Computing the Reliability of Shallow Foundations with Spatially Distributed Measurements

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Abstract. In many geotechnical projects, field data is used to determine the soil parameters. In most instances, however, the statistical analysis is performed ad hoc and the spatial distribution of this data is not (expclitly) accounted for. A more formal statistical approach allows to make better use of the data and combine it in a consistent manner with other information on soil parameters. In particular, Bayesian analysis enables combining information from different sources to learn parameters and models of engineering systems, and facilitates a spatial modeling. In this paper, we apply the Bayesian concept to learn the spatial probability distribution of the friction angle of a silty soil using outcomes of direct shear tests at different locations; we then use the derived distribution to compute the reliability of a shallow foundation. We employ two different approaches for constructing the spatial probabilistic model of the friction angle. Both approaches account for the spatial variability of the soil parameter. In the first approach, we apply a single random variable for modelling the soil property within the area of interest. The inherent spatial variability of the parameter is described by the distribution of the random variable and we use the measurements to update the parameter of this distribution. We adopt the simplifying assumption of a highly fluctuating soil and use the distribution of the mean of the friction angle in conjunction with an analytical model for the bearing capacity to update the reliability of the shallow foundation. The second approach consists of modelling the spatial variability explicitly through a random field model and using the measurements to directly update the random field. Thereby, we employ a finite element model of the soil to assess the reliability of shallow foundation.

Keywords. Bayesian updating, bearing capacity, silty soil, random fields, finite elements, reliability.

1. Introduction

Geotechnical engineers are usually faced with large uncertainties on site conditions. To assess accurately the geotechnical performance, it is necessary to combine information from different sources (expert knowledge, information from literature and in situ measurements). Bayesian updating offers a consistent means for combining these information to learn the probabilistic model of uncertain parameters (Straub and Papaioannou 2014). Thereby, a prior distribution reflecting the prior knowledge on site conditions is updated with measurements or other data to a posterior distribution. The derived distribution can be further used for reliability and risk assessment.

Soil properties are varying in space, even within one soil type (Baecher and Christian 2008). It is, therefore, imperative that available data and information are combined with models of spatially variable parameters in a consistent manner. In this paper, we perform Bayesian updating of the reliability of a shallow foundation in a silty soil using measurements of the friction angle from direct shear tests of soil probes taken at different locations. We employ two different models to address the spatial variability of the friction angle: a simplified model that involves a single random variable and a detailed random field model.

2. Bayesian analysis

Let $\mathbf{X}$ denote the vector of the random variables, representing the uncertain soil parameters. Any failure event of interest F can be expressed in terms of a limit state function $g(\mathbf{X})$, which typically depends on the outcome of a geotechnical model, such that $F = \{g(\mathbf{X}) \leq 0\}$. An appropriate prior probabilistic model of the random variables $\mathbf{X}$ is constructed through assessing information available prior to on site investigations. Aside from reflecting all prior knowledge (or lack of knowledge), the prior probability density function (PDF) of $\mathbf{X}$, denoted by $f'_{\mathbf{X}}$, should incorporate the inherent spatial variability of the soil parameters (e.g. Rackwitz 2000). Having established the prior distribution, the prior probability of failure before including site-specific data is obtained as:

$$\Pr(F) = \int_{g(\mathbf{x})\leq 0} f'_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{1}$$

The corresponding reliability index is $\beta' = -\Phi^{-1}[\Pr(F)]$, where Φ^{-1} is the inverse of the standard normal distribution function. Eq. (1) can be solved by application of any of the well-established structural reliability methods (e.g. Phoon 2008).

During the construction process, additional data become available, providing information on $\mathbf{X}$ either directly or indirectly. For example, a direct shear test of a soil probe provides direct information on the value of the friction angle at a specific location, while a measurement of the settlement of a foundation provides indirect information on the soil properties through the geotechnical model. Measurement events Z are described by the likelihood function $L(\mathbf{x})$. The likelihood of a measurement is defined as being proportional to the conditional probability of the measurement given a parameters state:

$$L(\mathbf{x}) \propto \Pr(Z|\mathbf{X} = \mathbf{x}) \tag{2}$$

If multiple measurements $Z_1, \dots, Z_m$ are available, likelihood functions $L_1, \dots, L_m$ are established for each measurement individually. If all measurements are independent for a given parameter state, then the joint likelihood of all measurements is obtained as:

$$L(\mathbf{x}) = \prod_{i=1}^{m} L_i(\mathbf{x}) \tag{3}$$

The impact of the measurements on the random parameters $\mathbf{X}$ is quantified through computing the posterior PDF $f''_{\mathbf{X}}$, i.e. the conditional PDF of $\mathbf{X}$ given the measurement outcome. $f''_{\mathbf{X}}$ is obtained through Bayes' rule:

$$f''_{\mathbf{X}}(\mathbf{x}) = aL(\mathbf{x})f'_{\mathbf{X}}(\mathbf{x}) \tag{4}$$

where a is a proportionality constant, which ensures that $f''_{\mathbf{X}}(\mathbf{x})$ integrates to one. Application of Eq. (4) is not always straightforward. In some situations, it is possible to obtain an analytical expression for $f''_{\mathbf{X}}$ in terms of a known distribution model. This occurs when the prior and likelihood are described by so-called conjugate distributions (e.g. Ang and Tang 2007). However, in most cases the posterior PDF is evaluated numerically, either by gradient-based approximations or by sampling approaches. Straub and Papaioannou (2014) provide a review of different methods for solving the Bayesian updating problem.

After estimating the posterior distribution of the parameters, the conditional probability of failure $\Pr(F|Z)$ can be obtained by replacing the prior PDF with the posterior PDF in Eq. (1). If the posterior PDF is available in analytical form, the evaluation of $\Pr(F|Z)$ can be carried out with the classical structural reliability methods. Alternatively, the reliability can be updated directly by application of the approach introduced in Straub (2011) and applied to geotechnical engineering in Papaioannou and Straub (2012). This approach is based on describing the measurement through a limit state function and solving two structural reliability problems.

3. Modeling spatially variable parameters

As mentioned earlier, the inherent spatial variability of the soil parameters needs to be addressed in a geotechnical reliability assessment. Spatial variability can be modeled in two fundamentally different ways.

In the first approach, the soil property within an area is modeled with a single random variable X. That is, the property at a specific location is not explicitly modeled and the inherent variability of the soil property within the area is represented by the PDF of X, f_X. This corresponds to the classical statistical approach, which is based on modeling the variability within a population with a random variable. This variability cannot be reduced with measurements, however the parameters $\boldsymbol{\theta}$ of the distribution f_X can be learned. This can be achieved by defining a prior distribution $f'_{\boldsymbol{\theta}}$ on $\boldsymbol{\theta}$ and then updating the distribution with samples of X. Noting that f_X is defined conditional on the parameters $\boldsymbol{\theta}$, which are uncertain, the distribution of X at each location, the so-called predictive distribution, can be obtained by marginalizing the joint PDF of X and $\boldsymbol{\theta}$.

In the context of reliability analysis, this approach facilitates the application of analytical geotechnical models that do not involve explicit spatial modeling. Parameters of such models usually refer to averages of a soil property over a failure surface. Spatial averaging is usually accounted for by reducing the variance of the inherent variability of X, through application of the variance reduction function (Rackwitz 2000). In the case of highly fluctuating soils, the variance of the spatial average of the soil property vanishes. In such cases the reliability can be calculated by considering only the uncertainty in the mean of the soil property.

The second modeling approach of spatially variable properties is to model the property at each location explicitly. In this approach, the property is modeled by a random field $X(z)$, which represents a random variable at each location z (Rackwitz 2000; Baecher and Christian 2008). The random field is usually modeled by the marginal distribution at each location and the auto-correlation coefficient function. In order to numerically represent the continuous random field $X(z)$, it is necessary to discretize it with a finite set of random variables, e.g. by application of the Karhunen-Loève expansion. Once the prior random field is established, measurements of X at specific locations can be used to update the random field or the random variables in its discrete representation.

4. Bayesian updating of foundation reliability

We illustrate the concepts of Bayesian analysis to the reliability assessment of a shallow foundation in silty soil. We consider a centrically loaded rigid strip footing with dimensions $B = 3\text{m}$ and $D = 1\text{m}$, as shown in Figure 1. This example is modified from Oberguggenberger and Fellin (2002).

The limit state function describing failure of the foundation is:

$$g(\mathbf{X}) = q_u - \frac{V}{B} \tag{5}$$

where q_u is the ultimate bearing capacity and V is the applied load. For simplicity, we assume a deterministic load $V = 1000\,\text{kN/m}$. The cohesion of the silty soil is close to zero and can be neglected. The unit weight of the soil is $\gamma = 19.8\,\text{kN/m}^3$. The uncertainty in the friction angle φ is modeled with the two different approaches discussed in Section 3. We update the reliability of the foundation using measurement outcomes from direct shear tests.

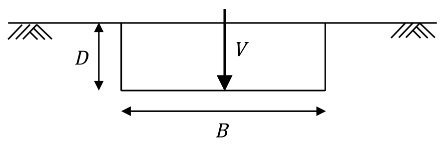

Figure 1. Foundation in silty soil.

4.1. Random variable approach

In this approach, we model the inherent variability of the friction angle with a single random variable with uncertain distribution parameters. We employ the lognormal distribution, which is a common choice for the probabilistic modeling of geotechnical properties (e.g. Griffiths and Fenton 2001). Assuming no mean trend, the conditional PDF of φ at each spatial location given the distribution parameters $\boldsymbol{\theta}$ reads:

$$f_\varphi(\varphi|\boldsymbol{\theta}) = \frac{1}{\varphi\zeta_\varphi\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln\varphi - \lambda_\varphi}{\zeta_\varphi}\right)\right] \tag{6}$$

where $\boldsymbol{\theta} = [\lambda_\varphi; \zeta_\varphi]$ are the mean and standard deviation of the underlying normal distribution of $\ln\varphi$, which can be expressed in terms of the mean μ_φ and coefficient of variation (CV) δ_φ of φ as follows:

$$\zeta_\varphi = \sqrt{\ln\left(1 + \delta_\varphi^2\right)} \tag{7}$$

$$\lambda_\varphi = \ln\mu_\varphi - \frac{1}{2}\zeta_\varphi^2 \tag{8}$$

The CV is modeled as constant in space and taken as $\delta_\varphi = 0.15$, which agrees with the typical CV of inherent variability of the friction angle of silty soils (Phoon and Kulhawy 1999). We assume that previous measurements on similar soils in the vicinity have indicated that the mean μ_φ is commonly between 25° and 31°.

These values are taken as the 10 and 90% quantiles of μ_φ. We model the prior distribution of μ_φ with a lognormal distribution and we evaluate its parameters λ'_{μ_φ} and ζ'_{μ_φ} by matching the 10 and 90% quantiles to the aforementioned values. The prior mean of μ_φ is $27.94°$ and its prior CV is 0.084. From Eq. (8), the prior distribution of λ_φ will be a normal distribution with parameters $\mu'_{\lambda_\varphi} = \lambda'_{\mu_\varphi} - \frac{1}{2}\zeta_\varphi^2$ and $\sigma'_{\lambda_\varphi} = \zeta'_{\mu_\varphi}$.

Direct shear tests of soil probes taken at certain locations in the area of the foundation resulted in the following values of the friction angle: $\varphi_1 = 25.6°$, $\varphi_2 = 25.5°$, $\varphi_3 = 24°$. These values are taken exemplarily from Oberguggenberger and Fellin (2002). In the present framework, these measurements correspond to samples of φ and can be used to update the distribution of λ_φ. The likelihood $L_i(\lambda_\varphi)$ of each sample φ_i is proportional to the probability of the sample given λ_φ and is obtained by replacing φ with φ_i in Eq. (6). Assuming independence between samples, the joint likelihood describing all three samples is given according to Eq. (3) as

$$L(\lambda_\varphi) = \prod_{i=1}^{3} L_i(\lambda_\varphi) \tag{9}$$

The posterior PDF $f''_{\lambda_\varphi}(\lambda_\varphi)$ of λ_φ is obtained following Eq. (4). For the particular choice of the prior distribution of μ_φ, the resulting normal prior distribution of λ_φ is the conjugate of the lognormal distribution of the underlying random variable φ (e.g. Ang and Tang 2007). Hence, the posterior PDF of λ_φ has the same analytical form as its prior; it is the normal PDF with parameters μ''_{λ_φ} and $\sigma''_{\lambda_\varphi}$ that can be evaluated using closed form expressions (e.g. Ang and Tang 2007). The posterior marginal PDF of φ at each spatial location can be evaluated by integrating out the distribution parameter λ_φ from the joint PDF of φ and λ_φ, i.e.

$$f_\varphi(\varphi) = \int_{-\infty}^{\infty} f_\varphi(\varphi|\lambda_\varphi)\, f''_{\lambda_\varphi}(\lambda_\varphi)\, d\lambda_\varphi \tag{10}$$

Eq. (10) is the predictive distribution of φ; here it is a lognormal distribution with parameters μ''_{λ_φ} and $\sqrt{\sigma''^{\,2}_{\lambda_\varphi} + \zeta_\varphi^2}$.

Because the variability of the friction angle is modeled with a single random variable, it is possible to evaluate the bearing capacity of the foundation in terms of the analytical bearing capacity factors, as follows:

$$q_u = qN_q + \frac{1}{2}\gamma B N_\gamma \tag{11}$$

where $N_q = \exp(\pi \tan\varphi)\tan^2(45° + \varphi/2)$, $N_\gamma = 2(N_q + 1)\tan\varphi$ and $q = \gamma D$ (e.g. Das 2009). Application of Eq. (11) requires the reduction of the inherent variability of φ to account for spatial averaging. However, if we assume a highly fluctuating soil, we can represent the variability of φ with the one of its mean μ_φ. This is an (non-conservative) approximation, which becomes exact for a soil with a scale of fluctuation of zero (Griffiths and Fenton 2001).

Replacing φ with μ_φ in Eq. (11), the a priori probability of failure is evaluated as $\Pr(F) = F'_{\mu_\varphi}(\varphi_F)$, where F'_{μ_φ} is the (lognormal) prior cumulative distribution function (CDF) of μ_φ, and $\varphi_F = 21.23°$ is the value of the friction angle for which $g = 0$. From Eq. (8), the posterior distribution of μ_φ conditional on the measurements is again lognormal with parameters $\lambda''_{\mu_\varphi} = \mu''_{\lambda_\varphi} + \frac{1}{2}\zeta_\varphi^2$ and $\zeta''_{\mu_\varphi} = \sigma''_{\lambda_\varphi}$, which corresponds to a posterior mean of μ_φ of $26.62°$ and posterior CV of 0.06. The posterior probability of failure is obtained as $\Pr(F|Z) = F''_{\mu_\varphi}(\varphi_F)$.

The prior, likelihood and posterior distribution of μ_φ are shown in Figure 2. The first row of Table 1 shows the prior and posterior failure probabilities and corresponding reliability indices computed with this approach. Although the measurements resulted in values lower than the prior mean of μ_φ, the posterior reliability is significantly higher than the prior. This is because of the reduced uncertainty, which can be observed in Figure 2.

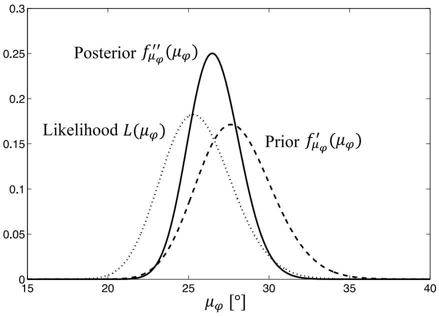

Figure 2. Prior PDF of μ_φ, posterior PDF of μ_φ and (normalized) joint likelihood describing the measurements.

4.2. *Random field approach*

In the second modeling approach, the friction angle is modeled explicitly at each location through a random field. For simplicity, we neglect the variability in the horizontal direction and model the friction angle with a one-dimensional homogeneous random field $\varphi(z)$, where z denotes the coordinate in the vertical direction. We recall that φ was assumed to follow the lognormal distribution with uncertain parameter λ_φ and fixed parameter ζ_φ. The prior distribution of λ_φ is normal with parameters μ'_{λ_φ}, $\sigma'_{\lambda_\varphi}$. The spatial fluctuation of φ is modeled by the auto-correlation coefficient function. We adopt the following exponential model for the prior auto-correlation coefficient function conditional on λ_φ:

$$\rho'_\varphi(\Delta z|\lambda_\varphi) = \exp\left(-\frac{\Delta z}{l}\right) \tag{12}$$

where $\Delta z = |z_1 - z_2|$ is the distance between two points and l is the correlation length, chosen as $l = 2\text{m}$. The marginal distribution of $\varphi(z)$ must equal the prior predictive distribution, which, analogous to Eq. (10), is a lognormal distribution with parameters μ'_{λ_φ} and $\sqrt{\sigma'^{\,2}_{\lambda_\varphi} + \zeta^2_\varphi}$. The random field $\varphi(z)$ is defined as:

$$\varphi(z) = \exp\big(\lambda_\varphi + \zeta_\varphi U_\varphi(z)\big) \tag{13}$$

where $U_\varphi(z)$ is an underlying standard normal random field, whose auto-correlation function $\rho'_U(\Delta z)$ is assumed equal to the one of $\varphi(z)$, i.e. $\rho'_U(\Delta z) \approx \rho'_\varphi(\Delta z)$. This is a valid assumption for small δ_φ. Hence, $\ln\varphi$ will be normal with prior mean $\mu'_{\ln\varphi} = \mu'_{\lambda_\varphi}$ and prior auto-covariance function:

$$\Gamma'_{\ln\varphi}(\Delta z) = \sigma'^{\,2}_{\lambda_\varphi} + \zeta^2_\varphi \rho'_U(\Delta z) \tag{14}$$

We note that due to the uncertainty in λ_φ, the covariance of $\ln\varphi$ becomes $\sigma'^{\,2}_{\lambda_\varphi}$ as $\Delta z \to \infty$.

We assume that the measurements considered in Section 4.1 ($\varphi_1 = 25.6°$, $\varphi_2 = 25.5°$, $\varphi_3 = 24°$) are taken at locations $z_1 = 1\text{m}$, $z_2 = 3\text{m}$, $z_3 = 5\text{m}$ below the ground level. Neglecting the measurement uncertainty (as was also done in Section 4.1), the likelihood of the observations is given by:

$$L(\varphi(z)) = \prod_{i=1}^{3} \delta(\varphi(z_i) - \varphi_i) \tag{15}$$

where δ is the Dirac delta function. Following Eq. (13), the posterior distribution of $\ln\varphi$ given the measurements is normal and its posterior mean function $\mu''_{\ln\varphi}$ and auto-covariance function $\Gamma''_{\ln\varphi}$ are known analytically (e.g. Straub 2012). Hence the posterior marginal distribution of φ is again lognormal. Figure 3 shows the resulting posterior mean and CV of φ. The posterior random field is no longer homogeneous. Because of the assumption of no measurement error, the posterior CV is zero at the locations of the measurements and increases away from these locations.

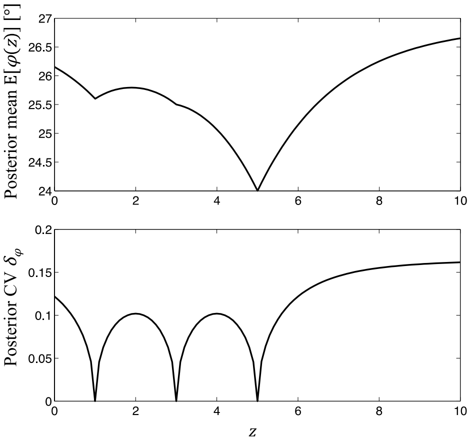

Figure 3. Posterior mean and posterior coefficient of variation (CV) of the friction angle.

In order to properly account for the spatial variability of the friction angle, the bearing capacity is evaluated with non-linear elasto-plastic finite element analysis, following Griffiths (1982). Since we account for the variability only in the vertical direction, we take advantage of the symmetry and model only one half of the soil profile. The finite element mesh, consisting of eight-node quadrilateral elements, is shown in Figure 4.

The prior and posterior reliability are evaluated by application of the line sampling method (Koutsourelakis et al. 2004). The results are shown in the second row of Table 1. The computed probabilities are considerably larger than the ones obtained with the random variable approach, reflecting the effect of the different assumptions on spatial correlation, which leads to a stronger spatial averaging in the case of the RV model.

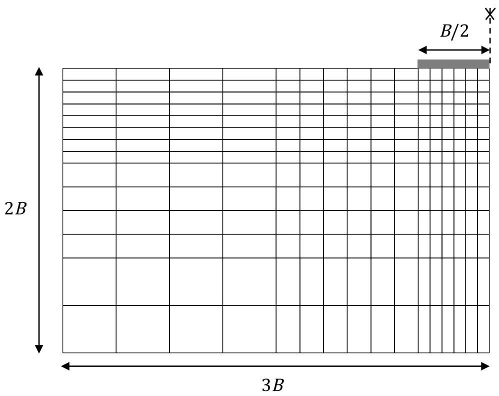

Figure 4. Finite element mesh used for evaluation of the bearing capacity.

5. Conclusion

We presented an application of Bayesian analysis for updating the reliability of a shallow foundation with measurements, considering the spatial variability of the soil. We demonstrated two different approaches for modeling the spatial variability: a random variable and a random field approach. We showed how data could be used to learn the distribution of soil properties modeled with any of the two approaches and how the derived posterior distributions could be employed to obtain the reliability of the foundation conditional on the data.

Table 1. Prior and posterior reliability for the two considered modeling approaches.

Modeling approach	**Prior**		**Posterior**	
	$\Pr(F)$	β'	$\Pr(F\|Z)$	β''
RV	6.21×10^{-4}	3.23	9.40×10^{-5}	3.73
RF	5.8×10^{-2}	1.57	1.32×10^{-3}	3.01

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