Optimal LQG Control under Delay-dependent Costly Information

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Abstract— In the design of closed-loop networked control systems (NCSs), induced transmission delay between sensors and the control station is an often-present issue which compromises control performance and may even cause instability. A very relevant scenario in which network-induced delay needs to be investigated is costly usage of communication resources. More precisely, advanced communication technologies, e.g. 5G, are capable of offering latency-varying information exchange for different prices. Therefore, induced delay becomes a decision variable. It is then the matter of decision maker's willingness to either pay the required cost to have low-latency access to the communication resource, or delay the access at a reduced price. In this article, we consider optimal price-based bi-variable decision making problem for single-loop NCS with a stochastic linear time-invariant system. Assuming that communication incurs cost such that transmission with shorter delay is more costly, a decision maker determines the switching strategy between communication links of different delays such that an optimal balance between the control performance and the communication cost is maintained. In this article, we show that, under mild assumptions on the available information for decision makers, the separation property holds between the optimal link selecting and control policies. As the cost function is decomposable, the optimal policies are efficiently computed.

I. INTRODUCTION

In the design of closed-loop NCSs where information is exchanged between sensors, controller and actuator over a limited-resource communication network, induced transmission delay plays a key role in characterizing control performance and stability properties [1], [2]. Day-by-day increase of data volume that needs to be exchanged urges access to fast and low-error communication infrastructure to support the stringent real-time requirements of such systems. This, however, imposes higher communication and computation costs, resulting in reconsideration of employing time-based sampling techniques with equidistant fixed temporal durations. Various approaches are developed to coordinate data exchange in NCSs with the aim of reducing the total sampling and communication rate. Effective techniques such as event-based sampling, scheduling, and network pricing are introduced leading to the reduction of communication and computational costs by restricting unnecessary data sampling. Having intermittent sampling, delay is induced in various parts of the networked system which may degrade control performance. Hence, such decision makers need to be carefully designed in order to preserve stability as well as providing required quality-of-control (QoC) guarantees.

Event-based control was introduced as a beneficial design framework to coordinate sampling of signals based on some urgency metrics, e.g. an action is executed only when some pre-defined events are triggered [3]. This idea received substantial attention and is further developed as a technique capable of significantly reducing sampling rate while preserving the required QoC [4]–[8]. The mentioned works, among many more, consider sporadic data sampling governed by real-time conditions of the control systems or the communication medium. Synthesis of optimal event-based strategies in NCSs are also addressed [9]–[11].

Data scheduling is employed by communication theorists for decades as an effective resource management technique [12], [13]. By emerging NCSs as integration of multiple control systems supported by communication networks, cross-layer scheduling attracted more attentions. The reason is scheduling induces delay and affects NCS stability and QoC, hence, scheduling approaches that take into account real-time conditions of control systems become popular [8], [14], [15].

Designing price mechanisms for multi-user networks, to guarantee quality-of-service (QoS) is popular in communication [16], [17]. In these works the goal is often set to maximize the QoS, which is a network-dependent utility expressed often in form of effective bandwidth requirements. In NCSs, however, QoC is of interest which additionally takes into account users' dynamics. Optimal communication pricing aiming at maximizing the QoC in NCSs has received less attention with a few exceptions, e.g. [18], [19].

In those mentioned works, delay is considered as an inevitable network-induced phenomena resulting from the employed sporadic sampling mechanisms. Novel communication technologies, e.g. 5G, offer not only "bandwidth" as the resource to pay for, but also real-time "latency". Users can decide to pay a higher price for lower latency or to delay data exchange at a reduced price. In such scenarios, the resulting induced delay plays as an explicit decision variable, i.e., users can optimize their utilities versus the communication price. In this article, we take the first steps in this direction by addressing the problem of joint optimal

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control and delay-dependent switching policies for a single-loop NCS with costly communication. The switching law determines the length of delay associated with the data sent over the network. We assume that every transmission incurs a cost determined by the associated delay, such that shorter delay incurs higher cost. Aggregating the LQG cost and delay-dependent communication cost over a finite horizon, we derive the optimal control and switching laws assuming that communication prices are known *apriori*. It is then shown that the optimal control and switching laws are separable in expectation, and thus can be computed offline. It guarantees the computational feasibility of our proposed approach.

II. PROBLEM STATEMENT

Consider an LTI control system, consisting of a physical plant $\mathcal{P}$ and a controller $\mathcal{C}$. The plant $\mathcal{P}$ is descried by

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the control signal executed at time k, and $w_k \in \mathbb{R}^n$ is the exogenous disturbance. The constant matrices $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$ describe drift matrix, and input matrix, with the pair (A, B) assumed to be controllable. The disturbance w_k is i.i.d with $w_k \sim \mathcal{N}(0, W)$ and the initial state x_0 is independent of $\{w_k\}$, and $x_0 \sim \mathcal{N}(0, \Sigma_0)$, where $W \succeq 0$ and $\Sigma_0 \succeq 0$ denoting the variances of the respective Gaussian distributions. For the purpose of simplicity, we assume that the sensor measurements are perfect copies of state values.

In this article we address a delay-dependent LQG problem. As shown in Fig. 1, there are D links with dealays $1, \ldots, D$ respectively. Selection of a transmission link decides the arrival time instance of data at the controller, i.e., controller update may be delayed. Each link has a known cost of operation that increases as delay decreases. Note that, in the NCS scenario illustrated in Fig. 1, the control unit which determines the switching policy of transmission links is a separate decision making unit with specific information structure, and must be distinguished from the plant controller.

Recall that classical optimal LQG control is a certainty equivalence control that uses state estimation based on regularly-sampled measurements. In this work, however, the arrival of measurements and consequently the estimation quality depends on the selected delay link. Thus, unlike standard optimal LQG where there is one controller that generates the control signal, here another control unit with an appropriate information structure exists and determines the optimal strategy to select the delay links. To take this into account, we first define the binary decision variable θ_k^i as follows:

$$\theta_k^i = \begin{cases} 1, & \text{link with } i \text{ step delay is selected at time } k \\ 0, & \text{link with } i \text{ step delay is not selected at time } k \end{cases}$$

Based on the above definition, if $\theta_k^i = 1$, the controller has access to system state x_k at time-step $k + i$. We assume the possibility of selecting more than one links at each time, i.e.

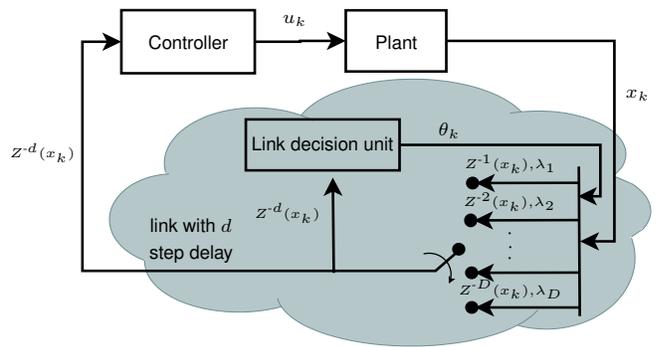$$\sum_{i=1}^{D} \theta_k^i \geq 1, \quad \forall\, k \in \{1, 2, \ldots\} \tag{2}$$

Fig. 1: Schematic of a closed-loop system with communication delay, where $Z^{-d}(x_k)$ means x_k will be received by control unit d time-steps later, at the expense of λ_d.

where, the finite variable $D \in \mathbb{N}$ denotes the maximum allowable delay. Each link with associated delay i is assigned a price, denoted by $\lambda_i \in \mathbb{R}^+$, to be paid if it is selected for transmission. Hence, at each time k, the switching decision θ_k, can be represented by a binary-valued vector as follows

$$\theta_k \triangleq [\theta_k^1, \ldots, \theta_k^D]^\mathsf{T}. \tag{3}$$

The prices for each communication link $i \in \{1, \ldots, D\}$ are denoted by λ_i, and are fixed *apriori* with the following order:

$$\lambda_1 > \lambda_2 > \ldots > \lambda_D > 0.$$

Remark 1: In this framework, a link with very large delay $D_{ol} \gg 1$ and cost $\lambda_{ol} = 0$ can be added such that a transmission becomes very unlikely. Theoretically, $D_{ol} \to \infty$, the system opts to be open-loop. In our scenario, however, system is forced to select at least one link, according to (2).

According to (3), the received state information at the controller at time-step k, denoted by $\mathcal{Y}_k$, is expressed as

$$\mathcal{Y}_k = \{\theta_{k-1}^1 x_{k-1}, \theta_{k-2}^2 x_{k-2}, \ldots, \theta_{k-D}^D x_{k-D}\} \tag{4}$$

where, $\theta_{-1}^i = \ldots = \theta_{-D}^i = 1$, for all i to represent equations compactly. The system possesses two decision makers; one decides the delay link via θ_k, and one computes control signal u_k. To define the information set and the associated σ-algebra available to each decision maker, we first introduce two sets $Y_k \triangleq \{\mathcal{Y}_0, \ldots, \mathcal{Y}_k\}$, and $U_k \triangleq \{u_0, \ldots, u_k\}$, containing the received state information, and control signals, up to and including time k, respectively. We now define the information sets $\mathcal{I}_k$ and $\bar{\mathcal{I}}_k$ at time k, respectively accessible for the switching and the plant controllers, as follows:

$$\mathcal{I}_k \triangleq \{Y_{k-1}, U_{k-1}, \cup_{t=1}^{k-1}\{\theta_t\}\}, \quad \bar{\mathcal{I}}_k \triangleq \{\mathcal{I}_k, \mathcal{Y}_k, \theta_k\}.$$

At every time k, the control and delay switching strategies are measurable functions of the σ-algebras generated by $\bar{\mathcal{I}}_k$, and $\mathcal{I}_k$, respectively, i.e., $u_k = g_k(\bar{\mathcal{I}}_k)$, and $\theta_k = s_k(\mathcal{I}_k)$. The order of decision making in one cycle of sampling is as follows: $\cdots \to \mathcal{I}_k \to \theta_k \to \bar{\mathcal{I}}_k \to u_k \to \mathcal{I}_{k+1} \to \cdots$. In general, the computation of the optimal control u_k^* requires the knowledge of the optimal θ_k^*. However, we show later that, under the introduced information structures, θ_k^* can be computed offline, and hence computation u_k^* will not require on-line update about θ_k^*. A possible implementation of this protocol is to send the preference of selecting the delay link to a

network manager (it is the communication service provider that offers different QoS (delay)) that, upon receiving the sensor data x_k, selects the preferred transmission link.

The cost function, that is jointly minimized by the two decision variables $g_k(\bar{\mathcal{I}}_k)$, and $s_k(\mathcal{I}_k)$, consists of an LQG part and communication cost. Within the finite horizon, the average cost function is stated by the following expectation

$$J(u,\theta) = \mathsf{E}\left[\sum_{t=0}^{T-1}\left[x_t^\top Q_1 x_t + u_t^\top R u_t + \theta_t^\top \Lambda\right] + x_T^\top Q_2 x_T\right],$$

where, $\Lambda \triangleq [\lambda_1, \ldots, \lambda_D]^\top$, $Q_1 \succeq 0$, $Q_2 \succeq 0$, and $R \succ 0$.

III. OPTIMAL CONTROL & SWITCHING POLICIES

The optimal control and switching strategies are the minimizing arguments of the latter average cost function, i.e.

$$(u^*, \theta^*) = \arg\min_{u,\theta} J(u,\theta), \tag{5}$$

where the average cost optimal value equals $J^* = J(u^*, \theta^*)$. In the sequel, we show that the problem (5) is separable in its arguments u, and θ and can be disjointly optimized offline. In fact, we show that the optimal control policy is linear, and independent from the sequence of link switching decisions θ, while the state estimation is a nonlinear function of θ.

A. Optimal control strategy

Knowing that $\mathcal{I}_k \subseteq \bar{\mathcal{I}}_k$, we can re-write $J(u,\theta)$ as:

$$J(u,\theta) = \tag{6}$$
$$\mathsf{E}\left[\mathsf{E}\left[\mathsf{E}\left[\sum_{t=0}^{T-1}[x_t^\top Q_1 x_t + u_t^\top R u_t + \theta_t^\top \Lambda] + x_T^\top Q_2 x_T\big|\bar{\mathcal{I}}_0\right]\big|\mathcal{I}_0\right]\right]$$

Thus, using the fact that u_k and θ_k are $\bar{\mathcal{I}}_k$ and $\mathcal{I}_k$ measurable:

$$\min_{\substack{u_{[0,T-1]}\\\theta_{[0,T-1]}}} J(u,\theta) = \mathsf{E}\left[\min_{\theta_{[0,T-1]}} \mathsf{E}\left[\min_{u_{[0,T-1]}} \mathsf{E}\left[C_0(u,\theta)|\bar{\mathcal{I}}_0\right]|\mathcal{I}_0\right]\right]$$

where, $C_k(u,\theta) = \sum_{t=k}^{T-1}[x_t^\top Q_1 x_t + u_t^\top R u_t + \sum_{i=1}^D \theta_t^i \lambda_i] + x_T^\top Q_2 x_T$. Moreover, we define the cost-to-go J_k^* as follows:

$$J_k^* = \min_{\theta_{[k,T-1]}} \mathsf{E}\left[\min_{u_{[k,T-1]}} \mathsf{E}\left[C_k(u,\theta)|\bar{\mathcal{I}}_0\right]|\mathcal{I}_0\right],$$

which reduces the optimization problem to the compact form

$$\min_{u_{[0,T-1]},\,\theta_{[0,T-1]}} J(u,\theta) = \mathsf{E}\left[J_0^*\right].$$

It then follows that $C_k(u,\theta) = V_k(u) + \sum_{t=k}^{T-1}\sum_{i=1}^D \theta_t^i \lambda_i$, where $V_k(u) = \sum_{t=k}^{T-1}[x_t^\top Q_1 x_t + u_t^\top R u_t] + x_T^\top Q_2 x_T$. It is easy to verify that $V_k^* = \min_{u_{[k,T-1]}} \mathsf{E}[V_k(u)|\bar{\mathcal{I}}_k]$ is a standard LQG cost-to-go. Having this, we state Theorem 1.

Theorem 1: Given the information set $\bar{\mathcal{I}}_k$, the optimal control policy $u_k^* = g_k^*(\bar{\mathcal{I}}_k)$, $k \in \{0, \ldots, T-1\}$, which minimizes $\mathsf{E}[V_k(u)|\bar{\mathcal{I}}_k]$, is a linear feedback law of the form

$$u_k^* = -(R + B^\top P_{k+1} B)^{-1} B^\top P_{k+1} A\, \mathsf{E}[x_k|\bar{\mathcal{I}}_k], \tag{7}$$

where, P_k is the solution of the following Riccati equation:

$$P_k = Q_1 + A^\top \big(P_{k+1} - P_{k+1} B(R + B^\top P_{k+1} B)^{-1} B^\top P_{k+1}\big) A,$$
$$P_T = Q_2.$$

Moreover, the optimal cost is $V_k^* = \mathsf{E}[x_k|\bar{\mathcal{I}}_k]^\top P_k\, \mathsf{E}[x_k|\bar{\mathcal{I}}_k] + \pi_k$, where for all $T > t \geq k$, π_k is expressed as

$$\pi_k = \mathsf{E}\left[e_k^\top P_k e_k + \sum_{t=k}^{T-1} e_t^\top \tilde{P}_t e_t|\bar{\mathcal{I}}_k\right] + \sum_{t=k+1}^T tr(P_t W)$$

with, $e_k = x_k - \mathsf{E}[x_k|\bar{\mathcal{I}}_k]$, $\tilde{P}_t = Q_1 + A^\top P_{t+1} A - P_t$.

Proof: See Appendix VI-A. ∎

From Theorem 1, $g_k^*(\bar{\mathcal{I}}_k) = L_k\, \mathsf{E}[x_k|\bar{\mathcal{I}}_k]$, with L_k independent of θ. This allows us to design the control law offline, while the estimator is θ-dependent (*Proposition* 1). This is intuitive as λ_i's are assumed to be state and time-independent.

B. Optimal Switching Strategy

Here we first show that the estimation at the controller is θ-dependent. It results in e_k being also θ-dependent, $\forall k > 0$.

Proposition 1: The estimator dynamics is θ-dependent s.t.

$$\hat{x}_k = \mathsf{E}[x_k|\bar{\mathcal{I}}_k] = \sum_{i=1}^{\min\{D,k+1\}} b_{i,k}\, \mathsf{E}[x_k|x_{k-i}, U_{k-1}], \tag{8}$$

where, $\forall k \geq 0, i \in \{1, \cdots, D\}$, $b_{i,k} \in \{0,1\}$. Moreover, $\sum_{i=1}^{\min\{D,k+1\}} b_{i,k} = 1$, and if $D > k$, then $\sum_{i=k+2}^D b_{i,k} = 0$.

Proof: Proof is presented in the Appendix VI-B ∎

Defining $t_m \triangleq \min\{D, k+1\}$, and initial condition $e_0 = x_0 - \mathsf{E}[x_0]$, and $w_{-1} = e_0$ for notational convenience; and knowing the noise realizations $\{w_{-1}, w_0, \cdots, w_{T-1}\}$ are mutually independent, it concludes from Proposition 1, that

$$e_k = x_k - \mathsf{E}[x_k|\bar{\mathcal{I}}_k] = \sum_{j=1}^{t_m}\sum_{i=j}^{t_m} b_{i,k} A^{j-1} w_{k-j}, \tag{9}$$

Defining $M_k = \mathsf{E}[e_k e_k^\top |\bar{\mathcal{I}}_k]$, it is straightforward to show

$$M_k = \sum_{i=1}^{t_m} c_{i,k} A^{i-1^\top} W_{k-i} A^{i-1},$$

where $W_{-1} = \mathsf{E}[e_0 e_0^\top] = \Sigma_0$, $W_0 = W_1 = \cdots = W$, and $c_{i,k} = \sum_{j=i}^{t_m} b_{j,k}$. Having this, one can easily show

$$V_k^* = \hat{x}_k^\top P_k \hat{x}_k + tr(P_k M_k) + \sum_{t=k}^{T-1} tr(\tilde{P}_t M_t) + \sum_{t=k+1}^T tr(P_t W).$$

Consequently, we can express J_0^* as follows:

$$J_0^* = \min_{\theta_{[0,T-1]}} \sum_{t=0}^{T-1}\left[\sum_{i=1}^{t_m} c_{i,t} tr(\tilde{P}_t A^{i-1^\top} W_{t-1} A^{i-1}) + \theta_t^\top \Lambda\right]$$
$$+ \hat{x}_0^\top P_0 \hat{x}_0 + \sum_{t=1}^T tr(P_t W) + tr(M_0 P_0), \tag{10}$$

Let us define two vectors γ_t and r_t, as in the following:

$$\gamma_t \triangleq [c_{1,t}, c_{2,t}, \cdots, c_{D,t}]^\top,$$
$$r_t \triangleq [tr(\tilde{P}_t W), tr(\tilde{P}_t A^\top W A), \cdots, tr(\tilde{P}_t A^{D-1^\top} W A^{D-1})]^\top.$$

Since the term $\hat{x}_0^\top P_0 \hat{x}_0 + \sum_{t=1}^T tr(P_t W) + tr(M_0 P_0)$ in (10), is independent of $\theta_{[0,T-1]}$, minimizing (10) is equivalent to

$$\tilde{J}_0^* = \min_{\theta_{[0,T-1]}} \sum_{t=0}^{T-1}\left[\gamma_t^\top r_t + \theta_t^\top \Lambda\right]. \tag{11}$$

After defining $(\gamma_k)_i$ to be the i-th component of the vector γ_k, the optimal strategy $\theta_{[0,T-1]}^*$ is the solution of the following mixed integer nonlinear programming (MINP):

$$\min_{\theta_{[0,T-1]}} \quad \sum_{k=0}^{T-1}\left[\theta_k^\mathsf{T}\Lambda + \gamma_k^\mathsf{T}r_k\right] \qquad (12)$$

subject to $(\gamma_k)_i = \sum_{j=i}^{D} b_{j,k}, \quad \sum_{i=1}^{D}\theta_k^i \geq 1$

$$b_{i,k} = \prod_{d=1}^{i-1}\prod_{j=1}^{d}(1-\theta_{k-d}^j)(\vee_{l=1}^i\theta_{k-i}^l)$$

$$\sum_{i=1}^{t_m} b_{i,k} = 1, \quad \sum_{i=k+2}^{D} b_{i,k} = 0$$

$$b_{i,k}\in\{0,1\}, \theta_k^i\in\{0,1\}, \ \forall k\in[0,T-1], i\in[1,D],$$

Remark 2: The optimal switching strategy $\theta_{[0,T-1]}^*$ is independent of the noise realizations and can be solved offline. This result is analogous to the conclusions of [20].

To significantly reduce the computational complexity of the MINP (12), we show that the derived MINP can be equivalently re-casted as a mixed integer linear program(MILP), by exploiting certain structure of the specific network setting. For this, by replacing $\sum_{i=1}^{D}\theta_k^i \geq 1$ with $\sum_{i=1}^{D}\theta_k^i = 1$ enables us to replace $\vee_{l=1}^i\theta_{k-i}^l$ in (12) by $\sum_{l=1}^i\theta_{k-i}^l$. Thus,

$$\min_{\theta_{[0,T-1]}} \quad \sum_{t=0}^{T-1}\left[\theta_t^\mathsf{T}\Lambda + \gamma_t^\mathsf{T}r_t\right] \qquad (13)$$

subject to $(\gamma_t)_i = \sum_{j=i}^{D} b_{j,t}, \quad \sum_{i=1}^{D}\theta_t^i = 1$

$$b_{i,t} = \prod_{d=1}^{i-1}\prod_{j=1}^{d}(1-\theta_{t-d}^j)(\sum_{l=1}^i\theta_{t-i}^l)$$

$$\sum_{i=1}^{t_m} b_{i,t} = 1, \quad \sum_{i=t+2}^{D} b_{i,t} = 0$$

$$b_{i,t}\in\{0,1\}, \theta_t^i\in\{0,1\}, \ \forall t\in[0,T-1], i\in[1,D].$$

Clearly, due to the conversion of an inequality constraint to an equality constraint, every feasible solution of (13) is a feasible solution for (12), and moreover the optimal value for (13) is no less than that of (12). Therefore, we only need to show that an optimal solution for (12) is a feasible solution for (13). To show this, we first claim that every θ which is feasible for (12) but not for (13) (i.e. $\sum_{i=1}^{D}\theta_k^i > 1$ for some k), there exists a $\tilde{\theta}$ which achieves a strictly lesser cost than θ. Let $1\leq i_1 < i_2 < \cdots < i_m \leq D$ be the indices such that $\theta_k^{i_n} = 1$. Now we construct a new $\tilde{\theta}_k$ such that $\tilde{\theta}_k^{i_1} = 1$, and $\tilde{\theta}_k^j = 0$, for all $j \neq i_1$. Thus, $\sum_{i=1}^{D}\tilde{\theta}_k^i = 1$, whereas, $\sum_{i=1}^{D}\theta_k^i > 1$. This is done for each k such that $\sum_{i=1}^{D}\theta_k^i > 1$. It can be verified that the cost $\sum_{t=0}^{T-1}\gamma_t^\mathsf{T}r_t$ remains the same while using $\theta_{[0,T-1]}$ or $\tilde{\theta}_{[0,T-1]}$; whereas $\sum_{t=0}^{T-1}\theta_t^\mathsf{T}\Lambda > \sum_{t=0}^{T-1}\tilde{\theta}_t^\mathsf{T}\Lambda$. Thus the optimal solution of (12) must be the optimal solution of (13). Relaxing the equality constraint of $b_{i,t}$ as $b_{i,t}\leq(\sum_{l=1}^i\theta_{t-i}^l)$ results in the following MILP which is equivalent to (12):

$$\min_{\theta_{[0,T-1]}} \quad \sum_{t=0}^{T-1}\left[\theta_t^\mathsf{T}\Lambda + \gamma_t^\mathsf{T}r_t\right] \qquad (14)$$

subject to $(\gamma_t)_i = \sum_{j=i}^{D} b_{j,t}, \quad b_{i,t} \leq \sum_{l=1}^i\theta_{t-i}^l$

$$\sum_{i=1}^{D}\theta_t^i = 1, \ \sum_{i=1}^{t_m} b_{i,t} = 1, \ \sum_{i=t+2}^{D} b_{i,t} = 0$$

$$b_{i,t}\in\{0,1\}, \theta_t^i\in\{0,1\}, \ \forall t\in[0,T-1], i\in[1,D].$$

Problem (14) is a relaxed version of (13), therefore, any optimal solution of (14) is also an optimal solution of (13) if it is a feasible solution for (13). At this point, it is trivial to verify that the optimal solution of (14) is a feasible (and hence optimal) solution for (13), and hence optimal for (12).

C. Communication Cost as a Constraint

So far, we have considered the cost function of the form

$$J = \min_{u,\theta} \mathsf{E}[J_{LQG} + J_{Comm}],$$

where, J_{Comm} is the communication cost. There are equivalent formulations of the this problem depending on the specific NCSs setup, e.g., constraint optimization problem:

$$\min_{u_{[k,T-1]},\ \theta_{[0,T-1]}} \quad \mathsf{E}\left[\sum_{t=0}^{T-1} x_t^\mathsf{T}Q_1 x_t + u_t^\mathsf{T}R u_t + x_T^\mathsf{T}Q_2 x_T \big| \mathcal{I}_k\right],$$

$$\text{s.t. } \mathsf{E}\left[\sum_{t=0}^{T-1}\theta_t^\mathsf{T}\Lambda\right] \leq b,$$

where $b \in \mathbb{R}^+$ is the budget; or, a bi-objective problem:

$$\min_{u_{[k,T-1]},\ \theta_{[0,T-1]}} \quad \{f_1(u,\theta), f_2(u,\theta)\}, \qquad (15)$$

with $f_1 = \mathsf{E}\left[\sum_{t=0}^{T-1} x_t^\mathsf{T}Q_1 x_t + u_t^\mathsf{T}R u_t + x_T^\mathsf{T}Q_2 x_T\right]$, and $f_2 = \mathsf{E}\left[\sum_{t=0}^{T-1}\theta_t^\mathsf{T}\Lambda\right]$. The solution of the bi-objective problem is characterized by Pareto frontier. Looking at Pareto curve for the section $f_2\leq b$, one obtains the solution of the constrained budget problem. Moreover, solving the constrained budget problem for all $b \geq 0$, the Pareto frontier for the bi-objective problem is obtained. The Pareto frontier for (15) can be constructed by optimizing the single objective function

$$\mathsf{E}\left[\sum_{t=0}^{T-1}\alpha\left[x_t^\mathsf{T}Q_1 x_t + u_t^\mathsf{T}R u_t + x_T^\mathsf{T}Q_2 x_T\right] + (1-\alpha)\theta_t^\mathsf{T}\Lambda\right], \quad (16)$$

for all $\alpha \in [0,1]$. Note that (16) is equivalent to (6) which can be solved following the discussion presented here.

Due to space restrictions some discussions were removed from this paper. Interested readers are directed to [21].

IV. SIMULATION RESULTS

Consider an NCS with unstable dynamics as:

$$x_{t+1} = \begin{bmatrix} 1.01 & 0 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.15 \end{bmatrix} u_t + \sqrt{1.5}w_t$$

where, $w_t, x_0 \sim \mathcal{N}(0, \mathbb{I}_2)$. The horizon T is set to be 100. There are 5 links with delays ranging from 1 to 5 time-steps and the corresponding prices are $[20, 13, 8, 2, 1]$. The optimal utilization of the links is shown in Fig. 2. For this choice of the parameters the network mainly uses the fastest (link 1) and the slowest (link 5) links. Only for few instances, the system utilizes the link 4 and the rest of the links are not used. Thus, we note that the measurements sent by the Link 5 is never used in estimation except towards the end. Thus, the system can remain open-loop for most of the time.

To assure our simulation setup accuracy, we set $\lambda_i = 0$ for all links, and we observe that only the fastest link is selected. Similarly, setting $\lambda_i \gg 1$, the system selects the slowest link,
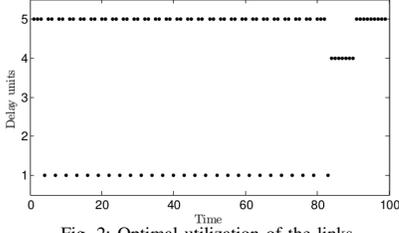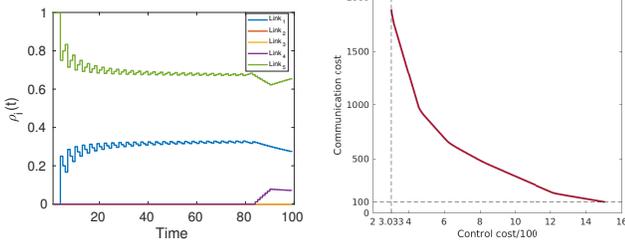
Fig. 2: Optimal utilization of the links

(a) Utilizations of different links over time: $\rho_i(t)$

(b) Pareto front of the bi-objective problem with $\Lambda = [20\ 13\ 8\ 2\ 1]$.

as the communication cost is exorbitantly high compared to the LQG cost. Similar profile is observed for all Λ, when disturbance is removed, and system becomes deterministic, so the only observation required is the initial state, and no need to send any measurement at all. However, the constraint $\sum_{i=1}^{D} \theta_k^i \geq 1$, forces the system to select the slowest link.

Let $\rho_i(t)$ be defined as $\rho_i(t) = \frac{\text{total utilization number of link } i}{t}$. In Fig. 3a, we observe that mostly two of the links (fastest and slowest) are utilized, while the rest are hardly used. This behavior is linked with the structure of the MILP (14), and studying it is beyond the scope of this article. However, this raises an interesting question for multiple systems scenario: How could the links be distributed among sub-systems so that the link utilization is fair? Also, we observe that $\rho_i(t)$ is very sensitive to the variations of λ_i. The design of prices Λ, as a time-varying or state-dependent variable, to achieve a desired utilization profile, is the subject of our future study.

In Fig. 3b we show the Pareto frontier of the bi-objective problem defined in (15). We notice that the minimum LQG cost (with fastest link being always selected) achievable for this set of parameters is 303.3 and the maximum LQG cost (with cheapest link being always selected) is 1503. The minimum communication cost is 100 (since cheapest link cost =1) which is associated with the maximum LQG cost.

V. CONCLUSION

In this article we address the problem of joint optimal LQG control and delay switching strategy in an NCS with a single stochastic LTI system. Assuming that the network utilization incurs cost, i.e. transmission with shorter delay is more costly, we derive the optimal delay switching profile. The overall cost function consisting of the LQG cost plus communication cost, is shown to be decomposable in expectation assuming *apriori* known prices. Having the separation property, the optimal laws can be computed offline as the solutions of an algebraic Riccati equation for the optimal control law, and a MILP, for the optimal switching profile.

VI. APPENDIX

A. Proof of Theorem 1

The LQG optimal value function at time $k+1$ is

$$V_{k+1}^* = \min_{u_{[k+1,T-1]}} \mathsf{E}\left[\sum_{t=k+1}^{T-1} x_t^\mathsf{T} Q_1 x_t + u_t^\mathsf{T} R u_t + x_T^\mathsf{T} Q_2 x_T | \bar{\mathcal{I}}_{k+1}\right].$$

Knowing that $\bar{\mathcal{I}}_k \subset \bar{\mathcal{I}}_{k+1}$, the law of total expectation yields

$$\mathsf{E}\left[V_{k+1}^* | \bar{\mathcal{I}}_k\right] = \min_{u_{[k+1,T-1]}} \mathsf{E}\left[\sum_{t=k+1}^{T-1} x_t^\mathsf{T} Q_1 x_t + u_t^\mathsf{T} R u_t + x_T^\mathsf{T} Q_2 x_T | \bar{\mathcal{I}}_k\right]$$

Therefore, it is straightforward to re-write V_k^* as follows:

$$V_k^* = \min_{u_{[k,T-1]}} \mathsf{E}[x_k^\mathsf{T} Q_1 x_k + u_k^\mathsf{T} R u_k + V_{k+1}^* | \bar{\mathcal{I}}_k]. \quad (17)$$

Assume that V_k^* can be expressed as follows:

$$V_k^* = \mathsf{E}[x_k | \bar{\mathcal{I}}_k]^\mathsf{T} P_k \mathsf{E}[x_k | \bar{\mathcal{I}}_k] + \pi_k \triangleq \hat{x}_k^\mathsf{T} P_k \hat{x}_k + \pi_k, \quad (18)$$

where, π_k will be derived later as a term independent of the control u_k. Having (18) assumed, (17) can be re-written as

$$V_k^* = \quad (19)$$
$$\min_{u_k} \mathsf{E}\left[x_k^\mathsf{T} Q_1 x_k + u_k^\mathsf{T} R u_k + \hat{x}_{k+1}^\mathsf{T} P_{k+1} \hat{x}_{k+1} + \pi_{k+1} | \bar{\mathcal{I}}_k\right].$$

We define the apriori state estimate $\hat{x}_k^- \triangleq \mathsf{E}\left[x_k | \bar{\mathcal{I}}_{k-1}\right] = A\hat{x}_k + Bu_k$. Due to the fact that

$$\mathsf{E}\left[\hat{x}_{k+1}^\mathsf{T} P_{k+1} \hat{x}_{k+1} | \bar{\mathcal{I}}_k\right] = \mathsf{E}\left[\hat{x}_{k+1}^{-\mathsf{T}} P_{k+1} \hat{x}_{k+1} | \bar{\mathcal{I}}_k\right] = \hat{x}_{k+1}^{-\mathsf{T}} P_{k+1} \hat{x}_{k+1}^-$$

then, (19) can be written as in the following:

$$V_k^* = \min_{u_k} \mathsf{E}[x_k^\mathsf{T} Q_1 x_k + u_k^\mathsf{T} R u_k | \bar{\mathcal{I}}_k]$$
$$+ \min_{u_k} \mathsf{E}[(A\hat{x}_k + Bu_k)^\mathsf{T} P_{k+1} (A\hat{x}_k + Bu_k) | \bar{\mathcal{I}}_k]$$
$$+ \xi_{k+1}^\mathsf{T} P_{k+1} \xi_{k+1} + \pi_{k+1}, \quad (20)$$

where, $\xi_{k+1} \triangleq \hat{x}_{k+1} - \hat{x}_{k+1}^-$. It is then simple to derive the optimal control u_k^*, minimizing (20), which is of the form

$$u_k^* = -(R + B^\mathsf{T} P_{k+1} B)^{-1} B^\mathsf{T} P_{k+1} A\hat{x}_k.$$

Plugging the optimal control u_k^* in (20), together with replacing x_k with its equivalent expression $e_k + \hat{x}_k$, result in

$$V_k^* = \mathsf{E}[\hat{x}_k^\mathsf{T}(\tilde{B}^\mathsf{T} R\tilde{B} + \tilde{A}^\mathsf{T} P_{k+1} \tilde{A} + Q_1)\hat{x}_k | \bar{\mathcal{I}}_k] \quad (21)$$
$$+ \mathsf{E}[e_k^\mathsf{T} Q_1 e_k + \xi_{k+1}^\mathsf{T} P_{k+1} \xi_{k+1} + \pi_{k+1} | \bar{\mathcal{I}}_k],$$

where, $\tilde{B}_k = \left(R + B^\mathsf{T} P_{k+1} B\right)^{-1} B^\mathsf{T} P_{k+1} A$, and $\tilde{A}_k = A - B\tilde{B}_k$. The equality (21) is ensured since

$$\mathsf{E}[e_k^\mathsf{T} Q_1 \hat{x}_k | \bar{\mathcal{I}}_k] = \mathsf{E}[\hat{x}_k^\mathsf{T} Q_1 e_k | \bar{\mathcal{I}}_k] =$$
$$\mathsf{E}[x_k^\mathsf{T} Q_1 \mathsf{E}[x_k | \bar{\mathcal{I}}_k] | \bar{\mathcal{I}}_k] - \mathsf{E}[(\mathsf{E}[x_k | \bar{\mathcal{I}}_k])^\mathsf{T} Q_1 (\mathsf{E}[x_k | \bar{\mathcal{I}}_k]) | \bar{\mathcal{I}}_k] = 0.$$

Comparing (21) with (18), the followings are concluded

$$\pi_k = \mathsf{E}\left[e_k^\mathsf{T} Q_1 e_k + \xi_{k+1}^\mathsf{T} P_{k+1} \xi_{k+1} + \pi_{k+1} | \bar{\mathcal{I}}_k\right]$$
$$= \mathsf{E}\left[\sum_{t=k}^{T-1} e_t^\mathsf{T} Q_1 e_t + e_T^\mathsf{T} Q_2 e_T + \sum_{t=k+1}^{T} \xi_t^\mathsf{T} P_t \xi_t | \bar{\mathcal{I}}_k\right]. \quad (22)$$

From definitions of ξ_k and e_k, it concludes for all $k \geq 1$, that

$$\xi_k + e_k = x_k - \hat{x}_k^- = Ae_{k-1} + w_{k-1}.$$

Knowing that $\mathsf{E}[\xi_k^\mathsf{T} P_k e_k | \bar{\mathcal{I}}_k] = \xi_k^\mathsf{T} P_k \mathsf{E}[e_k | \bar{\mathcal{I}}_k] = 0$, we obtain

$$\mathsf{E}[\xi_k^\mathsf{T} P_k \xi_k | \bar{\mathcal{I}}_k] + \mathsf{E}[e_k^\mathsf{T} P_k e_k | \bar{\mathcal{I}}_k] = \mathsf{E}[(\xi_k + e_k)^\mathsf{T} P_k (\xi_k + e_k) | \bar{\mathcal{I}}_k]$$
$$= \mathsf{E}[(Ae_{k-1} + w_{k-1})^\mathsf{T} P_k (Ae_{k-1} + w_{k-1}) | \bar{\mathcal{I}}_k]$$
$$= \mathsf{E}[e_{k-1}^\mathsf{T} A^\mathsf{T} P_k A e_{k-1} | \bar{\mathcal{I}}_k] + tr(P_k W).$$

Then it follows that

$$\mathsf{E}\left[\sum_{t=k+1}^{T} \xi_t^\mathsf{T} P_t \xi_t | \bar{\mathcal{I}}_k\right] = \sum_{t=k+1}^{T} tr(P_t W)$$
$$- \mathsf{E}\left[\sum_{t=k+1}^{T} e_t^\mathsf{T} P_t e_t | \bar{\mathcal{I}}_k\right] + \mathsf{E}\left[\sum_{t=k+1}^{T} e_{t-1}^\mathsf{T} A^\mathsf{T} P_t A e_{t-1} | \bar{\mathcal{I}}_k\right]$$

Finally, defining $\tilde{P}_t = Q_1 + A^\mathsf{T} P_{t+1} A - P_t$, for all $T > t \geq k$, the expression (22) for π_k can be re-written as

$$\pi_k = \mathsf{E}\left[\sum_{t=k}^{T-1} e_t^\mathsf{T}(Q_1 + A^\mathsf{T} P_{t+1} A)e_t + e_T^\mathsf{T} Q_2 e_T | \bar{\mathcal{I}}_k\right]$$
$$- \mathsf{E}\left[\sum_{t=k+1}^{T} e_t^\mathsf{T} P_t e_t | \bar{\mathcal{I}}_k\right] + \sum_{t=k+1}^{T} tr(P_t W)$$
$$= \mathsf{E}\left[e_k^\mathsf{T} P_k e_k + \sum_{t=k}^{T-1} e_t^\mathsf{T} \tilde{P}_t e_t | \bar{\mathcal{I}}_k\right] + \sum_{t=k+1}^{T} tr(P_t W).$$

B. Proof of Proposition 1

Consider two cases; $k \geq D$, and $k < D$. At any time $k \geq D$, the latest information the controller can have is x_{k-1}, only if $\theta_{k-1}^1 = 1$. If $\theta_{k-1}^1 = 0$, the latest information available is x_{k-2}, only if $\theta_{k-2}^1 \vee \theta_{k-2}^2 = 1$ ('$\vee$' is the logical OR operator). The algebraic representation of the logical constraint $\theta_{k-2}^1 \vee \theta_{k-2}^2 = 1$ is $\theta_{k-2}^1 + \theta_{k-2}^2 - \theta_{k-2}^1 \cdot \theta_{k-2}^2 = 1$. Similarly, we reach

$$\mathsf{E}[x_k | \bar{\mathcal{I}}_k] = \underbrace{\theta_{k-1}^1}_{b_{1,k}} \mathsf{E}[x_k | x_{k-1}, U_{k-1}] + \quad (23)$$

$$\underbrace{\left(1 - \theta_{k-1}^1\right)\left(\theta_{k-2}^1 \vee \theta_{k-2}^2\right)}_{b_{2,k}} \mathsf{E}[x_k | x_{k-2}, U_{k-1}] +$$

$$\underbrace{\left(1 - \theta_{k-1}^1\right)\left(1 - \theta_{k-2}^1\right)\left(1 - \theta_{k-2}^2\right)\left(\vee_{i=1}^3 \theta_{k-3}^i\right)}_{b_{3,k}} \mathsf{E}[x_k | x_{k-3}, U_{k-1}]$$

$$+ \quad \cdots \quad +$$

$$\underbrace{\prod_{d=1}^{D-1} \prod_{j=1}^{d}\left(1 - \theta_{k-d}^j\right)\left(\vee_{i=1}^D \theta_{k-D}^i\right)}_{b_{D,k}} \mathsf{E}[x_k | x_{k-D}, U_{k-1}]$$

For $k < D$, the oldest information the controller can have is x_0, only if $\vee_{i=1}^k \theta_0^i = 1$. Otherwise, if at time 0, the used link(s) had delay(s) greater than k, then x_0 is not available at time k, hence statistics of x_0 are used. Thus for $k < D$,

$$\mathsf{E}[x_k | \bar{\mathcal{I}}_k] = \theta_{k-1}^1 \mathsf{E}[x_k | x_{k-1}, U_{k-1}] +$$
$$\left(1 - \theta_{k-1}^1\right)\left(\theta_{k-2}^1 \vee \theta_{k-2}^2\right) \mathsf{E}[x_k | x_{k-2}, U_{k-1}] +$$
$$+ \quad \cdots \quad +$$
$$\prod_{d=1}^{k-1} \prod_{j=1}^{d}\left(1 - \theta_{k-d}^j\right)\left(\vee_{i=1}^k \theta_0^i\right) \mathsf{E}[x_k | x_0, U_{k-1}] +$$
$$\prod_{d=1}^{k} \prod_{j=1}^{d}\left(1 - \theta_{k-d}^j\right) \mathsf{E}[x_k | \bar{\mathcal{I}}_0, U_{k-1}]$$

For $k < D$, the same definition of $b_{1,k}, b_{2,k}, \cdots, b_{k,k}$ under-braced in (23) is used, while in addition, we define $b_{k+1,k} = \prod_{d=1}^{k} \prod_{j=1}^{d}(1 - \theta_{k-d}^j)$ and $b_{k+2,k} = b_{k+3,k} = \cdots = b_{D,k} = 0$. Finally, employing $\mathsf{E}[x_k | x_{-1}, U_{k-1}] \triangleq \mathsf{E}[x_k | \bar{\mathcal{I}}_0, U_{k-1}]$, the proof then readily follows.

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