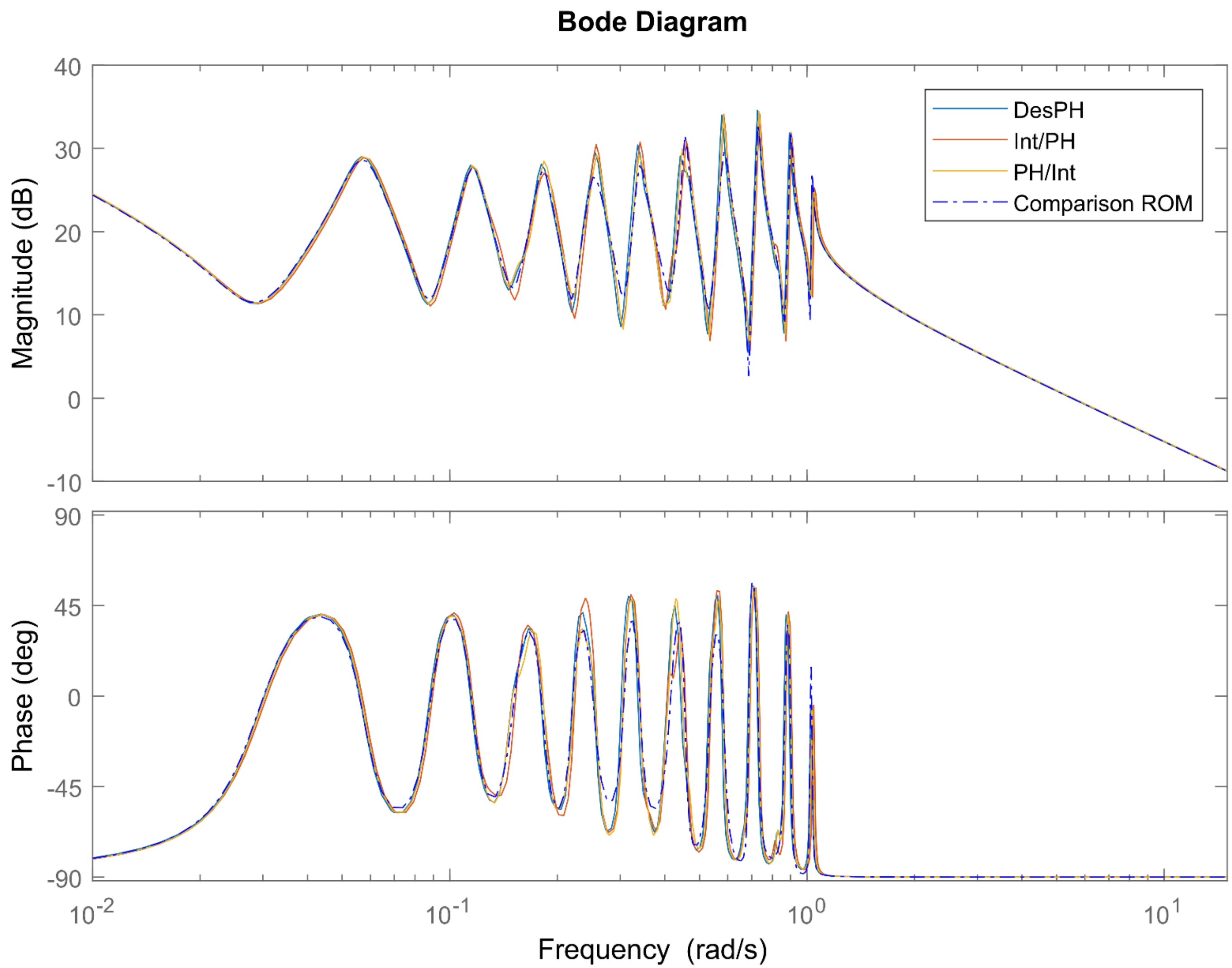
$w_1 = w_2 = 0.5$



Interpolation



Comparison


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
DesPH
Int/PH
PH/Int
Comparison ROM
40
30
20
10
0
-10
90
45
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}

Outlook: Reduction of non-linear PH-systems [7]

The PH system with nonlinear energy gradient

$$\dot{\boldsymbol{x}} = (\boldsymbol{J} - \boldsymbol{R})\nabla H(\boldsymbol{x}) + \boldsymbol{Bu}$$
$$\boldsymbol{y} = \boldsymbol{B}^T \nabla H(\boldsymbol{x})$$

Is reduced with $\boldsymbol{V}$ and $\boldsymbol{W}^T$. Additional approximation: $\frac{\partial}{\partial \boldsymbol{x}} H(\boldsymbol{V}\boldsymbol{x}_r) \approx \boldsymbol{W} \frac{\partial}{\partial \boldsymbol{x}_r} H_r(\boldsymbol{x}_r)$ with

$H_r(\boldsymbol{x}_r) = H(\boldsymbol{V}\boldsymbol{x}_r)$ and $\boldsymbol{W}^T\boldsymbol{V} = \boldsymbol{I}$ (biorthogonal).

$$\overbrace{\boldsymbol{W}^T\boldsymbol{V}}^{\boldsymbol{I}}\dot{\boldsymbol{x}} = \boldsymbol{W}^T(\boldsymbol{J} - \boldsymbol{R})\boldsymbol{W}\nabla H_r(\boldsymbol{x}_r) + \boldsymbol{W}^T\boldsymbol{Bu}$$
$$\boldsymbol{y} = \boldsymbol{B}^T\boldsymbol{W}\nabla H_r(\boldsymbol{x}_r)$$

NLPH-MOR

The reduction matrices are generated by Snapshots and orthogonalized:

$$\boldsymbol{V} = snap[\boldsymbol{x}_1 \quad \dots \quad \boldsymbol{x}_k] \qquad \boldsymbol{W} = snap[\nabla H(\boldsymbol{x}_1) \quad \dots \quad \nabla H(\boldsymbol{x}_k)]$$

Literature:

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[4] Gugercin, Polyuga, Beattie, van der Schaft: Structure-Preserving Tangential Interpolation for Model Reduction for Port-Hamiltonian Systems. Automatica 48 (2012), Seiten 1963-1974

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