Financial risk measures for a network of individual agents holding portfolios of light-tailed objects

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Abstract

We investigate a financial network of agents holding portfolios of independent light-tailed risky objects whose losses are asymptotically exponentially distributed with distinct tail parameters. We show that the asymptotic distributions of portfolio losses belong to the class of functional exponential mixtures. We also provide statements for Value-at-Risk and Expected Shortfall risk measures as well as for their conditional counterparts. We establish important qualitative differences in the asymptotic behavior of portfolio risks under a light tail assumption, compared to heavy tail settings, which have to be accounted for in practical risk management.

Keywords: Asymptotic exponential distribution, Expected Shortfall, Financial network, Risk management, Value-at-Risk

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1 Introduction

Studying a network of agents sharing financial risks by holding portfolios with different objects is of high relevance for both risk management and financial regulation. By monitoring a financial system, regulators or risk managers should assess risk exposures of different companies or business lines in order to determine capital reserves required in case of unexpectedly large losses. A regulator's or risk manager's assessment requires the following information: What are risk exposures of individual agents? Which are the dominant objects able to cause serious losses to agents or even to the entire system? How does the network structure affect the relationship between individual agent risks and the system risk?

We focus on a system where economic agents, e.g. insurance companies or investment funds, hold portfolios of risky objects forming a financial network. A possible structure of such a network is illustrated by a bipartite graph of agent-object relationships in Figure 1.1. As holding risky portfolios may lead to extreme losses, it is of immense importance to gain the corresponding asymptotic distributions of the portfolio losses in network contexts. Such results are of particular interest for risk managers and regulating authorities who should facilitate financial stability by monitoring both system and agents' losses. Moreover, they are required for computing of relevant risk measures, such as Value-at-Risk, Expected Shortfall, as well as Conditional Value-at-Risk and Expected Shortfall, cf. McNeil et al. [31], Adrian and Brunnermeier [1].

The effects of risk aggregation and risk sharing have been intensively studied in the current literature primarily for heavy-tailed risks with a power tail decay, see Embrechts et al. [13, 14], Kley et al. [23, 24], Lin et al. [26], Ly Vath et al. [27], Mainik and Rüschendorf [28], and Xia [33], among others. In many important settings, however, light-tailed distributions provide a suitable description of risks faced by insurance companies or financial institutions; such risks are faced for instance by household insurances or pension funds holding portfolios which are rebalanced in a monthly or quarterly frequency. For such essentially financial risks the family of generalized hyperbolic distributions with exponential tails is frequently used in mathematical finance literature (cf. Barndorff-Nielsen et al. [4], McNeil et al. [31]). Further studies dealing with light-tailed distributions are Asmussen and Albrecher [3], Behme et al. [5], Hernández and Junca [18], Kaas et al. [22], Kyprianou [25] in the insurance context, and Andersen et al. [2], Cont [9], Cont and Tankov [10] in the financial context.

Up to now, however, there are only few results on risk assessment for portfolios of objects with losses following light-tailed distributions. Jiang and Tang [21] study

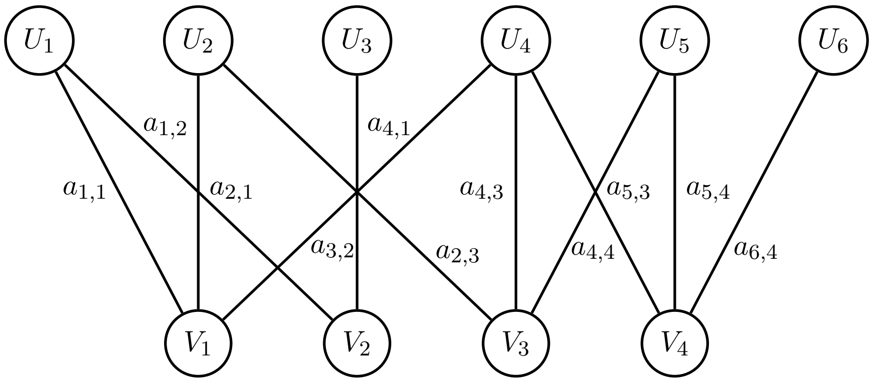

Figure 1.1: Bipartite graph for a network of 4 risky objects causing losses V_j held by 6 agents with risk exposures U_i. The portfolio weights are $a_{i,j}$ for agents $i \in \{1, \ldots, 6\}$ and objects $j \in \{1, \ldots, 4\}$.

the asymptotic behavior of losses for independently and identically exponentially distributed claims. Since they consider all claims with the same parameter of an exponential distribution, the aggregated claim (system loss) follows an Erlang distribution. Mitra and Resnick [32] analyze the sum of two losses with tail-equivalent distributions in the Gumbel max-domain of attraction, which contains distributions with exponential, Gaussian, and log-normal tails as special cases. They assume a certain bivariate dependence structure which leads to asymptotic independence. Farkas and Hashorva [15] consider portfolios of asymptotically Gaussian losses and derive limit results for the distribution of portfolio losses as well as the weak tail dependence coefficient (cf. Coles et al. [8]) for a pair of such portfolios. Dębicki et al. [12] investigate the distribution of losses in the Gumbel max-domain of attraction which are scaled by random factors. However, none of these papers study consequences of risk sharing in a network or system context.

We contribute to the current literature by exploring risk aggregation and risk sharing issues for portfolios of light-tailed losses in financial networks. Making assumptions about the full distribution of losses as e.g. a Gaussian or exponential, allows for many statements in explicit form. However, it is of high interest to generalize the analysis in the light tail setting by making assumptions only on the asymptotic behavior of losses. For this purpose, in the present paper we consider independent light-tailed object losses and only assume that they are asymptotically exponentially distributed. Since we impose no restriction on the finite behavior of losses, our framework appears to be rather general and flexible. Moreover, we allow for different risk classes with distinct tail parameters for the asymptotic distributions of losses on different objects, which is a more general setting than the commonly met assumption on tail-equivalent risks for either light- or heavy-tailed losses as e.g. in Jiang and Tang [21], Kley et al. [23, 24], Mitra and Resnick [32]. Hence, our study covers a broad class of light-tailed distributions in the context of portfolio risk sharing.

In order to quantify both agents' portfolio and system risks one requires statements concerning convolutions and mixtures of object loss distributions. For our general setting of asymptotically exponential distributions we obtain them in the form of functional exponential mixtures, where the mixing proportions are not constants but positive converging functions. Thus, we extend the findings for convolutions and mixtures of exponentially distributed random variables studied among others by Jewell [20] and McLachlan [30]. As functional exponential mixtures allow for novel, favorable representation of convolutions, this theoretical contribution may also be of interest beyond the financial network regulation context.

We start the discussion by deriving survival functions for the system loss defined as the sum of all object losses and for losses of individual agents holding portfolios with selected objects in Theorems 3.2 and 3.6. Then we analyze properties of the introduced functional mixture representations for the survival functions in Remark 3.3 and compare them with classical exponential mixtures. The important result for understanding extreme loss situations is presented in Theorem 4.1, where we show that the dominant impact on individual or system risk is determined by a single (distinct) object and that generically the risk-dominant object for the system does not coincide with those for individual agents. Moreover, we prove for our light tail setting that asymptotic behavior of individual and system risks is influenced not only by asymptotic but also by non-asymptotic behavior of object losses via their moment generating functions. Next, in Proposition 4.3 we apply our results in order to quantify the individual portfolio and system risks by computing popular quantile-based risk measures such as Value-at-Risk and Expected Shortfall. We also evaluate conditional risk measures for the network by deducing statements on the Conditional Value-at-Risks and the Conditional Expected Shortfalls in Theorems 5.3 and 5.5. Finally, we compare our findings for systems with light-tailed object losses to those with heavy-tailed ones. We point out the substantial qualitative differences in the stochastic behavior for these two settings and provide explanations for them. These differences underscore the importance of our analysis for a proper risk assessment for practical risk management and financial regulation.

Our paper is organized as follows. After introducing the framework of the study, notation and assumptions in Section 2, we derive in Section 3 the distributions of individual agent risks as well as of the system risk for object losses following asymptotically exponential distributions. For this purpose we develop a novel concept of functional exponential mixtures which appears to be useful in our analysis. In Section 4 we exploit our theoretical findings in order to analyze extreme loss situations and to present expressions on marginal quantile-based risk measures. In Section 5 we quantify the interdependence of individual and system risks within the network by deducing results on conditional risk measures. In Section 6 we summarize our findings for portfolios of light-tailed losses and compare them with those established for heavy-tailed ones. The proofs are summarized in Section 7.

2 Model framework: notation and assumptions

To formalize the framework for our investigation we introduce a system which consists of d objects and n agents for positive integers d and n. Object $j \in \underline{d} := \{1, \ldots, d\}$ causes a random *loss of size* $V_j > 0$ which is shared among the agents such that the *risk exposure of agent* i is given as

$$U_i = \sum_{j \in \underline{d}} a_{i,j}\, V_j\,, \quad i \in \underline{n} := \{1, \ldots, n\}\,, \tag{2.1}$$

where $a_{i,j}$ is the proportion of object j held by agent i. We denote indices referring to agents by $i \in \underline{n}$ and indices referring to objects by $j, k \in \underline{d}$. The *weights* for all d objects and n agents are collected into the matrix $A = (a_{i,j})_{i \in \underline{n}, j \in \underline{d}}$ of dimension $n \times d$, which is the weighted adjacency matrix to the bipartite graph as in Figure 1.1. The column-sums of A have to be less or equal to 1:

$$0 \le a_{i,j} \le 1 \quad \text{for all } i \in \underline{n},\ j \in \underline{d}\,; \qquad \sum_{i \in \underline{n}} a_{i,j} \le 1 \quad \text{for all } j \in \underline{d}\,.$$

The risk for object j is covered in total for the boundary case, where $\sum_{i \in \underline{n}} a_{i,j} = 1$. The system loss is defined as the sum of all object losses:

$$S = \sum_{j \in \underline{d}} V_j\,. \tag{2.2}$$

With $f(x) \sim g(x)$ for $x \to \infty$ we denote that functions $f(\cdot)$ and $g(\cdot)$ are asymptotically equivalent, i.e., $f(x)/g(x) \to 1$ for $x \to \infty$.

Throughout we meet the following assumption:

Assumption A The object losses V_j, $j \in \underline{d}$, are stochastically independent positive random variables and follow *asymptotic exponential ($\mathcal{AE}$) distributions*, i.e., the V_j have positive, continuous cumulative distribution functions $F_{V_j}(x)$, $x \ge 0$, and the survival functions $P\{V_j > x\} = 1 - F_{V_j}(x)$ satisfy asymptotically for $x \to \infty$:

$$P\{V_j > x\} \;\sim\; K_{V_j} \exp(-\lambda_j x)\,, \tag{2.3}$$

with pairwise distinct tail parameters $\lambda_j > 0$, i.e. $\lambda_j \neq \lambda_k$ for $j \neq k$, and factors $K_{V_j} \in (0, \infty)$, $j \in \underline{d}$. Without loss of generality, we assume

$$\lambda_1 < \lambda_2 < \cdots < \lambda_d. \tag{2.4}$$

Representation (2.3) implies that the light-tailed distribution of each object loss V_j with support $(0, \infty)$ is well-defined by the following two quantities: the positive *tail parameter* λ_j and the positive, continuous *factor function* $K_{V_j}(\cdot)$ with:

$$K_{V_j}(x) := P\{V_j > x\} \exp(\lambda_j x) , \quad x \geq 0 , \tag{2.5}$$

we write $V_j \in \mathcal{AE}(\lambda_j, K_{V_j}(\cdot))$. Hence, $K_{V_j}(\cdot)$ expresses the deviation from the exponential distribution, and Eqs. (2.3), (2.5) imply that $K_{V_j}(0) = 1$ and $\lim_{x\to\infty} K_{V_j}(x) = K_{V_j} \in (0, \infty)$. This flexible setting is of importance as it covers a broad class of light-tailed distributions. To the best of our knowledge, the class of $\mathcal{AE}$ distributions has not been investigated yet in the risk sharing context. In this paper we close this gap by deducing the results for individual portfolio risks and the system risk.

Our assumption concerning pairwise distinct tail parameters $\lambda_j \neq \lambda_k$ for $j \neq k$ is met for the analysis of mixtures of exponential distributions (see e.g. Bergel and Egídio dos Reis [6], McLachlan [30]), and we impose it now for studying mixtures of $\mathcal{AE}$ distributions. The losses with distinct tail parameters can be seen as caused by objects referring to different risk classes. This is in contrast to settings focused on tail-equivalent losses, as it is usually done for analyzing both light-tailed (cf. Jiang and Tang [21], Mitra and Resnick [32]) and heavy-tailed risks (cf. Kley et al. [23, 24]). Moreover, in Remark 4.2 below we explain how to handle those cases where some (or all) tail parameters λ_j coincide.

3 Risk of individual agents and of the system

We investigate the distributions of individual risk exposures U_i of agents $i \in \underline{n}$ from (2.1) and of the system loss S from (2.2) in terms of their survival functions $P\{U_i > x\}$ and $P\{S > x\}$. We show that they follow functional exponential mixture distributions and analyze their mixing proportion functions.

We start by studying the distribution of the system loss $S = \sum_{j\in\underline{d}} V_j$ which is a convolution of $\mathcal{AE}$ distributions. In the following remark we present the existing results on convolutions of exponential distributions.

Remark 3.1. Let the object losses V_j, $j \in \underline{d}$, $d \geq 2$, be exponentially distributed with densities $f_{V_j}(x) = \lambda_j \exp(-\lambda_j x)$, $x > 0$, and tail parameters $\lambda_j < \lambda_k$ for $j < k$.

The factor functions from (2.5) are constant $K_{V_j}(\cdot) \equiv 1$, hence, this is the special case $\mathcal{AE}(\lambda_j, 1)$ in our further analysis. Then the system loss S follows a so-called *generalized exponential mixture* distribution, whose survival function satisfies

$$P\{S > x\} \; = \; \sum_{j \in \underline{d}} \pi^*_{j,d} \, \exp(-\lambda_j x) \, , \quad x > 0 \, , \tag{3.1}$$

with the mixing proportions

$$\pi^*_{j,d} := \prod_{k \in \underline{d} \setminus \{j\}} \frac{\lambda_k}{\lambda_k - \lambda_j} \, . \tag{3.2}$$

The class of generalized exponential mixtures has been investigated (although not in a system risk context) e.g. by Jewell [20] and McLachlan [30]. This distribution class is also known as *generalized Erlang*, see e.g. Bergel and Egídio dos Reis [6]. The mixing proportions $\pi^*_{j,d}$, $j \in \underline{d}$, from (3.2) satisfy $\sum_{j \in \underline{d}} \pi^*_{j,d} = 1$ and alternate in sign: $\pi^*_{j,d}$ is positive for odd j and negative for even j. $\diamondsuit$

Now we consider convolutions of $\mathcal{AE}$ distributions and show in the following theorem that the system loss S for $\mathcal{AE}$ losses follows a *functional mixture* of exponential distributions where the mixing proportions are no longer constants $\pi_{j,d}$ but functions $\pi_{j,d}(\cdot)$. Moreover, it holds that the sum $\sum_{j \in \underline{d}} \pi_{j,d}(x)$ depends on $x > 0$ and is a positive value generically different to one.

Theorem 3.2. *Let Assumption A hold for a system of risky objects V_j, $j \in \underline{d}$. Then there exist continuous, positive functions $\pi_{j,d} : (0, \infty) \to (0, \infty)$, such that the survival function of the system loss $S = \sum_{j \in \underline{d}} V_j$ can be represented as:*

$$P\{S > x\} \; = \; \sum_{j \in \underline{d}} \pi_{j,d}(x) \; \exp(-\lambda_j x) \, , \qquad x > 0 \, . \tag{3.3}$$

The mixing proportion functions can be chosen recursively for $j \in \underline{k}$, $k \le d$:

$$\pi_{j,k}(x) := \begin{cases} \int_0^x \pi_{j,k-1}(x-y) \exp(\lambda_j y) \, \mathrm{d}F_{V_k}(y) & \text{for } j \in \underline{k-1} \, , \\ K_{V_k}(x) & \text{for } j = k \, , \end{cases} \tag{3.4}$$

where $K_{V_k}(\cdot)$ is defined as in Eq. (2.5). The mixing proportion functions $\pi_{j,d}(\cdot)$ are positive and bounded from above.

The order (2.4) of the tail parameters is important for building the recursion (3.4). Next, we interpret the result of Theorem 3.2 in the following remark and illustrate it in Figure 3.1.

Remark 3.3. (i) The mixing proportion functions $\pi_{j,d}(\cdot)$, $j \in \underline{d}$, in representation (3.3) for the convolution of V_j are *not uniquely* defined, not even up to asymptotic equivalence. Thus, the choice in (3.4) leads even in the special case of exponentially distributed losses to non-constant functions $\pi_{j,d}(\cdot)$ in contrast to the constant mixing proportions $\pi^*_{j,d}$ presented in (3.2); in general they differ in their sums because $\sum_{j\in\underline{d}} \pi^*_{j,d} = 1 \neq \sum_{j\in\underline{d}} \pi_{j,d}(x)$, and even asymptotically for $x \to \infty$ as $\pi_{j,d}(x) \not\to \pi^*_{j,d}$ for $j > 1$, cf. Theorem 4.1 below. Moreover, our choice in (3.4) guarantees that all mixing proportion functions are strictly positive. Hence, Theorem 3.2 shows that a generalized exponential mixture with *sign alternating* mixing proportions can be written as a functional exponential mixture with *positive* mixing proportions.

A functional mixture $P\{S > x\} = \sum_{j\in\underline{d}} \pi^*_{j,d}(x)\ \exp(-\lambda_j x)$, $x > 0$, which coincides for exponentially distributed losses with the generalized exponential mixture representation (3.1) with constant, sign-alternating mixing proportions, is given for $k \in \underline{d} \setminus \{1\}$ by the recursion:

$$\pi^*_{j,k}(x) := \begin{cases} \int_0^x \pi^*_{j,k-1}(u)\ \dfrac{\exp(-\lambda_j u)}{\exp((\lambda_k - \lambda_j)x) - 1}\ F_{V_k}(x - \mathrm{d}u) & \text{for } j \in \underline{k-1}\,, \\ K_{V_k}(x) - \sum_{l\in\underline{k-1}} \pi^*_{l,k}(x) & \text{for } j = k\,, \end{cases} \tag{3.5}$$

with initial value $\pi^*_{1,1}(x) = K_{V_1}(x)$. In contrast to the $\pi_{j,d}(\cdot)$ from (3.4), the recursion in (3.5) provides in the special case of exponentially distributed losses the product representation in (3.2) with constants $\pi^*_{j,d}(\cdot) \equiv \pi^*_{j,d}$.

(ii) We illustrate the meaning of functional mixtures by contrasting representations:

(A) $P\{S > x\} \ = \ \sum_{j\in\underline{d}} \pi^*_{j,d}\ \exp(-\lambda_j x)\,,$

(B) $P\{S > x\} \ = \ \sum_{j\in\underline{d}} \pi_{j,d}(x)\ \exp(-\lambda_j x)\,,$

for $x > 0$ with constant, real-valued $\pi^*_{j,d}$ from (3.2) and positive functions $\pi_{j,d}(\cdot)$ from (3.4). For the sake of the argument consider a system of three objects with exponentially distributed losses V_j with tail parameters $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. The survival function $P\{S > x\}$ is given by the aggregation of the three mixing components $\pi^*_{j,d} \exp(-\lambda_j x)$ or $\pi_{j,d}(x) \exp(-\lambda_j x)$, $j = 1, 2, 3$, either for (A) or for (B), respectively.

We plot the mixing components and the survival functions in Figure 3.1 all premultiplied with the term $\exp(\lambda_1 x)$ for a better visual presentation – such that the curves for the mixing components with $j = 1$ and, hence, for the survival function of S do not tend to the zero line. For representation (A), left, and (B), right, the figure displays the function $x \mapsto P\{S > x\} \exp(\lambda_1 x)$ in bold solid line which illustrates the behavior of the system risk as well as the functions $x \mapsto \pi^*_{j,d}\ \exp(-(\lambda_j - \lambda_1)x)$ and $x \mapsto \pi_{j,d}(x)\ \exp(-(\lambda_j - \lambda_1)x)$ in thin solid lines which illustrate the behavior

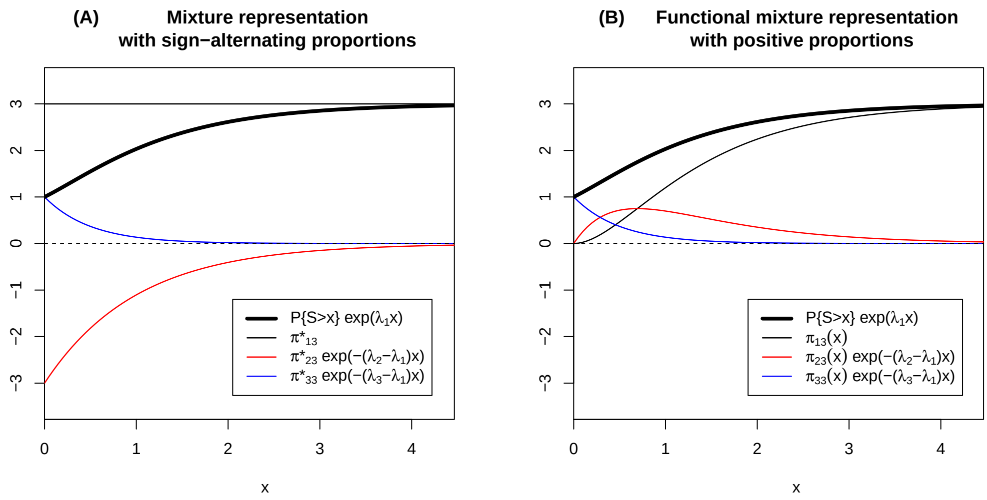

Figure 3.1: Comparison of established exponential mixture representation (A), left, and the functional mixture representation (B) introduced in Theorem 3.2, right, for the system risk $P\{S > x\}$; for more details see Remark 3.3(ii).

of the three mixing components.
We observe a quite different behavior of the mixing components for representations (A) and (B); in particular, in (B) all curves of the mixing components are located between the zero line and the survival curve, which allows us to evaluate the contribution of each component to the system risk. For extreme loss situations we find $\lim_{x\to\infty} P\{S > x\} \exp(\lambda_1 x) = 3 = \pi^*_{1,3} = \lim_{x\to\infty} \pi_{1,3}(x)$ as it follows from Theorem 4.1 below. $\diamondsuit$

Next, we deduce the survival function of the individual loss U_i for an arbitrary agent $i \in \underline{n}$ holding a portfolio of objects V_j with weights $a_{i,j}$, cf. (2.1), where we recall that Assumption A holds. Let D_i be the set of indices of the objects selected by agent i and d_i the number of those selected objects, i.e.,

$$D_i := \{j \in \underline{d} \mid a_{i,j} > 0\}\,, \qquad d_i := |D_i|\,. \tag{3.6}$$

Then the risk exposure $U_i = \sum_{j \in D_i} a_{i,j} V_j$ is the sum of the independent random variables $a_{i,j} V_j$ which all follow $\mathcal{AE}(\lambda_j / a_{i,j}, K_{V_j}(\cdot / a_{i,j}))$ distributions with tail parameter of $\lambda_j / a_{i,j}$ and factors $K_{V_j}(\cdot / a_{i,j}))$ for $j \in D_i$. We assume that the d_i tail parameters are pairwise distinct and order them as

$$\lambda_{i(1)}/a_{i,i(1)} < \cdots < \lambda_{i(d_i)}/a_{i,i(d_i)}\,. \tag{3.7}$$

Then, we obtain – analogously to the results for the system loss S presented in Theorem 3.2 – that:

Corollary 3.4. *The survival function of the individual loss* $U_i = \sum_{j\in \underline{d}} a_{i,j} V_j$ *for agent* $i \in \underline{n}$ *can be represented as:*

$$P\{U_i > x\} \;=\; \sum_{j\in D_i} \pi_{i,j,d_i}(x)\;\exp(-\lambda_j x/a_{i,j})\,, \quad x>0\,, \tag{3.8}$$

where the continuous, positive mixing proportion functions $\pi_{i,j,d_i} : (0,\infty) \to (0,\infty)$, $j \in D_i$, *can be chosen recursively for* $m \in \underline{k}$, $k \le d_i$ *as:*

$$\begin{aligned} &\pi_{i,i(m),k}(x) \\ &:= \begin{cases} \int_0^x \pi_{i,i(m),k-1}(x - a_{i,i(k)}y) \exp\left(\dfrac{a_{i,i(k)}\lambda_{i(m)}}{a_{i,i(m)}} y\right) \mathrm{d}F_{V_{i(k)}}(y) & \text{for } m \in \underline{k-1}\,, \\ K_{V_{i(k)}}(x/a_{i,i(k)}) & \text{for } m = k\,. \end{cases} \end{aligned} \tag{3.9}$$

For the asymptotic analysis of the mixing proportion functions for the system and the individual risks we require the following lemma concerning the object loss distributions.

Lemma 3.5. *Let object loss* V_k *follow an* $\mathcal{AE}(\lambda_k, K_{V_k}(\cdot))$ *distribution according to Assumption A. Then, the moment generating function* $\phi_{V_k}(t) := E[\exp(tV_k)]$ *of* V_k *exists, is finite for all* $t < \lambda_k$ *with singularity at* λ_k.

The mixing proportion functions for the distributions of individual and the system loss converge to positive finite limits, as we prove it in the following theorem.

Theorem 3.6. *Let Assumption A hold for a system of risky objects* V_j, $j \in \underline{d}$, *and let agent* $i \in \underline{n}$ *hold a portfolio with weights according to (3.7). Then, the mixing proportion functions* $\pi_{j,d}(\cdot)$ *from (3.4) and* $\pi_{i,j,d_i}(\cdot)$ *from (3.9) have finite and strictly positive limits for* $x \to \infty$. *These limits are given explicitly as:*

$$\pi_{j,d} \;:=\; \lim_{x\to\infty} \pi_{j,d}(x) = K_{V_j} \prod_{k=j+1}^{d} \phi_{V_k}(\lambda_j)\,, \tag{3.10}$$

$$\pi_{i,i(m),d_i} \;:=\; \lim_{x\to\infty} \pi_{i,i(m),d_i}(x) = K_{V_{i(m)}} \prod_{k=m+1}^{d_i} \phi_{V_{i(k)}}\Big(\frac{a_{i,i(k)}\lambda_{i(m)}}{a_{i,i(m)}}\Big), \tag{3.11}$$

with $j \in D_i$, $m \in \underline{d_i}$, *and the moment generating function* $\phi_{V_k}(\cdot)$ *of object loss* V_k:

$$\phi_{V_k}(\lambda_j) \;:=\; E[\exp(\lambda_j V_k)] = \int_0^\infty \exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y) \in (1,\infty)\,,\; j \in \underline{k-1}\,.$$

For $j = d$ *or* $m = d_i$ *the empty product in (3.10) or in (3.11) is set equal to 1 according to the common convention.*

Theorem 3.6 states that the survival functions of the system and the individual losses are asymptotically equivalent to non-functional exponential mixtures with positive mixing proportions: it holds that $P\{S > x\} \sim \sum_{j\in\underline{d}} \pi_{j,d} \exp(-\lambda_j x)$ and $P\{U_i > x\} \sim \sum_{j\in D_i} \pi_{i,j,d_i} \exp(-\lambda_j x/a_{i,j})$ for $x \to \infty$. We show that the limit values $\pi_{j,d}$ and π_{i,j,d_i} depend essentially on the moment generating functions $\phi_{V_k}(\cdot)$ of the object losses. Lemma 3.5 guarantees that the values $\phi_{V_k}(\lambda_j)$ for $j < k$ and $\phi_{V_{i(k)}}(a_{i,i(k)}\lambda_{i(m)}/a_{i,i(m)})$ for $m < k$ from the result in Theorem 3.6 are all finite and larger than one, see also (3.7).

For the special case of exponentially distributed losses, the result in Theorem 3.6 points out that the chosen mixing proportion functions $\pi_{j,d}(\cdot)$ for $P\{S > x\}$ differ even asymptotically from the generalized exponential mixing proportions $\pi^*_{j,d}$ given in (3.2) for $j > 1$.

4 Extreme losses in financial network and marginal risk measures

As extreme losses have the most adverse effects on the stability of financial systems, we derive in this section the asymptotic distributions of portfolio losses and provide statements for popular quantile-based risk measures such as Value-at-Risk and Expected Shortfall. As we have shown in Theorems 3.2 and 3.6, the distributions for the system loss S and the individual agents' exposures U_i can be written as functional exponential mixtures with positive mixing proportion functions $\pi_{j,d}(\cdot)$ and $\pi_{i,j,d_i}(\cdot)$ from (3.4) and (3.9), where the limits $\pi_{j,d} = \lim_{x\to\infty} \pi_{j,d}(x)$ and $\pi_{i,j,d_i} = \lim_{x\to\infty} \pi_{i,j,d_i}(x)$ exist, are finite and strictly positive, and, hence, do not influence the asymptotic tail decay.

In the following theorem we state an important result characterizing our framework of a network with $\mathcal{AE}$ object losses. We show that both system and individual agent's risk follow (as convolutions of $\mathcal{AE}$ distributions) again $\mathcal{AE}$ distributions. Moreover, in each case the asymptotic tail decay is determined by the object with the minimum tail parameter of all object losses V_j in the system or of all weighted losses $a_{i,j}V_j$ in the agent's portfolio, respectively:

$$\lambda_1 \;=\; \min_{j\in\underline{d}} \lambda_j \quad \text{and} \quad \mu_i \;:=\; \min_{j\in D_i} (\lambda_j/a_{i,j})\,. \tag{4.1}$$

Theorem 4.1. *Let Assumption A hold for a system of risky objects V_j, $j \in \underline{d}$, and let agent $i \in \underline{n}$ hold a portfolio with weights according to (3.7). Then:*

(i) the system loss $S = \sum_{j\in\underline{d}} V_j$ follows an $\mathcal{AE}(\lambda_1, K_S(\cdot))$ distribution with factor

function $K_S(x) := \pi_{1,d}(x)$*,* $x > 0$*. Asymptotically for* $x \to \infty$ *it holds that:*

$$P\{S > x\} \quad \sim \quad K_S \exp(-\lambda_1\, x)\,,$$

with $K_S := K_{V_1} \prod_{k=2}^{d} \phi_{V_k}(\lambda_1)$ *and the moment generating function* $\phi_{V_k}(\cdot)$ *of* V_k*;*

(ii) the individual loss $U_i = \sum_{j\in\underline{d}} a_{i,j} V_j$ *of agent* $i \in \underline{n}$ *follows an* $\mathcal{AE}(\mu_i, K_{U_i}(\cdot))$ *distribution with factor function* $K_{U_i}(x) := \pi_{i,i(1),d_i}(x)$*,* $x > 0$*. Asymptotically for* $x \to \infty$ *it holds that:*

$$P\{U_i > x\} \quad \sim \quad K_{U_i} \exp(-\mu_i\, x)\,,$$

where $K_{U_i} := K_{V_{i(1)}} \prod_{k=2}^{d_i} \phi_{V_{i(k)}}\Big(\frac{a_{i,i(k)}\lambda_{i(1)}}{a_{i,i(1)}}\Big)$ *with notation* $i(k)$ *from (3.7).*

For $d = 1$ *(or* $d_i = 1$*) the empty product is set equal to 1 according to the common convention.*

Theorem 4.1 proves that the class of $\mathcal{AE}$ distributions is closed under scaling and convolution, in contrast to the class of exponential distributions which does not satisfy this closure property under convolution.

Both system loss S and individual loss U_i of agent $i \in \underline{n}$ have asymptotically exponential tails, but with possibly different tail decays. The survival function of the system loss is asymptotically proportional to that of the object loss V_1 with minimum tail parameter λ_1. The asymptotic dominance of this object for the system risk is illustrated in Figure 3.1 which points out that – independently from the chosen mixture representation – the function $P\{S > x\}\exp(\lambda_1 x)$ converges to the positive limit $\pi_{1,d} = \lim_{x\to\infty} \pi_{1,d}(x)$. Analogously, the survival function of the individual loss U_i is asymptotically proportional to that of the weighted loss $a_{i,i(1)}V_{i(1)}$ with the agent's minimum tail parameter μ_i. If agent i does not select the most risky object V_1 in total; i.e., if $a_{i,1} < 1$ and, hence, $\lambda_1 < \mu_i$, the individual risk is asymptotically of lower order than the system risk:

$$P\{U_i > x\} \;=\; o(P\{S > x\}) \quad \text{for } \; x \to \infty\,. \tag{4.2}$$

In contrast to the tail parameter λ_1 of the system loss or the tail parameter μ_i of the individual loss, which are determined by only one (dominant) object, respectively, the factors K_S or K_{U_i} are influenced by all objects in the system or in the agent's portfolio. We show that the closer the tail parameters of the other objects are to that of the dominant object the larger is the value of the factor K_S or K_{U_i}, respectively.

Furthermore, in situations where the agent modifies his portfolio by adding or removing objects, the tail parameter of the agent's $\mathcal{AE}$ distribution remains unchanged as long as his dominant object remains the same. However, the asymptotic individual risk changes in terms of the factor K_{U_i}, when adding or removing objects.

Theorem 4.1 shows that asymptotic risks of individual agents and the system depend on the entire factor functions $K_{V_j}(\cdot)$ from the survival functions $P\{V_j > x\} = K_{V_j}(x)\exp(-\lambda_j x)$ of the object losses and not only on their limit values $\lim_{x\to\infty} K_{V_j}(x)$. This influence is determined by the moment generating functions $\phi_{V_k}(\cdot)$ of object losses V_k. The above result implies that in a system of independent $\mathcal{AE}$ distributed object losses the *asymptotic behavior* of system and individual agent risks is essentially influenced by the *non-asymptotic behavior* of the object loss distributions. This finding points out a qualitative difference to the established results for systems of heavy-tailed risks, where non-asymptotic behavior of the object loss distributions does not affect the asymptotic risks in the system.

In the following remark we comment on consequences of relaxing the assumption of pairwise distinct tail parameters in Assumption A.

Remark 4.2. If the tail parameters λ_j coincide for different objects j, then the distribution of the system loss S is a functional mixture of Erlang distributions such that:

$$P\{S > x\} \;=\; \sum_{k\in B} \pi_{k,b}(x)\, x^{b_k-1}\, \exp(-\lambda_k x)\,, \quad x > 0\,,$$

where $B \subset \underline{d}$ is a set of object indices satisfying $\{\lambda_j \mid j \in \underline{d}\} = \{\lambda_k \mid k \in B\}$ such that $\lambda_k < \lambda_l$ for $k < l$, $k, l \in B$. Moreover, $b := |B|$ denotes the number of pairwise distinct tail parameters in the system and $b_k := |\{j \in \underline{d} \mid \lambda_j = \lambda_k\}|$ denotes the number of tail parameters equal to λ_k.
The asymptotic results extending Theorem 4.1 lead to Erlang tails, in particular, we obtain for the system loss S the tail $\pi_{h,b}\, x^{b_h-1} \exp(-\lambda_1 x)$ where $h := \min B$. Hence, b_h is the number of *asymptotically dominant objects* with the minimum tail parameter λ_1. For the individual loss U_i this holds, respectively, with a set $B_i \subset D_i$ defined analogously to the set B above. $\diamondsuit$

Based on our results for the distributions of individual agent and system risks, we next provide statements for the marginal risk measures Value-at-Risk and Expected Shortfall which are of high importance in practice. Such results for light-tailed $\mathcal{AE}$ risks amend the corresponding analysis concerning risk measures for heavy-tailed loss distributions, provided among others by Das and Fasen-Hartmann [11], Ibragimov [19], Kley et al. [24], Mainik and Rüschendorf [28].

For level $\alpha \in (0, 1)$ we provide results on the Value-at-Risk for the loss from object $j \in \underline{d}$:

$$\operatorname{Obj} \operatorname{VaR}_j(\alpha) := \inf\{v \geq 0 \mid P\{V_j > v\} \leq 1-\alpha\}\,,$$

the individual Value-at-Risk for agent $i \in \underline{n}$:

$$\operatorname{Ind} \operatorname{VaR}_i(\alpha) := \inf\{u \geq 0 \mid P\{U_i > u\} \leq 1-\alpha\}\,,$$

and the system Value-at-Risk:

$$\operatorname{Sys} \operatorname{VaR}(\alpha) := \inf\{s \geq 0 \mid P\{S > s\} \leq 1-\alpha\}.$$

Moreover, we prove statements for the corresponding Expected Shortfalls which can be interpreted as the expected losses in extreme situations, where a given Value-at-Risk is exceeded. For level $\alpha \in (0,1)$ we investigate the Expected Shortfall for the loss of object $j \in \underline{d}$:

$$\operatorname{Obj} \operatorname{ES}_j(\alpha) := E\left[\, V_j \mid V_j > \operatorname{Obj} \operatorname{VaR}_j(\alpha) \,\right];$$

the individual Expected Shortfall for agent $i \in \underline{n}$:

$$\operatorname{Ind} \operatorname{ES}_i(\alpha) := E\left[\, U_i \mid U_i > \operatorname{Ind} \operatorname{VaR}_i(\alpha) \,\right];$$

and the system Expected Shortfall:

$$\operatorname{Sys} \operatorname{ES}(\alpha) := E\left[\, S \mid S > \operatorname{Sys} \operatorname{VaR}(\alpha) \,\right].$$

As we show in the following results, the asymptotic behavior of quantile-based risk measures in a system of $\mathcal{AE}$ losses is determined by the minimum tail parameter among those in the agent's portfolio μ_i or among all objects in the system λ_1 as defined in (4.1).

Proposition 4.3. *Let Assumptions A hold for a system of risky objects V_j, $j \in \underline{d}$, and let agent $i \in \underline{n}$ hold a portfolio with weights according to (3.7). Then:*

(i) for the Value-at-Risks it holds asymptotically as $\alpha \uparrow 1$ that:

$$\begin{aligned}
\operatorname{Obj} \operatorname{VaR}_j(\alpha) &\sim \frac{-\ln(1-\alpha)}{\lambda_j}\,, \quad j \in \underline{d}\,, \\
\operatorname{Ind} \operatorname{VaR}_i(\alpha) &\sim \frac{-\ln(1-\alpha)}{\mu_i}\,, \quad i \in \underline{n}\,, \\
\operatorname{Sys} \operatorname{VaR}(\alpha) &\sim \frac{-\ln(1-\alpha)}{\lambda_1}\,;
\end{aligned}$$

(ii) for the Expected Shortfalls it holds that:

$$\begin{aligned}
\lim_{\alpha\uparrow 1} (\mathrm{Obj}\,\mathrm{ES}_j(\alpha) - \mathrm{Obj}\,\mathrm{VaR}_j(\alpha)) &= 1/\lambda_j\,, \quad j \in \underline{d}\,,\\
\lim_{\alpha\uparrow 1} (\mathrm{Ind}\,\mathrm{ES}_i(\alpha) - \mathrm{Ind}\,\mathrm{VaR}_i(\alpha)) &= 1/\mu_i\,, \quad i \in \underline{n}\,,\\
\lim_{\alpha\uparrow 1} (\mathrm{Sys}\,\mathrm{ES}(\alpha) - \mathrm{Sys}\,\mathrm{VaR}(\alpha)) &= 1/\lambda_1\,.
\end{aligned}$$

In the following remark we interpret our results on the asymptotic behavior of the Value-at-Risks in a system of $\mathcal{AE}$ object losses.

Remark 4.4. (i) Proposition 4.3 states that all Value-at-Risks tend to infinity if the level α tends to 1 with the same logarithmic rate, so that they are asymptotically proportional to each other and differ only in their proportionality factors. The latter is in each case the reciprocal of the tail parameter for the respective dominant object: $1/\mu_i$ for the agent's portfolio and $1/\lambda_1$ for the system.
(ii) The asymptotic Value-at-Risks are not influenced by the number of objects selected by agent i or contained in the whole system. Hence, the individual agent's Value-at-Risk remains asymptotically unchanged if he modifies the portfolio by adding or removing objects as long as the dominant object (with the minimum value of $\lambda_j/a_{i,j}$) remains.
(iii) Another remarkable property proven in Proposition 4.3 is that the asymptotic Value-at-Risks depend on the marginal distributions $P\{V_j > x\} = K_{V_j}(x)\exp(-\lambda_j x)$, $x \geq 0$, of the object losses only by the tail parameter of the respective dominant object; i.e., the asymptotic Value-at-Risks are independent of the factor functions $K_{V_j}(\cdot)$ and their limits K_{V_j}. This is qualitatively different compared to the Value-at-Risk results for heavy-tailed loss distributions, more details are provided in Section 6, part (III). $\diamondsuit$

Proposition 4.3 implies that the Expected Shortfalls for objects, agents as well as for the system are asymptotically equivalent to the respective Value-at-Risks. Moreover, it specifies the asymptotic equivalence as it states that the distance between Expected Shortfall and Value-at-Risk converges to a finite, non-zero limit which depends on the respective dominant object.

Corollary 4.5. *For the Expected Shortfalls it holds for $\alpha \uparrow 1$ that:*

$$\begin{aligned}
\mathrm{Obj}\,\mathrm{ES}_j(\alpha) &\sim \mathrm{Obj}\,\mathrm{VaR}_j(\alpha) \sim \frac{-\ln(1-\alpha)}{\lambda_j}\,,\\
\mathrm{Ind}\,\mathrm{ES}_i(\alpha) &\sim \mathrm{Ind}\,\mathrm{VaR}_i(\alpha) \sim \frac{-\ln(1-\alpha)}{\mu_i}\,,\\
\mathrm{Sys}\,\mathrm{ES}(\alpha) &\sim \mathrm{Sys}\,\mathrm{VaR}(\alpha) \sim \frac{-\ln(1-\alpha)}{\lambda_1}\,.
\end{aligned}$$

Due to Corollary 4.5, all properties described in Remark 4.4 for Value-at-Risks are also valid for the Expected Shortfalls.

5 Conditional risk measures

To asses the riskiness of a system we quantify not only the marginal risks of individual agents or of the system, but also their interdependence within the network by considering conditional risk measures. Such statements are of particular relevance for regulators of a financial system which monitor its stability.

In this section we provide results on the Conditional Value-at-Risks (CoVaR) for a network of agents sharing $\mathcal{AE}$ objects in Theorem 5.3 as well as on the Conditional Expected Shortfalls (CES) in Theorem 5.5. We focus on expressions based on asymptotic statements for $P(S > s \mid U_i > u)$ and analogues, where the conditioning is on the stress event that a loss exceeds a given threshold.

For these statements, we consider equally weighted portfolios usually used in the relevant literature, see e.g. Brechmann et al. [7], Geluk et al. [16], Ibragimov [19]. The agent i selects some objects and holds the same proportion of each selected object in his portfolio, i.e.,

$$a_{i,j} \;=\; a_i \;\;\text{for all } j \in D_i \;\;\text{and some }\; 0 < a_i \leq 1\,. \tag{5.1}$$

As a consequence, we can simplify the notation from (4.1) using (3.7) as follows:

$$\lambda_1 \;=\; \min_{j\in\underline{d}} \lambda_j \qquad \text{and} \qquad \lambda_{i(1)} \;=\; \min_{j\in D_i} \lambda_j\,. \tag{5.2}$$

We first compute the joint probability of individual and system losses in the following proposition.

Proposition 5.1. *Let Assumption A hold for a system of risky objects V_j, $j \in \underline{d}$, and let agent $i \in \underline{n}$ hold a portfolio with weights according to (5.1). Then it follows:*

(i) The joint probability of agent's i exposure U_i and the system loss S has a functional exponential mixture representation in one of the arguments u or s, respectively:

$$P(U_i > u, S > s) = \begin{cases} \sum_{k\in\underline{d}} b_{k,u/a_i}(s)\exp(-\lambda_k s) & \textit{for } u \leq a_i s\,, \\ \sum_{j\in D_i} b_j(u/a_i)\exp(-\lambda_j u/a_i) & \textit{for } u > a_i s\,, \end{cases}$$

with bounded functions $b_{k,r} : [0,\infty) \to (-\infty,\infty)$, $k \in \underline{d}$, $r \in [0,\infty)$, whose precise form is given in (7.13)*, and $b_j : [0,\infty) \to (-\infty,\infty)$, $j \in D_i$, as in* (7.17)*.*

Asymptotically it holds that:

(ii) for $u > 0$ fixed and $s \to \infty$:

$$P(U_i > u, S > s) \sim C_i(u/a_i)\, \exp(-\lambda_1 s)\,, \tag{5.3}$$

with coefficient

$$C_i(u/a_i) := \begin{cases} \pi_{1,d} & \text{for } \lambda_1 = \lambda_{i(1)}\,, \\ \pi_{1,d} \;-\; H_i\Big(\dfrac{u}{a_i}\Big)\, K_{V_1} \displaystyle\prod_{k \in \underline{d} \setminus D_i, k \neq 1} \phi_{V_k}(\lambda_1) & \text{for } \lambda_1 < \lambda_{i(1)}\,, \end{cases}$$

where $\pi_{1,d}$ has been defined in (3.10) *and $\phi_{V_k}(\cdot)$ is the moment generating function of V_k and*

$$\begin{aligned} H_i\Big(\frac{u}{a_i}\Big) :=& \int_0^{u/a_i} \exp(\lambda_1 z) \mathrm{d}F_{\sum_{j \in D_i} V_j}(z) \\ =& 1 - \sum_{j \in D_i} \pi_{j,d_i}\big(\frac{u}{a_i}\big) \exp\big(-(\lambda_j - \lambda_1)\frac{u}{a_i}\big) \\ &+ \lambda_1 \sum_{j \in D_i} \int_0^{u/a_i} \pi_{j,d_i}(z) \exp(-(\lambda_j - \lambda_1) z) \mathrm{d}z\,, \end{aligned}$$

where $\pi_{j,d_i}(\cdot)$ is defined analogously to $\pi_{j,d}(\cdot)$ in (3.4) but on the subset $D_i \subseteq \underline{d}$.

(iii) for $s > 0$ fixed and $u \to \infty$:

$$P(U_i > u, S > s) \sim \pi_{i(1),d_i}\, \exp(-\lambda_{i(1)} u/a_i)\,, \tag{5.4}$$

where $\lambda_{i(1)}$ has been defined in (5.2) *and $\pi_{i(1),d_i} = K_{V_{i(1)}} \prod_{k \in D_i, k \neq i(1)} \phi_{V_k}(\lambda_{i(1)})$.*

Note that in part (iii) it holds $\pi_{i(1),d_i} = \pi_{i,i(1),d_i}$ as defined in (3.11), where the index i has become redundant due to assumption (5.1).

The asymptotic form of the joint probability $P(U_i > u, S > s)$ for $u \to \infty$ in (5.4) is independent of the fixed threshold value s, and the asymptotic form for $s \to \infty$ in (5.3) depends on the threshold value u/a_i only if agent i does not hold the risk-dominant object with tail parameter λ_1 in his portfolio.

In the following remark we comment on the techniques of proof for gaining our results for functional mixtures in this section.

Remark 5.2. In the proof of Proposition 5.1 the integrals with respect to the considered functional mixture distributions in our network of $\mathcal{AE}$ losses need more sophisticated treatment compared to those needed for classical mixtures with constant mixing proportions. The reason for this is that the functions $H_j(x) := \pi_j(x)F_{V_j}(x)$ do in general not define a measure due to the violation of monotonicity. Consequently, $\int g(z)\mathrm{d}F_{\sum_j V_j}(z) = \sum_j \int g(z)\mathrm{d}H_j(z)$ is not true for the *functional mixtures* $1 - F_{\sum_j V_j}(z) = \sum_j \pi_{j,d}(z)\exp(\lambda_j z)$ with non-constant $\pi_{j,d}(\cdot)$. However, for some measure defining function G the following integration by parts formula is still true:

$$\int_a^b G(z)\,\mathrm{d}F_{\sum_j V_j}(z) = \int_a^b P\{\sum_j V_j > z\}\,\mathrm{d}G(z) - P\{\sum_j V_j > z\}\,G(z)\Big|_a^b ,$$

which will be very convenient for the functional mixtures under consideration. $\diamondsuit$

Next, for level $\alpha \in (0,1)$ and bounds $u, s \in (0,\infty)$ we evaluate the Conditional Value-at-Risk for an individual agent $i \in \underline{n}$ in a systemic stress situation:

$$\mathrm{Ind\,CoVaR}_{U_i|S>s}(\alpha) := \inf\{u \geq 0 \mid P(U_i > u \mid S > s) \leq 1-\alpha\},$$

and the system Conditional Value-at-Risk for a situation where agent $i \in \underline{n}$ is in financial distress:

$$\mathrm{Sys\,CoVaR}_{S|U_i>u}(\alpha) := \inf\{s \geq 0 \mid P(S > s \mid U_i > u) \leq 1-\alpha\}.$$

A typical choice for the bounds is $u = \mathrm{Ind\,VaR}(\beta)$, $s = \mathrm{Sys\,VaR}(\beta)$ for some $\beta \in (0,1)$.

Additionally, as we are interested in the asymptotic analysis of risks, we modify the notion CoVaR by introducing proportionally increasing thresholds for both stress events $\{S > s\}$ and $\{U_i > \theta s\}$ with the scaling factor $\theta \in (0, a_i)$:

$$\begin{aligned} \mathrm{Ind\,CoVaR}_{U_i|S}(\alpha,\theta) &:= \inf\{s \geq 0 \mid P(U_i > \theta s \mid S > s) \leq 1-\alpha\}, \\ \mathrm{Sys\,CoVaR}_{S|U_i}(\alpha,\theta) &:= \inf\{s \geq 0 \mid P(S > s \mid U_i > \theta s) \leq 1-\alpha\}. \end{aligned}$$

Theorem 5.3. *Let Assumption A hold for a system of risky objects V_j, $j \in \underline{d}$, and let agent $i \in \underline{n}$ hold a portfolio with weights according to (5.1). Then for the Conditional Value-at-Risks with fixed thresholds $u, s \in (0,\infty)$ or with scaling factor $\theta \in (0, a_i)$ it holds asymptotically as level $\alpha \uparrow 1$ that:*

(i)

$$\begin{aligned} \mathrm{Ind\,CoVaR}_{U_i|S>s}(\alpha) &\sim \frac{-a_i \ln(1-\alpha)}{\lambda_{i(1)}} \sim \mathrm{Ind\,VaR}(\alpha), \\ \mathrm{Sys\,CoVaR}_{S|U_i>u}(\alpha) &\sim \frac{-\ln(1-\alpha)}{\lambda_1} \sim \mathrm{Sys\,VaR}(\alpha) \end{aligned}$$

(ii)

$$\begin{aligned} \mathrm{Ind\,CoVaR}_{U_i|S}(\alpha,\theta) &\sim \frac{-a_i \ln(1-\alpha)}{\theta(\lambda_{i(1)} - \lambda_1)} \quad \text{for } \lambda_1 < \lambda_{i(1)}, \\ \mathrm{Sys\,CoVaR}_{S|U_i}(\alpha,\theta) &\sim \frac{-a_i \ln(1-\alpha)}{(a_i - \theta)\lambda_1}. \end{aligned}$$

Our results in Theorem 5.3(i) show that for fixed thresholds in the conditional events $\{S > s\}$ or $\{U_i > u\}$ the asymptotic CoVaRs are independent of those thresholds s, u and behave equivalent to the respective unconditional VaRs. To measure the asymptotic dependence between individual and the system risk we provide in Theorem 5.3(ii) the modification of CoVaR based on conditional events with proportional increasing thresholds described by the factor θ. Here the system's influence on the individual agent's risk is determined by his portfolio weight a_i and by the difference between the risk-dominant tail parameters $\lambda_{i(1)}$ in the agent's portfolio and λ_1 in the entire system. Note that if agent i holds the dominant object with parameter λ_1, then the conditional distribution $P(U_i > \theta s \mid S > s)$ is degenerated as $P(U_i > \theta s, S > s) \sim P(S > s)$ for $\theta \in (0, a_i)$, $s \to \infty$ and, hence, a result for $\mathrm{Ind\,CoVaR}_{U_i|S}(\alpha, \theta)$ for $\lambda_1 = \lambda_{i(1)}$ does not exist. In contrast, the system's CoVaR given an agent in trouble is asymptotically proportional to the system's VaR, whereby it depends on both portfolio weight a_i and distance $a_i - \theta$.

Remark 5.4. Note that our definition of CoVaR is different from those of Adrian and Brunnermeier [1] where the conditioning loss is supposed to hit exactly some (high) value; i.e., the conditional events are of the form $\{S = s\}$ or $\{U_i = u\}$. However, as pointed out in Girardi and Ergün [17] and Mainik and Schaanning [29], this exact conditioning is rather restrictive. We follow the latter papers in conditioning on stress scenarios $\{S > s\}$ or $\{U_i > u\}$, which are more realistic settings in the CoVaR analysis.
In our system of independent $\mathcal{AE}$ losses the probability of the system loss $S > s$, given the exact loss value $u > 0$ of an agent's portfolio is given by:

$$P(S > s \,|\, U_i = u) \;=\; P\Big(\sum_{k\in \underline{d}\setminus D_i} V_k > s - \frac{u}{a_i}\Big) \sim \overline{C_i}\Big(\frac{u}{a_i}\Big)\exp(-\lambda_{\bar{i}(1)} s)\,,\; s \to \infty\,,$$

with $\overline{C_i}(u/a_i) := K_{V_{\bar{i}(1)}} \prod_{k\in \underline{d}\setminus D_i, k\neq \bar{i}(1)} \phi_{V_k}(\lambda_{\bar{i}(1)}) \exp(\lambda_{\bar{i}(1)} u/a_i)$, where $\bar{i}(1)$ is the the index of the minimum tail parameter for objects in $\underline{d} \setminus D_i$. This shows that only the objects *not* held by agent i influence the asymptotic conditional probability of the system loss. Hence, asymptotic results when conditioning on $\{U_i = u\}$ neglect potential severe object losses within the agent's portfolio. Moreover, this leads to $P(S > s \mid U_i = u) = o(P(S > s))$, $s \to \infty$, when $\lambda_1 < \lambda_{\bar{i}(1)}$. These problems can be avoided by using CoVaRs based on stress scenarios $\{U_i > u\}$ which lead due to result (5.3) for $s \to \infty$ to:

$$P(S > s \mid U_i > u) \sim C_i\Big(\frac{u}{a_i}\Big)\, \exp(-\lambda_1 s) \Big/ \sum_{j\in D_i} \pi_{j,d_i}\Big(\frac{u}{a_i}\Big) \exp\Big(-\lambda_j\Big(\frac{u}{a_i}\Big)\Big)\,.$$

$\diamondsuit$

Now we provide our results on the Conditional Expected Shortfalls in the network context, namely the individual Conditional Expected Shortfall of agent $i \in \underline{n}$:

$$\mathrm{Ind\,CES}_{U_i|S}(\alpha) \;:=\; E\left[\, U_i \mid S > \mathrm{Sys\,VaR}(\alpha) \,\right],$$

and the system Conditional Expected Shortfall:

$$\mathrm{Sys\,CES}_{S|U_i}(\alpha) \;:=\; E\left[\, S \mid U_i > \mathrm{Ind\,VaR}_i(\alpha) \,\right].$$

These are the two most practically important CES measures: $\mathrm{Ind\,CES}_{U_i|S}$ is the expected loss of agent i given that the financial system is in distress, and, hence, it can be used for comparing individual risks in a systemic crisis situation. Accordingly, $\mathrm{Sys\,CES}_{S|U_i}$ is the expected loss of the financial system given that agent i faces a high loss, and, hence, it is of a regulator's system stability interest given the agent i gets in trouble. Note that CES statements on $E[\, U_k \mid U_i > \mathrm{Ind\,VaR}_i(\alpha)\,]$ for two distinct agents k and i are of less practical relevance, as the (possibly competing) agents usually do not know the portfolio compositions of each other. Hence, we concentrate here on the CES results for S and U_i.

Theorem 5.5. *Let Assumption A hold for a system of risky objects V_j, $j \in \underline{d}$, and let agent $i \in \underline{n}$ hold a portfolio with weights according to (5.1). Then it follows for the Conditional Expected Shortfalls as level $\alpha \uparrow 1$:*

(i) in case $\lambda_1 = \lambda_{i(1)}$, where the risk dominant object is in the agent's portfolio, it holds that:

$$\mathrm{Ind\,CES}_{U_i|S}(\alpha) \;\sim\; -a_i \,\ln(1-\alpha)/\lambda_1 \;\sim\; a_i\,\mathrm{Sys\,ES}(\alpha),$$

otherwise, for $\lambda_1 < \lambda_{i(1)}$, it holds that:

$$\lim_{\alpha\uparrow 1} \mathrm{Ind\,CES}_{U_i|S}(\alpha) \;=\; a_i\,\frac{\pi_{\bar{i}(1),(d-d_i)}}{\pi_{1,d}} \sum_{j\in D_i} \pi_{j,d_i}\,\frac{\lambda_j}{(\lambda_j-\lambda_1)^2},$$

where $\pi_{\bar{i}(1),(d-d_i)} = K_{V_{\bar{i}(1)}} \prod_{k\in \underline{d}\setminus D_i, k\neq \bar{i}(1)} \phi_{V_k}(\lambda_{\bar{i}(1)})$ with $\lambda_{\bar{i}(1)} = \min_{k\in\underline{d}\setminus D_i} \lambda_k$ and $\pi_{j,d_i} = K_{V_j}\prod_{k\in D_i, k>j}\phi_{V_k}(\lambda_j)$;

(ii) for the system CES, it holds that:

$$\mathrm{Sys\,CES}_{S|U_i}(\alpha) \;\sim\; \frac{-\ln(1-\alpha)}{\lambda_{i(1)}} \;\sim\; \frac{1}{a_i}\,\mathrm{Ind\,ES}_i(\alpha).$$

Theorem 5.5(i) points out an interesting phenomenon: the individual Conditional Expected Shortfall of agent i depends qualitatively on whether he holds a proportion of the risk-dominant object in his portfolio or not. In the first case his CES is

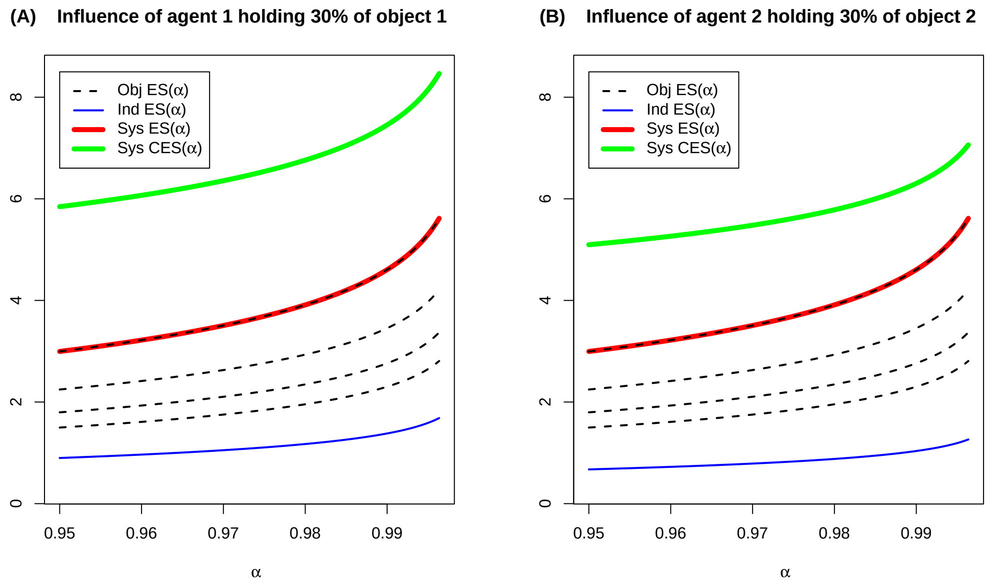

Figure 5.1: Asymptotic Expected Shortfalls for object, individual agent and system losses as well as system's Conditional Expected Shortfall given a large agent's loss for a portfolio with the riskiest object in (A) or the second-riskiest object in (B); for more details see Remark 5.6.

asymptotically proportional to the system's ES, in the second case it converges to a positive, finite limit. This implies that in a systemic crisis situation the individual expected loss of an agent holding the risk-dominant object increases proportionally to the Expected Shortfall of the system, i.e. the individual risk is unbounded; whereas agents avoiding the risk-dominant (i.e., the most "toxic") object has a finite; i.e. bounded, risk during a systemic crisis.

Theorem 5.5(ii) gives a measure for the influence of the individual agent's risk on the system risk, reflecting the impact of this agent on the entire system stability. The Conditional Expected Shortfall of the system, given an agent is in distress, is asymptotically proportional to the individual agent's Expected Shortfall. It depends *only* on the dominant tail parameter of the agent and, remarkably, is not affected by the chosen portfolio weight a_i.

Remark 5.6. In Figure 5.1 we visualize quantile-based risk measures for a system of $\mathcal{AE}$ object losses to illustrate the impact of an agent's portfolio structure on the system risk. For a four object system with tail parameters $\lambda_1 = 1$, $\lambda_2 = 4/3$, $\lambda_3 = 5/3$, $\lambda_4 = 2$ the asymptotic $\operatorname{Obj}\operatorname{ES}_j(\alpha)$, $j = 1, 2, 3, 4$ are displayed in thin, dotted lines, and the asymptotic $\operatorname{Sys}\operatorname{ES}(\alpha)$ – which coincides with the asymptotic $\operatorname{Obj}\operatorname{ES}_1(\alpha)$ – in a thick, red line. Moreover, in plot (A) we consider agent 1 holding his dominant object 1 with $a_{1,1} = 0.3$ and, hence, $\mu_1 = \min_{j \in D_1}(\lambda_1/a_{1,1}) = 0.3\,\lambda_1 =$

0.3, cf. (4.1); whereas in plot (B), we presume agent 2 holding his dominant object 2 with $a_{2,2} = 0.3$ and $\mu_2 = 0.3\,\lambda_2 = 0.4$. Their asymptotic Expected Shortfalls are plotted in (A) and (B) in thin, blue lines. Note that, although agent 1 holds the riskiest object of the system and agent 2 only the second-riskiest one, their individual Expected Shortfall functions in (A) and (B) are rather similar. However, by plotting $\mathrm{Sys\,CES}_{S|U_i}(\cdot)$ in thick, green lines we observe a large difference in the expected system loss depending on whether losses of agent 1 in (A) or of agent 2 in (B) are critical for the system. $\diamondsuit$

6 Comparison of results for light and heavy tails

Finally, we contrast our results for a system of light-tailed $\mathcal{AE}$ risks with established results under the assumption of heavy-tailed risks. The settings in Kley et al. [23, 24] are suitable for such a comparison as they investigate risky systems of a similar structure. The essential difference from our setting is that the object losses are assumed there to be heavy-tailed, in particular they are asymptotically Pareto ($\mathcal{AP}$) distributed with the same tail parameter $\gamma > 0$ for all objects. Hence, we compare the results for systems with independent object losses V_j, $j \in \underline{d}$, for two settings satisfying for $x \to \infty$:

$$\textbf{Light tail } \mathcal{AE}(\lambda_j) \textbf{ -assumption:} \quad P\{V_j > x\} \sim K_{V_j} \exp(-\lambda_j x)\,,$$

$$\textbf{Heavy tail } \mathcal{AP}(\gamma) \textbf{ -assumption:} \quad P\{V_j > x\} \sim K_{V_j}\, x^{-\gamma}\,.$$

Our comparison is focussed on the three main issues: (I) tail parameters, (II) individual agent and system loss distributions, (III) risk measures.

(I) We start by underscoring the different role of the tail parameters λ_j and γ. They both describe the tail decay of the survival functions for object losses, however, act differently as scale and shape parameters, respectively. Scaling a loss with some weight $a > 0$ changes the tail decay for $\mathcal{AE}$ losses but not for $\mathcal{AP}$ losses:

$$\begin{aligned} V \in \mathcal{AE}(\lambda) \quad &\Rightarrow \quad a\,V \in \mathcal{AE}(\lambda/a)\,, \\ V \in \mathcal{AP}(\gamma) \quad &\Rightarrow \quad a\,V \in \mathcal{AP}(\gamma)\,. \end{aligned}$$

This difference is an essential one: in an $\mathcal{AP}$-setting Kley et al. [23, 24] consider the same tail parameter γ for all objects because losses with larger tail parameter are asymptotically negligible. This is not the case for an $\mathcal{AE}$-setting: our Theorem 4.1 proves that object losses which are asymptotically negligible for the system can be dominant for agents' risk exposures, as scaling (by holding only proportions of the risky objects in the portfolio) changes the tail decay.

(II) Next, we compare individual and system risks for both $\mathcal{AE}$- and $\mathcal{AP}$-settings. For $\mathcal{AE}$ object losses we prove that the individual agent's risk is determined asymptotically by the dominant object in his portfolio. The survival functions of the agents' risk exposures U_i have generically distinct tail decays, which can differ from those of the system loss. This means that some individual risks are asymptotically of lower order compared to others and, in particular, to the system risk:

$$\frac{P\{U_i > x\}}{P\{S > x\}} \to 0 \quad \text{for} \quad x \to \infty\,,$$

as we have shown in Theorem 4.1, see also (4.2), with the only exception of the special case that agent i selects the most risky object V_1 alone with $a_{i,1} = 1$.

In contrast, for an $\mathcal{AP}$-setting individual and system risks are asymptotically proportional, see Theorem 3.2 in Kley et al. [23]:

$$\frac{P\{U_i > x\}}{P\{S > x\}} \to \ \text{const} \ > 0 \quad \text{for} \quad x \to \infty\,.$$

Moreover, our analysis reveals another fundamental difference between the risk distribution in light tail $\mathcal{AE}$- or heavy tail $\mathcal{AP}$-settings. In Theorem 3.6 we prove that in an $\mathcal{AE}$-setting, for both system and individual agent's losses, their asymptotic risk distributions depend not only on the asymptotic but also on the non-asymptotic behavior of the object loss distributions in terms of their moment generating functions. Whereas in an $\mathcal{AP}$-setting the evaluation of asymptotic risk distributions requires only the asymptotic distributions of object losses.

(III) Finally, we compare the results on marginal and conditional risk measures. For convenience, we focus on portfolios with equal proportions for all selected risky objects as defined in (5.1). In both $\mathcal{AE}$- and $\mathcal{AP}$- settings, Value-at-Risk and Expected Shortfall for losses of objects, individual agents and of the entire system are asymptotically proportional to each other within each setting, respectively. They tend to infinity for level $\alpha \uparrow 1$ with logarithmic rates for an $\mathcal{AE}$-setting (as we have proven in Proposition 4.3) and with power rates for an $\mathcal{AP}$-setting (cf. Cor. 3.7 in Kley et al. [23]) as:

$$\begin{aligned}
\mathrm{Obj\,VaR}_j(\alpha) &\sim \quad K_{V_j}^{1/\gamma}\,(1-\alpha)^{-1/\gamma}\,,\\
\mathrm{Ind\,VaR}_i(\alpha) &\sim \quad a_i\,\Big(\sum_{j\in D_i} K_{V_j}\Big)^{1/\gamma}\,(1-\alpha)^{-1/\gamma}\,,\\
\mathrm{Sys\,VaR}(\alpha) &\sim \quad \Big(\sum_{j\in \underline{d}} K_{V_j}\Big)^{1/\gamma}\,(1-\alpha)^{-1/\gamma}\,.
\end{aligned}$$

Moreover, in an $\mathcal{AP}$-setting the Value-at-Risk for objects, agents or the system is asymptotically proportional to the respective Expected Shortfall with the proportionality factor $\gamma/(\gamma-1)$ for $\gamma > 1$, see Cor. 3.8 in Kley et al. [23]. In an $\mathcal{AE}$-setting

Value-at-Risk and Expected Shortfall are asymptotically equivalent, more precisely, Proposition 4.3(ii) proves that their difference converges to a finite, non-zero limit. Furthermore, in an $\mathcal{AE}$-setting Value-at-Risks and Expected Shortfalls are independent of the factors K_{V_j} in contrast to those in an $\mathcal{AP}$-setting.

Finally, we compare light tail $\mathcal{AE}$- and heavy tail $\mathcal{AP}$-settings by analyzing the Conditional Expected Shortfalls within a financial network. For an $\mathcal{AP}$-setting, Kley et al. [24] show for heavy-tailed risks with finite mean, characterized by $\gamma > 1$, that $\mathrm{Sys\,CES}_{S|U_i}(\alpha) = E[S \mid U_i > \mathrm{Ind\,VaR}_i(\alpha)]$ is for $\alpha \uparrow 1$ asymptotically proportional to the unconditional individual Expected Shortfall $\mathrm{Ind\,ES}_i(\alpha)$. This is also valid for an $\mathcal{AE}$-setting as we derive in Theorem 5.5(ii). The situation is very different if we consider the impact of a systemic crisis on the individual agent's risk: In the heavy-tail $\mathcal{AP}$-setting the agent's $\mathrm{Ind\,CES}_{U_i|S}(\alpha) = E[U_i \mid S > \mathrm{Sys\,VaR}(\alpha)]$ always tends for $\alpha \uparrow 1$ to infinity as:

$$\mathrm{Ind\,CES}_{U_i|S}(\alpha) \quad \sim \quad a_i \frac{\gamma}{\gamma - 1} \Big(\sum_{j \in D_i} K_{V_j} \Big) \Big(\sum_{j \in \underline{d}} K_{V_j} \Big)^{1/\gamma - 1} (1 - \alpha)^{-1/\gamma} .$$

This contrasts with our result in Theorem 5.5(i) which states for an $\mathcal{AE}$-setting that $\mathrm{Ind\,CES}_{U_i|S}(\alpha)$ depends on the agent's portfolio composition. In particular, it increases proportionally to the Expected Shortfall of the system if the agent holds the risk-dominant object in his portfolio, but for all portfolios without the risk-dominant object it converges to a finite limit.

In summary we have established substantial differences in heavy tail and light tail settings, which have to be accounted for in risk management and regulatory decisions.

7 Proofs

Proof of Lemma 3.5. Since the object loss V_k, $k \in \underline{d}$ are positive random variables, its moment generating function $\phi_{V_k}(\cdot)$ exists at least for all $t \leq 0$. For $t \in (0, \lambda_k)$ we calculate

$$\begin{aligned} \phi_{V_k}(t) &= E[\exp(tV_k)] \;=\; 1 + \int_0^\infty P\{\exp(tV_k) - 1 > x\} \,\mathrm{d}x \\ &= 1 + \int_1^\infty K_{V_k}(\ln y/t) y^{-\lambda_k/t} \,\mathrm{d}y \;=\; 1 + K_{V_k}(\xi) \frac{t}{\lambda_k - t} \end{aligned}$$

for some $\xi \in (0, \infty)$ by the mean value theorem. Since the $K_{V_k}(\cdot)$, $k \in \underline{d}$ are continuous and strictly positive with $K_{V_k}(0) = 1$ and $\lim_{x \to \infty} K_{V_k}(x) = K_{V_k} \in (0, \infty)$, it follows that the moment generating function $\phi_{V_k}(\cdot)$ is finite for all $t < \lambda_k$ with a singularity at λ_k. $\square$

Proof of Theorems 3.2, 3.6, and Corollary 3.4. The results are proven by induction using that the convolution of $k \geq 2$ object losses can be calculated from the convolution of $(k-1)$ object losses recursively for $x > 0$ as follows:

$$P\Big\{\sum_{j\in\underline{k}} V_j > x\Big\} \;=\; P\{V_k > x\} + \int_0^x P\Big\{\sum_{j\in\underline{k-1}} V_j > x-y\Big\}\,\mathrm{d}F_{V_k}(y)\,. \tag{7.1}$$

For $d = 1$ object we obtain directly: $P\{S > x\} = P\{V_1 > x\} = \pi_{1,1}(x)\exp(-\lambda_1 x)$ where $\pi_{1,1}(x) := K_{V_1}(x)$, $x > 0$, is bounded away from zero and infinity, cf. definition of $K_{V_1}(\cdot)$ in Eq. (2.5) and proof of Lemma 3.5. Assumption A implies $\lim_{x\to\infty}\pi_{1,1}(x) = K_{V_1} \in (0,\infty)$.

Now assume that the results of Theorems 3.2, 3.6 on S are valid for $d{=}k{-}1$ objects for some $k \geq 2$, i.e. it holds that $P\{\sum_{j\in\underline{k-1}} V_j > x\} = \sum_{j\in\underline{k-1}}\pi_{j,k-1}(x)\exp(-\lambda_j x)$ with positive functions $\pi_{j,k-1}(x)$, $x > 0$ of form (3.4), which are bounded from above and have limits $\pi_{j,k-1} := \lim_{x\to\infty}\pi_{j,k-1}(x) = K_{V_j}\prod_{l=j+1}^{k-1}\phi_{V_l}(\lambda_j) \in (0,\infty)$ for all $j \in \underline{k-1}$, where $\phi_{V_l}(\cdot)$ denotes the moment generating function of V_l. Then Eq. (7.1) implies for the convolution of k object losses:

$$\begin{aligned}
&P\Big\{\sum_{j\in\underline{k}} V_j > x\Big\}\\
&= K_{V_k}(x)\exp(-\lambda_k x) + \sum_{j\in\underline{k-1}}\int_0^x \pi_{j,k-1}(x-y)\exp(-\lambda_j(x-y))\,\mathrm{d}F_{V_k}(y)\\
&= \sum_{j\in\underline{k}}\pi_{j,k}(x)\;\exp(-\lambda_j x)\,,
\end{aligned}$$

with functions

$$\pi_{j,k}(x) \;:=\; \int_0^x \pi_{j,k-1}(x-y)\exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y)\,,\quad j\in\underline{k-1}\,, \tag{7.2}$$

$$\pi_{k,k}(x) \;:=\; K_{V_k}(x)\,. \tag{7.3}$$

The mean value theorem implies that there exist values $\xi_j(x) \in (0,x)$, $j \in \underline{k-1}$, such that:

$$\pi_{j,k}(x) \;=\; \pi_{j,k-1}(x-\xi_j(x))\int_0^x \exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y) \;=:\; \pi_{j,k-1}(x-\xi_j(x))\,I_{j,k}(x)\,. \tag{7.4}$$

Let x_0 be an arbitrary but fixed value with $0 < x_0 < x$, then it holds for all $j \in \underline{k-1}$ that:

$$\lim_{x\to\infty} \int\limits_{x-x_0}^{x} \pi_{j,k-1}(x-y)\exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y) \quad = \quad 0\,, \tag{7.5}$$

which follows from $1 - F_{V_k}(x) \sim K_{V_k}\exp(-\lambda_k x)$, $x \to \infty$, where $\lambda_k > \lambda_j$ for all $j \in \underline{k-1}$ (cf. Assumption A) and from the boundedness of $\pi_{j,k-1}(\cdot)$ described above. The functions $I_{j,k}(x)$, $j \in \underline{k-1}$, from (7.4) are strictly positive for $x > 0$ (because of the positive integrands) and converge for $x \to \infty$ to the finite values $\phi_{V_k}(\lambda_j)$ of the moment generating function of V_k, where the finiteness is proven in Lemma 3.5. Hence, the $I_{j,k}(\cdot)$ and (consequently by Eq. (7.4)) the $\pi_{j,k}(\cdot)$, $j \in \underline{k-1}$, are bounded from above. With (7.5) it follows for $j \in \underline{k-1}$ and $x \to \infty$:

$$\begin{aligned}
\pi_{j,k}(x) \quad &\sim \quad \int\limits_{0}^{x-x_0} \pi_{j,k-1}(x-y)\exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y) \\
&= \quad \pi_{j,k-1}(x-\xi_j(x-x_0)) \int\limits_{0}^{x-x_0} \exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y)\,,
\end{aligned} \tag{7.6}$$

for arbitrary $0 < x_0 < x$ and values $\xi_j(x-x_0) \in (0, x-x_0)$. In the asymptotic analysis for $x \to \infty$ we can choose the value of x_0 and, hence, the value $x - \xi_j(x - x_0) \in (x_0, x)$ arbitrarily large. Together with $\lim_{x\to\infty}\pi_{j,k-1}(x) = \pi_{j,k-1} \in (0,\infty)$, we obtain for $j \in \underline{k-1}$ and $x \to \infty$:

$$\pi_{j,k}(x) \quad \sim \quad \pi_{j,k-1}\int\limits_{0}^{x} \exp(\lambda_j y)\,\mathrm{d}F_{V_k}(y) \;\sim\; \pi_{j,k-1}\,\phi_{V_k}(\lambda_j)\,.$$

Consequently, the mixing proportion functions $\pi_{j,k}(\cdot)$, $j \in \underline{k}$, from Eqs. (7.2) – (7.3) converge:

$$\lim_{x\to\infty}\pi_{j,k}(x) \;=\; \pi_{j,k-1}\,\phi_{V_k}(\lambda_j) \;=\; K_{V_j}\prod_{l=j+1}^{k}\phi_{V_l}(\lambda_j) \;=:\; \pi_{j,k} \in (0,\infty)\,, \tag{7.7}$$

where we apply $\pi_{j,k-1} = K_{V_j}\prod_{l=j+1}^{k-1}\phi_{V_l}(\lambda_j)$, $j \in \underline{k-1}$, as given above. For $j = k$ the empty product in (7.7) is set equal to 1 according to the common convention. Hence, the statements for convolution of $d = k$ object losses are deduced from those for $d = k-1$ objects.

Altogether, the results in Theorems 3.2 and 3.6 on S are proven, and the results in Corollary 3.4 and Theorem 3.6 on U_i follow analogously. □

Theorem 4.1 follows from Theorem 3.6.

Proof of Proposition 4.3. Theorem 4.1 gives for $i \in \underline{n}$ and $x \to \infty$:

$$P\{U_i > x\} \;\sim\; K_{U_i} \exp(-\mu_i x) \;=:\; P_i(x)\,,$$

with constant $K_{U_i} > 0$ given in statement (ii) of Theorem 4.1 and μ_i from (4.1). Hence, for the inverse it follows that:

$$P_i^{-1}(y) \;=\; \frac{\ln(K_{U_i}) - \ln(y)}{\mu_i} \;\sim\; \frac{-\ln(y)}{\mu_i}, \quad y \downarrow 0\,.$$

Inserting $y = 1 - \alpha$ gives the asymptotic result for $\mathrm{Ind\,VaR}_i(\alpha)$. The Value-at-Risks for object and system losses can be obtained analogously. Hence, the results in statement (i) of Proposition 4.3 are proven.
To deduce the results in statement (ii), we use the functional mixture representation provided in Theorem 3.6, cf. also Eq. (3.8), and obtain for $i \in \underline{n}$ and $u \to \infty$:

$$\begin{aligned}
&\int_u^\infty P\{U_i > x\}\,\mathrm{d}x \;=\; \sum_{j\in D_i} \int_u^\infty \pi_{i,j,d_i}(x) \exp(-\lambda_j x / a_{i,j})\,\mathrm{d}x \\
\sim\;& \sum_{j\in D_i} \pi_{i,j,d_i} \int_u^\infty \exp(-\lambda_j x/a_{i,j})\,\mathrm{d}x \;=\; \sum_{j\in D_i} \frac{a_{i,j}\pi_{i,j,d_i}}{\lambda_j} \exp(-\lambda_j u/a_{i,j}) \\
\sim\;& \frac{K_{U_i}}{\mu_i} \exp(-\mu_i u)\,, \qquad\qquad (7.8)
\end{aligned}$$

with $\pi_{i,j,d_i} = \lim_{x\to\infty} \pi_{i,j,d_i}(x)$. The last step in (7.8) follows from Theorem 4.1(ii) with constant $K_{U_i} > 0$ given there and μ_i from (4.1). Hence, we obtain for $u \to \infty$:

$$\begin{aligned}
E[U_i \,|\, U_i > u] - u \;&=\; \frac{E[(U_i - u)\,\mathbf{1}_{\{U_i - u > 0\}}]}{P\{U_i > u\}} \;=\; \int_u^\infty \frac{P\{U_i > x\}}{P\{U_i > u\}}\,\mathrm{d}x \\
&\sim\; \frac{K_{U_i}/\mu_i \;\exp(-\mu_i u)}{K_{U_i} \exp(-\mu_i u)} \;=\; \frac{1}{\mu_i}\,,
\end{aligned}$$

with indicator function $\mathbf{1}_{\{\cdot\}}$. The result in statement (ii) of Proposition 4.3 on $\mathrm{Ind\,ES}_i$ follows by inserting $u = \mathrm{Ind\ VaR_i}(\alpha)$; the other results in (ii) can be obtained as the following special cases: on $\mathrm{Obj\,ES}_j$ with $D_i = \{j\}$, $a_{i,j} = 1$, $\mu_i = \lambda_j$; on $\mathrm{Sys\,ES}$ with $D_i = \underline{d}$, $a_{i,j} = 1$ for all $j \in \underline{d}$, $\mu_i = \lambda_1$. □

Proof of Proposition 5.1. Consider some agent $i \in \underline{n}$ and partition the objects $\{V_j, j \in \underline{d}\}$ into the two subsets $M_i := \{V_j, j \in D_i\}$ of the objects selected by agent i and $\overline{M}_i := \{V_k, k \in \underline{d} \setminus D_i\}$ of the not-selected objects. Accordingly, we define two random variables $W_i := U_i/a_i = \sum_{j\in D_i} V_j$ and $\overline{W}_i := S - W_i = \sum_{k\in \underline{d}\setminus D_i} V_k$ which are stochastically independent and follow $\mathcal{AE}$ distributions given for $x > 0$ by (cf.

(3.3)):

$$P\{W_i > x\} = \sum_{j\in D_i} \pi_{j,d_i}(x)\exp(-\lambda_j x)\,, \tag{7.9}$$

$$P\{\overline{W}_i > x\} = \sum_{k\in(\underline{d}\setminus D_i)} \pi_{k,(d-d_i)}(x)\exp(-\lambda_k x). \tag{7.10}$$

Here $\pi_{j,d_i}(x)$, $j \in D_i$, are the mixing proportion functions corresponding to the sub-system formed by object set M_i, while $\pi_{k,(d-d_i)}(x)$, $k \in \underline{d}\setminus D_i$, are those corresponding to the sub-system formed by object set $\overline{M}_i$.

For $0 \le u \le a_i s$ we obtain with (7.9) and (7.10) for some mean value $\overline{\xi}_{u/a_i}(s) \in (u/a_i, s)$ that:

$$\begin{aligned}
& P(U_i > u, S > s) \;=\; P(W_i > u/a_i, W_i + \overline{W}_i > s) \\
=\;& P(W_i > s) + \int_{u/a_i}^{s} P(\overline{W}_i > s - z)\mathrm{d}F_{W_i}(z) \;=\; \sum_{j\in D_i} \pi_{j,d_i}(s)\exp(-\lambda_j s) \\
& + \sum_{k\in\underline{d}\setminus D_i} \pi_{k,(d-d_i)}(s - \overline{\xi}_{u/a_i}(s)) \int_{u/a_i}^{s} \exp(-\lambda_k(s-z))\mathrm{d}F_{W_i}(z).
\end{aligned} \tag{7.11}$$

For the integral in (7.11), by partial integration, (7.9), and again the mean value theorem we obtain for some $\xi_{u/a_i}(s) \in (u/a_i, s)$:

$$\begin{aligned}
& \int_{u/a_i}^{s} \exp(-\lambda_k(s-z))\mathrm{d}F_{W_i}(z) \;=\; \sum_{j\in D_i} \pi_{j,d_i}(z)\exp(-\lambda_j z)\exp(-\lambda_k(s-z))\Big|_{u/a_i}^{s} \\
& - \sum_{j\in D_i} \int_{u/a_i}^{s} \pi_{j,d_i}(z)\exp(-\lambda_j z)\lambda_k\exp(-\lambda_k(s-z))\mathrm{d}z \\
=\;& \sum_{j\in D_i} \exp(-\lambda_j s)\Big(\pi_{j,d_i}(s) - \frac{\lambda_k}{\lambda_k - \lambda_j}\pi_{j,d_i}(\xi_{u/a_i}(s))\Big) \\
& + \exp(-\lambda_k s)\sum_{j\in D_i}\Big[\exp\Big((\lambda_k - \lambda_j)\frac{u}{a_i}\Big)\Big(\frac{\lambda_k}{\lambda_k - \lambda_j}\pi_{j,d_i}(\xi_{u/a_i}(s)) - \pi_{j,d_i}\Big(\frac{u}{a_i}\Big)\Big)\Big].
\end{aligned}$$

Consequently, we obtain with (7.11) that the joint distribution of U_i and S has a functional exponential mixture representation:

$$P(U_i > u, S > s) \;=: \sum_{j\in\underline{d}} b_{j,u/a_i}(s)\exp(-\lambda_j s), \tag{7.12}$$

where

$$b_{j,u/a_i}(s) = \begin{cases} \pi_{j,d_i}(s) + \sum\limits_{k\in\underline{d}\setminus D_i} \pi_{k,(d-d_i)}(s-\overline{\xi}_{u/a_i}(s)) \\ \qquad \times\Big(\pi_{j,d_i}(s) - \dfrac{\lambda_k}{\lambda_k-\lambda_j}\pi_{j,d_i}(\xi_{u/a_i}(s))\Big) & \text{for } j\in D_i\,, \\ \sum\limits_{k\in D_i} \exp\big((\lambda_j-\lambda_k)\dfrac{u}{a_i}\big)\pi_{j,(d-d_i)}(s-\overline{\xi}_{u/a_i}(s)) \\ \qquad \times\Big(\dfrac{\lambda_j}{\lambda_j-\lambda_k}\pi_{k,d_i}(\xi_{u/a_i}(s)) - \pi_{k,d_i}\big(\dfrac{u}{a_i}\big)\Big) & \text{for } j\in\underline{d}\setminus D_i\,, \end{cases} \tag{7.13}$$

where $b_{j,u/a_i} : [0,\infty) \to (-\infty,\infty)$, $j \in \underline{d}$, are bounded functions.
Recall from Eq. (2.4) that the tail parameters are ordered such that $\lambda_j < \lambda_k$ for $j < k$. Hence, representation (7.12) implies that for the asymptotic analysis of $P(U_i > u, S > s)$ only the asymptotic behavior of the function $b_{1,u/a_i}(\cdot)$ in the dominant term $\exp(-\lambda_1 s)$ matters and depends on whether the most risky object is in the agent i portfolio or not. Denote by $\lambda_{i(1)} = \min_{j\in D_i}\lambda_j$, then we distinguish two cases:

$$\text{Case I. } \lambda_1 = \lambda_{i(1)} :\ 1\in D_i, \qquad \text{Case II. } \lambda_1 < \lambda_{i(1)}:\ \ 1\in\underline{d}\setminus D_i,$$

We start with Case II, where $\lambda_1 < \lambda_j$ for all $j \in D_i$ and obtain the finite limit:

$$\lim_{s\to\infty}\int\limits_{u/a_i}^{s}\exp(\lambda_1 z)\mathrm{d}F_{W_i}(z) = \prod_{j\in D_i}\phi_{V_j}(\lambda_1) - \int\limits_{0}^{u/a_i}\exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)\ \in(0,\infty)\,, \tag{7.14}$$

with moment generating function $\phi_{W_i}(\cdot) = \prod_{j\in D_i}\phi_{V_j}(\cdot)$. Now we go back to (7.11), noticing that the dominant term is the summand for $k = 1$ in the second term. The same argument as in (7.6) applies giving

$$\lim_{s\to\infty}\pi_{1,(d-d_i)}(s-\overline{\xi}_{u/a_i}(s)) = \pi_{1,(d-d_i)}\in(0,\infty).$$

Consequently, with (7.14) for $s\to\infty$:

$$\begin{aligned} P(U_i > u, S > s)\ &\sim\ \pi_{1,(d-d_i)}\exp(-\lambda_1 s)\int\limits_{u/a_i}^{s}\exp(\lambda_1 z)\mathrm{d}F_{W_i}(z) \qquad (7.15)\\ &\sim\ \pi_{1,(d-d_i)}\exp(-\lambda_1 s)\Big(\prod_{j\in D_i}\phi_{V_j}(\lambda_1) - \int\limits_{0}^{u/a_i}\exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)\Big)\\ &\sim\ \exp(-\lambda_1 s)\big(\pi_{1,d} - K_{V_1}\prod_{k\in\underline{d}\setminus D_i, k\neq 1}\phi_{V_k}(\lambda_1)\int\limits_{0}^{u/a_i}\exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)\big)\,, \end{aligned}$$

where in the last step we have adapted (3.10) to the situation, which gives $\pi_{1,(d-d_i)} = K_{V_1}\prod_{k\in\underline{d}\setminus D_i, k\neq 1}\ \phi_{V_k}(\lambda_1)$.

For Case I we start with the following partition:

$$
\begin{aligned}
P(U_i > u, S > s) \;&=\; P(W_i > u/a_i, W_i + \overline{W}_i > s) \\
= \int_0^\infty & P(W_i > \max(u/a_i, s-z))\mathrm{d}F_{\overline{W}_i}(z) \\
= \int_0^{s-u/a_i} & P(W_i > s-z)\mathrm{d}F_{\overline{W}_i}(z) \;+\; P(W_i > u/a_i)P(\overline{W}_i > s - u/a_i) \\
= \sum_{j\in D_i} & \exp(-\lambda_j s) \int_0^{s-u/a_i} \pi_{j,d_i}(s-z)\exp(\lambda_j z)\mathrm{d}F_{\overline{W}_i}(z) \\
& + \sum_{k\in\underline{d}\setminus D_i}\exp(-\lambda_k s)\sum_{j\in D_i}\pi_{j,d_i}(u/a_i)\pi_{k,(d-d_i)}(s-u/a_i)\exp((\lambda_k-\lambda_j)u/a_i).
\end{aligned}
$$

Since $1 \in D_i$ and $\lambda_1 < \lambda_k$ for all $k \in \underline{d} \setminus D_i$, the moment generating function $\phi_{\overline{W}_i}(\cdot) = \prod_{k\in\underline{d}\setminus D_i}\phi_{V_k}(\cdot)$ is finite at λ_1, such that for $s\to\infty$:

$$
\begin{aligned}
P(U_i > u, S > s) \;&\sim\; \exp(-\lambda_1 s)\int_0^{s-u/a_i}\pi_{1,d_i}(s-z)\exp(\lambda_1 z)\mathrm{d}F_{\overline{W}_i}(z) \\
\sim\; \exp(-\lambda_1 s)\,\pi_{1,d_i} & \prod_{k\in\underline{d}\setminus D_i}\phi_{V_k}(\lambda_1) \;=\; \pi_{1,d}\exp(-\lambda_1 s)\,,
\end{aligned}
\tag{7.16}
$$

where the limit value π_{1,d_i} follows as in (7.15).

Hereby, statement (i) for $u \le a_i s$ and statement (ii) in Proposition 5.1 are proven. Statement (i) for $u > a_i\, s$ and statement (iii) follows, since for $u > a_i s$:

$$
\begin{aligned}
P(U_i > u, S > s) \;&=\; P(W_i > u/a_i) = \sum_{j\in D_i}\pi_{j,d_i}(u/a_i)\exp(-\lambda_j u/a_i) \\
&\sim\; \pi_{i(1),d_i}\exp(-\lambda_{i(1)}u/a_i)\,, \quad \text{for } u\to\infty\,.
\end{aligned}
\tag{7.17}
$$

□

Proof of Theorem 5.3. From the results on the joint distribution of U_i and S as given in Proposition 5.1 it follows for the conditional probabilities that:

$$
\begin{aligned}
P(U_i > u \mid S > s) \;&\sim\; \pi_{i(1),d_i}\Big/\Big(\sum_{j\in\underline{d}}\pi_{j,d}(s)\exp(-\lambda_j s)\Big)\exp(-\lambda_{i(1)}u/a_i)\,, u\to\infty\,, \\
P(S > s \mid U_i > u) \;&\sim\; C_i\big(\frac{u}{a_i}\big)\Big/\Big(\sum_{j\in D_i}\pi_{j,d}\big(\frac{u}{a_i}\big)\exp\big(-\frac{\lambda_j u}{a_i}\big)\Big)\exp(-\lambda_1 s)\,, s\to\infty\,.
\end{aligned}
$$

Then the CoVaR results of statement (i) in Theorem 5.3 follow analogously to the VaR results in Proposition 4.3.

To prove the results of statement (ii) we analyze the functional exponential mixture representation (which follows from Eq. (7.12) in the proof of Proposition 5.1) such that for $s > 0$ and $\theta \in (0, a_i)$:

$$\begin{aligned} P(U_i > \theta s, S > s) \;&=\; \sum_{j \in D_i} b_{j,\theta s/a_i}(s) \exp(-\lambda_j s) \\ &+ \sum_{j \in D_i} \sum_{k \in \underline{d} \setminus D_i} B_{j,k}(s, \theta s/a_i) \exp(-(\lambda_j \theta/a_i + \lambda_k(1-\theta/a_i))s)\,, \end{aligned} \tag{7.18}$$

with $B_{j,k}(s, u/a_i) := b_{k,u/a_i}(s) \exp(-(\lambda_k - \lambda_j)u/a_i)$ which is a bounded function in both arguments s and u/a_i. Comparison of the exponents in (7.18) yields that $\lambda_j < \lambda_j \theta/a_i + \lambda_k(1-\theta/a_i)$ is equivalent to $\lambda_j < \lambda_k$, which implies:

$$P(U_i > \theta s, S > s) \sim \begin{cases} \pi_{1,d}\, \exp(-\lambda_1 s) & \text{for } \lambda_1 = \lambda_{i(1)}\,, \\ B_{i(1),1} \exp(-(\lambda_j \theta/a_i + \lambda_1(1-\theta/a_i))s) & \text{for } \lambda_1 < \lambda_{i(1)}\,. \end{cases} \tag{7.19}$$

Here we apply $\lim_{s\to\infty} b_{1,\theta s/a_i}(s) = \pi_{1,d}$ and $\lim_{s\to\infty} B_{i(1),1}(s, \theta s/a_i) = B_{i(1),1} \in (0,\infty)$. The quantile functions of $P(U_i > \theta s \mid S > s)$ and $P(S > s \mid U_i > \theta s)$ – which could be obtained from (7.19) – yield the CoVaR results in Theorem 5.3(ii). □

Proof of Theorem 5.5. We apply representation (7.18), see proof of Theorem 5.3 above, for $\theta s = u > 0$, and obtain that:

$$\begin{aligned} P(U_i > u)\, E[S \mid U_i > u] \;&=\; (u/a_i)\, P(W_i > u/a_i) + \int_{u/a_i}^{\infty} P(U_i > u, S > s)\mathrm{d}s \\ =\; u/a_i \sum_{j \in D_i} \pi_{j,d_i}(u/a_i) \exp(-\lambda_j u/a_i) \;&+\; \sum_{j \in D_i} \int_{u/a_i}^{\infty} \Big(b_{j,u/a_i}(s) \exp(-\lambda_j s) \\ &+ \sum_{k \in \underline{d} \setminus D_i} B_{j,k}(s, u/a_i) \exp((\lambda_k - \lambda_j)u/a_i) \exp(-\lambda_k s) \Big) \mathrm{d}s \\ \sim\; u/a_i\, \pi_{i(1),d_i}\, \exp(-\lambda_{i(1)}\, u/a_i)\,, &\quad \text{for } u \to \infty\,. \end{aligned}$$

By inserting $u = \mathrm{Ind\,VaR}_i(\alpha) \sim \mathrm{Ind\,ES}_i(\alpha)$, $\alpha \uparrow 1$ (see Corollary 4.5) this proves statement (ii) of Theorem 5.5.

The results in statement (i) can be deduced as follows: For $\lambda_1 = \lambda_{i(1)}$ we obtain with the mean values $\xi(s) \in (0, s)$, $\overline{\xi}(s) \in (s, \infty)$ that:

$$P(S > s)\, E[U_i \mid S > s] \;=\; \int\limits_0^{a_i\, s} P(U_i > u, S > s)\mathrm{d}u + \int\limits_{a_i\, s}^{\infty} P(U_i > u)\mathrm{d}u \tag{7.20}$$

$$\begin{aligned}
&= \sum_{j\in D_i} \exp(-\lambda_j s) b_{j,\xi(s)}(s) a_i s + \sum_{j\in D_i} a_i \exp(-\lambda_j s)\Big(\frac{\pi_{j,d_i}(\overline{\xi}(s))}{\lambda_j} \\
&\quad + \sum_{k\in \underline{d}\setminus D_i} \frac{B_{j,k}(s,\xi(s))}{\lambda_k - \lambda_j}\Big) - \sum_{k\in \underline{d}\setminus D_i} \exp(-\lambda_k s) \sum_{j\in D_i} \frac{a_i B_{j,k}(s,\xi(s))}{\lambda_k - \lambda_j} \\
&\sim a_i\, \pi_{1,d}\, s \exp(-\lambda_1 s)\,, \quad \text{for } s \to \infty\,,
\end{aligned}$$

where we apply $\lim_{s\to\infty} b_{j,\xi(s)}(s) = \pi_{1,d}$ which has be shown in the proof of Proposition 5.1, see (7.16). This gives statement (i) for case $\lambda_1 = \lambda_{i(1)}$.
To prove the corresponding result for case $\lambda_1 < \lambda_{i(1)}$, we further exploit the property that $P(U_i > u, S > s) \sim \pi_{1,(d-d_i)} \exp(-\lambda_1 s) \int_{u/a_i}^{s} \exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)$ holds in this case (see (7.15)), and obtain by changing the order of integrals that:

$$\begin{aligned}
&\int\limits_0^{a_i\, s} \pi_{1,(d-d_i)} \exp(-\lambda_1 s) \int\limits_{u/a_i}^{s} \exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)\, \mathrm{d}u \\
&= \pi_{1,(d-d_i)} \exp(-\lambda_1 s) \int\limits_0^{s} a_i\, z \exp(\lambda_1 z)\mathrm{d}F_{W_i}(z)\,.
\end{aligned} \tag{7.21}$$

Repeatedly applying integration by parts, we obtain for $s \to \infty$:

$$\begin{aligned}
&\int\limits_0^{s} z \exp(\lambda_1 z)\mathrm{d}F_{W_i}(z) \\
&= -\sum_{j\in D_i} \Big[\pi_{j,d_i}(s)\, s\, \exp(-(\lambda_j - \lambda_1)s) + \int\limits_0^{s} \pi_{j,d_i}(z)(1+\lambda_1 z) \exp(-(\lambda_j - \lambda_1)z)\mathrm{d}z\Big] \\
&\sim \sum_{j\in D_i} \pi_{j,d_i} \big[\exp(-(\lambda_j - \lambda_1)s)\big(-s - \frac{1+\lambda_1 s}{\lambda_j - \lambda_1} - \frac{\lambda_1}{(\lambda_j - \lambda_1)^2}\big) + \frac{1}{\lambda_j - \lambda_1} + \frac{\lambda_1}{(\lambda_j - \lambda_1)^2}\big] \\
&\to \sum_{j\in D_i} \pi_{j,d_i} \frac{\lambda_j}{(\lambda_j - \lambda_1)^2}\,.
\end{aligned}$$

Together with (7.20) and (7.21) we obtain for $s \to \infty$ that:

$$P(S > s)\, E[U_i \mid S > s] \sim \Big(a_i\, \pi_{1,(d-d_i)} \sum_{j\in D_i} \pi_{j,d_i} \frac{\lambda_j}{(\lambda_j - \lambda_1)^2}\Big) \exp(-\lambda_1 s)\,,$$

which gives the result for $\mathrm{Ind\,CES}_{U_i|S}(\alpha)$ in case $\lambda_1 < \lambda_{i(1)}$. Therefore the results of Theorem 5.5 are proven. □

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