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Symmetries and determination of heavy quark potentials in effective string theory

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Zusammenfassung

In dieser Dissertation untersuchen wir Symmetrieeigenschaften effektiver Feldtheorien (EFTs), besonders diejenigen, die schwere Quarks und Antiquarks beinhalten. Vor allem behandeln wir Raumzeitsymmetrien (i.e., Poincaré Invarianz) in Feldtheorien wie die non-relativistic QCD (NRQCD) und die potential NRQCD (pNRQCD). Weil diese Symmetrien deutlich in einer nichtlinearen Art und Weise realisiert sind, entstehen einige nicht triviale Beziehungen zwischen den Wilson Koeffizienten der non-relativistic EFTs. Zudem leiten wir einen analytischen Ausdruck der Wilson Koeffizienten, als die Potentiale der nicht perturbativen pNRQCD, her. Dabei verwenden wir die effektive Stringtheorie (EST), als ein effektives Modell des ,,QCD flux tube", das im nicht perturbativen Bereich gültig ist. Wir konstruktieren die EST für ein System eines gebundenen Zustandes zwischen einem schweren Quark und einem schweren Antiquark, und analysieren die Potentiale der führenden Ordnung (LO) und der nächsten führenden Ordnung (NLO) in der EST. Die explizite Realisierung der Poincaré Invarianz in dem Tief-Energie Bereich vereinfacht die Ausdrücke der Potentiale sowohl zur LO als auch zur NLO in der EST. Schließlich werden die hergeleiteten Potentiale mit den verfügbaren lattice-QCD Daten in den dazugehörigen Distanzen vergleichen.

Abstract

In this thesis, we investigate symmetric aspects of effective field theories (EFTs), especially the ones involving heavy quarks and heavy antiquarks. In particular, spacetime symmetries (i.e., Poincaré invariance) are studied in the context of non-relativistic QCD (NRQCD) and potential NRQCD (pNRQCD). As these fundamental spacetime symmeties are manifested in a non-linear fashion, there arise some non-trivial relations between the Wilson coefficients of the non-relativistic EFTs. Then, we investigate the analytic expressions of the Wilson coefficients of pNRQCD in the non-perturbative regime by utilizing the effective string theory (EST), which is an effective framework of the QCD flux tube model valid in the non-perturbative regime. We construct the EST suitable for the heavy quark-antiquark bound state from the symmetry of the system and calculate the potentials at leading order (LO) and next-to-leading order (NLO) within the EST power counting scheme. The explicit realization of the Poincaré invariance in the low-energy regime eventually simplifies the form of the potentials at LO as well as at NLO in the EST. Finally, the derived heavy quark potentials are compared to the available lattice QCD data.

Contents

Introduction **9**

1 Spacetime symmetries **15**

1.1 Spacetime symmetries in classical physics 15

1.1.1 Galilean invariance 15

1.1.2 Poincaré invariance 17

1.1.3 General coordinate invariance 19

1.2 Spacetime symmetries in quantum mechanics 20

1.2.1 Poincaré transformations 21

1.2.2 Little group formalism 22

1.3 Spacetime symmeries in quantum field theory 23

1.3.1 Poincaré transformations 23

1.3.2 Little group transformations 24

1.4 Spacetime symmetries in effective field theory 25

1.4.1 Effective field theory 26

1.4.2 Poincaré symmetries in effective field theory 27

2 Spacetime symmetries in effective theories of QCD **29**

2.1 Basics of QCD at high-energy 29

2.1.1 Degrees of freedom 29

2.1.2 Asymptotic freedom 30

2.2 Heavy quark effective theory 30

2.2.1 Convention 31

2.2.2 HQET Lagrangian 31

2.2.3 Heavy quark symmetry 33

2.2.4 Reparametrization invariance 34

2.3 Non-relativistic QCD 35

2.3.1 NRQCD Lagrangian 35

2.4 Symmetries in NRQCD 36

2.4.1 Discrete symmetries 36

2.4.2 Poincaré transformation 37

2.4.3 Invariance of the Lagrangian 43

2.4.4 Noether Charge 45

2.4.5 The four-fermion Lagrangian . . . 47
2.5 Summary and discussion . . . 49

3 Spacetime symmetries in potential NRQCD **51**
3.1 An effective field theory for heavy quarkonia . . . 51
3.1.1 Hierarchy of scales . . . 51
3.1.2 Lagrangian . . . 52
3.2 Symmetry transformations in pNRQCD . . . 54
3.2.1 Discrete symmetries . . . 54
3.2.2 Spacetime symmetries . . . 54
3.2.3 Redundancies from Poincaré transformations . . . 57
3.2.4 Field redefinitions by unitary transformation . . . 57
3.3 Poincaré invariance in pNRQCD . . . 63
3.3.1 Singlet sector . . . 63
3.3.2 Octet sector . . . 65
3.3.3 Singlet-octet sector . . . 67
3.4 Summary and discussion . . . 69

4 Heavy quark-antiquark potentials **73**
4.1 Matching between NRQCD and pNRQCD . . . 73
4.1.1 The leading order potential . . . 75
4.1.2 The potentials at $\mathcal{O}(1/M)$ and $\mathcal{O}(1/M^2)$ . . . 77
4.2 Heavy quark potentials . . . 79
4.2.1 Weak-coupling regime . . . 79
4.2.2 Strong coupling regime . . . 80
4.2.3 QCD flux tube model . . . 80

5 Effective string theory **83**
5.1 Relativistic theory of strings . . . 83
5.2 Effective theory of long strings . . . 85
5.2.1 Hierarchy of scales and power counting . . . 86
5.2.2 Lagrangian construction . . . 87
5.3 Long strings and heavy quark potentials . . . 88
5.3.1 Boundary conditions . . . 88
5.3.2 Green's function . . . 88
5.3.3 QCD-string theory equivalence conjecture . . . 90
5.3.4 QCD-to-EST mapping . . . 91
5.3.5 Heavy quark potentials at LO in EST . . . 92
5.3.6 Poincaré invariance . . . 100
5.4 Heavy quark potentials at NLO . . . 103
5.4.1 Green's function at NLO . . . 103
5.4.2 QCD to EST mapping at NLO . . . 104
5.4.3 Gauge field insertions at NLO . . . 105
5.4.4 Regularizations . . . 107

5.4.5 NLO calculation of the potentials 109
5.4.6 Poincaré invariance 116
5.5 Comparison to lattice data 121
5.5.1 Summary and discussions 125

6 Conclusion and outlook **129**
6.1 Conclusion 129
6.1.1 Applications of Poincaré invariance 132
6.1.2 Applications of EST 132

Appendices **135**

A **137**
A.1 NLO calculation in the four-fermion sector 137
A.2 Gauge field insertions to the Wilson loop 142
A.2.1 Chromoelectric field insertions 143
A.2.2 Chromomagnetic field insertions 152
A.2.3 Chromoelectric and chromomagnetic field insertions 153
A.3 Hard cut-off regularization 155

Introduction

Effective field theory (EFT) has been a standard tool for particle and nuclear physics at the frontier of research during the last decades. In particle physics, EFT is a modern technique for investigating physics above the electroweak scale, namely the Beyond the Standard Model (BSM), or even Dark Matter (DM) theories. Up until now, there does not exist any well-established underlying theory to compare with the EFT for the parametrization of physics BSM at the currently available energy scale of the Large Hadron Collider. In hadronic and nuclear physics, on the other hand, an EFT framework plays a pivotal role in the understanding of the lower-energy regime. In this case, the corresponding EFTs can be matched to the underlying theory, Quantum Chromodynamics (QCD), at a given energy scale.

The Lagrangian of an EFT is given as an expansion in the heavy scale that has been integrated out; in the case of non-relativistic EFTs, this is the quark mass M. It contains all terms allowed by the symmetries of the EFT and can schematically be expressed as

$$\mathcal{L}_{\mathrm{EFT}} = \sum_n c_n \frac{\mathcal{O}_n}{M^{d_n-4}} \,, \tag{1}$$

where the operators $\mathcal{O}_n$, made up by the fields that describe the effective degrees of freedom, are of mass dimension d_n, and the c_n are scalar functions, called *matching* or *Wilson coefficients*, that have zero mass dimension. These coefficients contain all information from the higher energy scale through the *matching* to the underlying theory.

In this thesis, we focus on EFTs of QCD. There exists a wide variety of such EFTs depending on the physical processes, and among them we want to concentrate on Heavy Quark Effective Theory (HQET), non-relativistic QCD (NRQCD), and potential NRQCD (pNRQCD). While HQET [1–4] is a low-energy EFT for heavy-light mesons, NRQCD [5, 6] provides a non-relativistic effective description for the dynamics of heavy quarks and antiquarks, and pNRQCD [7, 8] is an effective theory for heavy quark-antiquark bound states (heavy quarkonium). In these EFTs, a hierarchy of scales is assumed such that the mass M of the heavy (anti)quark is much larger than any other relevant energy scale including Λ_{QCD}, which is the scale at which confinement takes place.

The Wilson coefficients of these EFTs have to be determined through a matching calculation to the underlying theory. Beyond leading order in the coupling or the expansion parameter, this can easily become technically involved. For this reason, one would like to exploit as much prior knowledge on the Wilson coefficients as possible before commencing the calculation.

Furthermore, due to the non-relativistic expansions, Poincaré invariance is no longer manifest in these EFTs. A physical system is symmetric under Poincaré transformations when the action is invariant under spacetime translations, rotations, and boost transformations. In the late 19th and early 20th century, Lorentz, Poincaré and Einstein realized this fundamental spacetime symmetry of nature. Consequently, the quantum theory of fields was built upon this symmetry along with the cluster decomposition principle [9]. On the other hand, Dirac, during the 1940's, showed that the terms of the quantum mechanical Hamiltonian satisfy nontrivial relations if one imposes the Poincaré algebra [10]. This discussion was further developed for interacting relativistic composite systems [11–15], where relations between the relativistic correction terms were derived using the Poincaré algebra. As these quantum mechanical systems can be generalized by the framework of non-relativistic EFTs, it is natural to expect that also some non-trivial relations between the Wilson coefficients of EFTs can be derived from Poincaré invariance. It is natural to assume this invariance, because the EFT is equivalent to the low-energy limit of the underlying relativistic quantum field theory, which is manifestly symmetric under Poincaré transformations.

Another symmetry has been found in low-energy EFTs of QCD, for instance in HQET or Soft Collinear Effective Theory (SCET) [16–19], which is called reparametrization invariance. In these EFTs, the momentum of high energy particles is separated into a large momentum of the heavy particle and a small residual momentum from the interaction with a light particle. This separation is to some extent arbitrary, as a shift of the residual momentum by a small amount (compensated by a shift in the large momentum) preserves the scale hierarchy. Performing this shift at the level of the Lagrangian leads to a number of non-trivial relations between the Wilson coefficients [20, 21], which turn out to be equivalent to the ones obtained from Poincaré invariance [22]. This tells us that a shift in the parametrization of the high energy momentum may be interpreted just as well as a change of the reference frame. Although the implementation of reparametrization invariance might be more straightforward, its applications are limited. Poincaré invariance, on the other hand, is a general principle that all quantum field theories have to observe.

In [22], a direct implementation of Poincaré invariance was applied to two of the non-relativistic EFTs of QCD, namely, NRQCD and pNRQCD. As one constructs all generators of the symmetry group in these EFTs, the generators corresponding to spacetime translations and rotations are obtained in the usual way from the associated conserved Noether currents. The generators of boosts, on the other hand, are derived from a general ansatz that includes all operators allowed by other symmetries (such as parity, charge conjugation, and time reversal) up to a certain order in the expansion. As all generators has to satisfy the commutation relations of the Poincaré algebra, one can obtain relations between the Wilson coefficients of the EFTs. While this approach works well up to certain orders in the expansion, going into higher order poses a non-trivial challenge in calculations.

Recently, an approach was suggested for deriving constraints in EFTs through Poincaré invariance, which employs Wigner's *induced representation* [23]. It was proposed in [24] that a free nonrelativistic field ϕ in the rest frame, which has a well defined transformation behavior under rotations R as $\phi(x) \to D[R]\phi(R^{-1}x)$, should transform under a generic Lorentz transformation Λ as

$$\phi(x) \to D[W(\Lambda, i\partial)]\phi(\Lambda^{-1}x)\,. \tag{2}$$

The *little group element* W is a particular rotation associated with Λ and the momentum, which in position space leads to a dependence of W on derivatives of the field ϕ. The resulting expression is then expanded to the same order as the Lagrangian.

Although this approach is valid for non-interacting theories, some issues arise in an interacting gauge theory. In an interacting theory, one might think that promoting the derivatives to covariant derivatives would be sufficient, but it introduces some ambiguity in how the covariant derivatives should be ordered. It is also necessary to introduce additional gauge field dependent operators to the boost in order to cancel some terms that would prevent the EFT Lagrangian from being Poincaré invariant. In the end, the constraints obtained in this way agree with the previous results in NRQCD and non-relativistic QED (NRQED) [22, 25], and the derivation is somewhat simpler [24], but this covariantization procedure for an interacting theory seems arbitrary to some extent.

Therefore in this thesis, instead using these methods, we employ the full EFT approach for investigating the Poincaré invariance (under boost, in particular) in both NRQCD and pNRQCD: we include all possible terms in the boost generator that are allowed by the other symmetries of the theory (such as P, C, T) and assign a generic coefficient to each of them. Even though we start with the most general expression, we will exploit the possibility to redefine the effective fields in order to remove redundant terms from this ansatz. Since the boost generator for the field transformation has to satisfy the Poincaré algebra, we will also demonstrate how the usual commutation relations have to be implemented in the case of non-linear boost generators. Requiring all commutators of the Poincaré algebra to be satisfied will lead to additional constraints on the coefficients of the terms in the boost generator as well as the Wilson coefficients. In the end, applying the transformation by the constructed boost generator to the theory, we obtain some non-trivial relations between the Wilson coefficients; in pNRQCD, such derived relations from the Poincaré invariance are between the heavy quark potentials.

In the weakly-coupled regime of QCD, one can calculate the analytic expressions of the heavy quark potentials in a perturbative way [26–30], but at the long-distance scale (strongly-coupled case), determination of the potentials in a perturbative fashion is impossible due to color confinement [31]. Color confinement in QCD is one of the greatest challenges in the modern particle physics community. Around and below the hadronic scale $\Lambda_{\text{QCD}} \sim 200$ MeV, the conventional perturbative approach in non-Abelian gauge theory for describing color interactions between quarks and gluons is no longer a feasible theoretical framework, because the expansion parameter α_S exceeds the weak coupling limit. This manifests in experiments, such that only composite forms of particles (mesons and baryons) are detected via jets instead of singular quarks or gluons, due to hadronization; i.e., the detected particles are color-neutral objects.

Since the realization of color confinement in QCD, it was proposed that the dynamics of quark-antiquark bound states at long distance can be described by a flux tube model [32], in which the quark and antiquark is connected by an open string. The attractive force between the pair increases as the separation distance increases (for which the length scale is greater than the confining distance), thereby forming a flux in the shape of a tube due to the increase of the energy density between them. This suggestion has been verified by lattice QCD simulations [33–43]. The heavy quark and antiquark in this formulation are treated as static objects, while the gluonic interaction between the pair is described by vibrational modes of the string. Since the two ends of the string are fixed at the position of the pair, only the transversal modes of the string act as the dynamical degrees of freedom. Several years after Nambu's suggestion of a flux tube model, Kogut and Parisi extrapolated this idea further, such that the shape of the spin-spin interaction part of the potential was explicitly shown [44], which was also confirmed by lattice simulations [45–49].

In this line of investigation, a significant progress concerning the analysis of the long-distance heavy quark potential has been made during the last few decades. Potential terms of the heavy quark-antiquark bound state in the static limit were shown to be equivalent to the Wilson loop expectation value (and the gauge field insertions therein) via matching calculation between NRQCD and pNRQCD [50, 51]. Based on this result as well as on the *Wilson loop-string partition function equivalence conjecture* [52–55], a few other heavy quark potentials were directly computed through the effective string picture [56]. Recently, Brambilla et. al. have calculated all of the heavy quark potentials up to leading order (LO) of the effective string theory (EST) power counting [57]. Full summation of the heavy quark potential was compared to lattice simulations in order to constrain some of the parameters, which arise from the effective string picture itself. As it was pointed out there, however, this leading order calculation is not fully inclusive because some of the terms from next-to-leading order (NLO) calculation might be of the same order as the leading order terms of the EST. In other words, some terms arising in the EST calculation at NLO can alter the leading order coefficients of the potentials. It is, therefore, necessary for us to employ the proper EFT systematics of the string picture, so that not only the higher order suppression terms are understood, but all of the missing terms of LO can also be acquired. Furthermore, a recent comparison between the analytic result of the potentials at long-distance via EST and LQCD data was presented in [58], but the discrepancy is not negligible. We estimate that the subleading contributions to the potential will improve this discrepancy, and this is one of the major subjects of the thesis.

Therefore, for the detailed discussions of the Poincaré invariance in low-energy EFTs of QCD and long-distance heavy quark potentials, this thesis is organized as follows: in Chapter 1, we discuss basic principles of spacetime symmetries in classical physics, quantum mechanics, and quantum field theory. In particular, we discuss Wigner's induced representation [23], from which one can derive a boost transformation of an effective field in a non-linear fashion. We pose some issues and challenges in employing the induced representation in applying to the case of an EFT of an interacting gauge theory. In Chapter 2, we discuss basic features of QCD, including degrees of freedom and asymptotic freedom [59–61], and then, HQET and NRQCD are introduced. Symmetries of these two EFTs are discussed, Poincaré invariance in particular. As for NRQCD, we present a full EFT approach for constructing a suitable boost generator, and using the Poincaré algebra condition between two boost generators and the Poincaré invariance of the NRQCD Lagrangian, we obtain not only the constraints between the Wilson coefficients, but also constraints on the generic coefficients of the boost generator. In Chapter 3, we apply the same EFT approach to construct boost generators for both singlet and octet fields, and we use the field redefinitions to eliminate some redundant terms of the boost. From the same approach as in the NRQCD case, we obtain some non-trivial relations between the Wilson coefficients. In Chapter 4, we briefly discuss the matching procedure between NRQCD and pNRQCD, as well as some general features of the heavy quark potentials in both weakly-coupled and strongly-coupled regimes of pNRQCD. In order to analyze the potentials in long-distance regime, we introduce the EST in Chapter 5. Here, we derive the Nambu-Goto action from a minimal area law spanned by a string, from which we obtain the EST action by imposing some physical conditions. Introducing the QCD-to-EST mapping, we calculate the long-distance heavy quark potentials in LO and NLO in the EST power counting. Due to the Poincaré invariance of QCD, which gives some non-trivial relations between potentials, leads to constraints between several parameters arising from the EST calculations. Finally, we compare the simplified expressions of the potentials to the available LQCD data. Conclusion and outlook are presented in Chapter 6. In Appendix A, detailed derivation of constraints between the Wilson coefficients at NLO in the four-fermion sector and the calculation of gauge field insertions to the Wilson loop expectation value are presented.

Chapter 1

Spacetime symmetries

In this chapter, we briefly discuss spacetime symmetries of classical physics, quantum mechanics, and quantum field theory. In addition, we show the transformations of a non-relativistic effective field under the Poincaré group in both non-interacting and interacting cases.

1.1 Spacetime symmetries in classical physics

In this section, we briefly discuss the development in our understanding of the symmetries in space and time, from Galilean to general coordinate invariance in classical physics.

1.1.1 Galilean invariance

During the Newtonian mechanics era, space and time were perceived as two separate entities of the nature, Time was considered a parameter for the spatial trajectory of an object, while the spatial trajectory does not affect the flow of the time parameter. For instance, consider two inertial reference frames S and S', where S uses the coordinate[1] (t, x, y, z) and S' uses (t', x', y', z'). If both frames are moving along the x and x' axis, respective, then two of these frames are related by the following set of equations

$$\begin{aligned} t' &= t\,, \\ x' &= x - vt\,, \\ y' &= y\,, \\ z' &= z\,, \end{aligned} \tag{1.1}$$

where v is the relative velocity between these frames. The first relation, $t' = t$ shows the universality of time, independent of the relative motion of the frame. One can put this in a more general setting. The line element, which measures the distance in

[1] We use Cartesian coordinates as a simple example.

three-dimensional space, of the Newtonian system is given by

$$ds^2 = \sum_{i=1}^{3} dx_i^2 \,, \tag{1.2}$$

and one can easily see that the line element is independent of the spatial coordinate system of the reference frame being used. This is called the *Galilean invariance*. No matter on which inertial reference we are at, the distance measure does not change.

The symmetry mentioned in the previous paragraph is the so-called invariance under *boosts*. There are two more components of the Galilean symmetries: translations and rotations. All these transformations are mathematically written as

$$(t, \mathbf{x}) \to (t, \mathbf{x} + \mathbf{v}t) \,, \tag{1.3}$$
$$(t, \mathbf{x}) \to (t + a, \mathbf{x} + \mathbf{b}) \,, \tag{1.4}$$
$$(t, \mathbf{x}) \to (t, \mathcal{R}\mathbf{x}) \,, \tag{1.5}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{v} \in \mathbb{R}^3$, $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^3$, and $\mathcal{R}$ is the rotation operator, which maps a three-vector to another three-vector: $\mathcal{R} : \mathbb{R}^3 \to \mathbb{R}^3$. It is important to note that the equations of motion from the action principle of classical mechanics are invariant under these transformations.

In a group theoretic term (Lie group), the Galilean group has ten dimensions: there are three-dimensional rotations, three-dimensional boosts, and three-dimensional spatial translations as well as one-dimensional time translation. Each of these group transformation is caused by generators of group. As we denote H a generator of time translation, $\mathbf{P}$ a generator of spatial translations, $\mathbf{J}$ a generator of spatial rotations, and $\mathbf{K}$ a generator of boosts, a set of commutation relations hold between the generators

$$[H, H] = 0 \,, \tag{1.6}$$
$$[H, \mathbf{P}_i] = 0 \,, \tag{1.7}$$
$$[\mathbf{P}_i, \mathbf{P}_j] = 0 \,, \tag{1.8}$$
$$[\mathbf{J}_i, H] = 0 \,, \tag{1.9}$$
$$[\mathbf{K}_i, \mathbf{K}_j] = 0 \,, \tag{1.10}$$
$$[\mathbf{J}_i, \mathbf{J}_j] = i\epsilon_{ijk}\mathbf{J}_k \,, \tag{1.11}$$
$$[\mathbf{J}_i, \mathbf{P}_j] = i\epsilon_{ijk}\mathbf{P}_k \,, \tag{1.12}$$
$$[\mathbf{J}_i, \mathbf{K}_j] = i\epsilon_{ijk}\mathbf{K}_k \,, \tag{1.13}$$
$$[\mathbf{K}_i, H] = -i\mathbf{P}_i \,, \tag{1.14}$$
$$[\mathbf{K}_i, \mathbf{P}_j] = 0 \,, \tag{1.15}$$

where ϵ_{ijk} is a totally antisymmetric rank-3 tensor. In the next section, we will see how these commutation relations of the Lie algebra change in a more general setting of space and time.

1.1.2 Poincaré invariance

At the turn of twentieth century, our perception of space and time encountered a great revolution. Until the time of Hendrik Lorentz, Albert Einstein, and Henri Poincaré, time was considered an absolute entity, which is not affected by any other variables of space. This is well represented by the example of Eq. (1.1). However, this was only valid for the case of the relative velocity v much smaller than the speed of light c. This realization was achieved through the classical theory of electromagnetism. As people tried to measure the speed of light relative to the presumed substance in the atmosphere [62], the so-called "luminiferous aether", such substance was not found. In other words, the notion of the relative speed does not apply to the speed of light. Thus based upon the mathematical framework of Henri Poincaré's Lorentz transformation [63], Einstein postulated [64] two principles: (i) the speed of light in vacuum c is the same for all observers, regardless of the motion of the light source, and (ii) the laws of physics are invariant in all inertial systems (i.e., non-accelerating frames of reference). These two postulates negate the presumed notion of "absolute frame", which was the predominant concept before the relativity era by Einstein.

Suppose there are two reference frames S which has coordinates (t, x, y, z) and S' with coordinates (t', x', y', z'). If the relative speed between these two frames (along the x- and x'-axis) is v, then the relation between the coordinates is the modified expresstion of Eq. (1.1)

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) , \\ x' &= \gamma\left(x - vt\right) , \\ y' &= y , \\ z' &= z , \end{aligned} \tag{1.16}$$

where the Lorentz factor γ is given by

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} . \tag{1.17}$$

One can reproduce Eq. (1.1) from Eq. (1.16) by taking the limit $c \to \infty$, which is the case for a non-relativistic relative speed between the frames, where $v \ll c$. The remarkable implication of the Lorentz transformation is the fact that time is no longer an independent parameter. If the relative velocity between two inertial reference frames is relativistic, $v \lesssim c$, the relation between two different time frames becomes non-trivial. Furthermore, as we observe in the expression of the Lorentz factor, there is a speed limit in the relative velocity, otherwise, γ becomes imaginary, and we encounter an issue of causality[2]. The consequence of this special theory of relativity [64] is elucidated by the phenomenon like time dilation, length contraction, and simultaneity[3].

[2] In fact, a hypothetical object which exhibits the causality problem is called the *tachyon*.

[3] More detailed discussions on the consequence of the special relativity are found in standard textbooks, such as in [65].

As the notion of absolute time is changed, the definition of distance in space and time changes as well. The line element we discussed in the Galilean symmetry, Eq. (1.2), now incorporates the time element as well, which is given by

$$ds^2 = -c^2 dx_0^2 + \sum_{i=1}^{3} dx_i^2\,, \tag{1.18}$$

where we denote $x_0 = t$. Using the convention $c = 1$ (i.e., the relative velocity is compared to the speed of light by $v < 1$), this can more concisely be written as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu\,, \tag{1.19}$$

where $\eta_{\mu\nu}$ is a diagonal four-by-four matrix with the diagonal entries $(-1, 1, 1, 1)$. This notation elucidates the unification of space and time on the equal footing: we call it a *spacetime*. The matrix (or a symmetric rank-2 tensor) $\eta_{\mu\nu}$ is called the *Minkowski* metric in four-dimensional spacetime. The metric determines the distance and curvature of the four-dimensional Minkowski spacetime; note that, the curvature is in this case zero, meaning that the Minkowski spacetime is flat. In this thesis, we will be using an opposite convention of the metric: η would be the Minkowski metric with *mostly minus* convention; i.e., $(1, -1, -1, -1)$ on the diagonal entries.

In a group theoretic term, the symmetry group of the Lorentz transformations consists of generators of dimension six. The generator for rotations $\mathbf{J}$ has three components, the boost generator $\mathbf{K}$ has three components. Furthermore, there are four additional components generator for spacetime translation, H and $\mathbf{P}$, which are responsible for time and space translations, respectively. The generators for the Lorentz group and the generators for spacetime translations altogether are the generators of the so-called *Poincaré* group, and the generators satisfy the following commutation relations:

$$[H, H] = 0\,, \tag{1.20}$$
$$[H, \mathbf{P}_i] = 0\,, \tag{1.21}$$
$$[\mathbf{P}_i, \mathbf{P}_j] = 0\,, \tag{1.22}$$
$$[\mathbf{J}_i, H] = 0\,, \tag{1.23}$$
$$[\mathbf{K}_i, \mathbf{K}_j] = -i\epsilon_{ijk}\mathbf{J}_k\,, \tag{1.24}$$
$$[\mathbf{J}_i, \mathbf{J}_j] = i\epsilon_{ijk}\mathbf{J}_k\,, \tag{1.25}$$
$$[\mathbf{J}_i, \mathbf{P}_j] = i\epsilon_{ijk}\mathbf{P}_k\,, \tag{1.26}$$
$$[\mathbf{J}_i, \mathbf{K}_j] = i\epsilon_{ijk}\mathbf{K}_k\,, \tag{1.27}$$
$$[\mathbf{K}_i, H] = -i\mathbf{P}_i\,, \tag{1.28}$$
$$[\mathbf{K}_i, \mathbf{P}_j] = -iH\delta_{ij}\,. \tag{1.29}$$

Comparing to the commutations in the Galilean group, Eqs. (1.6) - (1.15), we observe some non-trivial modifications, especially of the boost generators. Compare Eq. (1.10) to Eq. (1.24), and Eq. (1.15) to Eq. (1.29). A successive action of the Lorentz boost, generated by the generator $\mathbf{K}$, does not commute, but it is non-trivially related to the rotation, which is generated by **J**. Also, the last commutation relation, Eq. (1.29), tells us that the actions of boosting the reference frame and spatial translations are no longer independent to each other, but it is now related by the time translation. This is the mathematical description that space and time are no longer independent entities. We will see the commutation relation between the boost generators, Eq. (1.24), again in the next two chapters.

1.1.3 General coordinate invariance

Unification of the space and time with the corresponding symmetries do not end here. Symmetries of the special theory of relativity [64] are only limited to the case of inertial reference frames, and Einstein, after ten years of the special relativity, extended the notion of spacetime to the case of accelerating frames [66], (or in a mathematically speaking, to the differentiable manifold equipped with a generic metric, which is the Riemannian manifold). In his general theory of relativity, the motion of a point particle (or a body) under gravitation is described by a geodesics on the given surface of the geometric background. In other words, the force of gravity is dictated by the shape of the spacetime on which the object is living. This is mathematically described by the much celebrated Einstein equations of motion

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.30}$$

where $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the Ricci tensor and the Ricci scalar, respectively, and $T_{\mu\nu}$ is the stress-energy tensor for any particle or energy being embedded on the given background $g_{\mu\nu}$. Also, G is the Newton's constant, which is also given in Newton's universal law of gravitation. Here, $g_{\mu\nu}$ is the solution of the equations, such that the Ricci tensor and scalars are expressed in terms of this metric. In other words, $g_{\mu\nu}$ is responsible for measuring distance and curvature of the background spacetime, such that the line element is then changed by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu\,. \tag{1.31}$$

Unlike in Eqs. (1.2) and (1.19), the metric tensor is now spacetime dependent; this tells us that the background spacetime features some non-trivial curvature. It is important to notice from Eq. (1.31) that the change of coordinate system, $x \to \tilde{x}$, does not change the line element:

$$\begin{aligned} ds'^2 &= g_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu \\ &= g_{\mu\nu}(\tilde{x})\frac{\partial\tilde{x}^\mu}{\partial x^\alpha}\frac{\partial\tilde{x}^\nu}{\partial x^\beta}dx^\alpha dx^\beta \\ &= g_{\alpha\beta}(x)dx^\alpha dx^\beta = ds^2\,, \end{aligned} \tag{1.32}$$

where the last line is due to the Jacobian map. In other words, the given geometry of spacetime is independent of the choice of coordinates, and this symmetry is called the *general coordinate invariance*. This is the key symmetric feature of the general theory of relativity. We can see this symmetry more vividly from the action principle. As Eq. (1.30) is derived by solving the equations of motion from the action, the so-called Einstein-Hilbert action, we also see that the action itself is invariant under the coordinate transformation (by setting $T_{\mu\nu} = 0$ for vacuum and normalizing the Newton's constant $G = 1$)

$$S_{\mathrm{EH}} = \int d^4x \sqrt{-g}\, \mathcal{R}\,, \tag{1.33}$$

where g is the determinant of the metric $g_{\mu\nu}$. Replacing x by a new coordinate $\tilde{x}$ would not change the action overall.

It is important to note that the relativity in an inertial reference frame is merely a special case (as the name indicates) of the general theory, such that if $g_{\mu\nu} \to \eta_{\mu\nu}$, the Einstein-Hilbert action vanishes, and we recover the expression of the line element as in Eq. (1.19). Furthermore, at the local level of the generic spacetime, one can approximate the curvature to be flat (Minkowski) due to the weak equivalence principle. This can mathematically be utilized by the Riemann normal coordinates, such as in [67, 68].

If there is a small fluctuation around the given spacetime, one can apply a perturbative method to analyze the dynamics of the small fluctuation. Let us call $h_{\mu\nu}$ a small fluctuation around the background $\tilde{g}_{\mu\nu}$, such that $|h_{\mu\nu}| \ll |g_{\mu\nu}|$, then one can expand the action, Eq. (1.33) as the following expression with respect to $h_{\mu\nu}$

$$\begin{aligned} S = \frac{1}{2} \int d^4x \sqrt{-g}\ \Big[&(\nabla_\mu h^{\mu\nu})(\nabla_\nu h) - (\nabla_\mu h^{\rho\sigma})(\nabla_\rho h^\mu_\sigma) \\ &+ \frac{1}{2}(\nabla_\mu h^{\rho\sigma})(\nabla_\nu h_\rho \sigma) - \frac{1}{2} g^{\mu\nu} (\nabla_\mu h)(\nabla_\nu h) \Big]\,, \end{aligned} \tag{1.34}$$

where ∇ is a covariant derivative associated with a curved background $g_{\mu\nu}$. This is the action of the gravitational field on a curved background, and the detection of this fluctuation (the gravitational wave) has recently been announced [69]. The general coordinate invariance of the Einstein-Hilbert action is now translated into the invariance under the field transformation as

$$h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu\,, \tag{1.35}$$

in which ξ_μ is a generic vector field. This is also called the gauge symmetry in field theories. It is interesting to see that the general coordinate invariance at the level of background spacetime is now rewritten in the form of gauge symmetry. If we go into the asymptotic limit of the spacetime, where $g_{\mu\nu} \to \eta_{\mu\nu}$, this symmetry transformation is reminiscent of the gauge symmetry of the electromagnetic field.

1.2 Spacetime symmetries in quantum mechanics

After exploring spacetime symmetries in classical physics, now we discuss symmetries in quantum mechanics. Since we will eventually be discussing symmetries in quantum field theory, it is a useful guide to observe how quantum states transform under the Poincaré transformations. In particular, we briefly discuss Wigner's little group formalism [23].

1.2.1 Poincaré transformations

In quantum mechanics, physical states are represented by vectors $\Phi_{p,\sigma}$ (with momentum p and spin σ) and $\Psi_{p',\sigma'}$ in Hilbert space [9], which satisfies the following conditions

$$\begin{aligned}(\Phi_{p,\sigma},\Psi_{p',\sigma'}) &= (\Psi_{p',\sigma'},\Phi_{p,\sigma})^* \,,\\ (\Phi_{p,\sigma},c_1\Psi_{p'_1,\sigma'_1}+c_2\Psi_{p'_2,\sigma'_2}) &= c_1(\Phi_{p,\sigma},\Psi_{p'_1,\sigma'_1})+c_2(\Phi_{p,\sigma},\Psi_{p'_2,\sigma'_2})\,,\\ (d_1\Phi_{p_1,\sigma_1}+d_2\Phi_{p_2,\sigma_2},\Psi_{p',\sigma'}) &= d_1^*(d_1\Phi_{p_1,\sigma_1},\Psi_{p',\sigma'})+d_2^*(\Phi_{p_2,\sigma_2},\Psi_{p',\sigma'})\,,\end{aligned} \tag{1.36}$$

where $c_1, c_2 \in \mathbb{C}$ and $d_1, d_2 \in \mathbb{C}$. Also, For such state vectors, transformations under Poincaré group are given by a unitary operator[4] $U(\Lambda, a)$ such that

$$(U\Phi_{p,\sigma},U\Psi_{p',\sigma'}) = (\Phi_{p,\sigma},\Psi_{p',\sigma'})\,. \tag{1.37}$$

Here Λ is a Lorentz transformation matrix (which is either boosts or rotations) acting on the coordinates, and a is for translations in spacetime:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu\,. \tag{1.38}$$

For the continuous symmetry transformation like Poincaré transformations, one can take an infinitesimal expansion of this unitary operator by expanding its argument Λ (Lorentz transformations) and a (spacetime translations)

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \quad \text{and} \quad a^\mu = \epsilon^\mu\,. \tag{1.39}$$

Plugging these into $U(\Lambda, a)$, the unitary operator is expanded by

$$U(1+\omega,\epsilon) = 1 + \frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta} - i\epsilon_\rho P^\rho + \mathcal{O}(\omega^2;\epsilon^2)\,, \tag{1.40}$$

where $J^{\alpha\beta}$ and P^μ are operators independent of the parameters ω and ϵ. Since ω is antisymmetric, we find the operator J is antisymmetric as well, $J^{\alpha\beta} = -J^{\beta\alpha}$. This implies that the operator has six independent components. It turns out that three components of the operator are responsible for the rotation $\mathbf{J} = \{J^{23}, J^{31}, J^{12}\}$ and the other three components are boosts $\mathbf{K} = \{J^{01}, J^{02}, J^{03}\}$. Then naturally, the other operator is for spacetime translations, $P^\rho = \{H, P^1, P^2, P^3\}$. We saw in the previous section that these operators observe a set of commutation reltaions:

$$[H,H] = 0\,, \tag{1.41}$$
$$[H,\mathbf{P}_i] = 0\,, \tag{1.42}$$
$$[\mathbf{P}_i,\mathbf{P}_j] = 0\,, \tag{1.43}$$
$$[\mathbf{J}_i,H] = 0\,, \tag{1.44}$$
$$[\mathbf{K}_i,\mathbf{K}_j] = -i\epsilon_{ijk}\mathbf{J}_k\,, \tag{1.45}$$
$$[\mathbf{J}_i,\mathbf{J}_j] = i\epsilon_{ijk}\mathbf{J}_k\,, \tag{1.46}$$

[4]This is due to Wigner's theorem [70].

$$[\mathbf{J}_i, \mathbf{P}_j] = i\epsilon_{ijk}\mathbf{P}_k \,, \tag{1.47}$$
$$[\mathbf{J}_i, \mathbf{K}_j] = i\epsilon_{ijk}\mathbf{K}_k \,, \tag{1.48}$$
$$[\mathbf{K}_i, H] = -i\mathbf{P}_i \,, \tag{1.49}$$
$$[\mathbf{K}_i, \mathbf{P}_j] = -iH\delta_{ij} \,. \tag{1.50}$$

As the momentum operator acts on the state vector Φ we observe $P^\mu \Phi_{p,\sigma} = p^\mu \Phi_{p,\sigma}$, we extrapolate to the case of the unitary operator associated with spacetime translations,

$$U(1,\epsilon)\Phi_{p,\sigma} = e^{-i\epsilon\cdot p}\Phi_{p,\sigma} \,. \tag{1.51}$$

In case of the unitary operator associated with Lorentz transformations Λ (which could be either rotations or boosts), it transforms a momentum eigenstate with momentum p to another state vector with momentum Λp due to the relation

$$\begin{aligned} P^\mu U(\Lambda,0)\Phi_{p,\sigma} &= U(\Lambda,0)U^{-1}(\Lambda,0)P^\mu U(\Lambda,0)\Phi_{p,\sigma} \\ &= U(\Lambda,0)(\Lambda^{-1})_\rho{}^\mu P^\rho \Phi_{p,\sigma} \\ &= \Lambda^\mu{}_\rho p^\rho U(\Lambda,0)\Phi_{p,\sigma} \,, \end{aligned} \tag{1.52}$$

where the second equality is due to the identity

$$U(\Lambda,a)P^\rho U^{-1}(\Lambda,a) = \Lambda_\mu{}^\rho P^\mu \,. \tag{1.53}$$

Thus, one can rewrite this eigenstate as a linear combination of new state vectors

$$U(\Lambda,0)\Phi_{p,\sigma} = \sum_{\sigma'} C_{\sigma',\sigma}(\Lambda,p)\Phi_{\Lambda p,\sigma} \,, \tag{1.54}$$

where the coefficients $C_{\sigma',\sigma} \in \mathbb{C}$.

1.2.2 Little group formalism

Eq. (1.54) is a generic expression for the Lorentz transformations of a one-particle state vector[5]. We can analyze its more concrete expression by choosing a particular reference frame. Suppose we choose a four-momentum vector k^μ of a particle (in particular, k^μ can be a rest frame of the particle), and one can boost this vector into a generic four-momentum vector p^μ by taking a Lorentz transformation

$$p^\mu = L^\mu{}_\nu(p)k^\nu \,, \tag{1.55}$$

where $L(p)$ is the standard Lorentz transformation, which depends on the momentum p to which we are boosting. Then one can express the generic Lorentz transformation of a generic state vector in terms of the standard Lorentz transformation as well as the chosen state vector. As the generic state vector and a chosen one are related by

$$\Phi_{p,\sigma} = U(L(p),0)\Phi_{k,\sigma} \,, \tag{1.56}$$

[5] In particular, there are not much information given for $C_{\sigma,\sigma'}$

the Lorentz transformations of the a generic state vector are given by

$$\begin{aligned} U(\Lambda,0)\Phi_{p,\sigma} &= U(\Lambda,0)U(L(p),0)\Phi_{k,\sigma} \\ &= U(L(\Lambda p),0)U(L^{-1}(\Lambda p)\Lambda L(p),0)\Phi_{k,\sigma} \\ &\equiv U(L(\Lambda p),0)U(W(\Lambda,p),0)\Phi_{k,\sigma}\,, \end{aligned} \tag{1.57}$$

in which

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p)\,, \tag{1.58}$$

is the so-called *little group element.* This maps the chosen vector k to k. From the generic expression of the Lorentz transformations of the states, Eq. (1.54), one can see that the fixed state vector transforms under the little group by

$$U(W,0)\Phi_{k,\sigma} = \sum_{\sigma'} D_{\sigma',\sigma}(W)\Phi_{k,\sigma'}\,, \tag{1.59}$$

where $D_{\sigma',\sigma}(W)$ is the representation of the little group. Then, by inserting Eq. (1.59) into Eq. (1.57), we obtain the finalized expression of the Lorentz transformation of the generic state vector in terms of the representation of the little group

$$U(\Lambda,0)\Phi_{p,\sigma} = \sum_{\sigma'} D_{\sigma',\sigma}(W(\Lambda,p))\Phi_{\Lambda p,\sigma}\,. \tag{1.60}$$

As we compare this to Eq. (1.54), there is a substantial improvement on the right-hand side. The generic notation for the coefficients in Eq. (1.54) is now replaced by the representation of the little group. It is left that one has to find the expression of the representation. This method is called the Wigner's induced representations [23]. We will make use of Eq. (1.60) for the case of quantum field theory, in the next section.

1.3 Spacetime symmeries in quantum field theory

Quantum field theory is built upon Poincaré invariance and cluster decomposition principle. Thus, it is obvious that the field theory would be invariant under Poincaré transformations.

1.3.1 Poincaré transformations

As the action of a generic quantum field theory is symmetric under the Poincaré group, we observe that a (free) quantum field transforms under the Lorentz group as

$$\phi_a(x) \to M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x)\,, \tag{1.61}$$

where $M(\Lambda)$ is a finite dimensional representation of the Lorentz group, and index a is for the representation of the field (scalar, spinor, vector, and so on). This is analogous to the expression in quantum mechanics, Eq. (1.54). Then in the infinitesimal form, the transformation of the field under the Poincaré group is written as

$$\phi \to [1 + i(a_0 h - \mathbf{a}\cdot\mathbf{p} - \boldsymbol{\theta}\cdot\mathbf{j} + \boldsymbol{\eta}\cdot\mathbf{k})]\phi\,, \tag{1.62}$$

in which $a_0, \mathbf{a}, \boldsymbol{\theta}$, and $\boldsymbol{\eta}$ are the infinitesimal parameters of time and spatial translations, rotations and boost, respectively. The generators of the Poincaré group are written by

$$h = i\partial_t\,, \tag{1.63}$$
$$\mathbf{p} = -i\boldsymbol{\nabla}\,, \tag{1.64}$$
$$\mathbf{j} = \mathbf{r}\times\mathbf{p} + \boldsymbol{\Sigma}\,, \tag{1.65}$$
$$\mathbf{k} = \mathbf{r}h - t\mathbf{p} \pm i\boldsymbol{\Sigma}\,, \tag{1.66}$$

where $\boldsymbol{\Sigma}$ is the (2s+1)-dimensional matrix generators of the spin-s representation of the rotations. In this thesis, we work on the Poincaré transformations of the effective field whose mass is heavy. The transformations given above, Eqs. (1.61) and (1.62), are not the most suitable representation in such cases.

1.3.2 Little group transformations

In the field theories with the mass scale M, it is useful to fix a reference frame[6] v (which is time like) such that a corresponding standard Lorentz transformation maps v into a generic vector w:

$$L(w,v)^{\mu}{}_{\nu}v^{\nu} = w^{\mu}\,. \tag{1.67}$$

Then by setting $v = k/M$ and $w = p/M$ (such that $k^2 = p^2 = M^2$), we realize that $L(w,v)$ is a rotation in the plane of $v = k/M$ and $w = p/M$, which is given by [20]

$$\begin{aligned} L(w,v)^{\mu}{}_{\nu} =& g^{\mu}{}_{\nu} - \frac{1}{1+v\cdot w}(w^{\mu}w_{\nu} + v^{\mu}v_{\nu}) + w^{\mu}v_{\nu} - v^{\mu}w_{\nu} \\ &+ \frac{v\cdot w}{1+v\cdot w}(w^{\mu}v_{\nu} + v^{\mu}w_{\nu})\,, \end{aligned} \tag{1.68}$$

$$L_{1/2}(w,v) = \frac{1+\not{w}\not{v}}{\sqrt{2(1+v\cdot w)}}\,, \tag{1.69}$$

in which the first equation is for the vector representation[7], and the second one is the spinor representation. In the effective field theories with heavy mass scale, all the other components of the little group elements are intact (i.e., expressions are identical to the generic case), but the only components that show major change are the boosts. Thus, let us focus on the little group element associated with boosts.

For the infinitesimal boost (with parameter η)

$$\mathcal{B}(v+\eta, v)v = v + \eta\,, \tag{1.70}$$

[6] A rest frame of the particle $v = (1,0,0,0)$ is a usual choice.

[7] Here, the metric g is the Minkowski metric.

the transformed vector shows $1 = v^2 = (v+\eta)^2$ and $v \cdot \eta = \mathcal{O}(\eta^2)$. The concrete expression of the infinitesimal boosts in the case of a vector and a spinor representations are given by

$$\mathcal{B}(\eta)^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} - (v^{\mu}\eta_{\nu} - \eta^{\mu}v_{\nu}) + \mathcal{O}(\eta^2)\,, \tag{1.71}$$

$$\mathcal{B}_{1/2}(\eta) = 1 + \frac{1}{2}\not{\eta}\not{v} + \mathcal{O}(\eta^2)\,, \tag{1.72}$$

and using the definition of the little group element, Eq. (1.58), as well as Eqs. (1.68) and (1.69), we finally obtain the expression of the little group element in the case of the infinitesimal boosts

$$W(\mathcal{B}(\eta), p) = L^{-1}(\mathcal{B}(\eta)p)\mathcal{B}(\eta)L(p) \tag{1.73}$$

$$= 1 + \frac{i}{2}\left[\frac{1}{M + v\cdot p}(\eta^{\alpha}p_{\perp}^{\beta} - p_{\perp}^{\alpha}\eta^{\beta})\mathcal{J}_{\alpha\beta}\right] + \mathcal{O}(\eta^2) \tag{1.74}$$

where $p_{\perp}^{\beta} \equiv p^{\beta} - (v\cdot p)p^{\beta}$, and

$$\mathcal{J}_{1/2}^{\alpha\beta} = \frac{i}{4}[\gamma^{\alpha}, \gamma^{\beta}]\,, \tag{1.75}$$

$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g^{\alpha}{}_{\mu}g^{\beta}{}_{\nu} - g^{\beta}{}_{\mu}g^{\alpha}{}_{\nu})\,. \tag{1.76}$$

We can use this expression of the little group for the field transformation under the boosts

$$\phi_a \to \left[1 + i\boldsymbol{\eta}\cdot\left(i\mathbf{r}\partial_0 + it\boldsymbol{\partial} \pm i\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\right)\right]_{ab}\phi_b\,, \tag{1.77}$$

in which the rest frame $v = (1,0,0,0)$ is chosen. From this it is clear that the boost generator is

$$\mathbf{k} = i\mathbf{r}\partial_0 + it\boldsymbol{\partial} \pm i\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\,. \tag{1.78}$$

Comparing to the generic expression of the boost generator, Eq. (1.66), now we have a non-linear expression. These two are supposed to deliver the same results, but the difference is that we are now referring to a particular reference frame $v = (1,0,0,0)$, whereas Eq. (1.66) is a generic expression. This is the method of the induced representation [23] in quantum field theory.

1.4 Spacetime symmetries in effective field theory

In this section, we want to apply the little group transformations to effective field theories (EFTs). First we briefly introduce the concept of EFTs and show how the non-linear transformations under the Poincaré group (especially boosts) are implemented in the case of EFTs involving heavy mass scale M. Also, we discuss the transformations of the interacting fields, in which the covariantization procedure is non-trivial.

1.4.1 Effective field theory

Effective field theory (EFT) has been a standard tool for studying problems involving multiple widely-separated energy scales in particle and nuclear (or even gravitational) physics during the last decades. In particle physics, EFT is a modern technique for investigating physics above the electroweak scale, namely the Beyond the Standard Model (BSM) or even Dark Matter (DM) theories. Up until now, there does not exist any well-established underlying theory to compare with the EFT for the parametrization of physics BSM at the currently available energy scale ($\sim$ 1TeV) of the Large Hadron Collider. In hadronic and nuclear physics, on the other hand, an EFT framework plays a pivotal role in the understanding of the lower-energy regime. In this case, the corresponding EFTs can be matched to the underlying theory, Quantum Chromodynamics (QCD), at a given energy scale. Since in the construction of an EFT one is not bound to only renormalizable operators, the number of free parameters increases rapidly when going to higher orders in the expansion.

The Lagrangian of an EFT is given as an expansion in the heavy scale that has been integrated out, which is M. It contains all terms allowed by the symmetries of the EFT (such as charge, parity, time reversal, and gauge invariance) and can schematically be expressed as

$$\mathcal{L}_{\rm EFT} = \sum_n c_n \frac{\mathcal{O}_n}{M^{d_n-4}}\,, \tag{1.79}$$

where the operators $\mathcal{O}_n$, made up by the fields that describe the effective degrees of freedom, are of mass dimension d_n, and the c_n are scalar functions, called *matching* or *Wilson coefficients*, that have zero mass dimension. These coefficients contain all information from the higher-energy scale through the *matching* to the underlying theory. In other words, the high-energy information (at the energy scale above M) is factorized from the low-energy ones[8] (at energy scale μ which is much below the heavy scale M) by this expansion.

Such expansion is most useful in the theories of quarks and hadrons, QCD, as there are six flavors of quarks with different mass scales as well as the hadronic scale $\Lambda_{\rm QCD}$. In this thesis, we focus on non-relativistic EFTs of QCD. There exists a wide variety of EFTs of QCD depending on the physical processes, and among them we concentrate on Heavy Quark Effective Theory (HQET), non-relativistic QCD (NRQCD), and potential NRQCD (pNRQCD). While HQET [1–4] is a low-energy EFT for heavy-light mesons, NRQCD [5, 6] provides a non-relativistic effective description for the dynamics of heavy quarks and antiquarks, and pNRQCD [7, 8] is an effective theory for heavy quark-antiquark bound states (heavy quarkonium). In these EFTs, a hierarchy of scales is assumed such that the mass M of the heavy (anti)quark is much larger than any other relevant energy scale including $\Lambda_{\rm QCD}$, which is the scale at which confinement takes place.

Keeping these applications in mind, let us first discuss how a free effective field would transform under the Poincaré transformations.

[8] The dynamics at low-energy is encoded in the effective degrees of freedom.

1.4.2 Poincaré symmetries in effective field theory

Free theories

In the non-interacting effective field theories with heavy mass scale M, one can directly apply the little group transformations by expanding Eq. (1.78) with respect to $\mathbf{p}/M \ll 1$. As we take the non-relativistic normalization of the free field ϕ

$$\phi''_a = \left(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\right)^{-1/4} e^{iMt}\phi_a\,, \tag{1.80}$$

then the normalized field ϕ'' transforms under the boost, Eq (1.78), as

$$\phi''_a(x) \to \Bigg\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}}{2M} + \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}\boldsymbol{\partial}^2}{4M^3} + \frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{\partial}}{2M}\left[1 + \frac{\boldsymbol{\partial}^2}{4M^2} + \mathcal{O}(1/M^4)\right]\Bigg\}\,\phi''_a(\mathcal{B}^{-1}x)\,. \tag{1.81}$$

This is the boost transformations of the non-interacting and non-relativistic field[9]. However, in order to apply this transformation to the non-relativistic EFTs, we need a corresponding expression for the interacting case.

Interacting theories

For interacting theories, one might think that replacing partial derivatives with gauge covariant derivatives (preserving the gauge symmetry) would be sufficient [24]

$$\phi''_a(x) \to \Bigg\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{D}}{2M} + \frac{i\boldsymbol{\eta}\cdot\boldsymbol{D}\boldsymbol{D}^2}{4M^3} + \frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{D}}{2M}\left[1 + \frac{\boldsymbol{D}^2}{4M^2} + \mathcal{O}(g;1/M^4)\right]\Bigg\}\,\phi''_a(\mathcal{B}^{-1}x)\,. \tag{1.82}$$

However, this implementation introduces some ambiguity in how the covariant derivatives should be ordered (in a non-Abelian gauge theory like QCD). It is also necessary to introduce additional gauge field dependent operators to the boost generator in order to cancel some terms that would prevent the corresponding EFT action from being invariant. Once the transformation is covariantized with respect to the gauge group, then there has to be a generic coefficient to each term involving a gauge field. In other words, a direct correspondence to the Wigner's induced representation for the Poincaré transformations is only limited to the case of a non-interacting field theory. In the interacting theories, some modifications to such expression, Eq. (1.82), is inevitable in order to cope with the Poincaré invariance of the corresponding action. In the next two chapters, we want to address these issues by employing the most generalized EFT approach to derive the proper implementation of the Poincaré transformations in low-energy EFTs of QCD. Before proceeding, we briefly discuss the Wilson coefficients that appear in the EFT actions.

[9] Rotations as well as the spacetime translations are identical to the relativistic field theories.

Wilson coefficients

Wilson coefficients of the EFTs are the scalar functions (with zero mass dimension), which contain information from the higher-enery scale. They have to be determined through a matching calculation to the underlying theory. Beyond leading order in the coupling or the expansion parameter, the matching calculation can easily become technically involved, and even more so for a theory with more than one expansion parameter (such as pNRQCD). For this reason, one would like to exploit as much prior knowledge on the Wilson coefficients as possible before commencing the calculation.

Also, Poincaré invariance is no longer manifest in the actions of the non-relativistic EFTs, as it is schematically shown in Eq. (1.79). Before quantum field theory was fully established, Poincaré symmetry in a quantum mechanical system was already under discussion during the 1940's [10]. Dirac showed that the terms of the quantum mechanical Hamiltonian satisfy non-trivial relations if one imposes the Poincaré algebra. This discussion was further extrapolated and developed for interacting relativistic composite systems [11–15], where relations between the relativistic correction terms were derived using the Poincaré algebra. As these quantum mechanical systems can be generalized by the framework of EFTs, it is natural to expect that also some non-trivial relations between the Wilson coefficients of non-relativistic EFTs can be deduced in a systematic way from Poincaré invariance. It is well justified to assume this invariance, as the EFT is equivalent to the low-energy limit of the underlying relativistic quantum field theory, which is symmetric under Poincaré transformations.

In the next chapter, we investigate the Poincaré transformations (boosts, in particular) of the non-relativistic effective fields of QCD with employing a full EFT approach, where one includes all possible terms in the boost transformation that are allowed by the other symmetries of the theory (such as charge, parity, and time reversal) and gives a generic coefficient to each of them.

Chapter 2

Spacetime symmetries in effective theories of QCD

In this chapter, we discuss effective field theories of QCD involving heavy quarks, which are heavy quark effective theory (HQET) and non-relativistic QCD (NRQCD). Reparametrization in HQET is briefly discussed, and Poincaré transformation of the heavy quark field of NRQCD is discussed in detail. From these symmetry transformations, we obtain some non-trivial constraining equations between the Wilson coefficients.

2.1 Basics of QCD at high-energy

In the Standard Model of particle physics, Quantum Chromodynamics (QCD) is a relativistic quantum field theory of color interactions between quarks and gluons, which constitute composite particles such as protons and neutrons. Within the gauge group of SU(3)×SU(2)× U(1) in the Standard Model, QCD is the non-abelian gauge theory with the gauge group SU(3). In this section, we discuss some basic features of QCD in the high-energy regime.

2.1.1 Degrees of freedom

QCD Lagrangian is given by the quark sector and a gluon sector, which are represented by the Dirac and the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} \,, \tag{2.1}$$

where the summation index f stands for the flavors of the quark (thus, m_f is the quark mass with flavor f) and the covariant derivative acting on the quark field ψ is defined by

$$D_\mu \psi = \partial_\mu \psi - igA^a_\mu T^a \psi \,, \tag{2.2}$$

and the field strength term is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,. \tag{2.3}$$

Also, T^a is the generator of the SU(3) gauge group, which satisfies the commutation relation

$$[T^a, T^a] = i f^{abc} T_c \,, \tag{2.4}$$

where f^{abc} is a structure constant[1] of SU(3) group. As the relativistic quantum field theory is built upon the Poincaré symmetry, thus, $\mathcal{L}_{\rm QCD}$ is manifestly invariant under the Poincaré transformations.

2.1.2 Asymptotic freedom

One of the most important features of QCD is the asymptotic freedom [59–61]. At one-loop level, the running coupling parameter α_S is given by solving the differential equation

$$\mu^2 \frac{d\alpha_S}{d\mu^2} = -\frac{1}{4\pi}\left(11 - \frac{2N_f}{3}\right)\alpha_S^2 \tag{2.5}$$

where N_f is the number of quark flavors. By solving this renormalization equation, we obtain expression of the running coupling in terms of the energy scale μ at one loop

$$\alpha_S(\mu) = \frac{12\pi}{(33 - 2N_f)\ln(\mu^2/\Lambda^2_{\rm QCD})} \,, \tag{2.6}$$

where $\Lambda_{\rm QCD}$ is the hadronic scale in QCD. This implies that if $N_f < 17$, the strength of the coupling decrease when the scale μ increases, and the perturbative description is more accurate. This feature is called the asymptotic freedom. The running coupling behavior is shown[2] in Fig. 2.1.

On the other hand, when the scale reaches near $\Lambda_{\rm QCD}$, the coupling α_S becomes close to unity or exceeds the weak-coupling limit, and the perturbative description breaks down. This is called the color confinement in QCD [31]. We discuss more about the color confinement in the chapter of heavy quark potentials in the non-perturbative regime.

2.2 Heavy quark effective theory

Heavy quark effective theory (HQET) [1–4] is a theory involving a heavy quark and a light quark, which constitute hadrons such as B mesons. The heavy quark mass M is much greater than the hadronic scale $\Lambda_{\rm QCD}$ and the light quark masses, while the momentum between the heavy and light quarks as well as the gluon scales around at $\Lambda_{\rm QCD}$. This implies that HQET is obtained by an expansion in the powers of $\Lambda_{\rm QCD}/M$.

[1] This structure constant is totally antisymmetric.

[2] Q in FIG. 2.1 corresponds to μ in our expressions.

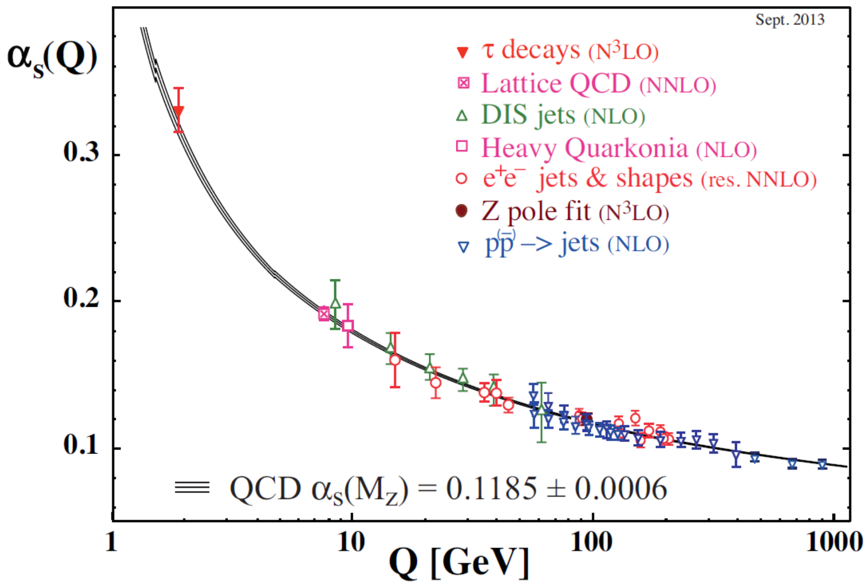


Figure 2.1: Running coupling parameter α_S [71].

2.2.1 Convention

For the rest of this chapter, we use the convention

$$D_0 = \partial_0 + igA_0\,, \quad \text{and} \quad \boldsymbol{D} = \boldsymbol{\nabla} - ig\boldsymbol{A}\,, \tag{2.7}$$

for the sign of the coupling constant g in the covariant derivatives, from which one obtains the chromoelectric and chromomagnetic fields as

$$\boldsymbol{E} = \frac{1}{ig}\,[D_0, \boldsymbol{D}]\;, \quad \text{and} \quad \boldsymbol{B} = \frac{i}{2g}\,\{\boldsymbol{D}\times, \boldsymbol{D}\}\;. \tag{2.8}$$

And $\boldsymbol{\sigma}$ denotes the three Pauli matrices written together as a vector. The commutator or anticommutator with a cross product is defined as

$$\big[\boldsymbol{X}\times, \boldsymbol{Y}\big] = \boldsymbol{X}\times\boldsymbol{Y} - \boldsymbol{Y}\times\boldsymbol{X} \qquad \text{and} \qquad \big\{\boldsymbol{X}\times, \boldsymbol{Y}\big\} = \boldsymbol{X}\times\boldsymbol{Y} + \boldsymbol{Y}\times\boldsymbol{X}\,, \tag{2.9}$$

and equivalently for the dot product[3].

2.2.2 HQET Lagrangian

HQET Lagrangian is derived by integrating out the heavy quark mass scale from the QCD Lagrangian. As a heavy quark Q interacts with light degrees of freedom at Λ_{QCD},

[3]Because of the antisymmetry of the cross product, the roles of commutator and anticommutator are actually reversed: $\{\boldsymbol{X}\times, \boldsymbol{Y}\}_i = \epsilon_{ijk}[X_j, Y_k]$ and $[\boldsymbol{X}\times, \boldsymbol{Y}]_i = \epsilon_{ijk}\{X_j, Y_k\}$.

terms are expanded with respect to $\Lambda_{\rm QCD}/M$. The four-momentum of the heavy quark is parametrized by

$$p^\mu = Mv^\mu + k^\mu \,, \tag{2.10}$$

where v^μ is the light-like ($v^2 = 1$) four-vector, which represents the reference frame of the heavy particle, and k^μ is the residual momentum at the scale of $\Lambda_{\rm QCD}$. The residual momentum is due to the interaction between the heavy particle and the light degrees of freedom. We redefine the heavy quark field by

$$\psi(x) = e^{-iMv\cdot x}[h_v(x) + H_v(x)] \,, \tag{2.11}$$

where

$$h_v(x) = e^{iMv\cdot x}\frac{1+\not{v}}{2}\psi(x) \,, \quad \text{and} \quad H_v(x) = e^{iMv\cdot x}\frac{1-\not{v}}{2}\psi(x) \,, \tag{2.12}$$

in which h_v represents a heavy particle field, and H_v is a heavy anti-particle field; i.e., the projection operator $(1\pm\not{v})/2$ projects out the particle and anti-particle components, respectively, and the following properties hold:

$$\not{v}h_v = h_v \,, \quad \text{and} \quad \not{v}H_v = -\not{v}H_v \,. \tag{2.13}$$

As we insert Eq. (2.12) into the QCD Lagrangian, we obtain

$$\mathcal{L}_{\rm HQET} = \overline{h}_v \, iv\cdot D h_v - \overline{H}_v(iv\cdot + 2M)H_v + \overline{h}_v \, i\not{D}_\perp H_v + \overline{H}_v i\not{D}_\perp h_v \,, \tag{2.14}$$

where the orthogonal derivative is defined by

$$D_\perp^\mu \equiv D^\mu - v^\mu(v\cdot D) \,. \tag{2.15}$$

As we solve the equations of motion of H_v

$$H_v = \frac{1}{iv\cdot D + 2M} i\not{D}_\perp h_v \,, \tag{2.16}$$

which is suppressed by $\Lambda_{\rm QCD}/M$. Thus, as we insert this back into the Lagrangian, Eq. (2.14) and expanding the terms up to $1/M$, we obtain the Lagrangian of the heavy quark sector up to $1/M$ (h on the superscript denotes the heavy quark sector)

$$\mathcal{L}^h_{\rm HQET} = \overline{h}_v \left(iv\cdot D - \frac{\not{D}_\perp \not{D}_\perp}{2M} + \mathcal{O}(1/M^2) \right) h_v \,, \tag{2.17}$$

which is valid at tree level. In fact, one can rewrite the $1/M$ term using the Clifford algebra

$$\not{D}_\perp \not{D}_\perp = D_\perp^2 + \frac{\sigma_{\mu\nu} g F_\perp^{\mu\nu}}{2} \,, \tag{2.18}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\,. \tag{2.19}$$

Furthermore, if we specify the heavy particle reference frame as the rest frame $v = (1, 0, 0, 0)$, the HQET Lagrangian is given by

$$\mathcal{L}^h_{\text{HQET}} = \overline{h}\left(iD_0 + \frac{\boldsymbol{D}^2}{2M} + \frac{\boldsymbol{S}\cdot g\boldsymbol{B}}{M} + \mathcal{O}(1/M^2)\right)h_v\,, \tag{2.20}$$

We observe that the second operator is responsible for the kinetic energy of the recoiling heavy quark and the second term is the spin-magnetic interaction part (where $\mathbf{S} = \boldsymbol{\sigma}/2$). In a similar fashion, we insert Eq. (2.16) into the Lagrangian and expanding it up to order $1/M^2$, and choosing the reference frame $v = (1, 0, 0, 0)$ we obtain the heavy quark sector of the HQET Lagrangian up to $1/M^2$.

While Eq. (2.20) is the expression only at tree level, if we include the loop corrections from the matching to QCD, the HQET Lagrangian up to $1/M^2$ is derived as follows [25]:

$$\begin{aligned}\mathcal{L}_{\text{HQET}} =& \overline{h}_v\Big[iD_0 - \frac{c_2}{2M}\boldsymbol{D}^2 - \frac{c_F}{2M}g\boldsymbol{B}\cdot\boldsymbol{\sigma} \\ & - \frac{c_D}{8M^2}[\boldsymbol{D}\cdot, g\boldsymbol{E}] + \frac{ic_S}{8M^2}[\boldsymbol{D}\times, g\boldsymbol{E}]\cdot\boldsymbol{\sigma} + \mathcal{O}(1/M^3)\Big]h_v \\ & + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{light}}\end{aligned} \tag{2.21}$$

where c_2, c_F, c_D, and c_S are the Wilson coefficients to be matched to the underlying theory, and the Yang-Mills sector is [72–74]

$$\mathcal{L}_{\text{YM}} = -\frac{d_1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \frac{d_2}{M^2}F^a_{\mu\nu}D^2F^{\mu\nu,a} + \frac{d_3}{M^2}gf_{abc}F^a_{\mu\nu}F^b_{\mu\alpha}F^c_{\nu\alpha} + \mathcal{O}(1/M^4)\,. \tag{2.22}$$

d_1, d_2, and d_3 are also the Wilson coefficients. The light quark sector $\mathcal{L}_{\text{light}}$ contains $\sum_l \bar{q}_l i\not{D} q_l$ (where q_l is the light quark field with the flavor index l) and $1/M$ suppressed corrections, but these are highly suppressed contributions, so we neglect these corrections here.

2.2.3 Heavy quark symmetry

One can see from the heavy quark sector of the HQET Lagrangian, Eq. (2.21), if one takes the limit $M \to \infty$, the Lagrangian would only contain the first term

$$\lim_{M\to\infty}[\mathcal{L}_{\text{HQET}}] = \overline{h}_v iD_0 h_v\,. \tag{2.23}$$

This implies that all flavors the heavy quarks behave the same way as far as only strong interactions are concerned, and this symmetry is called *heavy quark symmetry* [75]. Furthermore, there is also spin symmetry here because at this limit, there is no spin-dependent terms like $\frac{c_F}{2M}g\boldsymbol{B}\cdot\boldsymbol{\sigma}$ in Eq. (2.21). In other words, the $U(N_f)$ symmetry (where N_f is the number of the heavy quark flavor) along with SU(2) symmetry for the spin are embedded in the $U(2N_f)$ group, and this is called the *heavy quark spin symmetry.*

2.2.4 Reparametrization invariance

As it is shown in Eq. (2.10), we parametrize the heavy quark momentum in terms of the reference frame of the particle v^μ and the residual momentum k^μ. If one chooses v^μ a particular frame (for instance, $v = (1, 0, 0, 0)$ as we have done in deriving Eq. (2.21)), it might seem that the Poincaré invariance is violated because there is a preferred inertial reference frame. However, the choice of the reference frame v^μ is arbitrary up to redefinitions of order $\Lambda_{\rm QCD}/M$. One can explicitly show this by reparametrizing the reference frame as well as the residual momentum

$$v^\mu \to v'^\mu = v^\mu + \frac{q^\mu}{M}\,, \quad \text{and} \quad k^\mu \to k'^\mu = k^\mu - q^\mu\,, \tag{2.24}$$

where q is of order $\Lambda_{\rm QCD}$. Square of the light-like reference frame $v'^2 = 1$ requires $v \cdot q = 0$. Under this reparametrization, the heavy quark field h_v transforms into h'_v, so does the corresponding Lagrangian. As the heavy quark field with respect to the new reference frame v' has to preserve the property in Eq. (2.13), we want to express $h_{v'}$ in terms of the field with respect to the old reference frame h_v. Denoting $h_{v'} = h_v + \delta h_v$ (where δh_v is counted as $\mathcal{O}(1/M)$ suppression), Eq. (2.13) is rewritten with using Eq. (2.24)

$$\left(\not{v} + \frac{\not{q}}{M}\right)(h_v + \delta h_v) = h_v + \delta h_v\,, \tag{2.25}$$

and at $\mathcal{O}(1/M)$, following relation holds

$$(1 - \not{v})\,\delta h_v = \frac{\not{q}}{M} h_v\,. \tag{2.26}$$

One can choose δh_v as

$$\delta h_v = \frac{\not{q}}{2M} h_v\,. \tag{2.27}$$

Overall, the heavy quark field transforms by

$$h_v \to e^{iq\cdot x}\left(1 + \frac{\not{q}}{2M}\right) h_v\,, \tag{2.28}$$

in which the factor $e^{iq\cdot x}$ is for the reparametrization of the residual momentum, $k^\mu \to k^\mu - q^\mu$. When we apply Eqs. (2.24) and Eq. (2.28), to the Lagrangian

$$\mathcal{L}^h_{\rm HQET} = \overline{h}_v\left(iv\cdot D - c_2\frac{D_\perp^2}{2M} - c_F\frac{\sigma_{\alpha\beta} g F^{\alpha\beta}}{4M} + \mathcal{O}(1/M^2)\right) h_v\,, \tag{2.29}$$

it transforms to

$$(\mathcal{L}^h_{\rm HQET})' = \mathcal{L}^h_{\rm HQET} + (1 - c_2)\overline{h}_v i\frac{q\cdot D}{M} h_v + \mathcal{O}(1/M^2)\,. \tag{2.30}$$

As the Lagrangian is to be invariant under the reparametrization, the Wilson coefficient c_2 has to be unity. This is called the *reparametrization invaraince* [20, 21]. One can implement these transformations, Eqs. (2.24) and (2.28), at higher orders of the $1/M^2$ expansion in the HQET Lagrangian, and the invariance gives another constraint on the Wilson coefficients [25]:

$$c_S = 2c_F - 1\,, \tag{2.31}$$

in which c_S and c_F appear in Eq. (2.21).

One can derive these constraining equations between the Wilson coefficients by using the Poincaré invariance. This is not surprising, since they are closely related: a shift in the parametrization of the momentum may be interpreted just as well as a change of the reference frame. Whereas the implementation of reparametrization invariance might be more straightforward in calculations, its applications are limited only to the theories which can be parametrized like in Eq. (2.24). On the other hand, Poincaré invariance is a general principle that all quantum field theories have to obey. We will discuss the implementation of the Poincaré invariance in NRQCD in the next section.

2.3 Non-relativistic QCD

Non-relativistic QCD (NRQCD) is an effective theory of QCD involving a heavy quark and a heavy antiquark, which constitute a heavy meson like charmonium or bottomonium. [5, 6]. There are four relevant scales in NRQCD: heavy (anti-)quark mass M, relative momentum between a heavy quark and a heavy antiquark $p \sim Mv$ (where v is the relative velocity), relative kinetic energy Mv^2, and the hadronic scale Λ_{QCD}. As the scale M is heavy, the relative velocity between the quark and the antiquark is non-relativistic, so the hierarchy of scales in NRQCD is given by $M \gg Mv \gg Mv^2$, Λ_{QCD}.

2.3.1 NRQCD Lagrangian

NRQCD Lagrangian is obtained from QCD after integrating out the scale of the heavy quark mass M [5, 6]. The effective degrees of freedom are non-relativistic Pauli spinor fields ψ and χ, where ψ annihilates a heavy quark and χ creates a heavy antiquark, as well as gluon fields A_μ and light quark fields q_l with four-momenta constrained to take values much smaller than M. Its Lagrangian up to $\mathcal{O}\left(M^{-2}\right)$ is given by

$$\begin{aligned}
\mathcal{L}_{\mathrm{NRQCD}} = \psi^\dagger &\left\{ iD_0 + \frac{c_2}{2M}\boldsymbol{D}^2 + \frac{c_F}{2M} g\boldsymbol{B}\cdot\boldsymbol{\sigma} + \frac{c_D}{8M^2}\left[\boldsymbol{D}\cdot, g\boldsymbol{E}\right] + \frac{ic_S}{8M^2}\left[\boldsymbol{D}\times, g\boldsymbol{E}\right]\cdot\boldsymbol{\sigma} \right\}\psi \\
&+ \chi^\dagger \left\{ iD_0 - \frac{c_2}{2M}\boldsymbol{D}^2 - \frac{c_F}{2M} g\boldsymbol{B}\cdot\boldsymbol{\sigma} + \frac{c_D}{8M^2}\left[\boldsymbol{D}\cdot, g\boldsymbol{E}\right] + \frac{ic_S}{8M^2}\left[\boldsymbol{D}\times, g\boldsymbol{E}\right]\cdot\boldsymbol{\sigma} \right\}\chi \\
&+ \frac{1}{M^2}\Big\{ f_1(^1S_0)\,\psi^\dagger\chi\chi^\dagger\psi + f_1(^3S_1)\,\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi \\
&+ f_8(^1S_0)\,\psi^\dagger T^a\chi\chi^\dagger T^a\psi + f_1(^3S_1)\,\psi^\dagger\boldsymbol{\sigma}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi \Big\}
\end{aligned}$$

$$
- \frac{d_1}{4} F^a_{\mu\nu} F^{\mu\nu,a} + \frac{d_2}{M^2} F^a_{\mu\nu} D^2 F^{\mu\nu,a} + \frac{d_3}{M^2} g f_{abc} F^a_{\mu\nu} F^b_{\mu\alpha} F^c_{\nu\alpha} + \mathcal{L}_{\text{light}}
+ \mathcal{O}(1/M^4)\,, \tag{2.32}
$$

where the bilinear part of the heavy quark sector is identical to Eq. (2.21) in HQET. The Wilson coefficients c_2, c_F, c_D, and c_S of the heavy quark sector can be calculated by matching in perturbation theory to QCD. They are functions of α_S and depend logarithmically on the scale that is being integrated out: i.e., the coefficients are functions of $\ln(M/\mu)$, where the factorization scale μ is smaller than M. Note that all degrees of freedom in this Lagrangian have four momenta smaller than μ. The coefficients f's to the four-fermion terms will be discussed in Sec. 2.4.5. Furthermore, d's are the Wilson coefficients of the Yang-Mills sector [72–74]. $\mathcal{L}_{\text{light}}$ is the light quark sector which contains $\sum_l \bar{q}_l i \not{D} q_l$ plus $1/M$ correction terms, which are highly suppressed.

We have made use of the equations of motion[4] to remove all higher time derivatives and also removed the constant term $-M\psi^\dagger\psi + M\chi^\dagger\chi$ through the field redefinitions, $\psi \to e^{-iMt}\psi$ and $\chi \to e^{iMt}\chi$.

2.4 Symmetries in NRQCD

In this section, we discuss symmetries in NRQCD, including discrete symmetries like parity, charge conjugation, and time reversal (P, C, T), as well as Poincaré invariance. For the latter case, we take a general approach, in such a way that we allow all possible terms for the generator of the Poincaré group (boost, in particular) as long as the terms behave properly under the discrete symmetry (P, C, T) transformations. After constructing the boost generator, we apply it to the NRQCD Lagrangian. Due to the invariance of the theory after the transformation, we obtain constraints between Wilson coefficients as well as on the generic coefficients of the generator.

2.4.1 Discrete symmetries

Under the parity, charge conjugation, and time reversal, the coordinates and fields transform as the following list:

$$
(t, \boldsymbol{r}) \xrightarrow{P} (t, -\boldsymbol{r})\,, \qquad (t, \boldsymbol{r}) \xrightarrow{C} (t, \boldsymbol{r})\,, \qquad (t, \boldsymbol{r}) \xrightarrow{T} (-t, \boldsymbol{r})\,, \tag{2.33}
$$

$$
\psi \xrightarrow{P} \psi\,, \qquad \psi \xrightarrow{C} -i\sigma_2\chi^*\,, \qquad \psi \xrightarrow{T} i\sigma_2\psi\,, \tag{2.34}
$$

$$
\chi \xrightarrow{P} -\chi\,, \qquad \chi \xrightarrow{C} i\sigma_2\psi^*\,, \qquad \chi \xrightarrow{T} i\sigma_2\chi\,, \tag{2.35}
$$

$$
D_0 \xrightarrow{P} D_0\,, \qquad D_0 \xrightarrow{C} D_0^*\,, \qquad D_0 \xrightarrow{T} -D_0\,, \tag{2.36}
$$

$$
\boldsymbol{D} \xrightarrow{P} \boldsymbol{D}\,, \qquad \boldsymbol{D} \xrightarrow{C} \boldsymbol{D}^*\,, \qquad \boldsymbol{D} \xrightarrow{T} \boldsymbol{D}\,, \tag{2.37}
$$

$$
\boldsymbol{E} \xrightarrow{P} -\boldsymbol{E}\,, \qquad \boldsymbol{E} \xrightarrow{C} -\boldsymbol{E}^*\,, \qquad \boldsymbol{E} \xrightarrow{T} \boldsymbol{E}\,, \tag{2.38}
$$

[4]This is equivalent to performing certain field redefinitions, as shown in [76].

$$\boldsymbol{B} \xrightarrow{P} \boldsymbol{B}\,, \qquad \boldsymbol{B} \xrightarrow{C} -\boldsymbol{B}^*\,, \qquad \boldsymbol{B} \xrightarrow{T} -\boldsymbol{B}\,, \tag{2.39}$$

and we expect the boosted fields to transform in exactly the same way under these discrete symmetries, i.e.,

$$P\phi' = P(1 - i\boldsymbol{\eta}\cdot\boldsymbol{K})\phi = (1 - i(-\boldsymbol{\eta})\cdot(P\boldsymbol{K}))P\phi\,, \tag{2.40}$$
$$C\phi' = C(1 - i\boldsymbol{\eta}\cdot\boldsymbol{K})\phi = (1 - i\boldsymbol{\eta}\cdot(C\boldsymbol{K}))C\phi\,, \tag{2.41}$$
$$T\phi' = T(1 - i\boldsymbol{\eta}\cdot\boldsymbol{K})\phi = (1 + i(-\boldsymbol{\eta})\cdot(T\boldsymbol{K}))T\phi\,, \tag{2.42}$$

where we have also reversed the direction of the infinitesimal velocity $\boldsymbol{\eta}$ for P and T[5]. We take from this that the boost generators for the heavy quark and antiquark fields, $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$, respectively, need to satisfy

$$P\boldsymbol{k}_\psi = -\boldsymbol{k}_\psi\,, \qquad C\boldsymbol{k}_\psi = -\sigma_2\boldsymbol{k}_\chi^*\sigma_2\,, \qquad T\boldsymbol{k}_\psi = \sigma_2\boldsymbol{k}_\psi\sigma_2\,, \tag{2.43}$$
$$P\boldsymbol{k}_\chi = -\boldsymbol{k}_\chi\,, \qquad C\boldsymbol{k}_\chi = -\sigma_2\boldsymbol{k}_\psi^*\sigma_2\,, \qquad T\boldsymbol{k}_\chi = \sigma_2\boldsymbol{k}_\chi\sigma_2\,, \tag{2.44}$$

where the expressions on the left-hand sides mean that the transformed fields and coordinates according to Eqs. (2.33)-(2.39) are to be inserted into the explicit expressions for $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$.

2.4.2 Poincaré transformation

Before going into details, we need to clarify our notion of transformation. In general, performing a field transformation means to replace any field ϕ as well as its derivatives in the Lagrangian or other field-dependent objects by a new field ϕ', called the transformed field, and its derivatives. In the quantized theory, this corresponds to a change of variables $\phi \to \phi'$ in the path integral. The transformation constitutes a symmetry if the action remains invariant under this change of variables (in the quantized theory also the path integral measure needs to be considered).

In the case of coordinate transformations, the value of the field itself is not changed, it is just associated with a different spacetime coordinate. In the case of non-scalar fields, also the orientation with respect to the coordinate axes needs to be adjusted. So we can write

$$\phi_i(x) \xrightarrow{\mathcal{T}} \phi'_i(x) \equiv \mathcal{T}^{(R)}_{ij}\phi_j\left(\mathcal{T}^{-1}x\right)\,, \tag{2.45}$$

where $\mathcal{T}$ denotes a generic spacetime transformation and the representation R corresponds to the spin of the field ϕ. Note that the arrow in Eq. (2.45) represents the change of the *function* ϕ to ϕ', not its individual value at a certain position in spacetime; thus, the coordinate x is not assumed to have the same value on both sides of the arrow, and it should rather be considered as an index, just like the vector or spin index i does not imply the same polarization axis before and after the transformation. Instead the relation between the two fields is given on the right-hand side of Eq. (2.45): the value of the transformed field at position x is given by the value of the original field at

[5]Also remember that T takes the complex conjugate of numerical coefficients.

the same point, which in the old coordinates corresponds to $\mathcal{T}^{-1}x$, while its orientation is also adapted to the new polarization axes by $\mathcal{T}^{(R)}$. Also note that the notion of active or passive transformations[6] does not affect the form of Eq. (2.45), it just changes the sign of the generators of $\mathcal{T}$. For the record, we will assume passive transformations.

Spacetime translations

The spacetime translations act only on the coordinates, shifting the origin by a constant vector a^μ. The transformed field in the new coordinate system corresponds to the original field at the coordinates before the transformation. The form of the translation generator P_μ for a generic field $\phi(x)$ can then be obtained from a Taylor expansion to first order:

$$\phi(x) \xrightarrow{P_\mu} \phi'(x) = \phi(x+a) = \left[1 + a^\mu \partial_\mu + \mathcal{O}\left(a^2\right)\right] \phi(x) \equiv \left[1 - i a^\mu P_\mu + \mathcal{O}\left(a^2\right)\right] \phi(x) \,. \tag{2.46}$$

From this we take $P_\mu = i\partial_\mu$, or in non-relativistic notation $P_0 = i\partial_0$ and $\boldsymbol{P} = -i\boldsymbol{\nabla}$. This is already the final form of the translation generator for the light quark and gluon fields, but for the heavy (anti)quark fields we need to include the effect of the field redefinitions performed to remove the mass term in the Lagrangian. This modifies the generator to

$$P_\mu = e^{\pm iMt}(i\partial_\mu)e^{\mp iMt} = i\partial_\mu \pm \delta_{\mu 0} M \,, \tag{2.47}$$

so $P_0\psi = (i\partial_0 + M)\psi$ and $P_0\chi = (i\partial_0 - M)\chi$.

Rotations act both on the coordinates and on the field components. The coordinates are transformed under infinitesimal rotations such that $\boldsymbol{r}$ in the new coordinate system corresponds to $\boldsymbol{r}+\boldsymbol{\alpha}\times\boldsymbol{r}$ in the old, where the direction of $\boldsymbol{\alpha}$ gives the rotation axis and its absolute value gives the infinitesimal rotation angle. The components of the Pauli spinor fields are rotated with the Pauli matrix $\boldsymbol{\sigma}/2$, while the gauge fields transform as vectors, whose behavior follows directly from the coordinate transformations and Eq. (2.45):

$$A_0(x) \xrightarrow{J} A_0'(x) = A_0(x) + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), A_0(x)\right] \equiv \left(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{j}_0\right) A_0(x) \,, \tag{2.48}$$

$$\psi(x) \xrightarrow{J} \psi'(x) = \left(1 + \frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\right)\psi(x) + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), \psi(x)\right] \equiv \left(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{j}_{1/2}\right)\psi(x) \,, \tag{2.49}$$

$$\chi(x) \xrightarrow{J} \chi'(x) = \left(1 + \frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\right)\chi(x) + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), \chi(x)\right] \equiv \left(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{j}_{1/2}\right)\chi(x) \,, \tag{2.50}$$

$$\boldsymbol{A}(x) \xrightarrow{J} \boldsymbol{A}'(x) = \boldsymbol{A}(x) - \boldsymbol{\alpha}\times\boldsymbol{A}(x) + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), \boldsymbol{A}(x)\right] \equiv \left(1 + i\boldsymbol{\alpha}\cdot\boldsymbol{j}_1\right)\boldsymbol{A}(x) \,, \tag{2.51}$$

where again we have written universal term $\boldsymbol{r}\times(-i\boldsymbol{\nabla})$ for the coordinate transformations in the form of a commutator. We use a capital $\boldsymbol{J}$ to denote the generators of rotations in general, and a lowercase $\boldsymbol{j}$ with an index for a particular representation.

[6] An active transformation changes the position or orientation of a physical object with respect to a fixed coordinate system, while a passive transformation keeps the object fixed and changes the coordinates.

Again, we can convert the transformation of the gauge field A_μ into a relation for the covariant derivatives:

$$D_0' = \partial_0 + igA_0' = D_0 + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), D_0\right], \tag{2.52}$$
$$\boldsymbol{D}' = \boldsymbol{\nabla} - ig\boldsymbol{A}' = \boldsymbol{D} - \boldsymbol{\alpha}\times\boldsymbol{D} + \left[\boldsymbol{\alpha}\cdot(\boldsymbol{r}\times\boldsymbol{\nabla}), \boldsymbol{D}\right]. \tag{2.53}$$

From these, it also follows that the chromoelectric and chromomagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ transform as vectors under rotations.

Boost

Boost transformations are a priori not defined for the heavy (anti)quark fields. We will investigate how such a transformation can be constructed for the heavy quark fields, but this will no longer be linear in the fields. Gluons and light quarks are still relativistic fields and transform in the usual way under boosts. We will not discuss the light quark fields (they do not appear in any operator of interest in this discussion), but since the NRQCD Lagrangian is written in an explicitly non-relativistic fashion, we will also discuss the transformations of gluon fields, distinguishing between their space and time components.

The coordinates $(t, \boldsymbol{r})$ in a reference frame moving with the infinitesimal velocity $\boldsymbol{\eta}$ correspond to $(t+\boldsymbol{\eta}\cdot\boldsymbol{r}, \boldsymbol{r}+\boldsymbol{\eta}t)$ in a resting frame, where we will always neglect terms of $\mathcal{O}\left(\eta^2\right)$ or higher. The gluons are described by vector fields, whose transformations are identical to those of the coordinates, so Eq. (2.45) implies that

$$A_0'(t,\boldsymbol{r}) = A_0(t+\boldsymbol{\eta}\cdot\boldsymbol{r}, \boldsymbol{r}+\boldsymbol{\eta}t) - \boldsymbol{\eta}\cdot\boldsymbol{A}(t,\boldsymbol{r}), \tag{2.54}$$
$$\boldsymbol{A}'(t,\boldsymbol{r}) = \boldsymbol{A}(t+\boldsymbol{\eta}\cdot\boldsymbol{r}, \boldsymbol{r}+\boldsymbol{\eta}t) - \boldsymbol{\eta}A_0(t,\boldsymbol{r}). \tag{2.55}$$

It is convenient to perform a Taylor expansion to first order in $\boldsymbol{\eta}$ on all fields with transformed coordinates, in order to consistently work only with at most linear terms of this infinitesimal parameter. Since the gluon fields never appear individually in the Lagrangian but always inside covariant derivatives, we can also write the boost transformations explicitly for those:

$$D_0' = \partial_0 + igA_0' = D_0 + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot\boldsymbol{r}\partial_0, D_0] + \boldsymbol{\eta}\cdot\boldsymbol{D}, \tag{2.56}$$
$$\boldsymbol{D}' = \boldsymbol{\nabla} - ig\boldsymbol{A}' = \boldsymbol{D} + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot\boldsymbol{r}\partial_0, \boldsymbol{D}] + \boldsymbol{\eta}D_0. \tag{2.57}$$

Notice that the sign of the last terms has changed compared to Eqs. (2.54) and (2.55), which is a consequence of the fact that D_0 and $\boldsymbol{D}$ in the non-relativistic notation involve the gauge fields with opposite signs. The commutator in the middle terms serves two purposes: first, the commutator with the gauge field ensures that the derivatives (from the Taylor expansion) act exclusively on the gauge field and not on any other field that may be present in the Lagrangian. Second, the commutator with the derivative can easily be calculated and exactly cancels the derivative in the last term, ensuring that overall the derivatives on both sides of Eqs. (2.56), and (2.57) agree[7]. Finally, the transformations for the chromoelectric and chromomagnetic fields follow directly from their expressions in terms of the covariant derivatives:

$$\boldsymbol{E}' = \boldsymbol{E} + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot \boldsymbol{r}\partial_0, \boldsymbol{E}] + \boldsymbol{\eta}\times\boldsymbol{B}\,, \tag{2.58}$$

$$\boldsymbol{B}' = \boldsymbol{B} + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot \boldsymbol{r}\partial_0, \boldsymbol{B}] - \boldsymbol{\eta}\times\boldsymbol{E}\,. \tag{2.59}$$

In the following, we will use $\boldsymbol{K}$ to denote the generator for boosts as an operator in general, which may act on any kind of field, and $\boldsymbol{k}_\phi$ for the explicit expression for the field ϕ:

$$\phi(x) \xrightarrow{K} \phi'(x) = (1 - i\boldsymbol{\eta}\cdot\boldsymbol{K})\phi(x) = (1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_\phi)\phi(x)\,. \tag{2.60}$$

Since each field has the same coordinate transformations, the term $it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0$ appears in any $\boldsymbol{k}_\phi$, so we can write

$$\boldsymbol{k}_\phi = it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0 + \hat{\boldsymbol{k}}_\phi\,. \tag{2.61}$$

Now $\hat{\boldsymbol{k}}_\phi$ denotes the part of the boost transformation acting only on the components of ϕ and not the coordinates. The previously introduced notation of writing the coordinate transformations as commutators is particularly convenient when considering transformations of products of fields, as by the product rule of commutators we can write

$$\begin{aligned}\phi_1'\phi_2' &= \phi_1\phi_2 + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot\boldsymbol{r}\partial_0, \phi_1]\phi_2 + \phi_1[\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot\boldsymbol{r}\partial_0, \phi_2] \\ &\quad + (-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_1\phi_1)\phi_2 + \phi_1(-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_2\phi_2) \\ &= \phi_1\phi_2 + [\boldsymbol{\eta}\cdot t\boldsymbol{\nabla} + \boldsymbol{\eta}\cdot\boldsymbol{r}\partial_0, \phi_1\phi_2] + (-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_1\phi_1)\phi_2 + \phi_1(-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_2\phi_2)\,.\end{aligned} \tag{2.62}$$

In this way, the coordinate transformations can always be decoupled from the component transformations.

For relativistic fields, $\hat{\boldsymbol{k}}_\phi$ is some constant matrix (as it is shown in Eq. (1.66)), but for the heavy (anti)quark fields, which are non-relativistic, it takes the form of a function depending on the fields or their derivatives. Apart from the coordinate transformations, all derivatives have to be covariant, so we can write

$$\psi(x) \xrightarrow{K} \psi'(x) = \Big(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_\psi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x)\Big)\psi(x)\,, \tag{2.63}$$

$$\chi(x) \xrightarrow{K} \chi'(x) = \Big(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_\chi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x)\Big)\chi(x)\,. \tag{2.64}$$

In principle, $\hat{\boldsymbol{k}}_\phi$ depends on the coordinates only implicitly through the fields; however, the field redefinitions we have performed in order to remove the heavy mass terms from the Lagrangian also affect the boost transformations. So instead of the usual coordinate transformations generated by $it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0$, we have

$$e^{\pm iMt}(it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0)e^{\mp iMt} = it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0 \pm M\boldsymbol{r}\,, \tag{2.65}$$

[7]As stated above, we replace derivatives of fields in the Lagrangian by derivatives of the transformed fields: $\partial_x\phi(x) \xrightarrow{\mathcal{T}} \partial_x\phi'(x)$. The typical transformation of derivatives as vectors arises only when the transformed fields are replaced by the right-hand side of Eq. (2.45).

and we will include the terms $M\boldsymbol{r}$ and $-M\boldsymbol{r}$ in the definitions of $\hat{\boldsymbol{k}}_\psi$ and $\hat{\boldsymbol{k}}_\chi$, respectively.

Then the most general expressions for $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$ satisfying these conditions up to $\mathcal{O}(M^{-3})$ are given by[8] [77]

$$\begin{aligned}
\boldsymbol{k}_\psi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x) = {} & it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0 + M\boldsymbol{r} - \frac{k_D}{2M}\boldsymbol{D} - \frac{ik_{DS}}{4M}\boldsymbol{D}\times\boldsymbol{\sigma} + \frac{k_E}{8M^2}g\boldsymbol{E} \\
& + \frac{ik_{ES}}{8M^2}g\boldsymbol{E}\times\boldsymbol{\sigma} - \frac{k_{D3}}{4M^3}\boldsymbol{D}(\boldsymbol{D}^2) - \frac{ik_{D3S}}{16M^3}(\boldsymbol{D}\times\boldsymbol{\sigma})(\boldsymbol{D}^2) \\
& + \frac{ik_{B1}}{16M^3}[\boldsymbol{D}\times, g\boldsymbol{B}] + \frac{ik_{B2}}{16M^3}\{\boldsymbol{D}\times, g\boldsymbol{B}\} + \frac{k_{BS1}}{16M^3}[\boldsymbol{D},(g\boldsymbol{B}\cdot\boldsymbol{\sigma})] \\
& + \frac{k_{BS2}}{16M^3}\{\boldsymbol{D},(g\boldsymbol{B}\cdot\boldsymbol{\sigma})\} + \frac{k_{BS3}}{16M^3}[(\boldsymbol{D}\cdot\boldsymbol{\sigma}), g\boldsymbol{B}] \\
& + \frac{k_{BS4}}{16M^3}\{(\boldsymbol{D}\cdot\boldsymbol{\sigma}), g\boldsymbol{B}\} + \frac{k_{BS5}}{16M^3}\{\boldsymbol{D}\cdot, g\boldsymbol{B}\}\,\boldsymbol{\sigma}\,,
\end{aligned} \tag{2.66}$$

$$\begin{aligned}
\boldsymbol{k}_\chi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x) = {} & it\boldsymbol{\nabla} + i\boldsymbol{r}\partial_0 - M\boldsymbol{r} + \frac{k_D}{2M}\boldsymbol{D} + \frac{ik_{DS}}{4M}\boldsymbol{D}\times\boldsymbol{\sigma} + \frac{k_E}{8M^2}g\boldsymbol{E} \\
& + \frac{ik_{ES}}{8M^2}g\boldsymbol{E}\times\boldsymbol{\sigma} + \frac{k_{D3}}{4M^3}\boldsymbol{D}(\boldsymbol{D}^2) + \frac{ik_{D3S}}{16M^3}(\boldsymbol{D}\times\boldsymbol{\sigma})(\boldsymbol{D}^2) \\
& - \frac{ik_{B1}}{16M^3}[\boldsymbol{D}\times, g\boldsymbol{B}] - \frac{ik_{B2}}{16M^3}\{\boldsymbol{D}\times, g\boldsymbol{B}\} - \frac{k_{BS1}}{16M^3}[\boldsymbol{D},(g\boldsymbol{B}\cdot\boldsymbol{\sigma})] \\
& - \frac{k_{BS2}}{16M^3}\{\boldsymbol{D},(g\boldsymbol{B}\cdot\boldsymbol{\sigma})\} - \frac{k_{BS3}}{16M^3}[(\boldsymbol{D}\cdot\boldsymbol{\sigma}), g\boldsymbol{B}] \\
& - \frac{k_{BS4}}{16M^3}\{(\boldsymbol{D}\cdot\boldsymbol{\sigma}), g\boldsymbol{B}\} - \frac{k_{BS5}}{16M^3}\{\boldsymbol{D}\cdot, g\boldsymbol{B}\}\,\boldsymbol{\sigma}\,.
\end{aligned} \tag{2.67}$$

Temporal derivatives do not appear (aside from the coordinate transformations), because we assume that they have already been substituted through the equations of motion. Also terms that do not transform as a vector under rotations have already been removed, as they would violate one of the relations of the Poincaré algebra. In the following, it will be sufficient to discuss only the heavy quark sector, since the antiquark sector follows directly from charge conjugation.

The non-linear boost transformations constructed in this way have to satisfy the Poincaré algebra

$$\begin{aligned}
&\big[P_0, P_i\big] = 0\,, && \big[P_0, J_i\big] = 0\,, && \big[P_0, K_i\big] = -iP_i\,, \\
&\big[P_i, P_j\big] = 0\,, && \big[P_i, J_j\big] = i\epsilon_{ijk}P_k\,, && \big[P_i, K_j\big] = -i\delta_{ij}P_0\,, \\
&\big[J_i, J_j\big] = i\epsilon_{ijk}J_k\,, && \big[K_i, J_j\big] = i\epsilon_{ijk}K_k\,, && \big[K_i, K_j\big] = -i\epsilon_{ijk}J_k\,,
\end{aligned} \tag{2.68}$$

where P_0 is the generator of time translations, $\boldsymbol{P}$ is the generator of space translations, and $\boldsymbol{J}$ is the generator for rotations[9].

[8] Note that, in particular, $\hat{\boldsymbol{k}}_{\psi/\chi} = \pm i\boldsymbol{\sigma}$ are not allowed, even though they would satisfy all commutators of the Poincaré algebra, because they do not reproduce the right P or T transformation behavior. They would be appropriate for Weyl spinors, but here we deal with Pauli spinors.

[9] We are not using covariant notation here and in the rest of this paper; unless otherwise specified, we write all vector indices as lower indices.

The commutation relations of the Poincaré algebra not involving a boost generator are trivially satisfied, but for the boost they provide non-trivial information. It is also straightforward to check that the commutators of a boost generator with the generators of spacetime translations or rotations are satisfied. The commutator of two boosts, however, gives new constraints on the parameters from the boost generators.

Conceptually, the commutator between any two transformations is defined as the difference in performing them in opposite orders. In the case of linear transformations, this can be written simply as the commutator of two matrices, but in these non-linear transformations corresponding to the boost of a heavy (anti)quark field, one has to be careful to express the second transformation in terms of fields which have already undergone the first transformation.

It will be more convenient to calculate the commutator of two infinitesimal boost transformations rather than the commutator of two boost generators:

$$[1 - i\boldsymbol{\xi}\cdot\boldsymbol{K}, 1 - i\boldsymbol{\eta}\cdot\boldsymbol{K}] = i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{J}\,, \tag{2.69}$$

since then we may apply the left-hand side directly to the heavy quark field and write the two successive boosts as

$$\psi(x) \xrightarrow{K_\eta} \psi'_\eta(x) = \Big(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_\psi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x)\Big)\psi(x)\,, \tag{2.70}$$

$$\psi'_\eta(x) \xrightarrow{K_\xi} \psi''_{\xi\eta}(x) = \Big(1 - i\boldsymbol{\xi}\cdot\boldsymbol{k}_\psi(\boldsymbol{D}'_\eta,\boldsymbol{E}'_\eta,\boldsymbol{B}'_\eta,\psi'_\eta,\chi'_\eta,x)\Big)\psi'_\eta(x)\,. \tag{2.71}$$

Expanding the commutator to linear order in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ gives

$$\begin{aligned}
&[1 - i\boldsymbol{\xi}\cdot\boldsymbol{K}, 1 - i\boldsymbol{\eta}\cdot\boldsymbol{K}]\psi(x) = \psi''_{\xi\eta}(x) - \psi''_{\eta\xi}(x)\\
&= \Big(1 - i\boldsymbol{\xi}\cdot\boldsymbol{k}_\psi(\boldsymbol{D}'_\eta,\boldsymbol{E}'_\eta,\boldsymbol{B}'_\eta,\psi'_\eta,\chi'_\eta,x)\Big)\psi'_\eta(x) - \Big(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_\psi(\boldsymbol{D}'_\xi,\boldsymbol{E}'_\xi,\boldsymbol{B}'_\xi,\psi'_\xi,\chi'_\xi,x)\Big)\psi'_\xi(x)\\
&= (\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{r}\times\boldsymbol{\nabla})\psi(x) - \Big[\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\psi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x), \boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\psi(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x)\Big]\,\psi(x)\\
&\quad - i\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\psi(\boldsymbol{D}+\boldsymbol{\eta}D_0, \boldsymbol{E}+\boldsymbol{\eta}\times\boldsymbol{B}, \boldsymbol{B}-\boldsymbol{\eta}\times\boldsymbol{E}, (1-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\psi)\psi, (1-i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\chi)\chi, x)\,\psi(x)\\
&\quad + i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\psi(\boldsymbol{D}+\boldsymbol{\xi}D_0, \boldsymbol{E}+\boldsymbol{\xi}\times\boldsymbol{B}, \boldsymbol{B}-\boldsymbol{\xi}\times\boldsymbol{E}, (1-i\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\psi)\psi, (1-i\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\chi)\chi, x)\,\psi(x)\,,
\end{aligned} \tag{2.72}$$

where in the last two lines only linear orders of $\boldsymbol{\xi}$ or $\boldsymbol{\eta}$ are supposed to be kept. These last two lines contain new terms (compared to the naive application of the commutator in the previous line) arising from the non-linear nature of the boost transformation.

Inserting the explicit expression of Eq. (2.66) into Eq. (2.72), we obtain the following expression at $\mathcal{O}(M^{-2})$:

$$\begin{aligned}
\psi''_{\xi\eta}(x) - \psi''_{\eta\xi}(x) = i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\Big\{&\boldsymbol{r}\times(-i\boldsymbol{\nabla}) + \frac{k_{DS}}{2}\boldsymbol{\sigma} + \frac{ik_{DS}}{2M}\boldsymbol{\sigma}D_0 + \frac{k_{D3S}}{4M^2}\boldsymbol{\sigma}(\boldsymbol{D}^2)\\
&+ \frac{1}{16M^2}(k_{DS}^2 - k_{D3S})\,\{\boldsymbol{D},\boldsymbol{D}\cdot\boldsymbol{\sigma}\}\\
&- \frac{1}{16M^2}(4k_E - 4k_D^2 + k_{DS}^2 + 8k_{D3} + 4k_{B1})\,g\boldsymbol{B}
\end{aligned}$$

$$
- \frac{i}{16M^2}(2k_{ES} - 2k_D k_{DS} + k_{D3S} - 2k_{BS4} + 2k_{BS5})\,(g\boldsymbol{B}\times\boldsymbol{\sigma})\Big\}\,\psi(x)\,. \tag{2.73}
$$

In order to replace the temporal derivative, we use the equation of motion for the heavy quark field at $\mathcal{O}(M^{-1})$

$$
iD_0\psi(x) = \left\{-\frac{c_2}{2M}\boldsymbol{D}^2 - \frac{c_F}{2M}g\boldsymbol{B}\cdot\boldsymbol{\sigma}\right\}\psi(x)\,. \tag{2.74}
$$

Then, the resulting expression has to satisfy the commutation relation of Eq. (2.69), such that

$$
\begin{aligned}
i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\Big\{&\boldsymbol{r}\times(-i\boldsymbol{\nabla}) + \frac{k_{DS}}{2}\boldsymbol{\sigma}\\
&+ \frac{1}{4M^2}(k_{D3S} - k_{DS}c_2)\boldsymbol{\sigma}(\boldsymbol{D}^2) + \frac{1}{16M^2}(k_{DS}^2 - k_{D3S})\{\boldsymbol{D},(\boldsymbol{D}\cdot\boldsymbol{\sigma})\}\\
&- \frac{1}{16M^2}(4k_{DS}c_F + 4k_E - 4k_D^2 + k_{DS}^2 + 8k_{D3} + 4k_{B1})\,g\boldsymbol{B}\\
&- \frac{i}{16M^2}(4k_{DS}c_F + 2k_{ES} - 2k_D k_{DS} + k_{D3S} - 2k_{BS4} + 2k_{BS5})(g\boldsymbol{B}\times\boldsymbol{\sigma})\Big\}\,\psi(x)\\
\stackrel{!}{=} i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\Big\{&\boldsymbol{r}\times(-i\boldsymbol{\nabla}) + \frac{1}{2}\boldsymbol{\sigma}\Big\}\,\psi(x)\,.
\end{aligned} \tag{2.75}
$$

This gives the following relations:

$$
k_{DS} = 1\,, \quad k_{D3S} = 1\,, \quad c_2 = 1\,, \tag{2.76}
$$

$$
c_F + k_E - k_D^2 + 2k_{D3} + k_{B1} = -\frac{1}{4}\,, \tag{2.77}
$$

$$
2c_F + k_{ES} - k_D - k_{BS4} + k_{BS5} = -\frac{1}{2}\,. \tag{2.78}
$$

Note that already two of the Wilson coefficients, c_2 and c_F, are fixed or constrained in terms of the parameters from the boost generator, and two of the coefficients from boost generator k_{DS} and k_{D3S} are fixed to unity.

2.4.3 Invariance of the Lagrangian

Now that we have constructed a non-linear boost transformation for the heavy (anti)quark fields that satisfies the Poincaré algebra, we can proceed to check which conditions need to be satisfied in order for the NRQCD Lagrangian to be invariant under this transformation. We start with bilinear terms in the heavy quark sector. The Lagrangian at $\mathcal{O}(M^{-2})$ was already given in Eq. (2.32), but in order to study the transformed Lagrangian at this order, we also need to include $\mathcal{O}(M^{-3})$ terms that contain a derivative[10]:

$$
\mathcal{L}^{(3)}\Big|_{\boldsymbol{D}} = \psi^\dagger\Big\{\frac{c_4}{8M^3}(\boldsymbol{D}^2)^2 + \frac{c_{W1}}{8M^3}\{\boldsymbol{D}^2, g\boldsymbol{B}\cdot\boldsymbol{\sigma}\} - \frac{c_{W2}}{4M^3}D_i(g\boldsymbol{B}\cdot\boldsymbol{\sigma})D_i
$$

[10]The term $-iM\boldsymbol{r}$ from the boost transformation adds a power of M to the $\mathcal{O}(M^{-3})$ Lagrangian, but the commutator with this term vanishes unless there appears a derivative.

$$+\frac{c_{p'p}}{16M^3}\Big\{(\boldsymbol{D}\cdot\boldsymbol{\sigma}),\big\{\boldsymbol{D}\cdot,g\boldsymbol{B}\big\}\Big\}+\frac{ic_M}{8M^3}\Big\{\boldsymbol{D}\cdot,\big\{\boldsymbol{D}\times,g\boldsymbol{B}\big\}\Big\}\Bigg\}\psi\,. \tag{2.79}$$

Strictly speaking, it is not the Lagrangian that needs to be invariant under a transformation but the action. Thus, as for the invariance of the Lagrangian, we mean that the difference between transformed and original Lagrangian is at most an overall derivative, which we denote by $\partial_\mu\Delta^\mu\mathcal{L}$. Such terms are often implicitly omitted, as all they contribute to the action is a vanishing surface term. We will include them here for the completeness in argument, and because they play a role in the calculation of conserved Noether currents and charges.

With the Lagrangian defined in Eq. (2.32) and (2.79), we obtain the following transformation behavior at $\mathcal{O}(M^{-2})$:

$$\begin{aligned}
\partial_\mu\Delta^\mu\mathcal{L} &= \mathcal{L}\big(\boldsymbol{D}',\boldsymbol{E}',\boldsymbol{B}',\psi',\chi',x\big)-\mathcal{L}\big(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},\psi,\chi,x\big)\\
&= \boldsymbol{\eta}\cdot(\boldsymbol{r}\partial_0+t\boldsymbol{\nabla})\mathcal{L}+\boldsymbol{\eta}\cdot\psi^\dagger\Bigg\{i(1-c_2)\boldsymbol{D}+\frac{1}{2M}(c_2-k_D)\big\{D_0,\boldsymbol{D}\big\}\\
&\quad+\frac{1}{4M}(k_{DS}-2c_F+c_S)g\boldsymbol{E}\times\boldsymbol{\sigma}+\frac{i}{4M^2}(c_2k_D-c_4)\big\{\boldsymbol{D},\boldsymbol{D}^2\big\}+\frac{1}{8M^2}(c_D+k_E)\big[D_0,\boldsymbol{E}\big]\\
&\quad+\frac{i}{8M^2}(c_S+k_{ES})\big\{D_0,g\boldsymbol{E}\big\}\times\boldsymbol{\sigma}+\frac{1}{8M^2}(2c_M-c_D+c_Fk_{DS})\big\{\boldsymbol{D}\times,g\boldsymbol{B}\big\}\\
&\quad+\frac{i}{8M^2}(c_S-c_Fk_{DS}-c_{p'p})\big\{\boldsymbol{D}\cdot,g\boldsymbol{B}\big\}\boldsymbol{\sigma}+\frac{i}{8M^2}(c_Fk_{DS}-c_2k_{DS}-c_{p'p})\big\{(\boldsymbol{D}\cdot\boldsymbol{\sigma}),g\boldsymbol{B}\big\}\\
&\quad+\frac{i}{8M^2}(c_2k_{DS}+2c_Fk_D-c_S-2c_{W1}+2c_{W2})\big\{\boldsymbol{D},(g\boldsymbol{B}\cdot\boldsymbol{\sigma})\big\}\Bigg\}\psi\\
&\quad+\boldsymbol{\nabla}\cdot\boldsymbol{\eta}\,\psi^\dagger\left\{\frac{k_D}{2M}D_0-\frac{ic_2k_D}{4M^2}\boldsymbol{D}^2-\frac{ic_Fk_D}{4M^2}g\boldsymbol{B}\cdot\boldsymbol{\sigma}\right\}\psi\\
&\quad+\boldsymbol{\nabla}\cdot\boldsymbol{\eta}\times\psi^\dagger\left\{\frac{ik_{DS}}{4M}D_0\boldsymbol{\sigma}+\frac{c_2k_{DS}}{8M^2}(\boldsymbol{D}^2)\boldsymbol{\sigma}+\frac{c_Fk_{DS}}{8M^2}g\boldsymbol{B}+\frac{ic_Fk_{DS}}{8M^2}g\boldsymbol{B}\times\boldsymbol{\sigma}\right\}\psi\,.
\end{aligned} \tag{2.80}$$

The spatial derivative terms in the last two lines arise from covariant derivatives in the boost transformation of $\psi^\dagger$ through $(\boldsymbol{D}\psi)^\dagger=\boldsymbol{\nabla}\psi^\dagger-\psi^\dagger\boldsymbol{D}$.

All terms which are not overall derivatives have to vanish, otherwise the Lagrangian (or rather the action) would not be invariant. From this, the following constraints on the coefficients are obtained:

$$c_2=1\,,\quad k_D=1\,,\quad c_4=1\,,\quad k_E=-c_D\,,\quad k_{ES}=-c_S\,,\quad k_{DS}=1\,, \tag{2.81}$$

$$c_S=2c_F-1\,,\quad 2c_M=c_D-c_F\,,\quad c_{p'p}=c_F-1\,,\quad c_{W2}=c_{W1}-1\,. \tag{2.82}$$

These coincide with the constraints derived in HQET via reparametrization invariance [25]. We also see that they are consistent with the constraints obtained from the Poincaré algebra relation between two boost generators. By combining both results, Eqs. (2.76), (2.77), (2.78), (2.81), and (2.82), we can simplify the remaining constraints to:

$$k_{B1}=c_D-c_F+\frac{3}{4}-2k_{D3}\,,\quad k_{BS5}=k_{BS4}-\frac{1}{2}\,. \tag{2.83}$$

Note that the relations $k_{DS} = 1$ and $c_2 = 1$ are obtained at different orders in $1/M$ respectively, depending on whether we use the commutator of two boosts or the invariance of the Lagrangian. Ultimately, if one were able to do an all orders calculation, each calculation by itself should give all constraints, because the Wilson coefficients of the Lagrangian enter the calculation of the commutator of two boosts through the equations of motion. At any finite order, however, both calculations provide complementary information to each other.

The boost parameter k_{D3} has not been fixed yet. Its value can be easily derived from a term in $\partial_\mu \Delta^\mu \mathcal{L}$ at $\mathcal{O}\left(M^{-3}\right)$,

$$\psi^\dagger \frac{1}{8M^3}(c_4 - k_{D3})\Big\{\{D_0, \boldsymbol{\eta}\cdot\boldsymbol{D}\}, \boldsymbol{D}^2\Big\}\psi\,, \tag{2.84}$$

where $c_4 = 1$ from Eq. (2.81). This term cannot be canceled by any other term in the transformed Lagrangian because those contain at least one gluon field $\boldsymbol{E}$ or $\boldsymbol{B}$. Thus, the constraint $k_{D3} = 1$ is obtained at $\mathcal{O}(M^{-3})$ and can be used to further simplify Eq. (2.83) to

$$k_{B1} = c_D - c_F - \frac{5}{4}\,. \tag{2.85}$$

If we compare this result for the boost to the transformation used in [24][11], we observe that they are the same. It turns out that the generic boost coefficients k_D, k_{DS}, k_{D3}, and k_{D3S}, which were implicitly assumed to be equal to 1 in the method inspired by the induced representation [24], are in fact constrained to exactly this value by either the invariance of the Lagrangian itself or the relation in the Poincaré algebra for the commutator of two boosts. While this may not be a general proof that the assumptions of [24] remain valid also at higher orders in $1/M$, at least we have shown that at this order there is no contradiction between the two approaches.

2.4.4 Noether Charge

Now that the boost transformations of the heavy quark and antiquark fields are determined, we can obtain the corresponding Noether charge $\boldsymbol{\mathcal{K}}$:

$$\begin{aligned}
\boldsymbol{\mathcal{K}} &= \int d^3r \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}(-i\boldsymbol{k}_\phi\phi_i) - \Delta^0\mathcal{L}\right] \\
&= \int d^3r \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}(t\boldsymbol{\nabla} + \boldsymbol{r}\partial_0)\phi_i + \psi^\dagger\hat{\boldsymbol{k}}_\psi\psi + \chi^\dagger\hat{\boldsymbol{k}}_\chi\chi - \boldsymbol{\Pi}^a A_0^a - \boldsymbol{r}\mathcal{L}\right] \\
&= -\,t\boldsymbol{\mathcal{P}} + \int d^3r \left[\boldsymbol{r}h + \psi^\dagger\hat{\boldsymbol{k}}_\psi\psi + \chi^\dagger\hat{\boldsymbol{k}}_\chi\chi\right] \\
&= -\,t\boldsymbol{\mathcal{P}} + \frac{1}{2}\int d^3r \left\{\boldsymbol{r}, h + M\psi^\dagger\psi - M\chi^\dagger\chi\right\} - \int d^3r\,\psi^\dagger\left[\frac{i}{4M}\boldsymbol{D}\times\boldsymbol{\sigma} + \frac{c_D}{8M^2}g\boldsymbol{E}\right]\psi
\end{aligned}$$

[11]Ref. [24] is in the context of NRQED instead of NRQCD, but the two calculations are analogous at low orders in $1/M$.

$$+\int d^3r\, \chi^\dagger \left[\frac{i}{4M}\boldsymbol{D}\times\boldsymbol{\sigma} - \frac{c_D}{8M^2} g\boldsymbol{E}\right]\chi + \mathcal{O}(M^{-3})\,, \tag{2.86}$$

where ϕ_i stands for all three types of field ψ, χ, and $\boldsymbol{A}$, while $\boldsymbol{\Pi}$ is the canonical momentum field associated to $\boldsymbol{A}$:

$$\Pi_i^a = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_i^a)} = -E_i^a + \mathcal{O}(M^{-2})\,. \tag{2.87}$$

In addition, $\boldsymbol{\mathcal{P}}$ is the Noether charge associated with spatial translations

$$\begin{aligned}\boldsymbol{\mathcal{P}} &= \int d^3r \left\{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}(-\boldsymbol{\nabla})\phi_i\right\} \\ &= \int d^3r \left(\psi^\dagger(-i\boldsymbol{D})\psi + \chi^\dagger(-i\boldsymbol{D})\chi - \mathrm{Tr}\big[\boldsymbol{\Pi}\times,\boldsymbol{B}\big]\right)\,,\end{aligned} \tag{2.88}$$

where the equations of motion, $\big[\boldsymbol{D}\cdot,\boldsymbol{\Pi}\big] = -\left(\psi^\dagger gT^a\psi + \chi^\dagger gT^a\chi\right)T^a$, have been used in order to make the expression explicitly gauge invariant[12]. The Hamiltonian density h is given through the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \int d^3r \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}\partial_0\phi_i - \mathcal{L}\right) = \int d^3r \left(\psi^\dagger h_\psi\psi + \chi^\dagger h_\chi\chi + \mathrm{Tr}\big[\boldsymbol{\Pi}^2 + \boldsymbol{B}^2\big]\right) \\ &\equiv \int d^3r\, h\,,\end{aligned} \tag{2.89}$$

where h_ψ and h_χ are defined through the Lagrangian,

$$\mathcal{L} = \psi^\dagger(iD_0 - h_\psi)\psi + \chi^\dagger(iD_0 - h_\chi)\chi + \mathrm{Tr}\big[\boldsymbol{E}^2 - \boldsymbol{B}^2\big]\,, \tag{2.90}$$

and we have made use of the Gauss's law again. The initial expression $(\partial\mathcal{L}/\partial(\partial_0\phi_i)\cdot\partial_0\phi_i) - \mathcal{L}$ and the Hamiltonian density, as defined in the final expression, differ by a derivative term, $-\boldsymbol{\nabla}\cdot(\boldsymbol{\Pi}^a A_0^a)$, which vanishes in $\mathcal{H}$, but gives a contribution to $\boldsymbol{\mathcal{K}}$ that exactly cancels the $\boldsymbol{\Pi}^a A_0^a$ term in Eq. (2.86). In the last expression for $\boldsymbol{\mathcal{K}}$, we have replaced $\boldsymbol{r}h$ by $\frac{1}{2}\{\boldsymbol{r},h\} - \frac{1}{2}[h,\boldsymbol{r}]$ in order to obtain an explicitly hermitian expression; the antihermitian terms from $\psi^\dagger\hat{\boldsymbol{k}}_\psi\psi$ and $\chi^\dagger\hat{\boldsymbol{k}}_\chi\chi$ cancel against $\frac{1}{2}[h,\boldsymbol{r}]$.

The Noether charge $\boldsymbol{\mathcal{K}}$ corresponds exactly to the boost operator of the quantized theory obtained in [22] and extends it up to $\mathcal{O}(M^{-2})$. Note that the field redefinitions which remove the $\mathcal{O}(M)$ terms from the Lagrangian have not been performed in [22], so their definition of h differs from ours by $M\psi^\dagger\psi - M\chi^\dagger\chi$, which thus appears explicitly in our expression for $\boldsymbol{\mathcal{K}}$. Accordingly, the generators for time translations are given by $i\partial_0 \pm M$ after the redefinition of ψ and χ, so the proper Noether charge of time translations is given by the above Hamiltonian $\mathcal{H}$ plus $M\int d^3r(\psi^\dagger\psi - \chi^\dagger\chi)$, which coincides with the expression in [22].

[12] In fact, the equations of motion are the Gauss's law.

2.4.5 The four-fermion Lagrangian

We now turn to the four-fermion part of the NRQCD Lagrangian, or more specifically the part consisting of two heavy quark and two heavy antiquark fields. The lowest order terms of the Lagrangian are given by

$$\mathcal{L}^{(2)}\Big|_{4f} = \frac{1}{M^2}\Big\{ f_1({}^1S_0)\,\psi^\dagger\chi\chi^\dagger\psi + f_1({}^3S_1)\,\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi \\ + f_8({}^1S_0)\,\psi^\dagger T^a\chi\chi^\dagger T^a\psi + f_1({}^3S_1)\,\psi^\dagger\boldsymbol{\sigma}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi\Big\}\,. \tag{2.91}$$

These Wilson coefficients f are related by Poincaré invariance to the coefficients of the next order four-fermion Lagrangian, which in this case is $\mathcal{O}(M^{-4})$ [78]. It is straightforward to see that the $\mathcal{O}(M)$ terms of $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$ cancel each other in the boost transformation of the leading order part of this Lagrangian, so the first constraints will be obtained at $\mathcal{O}(M^{-3})$.

The $\mathcal{O}(M^{-4})$ Lagrangian will contribute only with the $\mathcal{O}(M)$ terms of $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$, which are given by $\pm M\boldsymbol{r}$. Since the boost of operators with two left-right derivatives, like $\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\chi\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\psi$ (see Eq. (2.93) for the definition of left-right derivatives), or with a chromomagnetic field $\boldsymbol{B}$, cancels at $\mathcal{O}(M)$, only operators with at least one "center-of-mass" derivative (i.e., a derivative acting on two heavy fields like $\boldsymbol{\nabla}\chi^\dagger\psi$) give non-vanishing contributions. Including only such terms, the four-fermion part of the Lagrangian at $\mathcal{O}(M^{-4})$ is given by

$$\begin{aligned}\mathcal{L}^{(4)}\Big|_{4f,\,cm} = &-\frac{if_{1\,cm}}{2M^4}\big(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})\chi\cdot\boldsymbol{\nabla}\chi^\dagger\psi + (\boldsymbol{\nabla}\psi^\dagger\chi)\cdot\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})\psi\big)\\ &-\frac{if_{8\,cm}}{2M^4}\big(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})T^a\chi\cdot\boldsymbol{D}^{ab}\chi^\dagger T^b\psi + (\boldsymbol{D}^{ab}\psi^\dagger T^b\chi)\cdot\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})T^a\psi\big)\\ &+\frac{if'_{1\,cm}}{2M^4}\big(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\chi\cdot(\boldsymbol{\nabla}\times\chi^\dagger\boldsymbol{\sigma}\psi) + (\boldsymbol{\nabla}\times\psi^\dagger\boldsymbol{\sigma}\chi)\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\psi\big)\\ &+\frac{if'_{8\,cm}}{2M^4}\big(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\chi\cdot(\boldsymbol{D}^{ab}\times\chi^\dagger\boldsymbol{\sigma}T^b\psi) + (\boldsymbol{D}^{ab}\times\psi^\dagger\boldsymbol{\sigma}T^b\chi)\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\psi\big)\\ &+\frac{g_{1a\,cm}}{M^4}\Big(\nabla_i\psi^\dagger\sigma_j\chi\Big)\Big(\nabla_i\chi^\dagger\sigma_j\psi\Big) + \frac{g_{8a\,cm}}{M^4}\Big(D_i^{ab}\psi^\dagger\sigma_jT^b\chi\Big)\Big(D_i^{ac}\chi^\dagger\sigma_jT^c\psi\Big)\\ &+\frac{g_{1b\,cm}}{M^4}\Big(\boldsymbol{\nabla}\cdot\psi^\dagger\boldsymbol{\sigma}\chi\Big)\Big(\boldsymbol{\nabla}\cdot\chi^\dagger\boldsymbol{\sigma}\psi\Big) + \frac{g_{8b\,cm}}{M^4}\Big(\boldsymbol{D}^{ab}\cdot\psi^\dagger\boldsymbol{\sigma}T^b\chi\Big)\Big(\boldsymbol{D}^{ac}\cdot\chi^\dagger\boldsymbol{\sigma}T^c\psi\Big)\\ &+\frac{g_{1c\,cm}}{M^4}\Big(\boldsymbol{\nabla}\psi^\dagger\chi\Big)\cdot\Big(\boldsymbol{\nabla}\chi^\dagger\psi\Big) + \frac{g_{8c\,cm}}{M^4}\Big(\boldsymbol{D}^{ab}\psi^\dagger T^b\chi\Big)\cdot\Big(\boldsymbol{D}^{ac}\chi^\dagger T^c\psi\Big)\,,\end{aligned} \tag{2.92}$$

where covariant derivatives with color indices are understood in the adjoint representation. The left-right derivatives are defined as follows:

$$\psi^\dagger(\overleftrightarrow{\boldsymbol{D}})^nT\chi = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\Big(\boldsymbol{D}^k\psi\Big)^\dagger T\Big(\boldsymbol{D}^{n-k}\chi\Big)\,, \tag{2.93}$$

where the order of the covariant derivatives on the right-hand side is the same in each term, and T stands for either the unit or a color matrix. In particular, it follows from this expression that $\overleftrightarrow{\boldsymbol{D}}$ does not act on any field outside of $\psi^\dagger$ and χ. Thus, we obtain the following expression at $\mathcal{O}(M^{-3})$ after taking the boost transformation:

$$\begin{aligned}
\partial_\mu \Delta^\mu \mathcal{L}^{(4f)} = &- \frac{1}{2M^3}\left(f_1(^1S_0) + 4g_{1c\,cm}\right)\left[(\boldsymbol{\eta}\cdot i\boldsymbol{\nabla}\psi^\dagger\chi)\chi^\dagger\psi + h.c.\right] \\
&- \frac{1}{2M^3}\left(f_8(^1S_0) + 4g_{8c\,cm}\right)\left[(\boldsymbol{\eta}\cdot i\boldsymbol{D}^{ab}\psi^\dagger T^b\chi)\chi^\dagger T^b\psi + h.c.\right] \\
&+ \frac{1}{4M^3}\left(f_1(^1S_0) - f_{1\,cm}\right)\left[\psi^\dagger\boldsymbol{\eta}\cdot(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})\chi\chi^\dagger\psi + h.c.\right] \\
&+ \frac{1}{4M^3}\left(f_8(^1S_0) - f_{8\,cm}\right)\left[\psi^\dagger\boldsymbol{\eta}\cdot(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})T^a\chi\chi^\dagger T^a\psi + h.c.\right] \\
&- \frac{1}{2M^3}\left(f_1(^3S_1) + 4g_{1a\,cm}\right)\left[(\boldsymbol{\eta}\cdot i\boldsymbol{\nabla}\psi^\dagger\sigma_i\chi)\chi^\dagger\sigma_i\psi + h.c.\right] \\
&- \frac{1}{2M^3}\left(f_8(^3S_1) + 4g_{8a\,cm}\right)\left[(\boldsymbol{\eta}\cdot i\boldsymbol{D}^{ab}\psi^\dagger\sigma_i T^b\chi)\chi^\dagger\sigma_i T^b\psi + h.c.\right] \\
&+ \frac{1}{4M^3}\left(f_1(^3S_1) - f'_{1\,cm}\right)\left[\psi^\dagger(\boldsymbol{\eta}\times\overleftrightarrow{\boldsymbol{D}})\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi + h.c.\right] \\
&+ \frac{1}{4M^3}\left(f_8(^3S_1) - f'_{8\,cm}\right)\left[\psi^\dagger(\boldsymbol{\eta}\times\overleftrightarrow{\boldsymbol{D}})T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi + h.c.\right] \\
&+ \frac{2}{M^3}g_{1b\,cm}\left[\psi^\dagger(\boldsymbol{\eta}\cdot\boldsymbol{\sigma})\chi(i\boldsymbol{\nabla}\cdot\chi^\dagger\boldsymbol{\sigma}\psi) + h.c.\right] \\
&+ \frac{2}{M^3}g_{8b\,cm}\left[\psi^\dagger(\boldsymbol{\eta}\cdot\boldsymbol{\sigma})T^a\chi(i\boldsymbol{D}^{ab}\cdot\chi^\dagger\boldsymbol{\sigma}T^b\psi) + h.c.\right]\,, \qquad (2.94)
\end{aligned}$$

where we have neglected the terms from the coordinate transformations. As none of these terms has the form of an overall derivative, all coefficients have to be equal to zero, which implies:

$$g_{1a\,cm} = -\frac{1}{4}f_1(^3S_1), \quad g_{1c\,cm} = -\frac{1}{4}f_1(^1S_0), \quad g_{8a\,cm} = -\frac{1}{4}f_8(^3S_1), \quad g_{8c\,cm} = -\frac{1}{4}f_8(^1S_0), \qquad (2.95)$$

$$f_{1\,cm} = \frac{1}{4}f_1(^1S_0), \qquad f'_{1\,cm} = \frac{1}{4}f_1(^3S_1), \qquad f_{8\,cm} = \frac{1}{4}f_8(^1S_0), \qquad f'_{8\,cm} = \frac{1}{4}f_8(^3S_1), \qquad (2.96)$$

$$g_{1b\,cm} = g_{8b\,cm} = 0\,. \qquad (2.97)$$

These relations were first derived in [78] and later confirmed in [79] for the singlet sector, which at this order is equivalent to NRQED.

At $\mathcal{O}(M^{-4})$, the boost generators contain terms involving the heavy (anti)quark fields themselves. This is a novel feature if one follows the line of argument in [79], where the appearance of gauge field operators has been explained with the ambiguities related to the ordering of derivatives when one promotes them to covariant derivatives. Of course, an argument could be made based on the fact that gauge fields and heavy (anti)quark fields are related through the equations of motions, but in the EFT approach used in this paper the appearance of heavy (anti)quark fields in the boost generators is only natural and requires no further justification. It turns out, however, that none of these terms leads to new constraints on the Wilson coefficient, at least not at the next order in the $1/M$ expansion, because any term in the boosted Lagrangian or the commutator of two boosts that is supposed to vanish can be made to do so just by adjusting the boost parameters. This is why we reserve the details of this calculation to Appendix A.1.

2.5 Summary and discussion

In this chapter, we have discussed basics of QCD at high-energy (degrees of freedom and asymptotic freedom) and introduced low-energy EFTs of QCD involving heavy quarks, HQET and NRQCD, in particular. As the HQET is parametrized by a reference frame of the heavy particle and its residual momentum (which is due to the recoil from a light quark), it is natural that the theory is invariant under reparametrization of the rest frame and the residual momentum. Such invariance leads to constraints between its Wilson coefficients. In fact, one can view this invariance from another perspective: reparametrization invariance implies that the theory is invariant under the choice of the reference frame of the particle, which, in fact, is the Poincaré invariance of the theory.

Then, we have investigated the Poincaré invariance of another low-energy EFT of QCD, namely NRQCD. While the non-relativistic fields transform under spacetime translations and rotations in a usual way, their transformation under boost is non-trivial. We investigated boost transformations by starting from the most general form allowed by charge conjugation, parity, and time reversal, while exploiting the freedom to remove redundant terms through field redefinitions. Relations between the Wilson coefficients were derived when we applied those expressions to the corresponding Lagrangian up to a certain order in the expansion, and required that they leave the corresponding action invariant as well as satisfy the Poincaré algebra. The results confirm known relations from the literature [22, 24, 78], in both NRQCD, and can be found in Eqs. (2.81), (2.82), (2.95), (2.96), and (2.97).

While at present we have not obtained new relations between the Wilson coefficients for NRQCD, we still would like to point out the following benefits of our approach and advantages over previous treatments. The derivation of the boost transformation via the induced representation[13] in [24] provides a somewhat intuitive understanding for the form of several (but not all) terms appearing in the boost generator; however, it fails to provide a convincing argument why one should not assign a matching coefficient to these terms in order to account for possible high energy effects beyond the factorization scale, just as one does for any other EFT operator. In addition, it does not specify how the necessary extension of the boost beyond the form given by the induced representation should be performed in cases where more than one suitable operator is available.

We have addressed both of these issues: using a general ansatz with only minimal assumptions (i.e., including every possible operator with a matching coefficient), we have shown unambiguously (and with only little additional computational effort) that the

[13] We have briefly discussed it in Sec. 1.4.2.

symmetries and the Poincaré algebra are sufficient to determine all constraints between Wilson coefficients and to fix the form of the boost generator in NRQCD.

In the next chapter, we apply this EFT method to another low-energy EFT of QCD, namely potential NRQCD (pNRQCD), which is an EFT for a heavy quarkonium (heavy quark-antiquark bound state).

Chapter 3

Spacetime symmetries in potential NRQCD

In this chapter, we introduce another low-energy effective theory of QCD involving heavy quarkonium (bound states of a heavy quark and a heavy antiquark), namely potential NRQCD (pNRQCD). We discuss its hierarchy of scales, both in the weakly-coupled and strongly-coupled cases, and the corresponding actions. In pNRQCD the Wilson coefficients appear in the form of potentials. We then investigate the Poincaré transformations in the most general way, as it was shown in the NRQCD section of the last chapter. From this approach, we obtain suitable expressions of the generators of Poincaré group, boost generator in particular. We apply the transformations to bilinear parts of the singlet, octet, and singlet-octet sector of the pNRQCD action, and its symmetry under the Poincaré group yields some non-trivial relations between the Wilson coefficients.

3.1 An effective field theory for heavy quarkonia

While NRQCD provides a useful tool for analyzing production or annihilation process involving a heavy quarkonium (such as a charmonium or bottomonium), it is less suited to explain the mass splittings or spectrum of the bound states. In this case, potential non-relativistic QCD (pNRQCD) gives a valid description of transitions or mass splittings of the heavy quarkonium states [8, 80]. PNRQCD is a low-energy EFT obtained from NRQCD after integrating out the scale of the relative momentum between a heavy quark and an antiquark. In this section, we discuss the hierarchy of scales and the Lagrangian of pNRQCD.

3.1.1 Hierarchy of scales

As we have seen in the previous chapters, the Lagrangian of an EFT which involves a heavy scale is organized as an expansion in the scale that has been integrated out, which is the quark mass M. The schematic form of its Lagrangian is given in Eq. (1.79).

In the case of pNRQCD one proceeds by integrating out the next relevant scale, which is the scale of the momentum transfer between the heavy quark and the heavy antiquark, and the originating matching coefficients play the role of potentials. Then, one should consider the relation between the three different scales (apart from the heavy scale M) arising in heavy quarkonium systems: $p \sim Mv, E \sim Mv^2$, and $\Lambda_{\rm QCD}$, where v is the relative velocity, p is the relative momentum, and E is the binding energy between the quark and the antiquark. For quarkonium systems, in which the hierarchy $Mv \gg \Lambda_{\rm QCD}$ is satisfied (which are the lowest quarkonium systems in bottomonium and charmonium with a typical radius smaller than the inverse of the confinement scale), the integration of the relative momentum can be done in perturbation theory. On the other hand, if the binding energy is less than the typical hadronic scale $\Lambda_{\rm QCD} \gg Mv^2$ (which is the strong-coupling regime), then $\Lambda_{\rm QCD}$ may be integrated out as well. The resulting effective Lagrangian is somewhat simpler than in weakly coupled pNRQCD (for which $\Lambda_{\rm QCD} \lesssim Mv^2$) due to the absence of colored degrees of freedom, although the matching is more complicated and in general non-perturbative[1]. The weak-coupling case in particular shows a proliferation of Wilson coefficients as the number of effective operators increases.

3.1.2 Lagrangian

The pNRQCD Lagrangian is obtained from NRQCD after integrating out the scale of the relative momentum Mv between a heavy quark and a heavy antiquark. In weakly-coupled pNRQCD, we also assume that this scale is much larger than $\Lambda_{\rm QCD}$, which implies that the matching can be carried out in a perturbative way. Since the relative momentum scale is of the same order as the inverse of the quark-antiquark distance, $Mv \sim 1/r$, integrating out this scale corresponds to a multipole expansion. The effective degrees of freedom are now heavy quarkonium fields instead of separate heavy quark and antiquark fields, since after integrating out the scale $1/r$, the other degrees of freedom (ultrasoft gluons and light quarks) can no longer resolve the individual fields. The heavy quark and the antiquark can form either a color singlet or an octet state in SU(3), so the quarkonium fields of pNRQCD appear as singlet S and octet O^a. These fields are the only ones that can depend[2] on the relative distance $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ as well as the center-of-mass coordinate $\boldsymbol{R} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$, while all other fields depend only on $\boldsymbol{R}$. Then the Lagrangian of weakly-coupled pNRQCD can schematically be written as

$$\begin{aligned}\mathcal{L}_{\rm pNRQCD}^{\rm weak} = &\int d^3r \, {\rm Tr}\left[{\rm S}^\dagger (i\partial_0 - h_S){\rm S} + O^{a\,\dagger}(iD_0^{ab} - h_O^{ab})O^b - ({\rm S}^\dagger h_{SO}^a O^a + h.c.)\right] \\ &+ \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm light}\,,\end{aligned} \tag{3.1}$$

where $h.c.$ stands for the hermitian conjugate of the last term, and the trace refers to the spin indices of the quarkonium fields. The explicit expressions of h_S, h_O, and h_{SO} are not immediately required for the following discussions, so we postpone them until

[1]We discuss the strong coupling regime in the next chapters.

[2]$\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the spatial coordinates of the heavy quark and the heavy antiquark, respectively.

they become relevant; h_S and h_O are found in Eqs. (3.52), (3.56), respectively, and h_{SO} is given by Eqs. (3.59) and (3.60). As usual, they contain all derivative or interaction terms allowed by the symmetries. In addtion, $\mathcal{L}_{\rm YM}$ is the ultrasoft gluon sector, and $\mathcal{L}_{\rm light}$ is the light quark sector.

Sometimes it is more convenient to write the quarkonium fields as matrices in color space:

$$S = \frac{1}{\sqrt{3}} \mathrm{S}\, \mathbb{1}\,, \qquad O = \sqrt{2} O^a T^a\,, \tag{3.2}$$

so that the Lagrangian is rewritten as

$$\begin{aligned} \mathcal{L}^{\rm weak}_{\rm pNRQCD} = \int d^3r\, \mathrm{Tr}\, \Big[& S^\dagger (i\partial_0 - h_S) S + O^\dagger i D_0 O - \Big(O^\dagger h_O O + c.c. \Big) \\ & - \Big(S^\dagger h_{SO} O + h.c. \Big) \Big] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{\rm light} \end{aligned} \tag{3.3}$$

where the trace here is understood both in spin and in color spaces, and *c.c.* stands for the charge conjugate of the preceding term within the parenthesis. The coefficients for the matrices are given such that the trace over two fields is properly normalized, and the covariant derivatives are understood as commutators with all terms to their right.

On the other hand, when the hierarchy of scales is given by $Mv \gtrsim \Lambda_{\rm QCD}$ or $\Lambda_{\rm QCD} \gg Mv^2$, the theory enters the strong coupling regime. In this case, the pNRQCD Lagrangian is obtained after integrating out the hadronic scale $\Lambda_{\rm QCD}$, which means that all colored degrees of freedom are absent [80]:

$$\mathcal{L}^{\rm strong}_{\rm pNRQCD} = \int d^3R \int d^3r S^\dagger \left(i\partial_0 - h_s \right) S\,, \tag{3.4}$$

where

$$h_s \equiv \frac{\mathbf{p}_1^2}{2M_1} + \frac{\mathbf{p}_2^2}{2M_2} + V_S\,. \tag{3.5}$$

$M_{1,2}$ is the mass of the heavy quark (or antiquark, respectively), $\mathbf{p}_{1,2}$ is the momentum of the quark (or the antiquark), and V_S is the singlet potential in the long-distance regime. In this chapter, we are focusing on deriving constraints between the Wilson coefficients of the weakly-coupled pNRQCD Lagrangian, so any superscript on the Lagrangian will be omitted in the following discussions.

The matching between NRQCD and weakly-coupled pNRQCD is performed through interpolating fields

$$\begin{aligned} & \chi^\dagger(\boldsymbol{R} - \boldsymbol{r}/2) \phi(\boldsymbol{R} - \boldsymbol{r}/2, \boldsymbol{R} + \boldsymbol{r}/2) \psi(\boldsymbol{R} + \boldsymbol{r}/2) \\ & \to Z_S^{(0)}(r) \mathrm{S}(\boldsymbol{r}, \boldsymbol{R}) + Z_O^{(2)}(r) r\boldsymbol{r} \cdot g\boldsymbol{E}^a(\boldsymbol{R}) O^a(\boldsymbol{r}, \boldsymbol{R}) + \mathcal{O}(r^3)\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} & \chi^\dagger(\boldsymbol{R} - \boldsymbol{r}/2) \phi(\boldsymbol{R} - \boldsymbol{r}/2, \boldsymbol{R}) T^a \phi(\boldsymbol{R}, \boldsymbol{R} + \boldsymbol{r}/2) \psi(\boldsymbol{R} + \boldsymbol{r}/2) \\ & \to Z_O^{(0)}(r) O^a(\boldsymbol{r}, \boldsymbol{R}) + Z_S^{(2)}(r) r\boldsymbol{r} \cdot g\boldsymbol{E}^a(\boldsymbol{R}) \mathrm{S}(\boldsymbol{r}, \boldsymbol{R}) + \mathcal{O}(r^3)\,. \end{aligned} \tag{3.7}$$

where the Wilson line ϕ acts as a gauge link from the heavy quark position to that of the heavy antiquark. Correlators of those interpolating fields in both theories are supposed to give the same result, which determines the matching coefficients Z.

3.2 Symmetry transformations in pNRQCD

3.2.1 Discrete symmetries

The interpolating fields determine how the coordinate dependence of the quarkonium fields behaves under the spacetime symmetries as well. In fact, in the limit $g \to 0$ one can neglect the Wilson lines and just determine the transformation of singlet and octet from different color projections of $Q = \psi\chi^\dagger$. The coordinate transformations do not depend on the color representation, so we use Q for both singlet and octet.

First, we give here the transformations under the discrete symmetries:

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{P} -Q(t,-\boldsymbol{r},-\boldsymbol{R})\,, \tag{3.8}$$

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{C} \sigma_2 Q^T(t,-\boldsymbol{r},\boldsymbol{R})\sigma_2\,, \tag{3.9}$$

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{T} \sigma_2 Q(-t,\boldsymbol{r},\boldsymbol{R})\sigma_2\,, \tag{3.10}$$

where the transpose on the charge conjugated field refers both to color and spin space. Also note that charge conjugation exchanges the positions of the quark and the antiquark fields, so that $\boldsymbol{r}$ goes to $-\boldsymbol{r}$.

Then, we list here how the boost generators, $\boldsymbol{k}_Q$, are required to behave under the discrete symmetries, parity, charge conjugation, and time reversal:

$$P\boldsymbol{k}_Q = -\boldsymbol{k}_Q\,, \qquad C\boldsymbol{k}_Q = \sigma_2\boldsymbol{k}_Q^T\sigma_2\,, \qquad T\boldsymbol{k}_Q = \sigma_2\boldsymbol{k}_Q\sigma_2\,. \tag{3.11}$$

Note that P changes the sign of both $\boldsymbol{r}$ and $\boldsymbol{R}$, C changes the sign of only $\boldsymbol{r}$, T changes the sign of t and takes the complex conjugate. For the singlet field, the transposition inherent to the C transformation is trivial in color space, while for the octet field in matrix notation, we have to write the boost generator in two separate parts:

$$O \xrightarrow{K} O' = O - i\boldsymbol{\eta}\cdot\left(\boldsymbol{k}_O^{(A)}O + O\boldsymbol{k}_O^{(B)}\right)\,. \tag{3.12}$$

These two parts are exchanged under C as $\boldsymbol{k}_O^{(A),(B)} \xrightarrow{C} \sigma_2\left(\boldsymbol{k}_O^{(B),(A)}\right)^T\sigma_2$.

3.2.2 Spacetime symmetries

Spacetime translations

We continue to work with the analogy between a quarkonium field Q and the unconnected heavy quark pair $\psi\chi^\dagger$. Time translation is straightforward in pNRQCD; ψ and $\chi^\dagger$ are evaluated at the same time, so the time argument of the quarkonium fields is shifted in the same way. The additional mass terms introduced through the field redefinitions of ψ and χ add up, which gives the following transformation

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{P_0} Q'(t,\boldsymbol{r},\boldsymbol{R}) = (1-2iMa_0)Q(t,\boldsymbol{r},\boldsymbol{R}) + \left[a_0\partial_0, Q(t,\boldsymbol{r},\boldsymbol{R})\right]\,. \tag{3.13}$$

We have implicitly assumed that the quark and the antiquark fields have the same mass M, so that the generator of time translation is $P_0 = i\partial_0 + 2M$.

Spatial translations act only on the center-of-mass coordinate $\boldsymbol{R}$; both the heavy quark and antiquark are shifted by the same amount, so the relative coordinate remains unaffected. This means

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{P_i} Q'(t,\boldsymbol{r},\boldsymbol{R}) = Q(t,\boldsymbol{r},\boldsymbol{R}) + \left[\boldsymbol{a}\cdot\boldsymbol{\nabla}_R, Q(t,\boldsymbol{r},\boldsymbol{R})\right], \tag{3.14}$$

with the generator for space translations $\boldsymbol{P} = -i\boldsymbol{\nabla}_R$.

Under rotations, both the center-of-mass and the relative coordinates transform in the same way. The component transformations of ψ and χ lead to a commutator with the quarkonium fields and the sigma matrices:

$$Q(t,\boldsymbol{r},\boldsymbol{R}) \xrightarrow{J} Q'(t,\boldsymbol{r},\boldsymbol{R}) = Q(t,\boldsymbol{r},\boldsymbol{R}) + \left[\boldsymbol{\alpha}\cdot\left(\boldsymbol{R}\times\boldsymbol{\nabla}_R + \boldsymbol{r}\times\boldsymbol{\nabla}_r + \frac{i}{2}\boldsymbol{\sigma}\right), Q(t,\boldsymbol{r},\boldsymbol{R})\right]. \tag{3.15}$$

With the convention for the sigma matrices from Eq. (3.23), this gives the generator of rotations as $\boldsymbol{J} = \boldsymbol{R}\times(-i\boldsymbol{\nabla}_R) + \boldsymbol{r}\times(-i\boldsymbol{\nabla}_r) + \left(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}\right)/2$. From this, it is straightforward to see that

$$Q_1 = \frac{1}{\sqrt{2}}\mathrm{Tr}[Q] \qquad \text{and} \qquad \boldsymbol{Q}_3 = \frac{1}{\sqrt{2}}\mathrm{Tr}[\boldsymbol{\sigma}Q] \tag{3.16}$$

transform as a singlet (scalar) and triplet (vector) respectively under rotations (the trace is understood only in spin space). We can decompose the matrix valued quarkonium field as

$$Q = \frac{1}{\sqrt{2}}Q_1\mathbb{1} + \frac{1}{\sqrt{2}}\boldsymbol{Q}_3\cdot\boldsymbol{\sigma}. \tag{3.17}$$

The bilinears in the Lagrangian then give

$$\mathrm{Tr}\left[Q^\dagger Q\right] = Q_1^\dagger Q_1 + \boldsymbol{Q}_3^\dagger\cdot\boldsymbol{Q}_3, \tag{3.18}$$

$$\mathrm{Tr}\left[Q^\dagger\frac{i\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)}{2}Q\right] = \boldsymbol{Q}_3^\dagger\times\boldsymbol{Q}_3, \quad \text{and} \quad \mathrm{Tr}\left[Q^\dagger\frac{\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}}{2}Q\right] = \boldsymbol{Q}_3^\dagger Q_1 + Q_1^\dagger\boldsymbol{Q}_3. \tag{3.19}$$

Boosts

Under boosts, the coordinate transformations are composed of the individual boosts of the heavy (anti)quark fields at $\boldsymbol{x}_1 = \boldsymbol{R} + \boldsymbol{r}/2$ and $\boldsymbol{x}_2 = \boldsymbol{R} - \boldsymbol{r}/2$:

$$\begin{aligned}
&\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2) \xrightarrow{K} \psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2) - iM\boldsymbol{\eta}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\\
&\quad + \left[\boldsymbol{\eta}\cdot(t\boldsymbol{\nabla}_1 + \boldsymbol{x}_1\partial_0), \psi(t,\boldsymbol{x}_1)\right]\chi^\dagger(t,\boldsymbol{x}_2) + \psi(t,\boldsymbol{x}_1)\left[\boldsymbol{\eta}\cdot(t\boldsymbol{\nabla}_2 + \boldsymbol{x}_2\partial_0), \chi^\dagger(t,\boldsymbol{x}_2)\right] + \dots\\
&= (1 - 2iM\boldsymbol{\eta}\cdot\boldsymbol{R})\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2) + \left[\boldsymbol{\eta}\cdot(t\boldsymbol{\nabla}_R + \boldsymbol{R}\partial_0), \psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\right]
\end{aligned}$$

$$+\frac{1}{2}(\boldsymbol{\eta}\cdot\boldsymbol{r})\left(\big[\partial_0,\psi(t,\boldsymbol{x}_1)\big]\chi^\dagger(t,\boldsymbol{x}_2)-\psi(t,\boldsymbol{x}_1)\big[\partial_0,\chi^\dagger(t,\boldsymbol{x}_2)\big]\right)+\dots\,, \tag{3.20}$$

where the ellipsis in the last line stands for all terms of the boost transformation that are not related to the coordinate transformation (this is shown in Eq. (3.22)). The first two terms on the right-hand side of the equality sign correspond to the usual coordinate transformations under boosts for a scalar field with mass $2M$, where only the center-of-mass coordinate participates in the boost and the relative distance remains unaffected. The third term on the right-hand side, the time derivatives acting on the quark and antiquark fields, cannot be written as one derivative acting on the whole quarkonium field because they have opposite sign. However, these time derivatives can be replaced by spatial derivatives through the equations of motion:

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{\eta}\cdot\boldsymbol{r})\left(\big[\partial_0,\psi(t,\boldsymbol{x}_1)\big]\chi^\dagger(t,\boldsymbol{x}_2)-\psi(t,\boldsymbol{x}_1)\big[\partial_0,\chi^\dagger(t,\boldsymbol{x}_2)\big]\right)\\&=(\boldsymbol{\eta}\cdot\boldsymbol{r})\left[\frac{i}{4M}(\boldsymbol{\nabla}_1^2-\boldsymbol{\nabla}_2^2),\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\right]+\mathcal{O}(M^{-3})\\&=(\boldsymbol{\eta}\cdot\boldsymbol{r})\left[\frac{i}{2M}\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r,\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\right]+\mathcal{O}(M^{-3})\,.\end{aligned} \tag{3.21}$$

Thus these terms give corrections of order $1/M$ and higher.

The other terms in the boost transformation of the quark and antiquark fields in the $g\to 0$ limit can also be rewritten in terms of the center of mass and relative coordinates, $\boldsymbol{R}$ and $\boldsymbol{r}$:

$$\begin{aligned}&\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\xrightarrow{K}\dots+\frac{i}{2M}\big[\boldsymbol{\eta}\cdot(\boldsymbol{\nabla}_1+\boldsymbol{\nabla}_2),\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\big]\\&\quad-\frac{1}{4M}\big[(\boldsymbol{\eta}\times\boldsymbol{\nabla}_1)\cdot,\boldsymbol{\sigma}\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\big]+\frac{1}{4M}\big[(\boldsymbol{\eta}\times\boldsymbol{\nabla}_2)\cdot,\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\boldsymbol{\sigma}\big]+\mathcal{O}(M^{-2})\\&=\dots+\frac{i}{2M}\big[\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_R,\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\big]-\frac{1}{8M}\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\cdot\big[(\boldsymbol{\eta}\times\boldsymbol{\nabla}_R),\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\big]\\&\quad-\frac{1}{4M}\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\cdot\big[(\boldsymbol{\eta}\times\boldsymbol{\nabla}_r),\psi(t,\boldsymbol{x}_1)\chi^\dagger(t,\boldsymbol{x}_2)\big]+\mathcal{O}(M^{-3})\,.\end{aligned} \tag{3.22}$$

The ellipsis here denotes the terms concerning the coordinate transformation, Eq. (3.20). We have also introduced the convenient notation

$$\boldsymbol{\sigma}^{(1)}Q=\boldsymbol{\sigma}Q\qquad\text{and}\qquad\boldsymbol{\sigma}^{(2)}Q=-Q\boldsymbol{\sigma}\,. \tag{3.23}$$

This implies that $\boldsymbol{\sigma}^{(1)}$ acts on the spin of the heavy quark and $\boldsymbol{\sigma}^{(2)}$ acts on the spin of the heavy antiquark (they correspond to the respective generators of rotations). Since

$$\sigma_2\boldsymbol{\sigma}\sigma_2=-\boldsymbol{\sigma}^T\,,\quad\text{and}\quad\boldsymbol{\sigma}^TQ^T=(Q\boldsymbol{\sigma})^T \tag{3.24}$$

charge conjugation effectively exchanges $\boldsymbol{\sigma}^{(1)}\leftrightarrow\boldsymbol{\sigma}^{(2)}$.

From these expressions, we expect the boost generator in the $g\to 0$ limit to behave like [77]

$$\boldsymbol{k}_Q\overset{g\to 0}{=}it\boldsymbol{\nabla}_R+i\boldsymbol{R}\partial_0+2M\boldsymbol{R}-\frac{1}{4M}\boldsymbol{\nabla}_R-\frac{1}{4M}\big\{\boldsymbol{r},(\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r)\big\}$$

$$- \frac{i}{8M} \boldsymbol{\nabla}_R \times \left(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} \right) - \frac{i}{4M} \boldsymbol{\nabla}_r \times \left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \right) + \mathcal{O}(M^{-3}) \,. \qquad (3.25)$$

This limit is interesting for the ansatz we are going to make for the singlet and octet boost generators, since it determines which coefficients we expect to be of order $1+\mathcal{O}(\alpha_s)$. In the last two terms of the first line, we have used

$$\boldsymbol{r}(\boldsymbol{\nabla}_R \cdot \boldsymbol{\nabla}_r) = \frac{1}{2}\big\{\boldsymbol{r}, (\boldsymbol{\nabla}_R \cdot \boldsymbol{\nabla}_r)\big\} - \frac{1}{2}\boldsymbol{\nabla}_R \,, \qquad (3.26)$$

in order to obtain terms that are explicitly hermitian or antihermitian.

3.2.3 Redundancies from Poincaré transformations

In order to find the boost generators in pNRQCD, we will use the EFT approach again and write down the most general form allowed by the symmetries of the theory. However, it turns out that several terms in this ansatz are redundant. In other words, one can make a field redefinition that removes these terms from the boost generator without changing the form of the Lagrangian. Thus, there is no loss in generality if one chooses to work with a boost generator where these redundant terms are absent. We will work out appropriate field redefinitions in this section. Since we calculate the transformation of the Lagrangian up to orders $M^0 r^1$ and $M^{-1} r^0$ in the next section, it is necessary to include all terms of order $M^0 r^2$, $M^{-1} r^0$. We will use the notation $c^{(m,n)}$ for the Wilson coefficients of terms of order $M^{-m} r^n$.

3.2.4 Field redefinitions by unitary transformation

Singlet field

Even though we work with a general ansatz, some terms may be omitted from the beginning, which is similar to the construction of the pNRQCD Lagrangian. A term like $\boldsymbol{r} \cdot \boldsymbol{\nabla}_r$, for example, is neutral with respect to any symmetry and also the power counting. In principle one could add an infinite number of these terms to any operator in the Lagrangian, which implies that at each order in the power counting, one would have to match an infinite number of terms, making the construction of the EFT impossible. In comparison to NRQCD, however, one sees that each derivative appears with at least one power of $1/M$, so also in pNRQCD one can neglect any term where there are more derivatives than powers of $1/M$. The same argument applies to spin-dependent terms, where each sigma matrix has to be suppressed by a power of $1/M$. The only exception to this are the kinetic terms, where there is one derivative more than powers of $1/M$.

By extension, these rules also apply to the construction of the boost generators in the following way. Operators leading to terms in the transformation of the Lagrangian which would have to be canceled by derivative or spin terms with an insufficient $1/M$ suppression are immediately ruled out. Keeping this in mind and writing everything in terms of explicitly hermitian or antihermitian operators (where we stay close to the nomenclature in [22]), the most general ansatz for the boost generator of the singlet is given by [77]:

$$
\begin{aligned}
\boldsymbol{k}_S = {} & it\boldsymbol{\nabla}_R + i\boldsymbol{R}\partial_0 + 2M\boldsymbol{R} - \frac{k_{SD}^{(1,0)}}{4M}\boldsymbol{\nabla}_R - \frac{1}{4M}\left\{k_{Sa'}^{(1,0)}\boldsymbol{r}, (\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r)\right\} \\
& - \frac{1}{4M}\left\{k_{Sa''}^{(1,0)}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R), \boldsymbol{\nabla}_r\right\} - \frac{1}{4M}\left\{k_{Sa'''}^{(1,0)}\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\}\boldsymbol{\nabla}_R \\
& - \frac{1}{4M}\left\{\frac{k_{Sb}^{(1,0)}}{r^2}\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)r_i, (\nabla_r)_i\right\} - \frac{ik_{Sc}^{(1,0)}}{8M}\boldsymbol{\nabla}_R\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right) \\
& - \frac{ik_{Sd''}^{(1,0)}}{8Mr^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right) - \frac{ik_{Sd'''}^{(1,0)}}{8Mr^2}\left((\boldsymbol{r}\times\boldsymbol{\nabla}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r} \\
& - \frac{i}{8M}\left\{k_{Sa}^{(1,-1)}, \boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\} + \frac{i}{8M}\left[\frac{k_{Sb'}^{(1,-1)}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\times, \boldsymbol{\nabla}_r\right] \\
& - \frac{i}{8M}\left\{\frac{k_{Sb''}^{(1,-1)}}{r^2}\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)r_i, (\nabla_r)_i\right\} + \mathcal{O}\left(M^{-2}r^0, M^{-1}r^1, M^0r^3\right) .
\end{aligned}
\tag{3.27}
$$

Since the Wilson coefficients here depend on r, they have to be included inside the anticommutators with the derivative $\boldsymbol{\nabla}_r$. We have used the identity

$$
\delta_{ij}\epsilon_{klm} = \delta_{ik}\epsilon_{jlm} + \delta_{il}\epsilon_{kjm} + \delta_{im}\epsilon_{klj} \,, \tag{3.28}
$$

in order to eliminate several terms. A term like

$$
ik_{Sd'}^{(1,0)}(\boldsymbol{r}\times\boldsymbol{\nabla}_R)(\boldsymbol{r}\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}))/(8Mr^2) \,, \tag{3.29}
$$

for instance, is not linearly independent from other terms in this ansatz, because it is related to the operators of $k_{Sc}^{(1,0)}$, $k_{Sd''}^{(1,0)}$, and $k_{Sd'''}^{(1,0)}$ through this identity; this can be shown by multiplying Eq. (3.28) with

$$
r_i r_j(\nabla_R)_k(\sigma^{(1)}+\sigma^{(2)})_l \,. \tag{3.30}
$$

A similar situation is found for the operators of the coefficients, $k_{Sa}^{(1,-1)}$, $k_{Sb'}^{(1,-1)}$, and $k_{Sb''}^{(1,-1)}$.

Not all these terms in the boost generator, Eq. (3.27), are necessary if one exploits the freedom to perform field redefinitions. In other words, one can always redefine the fields as long as the symmetry properties of the fields are not altered. In order to keep the form of the Lagrangian intact after the field redefinitions, we will only consider unitary transformations

$$
\mathcal{U}_S = \exp[u_S] \,, \tag{3.31}
$$

where u_S is antihermitian, for which the new singlet field $\widetilde{S}$ is related to the old S via $S = \mathcal{U}_S\widetilde{S}$ [22]. The reason for choosing unitary transformations[3] in particular is that the time derivative from the leading term of the Lagrangian, Eq. (3.3), appears only in commutators,

$$\mathcal{U}_S^\dagger i\partial_0\mathcal{U}_S = i\partial_0 + [i\partial_0, u_S] + \frac{1}{2}[[i\partial_0, u_S], u_S] + \frac{1}{6}[[[i\partial_0, u_S], u_S], u_S] + \dots \,. \tag{3.32}$$

In this and the analogous expression for the redefinition of the octet field, the commutators with the time derivative either vanish, give a chromoelectric field, or time derivatives of gluon fields, which can be removed through the equations of motion (i.e., redefinitions of the gluon fields). Thus, a unitary transformation does not introduce new time derivatives in the Lagrangian. It will introduce other terms, but those will be of a form already present in the Lagrangian, so that their contributions can be absorbed in a redefinition of the Wilson coefficients.

In order to find a suitable unitary transformation, we need to look for terms which are antihermitian and P, C, and T invariant. Such terms can easily be found by multiplying the hermitian terms in $\boldsymbol{k}_S$ with $\boldsymbol{\nabla}_R/M$, which explains the nomenclature we use for $\mathcal{U}_S$:

$$\begin{aligned}
\mathcal{U}_S = \exp\Bigg[&-\frac{1}{4M^2}\left\{q^{(1,0)}_{Sa''}\boldsymbol{r}\cdot\boldsymbol{\nabla}_R, \boldsymbol{\nabla}_r\cdot\boldsymbol{\nabla}_R\right\} - \frac{1}{4M^2}\left\{q^{(1,0)}_{Sa'''}\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\}\boldsymbol{\nabla}_R^2 \\
&- \frac{1}{4M^2}\left\{\frac{q^{(1,0)}_{Sb}}{r^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)^2\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\} - \frac{iq^{(1,0)}_{Sd'''}}{8M^2r^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)\left((\boldsymbol{r}\times\boldsymbol{\nabla}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right) \\
&+ \frac{i}{8M^2}\left\{q^{(1,-1)}_{Sa}, (\boldsymbol{\nabla}_r\times\boldsymbol{\nabla}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\} \\
&- \frac{i}{8M^2}\left\{\frac{q^{(1,-1)}_{Sb'}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\times\boldsymbol{\nabla}_R)\cdot, \boldsymbol{\nabla}_r\right\} \\
&+ \frac{i}{8M^2}\left\{\frac{q^{(1,-1)}_{Sb''}}{r^2}\left((\boldsymbol{r}\times\boldsymbol{\nabla}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\} + \dots\Bigg]\,,
\end{aligned} \tag{3.33}$$

where the ellipsis stands for higher order terms in $1/M$, which do not affect the calculations of this paper. The coefficients q are free parameters.

We can work out the transformation of the new singlet field $\widetilde{S}$ under boosts in the following way:

$$\begin{aligned}
\widetilde{S}' &= \mathcal{U}'^\dagger_S S' = \mathcal{U}'^\dagger_S(1 - i\boldsymbol{\eta}\cdot\boldsymbol{k}_S)\mathcal{U}_S\widetilde{S} = \left[1 - \mathcal{U}_S^\dagger(i\boldsymbol{\eta}\cdot\boldsymbol{k}_S)\mathcal{U}_S + \left(\delta\mathcal{U}_S^\dagger\right)\mathcal{U}_S\right]\widetilde{S} \\
&\equiv \left(1 - i\boldsymbol{\eta}\cdot\widetilde{\boldsymbol{k}}_S\right)\widetilde{S}\,,
\end{aligned} \tag{3.34}$$

where

$$\delta\mathcal{U}_S^\dagger(\boldsymbol{\nabla}_R, \boldsymbol{E}, \boldsymbol{B}) = \left[\boldsymbol{\eta}\cdot(t\boldsymbol{\nabla}_R + \boldsymbol{R}\partial_0), \mathcal{U}_S^\dagger(\boldsymbol{\nabla}_R, \boldsymbol{E}, \boldsymbol{B})\right]$$

[3]Another reason is that unitary transformations keep Hermitian operators Hermitian and do not modify the Poincaré algebra.

$$+\mathcal{U}_S^\dagger(\boldsymbol{\nabla}_R+\boldsymbol{\eta}\partial_0,\boldsymbol{E}+\boldsymbol{\eta}\times\boldsymbol{B},\boldsymbol{B}-\boldsymbol{\eta}\times\boldsymbol{E})-\mathcal{U}_S^\dagger(\boldsymbol{\nabla}_R,\boldsymbol{E},\boldsymbol{B})\,, \quad (3.35)$$

with the second line expanded to linear order in $\boldsymbol{\eta}$.

The transformed boost generator $\widetilde{\boldsymbol{k}}_S$ has to be expanded to the same order as the original $\boldsymbol{k}_S$, for which one extra term remains:

$$\begin{aligned}\widetilde{\boldsymbol{k}}_S &= \boldsymbol{k}_S+\left[\hat{\boldsymbol{k}}_S,u_S\right]-u_S(\boldsymbol{\nabla}_R+\boldsymbol{\eta}\partial_0,\boldsymbol{E}+\boldsymbol{\eta}\times\boldsymbol{B},\boldsymbol{B}-\boldsymbol{\eta}\times\boldsymbol{E})+u_S(\boldsymbol{\nabla}_R,\boldsymbol{E},\boldsymbol{B})\\ &\quad+\frac{1}{2}\left[\left[\hat{\boldsymbol{k}}_S,u_S\right]-u_S(\boldsymbol{\nabla}_R+\boldsymbol{\eta}\partial_0,\boldsymbol{E}+\boldsymbol{\eta}\times\boldsymbol{B},\boldsymbol{B}-\boldsymbol{\eta}\times\boldsymbol{E})+u_S(\boldsymbol{\nabla}_R,\boldsymbol{E},\boldsymbol{B}),u_S\right]+\dots\\ &= \boldsymbol{k}_S+[2M\boldsymbol{R},u_S]+\mathcal{O}(M^{-2})\,. \end{aligned} \quad (3.36)$$

Inserting the explicit field redefinition from Eq. (3.33), we obtain

$$\begin{aligned}\widetilde{\boldsymbol{k}}_S &= \boldsymbol{k}_S+\frac{1}{2M}\left\{q_{Sa''}^{(1,0)}\boldsymbol{r},(\boldsymbol{\nabla}_R\cdot\boldsymbol{\nabla}_r)\right\}+\frac{1}{2M}\left\{q_{Sa''}^{(1,0)}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R),\boldsymbol{\nabla}_r\right\}\\ &\quad+\frac{1}{M}\left\{q_{Sa'''}^{(1,0)}\boldsymbol{r}\cdot,\boldsymbol{\nabla}_r\right\}\boldsymbol{\nabla}_R+\frac{1}{M}\left\{\frac{q_{Sb}^{(1,0)}}{r^2}\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)r_i,(\nabla_r)_i\right\}\\ &\quad-\frac{iq_{Sd'''}^{(1,0)}}{4Mr^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)+\frac{iq_{Sd'''}^{(1,0)}}{4Mr^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\nabla}_R\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right)\boldsymbol{r}\\ &\quad+\frac{i}{4M}\left\{q_{Sa}^{(1,-1)},\boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\}-\frac{i}{4M}\left[\frac{q_{Sb'}^{(1,-1)}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\times,\boldsymbol{\nabla}_r\right]\\ &\quad+\frac{i}{4M}\left\{\frac{q_{Sb''}^{(1,-1)}}{r^2}\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)r_i,(\nabla_r)_i\right\}+\mathcal{O}\left(M^{-2}\right)\,. \end{aligned} \quad (3.37)$$

These extra terms can be absorbed in the operators already present in Eq. (3.27), which changes the coefficients in the following way:

$$\begin{aligned}&\widetilde{k}_{Sa'}^{(1,0)}=k_{Sa'}^{(1,0)}-2q_{Sa''}^{(1,0)}\,, &&\widetilde{k}_{Sa''}^{(1,0)}=k_{Sa''}^{(1,0)}-2q_{Sa''}^{(1,0)}\,, &&\widetilde{k}_{Sa'''}^{(1,0)}=k_{Sa'''}^{(1,0)}-4q_{Sa'''}^{(1,0)}\,,\\ &\widetilde{k}_{Sb}^{(1,0)}=k_{Sb}^{(1,0)}-4q_{Sb}^{(1,0)}\,, &&\widetilde{k}_{Sd''}^{(1,0)}=k_{Sd''}^{(1,0)}+2q_{Sd'''}^{(1,0)}\,, &&\widetilde{k}_{Sd'''}^{(1,0)}=k_{Sd'''}^{(1,0)}-2q_{Sd'''}^{(1,0)}\,,\\ &\widetilde{k}_{Sa}^{(1,-1)}=k_{Sa}^{(1,-1)}-2q_{Sa}^{(1,-1)}\,, &&\widetilde{k}_{Sb'}^{(1,-1)}=k_{Sb'}^{(1,-1)}-2q_{Sb'}^{(1,-1)}\,, &&\widetilde{k}_{Sb''}^{(1,-1)}=k_{Sb''}^{(1,-1)}-2q_{Sb''}^{(1,-1)}\,.\end{aligned} \quad (3.38)$$

Seven of the free parameters q's in the unitary operator can be chosen in any convenient way. Comparing this to the expected result in the $g\to 0$ limit from Eq. (3.25), we choose to eliminate $\widetilde{k}_{Sa''}^{(1,0)}$, $\widetilde{k}_{Sa'''}^{(1,0)}$, $\widetilde{k}_{Sb}^{(1,0)}$, $\widetilde{k}_{Sd'''}^{(1,0)}$, $\widetilde{k}_{Sb'}^{(1,-1)}$, and $\widetilde{k}_{Sb''}^{(1,-1)}$, as well as to fix $\widetilde{k}_{Sa}^{(1,-1)}=1$. Then after dropping the tilde notation for the new field, the general boost transformation is simplified as follows [77]

$$\begin{aligned}\boldsymbol{k}_S &= it\boldsymbol{\nabla}_R+i\boldsymbol{R}\partial_0+2M\boldsymbol{R}-\frac{k_{SD}^{(1,0)}}{4M}\boldsymbol{\nabla}_R-\frac{1}{4M}\left\{k_{Sa'}^{(1,0)}\boldsymbol{r},(\boldsymbol{\nabla}_r\cdot\boldsymbol{\nabla}_R)\right\}\\ &\quad-\frac{ik_{Sc}^{(1,0)}}{8M}\boldsymbol{\nabla}_R\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)-\frac{ik_{Sd''}^{(1,0)}}{8Mr^2}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R)\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\end{aligned}$$

$$- \frac{i}{4M} \boldsymbol{\nabla}_r \times \left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)} \right) + \mathcal{O}\left(M^{-2} r^0, M^{-1} r^1, M^0 r^3 \right) , \tag{3.39}$$

in which only four coefficients, $k_{SD}^{(1,0)}$, $k_{Sa'}^{(1,0)}$, $k_{Sc}^{(1,0)}$, and $k_{Sd''}^{(1,0)}$, remain undetermined.

Octet field

In a similar fashion, one can proceed to determine the most general form of the boost transformation for the octet field. The main difference from the singlet is that all center-of-mass derivatives (except for the coordinate transformations) have to be replaced by covariant derivatives in the adjoint representation $\boldsymbol{D}^{ab} = \delta^{ab}\boldsymbol{\nabla}_R - f^{abc} g \boldsymbol{A}^c$ due to the color charge of the octet field. There are no relevant operators at order $M^0 r^2$ for the singlet field, but for the case of the octet, there arise two operators involving the chromoelectric field. There are no new terms at order $M^{-1} r^0$.

We choose now to write the color components of the octet field explicitly instead of the matrix notation, for which the boost transformation is written as

$$O^a \xrightarrow{K} O^{a\,\prime} = \left(\delta^{ab} - i \boldsymbol{\eta} \cdot \boldsymbol{k}_O^{ab} \right) O^b \,. \tag{3.40}$$

The parity transformation of the boost generator in component notation is the same as in matrix notation. For the charge conjugation and time reversal transformation, we introduce a sign factor through $(T^a)^T = (T^a)^* = (-)^a T^a$ (the double appearance of the color index a in the last equality does not imply its summation). With this the fields in the adjoint representation transform as

$$O^a \xrightarrow{C} \sigma_2 (-)^a O^a \sigma_2 \,, \qquad E^a \xrightarrow{C} -(-)^a E^a \,, \qquad B^a \xrightarrow{C} -(-)^a B^a \,, \tag{3.41}$$

$$O^a \xrightarrow{T} \sigma_2 (-)^a O^a \sigma_2 \,, \qquad E^a \xrightarrow{T} (-)^a E^a \,, \qquad B^a \xrightarrow{T} -(-)^a B^a \,. \tag{3.42}$$

So the boost generator in component notation has to transform like

$$\boldsymbol{k}_O^{ab} \xrightarrow{C} (-)^a (-)^b \sigma_2 \left(\boldsymbol{k}_O^{ab} \right)^T \sigma_2 \,, \qquad \boldsymbol{k}_O^{ab} \xrightarrow{T} (-)^a (-)^b \sigma_2 \boldsymbol{k}_O^{ab} \sigma_2 \,. \tag{3.43}$$

For the sign factors one can use the following identities:

$$(-)^a (-)^b \delta^{ab} = \delta^{ab} \,, \qquad (-)^a (-)^b (-)^c f^{abc} = -f^{abc} \,, \qquad (-)^a (-)^b (-)^c d^{abc} = d^{abc} \,, \tag{3.44}$$

which follow from the commutation relations of the color matrices.

Then, the most general ansatz for the boost generator for octets is [77]:

$$\begin{aligned}
\boldsymbol{k}_O^{ab} = {} & \delta^{ab} (it \boldsymbol{\nabla}_R + i \boldsymbol{R} \partial_0 + 2M \boldsymbol{R}) - \frac{k_{OD}^{(1,0)}}{4M} \boldsymbol{D}_R^{ab} + \frac{i}{8} f^{abc} k_{Oa}^{(0,2)} (\boldsymbol{r} \cdot g \boldsymbol{E}^c) \boldsymbol{r} + \frac{i}{8} f^{abc} k_{Ob}^{(0,2)} r^2 g \boldsymbol{E}^c \\
& - \frac{1}{4M} \left\{ k_{Oa'}^{(1,0)} \boldsymbol{r}, (\boldsymbol{\nabla}_r \cdot \boldsymbol{D}_r^{ab}) \right\} - \frac{1}{4M} \left\{ k_{Oa''}^{(1,0)} (\boldsymbol{r} \cdot \boldsymbol{D}_R^{ab}), \boldsymbol{\nabla}_r \right\} - \frac{1}{4M} \left\{ k_{Oa'''}^{(1,0)} \boldsymbol{r} \cdot, \boldsymbol{\nabla}_r \right\} \boldsymbol{D}_R^{ab} \\
& - \frac{1}{4M} \left\{ \frac{k_{Ob}^{(1,0)}}{r^2} \boldsymbol{r} (\boldsymbol{r} \cdot \boldsymbol{D}_R^{ab}) r_i, (\nabla_r)_i \right\} - \frac{i k_{Oc}^{(1,0)}}{8M} \boldsymbol{D}_R^{ab} \times \left(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)} \right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{ik^{(1,0)}_{Od''}}{8Mr^2}\left(\boldsymbol{r}\cdot\boldsymbol{D}^{ab}_R\right)\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right) - \frac{ik^{(1,0)}_{Od'''}}{8Mr^2}\left((\boldsymbol{r}\times\boldsymbol{D}^{ab}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\\
&- \frac{i\delta^{ab}}{8M}\left\{k^{(1,-1)}_{Oa},\boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\} + \frac{i\delta^{ab}}{8M}\left[\frac{k^{(1,-1)}_{Ob'}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\times,\boldsymbol{\nabla}_r\right]\\
&- \frac{i\delta^{ab}}{8M}\left\{\frac{k^{(1,-1)}_{Ob''}}{r^2}\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)r_i,(\nabla_r)_i\right\} + \mathcal{O}\left(M^{-2}r^0, M^{-1}r^1, M^0r^3\right)\,.
\end{aligned}
\tag{3.45}
$$

We can again perform a redefinition of the octet field through a unitary transformation

$$
\widetilde{O}^a = \mathcal{U}^{ab}_O O^b\,, \quad \text{with} \quad \mathcal{U}_O = \exp[u_O]\,, \tag{3.46}
$$

in order to reduce the number of undetermined coefficients in $\boldsymbol{k}_O$. For this transformation matrix, the same arguments apply as in the singlet case, so that we write the antihermitian operator u_O as:

$$
\begin{aligned}
u^{ab}_O = &- \frac{q^{(0,2)}_{Oa}}{32M}\big\{(\boldsymbol{r}\cdot\boldsymbol{D}_R),(\boldsymbol{r}\cdot g\boldsymbol{E})\big\}^{ab} + \frac{q^{(0,2)}_{Ob}}{32M}r^2\big\{\boldsymbol{D}_R\cdot,g\boldsymbol{E}\big\}^{ab}\\
&- \frac{1}{4M^2}\left\{q^{(1,0)}_{Oa''}(\boldsymbol{r}\cdot\boldsymbol{D}_R),(\boldsymbol{\nabla}_r\cdot\boldsymbol{D}_R)\right\}^{ab} - \frac{1}{4M^2}\left\{q^{(1,0)}_{Oa'''}\boldsymbol{r}\cdot,\boldsymbol{\nabla}_r\right\}(\boldsymbol{D}^2_R)^{ab}\\
&- \frac{1}{4M^2}\left\{\frac{q^{(1,0)}_{Ob}}{r^2}\left((\boldsymbol{r}\cdot\boldsymbol{D}_R)^2\right)^{ab}\boldsymbol{r}\cdot,\boldsymbol{\nabla}_r\right\}\\
&- \frac{iq^{(1,0)}_{Od'''}}{16M^2r^2}\left\{(\boldsymbol{r}\cdot\boldsymbol{D}_R),\left((\boldsymbol{r}\times\boldsymbol{D}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right\}^{ab}\\
&+ \frac{i}{8M^2}\left\{q^{(1,-1)}_{Oa},(\boldsymbol{\nabla}_r\times\boldsymbol{D}^{ab}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\}\\
&- \frac{i}{8M^2}\left\{\frac{q^{(1,-1)}_{Ob'}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\times\boldsymbol{D}^{ab}_R)\cdot,\boldsymbol{\nabla}_r\right\}\\
&+ \frac{i}{8M^2}\left\{\frac{q^{(1,-1)}_{Ob''}}{r^2}\left((\boldsymbol{r}\times\boldsymbol{D}^{ab}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\cdot,\boldsymbol{\nabla}_r\right\} + \dots\,,
\end{aligned}
\tag{3.47}
$$

where $\{A,B\}^{ab} = A^{ab'}B^{b'b} + B^{ab'}A^{b'b}$, and it is understood that $\boldsymbol{E}^{ab} = -if^{abc}\boldsymbol{E}^c$.

Just like in the singlet case, the new boost generator (which corresponds to the new octet field) after this transformation is given by:

$$
\begin{aligned}
\widetilde{\boldsymbol{k}}^{ab}_O &= \boldsymbol{k}^{ab}_O + \left[2M\boldsymbol{R},u^{ab}_O\right] + \mathcal{O}\left(M^{-2}\right)\\
&= \boldsymbol{k}^{ab}_O - \frac{i}{8}f^{abc}q^{(0,2)}_{Oa}(\boldsymbol{r}\cdot g\boldsymbol{E}^c)\boldsymbol{r} - \frac{i}{8}f^{abc}q^{(0,2)}_{Ob}r^2 g\boldsymbol{E}^c\\
&\quad + \frac{1}{2M}\left\{q^{(1,0)}_{Oa''}\boldsymbol{r},(\boldsymbol{\nabla}_r\cdot\boldsymbol{D}^{ab}_R)\right\} + \frac{1}{2M}\left\{q^{(1,0)}_{Oa''}(\boldsymbol{r}\cdot\boldsymbol{D}^{ab}_R),\boldsymbol{\nabla}_r\right\}\\
&\quad + \frac{1}{M}\left\{q^{(1,0)}_{Oa'''}\boldsymbol{r}\cdot,\boldsymbol{\nabla}_r\right\}\boldsymbol{D}^{ab}_R + \frac{1}{M}\left\{\frac{q^{(1,0)}_{Ob}}{r^2}\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{D}^{ab}_R)r_i,(\nabla_r)_i\right\}
\end{aligned}
$$

$$
- \frac{iq^{(1,0)}_{Od'''}}{4Mr^2}\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\cdot\boldsymbol{D}^{ab}_R) + \frac{iq^{(1,0)}_{Od'''}}{4Mr^2}\left((\boldsymbol{r}\times\boldsymbol{D}^{ab}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}
+ \frac{i\delta^{ab}}{4M}\left\{q^{(1,-1)}_{Oa},\boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right\} - \frac{i\delta^{ab}}{4M}\left[\frac{q^{(1,-1)}_{Ob'}}{r^2}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\boldsymbol{r}\times,\boldsymbol{\nabla}_r\right]
+ \frac{i\delta^{ab}}{4M}\left\{\frac{q^{(1,-1)}_{Ob''}}{r^2}\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)r_i,(\nabla_r)_j\right\} + \mathcal{O}(M^{-2})\,. \tag{3.48}
$$

This formally gives the same relations for the transformed boost coefficients as for the singlet, with the addition of the two coefficients for the chromoelectric field terms:

$$
\begin{aligned}
&\widetilde{k}^{(0,2)}_{Oa} = k^{(0,2)}_{Oa} - q^{(0,2)}_{Oa}\,, && \widetilde{k}^{(0,2)}_{Ob} = k^{(0,2)}_{Ob} - q^{(0,2)}_{Ob}\,, && \\
&\widetilde{k}^{(1,0)}_{Oa'} = k^{(1,0)}_{Oa'} - 2q^{(1,0)}_{Oa''}\,, && \widetilde{k}^{(1,0)}_{Oa''} = k^{(1,0)}_{Oa''} - 2q^{(1,0)}_{Oa''}\,, && \widetilde{k}^{(1,0)}_{Oa'''} = k^{(1,0)}_{Oa'''} - 4q^{(1,0)}_{Oa'''}\,, \\
&\widetilde{k}^{(1,0)}_{Ob} = k^{(1,0)}_{Ob} - 4q^{(1,0)}_{Ob}\,, && \widetilde{k}^{(1,0)}_{Od''} = k^{(1,0)}_{Od''} + 2q^{(1,0)}_{Od'''}\,, && \widetilde{k}^{(1,0)}_{Od'''} = k^{(1,0)}_{Od'''} - 2q^{(1,0)}_{Od'''}\,, \\
&\widetilde{k}^{(1,-1)}_{Oa} = k^{(1,-1)}_{Oa} - 2q^{(1,-1)}_{Oa}\,, && \widetilde{k}^{(1,-1)}_{Ob'} = k^{(1,-1)}_{Ob'} - 2q^{(1,-1)}_{Ob'}\,, && \widetilde{k}^{(1,-1)}_{Ob''} = k^{(1,-1)}_{Ob''} - 2q^{(1,-1)}_{Ob''}\,.
\end{aligned} \tag{3.49}
$$

We choose the parameters q to eliminate $\widetilde{k}^{(1,0)}_{Oa''}$, $\widetilde{k}^{(1,0)}_{Oa'''}$, $\widetilde{k}^{(1,0)}_{Ob}$, $\widetilde{k}^{(1,0)}_{Od'''}$, $\widetilde{k}^{(1,-1)}_{Ob'}$, $\widetilde{k}^{(1,-1)}_{Ob''}$, and the new terms $\widetilde{k}^{(0,2)}_{Oa}$ and $\widetilde{k}^{(0,2)}_{Ob}$, as well as to fix $\widetilde{k}^{(1,-1)}_{Oa} = 1$. Then after dropping the tilde notation, the general boost transformation is simplified as follows [77]

$$
\boldsymbol{k}^{ab}_O = \delta^{ab}(it\boldsymbol{\nabla}_R + i\boldsymbol{R}\partial_0 + 2M\boldsymbol{R}) - \frac{k^{(1,0)}_{OD}}{4M}\boldsymbol{D}^{ab}_R - \frac{1}{4M}\left\{k^{(1,0)}_{Oa'}\boldsymbol{r},(\boldsymbol{\nabla}_r\cdot\boldsymbol{D}^{ab}_R)\right\}
- \frac{ik^{(1,0)}_{Oc}}{8M}\boldsymbol{D}^{ab}_R\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right) - \frac{ik^{(1,0)}_{Od''}}{8Mr^2}\left(\boldsymbol{r}\cdot\boldsymbol{D}^{ab}_R\right)\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)
- \frac{i\delta^{ab}}{4M}\boldsymbol{\nabla}_r\times\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right) + \mathcal{O}\left(M^{-2}r^0, M^{-1}r^1, M^0r^3\right)\,, \tag{3.50}
$$

in which only four undetermined coefficients, $k^{(1,0)}_{OD}$, $k^{(1,0)}_{Oa'}$, $k^{(1,0)}_{Oc}$, and $k^{(1,0)}_{Od''}$, remain just like in the case of the singlet, Eq. (3.39). These coefficients as well as the ones from the singlet will be constrained in the next section.

3.3 Poincaré invariance in pNRQCD

In this sectoin, we apply the boost transformations generated by Eqs. (3.39) and (3.50) to singlet, octet, and singlet-octet sector of the pNRQCD Lagrangian. Due to the invariance of the actions, we obtain non-trivial relations between the Wilson coefficients.

3.3.1 Singlet sector

The general boost generators have to satisfy the commutation relation we discussed in the previous chapter, Eq. (2.69), which at leading order in $1/M$ corresponds to

$$
(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{R}\times\boldsymbol{\nabla}_R) - \left[\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_S, 2M\boldsymbol{\eta}\cdot\boldsymbol{R}\right] + \left[\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_S, 2M\boldsymbol{\xi}\cdot\boldsymbol{R}\right] + \mathcal{O}(M^{-1})
$$

$$
\begin{aligned}
&= (\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{R}\times\boldsymbol{\nabla}_R) + (\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(k_{Sa'}^{(1,0)}\boldsymbol{r}\times\boldsymbol{\nabla}_r) + \frac{ik_{Sc}^{(1,0)}}{2}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}) \\
&\quad - \frac{ik_{Sd''}^{(1,0)}}{2r^2}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\left(\boldsymbol{r}\times\left(\boldsymbol{r}\times\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right) + \mathcal{O}(M^{-1}) \\
&\stackrel{!}{=} (\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\left(\boldsymbol{R}\times\boldsymbol{\nabla}_R + \boldsymbol{r}\times\boldsymbol{\nabla}_r + \frac{i}{2}\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right). \qquad (3.51)
\end{aligned}
$$

This already fixes three further coefficients: $k_{Sa'}^{(1,0)} = k_{Sc}^{(1,0)} = 1$, and $k_{Sd''}^{(1,0)} = 0$.

The last remaining coefficient $k_{SD}^{(1,0)}$ is fixed when we apply the boost transformation to the singlet sector of the Lagrangian up to $\mathcal{O}(M^{-2})$ (we follow the notation from Ref. [22])

$$
\begin{aligned}
\mathcal{L}_{\text{pNRQCD}}^{(S)} = \int d^3r\,\mathrm{Tr}\Bigg[S^\dagger\Bigg(& i\partial_0 + \frac{1}{2M}\left\{c_S^{(1,-2)},\boldsymbol{\nabla}_r^2\right\} + \frac{c_S^{(1,0)}}{4M}\boldsymbol{\nabla}_R^2 - V_S^{(0)} - \frac{V_S^{(1)}}{M} + \frac{V_{rS}}{M^2} \\
&+ \frac{V_{P^2Sa}}{8M^2}\boldsymbol{\nabla}_R^2 + \frac{1}{2M^2}\left\{V_{p^2Sb},\boldsymbol{\nabla}_r^2\right\} + \frac{V_{L^2Sa}}{4M^2r^2}\left(\boldsymbol{r}\times\boldsymbol{\nabla}_R\right)^2 + \frac{V_{L^2Sb}}{4M^2r^2}\left(\boldsymbol{r}\times\boldsymbol{\nabla}_r\right)^2 \\
&- \frac{V_{S_{12}S}}{M^2r^2}\left(3(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(1)})(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(2)}) - r^2(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\right) - \frac{V_{S^2S}}{4M^2}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} \\
&+ \frac{iV_{LSSa}}{4M^2}(\boldsymbol{r}\times\boldsymbol{\nabla}_R)\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}) + \frac{iV_{LSSb}}{4M^2}(\boldsymbol{r}\times\boldsymbol{\nabla}_r)\cdot(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\Bigg)S\Bigg],
\end{aligned}
\qquad (3.52)
$$

where the subscripts a and b (later also c, d, and e) on the potentials are simply labels used to distinguish different operators in the same sector (such as different spin-orbit couplings here or several singlet-octet interactions below). The difference between the transformed Lagrangian and the original $\mathcal{L}_{\text{pNRQCD}}^{(S)}$ needs to be a total derivative term. We obtain [77]:

$$
\begin{aligned}
\partial_\mu\widehat{\Delta}^\mu\mathcal{L}^{(S)} = \int d^3r\,\mathrm{Tr}\Bigg[\boldsymbol{\eta}\cdot S^\dagger\Bigg(& i\left(1-c_S^{(1,0)}\right)\boldsymbol{\nabla}_R - \frac{1}{2M}\left(k_{SD}^{(1,0)} - c_S^{(1,0)}\right)\boldsymbol{\nabla}_R\partial_0 \\
&- \frac{i}{M}\left(V_{P^2Sa} + V_{L^2Sa} + \frac{1}{2}V_S^{(0)}\right)\boldsymbol{\nabla}_R + \frac{i}{Mr^2}\left(V_{L^2Sa} + \frac{r}{2}V_S^{(0)\prime}\right)\boldsymbol{r}(\boldsymbol{r}\cdot\boldsymbol{\nabla}_R) \\
&+ \frac{1}{2M}\left(V_{LSSa} + \frac{1}{2r}V_S^{(0)\prime}\right)\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\times\boldsymbol{r}\Bigg)S\Bigg],
\end{aligned}
\qquad (3.53)
$$

where the prime on V denotes derivative with respect to the relative distance r. None of these terms have the form of an overall derivative, so all coefficients have to vanish. This gives the following constraints:

$$
\begin{aligned}
k_{SD}^{(1,0)} &= c_S^{(1,0)} = 1, & V_{P^2Sa} &= \frac{r}{2}V_S^{(0)\prime} - \frac{1}{2}V_S^{(0)}, \\
V_{L^2Sa} &= -\frac{r}{2}V_S^{(0)\prime}, & V_{LSSa} &= -\frac{1}{2r}V_S^{(0)\prime}.
\end{aligned}
\qquad (3.54)
$$

These coincide with the results in the literature [22]. Note that with the last remaining boost coefficient $k_{SD}^{(1,0)}$ now fixed to unity, the boost generator for the singlet field up to this order is exactly the same as in the $g \to 0$ limit, Eq. (3.25). In other words, there are no loop corrections to any of the coefficients. It is important to remember, however, that this form of the boost generator has been a particular choice obtained through certain field redefinitions. Other choices are equally valid and may change the constraints derived above. Our choice corresponds to the one taken in [22].

3.3.2 Octet sector

The calculation of the commutator of two boosts is analogous to that of the singlet, Eq. (3.51), so we have

$$k_{Oa'}^{(1,0)} = k_{Oc}^{(1,0)} = 1\,, \quad \text{and} \quad k_{Od''}^{(1,0)} = 0\,, \tag{3.55}$$

for the octet. The only remaining coefficient from the boost generator $\boldsymbol{k}_O$ is then $k_D^{(1,0)}$.

In [22], the octet bilinear sector of the pNRQCD Lagrangian is given as (now in the matrix notation)

$$\begin{aligned}
\mathcal{L}_{\text{pNRQCD}}^{(O)} = \int d^3r\, \mathrm{Tr}\Bigg\{ O^\dagger \Bigg(& iD_0 + \frac{1}{2M}\left\{c_O^{(1,-2)}, \boldsymbol{\nabla}_r^2\right\} + \frac{c_O^{(1,0)}}{4M}\boldsymbol{D}_R^2 - V_O^{(0)} - \frac{V_O^{(1)}}{M} - \frac{V_{rO}}{M^2} \\
& + \frac{V_{P^2Oa}}{4M^2}\boldsymbol{D}_R^2 + \frac{1}{2M^2}\{V_{p^2Ob}, \boldsymbol{\nabla}_r^2\} + \frac{V_{L^2Oa}}{4M^2r^2}(\boldsymbol{r}\times\boldsymbol{D}_R)^2 + \frac{V_{L^2Ob}}{M^2r^2}(\boldsymbol{r}\times\boldsymbol{\nabla}_r)^2 \\
& - \frac{V_{S_{12}O}}{M^2r^2}\left(3\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(1)}\right)\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(2)}\right) - r^2\left(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}\right)\right) - \frac{V_{S^2O}}{4M^2}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)} \\
& + \frac{iV_{LSOa}}{4M^2}(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) + \frac{iV_{LSOb}}{2M^2}(\boldsymbol{r}\times\boldsymbol{\nabla}_r)\cdot\left(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}\right)\Bigg) O \\
& + \Bigg[O^\dagger \Bigg(\frac{V_{OO}^{(0,1)}}{2}\boldsymbol{r}\cdot g\boldsymbol{E} + \frac{iV_{OOa}^{(0,2)}}{8}\left[(\boldsymbol{r}\cdot\boldsymbol{D}_R), (\boldsymbol{r}\cdot g\boldsymbol{E})\right] + \frac{iV_{OOb}^{(0,2)}}{8}r^2\left[\boldsymbol{D}_R\cdot, g\boldsymbol{E}\right] \\
& + \frac{iV_{OOa}^{(1,0)}}{8M}\{\boldsymbol{\nabla}_r\cdot, \boldsymbol{r}\times g\boldsymbol{B}\} + \frac{c_F V_{OOb}^{(1,0)}}{2M}g\boldsymbol{B}\cdot\boldsymbol{\sigma}^{(1)} - \frac{V_{O\otimes Ob}^{(1,0)}}{2M}g\boldsymbol{B}\cdot\boldsymbol{\sigma}^{(2)} \\
& + \frac{V_{OOc}^{(1,0)}}{2Mr^2}(\boldsymbol{r}\cdot g\boldsymbol{B})\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(1)}\right) - \frac{V_{O\otimes Oc}^{(1,0)}}{2Mr^2}(\boldsymbol{r}\cdot g\boldsymbol{B})\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(2)}\right) + \frac{V_{OOd}^{(1,0)}}{2Mr}\boldsymbol{r}\cdot g\boldsymbol{E} \\
& - \frac{iV_{OO}^{(1,1)}}{8M}\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot, g\boldsymbol{B}\} \\
& + \frac{ic_S V_{OOa}^{(2,0)}}{16M^2}\left[\boldsymbol{D}_R\times, g\boldsymbol{E}\right]\cdot\boldsymbol{\sigma}^{(1)} - \frac{iV_{O\otimes Oa}^{(2,0)}}{16M^2}\left[\boldsymbol{D}_R\times, g\boldsymbol{E}\right]\cdot\boldsymbol{\sigma}^{(2)} \\
& + \frac{iV_{OOb'}^{(2,0)}}{16M^2r^2}\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot, g\boldsymbol{E}\}(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(1)}) - \frac{iV_{OOb''}^{(2,0)}}{16M^2r^2}\left\{\left((\boldsymbol{r}\times\boldsymbol{D}_R)\cdot\boldsymbol{\sigma}^{(1)}\right), (\boldsymbol{r}\cdot g\boldsymbol{E})\right\} \\
& - \frac{iV_{O\otimes Ob'}^{(2,0)}}{16M^2r^2}\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot, g\boldsymbol{E}\}(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(2)}) + \frac{iV_{O\otimes Ob''}^{(2,0)}}{16M^2r^2}\left\{\left((\boldsymbol{r}\times\boldsymbol{D}_R)\cdot\boldsymbol{\sigma}^{(2)}\right), (\boldsymbol{r}\cdot g\boldsymbol{E})\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{16M^2}\left\{V^{(2,0)}_{OOc'}(\boldsymbol{r}\cdot g\boldsymbol{E}),(\boldsymbol{\nabla}_r\cdot\boldsymbol{D}_R)\right\} + \frac{1}{16M^2}\left\{V^{(2,0)}_{OOc''}r_i gE_j,(\nabla_r)_j(D_R)_i\right\}\\
&+ \frac{1}{16M^2}\left\{V^{(2,0)}_{OOc'''}r_i gE_j,(\nabla_r)_i(D_R)_j\right\} + \frac{1}{16M^2}\left\{\frac{V^{(2,0)}_{OOd}}{r^2}r_ir_j(\boldsymbol{r}\cdot g\boldsymbol{E}),(\nabla_r)_i(D_R)_j\right\}\\
&- \frac{iV^{(2,0)}_{OOe}}{8M^2r}\left\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot,g\boldsymbol{B}\right\}\Bigg)\,O + c.c.\Bigg]\Bigg\}\,, \qquad (3.56)
\end{aligned}
$$

where *c.c.* refers to the charge conjugate of every term inside the square brackets. In the terms of order $M^{-1}r^1$ and $M^{-2}r^0$, we include only those that contain a covariant derivative acting on the octet field, because otherwise they do not contribute in the boost transformation at the order we are interested in[4].

Applying the boost operation to this Lagrangian, one obtains the following difference with respect to the original Lagrangian, which has to vanish [77]:

$$
\begin{aligned}
\partial_\mu\widehat{\Delta}^\mu\mathcal{L}^{(O)} = &\int d^3r\,\mathrm{Tr}\left\{O^\dagger\left(i\left(1-c_O^{(1,0)}\right)(\boldsymbol{\eta}\cdot\boldsymbol{D}_R) - \frac{1}{4M}\left(k^{(1,0)}_{OD}-c_O^{(1,0)}\right)\boldsymbol{\eta}\cdot\{D_0,\boldsymbol{D}_R\}\right.\right.\\
&- \frac{i}{M}\left(V_{P^2Oa}+V_{L^2Oa}+\frac{1}{2}k^{(1,0)}_{OD}V^{(0)}_O\right)(\boldsymbol{\eta}\cdot\boldsymbol{D}_R)\\
&+ \frac{i}{Mr^2}\left(V_{L^2Oa}+\frac{r}{2}V^{(0)\prime}_O\right)(\boldsymbol{\eta}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{D}_R)\\
&- \frac{1}{2M}\left(V_{LSOa}+\frac{1}{2r}V^{(0)\prime}_O\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\Bigg)O\\
&+ \Bigg[O^\dagger\Bigg(\frac{1}{2}\left(V^{(1,1)}_{OO}-V^{(0,1)}_{OO}\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{B}\\
&+ \frac{1}{4M}\left(c_SV^{(2,0)}_{OOa}-2c_FV^{(1,0)}_{OOb}+\frac{1}{2}V^{(0,1)}_{OO}+\frac{1}{2}\right)(\boldsymbol{\eta}\times g\boldsymbol{E})\cdot\boldsymbol{\sigma}^{(1)}\\
&- \frac{1}{4M}\left(V^{(2,0)}_{O\otimes Oa}-2V^{(1,0)}_{O\otimes Ob}+\frac{1}{2}V^{(0,1)}_{OO}-\frac{1}{2}\right)(\boldsymbol{\eta}\times g\boldsymbol{E})\cdot\boldsymbol{\sigma}^{(2)}\\
&- \frac{1}{4Mr^2}\left(V^{(2,0)}_{OOb'}-2V^{(1,0)}_{OOc}\right)((\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{E})\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(1)}\right)\\
&+ \frac{1}{4Mr^2}\left(V^{(2,0)}_{O\otimes Ob'}-2V^{(1,0)}_{O\otimes Oc}\right)((\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{E})\left(\boldsymbol{r}\cdot\boldsymbol{\sigma}^{(2)}\right)\\
&+ \frac{1}{4Mr^2}\left(V^{(2,0)}_{OOb''}+\frac{r}{2}V^{(0,1)\prime}_{OO}\right)\left((\boldsymbol{\eta}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}^{(1)}\right)(\boldsymbol{r}\cdot g\boldsymbol{E})\\
&- \frac{1}{4Mr^2}\left(V^{(2,0)}_{O\otimes Ob''}+\frac{r}{2}V^{(0,1)\prime}_{OO}\right)\left((\boldsymbol{\eta}\times\boldsymbol{r})\cdot\boldsymbol{\sigma}^{(2)}\right)(\boldsymbol{r}\cdot g\boldsymbol{E})\\
&- \frac{i}{8M}\left\{\left(V^{(2,0)}_{OOc'}+V^{(1,0)}_{OOa}\right)(\boldsymbol{r}\cdot g\boldsymbol{E}),(\boldsymbol{\eta}\cdot\boldsymbol{\nabla}_r)\right\}
\end{aligned}
$$

[4]Note that in Ref. [22] the identity, Eq. (3.28), was not used, so there a set of operators is included which are not linearly independent. In particular, there are two other potentials $V^{(2,0)}_{OOb'''}$ and $V^{(2,0)}_{O\otimes Ob'''}$, which we have neglected in order to work only with linearly independent operators. In the end, these potentials are found to be zero in [22] and the other constraints do not depend on them, so the results remain unchanged.

$$
\begin{aligned}
&- \frac{i}{8M}\left\{\left(V^{(2,0)}_{OOc''} - V^{(1,0)}_{OOa} + 2\right)(\boldsymbol{\eta}\cdot\boldsymbol{r}), (\boldsymbol{\nabla}_r\cdot g\boldsymbol{E})\right\} \\
&- \frac{i}{8M}\left\{V^{(2,0)}_{OOc'''}(\boldsymbol{\eta}\cdot g\boldsymbol{E})\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\} - \frac{i}{8M}\left\{\frac{V^{(2,0)}_{OOd}}{r^2}(\boldsymbol{\eta}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot g\boldsymbol{E})\boldsymbol{r}\cdot, \boldsymbol{\nabla}_r\right\} \\
&+ \frac{1}{2Mr}\left(V^{(2,0)}_{OOe} - V^{(1,0)}_{OOd}\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{B}\Big) O + c.c.\Big]\Big\}\,. \qquad (3.57)
\end{aligned}
$$

Again, all coefficients need to vanish, from which the following constraints are derived:

$$
\begin{aligned}
k^{(1,0)}_{OD} &= c^{(1,0)}_{O} = 1\,, & V_{P^2Oa} &= \frac{r}{2}V^{(0)\prime}_O - \frac{1}{2}V^{(0)}_O\,, \\
V_{L^2Oa} &= -\frac{r}{2}V^{(0)\prime}_O\,, & V_{LSOa} &= -\frac{1}{2r}V^{(0)\prime}_O\,, \\
c_S V^{(2,0)}_{OOa} &= 2c_F V^{(1,0)}_{OOb} - \frac{1}{2}V^{(0,1)}_{OO} - \frac{1}{2}\,, & V^{(2,0)}_{O\otimes Oa} &= 2V^{(1,0)}_{O\otimes Ob} - \frac{1}{2}V^{(0,1)}_{OO} + \frac{1}{2}\,, \\
V^{(2,0)}_{OOb'} &= 2V^{(1,0)}_{OOc}\,, & V^{(2,0)}_{O\otimes Ob'} &= 2V^{(1,0)}_{O\otimes Oc} \\
V^{(2,0)}_{OOb''} &= -\frac{r}{2}V^{(0,1)\prime}_{OO}\,, & V^{(2,0)}_{O\otimes Ob''} &= -\frac{r}{2}V^{(0,1)\prime}_{OO}\,, \\
V^{(2,0)}_{OOc'} &= -V^{(1,0)}_{OOa}\,, & V^{(2,0)}_{OOc''} &= V^{(1,0)}_{OOa} - 2\,, \\
V^{(2,0)}_{OOc'''} &= 0\,, & V^{(2,0)}_{OOd} &= 0\,, \\
V^{(2,0)}_{OOe} &= V^{(1,0)}_{OOd}\,, & V^{(1,1)}_{OO} &= V^{(0,1)}_{OO}\,. \qquad (3.58)
\end{aligned}
$$

They are in agreement with [22] once the linearly dependent operators are removed. Note that in [22], the same field redefinitions have been performed as in this paper. The boost coefficient $k^{(1,0)}_{OD}$ is fixed to unity, so the boost generator for the octet field coincides with the one expected from the $g \to 0$ limit with covariant derivatives in the center-of-mass coordinate.

3.3.3 Singlet-octet sector

Finally, moving on to the singlet-octet sector of the Lagrangian, several terms that appear in the octet-octet sector are absent due to cancellation by charge conjugate counterparts. In accordance with [22], its Lagrangian is then given by

$$
\begin{aligned}
\mathcal{L}^{(SO,\,h)}_{\text{pNRQCD}} = \int d^3r\,\mathrm{Tr}\Bigg[S^\dagger\Bigg(&V^{(0,1)}_{SO}\boldsymbol{r}\cdot g\boldsymbol{E} + \frac{c_F V^{(1,0)}_{SOb}}{2M} g\boldsymbol{B}\cdot\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \\
&+ \frac{V^{(1,0)}_{SOc}}{2Mr^2}(\boldsymbol{r}\cdot g\boldsymbol{B})\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right)\right) + \frac{V^{(1,0)}_{SOd}}{Mr}\boldsymbol{r}\cdot g\boldsymbol{E} \\
&- \frac{i}{4M}V^{(1,1)}_{SO}\left\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot, g\boldsymbol{B}\right\} + \frac{ic_S}{16M^2}V^{(2,0)}_{SOa}\left[\boldsymbol{D}_R\times, g\boldsymbol{E}\right]\cdot\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right) \\
&+ \frac{iV^{(2,0)}_{SOb'}}{16M^2r^2}\left\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot, g\boldsymbol{E}\right\}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{iV^{(2,0)}_{SOb''}}{16M^2r^2}\left\{\left((\boldsymbol{r}\times\boldsymbol{D}_R)\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right),(\boldsymbol{r}\cdot g\boldsymbol{E})\right\}\\
&- \left.\left.\frac{iV^{(2,0)}_{SOe}}{4M^2r}\{(\boldsymbol{r}\times\boldsymbol{D}_R)\cdot,g\boldsymbol{B}\}\right)O+h.c.\right], \qquad (3.59)
\end{aligned}
$$

where again we only include terms with covariant derivatives acting on the quarkonium fields of order $M^{-1}r^1$ and $M^{-2}r^0$, and we have neglected the linearly dependent operator with potential $V^{(2,0)}_{SOb'''}$. As all operators between the round brackets are hermitian, an index h to the Lagrangian is used. These are the only operators that are allowed in the pure singlet or octet sectors. In the singlet-octet sector, on the other hand, one may in principle also add antihermitian operators. Instead of canceling, they give terms of the form $S^\dagger aO - O^\dagger aS$, where a indicates the antihermitian operator. We are not aware of any argument that would exclude such terms a priori, so we give here also the singlet-octet Lagrangian for the antihermitian operators:

$$
\begin{aligned}
\mathcal{L}^{(SO,a)}_{\text{pNRQCD}} = &\int d^3r\,\mathrm{Tr}\left[S^\dagger\left(\frac{1}{2M}\left\{rV^{(1,0)}_{SOe},\boldsymbol{\nabla}_r\cdot g\boldsymbol{E}\right\}+\frac{iV^{(1,0)}_{SOf}}{2Mr}(\boldsymbol{r}\times g\boldsymbol{E})\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right.\right.\\
&-\frac{i}{4M^2}\left\{rV^{(2,0)}_{SOf}g\boldsymbol{B}\cdot,(\boldsymbol{\nabla}_r\times\boldsymbol{D}_R)\right\}+\frac{V^{(2,0)}_{SOg'}}{16M^2r}\left\{(\boldsymbol{r}\cdot g\boldsymbol{B}),\left(\boldsymbol{D}_R\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right\}\\
&+\frac{V^{(2,0)}_{SOg''}}{16M^2r}\left\{(\boldsymbol{r}\cdot\boldsymbol{D}_R),\left(g\boldsymbol{B}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right\}\\
&+\left.\left.\frac{V^{(2,0)}_{SOg'''}}{16M^2r}\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\{\boldsymbol{D}_R\cdot,g\boldsymbol{B}\}\right)O+h.c.\right], \qquad (3.60)
\end{aligned}
$$

Such terms were not considered in [22].

The new terms of the singlet-octet Lagrangian after the boost transformation are the following [77]:

$$
\begin{aligned}
\partial_\mu\widehat{\Delta}^\mu\mathcal{L}^{(SO,h)} = &\int d^3r\,\mathrm{Tr}\left[S^\dagger\left(\left(V^{(1,1)}_{SO}-V^{(0,1)}_{SO}\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{B}\right.\right.\\
&+\frac{1}{4M}\left(c_SV^{(2,0)}_{SOa}-2c_FV^{(1,0)}_{SOb}+V^{(0,1)}_{SO}\right)(\boldsymbol{\eta}\times g\boldsymbol{E})\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\\
&-\frac{1}{4Mr^2}\left(V^{(2,0)}_{SOb'}-2V^{(1,0)}_{SOc}\right)((\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{E})\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)\\
&+\frac{1}{4Mr^2}\left(V^{(2,0)}_{SOb''}+rV^{(0,1)\prime}_{SO}\right)\left((\boldsymbol{\eta}\times\boldsymbol{r})\cdot\left(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\cdot g\boldsymbol{E})\\
&+\left.\left.\frac{1}{Mr}\left(V^{(2,0)}_{SOe}-V^{(1,0)}_{SOd}\right)(\boldsymbol{\eta}\times\boldsymbol{r})\cdot g\boldsymbol{B}\right)O+h.c.\right], \qquad (3.61)
\end{aligned}
$$

$$
\partial_\mu\widehat{\Delta}^\mu\mathcal{L}^{(SO,a)} = \int d^3r\,\mathrm{Tr}\left[S^\dagger\left(\frac{1}{2M}\left\{r\left(V^{(2,0)}_{SOf}-V^{(1,0)}_{SOe}\right),(\boldsymbol{\eta}\times\boldsymbol{\nabla}_r)\cdot g\boldsymbol{B}\right\}\right.\right.
$$

$$
\begin{aligned}
&- \frac{i}{4Mr}\left(V_{SOg'}^{(2,0)} - 2V_{SOf}^{(1,0)}\right)\left(\boldsymbol{\eta}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)(\boldsymbol{r}\cdot g\boldsymbol{B}) \\
&- \frac{i}{4Mr}\left(V_{SOg''}^{(2,0)} + 2V_{SOf}^{(1,0)}\right)(\boldsymbol{\eta}\cdot\boldsymbol{r})\left(g\boldsymbol{B}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right) \\
&\left.\left.- \frac{iV_{SOg'''}^{(2,0)}}{4Mr}(\boldsymbol{\eta}\cdot g\boldsymbol{B})\left(\boldsymbol{r}\cdot\left(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)}\right)\right)\right) O + h.c.\right], \qquad (3.62)
\end{aligned}
$$

which leads to the constraints

$$
\begin{aligned}
&V_{SO}^{(0,1)} = V_{SO}^{(1,1)}\,, && c_S V_{SOa}^{(2,0)} = 2c_F V_{SOb}^{(1,0)} - V_{SO}^{(0,1)}\,, && V_{SOb'}^{(2,0)} = 2V_{SOc}^{(1,0)}\,, \\
&V_{SOb''}^{(2,0)} = -rV_{SO}^{(0,1)\prime}\,, && V_{SOe}^{(2,0)} = V_{SOd}^{(1,0)}\,, && V_{SOf}^{(2,0)} = V_{SOe}^{(1,0)}\,, \\
&V_{SOg'}^{(2,0)} = 2V_{SOf}^{(1,0)}\,, && V_{SOg''}^{(2,0)} = -2V_{SOf}^{(1,0)}\,, && V_{SOg'''}^{(2,0)} = 0\,. \qquad (3.63)
\end{aligned}
$$

Again, these are in agreement with [22] after performing the field redefinitions, except for the new potentials $V_{SOe}^{(1,0)}$, $V_{SOf}^{(1,0)}$, $V_{SOf}^{(2,0)}$, $V_{SOg'}^{(2,0)}$, $V_{SOg''}^{(2,0)}$, and $V_{SOg'''}^{(2,0)}$, which are the Wilson coefficients from $\mathcal{L}_{\text{pNRQCD}}^{(SO,\,a)}$.

3.4 Summary and discussion

In this chapter, we have investigated Poincaré symmetry (especially the boost transformation) of a low-energy EFTs of QCD involving a heavy quarkonium [77]. Apart from the generators of spacetime translations and rotations, we have constructed the boost generator for singlet and octet fields by starting from the most general expression which is allowed by C, P, T, while exploiting the freedom to remove redundant terms through field redefinitions. Relations between the Wilson coefficients were derived when we applied those expressions to the corresponding pNRQCD Lagrangian up to $\mathcal{O}(1/M^2)$. Furthermore, the requirement that the Poincaré algebra has to satisfy for the boost generators simplifies the expression of boosts. The results of the constraining equations between the Wilson coefficients confirm the known relations from the literature in pNRQCD [22]. These are found in Eqs. (3.54), (3.58), and (3.63). We have seen that several equivalent boost generators are available, and we have resolved this ambiguity by removing redundant terms via a field redefinition (a unitary transformation, in particular).

We would like to point out the merits of our approach and some advantages over the previous treatment in [22]. In [22], a direct implementation of Poincaré invariance was achieved and applied to pNRQCD (as well as NRQCD). As one constructs all generators of the symmetry group in these EFTs, the generators corresponding to spacetime translations and rotations are obtained in the usual way from the associated conserved Noether currents. The generators of boosts, on the other hand, are derived from a general ansatz that includes all operators allowed by other symmetries (such as parity, charge conjugation, and time reversal) up to a certain order in the expansion; in other words, the general principles of EFTs for the construction of the Lagrangian are also applied to the boost generators. By demanding that all generators satisfy the commutation relations of the Poincaré algebra, one can obtain relations between the Wilson coefficients of the EFTs.

While the calculation in [22] is also fully general, it requires a considerable amount of computational effort, making extensions to higher orders appears a formidable task. In contrast, the method presented in this thesis is rather straightforward, as it only involves replacements of fields by their boosted expressions instead of calculating commutators between field operators. As such, it seems also particularly suited for automatization in programs capable of symbolic manipulation.

Towards long-distance heavy quark potentials

The constraints between the Wilson coefficients derived by exploiting the spacetime symmetries of pNRQCD can be utilized for another EFT in involving a QCD flux tube model. This EFT, namely the effective string theory (EST), is valid in the non-perturbative regime [32, 44, 57, 81–83]. The EST provides an analytic description of the gluodynamics of a static quark-antiquark bound system at long-distance scales $r\Lambda_{\rm QCD} \gg 1$, with transversal vibrations of a string between a heavy quark and antiquark as the degrees of freedom. It has been hypothesized that the expectation value of a rectangular Wilson loop in the large time limit can be expressed in terms of the string partition function

$$\lim_{T\to\infty} \langle W_\square \rangle = Z \int \mathcal{D}\xi^1 \mathcal{D}\xi^2 \, e^{iS[\xi^1,\xi^2]} \,, \quad \text{where} \quad W_\square = P \exp\left\{ -ig \oint_{r\times T} dz^\mu A_\mu \right\} \,, \tag{3.64}$$

in which P is a path-ordering operator acting in color space, Z is a normalization constant of the string partition function, and ξ^i $(i = 1, 2)$ are the transversal string vibrations; the angular brackets around the rectangular Wilson loop denote the expectation value over the Yang-Mills action. The action $S[\xi^1, \xi^2]$ involves the string tension σ $(\sim \Lambda_{\rm QCD}^2)$ and derivatives of the string fields $\partial\xi$. The effective action is derived by expanding the Nambu-Goto action[5].

On the other hand, potentials from pNRQCD are expressed in terms of gauge field insertions to the Wilson loop expectation value in the large time limit [50, 51], so that one can find a set of one-to-one mappings from the heavy quark potentials to correlators of the string fields [57]. This mapping is restricted due to the symmetries of the physical system as well as the mass dimension of the gauge fields inserted to the expectation value. The potentials are then calculated by utilizing the mapping, from which some (dimensionful) parameters arise. Constraints on the potential terms due to the Poincaré invariance in pNRQCD lead to a reduction in the number of these parameters. Eventually, the heavy quark-antiquark potential at long distances is expressed by only two free parameters, which are the string tension and the heavy (anti)quark mass [57]; they are determined by lattice simulations. As a comparison between the potentials calculated by the EST and the corresponding lattice data [47, 48] shows some discrepancies [58],

[5] The static gauge is used for the expansion [84]

especially at shorter distance ranges, it is necessary to employ proper EFT systematics (power counting, symmetries, etc) in order to include all possible correction terms in the action as well as in the mapping, so that the comparison can be improved. This analysis will be presented in the next two chapters.

Chapter 4

Heavy quark-antiquark potentials

So far, we have discussed Poincaré invariance in low-energy EFTs of QCD, NRQCD and pNRQCD in particular, in the weak-coupling regime[1]. When a different hierarchy is given such as $Mv \gtrsim \Lambda_{\rm QCD}$ or $\Lambda_{\rm QCD} \gg Mv^2$, the theory enters into the strong coupling regime, and the Lagrangian has to be constructed differently: the singlet field is the only relevant heavy degree of freedom in this case. In this chapter, we will briefly discuss the matching between NRQCD and pNRQCD, for both weakly-coupled and strongly-coupled cases. Through this matching, one can establish explicit relations between the heavy quark potentials which appear in the form of Wilson coefficients in the pNRQCD Lagrangian and the gauge field insertions to the Wilson loop expectation value of NRQCD. Eventually, these relations provide a useful tool for deriving the analytic expressions of the potentials.

4.1 Matching between NRQCD and pNRQCD

As it was shown in the previous chapter, the weakly-coupled pNRQCD Lagrangian up to the bilinear order in both color singlet and octet fields is heuristically expressed[2] by [8, 80]

$$\begin{aligned}\mathcal{L}^{\rm weak}_{\rm pNRQCD} = &\int d^3r \, {\rm Tr}\left[S^\dagger\left(i\partial_0 - V_S(r) + \dots\right)S + O^\dagger(iD_0 - V_0(r) + \dots)O\right] \\ &+ gV_A(r)\,{\rm Tr}\left[O^\dagger \mathbf{r}\cdot\boldsymbol{E}S + S^\dagger\mathbf{r}\cdot\boldsymbol{E}O\right] + g\frac{V_B(r)}{2}{\rm Tr}\left[O^\dagger\mathbf{r}\cdot\boldsymbol{E}O + O^\dagger O\mathbf{r}\cdot\boldsymbol{E}\right] \\ &- \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} + \mathcal{L}_{\rm light}\,,\end{aligned} \tag{4.1}$$

where the ellipses indicate higher order terms in the $1/M$ or r expansion.

On the other hand, the strongly-coupled pNRQCD Lagrangian is expressed only in

[1] The hierarchy of scales is given by $M \gg p \sim Mv \gg E \sim Mv^2 \gtrsim \Lambda_{\rm QCD}$.

[2] Here we are using the matrix notation.

terms of the singlet field (as long as light quarks are ignored) [80]

$$\mathcal{L}_{\text{pNRQCD}}^{\text{strong}} = \int d^3r \, \text{Tr} \left[S^\dagger \left(i\partial_0 - h_s \right) S \right] , \tag{4.2}$$

where

$$h_s \equiv \frac{\mathbf{p}_1^2}{2M_1} + \frac{\mathbf{p}_2^2}{2M_2} + V_S(r) \, . \tag{4.3}$$

$M_{1,2}$ is the mass of the heavy quark (or antiquark, respectively), and $\mathbf{p}_{1,2}$ is the momentum of the quark (or the antiquark).

The singlet potential V_S is organized according to the $1/M$ expansion up to quadratic order[3] [8]

$$\begin{aligned} V(r) = & V^{(0)}(r) + \frac{2}{M} V^{(1,0)}(r) + \frac{1}{M^2} \left\{ \left[2 \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2} + \frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{r^2} \right] \mathbf{L}^2 \right. \\ & + \left[V_{LS}^{(2,0)}(r) + V_{L_2 S_1}^{(1,1)}(r) \right] \mathbf{L} \cdot \mathbf{S} + V_{S^2}^{(1,1)}(r) \left(\frac{\mathbf{S}^2}{2} - \frac{3}{4} \right) + V_{S_{12}}^{(1,1)} \mathbf{S}_{12}(\hat{r}) \\ & \left. + \left[2V_{\mathbf{p}^2}^{(2,0)}(r) + V_{\mathbf{p}^2}^{(1,1)}(r) \right] \mathbf{p}^2 + 2V_r^{(2,0)}(r) + V_r^{(1,1)}(r) \right\} + \mathcal{O}(1/M^3) \, . \end{aligned} \tag{4.4}$$

Here, the masses are assumed to be equal: $M_1 = M_2 = M$. The parenthesis from the superscripts of the V's indicate the order of the expansion in the inverse mass of the heavy quark M_1 and the heavy antiquark M_2.[4] As it is assumed that the masses of the heavy quark and the antiquark are equal, the potential is invariant under the exchange of masses as well as CP. In other words, the potentials are invariant under the exchange of indices of the parenthesis, e.g.,

$$V_{\mathbf{L}^2}^{(2,0)}(r) = V_{\mathbf{L}^2}^{(0,2)} \, . \tag{4.5}$$

The tensorial spin-spin interaction term is defined by

$$\mathbf{S}_{12}(\hat{r}) \equiv 3\hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_1 \hat{\mathbf{r}} \cdot \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \, , \tag{4.6}$$

where $\mathbf{S}_{1,2} = \boldsymbol{\sigma}_{1,2}/2$ is the spin operator of the heavy quark and the heavy antiquark, respectively. The matching, order by order in the $1/M$ expansion (up to quadratic order, in particular), gives explicit relations between the heavy quark potentials and gauge field insertions to the rectangular Wilson loop expectation value.

[3] From now on, we omit the singlet subscript S of the potential since the context is clear.

[4] The first slot denotes the order of the $1/M_1$ expansion and the second slot denotes the $1/M_2$ expansion, respectively.

4.1.1 The leading order potential

A relation between the leading order potential $V^{(0)}$ and the Wilson loop[5] is derived by matching the Green's functions from NRQCD and pNRQCD, at leading order in the $1/M$ expansion [50].

Let us consider a state made of a heavy quark-antiquark pair, represented by $\psi(\mathbf{x}_1)$ (which annihilates a heavy quark at the position $\mathbf{x}_1$) and χ (which creates a heavy antiquark at the position of $\mathbf{x}_2$) in NRQCD, connected by a Wilson line ϕ

$$\psi^\dagger(\mathbf{x}_1)\phi(\mathbf{x}_1,\mathbf{x}_2)\chi(\mathbf{x}_2)|0\rangle = \sum_n a_n(\mathbf{x}_1,\mathbf{x}_2)|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}\,, \tag{4.7}$$

where the Wilson line is defined by[6]

$$\phi(\mathbf{x}_1,\mathbf{x}_2;t) = P\exp\left\{ig\int_0^1 ds\ (\mathbf{x_1}-\mathbf{x}_2)\cdot\mathbf{A}(\mathbf{x}_2 - s(\mathbf{x}_2-\mathbf{x}_1),t)\right\}\,, \tag{4.8}$$

and $|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ is the quark-antiquark sector of the Fock space, which is spanned by

$$|\underline{n};\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)} \equiv \psi^\dagger(\mathbf{x}_1)\chi(\mathbf{x}_2)|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}\,. \tag{4.9}$$

Also, $|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)}$ is an eigenstate of $H^{(0)}$ with energy $E_n^{(0)}(\mathbf{x}_1,\mathbf{x}_2)$, which encodes the gluonic content of the state[7]. Note that the normalization of these states are given by

$$\begin{aligned} {}^{(0)}\langle m;\mathbf{x}_1,\mathbf{x}_2|n;\mathbf{x}_1,\mathbf{x}_2\rangle^{(0)} &= \delta_{mn}\,,\\ {}^{(0)}\langle \underline{m};\mathbf{x}_1,\mathbf{x}_2|\underline{n};\mathbf{y}_1,\mathbf{y}_2\rangle^{(0)} &= \delta_{mn}\delta^{(3)}(\mathbf{x}_1-\mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2-\mathbf{y}_2)\,. \end{aligned} \tag{4.10}$$

From the matching condition, this heavy quark-antiquark pair is related to the singlet field by

$$\chi^\dagger(\mathbf{x}_2,t)\phi(\mathbf{x}_2,\mathbf{x}_1;t)\psi(\mathbf{x}_1,t) = Z^{1/2}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)S(\mathbf{x}_1,\mathbf{x}_2;t)\,, \tag{4.11}$$

where $Z^{1/2}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)$ is the normalization factor which depends on the relative coordinate between $\mathbf{x}_1$ and $\mathbf{x}_2$ as well as on the momenta of the individual quark and the antiquark, $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Then the Green's function from the heavy quark-antiquark pair in NRQCD is given by

$$\begin{aligned} G_{\rm NRQCD} =& \langle 0|\chi^\dagger(\mathbf{x}_2,T/2)\phi(\mathbf{x}_2,\mathbf{x}_1;T/2)\psi(\mathbf{x}_1,T/2)\\ &\times \psi^\dagger(\mathbf{y}_1,-T/2)\phi(\mathbf{y}_1,\mathbf{y}_2;-T/2)\chi(\mathbf{y_2},-T/2)|0\rangle\,, \end{aligned} \tag{4.12}$$

and its $1/M$ expansion is written as

$$G_{\rm NRQCD} = G^{(0)}_{\rm NRQCD} + \frac{2}{M}G^{(1,0)}_{\rm NRQCD} + \dots\,, \tag{4.13}$$

[5]The Wilson loop is a SU(3) gauge invariant object whose explicit formulation will be presented in the paragraph below.

[6]P stands for a path-ordering operator in SU(3) color space.

[7]This state is annihilated by $\chi^\dagger$ and ψ.

in which $M_1 = M_2 = M$ is assumed so that there is a factor of two in front of the first order in $1/M$. We are eventually going to the static limit of the system, so we need to integrate out the heavy quark and antiquark fields. After integrating out ψ and χ, the leading order Green's function turns out to be the expectation value of a rectangular Wilson loop (over the Yang-Mills action)

$$G^{(0)}_{\text{NRQCD}} = \langle W_\Box \rangle \, \delta^{(3)}(\mathbf{x}_1 - \mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2 - \mathbf{y}_2) \,. \tag{4.14}$$

The rectangular Wilson loop is defined by

$$W_\Box \equiv P \exp \left\{ -ig \oint_{r\times T} dz^\mu A_\mu(z) \right\} \,. \tag{4.15}$$

Note that r is the relative distance between the quark and the antiquark. The diagram of the rectangular Wilson loop can be found in FIG. 4.1, where $a = (-T/2, -\mathbf{y}_2)$,

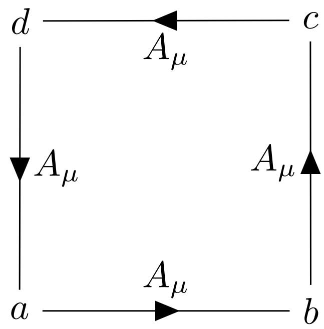


Figure 4.1: The diagram of the rectangular Wilson loop $W_\Box$, defined in Eq. 4.15.

$b = (-T/2, \mathbf{y}_1)$, $c = (T/2, \mathbf{x}_1)$, and $d = (T/2, \mathbf{x}_2)$. In other words, the time axis is along the vertical direction.

On the other hand, the Green's function for the singlet sector of pNRQCD is expressed by

$$G_{\text{pNRQCD}} = Z^{1/2} \exp\{-iTV\} Z^{\dagger 1/2} \delta^{(3)}(\mathbf{x}_1 - \mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2 - \mathbf{y}_2) \,, \tag{4.16}$$

where the expression of the singlet potential is given in Eq. (4.4). Thus, as we expand the singlet Green's function[8] in $1/M$

$$G_{\text{pNRQCD}} = G^{(0)}_{\text{pNRQCD}} + \frac{2}{M} G^{(1,0)}_{\text{pNRQCD}} + \dots \,, \tag{4.17}$$

the leading order is given in terms of the static potential from Eq. (4.4)

$$G^{(0)}_{\text{pNRQCD}} = Z_0 \exp\left\{ -igV^{(0)}(r)T \right\} \delta^{(3)}(\mathbf{x}_1 - \mathbf{y}_1)\delta^{(3)}(\mathbf{x}_2 - \mathbf{y}_2) \,. \tag{4.18}$$

[8]The factor of two in front of the $1/M$ term is again due to the invariance under the exchange of masses between the quark and the antiquark, as well as the CP.

As we compare Eq. (4.14) with Eq. (4.18) at zeroth order in $1/M$, we conclude that the following relation holds in the large-time limit

$$V^{(0)}(r) = \lim_{T\to\infty} \frac{i}{T} \ln\langle W_\Box\rangle \,. \tag{4.19}$$

4.1.2 The potentials at $\mathcal{O}(1/M)$ and $\mathcal{O}(1/M^2)$

In a similar fashion, the first order relativistic correction to the static potential is matched to the NRQCD counterpart, thereby yielding a relation between the Wilson coefficient and the gauge field insertions to the Wilson loop expectation value [50]

$$V^{(1,0)}(r) = -\frac{1}{2}\lim_{T\to\infty}\int_0^T dt\, t \langle\!\langle g\boldsymbol{E}_1(t)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c \,, \tag{4.20}$$

in which the double angular bracket is defined by $\langle\!\langle\ldots\rangle\!\rangle \equiv \langle\ldots W_\Box\rangle/\langle W_\Box\rangle$. The terms inside the double angular brackets of Eq. (4.20) are chromoelectric fields at the positions of the heavy quark (or the heavy antiquark) $\boldsymbol{E}_{1,2}(t) = \boldsymbol{E}(t,\pm r/2)$. The subscript c represents the connected part of the Wilson loop expectation value, which is defined by

$$\langle\!\langle O_1(t_1)O_2(t_2)\rangle\!\rangle_c = \langle\!\langle O_1(t_1)O_2(t_2)\rangle\!\rangle - \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)\rangle\!\rangle, \quad \text{for} \quad t_1 \geq t_2\,, \tag{4.21}$$

where $O_i(t_i)$ is a generic gauge field defined at time t_i.

Proceeding with the matching calculation for the potentials at $\mathcal{O}(1/M^2)$, the following relations hold for the second order correction in the $1/M$ expansion [50, 51]:

$$V^{(2,0)}_{\mathbf{L}^2}(r) = \frac{i}{4}(\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j)\int_0^\infty dt\, t^2 \langle\!\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_1(0)\rangle\!\rangle_c, \tag{4.22}$$

$$V^{(1,1)}_{\mathbf{L}^2}(r) = \frac{i}{2}(\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j)\int_0^\infty dt\, t^2 \langle\!\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_2(0)\rangle\!\rangle_c, \tag{4.23}$$

$$V^{(2,0)}_{\mathbf{p}^2}(r) = \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\int_0^\infty dt\, t^2 \langle\!\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_1(0)\rangle\!\rangle_c, \tag{4.24}$$

$$V^{(1,1)}_{\mathbf{p}^2}(r) = i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\int_0^\infty dt\, t^2 \langle\!\langle g\boldsymbol{E}^i_1(t) g\boldsymbol{E}^j_2(0)\rangle\!\rangle_c, \tag{4.25}$$

$$V^{(2,0)}_{LS}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r}\cdot\int_0^\infty dt\, t \langle\!\langle g\boldsymbol{B}_1(t)\times g\boldsymbol{E}_1(0)\rangle\!\rangle + \frac{c_S^{(1)}}{2r^2}\mathbf{r}\cdot(\boldsymbol{\nabla}_r V^{(0)}), \tag{4.26}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r}\cdot\int_0^\infty dt\, t \langle\!\langle g\boldsymbol{B}_1(t)\times g\boldsymbol{E}_2(0)\rangle\!\rangle, \tag{4.27}$$

$$V^{(1,1)}_{S^2}(r) = \frac{2c_F^{(1)}c_F^{(2)}}{3} i\int_0^\infty dt\, \langle\!\langle g\boldsymbol{B}_1(t)\cdot g\boldsymbol{B}_2(0)\rangle\!\rangle - 4(d_{sv} + d_{vv}C_F)\delta^{(3)}(\mathbf{r}), \tag{4.28}$$

$$V^{(1,1)}_{\mathbf{S}_{12}}(r) = \frac{c_F^{(1)}c_F^{(2)}}{4} i\mathbf{r}^i\mathbf{r}^j\int_0^\infty dt\left[\langle\!\langle g\boldsymbol{B}^i_1(t) g\boldsymbol{B}^j_2(0)\rangle\!\rangle - \frac{\delta^{ij}}{3}\langle\!\langle g\boldsymbol{B}_1(t)\cdot g\boldsymbol{B}_2(0)\rangle\!\rangle\right], \tag{4.29}$$

$$V^{(2,0)}_{r}(r) = \frac{\pi C_F\alpha_s c_D^{(1)\prime}}{2}\delta^{(3)}(\mathbf{r}) - \frac{ic_F^{(1)2}}{4}\int_0^\infty dt\, \langle\!\langle g\boldsymbol{B}_1(t)\cdot g\boldsymbol{B}_1(0)\rangle\!\rangle_c + \frac{1}{2}(\boldsymbol{\nabla}_r^2 V^{(2,0)}_{\mathbf{p}^2})$$

$$
\begin{aligned}
&- \frac{i}{2}\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \langle\!\langle g\boldsymbol{E}_1(t_1)\cdot g\boldsymbol{E}_1(t_2) g\boldsymbol{E}_1(t_3)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c \\
&+ \frac{1}{2}\left(\boldsymbol{\nabla}_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\!\langle g\boldsymbol{E}_1^i(t_1) g\boldsymbol{E}_1(t_2)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c\right) \\
&- d_3^{(1)\prime} f_{abc} \int d^2\mathbf{x} \lim_{T\to\infty} g\langle\!\langle F^a_{\mu\nu}(x) F^b_{\mu\alpha}(x) F^c_{\nu\alpha}(x)\rangle\!\rangle, \qquad (4.30)
\end{aligned}
$$

$$
\begin{aligned}
V_r^{(1,1)}(r) = &- \frac{1}{2}\left(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(1,1)}\right) + (d_{ss} + d_{vs} C_F)\delta^{(3)}(\mathbf{r}) \\
&- i\int_0^\infty dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 (t_2 - t_3)^2 \langle\!\langle g\boldsymbol{E}_1(t_1)\cdot g\boldsymbol{E}_1(t_2) g\boldsymbol{E}_2(t_3)\cdot g\boldsymbol{E}_2(0)\rangle\!\rangle_c \\
&+ \frac{1}{2}\left(\boldsymbol{\nabla}_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\!\langle g\boldsymbol{E}_1^i(t_1) g\boldsymbol{E}_2(t_2)\cdot g\boldsymbol{E}_2(0)\rangle\!\rangle_c\right) \\
&+ \frac{1}{2}\left(\boldsymbol{\nabla}_r^i \int_0^\infty dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^2 \langle\!\langle g\boldsymbol{E}_2^i(t_1) g\boldsymbol{E}_1(t_2)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c\right), \qquad (4.31)
\end{aligned}
$$

where C_F is the Casimir of the fundamental representation of SU(3); i.e., $C_F = 4/3$ and $c_F^{(1)}, c_F^{(2)}, c_S^{(1)}, c_D^{(1)\prime}, d_{sv}, d_{vv}, d_{vs}$ are the Wilson coefficients of NRQCD. Between the Wilson coefficients, the following relations hold [7, 25]

$$c_F^{(i)} = 1 + \mathcal{O}(\alpha_S)\,, \qquad (4.32)$$

$$c_S^{(i)} = 2c_F^{(i)} - 1\,, \qquad (4.33)$$

$$c_D^{(i)\prime} = 1 + \mathcal{O}(\alpha_S)\,, \qquad (4.34)$$

$$d_3^{(1)\prime} = \frac{\alpha_S}{720\pi} + \mathcal{O}(\alpha_S^2)\,, \qquad (4.35)$$

$$d_{sv} + d_{vv} C_F = \mathcal{O}(\alpha_S^2)\,, \qquad (4.36)$$

Furthermore, the connected part of the three- and four-gauge field insertions to the Wilson loop expectation value are defined by

$$
\begin{aligned}
\langle\!\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle\!\rangle_c = &\langle\!\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle\!\rangle - \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)O_3(t_3)\rangle\!\rangle_c \\
&- \langle\!\langle O_1(t_1)O_2(t_2)\rangle\!\rangle_c\langle\!\langle O_3(t_3)\rangle\!\rangle \\
&- \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)\rangle\!\rangle\langle\!\langle O_3(t_3)\rangle\!\rangle\,,
\end{aligned} \qquad (4.37)
$$

$$
\begin{aligned}
\langle\!\langle O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4)\rangle\!\rangle_c = &\langle\!\langle O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4)\rangle\!\rangle \\
&- \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)O_3(t_3)O_4(t_4)\rangle\!\rangle_c \\
&- \langle\!\langle O_1(t_1)O_2(t_2)\rangle\!\rangle_c\langle\!\langle O_3(t_3)O_4(t_4)\rangle\!\rangle_c \\
&- \langle\!\langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle\!\rangle_c\langle\!\langle O_4(t_4)\rangle\!\rangle \\
&- \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)\rangle\!\rangle\langle\!\langle O_3(t_3)O_4(t_4)\rangle\!\rangle_c \\
&- \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)O_3(t_3)\rangle\!\rangle_c\langle\!\langle O_4(t_4)\rangle\!\rangle \\
&- \langle\!\langle O_1(t_1)O_2(t_2)\rangle\!\rangle_c\langle\!\langle O_3(t_3)\rangle\!\rangle\langle\!\langle O_4(t_4)\rangle\!\rangle \\
&- \langle\!\langle O_1(t_1)\rangle\!\rangle\langle\!\langle O_2(t_2)\rangle\!\rangle\langle\!\langle O_3(t_3)\rangle\!\rangle\langle\!\langle O_4(t_4)\rangle\!\rangle\,,
\end{aligned} \qquad (4.38)
$$

where the inserted operators O_i ($i \in \{1, 2, 3, 4\}$) are defined at times $t_1 \geq t_2 \geq t_3 \geq t_4$.

4.2 Heavy quark potentials

The derived relations from the matching between NRQCD and pNRQCD, Eqs. (4.19), (4.20), and (4.22) - (4.31), give a useful tool for the determination of the potentials, both in the perturbative and in the non-perturbative regimes. We will briefly discuss the potentials in the peturbative regime and pose the challenge in deriving the analytic expressions of the potentials in the non-perturbative regime.

4.2.1 Weak-coupling regime

In the weak-coupling regime, the potentials are derived either by calculating the heavy quark-antiquark scattering amplitude [26–30] or by calculating the gauge field insertions to the Wilson loop expectation value [58]. As for the second method, from the relation from the leading order, Eq. (4.19), one can Taylor expand the Wilson loop (as $g \ll 1$), such that the first non-trivial term of the expectation value over the Yang-Mills action is proportional to the coupling α_S as well as inversely proportional to the distance between the pair. Hence,

$$V^{(0)}(r) = -\frac{C_F \alpha_S}{r}\,, \tag{4.39}$$

where the Casimir of the fundamental representation of SU(3), $C_F = 4/3$, is due to the path ordering operator P of the Wilson loop.

In a similar fashion, one can proceed with calculating the potentials at $1/M$ and $1/M^2$. The following is the result of the leading order calculations performed in [26–30, 58]

$$V^{(1,0)}(r) = -\frac{C_F C_A \alpha_S^2}{2r^2}\,, \tag{4.40}$$

$$V^{(2,0)}_{\mathbf{L}^2}(r) + V^{(1,1)}_{\mathbf{L}^2}(r) = \frac{C_F \alpha_S}{2r^3}\,, \tag{4.41}$$

$$V^{(2,0)}_{\mathbf{p}^2}(r) + V^{(1,1)}_{\mathbf{p}^2}(r) = -\frac{C_F \alpha_S}{r}\,, \tag{4.42}$$

$$V^{(2,0)}_{LS}(r) + V^{(1,1)}_{L_2 S_1}(r) = \frac{3 C_F \alpha_S}{2r^3}\,, \tag{4.43}$$

$$V^{(1,1)}_{S^2}(r) = \frac{4\pi C_F \alpha_S}{3}\delta^{(3)}(\mathbf{r})\,, \tag{4.44}$$

$$V^{(1,1)}_{S_{12}}(r) = \frac{C_F \alpha_S}{4r^3}\,, \tag{4.45}$$

$$2V^{(2,0)}_r + V^{(1,1)}_r(r) = \pi C_F \alpha_S \delta^{(3)}(\mathbf{r})\,, \tag{4.46}$$

where C_A is the Casimir of the adjoint representation of SU(3), $C_A = 3$.

4.2.2 Strong coupling regime

Color confinement

In the strong coupling regime, similar calculations of the potentials are not possible. Around and below the hadronic scale $\Lambda_{\rm QCD} \sim 200$ MeV, the conventional perturbative approach is no longer a feasible theoretical framework, because the expansion parameter, $\alpha_s = g^2/(4\pi)$, exceeds the weak coupling limit. A manifestation of the QCD in the strong coupling regime is called *color confinement* [31]. Color confinement manifests itself in experiments: only composite particles (mesons and baryons) are detected instead of quarks or gluons.

Lattice QCD

One can analyze the heavy quark potentials in the strong coupling regime by utilizing non-perturbative methods like lattice QCD (LQCD), which gives the numerical values for the gauge field insertions to the Wilson loop expectation value [45–49]. However, the analyses for the three- and four-gauge field insertions (which are found in the expressions of central potentials V_r, Eq. (4.30) and (4.31), are not easily obtained from this method. Our approach to resolve this difficulty is by employing the QCD *flux tube model* [32], which is a suitable theoretical framework for the static heavy quark-antiquark bound state at long distance.

4.2.3 QCD flux tube model

Historical development

Since the realization of color confinement in QCD [31], it was proposed that the dynamics of quark-antiquark bound states at long distance can be described by a flux tube model [32], in which the quark and antiquark is connected by an open string. The attractive force between the pair increases as the separation distance increases[9], thereby forming a flux in the shape of a tube due to the increase of the energy density between them. This suggestion has been verified by lattice QCD simulations [33–43]. FIG. 4.2 illustrates this feature by a lattice QCD simulation [41]. There are two sharp peaks in this figure, which are due to the presence of point sources (a static quark and an antiquark)[10]. The region between the sources is in the shape of a upper half of a tube. As the distance between two peaks becomes greater[11], the volume of the tube increases.

As for the analytic description of the flux tube model within the effective string framework, the heavy quark and antiquark are treated as static objects, while the gluonic interactions between the pair is described by the dynamics of the string. Since the two ends of the string are fixed at the position of the pair, only the transversal modes of

[9] In particular, the length scale is greater than the confining distance.

[10] Note that the height of the diagram represents energy density.

[11] Distance scale here is much greater than $\Lambda_{\rm QCD}^{-1}$

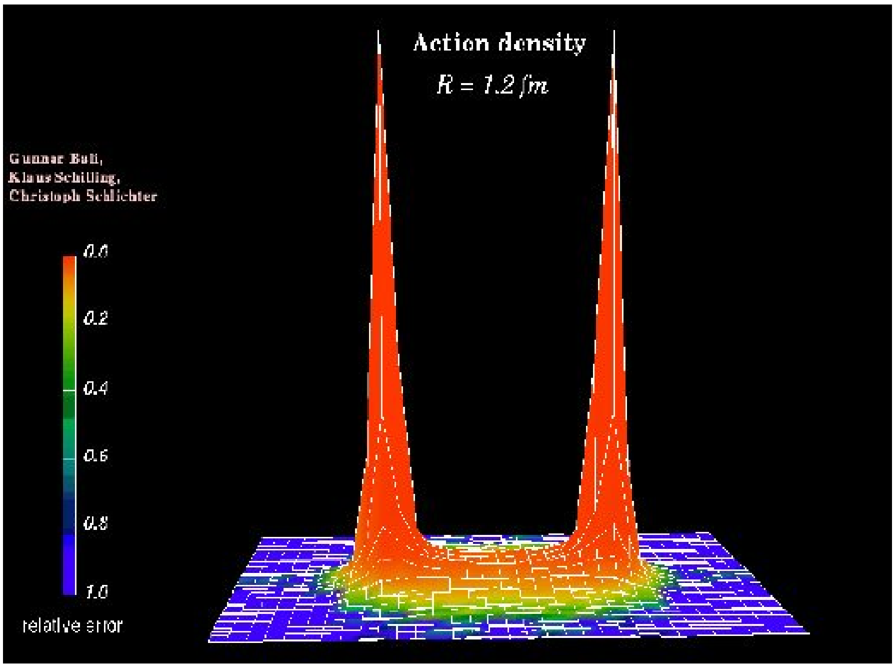


Figure 4.2: Lattice simulation of the QCD flux tube model taken from [41].

the string act as the dynamical degrees of freedom. Several years after Nambu's suggestion of a flux tube model [32], Kogut and Parisi developed this idea further, such that the shape of the spin-spin interaction part of the potential was explicitly derived [44], and later confirmed by lattice simulations [45–49].

Recent development

In this line of investigation, a significant progress concerning the analysis of the non-perturbative heavy quark potential has been made during the last few decades. Based on the relations between the potential terms of the heavy quark-antiquark bound state in the static limit and the Wilson loop expectation value (and the gauge field insertions therein) [50, 51], as well as on the *Wilson loop-string partition function equivalence conjecture* [52–55], a few other heavy quark potentials were directly computed through the effective string picture [56]. Recently, Brambilla et. al. have calculated all of the heavy quark potentials up to leading order (LO) of the effective string theory (EST) power counting [57]. Full summation of the heavy quark potentials was compared to lattice simulations in order to constrain some of the parameters, which arise from the effective string picture itself. As it was pointed out there [57], however, this leading order calculation is not fully inclusive because some of the terms from next-to-leading order (NLO) calculation might be of the same order as the leading order terms of the EST. In other words, some terms arising in the EST calculation at NLO can alter the leading order coefficients of the potentials. It is, therefore, necessary for us to employ the proper

EFT systematics of the string picture, so that not only the higher order suppression terms are understood, but all of the missing terms of LO can also be acquired. Furthermore, a recent comparison between the analytic result of the potentials at long-distance via EST and LQCD data was presented in [58], but the discrepancy is not negligible. We estimate that the subleading contributions to the potential will ameliorate this discrepancy, and this is the major goal of the second part of this thesis.

Towards the effective string theory

Therefore, in the next chapter, we will discuss about the EST in great detail. First, we will derive the relativistic string theory in four-dimensional spacetime (i.e., Nambu-Goto string theory), and from this we will naturally obtain the action for the EST by imposing some constraints (such as physical boundary conditions and hierarchy of scales). Based on the the symmetry of the system, a correspondence between the gauge field insertions to the Wilson loop expectation and the string counter part will be established, and from this we derive the analytic expressions of the heavy quark potentials in the non-perturbative regime. This calculation will firstly be presented at LO in the EST power counting, and this confirms the result presented in [57]. We then proceed further to include the NLO terms of the EST by exploiting the correspondence between the gague field insertions and the suitable string counter part. This will give the desired results, i.e., the NLO terms of the potentials. Finally, we will compare these results to the corresponding LQCD simulation data [47, 48].

Chapter 5

Effective string theory

In this chapter, we will derive the analytic expressions of the aforementioned heavy quark potentials using an effective framework of the QCD flux tube model, which is the effective string theory (EST). First, we discuss the generic description of the relativistic strings in four-dimensional spacetime, and from this the EST is derived by exploiting the hierarchy of scales, physical boundary conditions, and symmetries of the system. Based upon the equivalence conjecture between QCD and the EST, we utilize the EST to calculate the potentials at LO and NLO in the EST power counting scheme. At the end, a comparison between the analytic results of the potentials from the EST and the corresponding lattice QCD data is given.

5.1 Relativistic theory of strings

The relativistic theory of strings is constructed based on the minimal area law of the string action [85]. While a point particle (a zero-dimensional object) depends only on one parameter,[1] a vibrating string, a one-dimensional object, in $d-$dimensional space carries two parameters, which we denote with τ and λ. From this, it is easy to see that if the evolution of a point particle is parametrized by a world-line, a string traces out a two-dimensional surface, which is called a *world-sheet*. As this world-sheet is embedded in a higher dimensional space, the space in which the two-dimensional world-sheet lives is called the *target space*. Let us specify the dimension of the target space as four. Then the world-sheet is spanned by the coordinates of the string

$$\boldsymbol{\xi} = (\xi^0(\tau,\lambda), \xi^1(\tau,\lambda), \xi^2(\tau,\lambda), \xi^3(\tau,\lambda) \tag{5.1}$$

One can calculate the area of the world-sheet in terms of the string coordinates using a parallelogram. The infinitesimal area of a generic parallelogram is expressed in terms of two infinitesimal vectors $d\boldsymbol{v}_1$ and $d\boldsymbol{v}_2$ which are not parallel to each other (their relative angle is given by θ)

$$dA = |d\boldsymbol{v}_1||d\boldsymbol{v}_2||\sin\theta| = \sqrt{|d\boldsymbol{v}_1|^2|d\boldsymbol{v}_2|^2 - |d\boldsymbol{v}_1|^2|d\boldsymbol{v}_2|^2\cos^2\theta}$$

[1] It is normally written as a time variable.

$$= \sqrt{(d\boldsymbol{v}_1)^2(d\boldsymbol{v}_2)^2 - (d\boldsymbol{v}_1 \cdot d\boldsymbol{v}_2)^2}\,, \tag{5.2}$$

where the second line is due to the definition of the spatial dot product. In this fashion, one can rewrite the infinitesimal area spanned by the string coordinates. Let us assign the two infinitesimal vectors $d\boldsymbol{v}_{1,2}$ in terms of the string coordinates

$$d\boldsymbol{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial \tau} d\tau\,, \quad \text{and} \quad d\boldsymbol{v}_2 = \frac{\partial \boldsymbol{\xi}}{\partial \lambda} d\lambda\,, \tag{5.3}$$

then the infinitesimal area of the target space is given by

$$dA = d\tau d\lambda \sqrt{\left(\frac{\partial \boldsymbol{\xi}}{\partial \tau} \cdot \frac{\partial \boldsymbol{\xi}}{\partial \tau}\right)\left(\frac{\partial \boldsymbol{\xi}}{\partial \lambda} \cdot \frac{\partial \boldsymbol{\xi}}{\partial \lambda}\right) - \left(\frac{\partial \boldsymbol{\xi}}{\partial \tau} \cdot \frac{\partial \boldsymbol{\xi}}{\partial \lambda}\right)^2}\,. \tag{5.4}$$

However, as we specify the parameter τ as the (proper) time coordinate of the relativistic frame, this has to be written in a slightly different way, due to the metric signature of the four-dimensional Minkowski spacetime[2]. Instead of the dot product, let us use the Einstein convention of the summation over the four-dimensional spacetime, so that corresponding the infinitesimal surface is rewritten as

$$dA = d\tau d\lambda \sqrt{\left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \lambda}\right)^2 - \left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \tau}\right)\left(\frac{\partial \xi^\mu}{\partial \lambda}\frac{\partial \xi_\mu}{\partial \lambda}\right)}\,. \tag{5.5}$$

Then, the total area of the parallelogram is computed by taking the integral over the domain of the parameters

$$A = \int d\tau d\lambda \sqrt{\left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \lambda}\right)^2 - \left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \tau}\right)\left(\frac{\partial \xi^\mu}{\partial \lambda}\frac{\partial \xi_\mu}{\partial \lambda}\right)}\,. \tag{5.6}$$

Note that this quantity is Lorentz invariant since all of the Lorentz indices μ are contracted. The Nambu-Goto action for relativistic strings originates from this area argument. As the action is supposed to have zero mass dimension and the area has a negative two mass dimension, an extra parameter has to be incorporated to the action. This parameter is the so-called the *string tension* σ which is of the mass dimension two (which is in the unit of force), and the Nambu-Goto action is given by[3]

$$S = -\sigma \int d\tau d\lambda \sqrt{\left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \lambda}\right)^2 - \left(\frac{\partial \xi^\mu}{\partial \tau}\frac{\partial \xi_\mu}{\partial \tau}\right)\left(\frac{\partial \xi^\mu}{\partial \lambda}\frac{\partial \xi_\mu}{\partial \lambda}\right)}\,. \tag{5.7}$$

In fact, this expression can be written in a more concise way due to the reparametrization invariance. In other words, one can parametrize the area of the world-sheet surface

[2] We use mostly minus $(+,-,-,-)$ signature. From now on, it is implied that the target spacetime is the four-dimensional Minkowski spacetime.

[3] The minus sign in front of the string tension is a convention. If there is no minus sign in front, then the string tension will become a negative value.

in a different way choosing a different coordinate system, but the total surface area A does not change. This property is reminiscent of the general coordinate invariance of the general theory of relativity. In fact, the argument for the generalized expression of Eq. (5.7) refers to some notions of the general coordinate invariance. As the string world-sheet is living in the four-dimensional Minkowski spacetime, one can define an induced metric g from the string coordinates ξ and the target spacetime η

$$g_{ab} = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^a} \frac{\partial \xi^\beta}{\partial x^b} \,, \tag{5.8}$$

where x stands for the parameters on which the string coordinates depend. By the parametrization[4] of Eq. (5.7), $x^0 = \tau$ and $x^3 = \lambda$, so the induced metric is written by a two-by-two matrix form

$$g_{ab} = \begin{bmatrix} \frac{\partial \xi^\alpha}{\partial \tau} \frac{\partial \xi_\alpha}{\partial \tau} & \frac{\partial \xi^\alpha}{\partial \tau} \frac{\partial \xi_\alpha}{\partial \lambda} \\ \frac{\partial \xi^\alpha}{\partial \lambda} \frac{\partial \xi_\alpha}{\partial \tau} & \frac{\partial \xi^\alpha}{\partial \lambda} \frac{\partial \xi_\alpha}{\partial \lambda} \end{bmatrix} . \tag{5.9}$$

This metric measures the distance on the surface of the world-sheet spanned by the strings. Finally, we can rewrite the Naumbu-Goto action using the induced metric. The expression inside the square root in Eq. (5.7) is the determinant of the induced metric (with a minus sign attached to it)

$$S = -\sigma \int d\tau d\lambda \sqrt{-\det g} \,, \tag{5.10}$$

which is manifestly a reparametrization invariant notation. This is reminiscent of the general coordinate invariance of the general theory of relativity. The difference is that, whereas there is only one metric involved in the general relativity, which is a generic curved background spacetime, there are two metrics incorporated in the string theory: the Minkowski spacetime as the target spacetime and an induced metric, which measures distance and describes the curvature of the world-sheet spanned by the strings. One can generalize this notion of the string action for the case of higher dimensions in the parameter space as well as higher target spacetime dimensions, which can be useful in superstring theory[5].

In the next section, we are going to disucss the Nambu-Goto string theory with some physical boundary conditions imposed and exploiting the hierarchy of scales. This is a suitable tool for analytically describing the QCD flux tube model [32].

5.2 Effective theory of long strings

The effective theory of long strings (or the effective string theory; EST) originates from the Nambu-Goto string theory, Eqs. (5.7) and (5.10). We want to apply the EST to a

[4] The choice of x^3 instead of x^1 as one of the the coordinates of the parameter space becomes clear in the next section.

[5] We are using the string action with only two-dimensional parameter space (τ, λ) in the thesis.

heavy meson in the non-perturbative regime. If the heavy quark-antiquark pair is in the static limit[6], the energy flux between them can analytically be described by the EST in such a way that the gluodynamics between the quark-antiquark pair is replaced by the dynamics of a long-distance vibrating string. We utilize the EFT systematics to derive the EST action from the Nambu-Goto action. The EFT systematics comes from the hierarchy of scales, power counting schemes, and symmetries of the physical system. In this section, only the first non-trivial term of the EST action according to the EFT systematics is considered. This leading order term of the action gives non-trivial results to the calculations of the heavy quark potentials.

5.2.1 Hierarchy of scales and power counting

The long-distance regime is elucidated by the following hierarchy of scales

$$r\Lambda_{\mathrm{QCD}} \gg 1\,, \tag{5.11}$$

in which r is the distance between the heavy quark and the antiquark. Eq. (5.11) implies that the theory is valid only at energy scales much below the hadronic scale $\Lambda_{\mathrm{QCD}} \sim 200$ MeV. In the static limit, the gluodynamics between the heavy quark and the heavy antiquark is replaced by vibrational modes of the long string. As we indicate the string coordinates with ξ (see Eq. (5.7)), it is of the same order as the inverse of the hadronic scale $\xi \sim \Lambda_{\mathrm{QCD}}^{-1}$. On the other hand, a partial derivatve which acts on the string coordinates counts as the inverse interquark distance $\partial \sim 1/r$. Thus, it is clear that we are taking the derivative expansion

$$\partial\xi \sim \frac{1}{r\Lambda_{\mathrm{QCD}}} \ll 1\,, \tag{5.12}$$

and that the magnitude of the metric from the small fluctuations of the string becomes much smaller than one

$$|h_{ab}(x)| = \left| -\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^a}\frac{\partial\xi^\beta}{\partial x^b} \right| \ll 1\,. \tag{5.13}$$

Note that as compared with Eq. (5.8), h has been used for the notation of the metric by the string fluctuations, instead of g. This is due to the fact that we are exploiting the hierarchy of scales, Eq. (5.11), to the generic structure of the string action, and we need to differentiate between the generic induced metric g and the specified one h. In the asymptotic limit where the string fluctuations are negligible (i.e., $r \to \infty$), the induced metric g simply becomes a flat two-dimensional Minkowski metric because the small fluctuations of the string vanish $h \to 0$. Based on this power counting scheme, we can construct the effective action of the long strings.

[6] In other words, the pair is stationary.

5.2.2 Lagrangian construction

The EST action is obtained in the similar fashion as the linearization of the Einstein-Hilbert action in the general theory of relativity. As the magnitude of the metric from the string fluctuations at a generic spacetime point x is much less than the unity, Eq. (5.13), while the magnitude of the Minkowski metric equals to one, we can expand the (generic) induced metric g around the flat spacetime

$$\begin{aligned} g_{ab}(x) &= \eta_{ab}(x) + h_{ab}(x) \\ &= \eta_{ab}(x) - \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^a}\frac{\partial \xi^\beta}{\partial x^b}\,, \end{aligned} \tag{5.14}$$

which is a local expansion at a generic spacetime point x. While the metric signature of η_{ab} is $(1, -1)$, another Minkowski metric which contracts the indices of the string coordinates is the one from the four-dimensional target spacetime with the signature $(1, -1, -1, -1)$. As in the case of the general theory of relativity, in which the expansion is with respect to a small fluctuation around a given background spacetime[7], one can understand h in Eq. (5.14) as an infinitesimal string fluctuation around the string world-sheet[8]. Then, the asymptotic limit of this induced metric is clear: if the string fluctuation vanish, the induced metric simply becomes a two-dimensional Minkowskian metric. In other words, if the string fluctuation is negligible, the corresponding world-sheet is simply a two-dimensional plane of the parameters τ and λ.

Using Eq. (5.14), the Nambu-Goto action is given by

$$S = -\,\sigma\int d\tau d\lambda\sqrt{-\det\left(\eta_{ab} - \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^a}\frac{\partial \xi^\beta}{\partial x^b}\right)}\,. \tag{5.15}$$

where $a, b \in \{0, 3\}$, and the action is rewritten and expanded as

$$S = -\sigma\int d\tau d\lambda\left(1 + \frac{1}{2}\frac{\partial \xi^\alpha}{\partial x^a}\frac{\partial \xi_\alpha}{\partial x_a} + \dots\right)\,. \tag{5.16}$$

The ellipsis inside the parenthesis includes terms with higher orders in the derivative expansion (i.e., $(\partial\xi)^n$ for $n > 2$). In this thesis, we include only up to the Gaussian term of the action (the quadratic term with derivative acting on the string coordinates) to derive the EST Green's function. As for the derivation of the Green's function from the action, it is necessary to clarify some given boundary conditions as well as to observe symmetries of the physical system to which we are applying the EST. This will be discussed in the next section.

[7]Quantization of this fluctuation field around the background spacetime corresponds to the massless spin-2 field.

[8]The two-dimensional world-sheet is embedded in the four-dimensional target spacetime, which is accomodated by the Minkowski metric.

5.3 Long strings and heavy quark potentials

The EST action, Eq. (5.16), is still too generic to be applied to the physical system of a heavy quark-antiquark pair. One needs to impose some boundary conditions in light of the physical system for which we are utilizing the EST, to obtain an explicit description of a heavy meson in the non-perturbative regime. We discuss an equivalence relation between QCD at low-energy and the EST. Then, the long-distance heavy quark potentials are analytically calculated within the framework of the EST, yet only the first non-trivial terms (i.e., leading order terms) from the EST power counting are considered. Due to the equivalence relation between QCD and the EST, Poincaré invariance of QCD provides some constraints on the parameters arising in the EST. Thus, the analytic expressions of the long-distance heavy quark potentials are greatly simplified at the end.

5.3.1 Boundary conditions

Before applying the framework of EST to a heavy quark-antiquark pair, it is necessary to impose some boundary conditions to the string theory. Let us locate the heavy quark at the spatial coordinate $(0, 0, r/2)$ and the antiquark at $(0, 0, -r/2)$.[9] As we rename the proper time τ by the real time t, the open string which connects the quark-antiquark pair depends on the two-dimensional parameter space (t, z). Then, the open string is subject to the Dirichlet boundary conditions,

$$\xi^\mu(t, z = \pm r/2) = 0\,. \tag{5.17}$$

In other words, the string field ξ^μ has only two dynamical degrees of freedom: $\xi^1(t, z)$ and $\xi^2(t, z)$. They correspond to x and y coordinates of the string field in the target spacetime, respectively[10]. Then we can rewrite the effective action of the long string by (including only up to the Gaussian term)

$$S = -\sigma \int dtdz \left(1 - \frac{1}{2}\partial_a \xi^l \partial^a \xi^l\right)\,, \tag{5.18}$$

where $\partial_a = \partial/\partial x^a$ (for $a \in \{0, 3\}$), and $l \in \{1, 2\}$ is the spatial index of the string coordinates. Also note that the domain of the integration measure is given by $t \in [0, \infty)$ and $z \in [-r/2, r/2]$. By solving the equations of motion, we can derive the Green's function of the EST from this expression.

5.3.2 Green's function

The equation of motion from Eq. (5.18) is the two-dimensional Klein-Gordon equation[11]

$$\Box \xi^l(t, z) = 0\,. \tag{5.19}$$

[9] These positions are given in the Cartesian coordinate system (x, y, z).

[10] In other words, they are the transversal modes of the string field. This is the reason why we have chosen the indices of the induced metric, $a, b \in \{0, 3\}$.

[11] The argument presented in this paragraph is based on the unpublished notes of [56].

Then the Green's function of this equation is the Feynman propagator $G_F^{lm}(t,t';z,z')$ that satisfies the equations

$$\Box G_F^{lm} = -\frac{i}{\sigma}\delta^{lm}\delta(t-t')\delta(z-z')\,, \tag{5.20}$$

where δ^{lm} is a two-dimensional Kronecker delta, for $l,m \in \{1,2\}$. Due to Eq. (5.17), it is clear that the Green's function satisfies the same boundary conditions as well

$$G_F^{lm}(t,t';z,z')|_{z=\pm r/2} = 0\,. \tag{5.21}$$

Taking the Wick rotation, $t \to -it$, we have to solve the Poisson equations instead

$$\Delta G^{lm}(t,t';z,z') = -\frac{1}{\sigma}\delta^{lm}\delta(t-t')\delta(z-z')\,, \tag{5.22}$$

in which Δ is the two-dimensional Laplacian operator and G^{lm} is the Green's function in Euclidean spacetime; it clear that this Green's function satisfies the aforementioned boundary conditions[12]: $G^{lm}|_{z=\pm r/2} = 0$. Let us solve Eq. (5.22) by utilizing the eigenfunction equations for the Laplacian operator

$$\Delta\Psi_\rho(t,z) = \rho\Psi_\rho(t,z)\,, \quad \text{for} \quad \rho \in \mathbb{R}\,, \tag{5.23}$$

in which the eigenfunctions form the complete orthonormal basis:

$$\sum_\rho \Psi_\rho^*(t,z)\Psi_\rho(t',z') = \delta(t-t')\delta(z-z')\,. \tag{5.24}$$

Then, the Euclidean Green's function is re-written in terms of these eigenfunctions

$$G^{lm}(t,t';z,z') = -\frac{1}{\sigma}\delta^{lm}\sum_\rho \Psi_\rho^*(t,z)\Psi_\rho(t',z')\,. \tag{5.25}$$

We solve the eigenvalue equation, Eq. (5.23), by separation of variables as well as imposing the given boundary conditions, so that the normalized eigenfunctions are obtained

$$\Psi_\rho(t,z) = \frac{1}{\sqrt{\pi r}}e^{-ikt}\sin\left[\frac{n\pi}{r}\left(z-\frac{r}{2}\right)\right]\,, \tag{5.26}$$

with its eigenvalue $\rho = -k^2 - (n\pi/r)^2$, where $k \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, we finally derive the Green's functions in the Euclidean spacetime

$$G^{lm}(t,t';z,z') = \frac{\delta^{lm}}{\pi\sigma r}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{r}z-\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{r}z'-\frac{n\pi}{2}\right)\int_{-\infty}^{\infty}dk\frac{e^{-ik(t-t')}}{k^2+\left(\frac{n\pi}{r}\right)^2}\,. \tag{5.27}$$

[12]It is also understood that the Green's function is well-defined for $t \to \pm\infty$.

This expression is simplified by explicitly calculating the Fourier transform that appears in the equation

$$\int_{-\infty}^{\infty} dk \frac{e^{-ik(t-t')}}{k^2 + \left(\frac{n\pi}{r}\right)^2} = \frac{r}{n} e^{-\frac{n\pi}{r}|t-t'|}, \tag{5.28}$$

as well as by making use of the trigonometric relation for the infinite sum

$$\sin\left(\frac{n\pi}{r}z\right)\sin\left(\frac{n\pi}{r}z'\right) = \frac{1}{2}\left\{\cos\left[\frac{n\pi}{r}(z-z')\right] - \cos\left[\frac{n\pi}{r}(z+z')\right]\right\}, \tag{5.29}$$

so that the Green's function is re-written as

$$G^{lm}(t,t';z,z') = \frac{\delta^{lm}}{\pi\sigma r}\sum_{n=1}^{\infty}\frac{1}{2n}\mathrm{Re}\left(Z_-^n - Z_+^n\right), \tag{5.30}$$

where

$$Z_+ \equiv -e^{i\frac{\pi}{r}(z+z')}e^{-\frac{\pi}{r}|t-t'|}, \quad \text{and} \quad Z_- \equiv e^{i\frac{\pi}{r}(z-z')}e^{-\frac{\pi}{r}|t-t'|}. \tag{5.31}$$

As the infinite sum is realized as the Taylor expansion of a logarithm

$$\begin{aligned} G^{lm}(t,t';z,z') &= \frac{\delta^{lm}}{2\pi\sigma}\mathrm{Re}\left[\ln\left(\frac{1-Z_+}{1-Z_-}\right)\right] \\ &= \frac{\delta^{lm}}{2\pi\sigma}\ln\left|\frac{1-Z_+}{1-Z_-}\right|, \end{aligned} \tag{5.32}$$

the final expression Green's function is derived by substituting $Z_\pm$ into Eq. (5.32) and reorganizing terms within the logarithm[13] [56]

$$G^{lm}(t,t';z,z') = \frac{\delta^{lm}}{4\pi\sigma}\ln\left(\frac{\cosh[(t-t')\pi/r] + \cos[(z+z')\pi/r]}{\cosh[(t-t')\pi/r] - \cos[(z-z')\pi/r]}\right). \tag{5.33}$$

Eqs. (5.27) and (5.33) will be extensively used for the calculations of the heavy quark potentials in the non-perturbative regime.

5.3.3 QCD-string theory equivalence conjecture

A connection between QCD and the effective string theory is established by the *Wilson loop-string partition function equivalence conjecture* [52, 53]: in the large time limit, the gluodynamics between the heavy quark and the antiquark pair is replaced by vibrational modes of a long string, which is explicitly expressed by

$$\lim_{T\to\infty}\langle W_\square\rangle = Z\int \mathcal{D}\xi^1\mathcal{D}\xi^2 e^{i\,S[\xi^1,\xi^2]}, \tag{5.34}$$

[13]This derivation of the Green's function is based on the unpublished notes of [56].

where Z on the right-hand side of Eq. (5.34) is the normalization constant of the string partition function, and $\xi^{1,2}$ are two transverse modes of the string fields (or coordinates), whose distance scale are of order $1/\Lambda_{\rm QCD}$[14]. $W_\Box$ is the rectangular Wilson loop in QCD, and the angular bracket around denotes the expectation value with respect to the Yang-Mills action. Moreover, $S[\xi^1, \xi^2]$ is the effective action of the long string in four dimensional spacetime, which is given by Eq. (5.18). This conjecture is reminiscent of the well-known AdS/CFT correspondence [86, 87]. While AdS/CFT works with the superstring theory in $AdS_5 \times S^5$ spacetime, the EST in this thesis deals only with a long (open) string in four-dimensional Minkowski spacetime. Furthermore, since QCD is not conformal nor supersymmetric, the direct connection between the AdS/CFT correspondence and the QCD-string theory equivalence conjecture is still to be pursued [88].

5.3.4 QCD-to-EST mapping

Based upon the equivalence conjecture, Eq. (5.34), we can calculate the long distance potentials up to quadratic order in the $1/M$ expansion by utilizing the string variables ξ. In order to do so, a set of mappings between the gauge field insertions to the Wilson loop expectation value and the string correlator is needed for this calculation. The guiding principles of deriving the mappings are the global symmetry transformations and the dimensional counting of the system. We notice that the static heavy quark-antiquark pair is symmetric under the diatomic molecular group $D_{h\infty}$ with CP instead of the parity P [56]. Let us investigate how the gauge fields of QCD would transform under the relevant global transformations and compare it to the transformations of the string fields under the same group.

As the positions of the quark and the antiquark are aligned on the z-axis, the chromoelectric and chromomagnetic fields transform under the rotation around the z-axis R_z as

$$\begin{aligned} \boldsymbol{E}^i(t,z) &\xrightarrow{R_z} R^{ij}\boldsymbol{E}^j(t,z)\,, \\ \boldsymbol{B}^i(t,z) &\xrightarrow{R_z} R^{ij}\boldsymbol{B}^j(t,z)\,, \end{aligned} \tag{5.35}$$

where R^{ij} is the rotation matrix for $i, j \in \{1, 2, 3\}$. Under the reflection with respect to the xz-plane,

$$\begin{aligned} \boldsymbol{E}^i(t,z) &\xrightarrow{xz} \rho^{ij}\boldsymbol{E}^j(t,z)\,, \\ \boldsymbol{B}^i(t,z) &\xrightarrow{xz} -\rho^{ij}\boldsymbol{B}^j(t,z)\,, \end{aligned} \tag{5.36}$$

where $\rho^{ij} = \mathrm{diag}(1, -1, 1)$. Since the system is composed of the particle and the antiparticle, parity is replaced by CP, under which the gauge fields transform as

$$\begin{aligned} \boldsymbol{E}^i(t,z) &\xrightarrow{CP} (\boldsymbol{E}^i)^T(t,-z)\,, \\ \boldsymbol{B}^i(t,z) &\xrightarrow{CP} -(\boldsymbol{B}^i)^T(t,-z)\,, \end{aligned} \tag{5.37}$$

[14] In other words, the thickness of the QCD flux tube described by the strings is of order $\Lambda_{\rm QCD}^{-1}$, which is much smaller than the interquark distance r according to the hierarchy of scales, Eq. (5.11).

in which the transpose T is for the SU(3) color space matrices. Lastly, these fields transform under the time reversal T as

$$\begin{aligned} \boldsymbol{E}^i(t,z) &\xrightarrow{\mathrm{T}} (\boldsymbol{E}^i)^T(-t,z)\,, \\ \boldsymbol{B}^i(t,z) &\xrightarrow{\mathrm{T}} -(\boldsymbol{B}^i)^T(-t,z)\,. \end{aligned} \tag{5.38}$$

On the other hand, we observe that the string variables transform under the same group as the following:

$$\begin{aligned} \xi^i(t,z) &\xrightarrow{R_z} R^{ij}\xi^j(t,z)\,, \\ \xi^i(t,z) &\xrightarrow{xz} \rho^{ij}\xi^j(t,z)\,, \\ \xi^i(t,z) &\xrightarrow{CP} -\xi^j(t,-z)\,, \\ \xi^i(t,z) &\xrightarrow{\mathrm{T}} \xi^j(-t,z)\,. \end{aligned} \tag{5.39}$$

From the comparison between these two, the mappings between the gauge fields insertions to the string variables are constructed as [56, 57]

$$\begin{aligned} \langle\!\langle \dots \boldsymbol{E}^l_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \Lambda^2 \partial_z \xi^l(t, \pm r/2) \dots \rangle\,, \\ \langle\!\langle \dots \boldsymbol{B}^l_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \pm \Lambda' \epsilon^{lm} \partial_t \partial_z \xi^m(t, \pm r/2) \dots \rangle\,, \\ \langle\!\langle \dots \boldsymbol{E}^3_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \Lambda''^2 \dots \rangle\,, \\ \langle\!\langle \dots \boldsymbol{B}^3_{1,2}(t) \dots \rangle\!\rangle &= \langle \dots \pm \Lambda''' \epsilon^{lm} \partial_t \partial_z \xi^l(t, \pm r/2) \partial_z \xi^m(t, \pm r/2) \dots \rangle\,, \end{aligned} \tag{5.40}$$

in which $l, m \in \{1, 2\}$, and Λ, Λ', Λ'', and Λ''' are dimensionful parameters (which are of order Λ_{QCD}). The ellipses on the left- and right-hand sides of Eq. (5.40) represent additional gauge field insertions and string fields with derivatives, respectively. Notice that the third component of the chromoelectric field is mapped into a (dimensionful) constant Λ''^2 due to the parametrization of the physical system: as the heavy quark-antiquark pair is aligned on the z-axis, the third component of the chromoelectric field at the position of quark (or antiquark) is non-dynamical. Since the choromoelectric field has a mass dimension two, this has to be matched from the string side by inserting a dimensionful parameter Λ''^2. In a similar fashion, one can derive the mapping between the chromomagnetic field and the string field variables. In fact, the mapping for the chromomagnetic field can also be derived by using the electromagnetic duality transformation of the chromoelectric field [89].

5.3.5 Heavy quark potentials at LO in EST

Now we can derive the analytic expressions of the heavy quark potentials in terms of the string variables using the EST Green's function, Eq. (5.33), as well as the set of QCD-to-EST mappings, Eq. (5.40). Calculations are presented order by order in the $1/M$ expansion of the heavy quark potentials.

Static potential

Let us first look at the leading order potential. The static potential is related to the Wilson loop expectation value in the large time limit (from Eq. (4.19))

$$V^{(0)}(r) = \lim_{T\to\infty} \frac{i}{T} \ln\langle W_\square \rangle\,, \tag{5.41}$$

and the Wilson loop is related to the string partition function due to the conjecture, Eq (5.34). Then, the static potential is easily calculated using the first term of the EST action, Eq. (5.18),

$$\lim_{T\to\infty} \exp\left(-iV^{(0)}T\right) = \lim_{T\to\infty} \exp\left(-i\sigma \int_0^T dt \int_{-r/2}^{r/2} dz\right)\,, \tag{5.42}$$

from which it is clear that the static potential is linear in the interquark distance r

$$V^{(0)}(r) = \sigma r\,. \tag{5.43}$$

Note that this is the linear part of the well-known *Cornell potential* [90]. The value of the string tension itself is not determined within the EST because this is a fundamental parameter of the theory. The string tension is determined by comparing to the available lattice QCD data [91], and this will be discussed in the next section.

Potential at $\mathcal{O}(1/M)$

The first order correction in $1/M$ to the static potential is expressed in terms of a two-chromoelectric field insertion to the Wilson loop expectation value [50]

$$V^{(1,0)}(r) = -\frac{1}{2} \lim_{T\to\infty} \int_0^T dt\, t \langle\!\langle g\boldsymbol{E}_1(t)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c\,. \tag{5.44}$$

Using the QCD-to-EST mapping, Eq. (5.40), we can calculate this gauge field insertion in terms of the string variables. Let us decompose it into a transversal and longitudinal parts[15]

$$\langle\!\langle \boldsymbol{E}_1(t)\cdot \boldsymbol{E}_1(0)\rangle\!\rangle_c = \langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c + \langle\!\langle \boldsymbol{E}_1^3(t)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c\,. \tag{5.45}$$

First of all, the longitudinal part is mapped into a dimensionful parameter, and due to the definition of the connected part of the gauge field insertion, this quantity vanishes

$$\begin{aligned} \langle\!\langle \boldsymbol{E}_1^3(t)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c &= \langle\!\langle \boldsymbol{E}_1^3(t)\boldsymbol{E}_1^3(0)\rangle\!\rangle - \langle\!\langle \boldsymbol{E}_1^3(t)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle \\ &= \langle \Lambda''^4\rangle - \langle \Lambda''^2\rangle\langle \Lambda''^2\rangle = 0\,. \end{aligned} \tag{5.46}$$

[15]From now on the sum over the transversal mode is assumed, so we omit the summation symbol $\sum_{l=1}^{2}$.

On the other hand, the transversal part is non-trivial as it is mapped onto a dynamical quantity

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c &= \langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle - \langle\!\langle \boldsymbol{E}_1^l(t)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^l(0)\rangle\!\rangle \\
&= \Lambda^4\langle\partial_z\xi^l(t,r/2)\partial_z\xi^l(0,r/2)\rangle - \Lambda^4\langle\partial_z\xi^l(t,r/2)\rangle\langle\partial_z\xi^l(0,r/2)\rangle \\
&= \Lambda^4\partial_z\partial_{z'}\langle\xi(t,z)\xi(t',z')\rangle|_{z=z'=r/2}^{t'=0}\,,
\end{aligned}
\tag{5.47}
$$

where the third line is given by $\langle\xi^l\rangle = 0$. This is due to the Gaussianity of the EST action, Eq, (5.18). The correlator in the third line is in fact the Feynman propagator of the EST action, and by taking the Wick rotation, $t \to -it$, we can make use of the Green's function in the Euclidean spacetime, Eq. (5.33)

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c &= \Lambda^4\partial_z\partial_{z'}G(t,t';z,z')|_{z=z'=r/2}^{t'=0} \\
&= \frac{\pi\Lambda^4}{4\sigma r^2}\sinh^{-2}\left(\frac{\pi t}{2r}\right)\,.
\end{aligned}
\tag{5.48}
$$

Then, by inserting this expression into Eq. (5.45), we obtain the following expression of the potential [56, 57]

$$
V^{(1,0)}(r) = \frac{g^2\Lambda^4}{2\pi\sigma}\ln(\sigma r^2) + \mu_1\,,
\tag{5.49}
$$

in which μ_1 is a renormalization parameter that comes from the lower limit of the integration over time ($t \to 0$). This logarithmic behavior with r has already been confirmed with lattice data [47, 56].

Momentum-dependent (but spin-independent) potentials

Moving on with $\mathcal{O}(1/M^2)$ corrections, we can use the result of the gauge field insertions, Eq. (5.48), for momentum-dependent (but spin-independent) potentials

$$
\begin{aligned}
V_{\mathbf{L^2}}^{(2,0)}(r) =& \frac{i}{4}(\delta^{ij} - 3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j)\lim_{T\to\infty}\int_0^T dt\, t^2\langle\!\langle g\boldsymbol{E}_1^i(t)g\boldsymbol{E}_1^j(0)\rangle\!\rangle_c\,, \\
V_{\mathbf{p^2}}^{(2,0)}(r) =& \frac{i}{2}\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\, t^2\langle\!\langle g\boldsymbol{E}_1^i(t)g\boldsymbol{E}_1^j(0)\rangle\!\rangle_c\,.
\end{aligned}
\tag{5.50}
$$

Due to the parametrization of the system, the normal vector along the axis on which the heavy quark-antiquark pair is aligned is given by $\hat{\mathbf{r}} = (0,0,1)$, so that these potentials are rewritten as

$$
\begin{aligned}
V_{\mathbf{L^2}}^{(2,0)}(r) =& \frac{i}{4}\lim_{T\to\infty}\int_0^T dt\, t^2\left(\langle\!\langle g\boldsymbol{E}_1^l(t)g\boldsymbol{E}_1^l(0)\rangle\!\rangle_c - 2\langle\!\langle g\boldsymbol{E}_1^3(t)g\boldsymbol{E}_1^3(0)\rangle\!\rangle_c\right)\,, \\
V_{\mathbf{p^2}}^{(2,0)}(r) =& \frac{i}{2}\lim_{T\to\infty}\int_0^T dt\, t^2\langle\!\langle g\boldsymbol{E}_1^3(t)g\boldsymbol{E}_1^3(0)\rangle\!\rangle_c\,.
\end{aligned}
\tag{5.51}
$$

As it was shown in Eq. (5.46), the longitudinal part of the gauge field insertion vanishes in the EST, so just using Eq. (5.48) for the Wick rotated expression on the right-hand side, we derive the potentials

$$V^{(2,0)}_{\mathbf{L^2}}(r) = -\frac{g^2\Lambda^4}{6\sigma}r\,, \tag{5.52}$$

$$V^{(2,0)}_{\mathbf{p}^2}(r) = 0\,. \tag{5.53}$$

For $V^{(1,1)}_{\mathbf{L^2}}(r)$ and $V^{(1,1)}_{\mathbf{p}^2}(r)$, we need to calculate another part of the gauge field insertions, which is mapped onto the string variables and simplified by

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1(t)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c &= \langle\!\langle \boldsymbol{E}^l_1(t)\boldsymbol{E}^l_2(0)\rangle\!\rangle_c\\ &= \Lambda^4\partial_z\partial_{z'}\langle\xi(t,z)\xi(t',z')\rangle|^{t'=0}_{z=-z'=r/2}\,,\end{aligned} \tag{5.54}$$

and after the Wick rotation, we obtain the following expression

$$\begin{aligned}\langle\!\langle \boldsymbol{E}^l_1(t)\boldsymbol{E}^l_2(0)\rangle\!\rangle_c &= \Lambda^4\partial_z\partial_{z'}G(t,t';z,z')|^{t'=0}_{z=-z'=r/2}\\ &= -\frac{\pi\Lambda^4}{4\sigma r^2}\cosh^{-2}\left(\frac{\pi t}{2r}\right)\,.\end{aligned} \tag{5.55}$$

Then, the calculations of the momentum-dependent potentials are straightforward as we insert Eq. (5.55) into Eqs. (4.23) and (4.25)

$$V^{(1,1)}_{\mathbf{L^2}}(r) = \frac{g^2\Lambda^4}{6\sigma}r\,, \tag{5.56}$$

$$V^{(1,1)}_{\mathbf{p}^2}(r) = 0\,. \tag{5.57}$$

Spin-orbit potentials

Two of the spin-orbit potentials are given in terms of the insertions of the cross product between a chromomagnetic field and a chromoelectric field

$$V^{(2,0)}_{LS}(r) = -\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\,t\langle\!\langle g\boldsymbol{B}_1(t)\times g\boldsymbol{E}_1(0)\rangle\!\rangle_c\,, \tag{5.58}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\,t\langle\!\langle g\boldsymbol{B}_1(t)\times g\boldsymbol{E}_2(0)\rangle\!\rangle_c\,. \tag{5.59}$$

Due to the parametrization $\mathbf{r} = (0,0,1)$, these expressions are rewritten

$$V^{(2,0)}_{LS}(r) = -\frac{ic_F^{(1)}}{r}\lim_{T\to\infty}\int_0^T dt\,t\,\epsilon^{3lm}\langle\!\langle g\boldsymbol{B}^l_1(t)g\boldsymbol{E}^m_1(0)\rangle\!\rangle_c\,, \tag{5.60}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{ic_F^{(1)}}{r}\lim_{T\to\infty}\int_0^T dt\,t\,\epsilon^{3lm}\langle\!\langle g\boldsymbol{B}^l_1(t)g\boldsymbol{E}^m_2(0)\rangle\!\rangle_c\,, \tag{5.61}$$

in which ϵ^{lmn} is a totally antisymmetry rank-3 tensor. Thus, we have to compute the gauge field insertions on the right-hand sides in terms of the string variables. Definition of the connected part of the gauge field insertion is given and simplified by

$$\begin{aligned}\langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_{1,2}^m(0)\rangle\rangle_c &= \langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_{1,2}^m(0)\rangle\rangle - \langle\langle \boldsymbol{B}_1^l(t)\rangle\rangle\langle\langle \boldsymbol{E}_{1,2}^m(0)\rangle\rangle \\ &= \langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_{1,2}^m(0)\rangle\rangle\,,\end{aligned} \tag{5.62}$$

in which the second line is due to the mapping and the Gaussianity of the EST action, and using the mapping, Eq. (5.40), the expression is mapped onto

$$\begin{aligned}\langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_{1,2}^m(0)\rangle\rangle &= \Lambda'\Lambda^2\epsilon^{ln}\langle\partial_t\partial_z\xi^n(t,r/2)\partial_z\xi^m(0,\pm r/2)\rangle \\ &= \Lambda'\Lambda^2\epsilon^{ln}\partial_t\partial_z\partial_{z'}\langle\xi^n(t,z)\xi^m(t',z')\rangle\big|_{z=\pm z'=r/2}^{t'=0}\,.\end{aligned} \tag{5.63}$$

By taking the Wick rotation and using Eq. (5.33) for the string correlator, we obtain the analytic expressions of the insertions

$$\epsilon^{3lm}\langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_1^m(0)\rangle\rangle_c = \frac{i\pi^2\Lambda^2\Lambda'}{2\sigma r^3}\cosh\left(\frac{\pi t}{2r}\right)\sinh^{-3}\left(\frac{\pi t}{2r}\right)\,, \tag{5.64}$$

$$\epsilon^{3lm}\langle\langle \boldsymbol{B}_1^l(t)\boldsymbol{E}_2^m(0)\rangle\rangle_c = -\frac{i\pi^2\Lambda^2\Lambda'}{2\sigma r^3}\sinh\left(\frac{\pi t}{2r}\right)\cosh^{-3}\left(\frac{\pi t}{2r}\right)\,. \tag{5.65}$$

Then, the corresponding potentials are derived

$$V_{LS}^{(2,0)}(r) = -\frac{\mu_2}{r} - \frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2}\,, \tag{5.66}$$

$$V_{L_2S_1}^{(1,1)}(r) = -\frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2}\,, \tag{5.67}$$

where μ_2 is a renormalization parameter, which comes from the lower limit of the integration domain ($t\to 0$) just like μ_1. As we will see in the next section, this parameter can be constrained in terms of the fundamental parameter of the EST due to the symmetry of QCD.

Spin-spin interaction potentials

Spin-spin interaction part of the potentials are given by (using the parametrization $\mathbf{r}=(0,0,r)$)

$$\begin{aligned}V_{S^2}^{(1,1)}(r) =& \frac{2ic_F^{(1)}c_F^{(2)}}{3}\lim_{T\to\infty}\int_0^T dt\,\Big(\langle\langle g\boldsymbol{B}_1^l(t)g\boldsymbol{B}_2^l(0)\rangle\rangle_c + \langle\langle g\boldsymbol{B}_1^3(t)g\boldsymbol{B}_2^3(0)\rangle\rangle_c\Big) \\ &- 4(d_{sv}+d_{vv}C_f)\delta^{(3)}(\mathbf{r})\,,\end{aligned} \tag{5.68}$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{ic_F^{(1)}c_F^{(2)}}{4}\lim_{T\to\infty}\int_0^T dt\left(-\frac{1}{3}\langle\langle g\boldsymbol{B}_1^l(t)g\boldsymbol{B}_2^l(0)\rangle\rangle_c + \frac{2}{3}\langle\langle g\boldsymbol{B}_1^3(t)g\boldsymbol{B}_2^3(0)\rangle\rangle_c\right)\,. \tag{5.69}$$

Due to the Gaussianty of the action as well as the QCD-to-EST mapping, the transverse component is given by

$$\begin{aligned}\langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c =& -\Lambda'^2\epsilon^{lm}\epsilon^{ln}\langle\partial_t\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,-r/2)\rangle \\ =& -\Lambda'^2\delta^{mn}\partial_t\partial_z\partial_{t'}\partial_{z'}\langle\xi^m(t,z)\xi^n(t',z')\rangle|_{z=-z'=r/2}^{t'=0}\,,\end{aligned} \tag{5.70}$$

and the Wick rotated expression is derived

$$\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c = -\frac{\pi^3\Lambda'^2}{4\sigma r^4}\cosh^{-4}\left(\frac{\pi t}{2r}\right)\left[2-\cosh\left(\frac{\pi t}{r}\right)\right]\,. \tag{5.71}$$

On the other hand, the longitudinal component is decomposed and mapped onto the four-string field correlator

$$\begin{aligned}&\langle\!\langle \boldsymbol{B}_1^3(t)\boldsymbol{B}_2^3(0)\rangle\!\rangle_c \\ =&\langle\!\langle \boldsymbol{B}_1^3(t)\boldsymbol{B}_2^3(0)\rangle\!\rangle - \langle\!\langle \boldsymbol{B}_1^3(t)\rangle\!\rangle\langle\!\langle \boldsymbol{B}_2^3(0)\rangle\!\rangle \\ =& -\Lambda'''^2\epsilon^{lm}\epsilon^{np}\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,-r/2)\partial_z\xi^p(0,-r/2)\rangle \\ &+\Lambda'''^2\epsilon^{lm}\epsilon^{np}\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^m(t,r/2)\rangle\langle\partial_t\partial_z\xi^n(0,-r/2)\partial_z\xi^p(0,-r/2)\rangle \\ =& -\Lambda'''^2\epsilon^{lm}\epsilon^{np}\Big[\langle\partial_t\partial_z\xi^l(t,r/2)\partial_t\partial_z\xi^n(0,-r/2)\rangle\langle\partial_z\xi^m(t,r/2)\partial_z\xi^p(0,-r/2)\rangle \\ &+\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^p(0,-r/2)\rangle\langle\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,-r/2)\rangle\Big]\,,\end{aligned} \tag{5.72}$$

in which the third equality is given by the Wick contraction. After taking the Wick rotation and using Eq. (5.33), we obtain the analytic expression

$$\langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_2^3(0)\rangle\!\rangle_c = \frac{\pi^4\Lambda'''^2}{16\sigma^2 r^6}\cosh^{-6}\left(\frac{\pi t}{2r}\right)\,. \tag{5.73}$$

Taking the integral of Eqs. (5.71) and (5.73) over time, analytic expressions of the spin-spin potentials are derived [44]

$$V_{S^2}^{(1,1)}(r) = \frac{2\pi^2 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{45\sigma^2 r^5} - 4(d_{sv}+d_{vv}C_f)\delta^{(3)}(\mathbf{r})\,, \tag{5.74}$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{\pi^2 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{90\sigma^2 r^5}\,. \tag{5.75}$$

Note that the integral of Eq. (5.71) vanishes, thus, only the longitudinal component contributes to the potentials.

Central potentials

Lastly, we derive the central potentials V_r's. The expression of $V_r^{(2,0)}$ in terms of the gauge field insertion is given in Eq. (4.30)

$$V_r^{(2,0)}(r) = \frac{\pi C_f\alpha_s c_D^{(1)\prime}}{2}\delta^{(3)}(\mathbf{r}) - \frac{ic_F^{(1)2}}{4}\lim_{T\to\infty}\int_0^T dt\langle\!\langle g\boldsymbol{B}_1(t)\cdot g\boldsymbol{B}_1(0)\rangle\!\rangle_c + \frac{1}{2}(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(2,0)})$$

$$
\begin{aligned}
&-\frac{i}{2}\int_0^\infty dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3(t_2-t_3)^2\langle\!\langle g\boldsymbol{E}_1(t_1)\cdot g\boldsymbol{E}_1(t_2)g\boldsymbol{E}_1(t_3)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c\\
&+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\int_0^\infty dt_1\int_0^{t_1}dt_2(t_1-t_2)^2\langle\!\langle g\boldsymbol{E}_1^i(t_1)g\boldsymbol{E}_1(t_2)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c\right)\\
&-d_3^{(1)\prime}f_{abc}\int d^2\mathbf{x}\lim_{T\to\infty}g\langle\!\langle F^a_{\mu\nu}(x)F^b_{\mu\alpha}(x)F^c_{\nu\alpha}(x)\rangle\!\rangle\,. \qquad (5.76)
\end{aligned}
$$

We calculate the relevant gauge field insertions. The first one is of a dot product between two chromomagnetic fields. This is decomposed into a transversal and a longitudinal parts,

$$
\langle\!\langle \boldsymbol{B}_1(t)\cdot\boldsymbol{B}_1(0)\rangle\!\rangle_c=\langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c+\langle\!\langle \boldsymbol{B}_1^3(t)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c\,, \qquad (5.77)
$$

and their corresponding mappings onto the string variables are given by

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c=&\Lambda'^2\epsilon^{lm}\epsilon^{ln}\langle\partial_t\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,r/2)\rangle\,, \qquad (5.78)\\
\langle\!\langle \boldsymbol{B}_1^3(t)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c=&\Lambda''^2\epsilon^{lm}\epsilon^{np}\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,r/2)\partial_z\xi^p(0,r/2)\rangle\\
&-\Lambda'^2\epsilon^{lm}\epsilon^{np}\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^m(t,r/2)\rangle\langle\partial_t\partial_z\xi^n(0,r/2)\partial_z\xi^p(0,r/2)\rangle\\
=&\Lambda'^2\epsilon^{lm}\epsilon^{np}\Big[\langle\partial_t\partial_z\xi^l(t,r/2)\partial_t\partial_z\xi^n(0,r/2)\rangle\langle\partial_z\xi^m(t,r/2)\partial_z\xi^p(0r,2)\rangle\\
&+\langle\partial_t\partial_z\xi^l(t,r/2)\partial_z\xi^p(0,r/2)\rangle\langle\partial_z\xi^m(t,r/2)\partial_t\partial_z\xi^n(0,r/2)\rangle\Big]\,. \qquad (5.79)
\end{aligned}
$$

Taking the Wick rotation, we derive the following expressions for the two-magnetic field insertions

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c=&\frac{\pi^2\Lambda'^2}{4\sigma r^4}\sinh^{-4}\left(\frac{\pi t}{2r}\right)\left[2+\cosh\left(\frac{\pi t}{r}\right)\right]\,, \qquad (5.80)\\
\langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c=&\frac{\pi^4\Lambda'''^2}{16\sigma^2 r^6}\sinh^{-6}\left(\frac{\pi t}{2r}\right)\,. \qquad (5.81)
\end{aligned}
$$

Integrating these expressions over time, we obtain the following contribution to the potential

$$
-\frac{ic_F^{(1)2}}{4}\lim_{T\to\infty}\int_0^T dt\langle\!\langle g\boldsymbol{B}_1(-it)\cdot g\boldsymbol{B}_1(0)\rangle\!\rangle_c=\mu_3+\frac{\mu_4}{r^2}+\frac{\mu_5}{r^4}+\frac{\pi^3 c_F^{(1)2}g^2\Lambda'''^2}{60\sigma^2 r^5}\,, \qquad (5.82)
$$

where μ_3, μ_4, and μ_5 are renormalization parameters, which come from the lower limit of the time integral.

A four-chromoelectric field insertion is mapped onto the string variables and simplified due to the definition of the connected part of the expectation value

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(t_1)\cdot\boldsymbol{E}_1(t_2)\boldsymbol{E}_1(t_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c\\
=&\Lambda^8\Big[\langle\partial_z\xi^l(t_1,r/2)\partial_z\xi^m(t_3,r/2)\rangle\langle\partial_z\xi^l(t_2,r/2)\partial_z\xi^m(0,r/2)\rangle\\
&+\langle\partial_z\xi^l(t_1,r/2)\partial_z\xi^m(0,r/2)\rangle\langle\partial_z\xi^l(t_2,r/2)\partial_z\xi^m(t_3,r/2)\rangle\Big]\,, \qquad (5.83)
\end{aligned}
$$

and taking the Wick rotation, we obtain the analytic expression for it

$$\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_1(-it_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c \\
&=\frac{\pi^2\Lambda^8}{8\sigma^2r^4}\left\{\sinh^{-2}\left(\frac{\pi t_2}{2r}\right)\sinh^{-2}\left[\frac{\pi\,(t_1-t_3)}{2r}\right]+\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left[\frac{\pi\,(t_2-t_3)}{2r}\right]\right\}.
\end{aligned} \tag{5.84}$$

The time integral of this expression turns out to be a Riemann zeta function of argument three ζ_3 due to the following identity

$$\begin{aligned}
&\int_0^\infty dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\,(t_2-t_3)^2\left[\sinh^{-2}(t_2)\sinh^{-2}(t_1-t_3)+\sinh^{-2}(t_1)\sinh^{-2}(t_2-t_3)\right] \\
&=\zeta_3\,,
\end{aligned} \tag{5.85}$$

so that we derive the analytic expression

$$\begin{aligned}
&-\frac{i}{2}\int_0^\infty dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3(t_2-t_3)^2\langle\!\langle g\boldsymbol{E}_1(t_1)\cdot g\boldsymbol{E}_1(t_2)g\boldsymbol{E}_1(t_3)\cdot g\boldsymbol{E}_1(0)\rangle\!\rangle_c \\
&=-\frac{2\zeta_3g^4\Lambda^8}{\pi^3\sigma^2}r\,.
\end{aligned} \tag{5.86}$$

Note that a three-chromoelectric field insertion for $V_r^{(2,0)}$ vanishes due to the definition of the connected part of the expectation value as well as the QCD-to-EST mappings. Also, there is no corresponding mapping for the field strength insertion. Thus, the potential is given by summing all these terms

$$\begin{aligned}
V_r^{(2,0)}(r)=&-\frac{2\zeta_3g^4\Lambda^8}{\pi^3\sigma^2}r+\mu_3+\frac{\mu_4}{r^2}+\frac{\mu_5}{r^4}+\frac{\pi^3c_F^{(1)2}g^2\Lambda'''^2}{60\sigma^2r^5}+\frac{\pi C_f\alpha_sc_D^{(1)'}}{2}\delta^{(3)}(\mathbf{r}) \\
&-d_3^{(1)'}f_{abc}\int d^2\mathbf{x}\lim_{T\to\infty}g\langle\!\langle F^a_{\mu\nu}(x)F^b_{\mu\alpha}(x)F^c_{\nu\alpha}(x)\rangle\!\rangle\,.
\end{aligned} \tag{5.87}$$

As for $V_r^{(1,1)}(r)$, Eq. (4.31), we only have to calculate the four-chromoelectric field insertion because the ones involving three-chromoelectric field insertions vanish due to the mapping as well as the definition of the connected part of the Wilson loop expectation value. The four-field insertion is mapped and simplified by

$$\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(t_1)\cdot\boldsymbol{E}_1(t_2)\boldsymbol{E}_2(t_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c \\
=&\Lambda^8\Big[\langle\partial_z\xi^l(t_1,r/2)\partial_z\xi^m(t_3,-r/2)\rangle\langle\partial_z\xi^l(t_2,r/2)\partial_z\xi^m(0,-r/2)\rangle \\
&+\langle\partial_z\xi^l(t_1,r/2)\partial_z\xi^m(0,-r/2)\rangle\langle\partial_z\xi^l(t_2,r/2)\partial_z\xi^m(t_3,-r/2)\rangle\Big]\,,
\end{aligned} \tag{5.88}$$

and the Wick rotated expression is given by

$$\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_2(-it_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c$$

$$= \frac{\pi^2\Lambda^8}{8\sigma^2 r^4}\left\{\cosh^{-2}\left(\frac{\pi t_2}{2r}\right)\cosh^{-2}\left[\frac{\pi\left(t_1-t_3\right)}{2r}\right]+\cosh^{-2}\left(\frac{\pi t_1}{2r}\right)\cosh^{-2}\left[\frac{\pi\left(t_2-t_3\right)}{2r}\right]\right\}. \tag{5.89}$$

Its time integral also turns out to be of the Riemann zeta function of order three. Thus, the potential is given by

$$V_r^{(1,1)}(r) = \frac{\zeta_3 g^4\Lambda^8}{2\pi^3\sigma^2}r + (d_{ss}+d_{vs}C_f)\delta^{(3)}(\mathbf{r}). \tag{5.90}$$

All in all, these analytic results on the potentials agree with [57].

5.3.6 Poincaré invariance

In summary, we obtain the following list of the long-distance heavy quark potentials, which are calculated by utilizing the QCD-to-EST mapping, Eq. (5.40), as well as the EST Green's funciton, Eq. (5.33),

$$V^{(0)}(r) = \sigma r\,, \tag{5.91}$$

$$V^{(1,0)}(r) = \frac{g\Lambda^4}{2\pi\sigma}\ln\left(\sigma r^2\right)+\mu_1\,, \tag{5.92}$$

$$V^{(2,0)}_{\mathbf{L^2}}(r) = -\frac{g^2\Lambda^4}{6\sigma}r\,, \tag{5.93}$$

$$V^{(1,1)}_{\mathbf{L^2}}(r) = \frac{g^2\Lambda^4}{6\sigma}r\,, \tag{5.94}$$

$$V^{(2,0)}_{\mathbf{p^2}}(r) = 0\,, \tag{5.95}$$

$$V^{(1,1)}_{\mathbf{p^2}}(r) = 0\,, \tag{5.96}$$

$$V^{(2,0)}_{LS}(r) = -\frac{\mu_2}{r}-\frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2}\,, \tag{5.97}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2}\,, \tag{5.98}$$

$$V^{(1,1)}_{S^2}(r) = \frac{2\pi^2 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{45\sigma^2 r^5}-4(d_{sv}+d_{vv}C_f)\delta^{(3)}(\mathbf{r})\,, \tag{5.99}$$

$$V^{(1,1)}_{\mathbf{S}_{12}}(r) = \frac{\pi^2 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{90\sigma^2 r^5}\,, \tag{5.100}$$

$$V^{(2,0)}_{r}(r) = -\frac{2\zeta_3 g^4\Lambda^8}{\pi^3\sigma^2}r+\mu_3+\frac{\mu_4}{r^2}+\frac{\mu_5}{r^4}+\frac{\pi^3 c_F^{(1)2}g^2\Lambda'''^2}{60\sigma^2 r^5}+\frac{\pi C_f\alpha_s c_D^{(1)'}}{2}\delta^{(3)}(\mathbf{r})$$
$$-d_3^{(1)'}f_{abc}\int d^2\mathbf{x}\lim_{T\to\infty}g\langle\!\langle F^a_{\mu\nu}(x)F^b_{\mu\alpha}(x)F^c_{\nu\alpha}(x)\rangle\!\rangle\,, \tag{5.101}$$

$$V^{(1,1)}_{r}(r) = \frac{\zeta_3 g^4\Lambda^8 r}{2\pi^3\sigma^2}+(d_{ss}+d_{vs}C_f)\delta^{(3)}(\mathbf{r})\,, \tag{5.102}$$

where $c_F^{(1)}, c_F^{(2)}, c_D^{(1)\prime}, d_3^{(1)\prime}, d_{sv}, d_{vv}$ are the Wilson coefficients from NRQCD, and μ^i (for $i \in \{1, 2, \ldots, 5\}$) are renormalization parameters arising from time integrals. Furthermore, Λ's are the dimensionful parameters from the mapping. Besides the string tension σ, which is fundamental parameter of the EST, there are a number of free parameters appearing in these expressions, so it would be useful to reduce some of them before comparing to available lattice data. Symmetries play a crucial role for that. If the Poincaré invariance in QCD is manifest in the low-energy regime, especially in the EFTs like NRQCD and pNRQD, there arise some constraining equations between the heavy quark potentials. We are going to exploit these symmetries, thereby simplifying the expressions of the potentials.

The first constraining equation from the Poincaré invariance is the Gromes relation [92]:

$$\frac{1}{2r}\frac{dV^{(0)}}{dr} + V_{LS}^{(2,0)} - V_{L_2S_1}^{(1,1)} = 0\,, \tag{5.103}$$

from which a constraint is given on the renormalization parameter

$$\mu_2 = \frac{\sigma}{2}\,. \tag{5.104}$$

The second equation is concerning the momentum-dependent potentials [93]

$$\frac{r}{2}\frac{dV^{(0)}}{dr} + 2V_{\mathbf{L^2}}^{(2,0)} - V_{\mathbf{L^2}}^{(1,1)} = 0\,, \tag{5.105}$$

and a constraint is given on one of the dimensionful parameter from the QCD-to-EST mapping

$$g\Lambda^2 = \sigma\,. \tag{5.106}$$

In [50], an equation is given in the large time limit $-\boldsymbol{\nabla}_1 V^{(0)} = \langle\!\langle g\boldsymbol{E}_1\rangle\!\rangle$, and as well mapped this equation onto the string variable, we obtain another constraint on a dimensionful parameter

$$-\sigma = g\Lambda''^2\,. \tag{5.107}$$

Note that the spatial derivative $\boldsymbol{\nabla}_1$ denotes a derivative in the direction towards the position of the heavy quark; i.e., $\boldsymbol{\nabla}_r$. On the other hand, the gauge field insertion on the right-hand side boils down to the longitudianl component as the transversal components vanish due to the Gaussianity of the EST action, so that $\langle\!\langle g\boldsymbol{E}_1\rangle\!\rangle = g\Lambda''^2$. Inserting Eqs. (5.104), (5.106), and (5.107) into the expressions of the potentials, Eqs. (5.91), (5.92), (5.93), (5.94), (5.97), (5.98), (5.99), (5.100), (5.101), and (5.102), we obtain the simplified expressions:

$$V^{(0)}(r) = \sigma r\,, \tag{5.108}$$

$$V^{(1,0)}(r) = \frac{\sigma}{2\pi}\ln\left(\sigma r^2\right) + \mu_1\,, \tag{5.109}$$

$$V^{(2,0)}_{\mathbf{L}^2}(r) = -\frac{\sigma r}{6}\,, \tag{5.110}$$

$$V^{(1,1)}_{\mathbf{L}^2}(r) = \frac{\sigma r}{6}\,, \tag{5.111}$$

$$V^{(2,0)}_{\mathbf{p}^2}(r) = 0\,, \tag{5.112}$$

$$V^{(1,1)}_{\mathbf{p}^2}(r) = 0\,, \tag{5.113}$$

$$V^{(2,0)}_{LS}(r) = -\frac{\sigma}{2r} - \frac{c_F^{(1)} g\Lambda'}{r^2}\,, \tag{5.114}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{c_F^{(1)} g\Lambda'}{r^2}\,, \tag{5.115}$$

$$V^{(1,1)}_{S^2}(r) = \frac{2\pi^2 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{45\sigma^2 r^5} - 4(d_{sv} + d_{vv}C_f)\delta^{(3)}(\mathbf{r})\,, \tag{5.116}$$

$$V^{(1,1)}_{\mathbf{S}_{12}}(r) = \frac{\pi^2 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{90\sigma^2 r^5}\,, \tag{5.117}$$

$$\begin{aligned} V^{(2,0)}_{r}(r) = & -\frac{2\zeta_3 \sigma^2 r}{\pi^3} + \mu_3 + \frac{\mu_4}{r^2} + \frac{\mu_5}{r^4} + \frac{\pi^3 c_F^{(1)2} g^2 \Lambda'''^2}{60\sigma^2 r^5} + \frac{\pi C_f \alpha_s c_D^{(1)'}}{2}\delta^{(3)}(\mathbf{r}) \\ & - d_3^{(1)'} f_{abc} \int d^2\mathbf{x} \lim_{T\to\infty} g\langle\!\langle F^a_{\mu\nu}(x) F^b_{\mu\alpha}(x) F^c_{\nu\alpha}(x)\rangle\!\rangle\,, \end{aligned} \tag{5.118}$$

$$V^{(1,1)}_{r}(r) = \frac{\zeta_3 \sigma^2 r}{2\pi^3} + (d_{ss} + d_{vs}C_f)\delta^{(3)}(\mathbf{r})\,. \tag{5.119}$$

Note that the momentum-dependent (but spin-independent) potentials are greatly simplified after using the symmetries, especially the angular momentum-dependent contribution.

This analytic result of the heavy quark potentials gives a useful tool for calculating the heavy quarkonium spectrum. In order to do so, it is better to sum the potentials, order by order in the $1/M$ expansion. In the center-of-mass frame, in which the Hamiltonian of the system is $H = \mathbf{p}^2/M + V(r)$, and $V(r)$ is given by summing the potentials above [57]

$$\begin{aligned} V(r) = & V^{(0)}(r) + \frac{2}{M}V^{(1,0)}(r) + \frac{1}{M^2}\left\{\left[2\frac{V^{(2,0)}_{L^2}(r)}{r^2} + \frac{V^{(1,1)}_{L^2}(r)}{r^2}\right]\mathbf{L}^2\right. \\ & + \left[V^{(2,0)}_{LS}(r) + V^{(1,1)}_{L_2S_1}(r)\right]\mathbf{L}\cdot\mathbf{S} + V^{(1,1)}_{S^2}(r)\left(\frac{\mathbf{S}^2}{2} - \frac{3}{4}\right) + V^{(1,1)}_{S_{12}}\mathbf{S}_{12}(\hat{r}) \\ & \left. + \left[2V^{(2,0)}_{p^2}(r) + V^{(1,1)}_{p^2}(r)\right]\mathbf{p}^2 + 2V^{(2,0)}_{r}(r) + V^{(1,1)}_{r}(r)\right\} \\ \approx & \,\sigma r + \mu_1 + \frac{1}{M}\frac{\sigma}{\pi}\ln\left(\sigma r^2\right) + \frac{1}{M^2}\left(-\frac{\sigma}{6r}\mathbf{L}^2 - \frac{\sigma}{2r}\mathbf{L}\cdot\mathbf{S} - \frac{9\zeta_3\sigma^2 r}{2\pi^3}\right)\,, \end{aligned} \tag{5.120}$$

where we have truncated the terms at $\mathcal{O}(r^{-2})$. Note that analytic expression of the long-distance heavy quark potential is given only in terms of the string tension σ and the heavy quark mass M. Although this final result carries few parameters, several terms are subject to be modified at leading order in the $1/r$ expansion if NLO terms in the mapping are included (NLO in the power counting of the EST). For instance, V_r, which is written in terms of two-, three-, and four-gauge field insertions to the Wilson loop expectation value[16], can acquire corrections at linear order in r if the NLO terms are included. In other words, this leading order calculation is not fully inclusive at its respective order. Furthermore, as it was discussed in [58], a comparison between the analytic result of the momentum-dependent (but spin-independent) potentials and the LQCD data [47, 48] shows significant discrepancies, especially at the intermediate distance range. One can estimate that the discrepancies can be reduced by including subleading terms of the potentials within the EST calculation. Therefore, we will investigate this possibility and show how one can incorporate the subleading contributions of the effective string description. In the next section, a NLO calculation of the EST is presented as a first step towards this goal.

5.4 Heavy quark potentials at NLO

In this section, we are performing a similar calculation to the previous section, but at NLO in the EST power counting [94]. As we will see shortly, NLO contributions to the potentials come from the QCD-to-EST mapping at NLO. The calculations of the potentials from this mapping show divergences. We will discuss about the regualrization schemes for those, and after that full analytic expressions of the potentials are presented. In addition, just like in the case of LO calculation, we exploit the symmetry of QCD, Poincaré invariance, in order to reduce the number of free parameters arising from the mapping as well as renormalization ones from the evaluation of the potentials.

5.4.1 Green's function at NLO

There are two possible NLO contributions to the heavy quark potentials from the effective string description. The first possible contribution is from the inclusion of the NLO terms of the EST action, which are denoted by the ellipsis in Eq. (5.16), while keeping the leading order mapping, Eq. (5.40). In [55, 84, 95], it was shown that the only possible NLO terms of the EST action are proportional to $(\partial_a\xi^l\partial^a\xi^l)^2$ and $(\partial_a\xi^l\partial_b\xi^l)(\partial^a\xi^m\partial^b\xi^m)$, due to the open-closed string duality, the EST power counting, and symmetry of the EST action[17]. After the inclusion of these terms, the Green's function is derived by solving the equations of motion using a perturbative expansion. A back of the envelope calculation shows, however, that this NLO part of the correlator is $(\sigma r^2)^{-2}$ suppressed instead of the $(\sigma r^2)^{-1}$ because the perturbative expansion is given in terms of the six-string field correlator instead of the one from the four-string field. For instance, if we include $(\partial_a\xi^l\partial^a\xi^l)^2$ to the action, then the Green's function at spacetime point x and y

[16] Eqs. (4.30) and (4.31).

[17] In fact, these terms are found in the determinant of Eq. (5.15).

is given by

$$
\begin{aligned}
G(x,y) \sim & \langle 0|T\left\{\xi(x)\xi(y)\exp\left[-i\int_{-\infty}^{\infty} dt\,(\partial\xi\partial\xi)^2\right]\right\}|0\rangle \\
= & \langle 0|T\left\{\xi(x)\xi(y)\right\}|0\rangle - i\langle 0|T\left\{\xi(x)\xi(y)\int_{-\infty}^{\infty} dt\,(\partial\xi\partial\xi)^2\right\}|0\rangle + \langle 0|\mathcal{O}(\xi^8)|0\rangle\,,
\end{aligned}
\tag{5.121}
$$

where T stands for the time ordering operator. We omitted the indices for the string coordinates as well as the partial derivatives for simplicity[18]. Note that the second term on the second line of Eq. 5.121 is the six-field correlator, which is the first subleading contribution to the Green's function. A similar argument applies when we include $(\partial_a\xi^l\partial_b\xi^l)(\partial^a\xi^m\partial^b\xi^m)$ to the action. This estimation implies that the contribution from this part of the correlator has to be counted as next-to-next-to-leading order (NNLO) instead of NLO. Hence, this possibility is not considered in this section.

5.4.2 QCD to EST mapping at NLO

The second possible NLO contributions to the potentials come from the NLO part of the mapping. While keeping the same order terms of the action (i.e., only up to the Gaussian part), we exploit the QCD-to-EST mapping at NLO (denoted by NLO on the superscript), which is given by

$$
\begin{aligned}
\langle\!\langle \dots \boldsymbol{E}^l_{1,2}(t)\dots\rangle\!\rangle^{\mathrm{NLO}} &= \langle \dots \overline{\Lambda}^2 \partial_z\xi^l_{1,2}(t)(\partial\xi_{1,2})^2(t)\dots\rangle\,,\\
\langle\!\langle \dots \boldsymbol{B}^l_{1,2}(t)\dots\rangle\!\rangle^{\mathrm{NLO}} &= \langle \dots \pm\overline{\Lambda'}\epsilon^{lm}\partial_t\partial_z\xi^m_{1,2}(t)(\partial\xi_{1,2})^2(t)\dots\rangle\,,\\
\langle\!\langle \dots \boldsymbol{E}^3_{1,2}(t)\dots\rangle\!\rangle^{\mathrm{NLO}} &= \langle \dots \overline{\Lambda''}^2(\partial\xi_{1,2})^2(t)\dots\rangle\,,\\
\langle\!\langle \dots \boldsymbol{B}^3_{1,2}(t)\dots\rangle\!\rangle^{\mathrm{NLO}} &= \langle \dots \pm\overline{\Lambda'''}\epsilon^{lm}\partial_t\partial_z\xi^l_{1,2}(t)\partial_z\xi^m_{1,2}(t)(\partial\xi_{1,2})^2(t)\dots\rangle\,,
\end{aligned}
\tag{5.122}
$$

in which $\boldsymbol{E}_{1,2}(t) \equiv \boldsymbol{E}(t, \pm r/2)$ and $\xi_{1,2}(t) \equiv \xi(t, \pm r/2)$. The dimensionful parameters arising from this mapping (Λ's with a bar above) are compared to the ones from the LO mapping, Eq. (5.40), as

$$
\Lambda \geq \overline{\Lambda}, \quad \Lambda' \geq \overline{\Lambda'}, \quad \Lambda'' \geq \overline{\Lambda''}, \quad \text{and} \quad \Lambda''' \geq \overline{\Lambda'''}\,. \tag{5.123}
$$

Although these conditions on the parameters are not obvious in the first glance, they are, in fact, analogous to the Taylor expansion: the coefficients for the subleading part of the expansion are smaller than the coefficients of the leading terms[19]. This mapping differs from Eq. (5.40) by a factor of $(\partial\xi)^2$, which gives $(r\Lambda_{\mathrm{QCD}})^{-2}$ suppression in the EST power counting scheme.

[18] Note that Eq. 5.121 is only a heuristic estimation.
[19] Later, we will see that some of these conditions are met.

5.4.3 Gauge field insertions at NLO

With this mapping in mind, let us calculate the transversal component of two-chromoelectric field insertion as a simple example,

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\text{NLO}} =& \langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\text{NLO}} \\
=& \overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_a\xi_1^m(t)\partial^a\xi_1^m(t)\partial_z\xi_1^l(0)\rangle \\
&+ \Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\partial_b\xi_1^m(0)\partial^b\xi_1^n(0)\rangle \,.
\end{aligned}
\tag{5.124}
$$

The subscript c is absent on the right-hand side of the first equality, and this is due to the Gaussianity of the EST action. The second equality is given by the mapping, Eq. (5.122). After taking the Wick contraction as well as the Wick rotation, $t \to -it$, we obtain the following simplification of this gauge field insertion

$$
\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\text{NLO}} = 16\Lambda^2\overline{\Lambda}^2\partial_z\partial_{z'}G(t,t';z,z')|_{z=z'=r/2}^{t=t'} \times \partial_z\partial_{z'}G(t,t';z,z')|_{z=z'=r/2}^{t'=0} \,,
\tag{5.125}
$$

where $G(t,t';z,z')$ is the Green's function from Eq. (5.33) without the tensor indices,

$$
G(t,t';z,z') = \frac{1}{4\pi\sigma}\ln\left(\frac{\cosh\left[(t-t')\,\pi/r\right] + \cos\left[(z+z')\,\pi/r\right]}{\cosh\left[(t-t')\,\pi/r\right] - \cos\left[(z-z')\,\pi/r\right]}\right) \,.
\tag{5.126}
$$

Likewise, the other correlators are calculated and simplified by[20]

$$
\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\text{NLO}} = 4\overline{\Lambda''}^4\left[\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0}\right]^2 \,,
\tag{5.127}
$$

$$
\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\text{NLO}} = 16\Lambda^2\overline{\Lambda}^2\left[\partial_z\partial_{z'}G|_{z=z'=-r/2}^{t=t'} \times \partial_z\partial_{z'}G|_{z=-z'=r/2}^{t'=0}\right] \,,
\tag{5.128}
$$

$$
\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\text{NLO}} = 4\overline{\Lambda''}^4\left[\partial_z\partial_{z'}G|_{z=-z'=r/2}^{t'=0}\right]^2 \,,
\tag{5.129}
$$

$$
\begin{aligned}
\mathbf{r}\cdot\langle\!\langle \boldsymbol{B}_1(-it)\times\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\text{NLO}} =& r\left(\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\text{NLO}} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\text{NLO}}\right) \\
=& -4ir\Lambda^2\overline{\Lambda'}^2\left[\partial_t\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'} \times \partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0}\right. \\
&\left.+\partial_t\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0} \times \partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'}\right] \\
&-8ir\Lambda^2\overline{\Lambda'}^2\partial_t\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0} \times \partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'} \,,
\end{aligned}
\tag{5.130}
$$

$$
\begin{aligned}
\mathbf{r}\cdot\langle\!\langle \boldsymbol{B}_1(-it)\times\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\text{NLO}} =& r\left(\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_2^2(0)\rangle\!\rangle_c^{\text{NLO}} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_2^1(0)\rangle\!\rangle_c^{\text{NLO}}\right) \\
=& -4ir\Lambda^2\overline{\Lambda'}^2\left[\partial_t\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'} \times \partial_z\partial_{z'}G|_{z=-z'=r/2}^{t'=0}\right. \\
&\left.+\partial_t\partial_z\partial_{z'}G|_{z=-z'=r/2}^{t'=0} \times \partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'}\right]
\end{aligned}
$$

[20] $G = G(t,t';z,z')$ is implied from now on.

$$
- 8ir\Lambda'\overline{\Lambda}^2 \partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=-z'=r/2} \times \partial_z\partial_{z'}G|^{t=t'}_{z=z'=-r/2} \,, \tag{5.131}
$$

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}^l_1(-it)\boldsymbol{B}^l_1(0)\rangle\!\rangle^{\rm NLO}_c =& - 4\overline{\Lambda'}\Lambda' \Big[\partial_t\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2} \times \partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=z'=r/2} \\
&+\partial_t\partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=z'=r/2} \times \partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\Big] \\
&- 4\Lambda'\overline{\Lambda'} \Big[\partial_t\partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=z'=r/2} \times \partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2} \\
&+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2} \times \partial_t\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\Big] \,,
\end{aligned} \tag{5.132}
$$

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}^l_1(-it)\boldsymbol{B}^l_2(0)\rangle\!\rangle^{\rm NLO}_c =& - 4\overline{\Lambda'}\Lambda' \Big[\partial_t\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2} \times \partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=-z'=r/2} \\
&+\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2} \times \partial_t\partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=-z'=r/2}\Big] \\
&- 4\Lambda'\overline{\Lambda'} \Big[\partial_t\partial_z\partial_{t'}\partial_{z'}G|^{t'=0}_{z=-z'=r/2} \times \partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=-r/2} \\
&+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=-z'=r/2} \times \partial_t\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=-r/2}\Big] \,,
\end{aligned} \tag{5.133}
$$

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}^3_1(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle^{\rm NLO}_c =& - 4\overline{\Lambda''}^2\Lambda^4 \Big[\partial_{z_1}\partial_{z_2}G|_{z_1=z_2=r/2} \times \partial_{z_1}\partial_{z_3}G|^{t_3=0}_{z_1=z_3=r/2}\Big] \\
&+ 4\Lambda''^2\overline{\Lambda}^4 \Big[\partial_{z_1}\partial_{z_3}G|^{t_3=0}_{z_1=z_3=r/2}\Big]^2 \,,
\end{aligned} \tag{5.134}
$$

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}^3_1(-it_1)\boldsymbol{E}_2(-it_2)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle^{\rm NLO}_c =& - 4\overline{\Lambda''}^2\Lambda^4 \Big[\partial_{z_1}\partial_{z_2}G|^{z_1=r/2}_{z_2=-r/2} \times \partial_{z_1}\partial_{z_3}G|^{t_3=0}_{z_1=-z_3=r/2}\Big] \\
&+ 4\Lambda''^2\overline{\Lambda}^4 \Big[\partial_{z_1}\partial_{z_3}G|^{t_3=0}_{z_1=-z_3=r/2}\Big]^2 \,,
\end{aligned} \tag{5.135}
$$

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}^3_2(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle^{\rm NLO}_c =& - 4\overline{\Lambda''}^2\Lambda^2 \Big[\partial_{z_1}\partial_{z_2}G|^{z_1=-r/2}_{z_2=r/2} \times \partial_{z_1}\partial_{z_3}G|^{t_3=0}_{-z_1=z_3=r/2}\Big] \\
&+ 4\Lambda''^2\overline{\Lambda}^4 \Big[\partial_{z_1}\partial_{z_3}G|^{t_3=0}_{-z_1=z_3=r/2}\Big]^2 \,.
\end{aligned} \tag{5.136}
$$

And lastly, two of the four-chromoelectric field insertions to the Wilson loop expectation values are simplified and given by

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_1(-it_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle^{\rm NLO}_c \\
=& - 4\overline{\Lambda''}^2\Lambda''^2\Lambda^4\partial_{z_1}\partial_{z_3}G|_{z_1=z_3=r/2} \times \partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=z_4=r/2} + 4\Lambda''^4\overline{\Lambda}^4 \Big[\partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=z_4=r/2}\Big]^2 \\
&- 4\Lambda^4\Lambda''^2\overline{\Lambda''}^2 \Big[\partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=z_4=r/2} \times \partial_{z_2}\partial_{z_4}G|^{t_4=0}_{z_2=z_4=r/2}\Big] \,,
\end{aligned} \tag{5.137}
$$

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_2(-it_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle^{\rm NLO}_c \\
=& - 4\overline{\Lambda''}^2\Lambda''^2\Lambda^4 \Big[\partial_{z_1}\partial_{z_3}G|^{z_1=r/2}_{z_3=-r/2} \times \partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=-z_4=r/2}\Big] \\
&+ 4\Lambda''^4\overline{\Lambda}^4 \Big[\partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=-z_4=r/2}\Big]^2 \\
&- 4\Lambda^4\Lambda''^2\overline{\Lambda''}^2 \Big[\partial_{z_1}\partial_{z_4}G|^{t_4=0}_{z_1=-z_4=r/2} \times \partial_{z_2}\partial_{z_4}G|^{t_4=0}_{z_2=-z_4=r/2}\Big] \,.
\end{aligned} \tag{5.138}
$$

More detailed derivations of these expressions can be found in Appendix A.2.

As one tries to evaluate these expressions on the given space time points, some divergences appear. These divergences are due to the partial derivatives acting on the Green's function, and it being evaluated at the same spacetime points. It is natural to see these divergences because the EST is an effective description of a vibrating string, which connects the static heavy quark and the heavy antiquark in the long-distance regime. In other words, the EST is not able to provide any feasible description of the dynamics at scales above the hadronic scale Λ_{QCD}. We will analyze these divergence behaviors in great detail and show suitable regularization schemes for them.

5.4.4 Regularizations

There are three types of divergence that appear in the evaluation of a string correlator defined at equal spacetime points[21].

The first type is given by

$$\lim_{t'\to t}\lim_{z'\to z}\partial_z\partial_{z'}G(t,t';z,z')|_{z=\pm r/2}=\infty\,. \tag{5.139}$$

As we go back to the original derivation of the string field correlator in Euclidean spacetime, Eq. (5.27), the Green's function without the tensor indices is written as

$$G(t,t';z,z')=\frac{1}{\pi\sigma r}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{r}z-\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{r}z'-\frac{n\pi}{2}\right)\int_{-\infty}^{\infty}dk\frac{e^{-ik(t-t')}}{k^2+\left(\frac{n\pi}{r}\right)^2}\,, \tag{5.140}$$

which is identical to Eq. (5.33) except the rank-2 Kronecker delta δ^{lm}. Eq. (5.33) is simply derived by taking the infinite sum over n as well as integrating over the entire Fourier space $k\in(-\infty,\infty)$. By taking the partial derivatives with respect to the spatial coordinates z and z' on Eq. (5.140), its evaluation at the same spacetime points is rewritten

$$\partial_z\partial_{z'}G(t,t';z,z')|^{t=t'}_{z=z'=\pm r/2}=\frac{1}{\sigma r^2}\sum_{n=1}^{\infty}n\int_{-\infty}^{\infty}dy\,\frac{1}{y^2+1}\,. \tag{5.141}$$

The infinite sum over natural numbers on the right-hand side of Eq. (5.141) clearly diverges. This is due to the fact that we are taking into account the sum over infinitely many modes of the string vibration, which is a manifestation that the EST is a UV-divergent effective theory. Thus, it is necessary to employ a proper regularization scheme for that. The *zeta function regularization*, which is a common method in Bosonic string theory, gives a finite value of the sum [96, 97],

$$\sum_{n=1}^{\infty}n=-\frac{1}{12}\,. \tag{5.142}$$

[21] See Sec. 5.4.3 for the relevant expressions.

As the y integral on the right-hand side of Eq. (5.141) is evaluated

$$\int_{-\infty}^{\infty} dy \, \frac{1}{y^2+1} = \pi \,, \tag{5.143}$$

we obtain the regularized expression for Eq. (5.139)

$$\partial_z \partial_{z'} G(t,t';z,z')|_{z=z'=\pm r/2}^{t=t'} = -\frac{\pi}{12\sigma r^2} \,. \tag{5.144}$$

The second type of divergence comes from

$$\lim_{t' \to t} \lim_{z' \to z} \partial_t \partial_z \partial_{z'} G(t,t';z,z')|_{z=\pm r/2} = \infty \,, \tag{5.145}$$

whose divergent behavior is manifested by the infinite sum over the square of natural numbers

$$\partial_t \partial_z \partial_{z'} G(t,t';z,z')|_{z=z'=\pm r/2}^{t=t'} = \frac{\pi}{\sigma r^3} \sum_{n=1}^{\infty} n^2 \int_{-\infty}^{\infty} dy \left(\frac{-iy}{y^2+1} \right) \,. \tag{5.146}$$

Again by the zeta function regularization, the infinite sum over the square of natural numbers vanishes

$$\sum_{n=1}^{\infty} n^s = -\frac{B_{s+1}}{s+1} \,, \tag{5.147}$$

where B is the Bernoulli number, which vanishes for $s = 2$. Thus, this expression simply vanishes:

$$\partial_t \partial_z \partial_{z'} G(t,t';z,z')|_{z=z'=\pm r/2}^{t=t'} = 0 \,. \tag{5.148}$$

Also, note that $y/(y^2+1)$ is an odd function so that the integral over the entire domain $y \in (-\infty, \infty)$ vanishes as well. In a similar fashion[22],

$$\partial_z \partial_{t'} \partial_{z'} G|_{z=z'=\pm r/2}^{t=t'} = 0 \,. \tag{5.149}$$

The last type of divergence comes from the following expression:

$$\partial_t \partial_{t'} G(t,t';z,z')|_{z=z'=\pm r/2}^{t=t'} = \frac{1}{\pi \sigma r} \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} \left(\frac{n\pi}{r} \right) \sin[\epsilon]^2 \int_{-\infty}^{\infty} dy \left(\frac{y^2}{y^2+1} \right) \,. \tag{5.150}$$

While the infinite sum on the right-hand side of Eq. (5.150) vanishes as the infinitesimal parameter ϵ approaches to zero, the integral over y diverges. We use dimensional regularization for this divergent integral, which gives the finite result -2π. Thus, we see that this quantity vanishes as well,

$$\partial_t \partial_{t'} G(t,t';z,z')|_{z=z'=\pm r/2}^{t=t'} = 0 \,. \tag{5.151}$$

[22] In fact, this is due to the Schwarz's theorem: as the Green's function is continuous under second order partial derivatives, two distinct partial derivatives acting on the Green's function commute.

It appears that there are two different regularization schemes incorporated in this analysis. However, it turns out that both are identical to each other [98, 99]: the zeta function regularization is merely a discrete version of dimensional regularization, so our regularization scheme is consistent. Also, one can use a hard cut-off regularization scheme instead of the zeta function, by exploiting the hierarchy of scales of the EST. This method is presented in Appendix A.3.

In the next section, we will use Eqs. (5.144), (5.146), and (5.151) for the explicit NLO calculation of the gauge field insertions to the Wilson loop expectation value, Eqs. (5.125), (5.127), (5.128), (5.129), (5.130), (5.131), (5.132), (5.133), (5.134), (5.135), (5.136), (5.137), and (5.138). Full analytic expressions of the heavy potentials are presented at the end of the section.

5.4.5 NLO calculation of the potentials

Static potential

As we go back to the Wilson loop-string partition equivalence conjecture, Eq. (5.34), the NLO term of the static potential is derived by solving the string partition function [55]

$$\begin{aligned}\mathcal{Z}_0 =& Z\int \mathcal{D}\xi^1\mathcal{D}\xi^2 \exp\left\{-\sigma\int_0^T dt\int_{-r/2}^{r/2} dz\left(1-\frac{1}{2}\partial_a\xi^l\partial_a\xi^l\right)\right\} \\ =& \exp\left\{-\sigma rT-\mu T\right\}\eta(q)^{-2}\,,\end{aligned} \tag{5.152}$$

where $\eta(q)$ is defined by [100–102]

$$\eta(q)\equiv q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,,\quad \text{and}\quad q\equiv e^{-\pi T/r}\,. \tag{5.153}$$

As we expand in powers of q, this partition function is given in terms of the energy eigenvalues of the system

$$\mathcal{Z}_0=\sum_{n=0}^{\infty} w_n\, e^{-E_n^0 T} \tag{5.154}$$

where w_n are the positive weights, and the energy E_n^0 is given by[23]

$$E_n^0=\sigma r+\mu+\frac{\pi}{r}\left(-\frac{1}{12}+n\right)\,. \tag{5.155}$$

From this, we see that the ground state energy made of a linear term in r plus a constant and a term suppressed by $1/r$. As we compare this to the Wilson loop expectation value, the analytic expression of the static potential is finally derived

$$V^{(0)}=\sigma r+\mu-\frac{\pi}{12r}\,, \tag{5.156}$$

where the $\mathcal{O}(\sigma^{-1}r^{-2})$ suppressed term is the so called *Lüscher term* in a four-dimensional spacetime.

[23] The superscript "0" denotes the exact solution from the Gaussian action.

Potential at $\mathcal{O}(1/M)$

The NLO part of Eq. (4.20) is given in terms of the two-chromoelectric field insertion to the Wilson loop expectation value, calculated at NLO within the EST power counting,

$$V^{(1,0)}(r)\,|^{\rm NLO}= \frac{g^2}{2}\int_0^\infty dt\, t\left[\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} + \langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO}\right]\,, \quad (5.157)$$

in which the time variable has been Wick rotated, $t \to it$.

In the previous section, the gauge field insertions in the integrand on the right hand side of Eq. (5.157) are mapped and expressed in terms of the string fields, Eqs. (5.125) and (5.127). By using the zeta function regularization scheme, Eq. (5.144), we obtain the following analytic expressions for the gauge field insertions:

$$\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} = -\frac{\Lambda^2\overline{\Lambda}^2\pi^2}{3\sigma^2 r^4}\sinh^{-2}\left(\frac{\pi t}{2r}\right), \quad (5.158)$$

$$\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{NLO} = \frac{\pi^2\overline{\Lambda''}^4}{\sigma^2 r^4}\left[\cosh\left(\frac{\pi t}{r}\right)-1\right]^{-2}. \quad (5.159)$$

As we insert these expressions back into Eq. (5.157), the NLO contribution to the potential is analytically derived

$$V^{(1,0)}(r)|^{\rm NLO} = \mu_0^{(1,0)} + \frac{\mu_2^{(1,0)}}{r^2} - \frac{g^2(24\Lambda^2\overline{\Lambda}^2 + 13\overline{\Lambda''}^4)}{36\sigma^2 r^4}. \quad (5.160)$$

Here $\mu_0^{(1,0)}$ and $\mu_2^{(1,0)}$ are renormalization parameters that come from the time integral[24]. They are of mass dimension zero and two, respectively[25].

Momentum-dependent (but spin-independent) potentials

Moving onto the $1/M^2$ corrections to the static potential, momentum-dependent but spin-independent potentials such as $V_{\mathbf{L}^2}^{(2,0)}|^{\rm NLO}$ are easily obtained by taking time integral of Eqs. (5.158) and (5.159),

$$\begin{aligned} V_{\mathbf{L}^2}^{(2,0)}(r)|^{\rm NLO} &= \frac{g^2}{4}\int_0^\infty dt\, t^2 \langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} - \frac{g^2}{2}\int_0^\infty dt\, t^2\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO} \\ &= \mu_{\mathbf{L}^2,0}^{(2,0)} - \frac{\pi g^2\Lambda^2\overline{\Lambda}^2}{9\sigma^2 r} + \left(\frac{1}{3\pi}+\frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}, \end{aligned} \quad (5.161)$$

in which $\mu_{\mathbf{L}^2,0}^{(2,0)}$ is a renormalization parameter of mass dimension one. The other NLO contribution to the potential $V_{\mathbf{L}^2}$ is given by

$$V_{\mathbf{L}^2}^{(1,1)}(r)|^{\rm NLO} = \frac{g^2}{2}\int_0^\infty dt\, t^2\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} - g^2\int_0^\infty dt\, t^2\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO}, \quad (5.162)$$

[24] The time integral diverges at the lower boundary of its domain, $t \to 0$, which corresponds to the UV regime of the theory. This divergence is another manifestation of the fact that the EST is a UV-divergent EFT.

[25] The subscript of the parameter $\mu_i^{(1,0)}$ for $i \in \{0, 2\}$ denotes $1/r^i$ dependence of the term.

where the gauge field insertions are given in terms of the string fields by Eqs. (5.128) and (5.129). As these expressions diverge at NLO within the EST framework, we make use of Eq. (5.151) for the regularization, so that these correlators can analytically be written as

$$\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^2\Lambda^2\overline{\Lambda}^2}{3\sigma^2 r^4}\cosh^{-2}\left(\frac{\pi t}{2r}\right), \tag{5.163}$$

$$\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^2\overline{\Lambda''}^4}{2\sigma^2 r^4}\cosh^{-4}\left(\frac{\pi t}{2r}\right). \tag{5.164}$$

Thus, the analytic expression of this potential becomes

$$V_{\mathbf{L}^2}^{(1,1)}(r)|^{\rm NLO} = \frac{\pi g^2\Lambda^2\overline{\Lambda}^2}{9\sigma^2 r} + \frac{g^2\overline{\Lambda''}^4}{3\pi\sigma^2 r} - \frac{\pi g^2\overline{\Lambda''}^4}{18\sigma^2 r}. \tag{5.165}$$

As for the other set of momentum-dependent but spin-independent potentials like $V_{\mathbf{p}^2}$'s, the relevant gauge field insertion was already computed in Eq. (5.159)

$$\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^2\overline{\Lambda''}^4}{\sigma^2 r^4}\left[\cosh\left(\frac{\pi t}{r}\right)-1\right]^{-2}, \tag{5.166}$$

and plugging this into the expression for the time integral, Eq. (4.24), we obtain the analytic expression of the NLO contribution

$$\begin{aligned} V_{\mathbf{p}^2}^{(2,0)}(r)|^{\rm NLO} &= -\frac{g^2}{2}\int_0^\infty dt\, t^2 \langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO} \\ &= -\mu_{\mathbf{p}^2}^{(2,0)} + \left[\frac{1}{3\pi}+\frac{\pi}{9}\right]\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}, \end{aligned} \tag{5.167}$$

in which $\mu_{\mathbf{p}^2}^{(2,0)}$ is a renormalization parameter with mass dimension zero. Likewise, we compute the other relevant gauge field insertion as

$$\langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm NLO} = \frac{\pi^2\overline{\Lambda''}^4}{\sigma^2 r^4}\left[\cosh\left(\frac{\pi t}{r}\right)+1\right]^{-2}, \tag{5.168}$$

thereby deriving the other part of the momentum-dependent potential at NLO

$$\begin{aligned} V_{\mathbf{p}^2}^{(1,1)}(r)|^{\rm NLO} &= -g^2\int_0^\infty dt\, t^2 \langle\!\langle \boldsymbol{E}_1^3(-it)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} \\ &= \left[\frac{2}{3\pi}-\frac{\pi}{9}\right]\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}. \end{aligned} \tag{5.169}$$

Spin-orbit potentials

The first part of the spin-orbit contribution to the potential, evaluated at NLO is given by

$$V_{LS}^{(2,0)}(r)|^{\rm NLO} = \frac{ic_F^{(1)}g^2}{r}\int_0^\infty dt\,t\left[\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\rm NLO} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\rm NLO}\right]\,, \tag{5.170}$$

and the corresponding integrand on the right-hand side is mapped onto the string variables by Eq. (5.130). Its divergence is regularized by Eqs. (5.144) and (5.148), so that we obtain the following expression

$$\begin{aligned}\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\rm NLO} &= -\,\langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\rm NLO}\\ &= -\,\frac{i\pi^3(\Lambda^2\overline{\Lambda'}+2\Lambda'\overline{\Lambda}^2)}{6\sigma^2r^5}\sinh\left(\frac{\pi t}{r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)-1\right]^{-2}\,.\end{aligned} \tag{5.171}$$

Then, the analytic expression of the Eq. (5.170) is derived,

$$V_{LS}^{(2,0)}(r)|^{\rm NLO} = \frac{\mu_{LS,3}^{(2,0)}}{r^3} - \frac{c_F^{(1)}\pi g^2(\Lambda^2\overline{\Lambda'}+2\Lambda'\overline{\Lambda}^2)}{6\sigma^2r^4}\,, \tag{5.172}$$

where $\mu_{LS,3}^{(2,0)}$ is a renormalization parameter with mass dimension zero. The second part of the spin-orbit potential at NLO is given by

$$V_{L_2S_1}^{(1,1)}(r)|^{\rm NLO} = \frac{ic_F^{(1)}g^2}{r}\int_0^\infty dt\,t\left[\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_2^2(0)\rangle\!\rangle_c^{\rm NLO} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_2^1(0)\rangle\!\rangle_c^{\rm NLO}\right]\,. \tag{5.173}$$

The divergences, as the gauge field insertions are mapped onto the string variables, are regularized by Eqs. (5.144) and (5.148) so that the following expression is obtained:

$$\begin{aligned}&\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_2^2(0)\rangle\!\rangle_c^{\rm NLO} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_2^1(0)\rangle\!\rangle_c^{\rm NLO}\\ &= \frac{i\pi^3(\Lambda^2\overline{\Lambda'}+2\Lambda'\overline{\Lambda}^2)}{6\sigma^2r^5}\sinh\left(\frac{\pi t}{r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)+1\right]^{-2}\,.\end{aligned} \tag{5.174}$$

Thus, the NLO contribution of the second part of the spin-orbit potential is analytically derived to be

$$V_{L_2S_1}^{(1,1)}(r)|^{\rm NLO} = -\frac{c_F^{(1)}\pi g^2(\Lambda^2\overline{\Lambda'}+2\Lambda'\overline{\Lambda}^2)}{6\sigma^2r^4}\,. \tag{5.175}$$

Spin-spin interaction potentials

One of the spin-spin interaction parts of the potential at NLO is given by

$$V_{S^2}^{(1,1)}(r)|^{\rm NLO} = \frac{2c_F^{(1)}c_F^{(2)}g^2}{3}\int_0^\infty dt\left[\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c^{\rm NLO} + \langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_2^3(0)\rangle\!\rangle_c^{\rm NLO}\right]\,. \tag{5.176}$$

The first term inside the bracket on the right hand side of Eq. (5.176) is mapped onto the string variables by Eq. (5.133), which features two types of divergence, $\partial_z\partial_{z'}G$ and $\partial_t\partial_z\partial_{z'}G$. We regularize them by using Eqs. (5.144) and (5.148), such that the following expression is obtained,

$$\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^4\Lambda'\overline{\Lambda'}}{12\sigma^2 r^6}\cosh^{-4}\left(\frac{\pi t}{2r}\right)\left[\cosh\left(\frac{\pi t}{2r}\right)-2\right]. \tag{5.177}$$

On the other hand, the second term inside the bracket of Eq. (5.176) is of order $\sigma^{-3}r^{-6}$ in accordance with the power counting scheme, so this part is to be included in the NNLO contribution. Then it turns out that the time integral of the Eq. (5.177) is trivial

$$\int_0^\infty dt\cosh^{-4}\left(\frac{\pi t}{2r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)-2\right] = 0\,, \tag{5.178}$$

which implies that the NLO part of this potential vanishes: $V_{S^2}^{(1,1)}(r)|^{\rm NLO} = 0$. Due to the parametrization of our physical system, two of the spin-spin interaction parts of the potentials at NLO are related to each other by

$$V_{S_{12}}^{(1,1)}|^{\rm NLO} = -\frac{1}{8}V_{S^2}^{(1,1)}|^{\rm NLO}\,, \tag{5.179}$$

and this implies that $-\frac{1}{8}V_{S^2}^{(1,1)}|^{\rm NLO}$ vanishes as well. Therefore, we conclude

$$V_{S^2}^{(1,1)}(r)|^{\rm NLO} = 0, \quad \text{and} \quad V_{S_{12}}^{(1,1)}(r)|^{\rm NLO} = 0\,. \tag{5.180}$$

Central potentials

Lastly, V_r's at NLO are calculated. As for $V_r^{(2,0)}(r)|^{\rm NLO}$, one of the contributions is two-chromomagnetic field insertion

$$V_r^{(2,0)}(r)|^{\rm NLO} \ni -\frac{c_F^{(1)2}g^2}{4}\int_0^\infty dt\left[\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c^{\rm NLO} + \langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c^{\rm NLO}\right]. \tag{5.181}$$

It was shown in the previous paragraph that $\langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c^{\rm NLO}$ is of order $\sigma^{-3}r^{-6}$, which implies that this term is counted as a NNLO contribution within the EST power counting. The other term inside the bracket of Eq. (5.181) is mapped onto the string variables by Eq. (5.132), and its divergence is regularized by Eqs. (5.144) and (5.148), resulting in the following expression

$$\sum_{l=1}^{2}\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^4\Lambda'\overline{\Lambda'}}{12\sigma^2 r^6}\left[\cosh\left(\frac{\pi t}{r}\right)+2\right]\sinh^{-4}\left(\frac{\pi t}{2r}\right). \tag{5.182}$$

Thus, this contribution is analytically computed to be

$$V_r^{(2,0)}(r)|^{\rm NLO} \ni -\frac{c_F^{(1)2}g^2}{4}\int_0^\infty dt \sum_{l=1}^{2} \langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_1^l(0)\rangle\!\rangle_c^{\rm NLO} = -\frac{\mu_{r,2}^{(2,0)}}{r^2}\,, \tag{5.183}$$

in which $\mu_{r,2}^{(2,0)}$ is a renormalization parameter of mass dimension one.
The second contribution to the potential is straightforward as we use Eq. (5.167),

$$V_r^{(2,0)}(r)|^{\rm NLO} \ni \frac{1}{2}(\nabla_r^2 V_{p^2}^{(2,0)}|^{\rm NLO}) = \left(\frac{1}{3\pi}+\frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3}\,. \tag{5.184}$$

The third contribution comes from a three-chromoelectric field insertion,

$$V_r^{(2,0)}(r)|^{\rm NLO} \ni \frac{g^3}{2}\nabla_r^3\int_0^\infty dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}\,, \tag{5.185}$$

and its integrand is given by Eq. (5.134), which reads

$$\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}\\
&= -\frac{\pi^2\overline{\Lambda''}^2\Lambda^4}{4\sigma^2 r^4}\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left[\frac{\pi(t_1-t_2)}{2r}\right]+\frac{\pi^2\Lambda''^2\overline{\Lambda}^4}{4\sigma^2 r^4}\sinh^{-4}\left(\frac{\pi t_1}{2r}\right)\,.
\end{aligned}\tag{5.186}$$

Then, the analytic expression of this contribution is calculated as

$$\begin{aligned}
V_r^{(2,0)}(r)|^{\rm NLO} &\ni \frac{g^3}{2}\nabla_r^3\int_0^\infty dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}\\
&= -\frac{2g^3\overline{\Lambda''}^2\Lambda^4}{\sigma^2\pi^2 r}+\frac{2g^3\Lambda''^2\overline{\Lambda}^4}{3\sigma^2\pi^2 r}\,.
\end{aligned}\tag{5.187}$$

The last contribution to the potential comes from the four-chromoelectric field insertion, which is mapped onto the string variables as in Eq. (5.137). We evaluate this expression on the specified spatial coordinates ($z_1 = z_3 = z_4 = r/2$, $z_1 = z_4 = r/2$, or $z_1 = z_2 = z_4 = r/2$)

$$\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_1(-it_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}\\
&= -\frac{\pi^2\overline{\Lambda''}^2\Lambda''^2\Lambda^4}{4\sigma^2 r^4}\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left[\frac{\pi(t_1-t_2)}{2r}\right]+\frac{\pi^2\Lambda''^4\overline{\Lambda}^4}{4\sigma^2 r^4}\sinh^{-4}\left(\frac{\pi t_1}{2r}\right)\\
&\quad -\frac{\pi^2\Lambda^4\Lambda''^2\overline{\Lambda''}^2}{4\sigma^2 r^4}\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left(\frac{\pi t_2}{2r}\right)\,,
\end{aligned}\tag{5.188}$$

with the relevant integral being

$$-\frac{g^4}{2}\int_0^\infty dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 (t_2-t_3)^2 \langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_1(-it_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}$$

$$= \frac{ag^4\Lambda^4\Lambda''^2\overline{\Lambda''}^2 r}{4\pi^3\sigma^2} + \frac{\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{135\sigma^2} - \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2}\,. \tag{5.189}$$

The constant $a \simeq 7.08603$ comes from the time integral over the function, $\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left[\frac{\pi(t_1-t_2)}{2r}\right]$[26]. Therefore, this contribution to the potential is written altogether as

$$\begin{aligned} V_r^{(2,0)}|^{\rm NLO} =& \frac{ag^4\Lambda^4\Lambda''^2\overline{\Lambda''}^2 r}{4\pi^3\sigma^2} + \frac{\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{135\sigma^2} - \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2} - \frac{2g^3\overline{\Lambda''}^2\Lambda^4}{\sigma^2\pi^2 r} + \frac{2g^3\Lambda''^2\overline{\Lambda}^4}{3\sigma^2\pi^2 r} \\ &+ \left(\frac{1}{3\pi} + \frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3}, \quad \text{where} \quad a \simeq 7.08603. \end{aligned} \tag{5.190}$$

Note that the first three terms are of the same order as the leading order terms in Eq. (5.118), which means they contribute as the correction terms.

In a similar fashion, $V_r^{(1,1)}(r)|^{\rm NLO}$ is computed. The first contribution, using Eq. (5.169), is given by

$$V_r^{(1,1)}(r)|^{\rm NLO} \ni -\frac{1}{2}(\nabla_r^2 V_{p^2}^{(1,1)}|^{\rm NLO}) = \left(\frac{\pi}{9} - \frac{2}{3\pi}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3}, \tag{5.191}$$

while the ones with three-gauge field insertions, Eqs. (5.135) and (5.136), vanish when they are integrated over time,

$$\begin{aligned} 0 =& \frac{g^3}{2}\left(\nabla_r^3\int_0^\infty dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_2(-it_2)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\rm NLO}\right) \\ =& \frac{g^3}{2}\left(\nabla_r^3\int_0^\infty dt_1\int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle \boldsymbol{E}_2^3(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}\right)\,. \end{aligned} \tag{5.192}$$

Finally, the contribution with a four-chromoelectric field insertion, Eq. (5.138), is computed as

$$\begin{aligned} &- g^4\int_0^\infty dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 (t_2-t_3)^2 \langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_2(-it_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\rm NLO} \\ &= \frac{bg^4\overline{\Lambda''}^2\Lambda''^2\Lambda^4 r}{2\pi^3\sigma^2} - \frac{7\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{540\sigma^2} + \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2}\,, \end{aligned} \tag{5.193}$$

where $b \simeq 1.26521$ is obtained by numerically solving the time integral over the function $\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi t_2}{2r}\right]$. The potential altogether is then given by

$$\begin{aligned} V_r^{(1,1)}(r)|^{\rm NLO} =& \frac{bg^4\overline{\Lambda''}^2\Lambda''^2\Lambda^4 r}{2\pi^3\sigma^2} - \frac{7\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{540\sigma^2} + \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2} + \left(\frac{\pi}{9} - \frac{2}{3\pi}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3}\,, \\ &\text{where} \quad b \simeq 1.26521. \end{aligned} \tag{5.194}$$

Again, we observe that the first three terms are corrections to the leading order part given in Eq. (5.119).

[26]The integral has been obtained numerically.

5.4.6 Poincaré invariance

In summary, as the LO terms in the EST are already given[27] in Eqs. (5.91), (5.92), (5.93), (5.94), (5.95), (5.96), (5.97), (5.98), (5.99), (5.100), (5.101), and (5.102), we have derived the following list of potentials, which are computed up to NLO within the EST power counting scheme

$$V^{(0)}(r) = \sigma r + \mu - \frac{\pi}{12r}\,, \tag{5.195}$$

$$V^{(1,0)}(r) = \frac{g^2\Lambda^4}{2\pi\sigma}\ln(\sigma r^2) + \mu_1 + \mu_0^{(1,0)} - \frac{g^2(24\Lambda^2\overline{\Lambda}^2 + 13\overline{\Lambda''}^4)}{36\sigma^2 r^2} + \frac{\mu_2^{(1,0)}}{r^2}\,, \tag{5.196}$$

$$V^{(2,0)}_{\mathbf{L}^2}(r) = -\frac{g^2\Lambda^4}{6\sigma}r + \mu^{(2,0)}_{L^2,0} - \frac{\pi g^2\Lambda^2\overline{\Lambda}^2}{9\sigma^2 r} + \left(\frac{1}{3\pi} + \frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}\,, \tag{5.197}$$

$$V^{(1,1)}_{\mathbf{L}^2}(r) = \frac{g^2\Lambda^4}{6\sigma}r + \frac{\pi g^2\Lambda^2\overline{\Lambda}^2}{9\sigma^2 r} + \left(\frac{1}{3\pi} - \frac{\pi}{18}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}\,, \tag{5.198}$$

$$V^{(2,0)}_{\mathbf{p}^2}(r) = \left(\frac{1}{3\pi} + \frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r} - \mu^{(2,0)}_{p^2}\,, \tag{5.199}$$

$$V^{(1,1)}_{\mathbf{p}^2}(r) = \left(\frac{2}{3\pi} - \frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r}\,, \tag{5.200}$$

$$V^{(2,0)}_{LS}(r) = -\frac{\mu_2}{r} - \frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2} + \frac{\mu^{(2,0)}_{LS,3}}{r^3} - \frac{c_F^{(1)}\pi g^2(\Lambda^2\overline{\Lambda'} + 2\Lambda'\overline{\Lambda}^2)}{6\sigma^2 r^4}\,, \tag{5.201}$$

$$V^{(1,1)}_{L_2S_1}(r) = -\frac{c_F^{(1)}g^2\Lambda^2\Lambda'}{\sigma r^2} - \frac{c_F^{(1)}\pi g^2(\Lambda^2\overline{\Lambda'} + 2\Lambda'\overline{\Lambda}^2)}{6\sigma^2 r^4}\,, \tag{5.202}$$

$$V^{(1,1)}_{S^2}(r) = \frac{2\pi^3 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{45\sigma^2 r^5} - 4(d_{sv} + d_{vv}C_f)\delta^3(\mathbf{r})\,, \tag{5.203}$$

$$V^{(1,1)}_{S_{12}}(r) = \frac{\pi^3 c_F^{(1)}c_F^{(2)}g^2\Lambda'''^2}{90\sigma^2 r^5}\,, \tag{5.204}$$

$$\begin{aligned} V^{(2,0)}_{r}(r) = &-\frac{2\zeta_3 g^4\Lambda^8 r}{\pi^3\sigma^2} + \frac{a g^4\Lambda^4\Lambda''^2\overline{\Lambda''}^2 r}{4\pi^3\sigma^2} + \frac{\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{135\sigma^2} - \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2} + \mu_3 \\ &- \frac{2g^3\overline{\Lambda''}^2\Lambda^4}{\sigma^2\pi^2 r} + \frac{2g^3\Lambda''^2\overline{\Lambda}^4}{3\sigma^2\pi^2 r} + \frac{\mu_4}{r^2} + \left(\frac{1}{3\pi} + \frac{\pi}{9}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3} + \frac{\mu_5}{r^4} + \frac{\pi^3 c_F^{(1)2}g^2\Lambda'''^2}{60\sigma^2 r^5} \\ &+ \frac{\pi C_f\alpha_s c_D^{(1)'}}{2}\delta^{(3)}(\mathbf{r}) - d_3^{(1)'}f_{abc}\int d^3\mathbf{x}\lim_{T\to\infty} g\langle\!\langle F^a_{\mu\nu}(x)F^b_{\mu\alpha}(x)F^c_{\nu\alpha}(x)\rangle\!\rangle\,, \end{aligned} \tag{5.205}$$

$$\begin{aligned} V^{(1,1)}_{r}(r) = &-\frac{\zeta_3 g^4\Lambda^8 r}{2\pi^3\sigma^2} + \frac{b g^4\overline{\Lambda''}^2\Lambda''^2\Lambda^4 r}{2\pi^3\sigma^2} - \frac{7\pi g^4\Lambda''^4\overline{\Lambda}^4 r}{540\sigma^2} + \frac{g^4\Lambda''^4\overline{\Lambda}^4 r}{9\pi\sigma^2} \\ &+ \left(\frac{\pi}{9} - \frac{2}{3\pi}\right)\frac{g^2\overline{\Lambda''}^4}{\sigma^2 r^3} + (d_{ss} + d_{vs}C_f)\delta^{(3)}(\mathbf{r})\,, \end{aligned} \tag{5.206}$$

[27] These are the expressions before imposing the Poincaré invariance

where $a \simeq 7.08603$ and $b \simeq 1.26521$. Note that the static potential, Eq. (5.195), now contains the Lüscher term $-\pi/(12r)$ in four-dimensional spacetime [52, 53, 55].

As we utilize the mapping from QCD to the EST, Eqs. (5.40) and (5.122), there arise a number of undetermined (and dimensionful) parameters, Λ's, and $\overline{\Lambda}$'s, as well as renormalization parameters μ's. It was briefly mentioned in the previous sections that the number of these parameters can be reduced when the Poincaré invariance in QCD is taken into account in the low-energy regime [22]. It is important to note here that unlike the pQCD action, its low-energy effective descriptions do not show the explicit symmetry under the Poincaré transformations, especially under the boosts. However, as it was argued in [22, 77, 103, 104], the NRQCD and pNRQCD Lagrangians are invariant under the boost transformations if a certain set of constraints on the Wilson coefficients of these EFTs are satisfied. In other words, as the underlying theory preserves the Poincaré symmetry, its low-energy EFTs have to be invariant under the same symmetry group as well. The equations we will be using in the next paragraphs provide the constraints as a result of the Poincaré invariance in pNRQCD.

The first equation we apply is the *Gromes relation* [92], which relates the static potential to the spin-orbit ones

$$\frac{1}{2r}\frac{dV^{(0)}}{dr} + V_{LS}^{(2,0)} - V_{L_2S_1}^{(1,1)} = 0\,. \tag{5.207}$$

As we insert Eqs. (5.195), (5.201), and (5.202) into this equation, two constraints arise from it:

$$\mu_2 = \frac{\sigma}{2}, \quad \text{and} \quad \mu_{LS,3}^{(2,0)} = -\frac{\pi}{24}\,. \tag{5.208}$$

Comparing to the constraining equation for the potentials at LO, Eq. (5.104), one additional constraint on a renormalization parameter appears, which is due to the inclusion of NLO terms in the EST. Thus, the relevant spin-orbit potential is simplified

$$V_{LS}^{(2,0)}(r) = -\frac{\sigma}{2r} - \frac{c_F^{(1)} g^2\Lambda^2\Lambda'}{\sigma r^2} - \frac{\pi}{24r^3} - \frac{c_F^{(1)}\pi g^2(\Lambda^2\overline{\Lambda'} + 2\Lambda'\overline{\Lambda}^2)}{6\sigma^2 r^4}\,. \tag{5.209}$$

We observe that the third and fourth terms on the right-hand side are suppressed in the long range as compared to the LO result, Eq. (5.114).

The second equation from the Poincaré invariance gives a relation between the static potential and the momentum-dependent (but spin-independent) potentials [93]

$$\frac{r}{2}\frac{dV^{(0)}}{dr} + 2V_{\mathbf{L}^2}^{(2,0)} - V_{\mathbf{L}^2}^{(1,1)} = 0\,. \tag{5.210}$$

By plugging in Eqs. (5.195), (5.197), and (5.198), the following constraints are derived

$$\mu_{\mathbf{L}^2,0}^{(2,0)} = 0\,, \tag{5.211}$$

$$g\Lambda^2 = \sigma\,, \tag{5.212}$$

$$g\overline{\Lambda}^2 = \frac{\sigma}{8} + \left(\frac{1}{\pi^2} + \frac{5}{6}\right) \frac{g^2\overline{\Lambda''}^4}{\sigma} \,. \tag{5.213}$$

There was only Eq. (5.212) in the LO calculation, while here we have two additional constraints, Eqs. (5.211) and (5.213). In fact, one can use another relation [50],

$$-\boldsymbol{\nabla}_1 \left(V^{(0)}|^{\rm NLO}\right) = \langle\!\langle g\boldsymbol{E}_1\rangle\!\rangle^{\rm NLO} \,, \tag{5.214}$$

into which we insert Eqs. (5.122) and (5.195), that gives the following relation between the string tension and the parameters from the mapping

$$-\left(\sigma + \frac{\pi}{12r^2}\right) = g\Lambda''^2 + \frac{\pi g\overline{\Lambda''}^2}{6\sigma r^2} \,. \tag{5.215}$$

As we compare both sides order by order of in the $1/r$ expansion, this equation gives constraints on the following dimensionful parameters:

$$g\Lambda''^2 = -\sigma \,, \tag{5.216}$$

$$g\overline{\Lambda''}^2 = -\frac{\sigma}{2} \,. \tag{5.217}$$

Eq. (5.217) is a new result as compared to Eq. (5.107). Then, by inserting Eq. (5.217) into Eq. (5.213), we obtain

$$g\overline{\Lambda}^2 = \left(\frac{1}{3} + \frac{1}{4\pi^2}\right)\sigma \,. \tag{5.218}$$

Therefore, by inserting Eqs. (5.211), (5.212), (5.213), (5.216), and (5.217) into Eqs. (5.197) and (5.198), the potentials are simplified in the following way:

$$V_{\mathbf{L}^2}^{(2,0)} = -\frac{\sigma r}{6} + \left(\frac{11}{36\pi} + \frac{2\pi}{27}\right)\frac{1}{r} \,, \tag{5.219}$$

$$V_{\mathbf{L}^2}^{(1,1)} = \frac{\sigma r}{6} + \left(\frac{1}{9\pi} + \frac{5\pi}{216}\right)\frac{1}{r} \,, \tag{5.220}$$

which only depend on the string tension σ. Clearly the second terms on the right-hand sides are the suppression terms by the EST calculation at NLO.

Lastly, we use another constraining equation from the Poincaré invariance [22, 93]

$$-4V_{\mathbf{p}^2}^{(2,0)} + 2V_{\mathbf{p}^2}^{(1,1)} - V^{(0)} + r\frac{dV^{(0)}}{dr} = 0 \,, \tag{5.221}$$

into which we insert Eqs. (5.199) and (5.200) as well as make use of the given constraint on $\overline{\Lambda''}$, Eq. (5.217). This shows that two of the renormalization parameters are related to each other:

$$\mu_{\mathbf{p}^2}^{(2,0)} = \frac{\mu}{4} \,, \tag{5.222}$$

so that these momentum-dependent potentials are simplified as well:

$$V_{\mathbf{p}^2}^{(2,0)} = \left(\frac{1}{12\pi} + \frac{\pi}{36}\right)\frac{1}{r} - \frac{\mu}{4}\,, \tag{5.223}$$

$$V_{\mathbf{p}^2}^{(1,1)} = \left(\frac{1}{6\pi} - \frac{\pi}{36}\right)\frac{1}{r}\,. \tag{5.224}$$

While these two potentials were trivial from the EST calculation at LO, Eqs. (5.95) and (5.96), here we have not only non-trivial but also precise analytic prediction of the potentials. We will discuss about the implications and impact of these results in the next section.

Applying all these constraints arising from the Poincaré invariance in QCD, we have the final list of potentials, which include terms up to NLO in the EST power counting

$$V^{(0)}(r) =\sigma r + \mu - \frac{\pi}{12r}\,, \tag{5.225}$$

$$V^{(1,0)}(r) =\frac{\sigma}{2\pi}\ln(\sigma r^2) + \mu_1' - \left(\frac{5}{16} + \frac{1}{6\pi^2}\right)\frac{1}{r^2} + \frac{\mu_2^{(1,0)}}{r^2}\,, \tag{5.226}$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = -\frac{\sigma r}{6} + \left(\frac{11}{36\pi} + \frac{2\pi}{27}\right)\frac{1}{r}\,, \tag{5.227}$$

$$V_{\mathbf{L}^2}^{(1,1)}(r) =\frac{\sigma r}{6} + \left(\frac{1}{9\pi} + \frac{5\pi}{216}\right)\frac{1}{r}\,, \tag{5.228}$$

$$V_{\mathbf{p}^2}^{(2,0)}(r) = \left(\frac{1}{12\pi} + \frac{\pi}{36}\right)\frac{1}{r} - \frac{\mu}{4}\,, \tag{5.229}$$

$$V_{\mathbf{p}^2}^{(1,1)}(r) = \left(\frac{1}{6\pi} - \frac{\pi}{36}\right)\frac{1}{r}\,, \tag{5.230}$$

$$V_{LS}^{(2,0)}(r) = -\frac{\sigma}{2r} - \frac{c_F^{(1)} g\Lambda'}{r^2} - \frac{\pi}{24r^3} - \frac{c_F^{(1)}\pi g\overline{\Lambda'}}{6\sigma r^4} - \left(\frac{\pi}{9} + \frac{1}{12\pi}\right)\frac{c_F^{(1)} g\Lambda'}{\sigma r^4}\,, \tag{5.231}$$

$$V_{L_2S_1}^{(1,1)}(r) = -\frac{c_F^{(1)} g\Lambda'}{r^2} - \frac{c_F^{(1)}\pi g\overline{\Lambda'}}{6\sigma r^4} - \left(\frac{\pi}{9} + \frac{1}{12\pi}\right)\frac{c_F^{(1)} g\Lambda'}{\sigma r^4}\,, \tag{5.232}$$

$$V_{S^2}^{(1,1)}(r) =\frac{2\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{45\sigma^2 r^5} - 4(d_{sv} + d_{vv}C_f)\delta^3(\mathbf{r})\,, \tag{5.233}$$

$$V_{S_{12}}^{(1,1)}(r) =\frac{\pi^3 c_F^{(1)} c_F^{(2)} g^2 \Lambda'''^2}{90\sigma^2 r^5}\,, \tag{5.234}$$

$$\begin{aligned} V_r^{(2,0)}(r) = &\left\{-\frac{2\zeta_3}{\pi^3} + \frac{a}{8\pi^3} + \left(\frac{1}{3} + \frac{1}{4\pi^2}\right)^2\left(\frac{\pi}{135} - \frac{1}{9\pi}\right)\right\}\sigma^2 r + \mu_3 \\ &+ \left(\frac{25}{9} - \frac{1}{3\pi^2} - \frac{1}{8\pi^4}\right)\frac{1}{3\pi^2}\cdot\frac{\sigma}{r} + \frac{\mu_4 - \mu_{r,2}^{(2,0)}}{r^2} \\ &+ \left(\frac{1}{12\pi} + \frac{\pi}{36}\right)\frac{1}{r^3} + \frac{\mu_5}{r^4} + \frac{\pi^3 c_F^{(1)2} g^2 \Lambda'''^2}{60\sigma^2 r^5} \end{aligned}$$

$$+\frac{\pi C_f \alpha_s c_D^{(1)'}}{2}\delta^{(3)}(\mathbf{r}) - d_3^{(1)'} f_{abc}\int d^3\mathbf{x} \lim_{T\to\infty} g\langle\!\langle F^a_{\mu\nu}(x)F^b_{\mu\alpha}(x)F^c_{\nu\alpha}(x)\rangle\!\rangle\,, \tag{5.235}$$

$$V_r^{(1,1)}(r) = \left\{-\frac{\zeta_3}{2\pi^3} + \frac{b}{4\pi^3} + \left(\frac{1}{9\pi} - \frac{7\pi}{540}\right)\left(\frac{1}{3} + \frac{1}{4\pi^2}\right)^2\right\}\sigma^2 r + \left(\frac{\pi}{36} - \frac{1}{6\pi}\right)\frac{1}{r^3}$$
$$+ (d_{ss} + d_{vs}C_f)\,\delta^{(3)}(\mathbf{r})\,, \tag{5.236}$$

where $\mu_1' \equiv \mu_1 + \mu_1^{(1,0)}$, $a \simeq 7.08603$, and $b \simeq 1.26521$. Note that only three of the dimensionful parameters from the string mapping, $\Lambda', \overline{\Lambda'}$, and Λ''', remain unconstrained here, while all the other parameters are given in terms of the string tension σ. The string tension is the fundamental parameter of the EST, which is to be determined by comparing to lattice simulations.

All in all, the singlet potential in the center of mass frame can be written as

$$\begin{aligned}
V(r) =& V^{(0)}(r) + \frac{2}{M}V^{(1,0)}(r) + \frac{1}{M^2}\left\{\left[2\frac{V^{(2,0)}_{\mathbf{L}^2}(r)}{r^2} + \frac{V^{(1,1)}_{\mathbf{L}^2}(r)}{r^2}\right]\mathbf{L}^2\right.\\
&+ \left[V^{(2,0)}_{LS}(r) + V^{(1,1)}_{L_2S_1}(r)\right]\mathbf{L}\cdot\mathbf{S} + V^{(1,1)}_{S^2}(r)\left(\frac{\mathbf{S}^2}{2} - \frac{3}{4}\right) + V^{(1,1)}_{S_{12}}\mathbf{S}_{12}(\hat{r})\\
&\left.+ \left[2V^{(2,0)}_{\mathbf{p}^2}(r) + V^{(1,1)}_{\mathbf{p}^2}(r)\right]\mathbf{p}^2 + 2V_r^{(2,0)}(r) + V_r^{(1,1)}(r)\right\}\\
\approx& \sigma r + \mu - \frac{\pi}{12r} + \frac{2}{M}\left[\frac{\sigma}{2\pi}\ln\left(\sigma r^2\right) + \mu_1'\right]\\
&+ \frac{1}{M^2}\left\{-\frac{\sigma}{6r}\mathbf{L}^2 - \frac{\sigma}{2r}\mathbf{L}\cdot\mathbf{S} + \left[\left(\frac{1}{3\pi} + \frac{\pi}{36}\right)\frac{1}{r} - \frac{\mu}{2}\right]\mathbf{p}^2\right.\\
&+ \left[\frac{(a+b) - 18\zeta_3}{4\pi^3} + \left(\frac{\pi}{540} - \frac{1}{9\pi}\right)\left(\frac{1}{3} + \frac{1}{4\pi^2}\right)^2\right]\sigma^2 r\\
&\left.+\mu_3 + \left(\frac{50}{27\pi^2} - \frac{2}{9\pi^4} - \frac{1}{12\pi^6}\right)\frac{\sigma}{r}\right\}\\
\approx& \sigma r + \mu - \frac{\pi}{12r} + \frac{2}{M}\left[\frac{\sigma}{2\pi}\ln\left(\sigma r^2\right) + \mu_1'\right]\\
&+ \frac{1}{M^2}\left\{-\frac{\sigma}{6r}\mathbf{L}^2 - \frac{\sigma}{2r}\mathbf{L}\cdot\mathbf{S} + \left[\frac{0.19}{r} - \frac{\mu}{2}\right]\mathbf{p}^2 - 0.11\,\sigma^2 r + \mu_3 + \frac{0.19\,\sigma}{r}\right\}
\end{aligned} \tag{5.237}$$

in which μ is the parameter with mass dimension one (from $V^{(0)}$), μ_1' is the renormalization parameter with mass dimension two (from $V^{(1,0)}$), and μ_3 is the renormalization parameter with mass dimension three (from $V_r^{(2,0)}$). Note also that we have truncated the terms in Eq. (5.237) after the linear order in $1/r$. Comparing this expression to the previous result of the LO calculation, Eq. (5.120), we observe that some additional terms appear. The significant result is at order $1/M^2$: as it was predicted in [57], there is a modification to the coefficient of the linear term in r.

In the next section, we study the impact of these NLO terms from the EST calculation by comparing it to lattice (simulation) data.

5.5 Comparison to lattice data

Static potential

Let us start with a comparison between the analytic prediction of the static potential and its corresponding lattice data. The static potential in the EST carries a suppression term at $\mathcal{O}(1/r)$, which is the so-called *Lüscher term*, such that the potential is written as

$$V^{(0)}(r) = \sigma r + \mu - \frac{\pi}{12r} \,. \tag{5.238}$$

In general, the suppression term in $1/r$ is given by $-\pi/[d(d-1)r]$, in which d stands for the spacetime dimensions [52, 53, 55]. As we compare the linear part of the potential to the lattice data at larger distance [105], a numerical value of the string tension can be extracted:

$$\sigma_{\text{sim.}} = (1.38 \pm 0.04) r_0^{-2} \,, \tag{5.239}$$

where $r_0 = 0.5$ fm is the Sommer scale [106]. Since this value is consistent with the widely accepted value by Necco and Sommer [91],

$$\sigma_{\text{NS}} = 1.3882\, r_0^2 \,, \tag{5.240}$$

from now on we adopt this numerical value σ_{NS} for the rest of our analysis. A comparison between the LQCD data and the static potential is given in FIG. 5.1, in which the

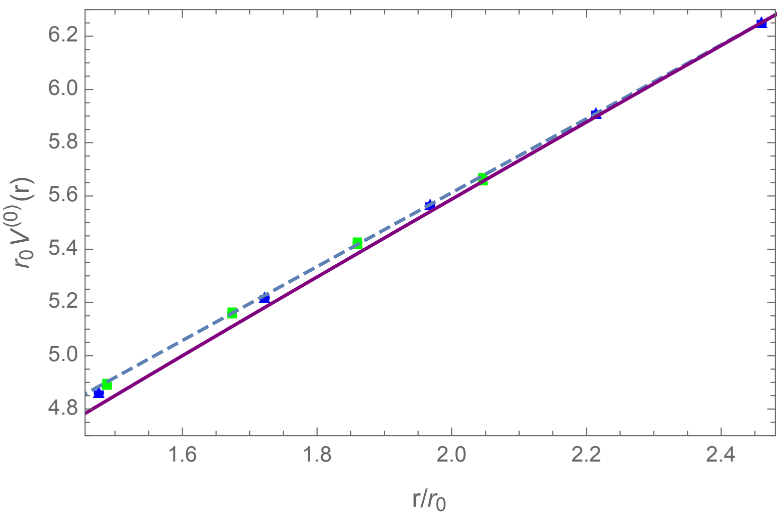


Figure 5.1: The static potential, normalized by the Sommer scale r_0, calculated at LO (dashed line) and NLO (bold line) in the EST power counting versus lattice data at $\beta = 5.85$ (blue points) and $\beta = 6.00$ [105] (green points). $\sigma_{\text{NS}} = 1.3882 r_0^2$ [55] was used as the string tension for the analytic expressions of the potentials.

blue points represent LQCD data at $\beta = 5.85$, while the green points are the data at $\beta = 6.00$. The dashed line is the string prediction of the linear potential and the bold line is the static potential including the Lüscher term. We have made the comparison at the distance range from $r/r_0 \sim 1.5$ because the validity of the EST[28] is supposed to be below the hadronic scale Λ_{QCD}. These two plots are adjusted in such a way that they coincide with the rightmost lattice point, which is given at $r/r_0 \sim 2.4$, because the EST is more accurate at a longer-distance regime. Also note that the interquark distance r has been normalized by the Sommer scale r_0. The data and plots agree in the longer-distance regime, but at the shorter distance range like $r/r_0 \sim 1.5$ it shows some discrepancies. FIG. 5.2 illustrates the discrepancies more clearly. We have subtracted

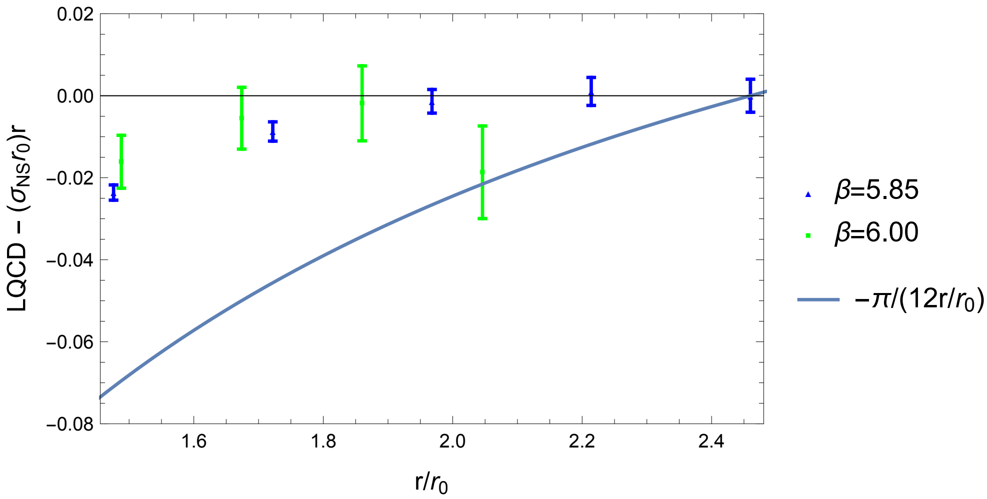


Figure 5.2: A comparison between LQCD data at $\beta = 5.85$ and $\beta = 6.00$ [105], which is subtracted by $(\sigma_{\text{NS}} r_0)r$ at the corresponding distances, and the Lüscher term of the static potential $-\pi/(12r/r_0)$ (also normalized by the Sommer scale r_0).

the linear potential $(\sigma_{\text{NS}} r_0)r$ from the lattice data points at the corresponding distances, and compared them to the NLO term of the static potential $-\pi r_0/(12r)$, which is the Lüscher term normalized by the Sommer scale. Although the data points show a general tendency of decrease in values at shorter distance, the discrepancy between data and the Lüscher term is roughly about 0.05 at the distance $r/r_0 \sim 1.5$. Since the LO plot (the horizontal line at 0) is closer to the LQCD data than the Lüscher term, it seems that inclusion of the NLO term worsens the comparison due to its sharp decrease at the shorter distance range. However, while the LO part does not show the general tendency of decrease in values at shorter distances, the NLO part resembles the general tendency of decrease. We estimate that this discrepancy at NLO can be decreased if higher order terms are included to the potential.

[28] Note that the inverse of the Sommer scale is comparable to the hadronic scale $\Lambda_{\text{QCD}} \sim 200$ MeV.

Potential at $\mathcal{O}(1/M)$

Moving onto the first order correction to the static potential, its analytic expression is given by Eq. (5.226):

$$V^{(1,0)}(r) = \frac{\sigma}{2\pi}\ln(\sigma r^2) + \mu_1' - \left(\frac{5}{16} + \frac{1}{6\pi^2} - \mu_2^{(1,0)}\right)\frac{1}{r^2}\,, \quad \text{where} \quad \mu_1' \equiv \mu_1 + \mu_1^{(1,0)}\,. \tag{5.241}$$

A comparison between the EST prediction and the LQCD data [47, 48] is illustrated in FIG. 5.3, in which $\sigma_{\rm NS}$[29] was used for the string tension σ. We have fitted the parameters

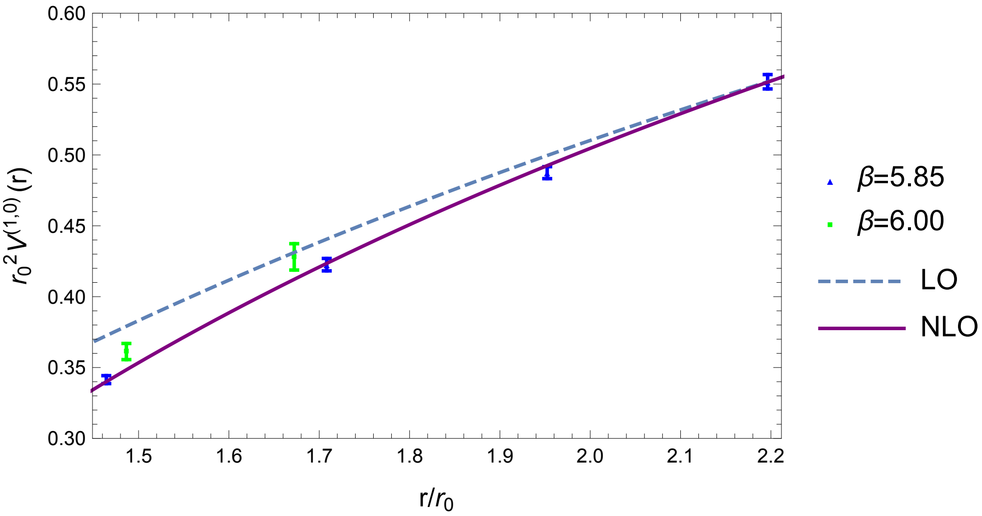


Figure 5.3: A comparison between the $\mathcal{O}(1/M)$ potential calculated by the EST at LO (dashed line) and NLO (bold line) and the lattice data [47, 48] at $\beta = 5.85$ and $\beta = 6.00$.

μ_1 and μ_1' for both the LO (dashed line) and the NLO (bold line) plots so that they coincide with the lattice point at the rightmost region, which is at $r/r_0 \sim 2.2$. Also, the renormalization parameter $\mu_2^{(1,0)}$ in $V^{(1,0)}(r)$ is determined by fitting the NLO expression of Eq. (5.241) to the lattice data points at $\beta = 5.85$ (where the most data points are available), yielding

$$\mu_2^{(1,0)} = 0.1715 \pm 0.004\,. \tag{5.242}$$

The analytic result which includes the NLO terms, Eq. (5.241), gives a better comparison to the lattice data, especially at the shorter-distance range $r/r_0 \sim 1.5$.

Potentials at $\mathcal{O}(1/M^2)$

As for the $\mathcal{O}(1/M^2)$ corrections to the static potential, the contributions that carry a minimal number of free parameters are the momentum-dependent but spin-independent

[29] Eq. (5.240)

potentials like $V_{\mathbf{p}^2}$'s and $V_{\mathbf{L}^2}$'s. Thus, comparing them to the available lattice data would be more significant because this can examine the consistency of the effective framework of the string theory in the first place. The analytic expressions of these momentum-dependent potentials are given in Eqs. (5.229), (5.230), (5.227), and (5.228):

$$V_{\mathbf{p}^2}^{(2,0)}(r) = \left(\frac{1}{12\pi} + \frac{\pi}{36}\right)\frac{1}{r} - \frac{\mu}{4}\,, \quad \text{and} \quad V_{\mathbf{p}^2}^{(1,1)}(r) = \left(\frac{1}{6\pi} - \frac{\pi}{36}\right)\frac{1}{r}\,, \tag{5.243}$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = -\frac{\sigma r}{6} + \left(\frac{11}{36\pi} + \frac{2\pi}{27}\right)\frac{1}{r}\,, \quad \text{and} \quad V_{\mathbf{L}^2}^{(1,1)}(r) = \frac{\sigma r}{6} + \left(\frac{1}{9\pi} + \frac{5\pi}{216}\right)\frac{1}{r}\,, \tag{5.244}$$

in which the free parameters are only σ and μ. As it was discussed in the previous section, this is due to the Poincaré invariance in QCD. In order to make the comparison to the LQCD data [47, 48], it is necessary to express these potentials in terms of the velocity-dependent potentials[30] like V_b, V_c, V_d, and V_e

$$V_{\mathbf{p}^2}^{(1,1)}(r) = -V_b(r) + \frac{2}{3}V_c(r)\,, \quad \text{and} \quad V_{\mathbf{L}^2}^{(1,1)}(r) = -V_c(r)\,, \tag{5.245}$$

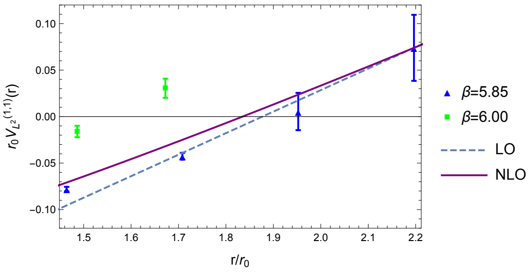
$$V_{\mathbf{p}^2}^{(2,0)}(r) = V_d(r) - \frac{2}{3}V_e(r)\,, \quad \text{and} \quad V_{\mathbf{L}^2}^{(2,0)}(r) = V_e(r)\,, \tag{5.246}$$

as the available simulation data correspond to these expressions. Using σ_{NS}, Eq. (5.240), and normalizing the distance r with respect to the Sommer scale r_0, our comparison to the LQCD data [47, 48] is illustrated in FIG. 5.4. The dashed line of each plot represents the string theory prediction of the momentum-dependent (but spin-independent) potentials including terms only up to leading order: the LO plots of FIG. 5.4a and 5.4b contain the linear parts $\propto (\sigma_{\text{NS}} r_0)r$ (plus some constants), whereas the LO plots of FIG. 5.4c and 5.4d only feature constant functions[31]. These plots are adjusted so that they coincide with the rightmost lattice data points.

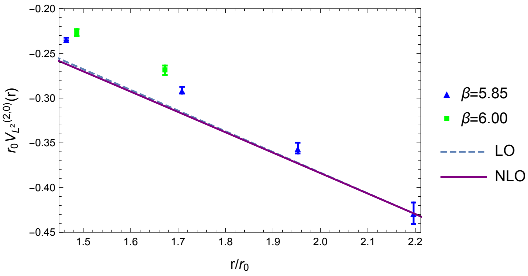
On the other hand, the bold lines denote the EST predictions of the potentials that contain terms up to NLO in the EST power counting scheme, and the constant contributions are adjusted in a similar fashion. The NLO plots of FIG. 5.4a and 5.4d show a sizable improvement with respect to the LO although there are some deviations remaining. FIG. 5.4b does not improve the comparison at shorter distance range $r/r_0 \sim 1.5$, and the improvement in FIG 5.4c is only marginal. However, the inclusion of NLO contributions show slight curvy patterns which are in accordance with the available lattice data points. We estimate that this can further be improved if next-to-next-to-leading order (NNLO) terms within the EST calculation are included.

Finally, it is important to note that the deviation between the lattice data and the Lüscher term of the static potential, FIG. 5.2, influences the deviations in these momentum-dependent potentials due to the constraining equations arising from the Poincaré invariance of QCD. This is the reason why the magnitude of deviations in FIG. 5.4a, 5.4b, 5.4c, and 5.4d are within the range of 0.05. One could reduce this deviation by performing a higher order calculation in the EST framework, and this will be discussed in the next chapter.

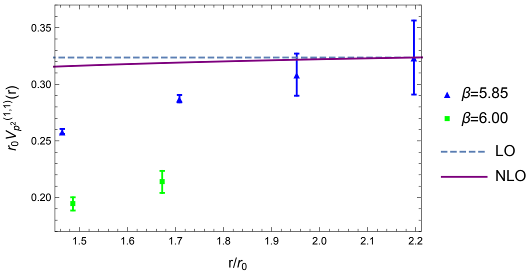
[30]See Sec. 2 in [47] for the definitions.
[31]See Eqs. (5.95) and (5.96).



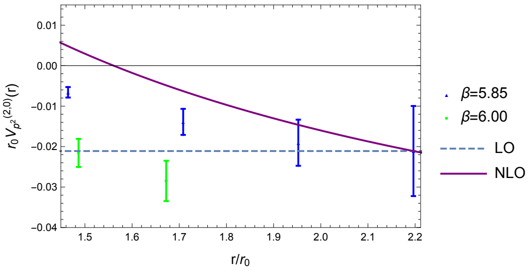
(a) $r_0 V^{(1,1)}_{\mathbf{L}^2}(r)$ versus lattice data



(b) $r_0 V^{(2,0)}_{\mathbf{L}^2}(r)$ versus lattice data



(c) $r_0 V^{(1,1)}_{\mathbf{p}^2}(r)$ versus lattice data



(d) $r_0 V^{(2,0)}_{\mathbf{p}^2}(r)$ versus lattice data

Figure 5.4: A comparison between the momentum-dependent (but spin-independent) potentials, which are calculated by the EST, and the lattice data [47, 48] at $\beta = 5.85$ and $\beta = 6.00$. The dashed lines are the potentials at LO (normalized by the Sommer scale r_0), Eq. (5.111), (5.110), (5.113), and (5.112) respectively, and bold lines are the results of the NLO calculation (also normalized by the Sommer scale).

5.5.1 Summary and discussions

In this chapter, we have studied the effective framework of a long string as a tool for calculating the potential terms between a static heavy quark-antiquark pair in the non-perturbative regime. There has already been substantial achievements in the understanding of the heavy quark potentials within the EST [56, 57], but only the leading order contributions were taken into account for the comparison to the available lattice data [45–48]. This causes a sizable discrepancy between the simulations and the analytic prediction [58]. In other words, the precision of the string theory calculation up to leading order shows some limitation at the intermediate distance range $r/r_0 \sim 1.5$. In order to improve the comparison to the corresponding LQCD data, we have computed NLO terms of the potentials by using the power counting scheme within the EST. There are two possible NLO contributions in the EST: (i) by including the NLO terms of the effective string action [55, 84, 95], or (ii) by including NLO terms to the QCD-to-EST mapping. It turns out that the first possibility results in the NNLO instead of the NLO contribution to the potentials. Thus, we have investigated the QCD-to-EST mapping up to NLO, and this yields the desired NLO terms to the potential as well as the terms of the same order as the ones from the leading order calculation of the EST. This is due to the fact that previous derivations of the LO terms within the EST are not fully inclusive by themselves, which was already pointed out in [57].

There arise some divergences as we derive the NLO terms of the potentials, which are similar to the self-energy terms of QED or QCD. The appearance of these divergences is natural because the EST is a UV-divergent effective theory. In other words, the theory is to be valid only at the energy scale much below the hadronic scale $\Lambda_{\text{QCD}} \sim 200$ MeV. We have addressed these issues by utilizing schemes like *zeta-function regularization* as well as *dimensional regularization*, which are in fact equivalent to each other [98, 99], as the zeta-function regularization is the discrete version of dimensional regularization.

After deriving the analytic expressions of potential terms up to NLO in the EST, we have utilized the constraining equations between the heavy quark potential terms (which arise from the Poincaré invariance in the low-energy EFTs of QCD [22, 92, 93]), so that the number of parameters arising in the QCD-to-EST mapping as well as some or all renormalization parameters originating from the time integral are reduced. The significant result of this calculation are the spin-independent but momentum-dependent potentials like V_{p^2}'s and V_{L^2}'s. They carry a minimum number of free parameters, which are the string tension and the renormalization parameters[32]. The string tension is a fundamental parameter of the EST, and it is to be determined by a comparison to LQCD data; also the renormalization parameters are to be fitted from the analytic expressions of these potentials to the corresponding lattice data.

In order to examine the consistency of our analytic results, we have compared our analytic results to LQCD data [45–48, 105] in a systematic way. First of all, the numerical value of the string tension was extracted by fitting the static potential to the most up-to-date data [105]. It turns out that the extracted value is consistent with the widely accepted one from the literature [91], so the literature value, $\sigma_{\text{NS}} = 1.3882\, r_0^2$, is used throughout the analysis. The $\mathcal{O}(1/M)$ correction to the potential is then fitted to the lattice data, such that the free parameter (a renormalization parameter) can be determined. Finally, we have compared the $\mathcal{O}(1/M^2)$ correction to the potential, especially the momentum-dependent (but spin-independent) potentials, to the data. This comparison shows some discrepancies. This originates from the discrepancy between the EST result of the static potential which includes the Lüscher term and the corresponding LQCD data. As the static potential is related to the momentum-dependent potentials due to Poincaré invariance [33], the discrepancy occurring for the static potential influences the comparison for the momentum-dependent potentials as well. Indeed the magnitude of the discrepancies for these potentials are comparable to the one from the static potential.

In addition, note that the LQCD data itself observes the Poincaré invariance at the distance range we are investigating[34]. This implies that the comparison between the analytic results from the EST and the LQCD data is based on the Poincaré invariance. Therefore, we conclude that our approach to the inclusion of NLO terms to the heavy

[32] They are constant terms of the potential.
[33] See Eq. (5.210) and (5.221).
[34] See Figure 5 in [47] for the detailed discussion.

quark potentials within the EST framework is consistent.

Chapter 6

Conclusion and outlook

6.1 Conclusion

In this thesis, we have studied symmetries of low-energy effective field theories, especially the ones involving heavy quarks and heavy antiquarks, whose mass scale is much greater than the hadronic scale of QCD, $\Lambda_{\rm QCD} \sim 200$ MeV. While a relativistic quantum field theory of color interactions (QCD) manifestly preserves Poincaré invariance, its low-energy counterpart, which is described by an EFT, does not exhibit the same invariance in a manifest fashion. As the EFT is the low-energy limit of its underlying theory, however, the EFT has also to observe the manifest Poincaré invariance, and we have studied this aspect of symmetry in great detail.

In order to elucidate this notion of spacetime symmetry, we have investigated the Poincaré transformations in non-relativistic EFTs of QCD, especially NRQCD [5, 6] and pNRQCD [8, 80]. In NRQCD, as the heavy quark mass is much greater than any other scales (including the relative momentum between the heavy quark and the antiquark), the EFT is non-relativistically expanded after integrating out the heavy mass scale. In the case of the bound states of a heavy quark and a heavy antiquark, pNRQCD is a suitable EFT, which is obtained by integrating out the relative momentum between the heavy quark and heavy antiquark. In the weakly-coupled case, this EFT is derived by taking a multipole expansion in the relative distance between the quark and antiquark since the momentum scales as an inverse of the distance. In both cases, we observe that the boost transformation of the fields is realized in a non-linear fashion unlike in the high-energy case.

Recently, there has been a suggestion [24] for deriving boost generator by using the Wigner's induced representation [23], but the application to the case of interacting low-energy EFTs like NRQED and NRQCD shows some ambiguity in including some gauge field dependent terms in the boost generator; in addition, there is an issue about fixing several coefficients of the terms on the boost generators.

In order to avoid such ambiguity, instead of referring to the induced representation, we have taken the full EFT approach in order to derive suitable expressions of boost generators, especially for NRQCD and pNRQCD. To be consistent with the basic principles of EFT (symmetries of the system and power counting) we have included all possible terms to the generator of boost, as far as parity, charge conjugation, and time reversal are concerned. Since the generators of the Poincaré group are to satisfy a set of commutation relations given by the Poincaré algebra, we were able to constrain some of the generic terms of the boost generator that we initially constructed. Furthermore, in the case of pNRQCD (weakly-coupled, in particular), we have exploited the freedom to redefine the fields to eliminate redundancies in our expression of the boost by utilizing unitary transformation of the fields. Then, as we apply the boost transformation of the theory, we have obtained some additional terms to the theory as comparing it to the original one. Since the low-energy EFTs are also supposed to be symmetric under the Poincaré transformations, these additional terms of the transformed theory have to vanish up to total derivatives. We have obtained non-trivial relations between the Wilson coefficients of the EFTs from this invariance, both in NRQCD and pNRQCD. The results are consistent with the literature [22, 24, 78]. Through these constraints on the coefficients, the low-energy limit of the underlying theory is manifestly Poincaré invariant. In other words, the fundamental spacetime symmetry of quantum field theory is not only manifestly observed at high-energy scales, but also in low-energy regimes as long as some constraints are met.

Poincaré invariance results in some non-trivial relations between the heavy quark-antiquark potentials in the case of pNRQCD. We have analyzed analytic expressions of these potentials in the long-distance regime, in which any kind of perturbative method breaks down due to color confinement [31]. We have investigated this non-perturbative nature by utilizing another EFT of a QCD flux tube model [32], namely the effective string theory (EST) [32, 44, 57, 81–83]. In accordance with the EST, the non-perturbative[1] gluodynamics between a heavy quark-antiquark pair in the static limit can be described by vibrational modes of a long string which connects the pair.

On the other hand, by the matching procedure between NRQCD and pNRQCD, one can find the relations between a Wilson loop expectation value as well as some gauge field insertions therein and the heavy quark potentials[2], order by order in $1/M$. The leading order heavy potential is given in terms of the expectation value of a rectangular Wilson loop in the large time limit [50], and the first and the second order corrections to the static potentials are shown to be equivalent to the gauge field insertions to the Wilson loop [50, 51]. One can derive the analytic expressions of the potentials either by calculating heavy quark-antiquark scattering amplitudes in the perturbative regime [26–30], or by explicitly computing the gauge field insertions to the Wilson loop expectation value [58]. In the strongly-coupled case, however, such calculations are not valid because the perturbative expansion parameter α_S exceeds the weak coupling limit.

For such case, based on the Wilson loop-string partition function equivalence conjecture [52, 53], we have employed the EST to derive the long-distance potentials. The

[1]Here, the non-perturbative regime is expressed by the hierachy of scales: $r\Lambda_{\rm QCD} \gg 1$, where r is the distance between the heavy quark-antiquark pair.

[2]We focus on the singlet potential in this thesis.

EST action is derived by imposing boundary conditions as well as utilizing the hierarchy of scales $r\Lambda_{\rm QCD} \gg 1$ in the Nambu-Goto action. We have included the first non-trivial term of the action, which is a Gaussian term, and derived the Green's function by imposing Dirichlet boundary conditions.

One can find a link between gauge field insertions to the Wilson loop and the EST by constructing a set of one-to-one mappings. The expressions of the mappings are restricted by the symmetry group of the system, which is the diatomic molecular group $D_{h\infty}$, and CP symmetry, as well as counting mass dimensions on both sides [56, 57]. We have extended the work of [57] to next-to-leading order (NLO) in order to calculate the potentials at NLO in the EST power counting scheme [94, 107].

Then, by using the QCD-to-EST mapping both at leading order (LO) and NLO as well as the Green's function from the EST action, we have analytically calculated the heavy quark potentials both up to LO and NLO in the EST power counting. In case of the NLO calculation, there arise some divergences due to the string correlators defined at the same spacetime points. Such divergences in higher orders of the EST power counting are natural because the EST is only an effective description of a long string; in other words, the theory is valid only at a scale much below $\Lambda_{\rm QCD}$. We have regularized these divergences by using zeta function regularization and dimensional regularization schemes. It might seem that we are using two different regularization schemes, but it turns out that both are equivalent [98, 99].

The derived potentials contain a number of parameters, the ones from the QCD-to-EST mapping or from time integrals for the potentials. We have utilized the non-trivial relations between the potentials derived from the Poincaré invariance of QCD [22, 50, 92, 93], in order to constrain some of these parameters. Then, we observe that some of the simplified expressions of the potentials have minimal dependence on the parameters (string tension and heavy quark mass), especially the momentum-dependent (but spin-independent) potentials [94].

We have compared our analytic results to available lattice data [45–48, 105] in a systematic way. The numerical value of the string tension was extracted by fitting the static potential to the most up-to-date data [105], but since the extracted value is consistent with the widely accepted one from the literature [91], we have used the value in [91] for the analysis. Then the potential at $\mathcal{O}(1/M)$ is fitted to the lattice data, such that some parameters are determined. Lastly, we have compared the momentum-dependent potentials (both at LO and NLO) to the LQCD data, but some discrepancies remain (FIG. 5.4a - 5.4d). In fact, such discrepancies originate from the discrepancy between the analytic expression of the static potential up to NLO in the EST power counting and the corresponding LQCD data (FIG. 5.2). Since the static potential is non-trivially related to the momentum-dependent potentials due to the Poincaré invariance in QCD, the discrepancy for the static potential is directly related to the discrepancies in the momentum-dependent potentials. We estimate that this comparison can be improved by proceeding with higher order calculations in the EST, and/or by improving on the lattice data.

6.1.1 Applications of Poincaré invariance

In view of the prospects of our general method of implementing Poincaré invariance in low-energy EFTs, we envisage numerous possible applications in EFTs. First, we expect that our method can be applied to theories of weakly-interacting massive particles (or WIMPs). There have been various suggestions about the properties of dark matter using WIMPs during the last few decades, such as supersymmetric dark matter (SUSY-DM), axions, sterile neutrinos, etc [108–110]. As the mass of the DM candidates is assumed to be greater than currently accessible energy scales, the study of DM production and annihilation at current LHC experiments is largely based on the method of nonrelativistic EFTs. The direct detection via nucleon-DM scattering processes was investigated in [111], where the operators were constructed based on Galilean invariance and the EFT formalism. Instead of Galilean invariance, one can construct the Lagrangian from Poincaré invariance, such as has been done in [112]. We are currently investigating if Poincaré invariance can be imposed on an EFT which was constructed according to Galilean invariance; after all, the Galilean invariant operators ultimately are to be embedded into a Poincaré invariant effective theory.

Also, one can apply the formalism developed and investigated in this thesis to the low-energy EFTs for SUSY-DM candidates. A large number of operators can be reduced by deriving suitable constraints on the Wilson coefficients, thereby simplifying the calculations of (possible) physical observables at the LHC experiments. More detailed arguments concerning WIMPs will be discussed in our future works.

6.1.2 Applications of EST

The EFT method we have presented in this paper opens up the way to a precision calculation of the long-distance heavy quark potentials, especially around the intermediate distance range $r/r_0 \sim 1.5$. Although the terms are included only up to NLO within the EST in this work, the scheme to calculate higher order terms is clear: NNLO terms can be derived by (i) including the NLO terms of the effective string action [55, 84, 95] so that the Green's function is derived via a perturbative expansion, or by (ii) exploiting the QCD-to-EST mapping up to NNLO. Some divergences will appear in both cases and they can be regularized by the schemes we discussed in Sec. (5.4.4). In the case of (i), one iterates the calculation of the potentials by utilizing the QCD-to-EST mapping at LO because the Green's function in this case is already $\mathcal{O}(\sigma^{-2}r^{-4})$ suppressed with respect to the one from the Gaussian action. Thus, only the mapping up to leading order is needed for the NNLO calculation. After exploiting constraints due to Poincaré invariance, it is important to check that the NNLO terms of the potentials, especially of the momentum-dependent potentials like V_{p^2}'s and V_{L^2}'s, reduce the deviations from the corresponding LQCD data. Subsequently, the N^3LO terms can be calculated by taking combinations of the perturbative expansion for the Green's function and the subleading orders of the QCD-to-EST mapping.

One can apply the result of our calculation of the static heavy quark potential to the analysis of the heavy quarkonium spectrum, as it was already discussed in [57, 113].

For the quarkonium spectrum, the analytic expression of the perturbative part of the potential is needed as well [26–30], which is valid at the short-distance range ($r \ll \Lambda_{\rm QCD}^{-1}$). While this amalgamation for the full-range potential works well in short and long distances separately, the intermediate distance range might be problematic as the strong coupling parameter $\alpha_S(1/r)$ encounters a singularity around the hadronic scale $1/r \sim \Lambda_{\rm QCD}$. In [113], the strong coupling was replaced by a free parameter a in order to circumvent this issue, and its value was extracted along with the string tension and the heavy quark masses (for both charm and bottom) by comparing the theoretical prediction of the quarkonium mass spectrum, which is calculated in quantum mechanical perturbation theory

$$\begin{aligned}\boldsymbol{M}(n^{2S+1}L_J) =& 2M_{c,b} + E_{nl}^{(0)} + \langle nl|V^{\rm NLO}(r)|nl\rangle + \sum_{m\neq n}^{\infty} \frac{|\langle nl|V^{\rm NLO}(r)|ml\rangle|^2}{E_{nl}^{(0)} - E_{ml}^{(0)}} \\ &+ \langle nljs|V^{\rm NNLO}(r)|nljs\rangle\,,\end{aligned} \tag{6.1}$$

where n, L, J, S are quantum numbers (principal, angular momentum, total angular momentum, spin, respectively) of the states, to the experimental data [71] as well as lattice simulation [114] values of the four different quantum states[3]. From this comparison, numerical values of the parameter a, the string tension σ, and the heavy quark masses $M_{c,b}$ are extracted. One can insert these values back into Eq. (6.1) in order to evaluate the heavy quarknoium spectrum while including the expressions of the long-distance potentials given by the string calculation. This can yield more precise values of the spectra since $V^{\rm NLO}$ and $V^{\rm NNLO}$ of Eq. (6.1) would contain suppression terms in $1/r$ which are more significant at the intermediate distance range $r/r_0 \sim 1.5$. This is under current investigation, and the results will be discussed in the upcoming paper.

Lastly, one can also utilize the framework of effective string theory to an analytic calculation of the potentials for baryons consisting of three heavy quarks such as *ccc*, *ccb*, or *bbb* [115]. Although the observation of such states is yet to come, this possibility has already been explored from the QCD point of view by using the Wilson area law [116, 117]. The Wilson loop formalism was then extended to the framework of pNRQCD for heavy baryons [118]. As it was presented in Sec. 4.1, potential terms are shown to be related to the gauge field insertions to the Wilson loop expectation value by the matching calculation between NRQCD and pNRQCD, in the long-distance regime. A Wilson loop for the three static heavy quarks, however, is more complicated than for the two-body case due to the permutation of three static sources. Also, the heavy quark potentials show a different position dependence because there are two independent relative distances between the three heavy quarks.

On the other hand, the effective framework of a long string for heavy baryons has also been explored [119], which was then compared to lattice data [43, 105, 120]. Here, the calculation of the leading fluctuation of the potential, which is analogous to the Lüscher term in the two-body case, is shown by using the minimal total length of the strings.

[3]Potentials, $V^{\rm NLO}$ and $V^{\rm NNLO}$, are organized according to three different power counting schemes, which are presented in chapter 5 of [113]. We use the third method for the counting of the potential.

In the long-distance limit, the connected strings feature a *Y-shaped* configuration: three strings, originating from the positions of the heavy quarks, are connected to a junction at the center. As the junction itself is also subject to some fluctuations, the effective action in this case consists not only of the string fields, but also of another field configuration for the position of the junction. Then the string fluctuation at leading order is given by solving the partition function [119].

From this point, one can start calculating relativistic corrections to the static potential by constructing a set of mappings from the gauge field insertions to the Wilson loop expectation value to the string correlators, which are in accordance with symmetry properties of the physical system as well as matching the mass dimensions of both sides. This will introduce some dimensionful parameters just like in Eqs. (5.40) and (5.122), eventually constrained by Poincaré invariance.

Appendices

Appendix A

A.1 NLO calculation in the four-fermion sector

At $\mathcal{O}\left(M^{-4}\right)$, one has to include also heavy (anti)quark fields in $\boldsymbol{k}_\psi$ and $\boldsymbol{k}_\chi$ [77]. The terms affecting the four-fermion Lagrangian given in Sec. 2.4.5 can be parametrized as follows:

$$
\begin{aligned}
\hat{\boldsymbol{k}}_\psi^{(2f)} = & \frac{a_{11}}{M^4}\overleftrightarrow{\boldsymbol{D}}\chi\chi^\dagger + \frac{a_{12}}{M^4}\chi\boldsymbol{\nabla}\chi^\dagger + \frac{a_{13}}{M^4}\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}} \\
& + \frac{a_{81}}{M^4}\overleftrightarrow{\boldsymbol{D}}T^a\chi\chi^\dagger T^a + \frac{a_{82}}{M^4}T^a\chi\boldsymbol{D}^{ab}\chi^\dagger T^b + \frac{a_{83}}{M^4}T^a\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a \\
& + \frac{ib_{11}}{M^4}\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma}\chi\chi^\dagger - \frac{ib_{12}}{M^4}\boldsymbol{\sigma}\chi\times\boldsymbol{\nabla}\chi^\dagger - \frac{ib_{13}}{M^4}\boldsymbol{\sigma}\chi\times\chi^\dagger\overleftrightarrow{\boldsymbol{D}} \\
& + \frac{ib_{14}}{M^4}\overleftrightarrow{\boldsymbol{D}}\chi\times\chi^\dagger\boldsymbol{\sigma} + \frac{ib_{15}}{M^4}\chi\boldsymbol{\nabla}\times\chi^\dagger\boldsymbol{\sigma} + \frac{ib_{16}}{M^4}\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma} \\
& + \frac{ib_{81}}{M^4}\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma}T^a\chi\chi^\dagger T^a - \frac{ib_{82}}{M^4}\boldsymbol{\sigma}T^a\chi\times\boldsymbol{D}^{ab}\chi^\dagger T^b - \frac{ib_{83}}{M^4}\boldsymbol{\sigma}T^a\chi\times\chi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a \\
& + \frac{ib_{84}}{M^4}\overleftrightarrow{\boldsymbol{D}}T^a\chi\times\chi^\dagger\boldsymbol{\sigma}T^a + \frac{ib_{85}}{M^4}T^a\chi\boldsymbol{D}^{ab}\times\chi^\dagger\boldsymbol{\sigma}T^b + \frac{ib_{86}}{M^4}T^a\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma}T^a \\
& + \frac{c_{11}}{M^4}(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})\chi\chi^\dagger\boldsymbol{\sigma} + \frac{c_{12}}{M^4}\sigma_i\chi\nabla_i\chi^\dagger\boldsymbol{\sigma} + \frac{c_{13}}{M^4}\sigma_i\chi\chi^\dagger\overleftrightarrow{D}_i\boldsymbol{\sigma} \\
& + \frac{c_{14}}{M^4}\overleftrightarrow{D}_i\boldsymbol{\sigma}\chi\chi^\dagger\sigma_i + \frac{c_{15}}{M^4}\boldsymbol{\sigma}\chi\nabla_i\chi^\dagger\sigma_i + \frac{c_{16}}{M^4}\boldsymbol{\sigma}\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma}) \\
& + \frac{c_{17}}{M^4}\overleftrightarrow{\boldsymbol{D}}\sigma_i\chi\chi^\dagger\sigma_i + \frac{c_{18}}{M^4}\sigma_i\chi\boldsymbol{\nabla}\chi^\dagger\sigma_i + \frac{c_{19}}{M^4}\sigma_i\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\sigma_i \\
& + \frac{c_{81}}{M^4}(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\chi\chi^\dagger\boldsymbol{\sigma}T^a + \frac{c_{82}}{M^4}\sigma_iT^a\chi D_i^{ab}\chi^\dagger\boldsymbol{\sigma}T^b + \frac{c_{83}}{M^4}\sigma_iT^a\chi\chi^\dagger\overleftrightarrow{D}_i\boldsymbol{\sigma}T^a \\
& + \frac{c_{84}}{M^4}\overleftrightarrow{D}_i\boldsymbol{\sigma}T^a\chi\chi^\dagger\sigma_iT^a + \frac{c_{85}}{M^4}\boldsymbol{\sigma}T^a\chi D_i^{ab}\chi^\dagger\sigma_iT^b + \frac{c_{86}}{M^4}\boldsymbol{\sigma}T^a\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a \\
& + \frac{c_{87}}{M^4}\overleftrightarrow{\boldsymbol{D}}\sigma_iT^a\chi\chi^\dagger\sigma_iT^a + \frac{c_{88}}{M^4}\sigma_iT^a\chi\boldsymbol{D}^{ab}\chi^\dagger\sigma_iT^b + \frac{c_{89}}{M^4}\sigma_iT^a\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\sigma_iT^a\,, \qquad \text{(A.1)}
\end{aligned}
$$

$$
\hat{\boldsymbol{k}}_\chi^{(2f)} = \hat{\boldsymbol{k}}_\psi^{(2f)}(\psi\leftrightarrow\chi)\,. \qquad \text{(A.2)}
$$

Here the definition of the left-right derivatives on the left hand side of $\chi\chi^\dagger$ is a bit trickier. We will understand them as

$$
\overleftrightarrow{\boldsymbol{D}}T\chi\chi^\dagger T\psi = T(\boldsymbol{D}\chi)\chi^\dagger T\psi + \boldsymbol{D}(T\chi\chi^\dagger T\psi)\,, \qquad \text{(A.3)}
$$

and implicitly perform integration by parts on the second term. The overall spatial derivatives introduced by this integration are irrelevant for everything that will be discussed in this paper, so we will ignore them. But this definition then also implies that the left-derivative part of $\overleftrightarrow{\boldsymbol{D}}$ acts also on the terms outside the bilinear in which it appears. The left-right derivatives on the right hand side of $\chi\chi^\dagger$ are defined as above and act only within their bilinear. As an example we give the boost transformation proportional to a_{11} and a_{13} due to the χ field in $\psi^\dagger\chi\chi^\dagger\psi$:

$$\psi^\dagger(\hat{\boldsymbol{k}}_\chi\chi)\chi^\dagger\psi = \frac{a_{11}}{M^4}(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\psi)\psi^\dagger\chi\chi^\dagger\psi - \frac{a_{11}}{M^4}\psi^\dagger\psi\psi^\dagger\chi(\boldsymbol{\nabla}\chi^\dagger\psi) + \frac{a_{13}}{M^4}\psi^\dagger\psi(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\chi)\chi^\dagger\psi + \dots . \tag{A.4}$$

When we now calculate the commutator of two boosts at $\mathcal{O}(M^{-3})$ and consider only the two-fermion part, we get some constraints on these boost coefficients a, b and c. As above, we see that at this order only the terms with a center-of-mass derivative do not cancel automatically, and none of the a coefficients can appear because they do not give terms antisymmetric in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

There are three contributions to this commutator, the first of which is

$$\begin{aligned}
&-\left[\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\psi^{(2f)}, M\boldsymbol{\eta}\cdot\boldsymbol{r}\right] + \left[\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\psi^{(2f)}, M\boldsymbol{\xi}\cdot\boldsymbol{r}\right]\\
&= -\frac{2i}{M^3}(b_{11}+b_{12}+b_{13})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\chi\chi^\dagger - \frac{2i}{M^3}(b_{14}+b_{15}+b_{16})\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\\
&\quad -\frac{2i}{M^3}(b_{81}+b_{82}+b_{83})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}T^a\chi\chi^\dagger T^a - \frac{2i}{M^3}(b_{84}+b_{85}+b_{86})T^a\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}T^a\\
&\quad +\frac{1}{M^3}(c_{11}+c_{12}+c_{13}-c_{14}-c_{15}-c_{16})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}\chi\times\chi^\dagger\boldsymbol{\sigma})\\
&\quad +\frac{1}{M^3}(c_{81}+c_{82}+c_{83}-c_{84}-c_{85}-c_{86})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}T^a\chi\times\chi^\dagger\boldsymbol{\sigma}T^a)\,.
\end{aligned}\tag{A.5}$$

The second contribution comes from the transformation of the χ fields inside $\hat{\boldsymbol{k}}_\psi^{(2f)}$

$$\begin{aligned}
&-i\boldsymbol{\xi}\cdot\hat{\boldsymbol{k}}_\psi^{(2f)}(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},(1+iM\boldsymbol{\eta}\cdot\boldsymbol{r})\chi,\psi) + i\boldsymbol{\eta}\cdot\hat{\boldsymbol{k}}_\psi^{(2f)}(\boldsymbol{D},\boldsymbol{E},\boldsymbol{B},(1+iM\boldsymbol{\xi}\cdot\boldsymbol{r})\chi,\psi)\Big|_{\mathcal{O}(\xi^1,\eta^1)}\\
&= \frac{2i}{M^3}(b_{11}-b_{12}+b_{13})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\chi\chi^\dagger + \frac{2i}{M^3}(b_{14}-b_{15}+b_{16})\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\\
&\quad +\frac{2i}{M^3}(b_{81}-b_{82}+b_{83})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}T^a\chi\chi^\dagger T^a + \frac{2i}{M^3}(b_{84}-b_{85}+b_{86})T^a\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}T^a\\
&\quad -\frac{1}{M^3}(c_{11}-c_{12}+c_{13}-c_{14}+c_{15}-c_{16})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}\chi\times\chi^\dagger\boldsymbol{\sigma})\\
&\quad -\frac{1}{M^3}(c_{81}-c_{82}+c_{83}-c_{84}+c_{85}-c_{86})(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}T^a\chi\times\chi^\dagger\boldsymbol{\sigma}T^a)\,.
\end{aligned}\tag{A.6}$$

The last contribution comes from the term $-\frac{1}{2M}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}D_0$, which has already been derived previously. When we use the equation of motion for $iD_0\psi$, this becomes in the two-fermion sector

$$-\frac{i(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}}{2M^3}\Big\{f_1(^1S_0)\,\chi\chi^\dagger + f_1(^3S_1)\,\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma} + f_8(^1S_0)\,T^a\chi\chi^\dagger T^a$$

$$
\begin{aligned}
&+ f_1(^3S_1)\,\boldsymbol{\sigma} T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma} T^a\Big\}\\
=&-\frac{if_1(^1S_0)}{2M^3}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\chi\chi^\dagger-\frac{if_8(^1S_0)}{2M^3}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma} T^a\chi\chi^\dagger T^a\\
&-\frac{if_1(^3S_1)}{2M^3}\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}-\frac{if_8(^3S_1)}{2M^3}T^a\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma} T^a\\
&-\frac{f_1(^3S_1)}{2M^3}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}\chi\times\chi^\dagger\boldsymbol{\sigma})-\frac{f_8(^3S_1)}{2M^3}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma} T^a\chi\times\chi^\dagger\boldsymbol{\sigma} T^a)\,. \qquad \text{(A.7)}
\end{aligned}
$$

The sum of these three contributions has to vanish, thus we have

$$
\begin{aligned}
0=&-\frac{i}{2M^3}\left(8b_{12}+f_1(^1S_0)\right)(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}\chi\chi^\dagger-\frac{i}{2M^3}\left(8b_{82}+f_8(^1S_0)\right)(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma} T^a\chi\chi^\dagger T^a\\
&-\frac{i}{2M^3}\left(8b_{15}+f_1(^3S_1)\right)\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma}-\frac{i}{2M^3}\left(8b_{85}+f_8(^3S_1)\right)T^a\chi\chi^\dagger(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma} T^a\\
&+\frac{1}{2M^3}\left(4c_{12}-4c_{15}-f_1(^3S_1)\right)(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma}\chi\times\chi^\dagger\boldsymbol{\sigma})\\
&+\frac{1}{2M^3}\left(4c_{82}-4c_{85}-f_8(^3S_1)\right)(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot(\boldsymbol{\sigma} T^a\chi\times\chi^\dagger\boldsymbol{\sigma} T^a)\,, \qquad \text{(A.8)}
\end{aligned}
$$

which fixes the two-fermion boost parameters to be

$$
b_{12}=-\frac{1}{8}f_1(^1S_0)\,,\quad b_{15}=-\frac{1}{8}f_1(^3S_1)\,,\quad b_{82}=-\frac{1}{8}f_8(^1S_0)\,,\quad b_{85}=-\frac{1}{8}f_8(^3S_1)\,,\quad \text{(A.9)}
$$

$$
c_{12}-c_{15}=\frac{1}{4}f_1(^3S_1)\,,\qquad c_{82}-c_{85}=\frac{1}{4}f_8(^3S_1)\,. \qquad \text{(A.10)}
$$

At $\mathcal{O}(M^{-4})$ there is no new information from the boost commutator. The $\mathcal{O}(M^{-5})$ terms of $\hat{\boldsymbol{k}}_\psi^{(2f)}$ can either contain two derivatives or one gluon field for dimensional reasons, but only the chromoelectric field has the right parity transformation behavior. So there can be no $\mathcal{O}(M^{-5})$ terms with derivatives, and therefore the commutator of the $\mathcal{O}(M^{-5})$ $\hat{\boldsymbol{k}}_\psi^{(2f)}$ with $M\boldsymbol{r}$ vanishes. The boost transformation of the fields inside $\hat{\boldsymbol{k}}_\psi^{(2f)}$ at $\mathcal{O}(M^{-4})$ gives only temporal derivatives, which have to be replaced through the equations of motion for ψ and χ and thus contribute only at $\mathcal{O}(M^{-5})$. And there are no four-fermion $\mathcal{O}(M^{-3})$ terms that could give a contribution at $\mathcal{O}(M^{-4})$ from $-\frac{1}{2M}(\boldsymbol{\xi}\times\boldsymbol{\eta})\cdot\boldsymbol{\sigma} D_0$.

In order to get the constraints from the boost transformation of $\mathcal{L}$ at $\mathcal{O}(M^{-4})$, we need all four-fermion terms at $\mathcal{O}(M^{-4})$, most of which can be found in [78], and all center-of-mass derivative terms at $\mathcal{O}(M^{-5})$, which were not included in [78].

$$
\begin{aligned}
\mathcal{L}^{(4f)}_{M^{-4}}=&-\frac{g_1(^1S_0)}{8M^4}\left(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}^2\chi\chi^\dagger\psi+\psi^\dagger\chi\chi^\dagger\overleftrightarrow{\boldsymbol{D}}^2\psi\right)\\
&-\frac{g_1(^3S_1)}{8M^4}\left(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}^2)\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi+\psi^\dagger\boldsymbol{\sigma}\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}^2)\boldsymbol{\sigma}\psi\right)\\
&-\frac{g_1(^3S_1,{}^1S_0)}{8M^4}\left(\frac{1}{2}\psi^\dagger\left\{(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma}),\overleftrightarrow{\boldsymbol{D}}\right\}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi-\frac{1}{3}\psi^\dagger\overleftrightarrow{\boldsymbol{D}}^2\chi\chi^\dagger\psi+h.c.\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{g_8(^1S_0)}{8M^4}\left(\psi^\dagger \overleftrightarrow{\boldsymbol{D}}^2 T^a\chi\chi^\dagger T^a\psi + \psi^\dagger\chi\chi^\dagger \overleftrightarrow{\boldsymbol{D}}^2 T^a\psi\right)\\
&- \frac{g_8(^3S_1)}{8M^4}\left(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}^2)\boldsymbol{\sigma}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi + \psi^\dagger\boldsymbol{\sigma}T^a\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}^2)T^a\boldsymbol{\sigma}\psi\right)\\
&- \frac{g_8(^3S_1,{}^1S_0)}{8M^4}\left(\frac{1}{2}\psi^\dagger\left\{(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma}),\overleftrightarrow{\boldsymbol{D}}\right\}T^a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T^a\psi\right.\\
&\left.-\frac{1}{3}\psi^\dagger\overleftrightarrow{\boldsymbol{D}}^2 T^a\chi\chi^\dagger T^a\psi + h.c.\right)\\
&- \frac{f_1(^1P_1)}{4M^4}\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\chi\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\psi\\
&- \frac{f_1(^3P_0)}{12M^4}\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})\psi\\
&- \frac{f_1(^3P_1)}{8M^4}\left(\psi^\dagger\overleftrightarrow{D}_i\sigma_j\chi\chi^\dagger\overleftrightarrow{D}_i\sigma_j\psi - \psi^\dagger\overleftrightarrow{D}_i\sigma_j\chi\chi^\dagger\overleftrightarrow{D}_j\sigma_i\psi\right)\\
&- \frac{f_1(^3P_2)}{4M^4}\left(\frac{1}{2}\psi^\dagger\overleftrightarrow{D}_i\sigma_j\chi\chi^\dagger\overleftrightarrow{D}_i\sigma_j\psi + \frac{1}{2}\psi^\dagger\overleftrightarrow{D}_i\sigma_j\chi\chi^\dagger\overleftrightarrow{D}_j\sigma_i\psi\right.\\
&\qquad\qquad\left.-\frac{1}{3}\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})\psi\right)\\
&- \frac{f_8(^1P_1)}{4M^4}\psi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\chi\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\psi\\
&- \frac{f_8(^3P_0)}{12M^4}\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\psi\\
&- \frac{f_8(^3P_1)}{8M^4}\left(\psi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\chi\chi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\psi - \psi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\chi\chi^\dagger\overleftrightarrow{D}_j\sigma_iT^a\psi\right)\\
&- \frac{f_8(^3P_2)}{4M^4}\left(\frac{1}{2}\psi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\chi\chi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\psi + \frac{1}{2}\psi^\dagger\overleftrightarrow{D}_i\sigma_jT^a\chi\chi^\dagger\overleftrightarrow{D}_j\sigma_iT^a\psi\right.\\
&\qquad\qquad\left.-\frac{1}{3}\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\chi\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\psi\right)\\
&- \frac{if_{1\,cm}}{2M^4}\big(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})\chi\cdot\boldsymbol{\nabla}\chi^\dagger\psi + (\boldsymbol{\nabla}\psi^\dagger\chi)\cdot\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})\psi\big)\\
&+ \frac{if'_{1\,cm}}{2M^4}\big(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}\chi\cdot(\boldsymbol{\nabla}\times\chi^\dagger\boldsymbol{\sigma}\psi) + (\boldsymbol{\nabla}\times\psi^\dagger\boldsymbol{\sigma}\chi)\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}\psi\big)\\
&- \frac{if_{8\,cm}}{2M^4}\big(\psi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})T^a\chi\cdot\boldsymbol{D}^{ab}\chi^\dagger T^b\psi + (\boldsymbol{D}^{ab}\psi^\dagger T^b\chi)\cdot\chi^\dagger(\overleftrightarrow{\boldsymbol{D}}\times\boldsymbol{\sigma})T^a\psi\big)\\
&+ \frac{if'_{8\,cm}}{2M^4}\big(\psi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\chi\cdot(\boldsymbol{D}^{ab}\times\chi^\dagger\boldsymbol{\sigma}T^b\psi) + (\boldsymbol{D}^{ab}\times\psi^\dagger\boldsymbol{\sigma}T^b\chi)\cdot\chi^\dagger\overleftrightarrow{\boldsymbol{D}}T^a\psi\big)\\
&+ \frac{g_{1a\,cm}}{M^4}\left(\nabla_i\psi^\dagger\sigma_j\chi\right)\left(\nabla_i\chi^\dagger\sigma_j\psi\right) + \frac{g_{8a\,cm}}{M^4}\left(D_i^{ab}\psi^\dagger\sigma_jT^b\chi\right)\left(D_i^{ac}\chi^\dagger\sigma_jT^c\psi\right)\\
&+ \frac{g_{1b\,cm}}{M^4}\left(\boldsymbol{\nabla}\cdot\psi^\dagger\boldsymbol{\sigma}\chi\right)\left(\boldsymbol{\nabla}\cdot\chi^\dagger\boldsymbol{\sigma}\psi\right) + \frac{g_{8b\,cm}}{M^4}\left(\boldsymbol{D}^{ab}\cdot\psi^\dagger\boldsymbol{\sigma}T^b\chi\right)\left(\boldsymbol{D}^{ac}\cdot\chi^\dagger\boldsymbol{\sigma}T^c\psi\right)\\
&+ \frac{g_{1c\,cm}}{M^4}\left(\boldsymbol{\nabla}\psi^\dagger\chi\right)\cdot\left(\boldsymbol{\nabla}\chi^\dagger\psi\right) + \frac{g_{8c\,cm}}{M^4}\left(\boldsymbol{D}^{ab}\psi^\dagger T^b\chi\right)\cdot\left(\boldsymbol{D}^{ac}\chi^\dagger T^c\psi\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{s_{1-8}({}^1S_0, {}^3S_1)}{2M^4} \Big(\psi^\dagger g\boldsymbol{B} \cdot \boldsymbol{\sigma}\chi\chi^\dagger\psi + \psi^\dagger\chi\chi^\dagger g\boldsymbol{B} \cdot \boldsymbol{\sigma}\psi \Big) \\
&+ \frac{s_{1-8}({}^3S_1, {}^1S_0)}{2M^4} \Big(\psi^\dagger g\boldsymbol{B}\chi \cdot \chi^\dagger\boldsymbol{\sigma}\psi + \psi^\dagger\boldsymbol{\sigma}\chi \cdot \chi^\dagger g\boldsymbol{B}\psi \Big) \\
&+ \frac{s_{8-8}({}^1S_0, {}^3S_1)}{2M^4} d^{abc} g\boldsymbol{B}^a \cdot \Big(\psi^\dagger\boldsymbol{\sigma}T^b\chi\chi^\dagger T^c\psi + \psi^\dagger T^b\chi\chi^\dagger\boldsymbol{\sigma}T^c\psi \Big) \\
&+ \frac{s_{8-8}({}^3S_1, {}^3S_1)}{2M^4} f^{abc} g\boldsymbol{B}^a \cdot \Big(\psi^\dagger\boldsymbol{\sigma}T^b\chi \times \chi^\dagger\boldsymbol{\sigma}T^c\psi \Big) \,. \qquad \text{(A.11)}
\end{aligned}
$$

For dimensional reasons the $\mathcal{O}(M^{-5})$ four-fermion Lagrangian can either contain three derivatives or one derivative and one gluon field. Parity allows only the combination of a chromoelectric field and a derivative. As stated above, only center-of-mass derivatives are relevant for this order of the boost transformation.

$$
\begin{aligned}
\mathcal{L}^{(4f)}_{M^{-5}\,cm} = &\frac{is_{1-8\,cm}}{2M^5} \Big(\psi^\dagger g\boldsymbol{E} \times \boldsymbol{\sigma}\chi \cdot \boldsymbol{\nabla}\chi^\dagger\psi - (\boldsymbol{\nabla}\psi^\dagger\chi) \cdot \chi^\dagger g\boldsymbol{E} \times \boldsymbol{\sigma}\psi \Big) \\
&- \frac{is'_{1-8\,cm}}{2M^5} \Big(\psi^\dagger g\boldsymbol{E}\chi \cdot (\boldsymbol{\nabla} \times \chi^\dagger\boldsymbol{\sigma}\psi) - (\boldsymbol{\nabla} \times \psi^\dagger\boldsymbol{\sigma}\chi) \cdot \chi^\dagger g\boldsymbol{E}\psi \Big) \\
&+ \frac{is_{8-8\,cm}}{2M^5} d^{abc} g\boldsymbol{E}^a \cdot \Big(\psi^\dagger\boldsymbol{\sigma}T^b\chi \times \boldsymbol{D}^{cd}\chi^\dagger T^d\psi + (\boldsymbol{D}^{bd}\psi^\dagger T^d\chi) \times \chi^\dagger\boldsymbol{\sigma}T^c\psi \Big) \\
&+ \frac{is'_{8-8\,cm}}{2M^5} f^{abc} gE^a_i \Big(\psi^\dagger\sigma_i T^b\chi(\boldsymbol{D}^{cd} \cdot \chi^\dagger\boldsymbol{\sigma}T^d\psi) + (\boldsymbol{D}^{bd} \cdot \psi^\dagger\boldsymbol{\sigma}T^d\chi)\chi^\dagger\sigma_i T^c\psi \Big) \,. \qquad \text{(A.12)}
\end{aligned}
$$

In principle, one can write more terms with a center-of-mass derivative, but those can be integrated by parts, neglecting overall derivatives, and they give a derivative that acts only on the chromoelectric field, such that

$$
if^{abc} g\boldsymbol{E}^a \cdot \Big(\psi^\dagger T^b\chi\boldsymbol{D}^{cd}\chi^\dagger T^d\psi + (\boldsymbol{D}^{bd}\psi^\dagger T^d\chi)\chi^\dagger T^c\psi \Big) = -(\boldsymbol{D}^{ad} \cdot g\boldsymbol{E}^d)\, if^{abc}\psi^\dagger T^b\chi\chi^\dagger T^c\psi \,. \qquad \text{(A.13)}
$$

These terms obviously do not contribute to the boost transformation of the Lagrangian at $\mathcal{O}(M^{-4})$. We therefore chose a minimal basis of operators where only the terms given above have explicit center-of-mass derivatives.

After a lengthy calculation of the boost transformation of the Lagrangian at $\mathcal{O}(M^{-4})$, we obtain the following constraints:

$$
a_{11} = \frac{1}{4} g_1({}^1S_0) \,, \quad a_{12} = g_{1c\,cm} \,, \quad a_{13} = \frac{1}{4} f_1({}^1P_1) \,,
$$

$$
a_{81} = \frac{1}{4} g_8({}^1S_0) \,, \quad a_{82} = g_{8c\,cm} \,, \quad a_{83} = \frac{1}{4} f_8({}^1P_1) \,,
$$

$$
b_{12} = -\frac{1}{2} f_{1\,cm} \,, \quad b_{15} = -\frac{1}{2} f'_{1\,cm} \,, \quad b_{82} = -\frac{1}{2} f_{8\,cm} \,, \quad b_{85} = -\frac{1}{2} f'_{8\,cm} \,,
$$

$$
b_{13} = -\frac{1}{2} f'_{1\,cm} + b_{14} \,, \quad b_{16} = -\frac{1}{2} f_{1\,cm} + b_{11} \,, \quad b_{83} = -\frac{1}{2} f'_{8\,cm} + b_{84} \,, \quad b_{86} = -\frac{1}{2} f_{8\,cm} + b_{81} \,,
$$

$$c_{11} = \frac{1}{8} g_1({}^3S_1, {}^3D_1)\,, \qquad c_{13} = \frac{1}{8}\left(f_1({}^3P_2) - f_1({}^3P_1)\right)\,,$$
$$c_{14} = \frac{1}{8} g_1({}^3S_1, {}^3D_1)\,, \qquad c_{16} = \frac{1}{12}\left(f_1({}^3P_0) - f_1({}^3P_2)\right)\,,$$
$$c_{17} = \frac{1}{12}\left(4g_1({}^3S_1) - g_1({}^3S_1, {}^3D_1)\right)\,, \qquad c_{19} = \frac{1}{8}\left(f_1({}^3P_1) + f_1({}^3P_2)\right)\,,$$
$$c_{81} = \frac{1}{8} g_8({}^3S_1, {}^3D_1)\,, \qquad c_{83} = \frac{1}{8}\left(f_8({}^3P_2) - f_8({}^3P_1)\right)\,,$$
$$c_{84} = \frac{1}{8} g_8({}^3S_1, {}^3D_1)\,, \qquad c_{86} = \frac{1}{12}\left(f_8({}^3P_0) - f_8({}^3P_2)\right)\,,$$
$$c_{87} = \frac{1}{12}\left(4g_8({}^3S_1) - g_8({}^3S_1, {}^3D_1)\right)\,, \qquad c_{89} = \frac{1}{8}\left(f_8({}^3P_1) + f_8({}^3P_2)\right)\,,$$
$$c_{15} = -c_{12}\,, \quad c_{18} = g_{1a\,cm}\,, \quad c_{85} = -c_{82}\,, \quad c_{88} = g_{8a\,cm}\,,$$
$$s_{1-8\,cm} - \frac{1}{2} s_{1-8}({}^1S_0, {}^3S_1) - \frac{c_S}{4} f_1({}^1S_0) - \frac{c_S}{4} f_8({}^1S_0) - 2b_{11} - 2b_{84} = 0\,,$$
$$s'_{1-8\,cm} - \frac{1}{2} s_{1-8}({}^3S_1, {}^1S_0) - \frac{c_S}{4} f_1({}^1S_0) - \frac{c_S}{4} f_8({}^1S_0) - 2b_{14} - 2b_{81} = 0\,,$$
$$s_{8-8\,cm} - \frac{1}{2} s_{8-8}({}^1S_0, {}^3S_1) - \frac{c_S}{4} f_8({}^1S_0) - b_{81} - b_{84} = 0\,,$$
$$s'_{8-8\,cm} + \frac{1}{2} s_{8-8}({}^3S_1, {}^3S_1) - \frac{c_S}{4} f_8({}^3S_1) - \frac{1}{16} g_8({}^3S_1, {}^3D_1) + c_{82} = 0\,. \qquad \text{(A.14)}$$

So far none of these constraints involves only Wilson coefficients of the Lagrangian, they rather define the boost parameters of $\hat{\boldsymbol{k}}_\psi^{(2f)}$ and $\hat{\boldsymbol{k}}_\chi^{(2f)}$. There remain two free parameters, c_{12} and one of either b_{11}, b_{14}, b_{81} or b_{84}. But if we combine them with the relations obtained from the commutator of two boosts, we get

$$c_{12} = \frac{1}{8} f_1({}^3S_1)\,, \quad c_{15} = -\frac{1}{8} f_1({}^3S_1)\,, \quad c_{82} = \frac{1}{8} f_8({}^3S_1)\,, \quad c_{85} = -\frac{1}{8} f_8({}^3S_1)\,,$$
$$s'_{8-8\,cm} + \frac{1}{2} s_{8-8}({}^3S_1, {}^3S_1) - \frac{2c_S - 1}{8} f_8({}^3S_1) - \frac{1}{16} g_8({}^3S_1, {}^3D_1) = 0\,. \qquad \text{(A.15)}$$

The last equation now gives a constraint on the Wilson coefficients without any parameters from the boost. The other relations we derived for b_{12}, b_{52}, b_{82} and b_{85} from the commutator of two boosts are consistent with the ones obtained from the transformation of the Lagrangian at $\mathcal{O}(M^{-4})$ and $\mathcal{O}(M^{-2})$.

A.2 Gauge field insertions to the Wilson loop

In this section, we present detailed calculation of the gauge field insertions to the Wilson loop expectation value via the QCD-to-EST mapping at NLO. This presentation is divided into three parts: (i) two-chromoelectric field insertion, (ii) two-chromomagnetic field insertion, and (iii) an insertion of a cross product between a chromoelectric and a chromomagnetic field. Results from these calculations are summarized and used in Sec. 5.4.5

A.2.1 Chromoelectric field insertions

Within the framework of effective string theory, a mapping from a two-chromomagnetic field insertion onto the Wilson loop expectation value, $\langle\!\langle \boldsymbol{E}_1(t)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}$, to the string variables is needed for the $1/M$ correction to the static potential, $V^{(1,0)}(r)$. As we decompose this expression into transversal (or dynamical) and longitudinal (or non-dynamical) parts, the transversal one is simplified unto:

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} &= \langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO} \\ &\quad - \langle\!\langle \boldsymbol{E}_1^l(t)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm LO} - \langle\!\langle \boldsymbol{E}_1^l(t)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO} \\ &= \langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO}\,, \end{aligned} \tag{A.16}$$

in which the second equality here is due to the Gaussianity of the EST action; i.e., a string correlator with odd number of the fields vanishes. Then this expression is mapped onto the following expression:

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^l(t)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO} &= \overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_a\xi_1^m(t)\partial^a\xi_1^m(t)\partial_z\xi_1^l(0)\rangle \\ &\quad + \Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\partial_b\xi_1^n(0)\partial^b\xi_1^n(0)\rangle\,, \end{aligned} \tag{A.17}$$

and each term on the right-hand side of the equation is simplified by the Wick contraction

$$\begin{aligned}\overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_a\xi_1^m(t)\partial^a\xi_1^m(t)\partial_z\xi_1^l(0)\rangle &= 2\overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_a\xi_1^m(t)\rangle\langle \partial^a\xi_1^m(t)\partial_z\xi_1^l(0)\rangle \\ &\quad + \overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\rangle\langle \partial_a\xi_1^m(t)\partial^a\xi_1^m(t)\rangle \\ &= 2\overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^m(t)\rangle\langle \partial_z\xi_1^m(t)\partial_z\xi_1^l(0)\rangle \\ &\quad + \overline{\Lambda}^2\Lambda^2\langle \partial_z\xi_1^m(t)\partial_z\xi_1^m(t)\rangle\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\rangle, \end{aligned} \tag{A.18}$$

$$\begin{aligned}\Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\partial_b\xi_1^n(0)\partial^b\xi_1^n(0)\rangle &= \Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\rangle\langle \partial_b\xi_1^n(0)\partial^b\xi_1^n(0)\rangle \\ &\quad + 2\Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_b\xi_1^n(0)\rangle\langle \partial_z\xi_1^l(0)\partial^b\xi_1^n(0)\rangle \\ &= \Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^l(0)\rangle\langle \partial_z\xi_1^n(0)\partial_z\xi_1^n(0)\rangle \\ &\quad + 2\Lambda^2\overline{\Lambda}^2\langle \partial_z\xi_1^l(t)\partial_z\xi_1^n(0)\rangle\langle \partial_z\xi_1^l(0)\partial_z\xi_1^n(0)\rangle\,. \end{aligned} \tag{A.19}$$

Note that the following identities were used for the Wick contraction,

$$\langle \partial_z\xi_1^{l,m}(t)\partial_t\xi_1^{l,m}(t)\rangle = 0\,, \tag{A.20}$$

$$\langle \partial_t\xi_1^{l,m}(t)\partial_t\xi_1^{l,m}(0)\rangle = 0\,, \tag{A.21}$$

$$\langle \partial_t\xi_1^{l,m}(t)\partial_t\xi_1^{l,m}(t)\rangle = 0\,, \tag{A.22}$$

$$\langle \partial_t\xi_1^{l,m}(0)\partial_t\xi_1^{l,m}(0)\rangle = 0\,, \tag{A.23}$$

where the third and fourth identities are due to dimensional function regularization, Eq. (5.151). By taking the Wick rotation ($t \to -it$), we obtain

$$\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO} = 8\overline{\Lambda}^2\Lambda^2\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'} \times \partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0}$$

$$
\begin{aligned}
&+ 8\Lambda^2\overline{\Lambda}^2 \partial_z \partial_{z'} G|^{t=t'}_{z=z'=r/2} \times \partial_z \partial_{z'} G|^{t'=0}_{z=z'=r/2} \\
&= 16\Lambda^2\overline{\Lambda}^2 \partial_z \partial_{z'} G|^{t=t'}_{z=z'=r/2} \times \partial_z \partial_{z'} G|^{t'=0}_{z=z'=r/2} \\
&= -\frac{\pi^2\Lambda^2\overline{\Lambda}^2}{3\sigma^2 r^4} \sinh^{-2}\left(\frac{\pi t}{2r}\right) ,
\end{aligned} \tag{A.24}
$$

where G is the Green's function, Eq. (5.33), without the tensor indices.

On the other hand, the longitudinal part of the two-chromoelectric field insertion is mapped onto the four-string field correlator, which is due to fact that the correlator associated with two string fields vanish by the definition of the connected part of the Wilson loop expectation value, Eq. (4.21). So its expression in terms of the string fields is given by

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(t) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\mathrm{NLO}} &= \langle\!\langle \boldsymbol{E}_1^3(t) \boldsymbol{E}_1^3(0) \rangle\!\rangle^{\mathrm{NLO}} \\
&\quad - \langle\!\langle \boldsymbol{E}_1^3(t) \rangle\!\rangle^{\mathrm{NLO}} \langle\!\langle \boldsymbol{E}_1^3(0) \rangle\!\rangle^{\mathrm{LO}} - \langle\!\langle \boldsymbol{E}_1^3(t) \rangle\!\rangle^{\mathrm{LO}} \langle\!\langle \boldsymbol{E}_1^3(0) \rangle\!\rangle^{\mathrm{NLO}} \\
&= \overline{\Lambda''}^4 \Big[\langle \partial_a \xi_1^l(t) \partial^a \xi_1^l(t) \partial_b \xi_1^m(0) \partial^b \xi_1^m(0) \rangle \\
&\quad - \langle \partial_a \xi_1^l(t) \partial^a \xi_1^l(t) \rangle \langle \partial_b \xi_1^m(0) \partial^b \xi_1^m(0) \rangle \Big] \\
&= 2\overline{\Lambda''}^4 \langle \partial_a \xi_1^l(t) \partial_b \xi_1^m(0) \rangle \langle \partial^a \xi_1^l(t) \partial^b \xi_1^m(0) \rangle \\
&= 2\overline{\Lambda''}^4 \langle \partial_z \xi_1^l(t) \partial_z \xi_1^m(0) \rangle \langle \partial_z \xi_1^l(t) \partial_z \xi_1^m(0) \rangle ,
\end{aligned} \tag{A.25}
$$

and after taking the Wick rotation, we obtain the expression of the longitudinal part

$$
\langle\!\langle \boldsymbol{E}_1^3(-it) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\mathrm{NLO}} = \frac{\pi^2 \overline{\Lambda''}^4}{\sigma^2 r^4} \left[\cosh\left(\frac{\pi t}{r}\right) - 1 \right]^{-2} . \tag{A.26}
$$

Likewise, the analytic expressions of $\langle\!\langle \boldsymbol{E}_1^l(-it) \boldsymbol{E}_2^l(0) \rangle\!\rangle_c^{\mathrm{NLO}}$ and $\langle\!\langle \boldsymbol{E}_1^3(-it) \boldsymbol{E}_2^3(0) \rangle\!\rangle_c^{\mathrm{NLO}}$ are derived

$$
\langle\!\langle \boldsymbol{E}_1^l(-it) \boldsymbol{E}_2^l(0) \rangle\!\rangle_c^{\mathrm{NLO}} = \frac{\pi^2\Lambda^2\overline{\Lambda}^2}{6\sigma^2 r^4} \cosh^{-2}\left(\frac{\pi t}{2r}\right) , \tag{A.27}
$$

$$
\langle\!\langle \boldsymbol{E}_1^3(-it) \boldsymbol{E}_2^3(0) \rangle\!\rangle_c^{\mathrm{NLO}} = \frac{\pi^2 \overline{\Lambda''}^4}{\sigma^2 r^4} \left[\cosh\left(\frac{\pi t}{r}\right) + 1 \right]^{-2} . \tag{A.28}
$$

As for V_r's, three- and four-chromoelectric field insertions are computed at the NLO of the EST power counting. Due to the definition of the connected part of the expectation value and Gaussianity of the string action, a three-chromoelectric field insertion is then decomposed

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1(t_2) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle_c^{\mathrm{NLO}} &= \langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1(t_2) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle^{\mathrm{NLO}} \\
&\quad - \langle\!\langle \boldsymbol{E}_1^3(t_1) \rangle\!\rangle^{\mathrm{NLO}} \langle\!\langle \boldsymbol{E}_1(t_2) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle^{\mathrm{LO}} \\
&\quad - \langle\!\langle \boldsymbol{E}_1^3(t_1) \rangle\!\rangle^{\mathrm{LO}} \langle\!\langle \boldsymbol{E}_1(t_2) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle^{\mathrm{NLO}} ,
\end{aligned} \tag{A.29}
$$

and this decomposition is then again divided into the transversal and longitudinal parts. The transversal part is decomposed and mapped to the string fields

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\mathrm{NLO}} - \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\mathrm{LO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\mathrm{NLO}} \\ =& \overline{\Lambda''}^2\Lambda^4\langle(\partial\xi_1)^2(t_1)\partial_z\xi_1^l(t_2)\partial_z\xi_1^l(0)\rangle \\ &+ \Lambda''^2\overline{\Lambda}^2\Lambda^2\langle\partial_z\xi_1^l(t_2)(\partial\xi_1)^2(t_2)\partial_z\xi_1^l(0)\rangle \\ &+ \Lambda''^2\Lambda^2\overline{\Lambda}^2\langle\partial_z\xi_1^l(t_2)\partial_z\xi_1^l(0)(\partial\xi_1)^2(0)\rangle \\ &- \overline{\Lambda''}^2\Lambda^4\langle(\partial\xi_1)^2(t_1)\rangle\langle\partial_z\xi_1^l(t_2)\partial_z\xi_1^l(0)\rangle \\ &- \Lambda''^2\overline{\Lambda}^2\Lambda^2\langle\partial_z\xi_1^l(t_2)(\partial\xi_1)^2(t_2)\partial_z\xi_1^l(0)\rangle \\ &- \Lambda''^2\Lambda^2\overline{\Lambda}^2\langle\partial_z\xi_1^l(t_2)\partial_z\xi_1^l(0)(\partial\xi_1)^2(0)\rangle \\ =& -2\overline{\Lambda''}^2\Lambda^4\langle\partial_z\xi_1^m(t_1)\partial_z\xi_1^l(t_2)\rangle\langle\partial_z\xi_1^m(t_1)\partial_z\xi_1^l(0)\rangle\,, \end{aligned} \tag{A.30}$$

and the longitudinal part is mapped onto the string fields

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{LO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}} \\ =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{LO}} \\ =& 2\Lambda''^2\overline{\Lambda}^4\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^m(0)\rangle\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^m(0)\rangle\,. \end{aligned} \tag{A.31}$$

Thus, the analytic expression of the three-chromoelectric field insertion, after taking the Wick rotation, is:

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1(-it_2)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& -4\overline{\Lambda''}^2\Lambda^4\partial_{z_1}\partial_{z_2}G(t_1,t_2;z_1,z_2)|_{z_1=z_2=r/2} \\ &\times\partial_{z_1}\partial_{z_3}G(t_1,t_3;z_1,z_3)|_{z_1=z_3=r/2}^{t_3=0} \\ &+ 4\Lambda''^2\overline{\Lambda}^4\left[\partial_{z_1}\partial_{z_3}G(t_1,t_3;z_1,z_3)|_{z_1=z_3=r/2}^{t_3=0}\right]^2 \\ =& -\frac{\pi^2\overline{\Lambda''}^2\Lambda^4}{4\sigma^2r^4}\sinh^{-2}\left(\frac{\pi t_1}{2r}\right)\sinh^{-2}\left(\frac{\pi(t_1-t_2)}{2r}\right) \\ &+ \frac{\pi^2\Lambda''^2\overline{\Lambda}^4}{4\sigma^2r^4}\sinh^{-4}\left(\frac{\pi t_1}{2r}\right)\,. \end{aligned} \tag{A.32}$$

Another three-chromoelectric field insertion which contributes to V_r, is decomposed as

$$\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2(t_2)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\mathrm{NLO}} = \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^l(t_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\mathrm{NLO}} + \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^3(t_2)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}\,, \tag{A.33}$$

where the first term on the right-hand side of Eq. (A.33) is further decomposed and mapped onto the string variables by

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^l(t_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^l(t_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm NLO} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^l(t_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm LO} \\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^l(t_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm NLO} \\ =& \overline{\Lambda''}^2\Lambda^4\Big[\langle(\partial\xi_1)^2(t_1)\partial_z\xi_2^l(t_2)\partial_z\xi_2^l(0)\rangle \\ &-\langle(\partial\xi_1)^2(t_1)\rangle\langle\partial_z\xi_2^l(t_2)\partial_z\xi_2^l(0)\rangle\Big] \\ =& -2\overline{\Lambda''}^2\Lambda^4\langle\partial_z\xi_1^m(t_1)\partial_z\xi_2^l(t_2)\rangle\langle\partial_z\xi_1^m(t_1)\partial_z\xi_2^l(0)\rangle\,. \end{aligned} \tag{A.34}$$

After the Wick rotation, its analytic expression is

$$\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_2^l(-it_2)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} = -\frac{\pi^2\overline{\Lambda''}^2\Lambda^4}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi(t_1-t_2)}{2r}\right]. \tag{A.35}$$

The second term on the right-hand side of Eq. (A.33) is similarly re-arranged and mapped to the string fields as

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^3(t_2)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^3(t_2)\rangle\!\rangle^{\rm LO} \\ =& 2\Lambda''^2\overline{\Lambda}^4\langle\partial_z\xi_1^l(t_1)\partial_z\xi_2^m(0)\rangle\langle\partial_z\xi_1^l(t_1)\partial_z\xi_2^m(0)\rangle\,, \end{aligned} \tag{A.36}$$

and its Wick rotation gives the following expression:

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_2^3(-it_2)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} =& 4\Lambda''^2\overline{\Lambda}^4\left[\partial_{z_1}\partial_{z_3}G(t_1,t_3;z_1,z_3)|_{z_1=-z_3=r/2}^{t_3=0}\right]^2 \\ =& \frac{\pi^2\Lambda''^2\overline{\Lambda}^4}{4\sigma^2r^4}\cosh^{-4}\left[\frac{\pi t_1}{2r}\right]. \end{aligned} \tag{A.37}$$

From Eqs. (A.35) and (A.37) the desired expression of the three-chromoelectric field insertion to the Wilson loop expectation value at NLO in the EST is obtained

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_2(-it_2)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\rm NLO} =& -\frac{\pi^2\overline{\Lambda''}^2\Lambda^4}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi(t_1-t_2)}{2r}\right] \\ &+\frac{\pi^2\Lambda''^2\overline{\Lambda}^4}{4\sigma^2r^4}\cosh^{-4}\left[\frac{\pi t_1}{2r}\right]. \end{aligned} \tag{A.38}$$

The last part of the three-chromoelectric field insertion at NLO in the EST is $\langle\!\langle \boldsymbol{E}_2^3(t_1)\boldsymbol{E}_1(t_2)\cdot \boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO}$, whose transversal component is simplified and mapped to the string fields as

$$\langle\!\langle \boldsymbol{E}_2^3(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} = \langle\!\langle \boldsymbol{E}_2^3(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO}$$

$$
\begin{aligned}
&- \langle\!\langle \boldsymbol{E}_2^3(t_1) \rangle\!\rangle^{\rm NLO} \langle\!\langle \boldsymbol{E}_1^l(t_2) \boldsymbol{E}_1^l(0) \rangle\!\rangle^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_2^3(t_1) \rangle\!\rangle^{\rm LO} \langle\!\langle \boldsymbol{E}_1^l(t_2) \boldsymbol{E}_1^l(0) \rangle\!\rangle^{\rm NLO} \\
=&\, \overline{\Lambda''}^2 \Lambda^4 \Big[\langle (\partial\xi_2)^2(t_1) \partial_z \xi_1^l(t_2) \partial_z \xi_1^l(0) \rangle \\
&- \langle (\partial\xi_2)^2(t_1) \rangle \langle \partial_z \xi_2^l(t_2) \partial_z \xi_2^l(0) \rangle \Big] \\
=&- 2\overline{\Lambda''}^2 \Lambda^4 \langle \partial_z \xi_2^m(t_1) \partial_z \xi_1^l(t_2) \rangle \langle \partial_z \xi_2^m(t_1) \partial_z \xi_1^l(0) \rangle \,, \qquad \text{(A.39)}
\end{aligned}
$$

and the Wick rotation gives

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_2^3(-it_1) \boldsymbol{E}_1^l(-it_2) \boldsymbol{E}_1^l(0) \rangle\!\rangle_c^{\rm NLO} =& - 4\overline{\Lambda''}^2 \Lambda^4 \partial_{z_1} \partial_{z_2} G(t_1, t_2; z_1, z_2)|_{z_2=r/2}^{z_1=-r/2} \\
&\times \partial_{z_1} \partial_{z_3} G(t_1, t_3; z_1, z_3)|_{-z_1=z_3=r/2}^{t_3=0} \,. \qquad \text{(A.40)}
\end{aligned}
$$

On the other hand, the longitudinal part is decomposed and mapped to the string variables

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_2^3(t_1) \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_2^3(t_1) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\rm NLO} \langle\!\langle \boldsymbol{E}_1^3(t_2) \rangle\!\rangle^{\rm LO} \\
=& 2\Lambda''^2 \overline{\Lambda}^4 \langle \partial_z \xi_2^l(t_1) \partial_z \xi_1^m(0) \rangle \langle \partial_z \xi_2^l(t_1) \partial_z \xi_1^m(0) \rangle \,, \qquad \text{(A.41)}
\end{aligned}
$$

and its Wick rotation is given by

$$
\langle\!\langle \boldsymbol{E}_2^3(-it_1) \boldsymbol{E}_1^3(-it_2) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\rm NLO} = 4\Lambda''^2 \overline{\Lambda}^4 \left[\partial_{z_1} \partial_{z_3} G(t_1, t_3; z_1, z_3)|_{-z_1=z_3=r/2}^{t_3=0} \right]^2 . \qquad \text{(A.42)}
$$

In fact, it turns out that this contribution is identical to the previous part:

$$
\langle\!\langle \boldsymbol{E}_2^3(t_1) \boldsymbol{E}_1(t_2) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle_c^{\rm NLO} = \langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_2(t_2) \cdot \boldsymbol{E}_2(0) \rangle\!\rangle_c^{\rm NLO} . \qquad \text{(A.43)}
$$

Finally, the four-chromoelectric field insertion is decomposed into the four components

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1(t_1) \cdot \boldsymbol{E}_1(t_2) \boldsymbol{E}_1(t_3) \cdot \boldsymbol{E}_1(0) \rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^l(t_3) \boldsymbol{E}_1^l(0) \rangle\!\rangle_c^{\rm NLO} \\
&+ \langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^3(t_3) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\rm NLO} \\
&+ \langle\!\langle \boldsymbol{E}_1^l(t_1) \boldsymbol{E}_1^l(t_2) \boldsymbol{E}_1^m(t_3) \boldsymbol{E}_1^m(0) \rangle\!\rangle_c^{\rm NLO} \\
&+ \langle\!\langle \boldsymbol{E}_1^l(t_1) \boldsymbol{E}_1^l(t_2) \boldsymbol{E}_1^3(t_3) \boldsymbol{E}_1^3(0) \rangle\!\rangle_c^{\rm NLO} . \qquad \text{(A.44)}
\end{aligned}
$$

Let us compute each term on the right-hand side of Eq. (A.44), one by one. The first term is further decomposed:

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^l(t_3) \boldsymbol{E}_1^l(0) \rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1) \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^l(t_3) \boldsymbol{E}_1^l(0) \rangle\!\rangle^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1) \rangle\!\rangle^{\rm NLO} \langle\!\langle \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^l(t_3) \boldsymbol{E}_1^l(0) \rangle\!\rangle_c^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1) \rangle\!\rangle^{\rm LO} \langle\!\langle \boldsymbol{E}_1^3(t_2) \boldsymbol{E}_1^l(t_3) \boldsymbol{E}_1^l(0) \rangle\!\rangle_c^{\rm NLO}
\end{aligned}
$$

$$
\begin{aligned}
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm LO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm LO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle_c^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm LO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle_c^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO}\,,
\end{aligned}
\tag{A.45}
$$

in which the fifth line of Eq. (A.45) vanishes due to the definition of the connected part of the Wilson loop expectation value, and its corresponding EST mapping turns out to be trivial:

$$
\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm LO} = 0\,. \tag{A.46}
$$

After decomposing the second and third lines of Eq. (A.45), the decomposition of the left hand side of Eq. (A.45) is then simplified and mapped to the string variables by

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^l(t_3)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^l(t_3)\boldsymbol{E}_1^l(0)\rangle\!\rangle^{\rm NLO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\rm LO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\rm NLO}\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm LO}\\
=& \overline{\Lambda''}^2\Lambda''^2\Lambda^4\langle(\partial\xi_1)^2(t_1)\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)\rangle\\
&+ \Lambda''^2\overline{\Lambda''}^2\Lambda^4\langle(\partial\xi_1)^2\,(t_2)\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)\rangle\\
&+ \Lambda''^4\overline{\Lambda}^2\Lambda^2\langle\partial_z\xi_1^l(t_3)\,(\partial\xi_1)^2\,(t_3)\partial_z\xi_1^l(0)\rangle\\
&+ \Lambda''^4\Lambda^2\overline{\Lambda}^2\langle\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)(\partial\xi_1)^2(0)\rangle\\
&- \overline{\Lambda''}^2\Lambda''\Lambda^4\langle(\partial\xi_1)^2(t_1)\rangle\langle\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)\rangle\\
&- \Lambda''^2\overline{\Lambda''}^2\Lambda^4\langle(\partial\xi_1)^2(t_2)\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)\rangle\\
&- \Lambda''^4\overline{\Lambda}^2\Lambda^2\langle\partial_z\xi_1^l(t_3)\,(\partial\xi_1)^2\,(t_3)\partial_z\xi_1^l(0)\rangle\\
&- \Lambda''^4\overline{\Lambda}^2\Lambda^2\langle\partial_z\xi_1^l(t_3)\partial_z\xi_1^l(0)(\partial\xi_1)^2(0)\rangle\\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm LO}\,.
\end{aligned}
\tag{A.47}
$$

The last line of the second equality of Eq. (A.47) is of order $\sigma^{-3}r^{-6}$ in accordance with the EST power counting, so it may be neglected at NLO. After several steps of calculation, the above expression becomes

$$
\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^l(t_3)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO} = -\,2\overline{\Lambda''}^2\Lambda''^2\Lambda^4\langle\partial_z\xi_1^m(t_1)\partial_z\xi_1^l(t_3)\rangle\langle\partial_z\xi_1^m(t_1)\partial_z\xi_1^l(0)\rangle\,, \tag{A.48}
$$

and then its Wick rotation gives the following analytic expression:

$$
\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1^3(-it_2)\boldsymbol{E}_1^l(-it_3)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\rm NLO}
$$

$$= -4\overline{\Lambda''}^2\Lambda''^2\Lambda^4\partial_{z_1}\partial_{z_3}G(t_1,t_3;z_1,z_3)|^{z_1=r/2}_{z_3=r/2}\times\partial_{z_1}\partial_{z_4}G(t_1,t_4,z_1,z_4)|^{t_4=0}_{z_1=z_4=r/2}$$
$$= -\frac{\pi^2\overline{\Lambda''}^2\Lambda''^2\Lambda^4}{4\sigma^2r^4}\sinh^{-2}\left[\frac{\pi t_1}{2r}\right]\sinh^{-2}\left[\frac{\pi(t_1-t_3)}{2r}\right]. \quad \text{(A.49)}$$

The second term on the right-hand side of Eq. (A.44) is decomposed as

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}}\\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}\\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}}\\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}\\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}}\\ &- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_3)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}}\\ &- \left[\langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^3(t_3)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle\right]^{\mathrm{NLO}},\end{aligned} \quad \text{(A.50)}$$

where the second line is further decomposed as

$$\langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} = \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_3)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}. \quad \text{(A.51)}$$

Likewise, the third line of Eq. (A.50) is decomposed as

$$\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}} = \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{LO}}, \quad \text{(A.52)}$$

so that Eq. (A.50) is simplified and mapped onto the string variables by

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& .\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(t_3)\rangle\!\rangle^{\mathrm{LO}}\\ =& 2\Lambda''^4\overline{\Lambda}^4\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^m(0)\rangle\langle\partial_z\xi_1^l(t_1)\xi_1^m(0)\rangle,\end{aligned} \quad \text{(A.53)}$$

and we obtain the final expression after taking the Wick rotation:

$$\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1^3(-it_2)\boldsymbol{E}_1^3(-it_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} = \frac{\pi^2\Lambda''^4\overline{\Lambda}^4}{4\sigma^2r^4}\sinh^{-4}\left(\frac{\pi t_1}{2r}\right). \quad \text{(A.54)}$$

As we move onto the third term on the right-hand side of the Eq. (A.44), its mapping to the EST corresponds to a six-string field correlator instead of four-string field, which implies that this term belongs to the NNLO contribution instead of NLO. Thus, only the fourth term on the right-hand side of Eq. (A.44) remains to be computed. It is decomposed in a similar fashion

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\mathrm{NLO}} =& \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}}\\ &- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle^{\mathrm{LO}}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\mathrm{NLO}}\end{aligned}$$

$$- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^3(t_3)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(0)\rangle\!\rangle^{\rm LO}$$
$$- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle^{\rm NLO}\,, \quad (A.55)$$

in which the last term on the right-hand side belongs to a NNLO contribution according to the EST power counting. Then the string mapping for the rest of the terms is

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_1^3(t_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO} &= \Lambda^4\Lambda''^2\overline{\Lambda''}^2\Big[\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^l(t_2)\,(\partial\xi_1)\,(0)\rangle \\ &\quad -\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^l(t_2)\rangle\langle(\partial\xi_1)^2(0)\rangle\Big]\,, \\ &= -\,2\Lambda^4\Lambda''^2\overline{\Lambda''}^2\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^m(0)\rangle\langle\partial_z\xi_1^l(t_2)\partial_z\xi_1^m(0)\rangle\,,\end{aligned} \quad (A.56)$$

and its Wick rotation gives the analytic expression:

$$\begin{aligned}&\langle\!\langle \boldsymbol{E}_1^l(-it_1)\boldsymbol{E}_1^l(-it_2)\boldsymbol{E}_1^3(-it_3)\boldsymbol{E}_1^3(0)\rangle\!\rangle_c^{\rm NLO} \\ &= -\,4\Lambda^4\Lambda''^2\overline{\Lambda''}^2\partial_{z_1}\partial_{z_4}G(t_1,t_4;z_1,z_4)|^{t_4=0}_{z_1=z_4=r/2}\times\partial_{z_2}\partial_{z_4}G(t_2,t_4;z_2,z_4)|^{t_4=0}_{z_2=z_4=r/2} \\ &= -\,\frac{\pi^2\Lambda^4\Lambda''^2\overline{\Lambda''}^2}{4\sigma^2r^4}\sinh^{-2}\left[\frac{\pi t_1}{2r}\right]\sinh^{-2}\left[\frac{\pi t_2}{2r}\right]\,.\end{aligned} \quad (A.57)$$

Finally, putting all these pieces together, Eqs. (A.49), (A.54), and (A.57), the full expression of the four-chromoelectric field insertion to the Wilson loop expectation value at NLO in the EST is given by

$$\begin{aligned}&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_1(-it_3)\cdot\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO} \\ &= -\,\frac{\pi^2\overline{\Lambda''}^2\Lambda''^2\Lambda^4}{4\sigma^2r^4}\sinh^{-2}\left[\frac{\pi t_1}{2r}\right]\sinh^{-2}\left[\frac{\pi(t_1-t_3)}{2r}\right] \\ &\quad -\,\frac{\pi^2\Lambda^4\Lambda''^2\overline{\Lambda''}^2}{4\sigma^2r^4}\sinh^{-2}\left[\frac{\pi t_1}{2r}\right]\sinh^{-2}\left[\frac{\pi t_2}{2r}\right] \\ &\quad +\,\frac{\pi^2\Lambda''^4\overline{\Lambda}^4}{4\sigma^2r^4}\sinh^{-4}\left[\frac{\pi t_1}{2r}\right]\,.\end{aligned} \quad (A.58)$$

The other part that contributes to the four-chromoelectric field insertion can be decomposed as

$$\begin{aligned}\langle\!\langle \boldsymbol{E}_1(t_1)\cdot\boldsymbol{E}_1(t_2)\boldsymbol{E}_2(t_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\rm NLO} &= \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} \\ &\quad +\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} \\ &\quad +\langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^m(t_3)\boldsymbol{E}_2^m(0)\rangle\!\rangle_c^{\rm NLO} \\ &\quad +\langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO}\,,\end{aligned} \quad (A.59)$$

in which the first term on the right-hand side is further decomposed and mapped to the string fields as follows

$$\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} = \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm NLO}$$

$$
\begin{aligned}
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^l(t_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle^{\rm LO} \\
=& \overline{\Lambda''}^2\Lambda''^2\Lambda^4\Big[\langle(\partial\xi_1)^2(t_1)\partial_z\xi_2^l(t_3)\partial_z\xi_2^l(0)\rangle \\
&- \langle(\partial\xi_1)^2(t_1)\rangle\langle\partial_z\xi_2^l(t_3)\partial_z\xi_2^l(0)\rangle\Big] + \mathcal{O}(\sigma^{-3}r^{-6}) \\
=& -2\overline{\Lambda''}^2\Lambda''^2\Lambda^4\langle\partial_z\xi_1^m(t_1)\partial_z\xi_2^l(t_3)\rangle\langle\partial_z\xi_1^m(t_1)\partial_z\xi_2^l(0)\rangle\,.
\end{aligned}
\tag{A.60}
$$

Taking the Wick rotation, we obtain its analytic expression:

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1^3(-it_2)\boldsymbol{E}_2^l(-it_3)\boldsymbol{E}_2^l(0)\rangle\!\rangle_c^{\rm NLO} \\
=& -4\overline{\Lambda''}^2\Lambda''^2\Lambda^4\partial_{z_1}\partial_{z_3}G(t_1,t_3;z_1,z_3)|_{z_3=-r/2}^{z_1=r/2}\times\partial_{z_1}\partial_{z_4}G(t_1,t_4;z_1,z_4)|_{z_1=-z_4=r/2}^{t_4=0} \\
=& -\frac{\pi^2\overline{\Lambda''}^2\Lambda''^2\Lambda^4}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi(t_1-t_3)}{2r}\right]\,.
\end{aligned}
\tag{A.61}
$$

The second term on the right-hand side of Eq. (A.59) is decomposed and simplified as

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^3(t_3)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm LO} \\
&- \left[\langle\!\langle \boldsymbol{E}_1^3(t_1)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_2^3(t_3)\rangle\!\rangle\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle\right]^{\rm NLO}\,, \\
=& \langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO}\langle\!\langle \boldsymbol{E}_1^3(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^3(t_3)\rangle\!\rangle^{\rm LO}\,,
\end{aligned}
\tag{A.62}
$$

so that its QCD-to-EST mapping is given by

$$
\langle\!\langle \boldsymbol{E}_1^3(t_1)\boldsymbol{E}_1^3(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} = 2\Lambda''^4\overline{\Lambda}^4\langle\partial_z\xi_1^l(t_1)\partial_z\xi_2^m(0)\rangle\langle\partial_z\xi_1^l(t_1)\partial_z\xi_2^m(0)\rangle\,.
\tag{A.63}
$$

Taking the Wick rotation, we obtain the final expression:

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^3(-it_1)\boldsymbol{E}_1^3(-it_2)\boldsymbol{E}_2^3(-it_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} =& 4\Lambda''^4\overline{\Lambda}^4\left[\partial_{z_1}\partial_{z_4}G(t_1,t_4;z_1,z_4)|_{z_1=-z_4=r/2}^{t_4=0}\right]^2 \\
=& \frac{\pi^2\Lambda''^4\overline{\Lambda}^4}{4\sigma^2r^4}\cosh^{-4}\left(\frac{\pi t_1}{2r}\right)\,.
\end{aligned}
\tag{A.64}
$$

The third term on the right-hand side of Eq. (A.59), which is purely of longitudinal components, is counted as a NNLO term in the EST, so it remains for us to compute the last term of Eq. (A.59). It is decomposed and mapped as

$$
\begin{aligned}
\langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^3(t_3)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm NLO} \\
&- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\boldsymbol{E}_2^3(t_3)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm LO} \\
&- \langle\!\langle \boldsymbol{E}_1^l(t_1)\boldsymbol{E}_1^l(t_2)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{E}_2^3(t_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle^{\rm NLO} \\
=& \Lambda^4\Lambda''^2\overline{\Lambda''}^2\Big[\langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^l(t_2)(\partial\xi_2)^2(0)\rangle \\
&- \langle\partial_z\xi_1^l(t_1)\partial_z\xi_1^l(t_2)\rangle\langle(\partial\xi_2)^2(0)\rangle\Big] + \mathcal{O}(\sigma^{-3}r^{-6}) \\
=& -2\Lambda^4\Lambda''^2\overline{\Lambda''}^2\langle\partial_z\xi_1^l(t_1)\partial_z\xi_2^m(0)\rangle\langle\partial_z\xi_1^l(t_2)\partial_z\xi_2^m(0)\rangle \\
&+ \mathcal{O}(\sigma^{-3}r^{-6})\,,
\end{aligned}
\tag{A.65}
$$

and after taking the Wick rotation, its analytic expression is:

$$
\langle\!\langle \boldsymbol{E}_1^l(-it_1)\boldsymbol{E}_1^l(-it_2)\boldsymbol{E}_2^3(-it_3)\boldsymbol{E}_2^3(0)\rangle\!\rangle_c^{\rm NLO} = -\frac{\pi^2\Lambda^4\Lambda''^2\overline{\Lambda''}^2}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi t_2}{2r}\right]. \tag{A.66}
$$

Therefore, this four-gauge field insertions at NLO in the EST is given by the summation of Eqs. (A.61), (A.64), and (A.66)

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{E}_1(-it_1)\cdot\boldsymbol{E}_1(-it_2)\boldsymbol{E}_2(-it_3)\cdot\boldsymbol{E}_2(0)\rangle\!\rangle_c^{\rm NLO} \\
=& -\frac{\pi^2\overline{\Lambda''}^2\Lambda''^2\Lambda^4}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi(t_1-t_3)}{2r}\right] \\
&-\frac{\pi^2\Lambda^4\Lambda''^2\overline{\Lambda''}^2}{4\sigma^2r^4}\cosh^{-2}\left[\frac{\pi t_1}{2r}\right]\cosh^{-2}\left[\frac{\pi t_2}{2r}\right] \\
&+\frac{\pi^2\Lambda''^4\overline{\Lambda}^4}{4\sigma^2r^4}\cosh^{-4}\left[\frac{\pi t_1}{2r}\right].
\end{aligned}
\tag{A.67}
$$

A.2.2 Chromomagnetic field insertions

Two-chromomagnetic field insertions to the Wilson loop expectation value at NLO in the EST is decomposed into two parts: transversal and longitudinal components. The transversal component is given by

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c^{\rm NLO} =& \langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_2^l(0)\rangle\!\rangle^{\rm NLO} - \langle\!\langle \boldsymbol{B}_1^l(t)\rangle\!\rangle^{\rm LO}\langle\!\langle \boldsymbol{B}_2^l(0)\rangle\!\rangle^{\rm NLO} \\
&-\langle\!\langle \boldsymbol{B}_1^l(t)\rangle\!\rangle^{\rm NLO}\langle\!\langle \boldsymbol{B}_2^l(0)\rangle\!\rangle^{\rm LO}\,, \\
=& \langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_2^l(0)\rangle\!\rangle^{\rm NLO}\,,
\end{aligned}
\tag{A.68}
$$

where the second equality is due to the Gaussianity of the string action. This expression is then mapped to the string variables as

$$
\langle\!\langle \boldsymbol{B}_1^l(t)\boldsymbol{B}_2^l(0)\rangle\!\rangle^{\rm NLO} = -\Lambda'\overline{\Lambda'}\,[\langle\partial_t\partial_z\xi_1^m(t)\partial_a\xi_1^p(t)\partial^a\xi_1^p(t)\partial_t\partial_z\xi_2^m(0)\rangle
$$

$$
\begin{aligned}
&+\langle \partial_t\partial_z\xi_1^m(t)\partial_t\partial_z\xi_2^m(0)\partial_a\xi_2^p(0)\partial^a\xi_2^p(0)\rangle] \\
=&\Lambda'\overline{\Lambda'}\,[2\langle\partial_t\partial_z\xi_1^m(t)\partial_z\xi_1^p(t)\rangle\langle\partial_z\xi_1^p(t)\partial_t\partial_z\xi_2^m(0)\rangle \\
&+\langle\partial_t\partial_z\xi_1^m(t)\partial_t\partial_z\xi_2^m(0)\rangle\langle\partial_z\xi_1^p(t)\partial_z\xi_1^p(t)\rangle \\
&+\langle\partial_t\partial_z\xi_1^m(t)\partial_t\partial_z\xi_2^m(0)\rangle\langle\partial_z\xi_2^p(0)\partial_z\xi_2^p(0)\rangle \\
&+2\langle\partial_t\partial_z\xi_1^m(t)\partial_t\xi_2^p(0)\rangle\langle\partial_t\partial_z\xi_2^m(0)\partial_z\xi_2^p(0)\rangle] \,,
\end{aligned} \tag{A.69}
$$

and after taking the Wick rotation, we obtain the final expression:

$$
\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c^{\rm NLO} = \frac{\pi^4\Lambda'\overline{\Lambda'}}{12\sigma^2r^6}\cosh^{-4}\left(\frac{\pi t}{2r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)-2\right]. \tag{A.70}
$$

On the other hand, the longitudinal component of the two-chromomagnetic field insertions is counted as NNLO according to the EST power counting

$$
\langle\!\langle \boldsymbol{B}_1^3(-it)\boldsymbol{B}_1^3(0)\rangle\!\rangle_c^{\rm NLO} \sim \mathcal{O}(\sigma^2r^{-6})\,. \tag{A.71}
$$

Thus, we only include Eq. (A.70) for the calculation of V_r at NLO.

A.2.3 Chromoelectric and chromomagnetic field insertions

As for the calculation of the spin-orbit potentials, Eqs. (4.26) and (4.27) at NLO, we need to compute a cross product between a chromoelectric and a chromomagnetic field inserted into the Wilson loop expectation value:

$$
\begin{aligned}
\mathbf{r}\cdot\langle\!\langle \boldsymbol{B}_1(t)\times\boldsymbol{E}_1(0)\rangle\!\rangle_c^{\rm NLO} &= \epsilon^{ijk}\mathbf{r}^i\langle\!\langle \boldsymbol{B}_1^j(t)\boldsymbol{E}_1^k(0)\rangle\!\rangle_c^{\rm NLO}\,, \\
&= r\left[\langle\!\langle \boldsymbol{B}_1^1(t)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\rm NLO} - \langle\!\langle \boldsymbol{B}_1^2(t)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\rm NLO}\right],
\end{aligned} \tag{A.72}
$$

from which it is clear that we need to compute $\langle\!\langle \boldsymbol{B}_1^i(t)\boldsymbol{E}_1^j(0)\rangle\!\rangle_c^{\rm NLO}$, for $i\neq j$. It is re-expressed and mapped onto the string variables as

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^i(t)\boldsymbol{E}_1^j(0)\rangle\!\rangle_c^{\rm NLO} =&\langle\!\langle \boldsymbol{B}_1^i(t)\boldsymbol{E}_1^j(0)\rangle\!\rangle^{\rm NLO} \\
=&\overline{\Lambda'}\Lambda^2\langle\epsilon^{ik}\partial_t\partial_z\xi_1^k(t)\partial_a\xi_1^l(t)\partial^a\xi_1^l(t)\partial_z\xi_1^j(0)\rangle \\
&+\Lambda'\overline{\Lambda}^2\langle\epsilon^{ik}\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^j(0)\partial_a\xi_1^l(0)\partial^a\xi_1^l(0)\rangle \\
=&-\overline{\Lambda'}\Lambda^2\epsilon^{ik}\Big[2\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^l(t)\rangle\langle\partial_z\xi_1^l(t)\partial_z\xi_1^j(0)\rangle \\
&+\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^j(0)\rangle\langle\partial_z\xi_1^l(t)\partial_z\xi_1^l(t)\rangle\Big] \\
&-\Lambda'\overline{\Lambda}^2\epsilon^{ik}\Big[\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^j(0)\rangle\langle\partial_z\rangle\langle\partial_z\xi_1^l(0)\partial_z\xi_1^l(0)\rangle \\
&+2\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^l(0)\rangle\langle\partial_z\xi_1^j(0)\partial_z\xi_1^l(0)\rangle\Big]\,,
\end{aligned} \tag{A.73}
$$

where the first equality is due to the Gaussianity of the EST action. After performing the Wick rotation, we obtain the following expression

$$
\langle\!\langle \boldsymbol{B}_1^i(-it)\boldsymbol{E}_1^j(0)\rangle\!\rangle_c^{\rm NLO} = -2i\overline{\Lambda'}\Lambda^2\epsilon^{ij}\left[\partial_t\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'}\times\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0}\right.
$$

$$
\begin{aligned}
&+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\Big]\\
&-2i\Lambda'\overline{\Lambda}^2\epsilon^{ij}\Big[\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=r/2}\\
&+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=r/2}\Big]\,,
\end{aligned}
\tag{A.74}
$$

from which, it is clear that

$$
\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\text{NLO}} = -\langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\text{NLO}}\,. \tag{A.75}
$$

Thus, the Wilson loop expectation value for the spin-orbit potential is computed as

$$
\begin{aligned}
&\langle\!\langle \boldsymbol{B}_1^1(t)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\text{NLO}} - \langle\!\langle \boldsymbol{B}_1^2(t)\boldsymbol{E}_1^1(0)\rangle\!\rangle_c^{\text{NLO}} = 2\langle\!\langle \boldsymbol{B}_1^1(t)\boldsymbol{E}_1^2(0)\rangle\!\rangle_c^{\text{NLO}}\\
=&-4i\Lambda^2\overline{\Lambda'}\Big[\partial_t\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=r/2}\Big]\\
&-8i\Lambda'\overline{\Lambda}^2\Big[\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=r/2}\Big]\\
=&-\frac{i\pi^3\Lambda^2\overline{\Lambda'}}{6\sigma^2r^5}\sinh\left(\frac{\pi t}{r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)-1\right]^{-2}-\frac{i\pi^3\Lambda'\overline{\Lambda}^2}{3\sigma^2r^5}\sinh\left(\frac{\pi t}{r}\right)\left[\cosh\left(\frac{\pi t}{r}\right)-1\right]^{-2}.
\end{aligned}
\tag{A.76}
$$

Likewise, another cross product between a chromomagnetic and a chromoelectric field which contributes to the potential is mapped to the string variables as

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^i(t)\boldsymbol{E}_2^j(0)\rangle\!\rangle_c^{\text{NLO}} =&\langle\!\langle \boldsymbol{B}_1^i(t)\boldsymbol{E}_2^j(0)\rangle\!\rangle^{\text{NLO}}\,,\\
=&\overline{\Lambda'}\Lambda^2\epsilon^{ik}\langle\partial_t\partial_z\xi_1^k(t)\partial_a\xi_1^l(t)\partial^a\xi_1^l(t)\partial_z\xi_2^j(0)\rangle\\
&+\Lambda'\overline{\Lambda}^2\epsilon^{ik}\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_2^j(0)\partial_a\xi_2^l(0)\partial^a\xi_2^l(0)\rangle\,,\\
=&-\overline{\Lambda'}\Lambda^2\epsilon^{ik}\Big[2\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_1^l(t)\rangle\langle\partial_z\xi_1^l(t)\partial_z\xi_2^j(0)\rangle\\
&+\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_2^j(0)\rangle\langle\partial_z\xi_1^l(t)\partial_z\xi_1^l(t)\rangle\Big]\\
&-\Lambda'\overline{\Lambda}^2\epsilon^{ik}\Big[\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_2^j(0)\rangle\langle\partial_z\xi_2^l(0)\partial_z\xi_2^l(0)\rangle\\
&+2\langle\partial_t\partial_z\xi_1^k(t)\partial_z\xi_2^l(0)\rangle\langle\partial_z\xi_2^j(0)\partial_z\xi_2^l(0)\rangle\Big]\,,
\end{aligned}
\tag{A.77}
$$

and after performing the Wick rotation, its analytic expression is given by

$$
\begin{aligned}
\langle\!\langle \boldsymbol{B}_1^i(-it)\boldsymbol{E}_2^j(0)\rangle\!\rangle_c^{\text{NLO}} =&-2i\Lambda^2\overline{\Lambda'}\epsilon^{ij}\Big[\partial_t\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\times\partial_z\partial_{z'}G|^{t'=0}_{z=-z'=r/2}\\
&+\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=-z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'}_{z=z'=r/2}\Big]\\
&-4i\Lambda'\overline{\Lambda}^2\epsilon^{ij}\Big[\partial_t\partial_z\partial_{z'}G|^{t'=0}_{z=-z'=r/2}\times\partial_z\partial_{z'}G|^{t=t'=0}_{z=z'=-r/2}\Big]\,.
\end{aligned}
\tag{A.78}
$$

Therefore, the final expression of the cross product inserted into the Wilson loop expectation value is given by

$$
\langle\!\langle \boldsymbol{B}_1^1(-it)\boldsymbol{E}_2^2(0)\rangle\!\rangle_c^{\text{NLO}} - \langle\!\langle \boldsymbol{B}_1^2(-it)\boldsymbol{E}_2^1(0)\rangle\!\rangle_c^{\text{NLO}} = \frac{i\pi^3\Lambda^2\overline{\Lambda'}}{6\sigma^2r^5}\sinh\left(\frac{\pi t}{r}\right)\left[1+\cosh\left(\frac{\pi t}{r}\right)\right]^{-2}
$$

$$
+\frac{i\pi^3\Lambda'\overline{\Lambda}^2}{3\sigma^2 r^5}\sinh\left(\frac{\pi t}{r}\right)\left[1+\cosh\left(\frac{\pi t}{r}\right)\right]^{-2}. \tag{A.79}
$$

A.3 Hard cut-off regularization

In this section we use a hard cut-off regularization scheme for the divergent expression of the gauge field insertions presented in Sec. 5.4.3. Eventually, the expressions after applying this regularization will coincide with the result after using the zeta function and dimensional regularization. Since the EST is a UV-divergent effective description, we need to explicitly implement the cut-off when we calculate the relevant gaug field insertions. The hierarchy of the scales is already given in Eq. (5.11).

First of all, as was presented in Sec. 5.4.3 as well as in Appendix A.2, a transversal component of the two chromoelectric field insertion is given in terms of the EST Green's function

$$
\langle\!\langle \boldsymbol{E}_1^l(-it)\boldsymbol{E}_1^l(0)\rangle\!\rangle_c^{\mathrm{NLO}} = 16\Lambda^2\overline{\Lambda}^2\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'}\times\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=0}, \tag{A.80}
$$

where G is given by Eq. (5.33) without the tensor δ^{lm}. This expression diverges due to $\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'}$. In other words, the theory does not have an arbitrarily high enough resolution to probe the physics evaluated at the same spacetime points. As the Fourier transform of the time variable corresponds to the energy, we can express this into a mathematical way. Let us introduce an infinitesimal parameter $\epsilon > 0$, which is of mass dimension minus one, and redefine the time evaluation of the Green's function by

$$
t' = t+\epsilon\,, \tag{A.81}
$$

so that we can define

$$
\partial_z\partial_{z'}G|_{z=z'=r/2}^{t=t'} = \lim_{\epsilon\to 0}\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=t+\epsilon}. \tag{A.82}
$$

As we substitute t' by $t+\epsilon$ when evaluating the two partial derivatives on the Green's function, we obtain

$$
\lim_{\epsilon\to 0}\partial_z\partial_{z'}G|_{z=z'=r/2}^{t'=t+\epsilon} = \lim_{\epsilon\to 0}\left\{-\frac{\pi}{12\sigma r^2}+\frac{1}{\pi\sigma\epsilon^2}+\frac{\pi^3\epsilon^2}{240\sigma r^4}+\mathcal{O}(\epsilon^3)\right\}. \tag{A.83}
$$

It is clear, by comparing this expression to the one obtained from the zeta-function regularization, that the finite part coincides with Eq. (5.139). Here the divergent part is given by the inverse square of the regulator ϵ. Note that this regulator is not arbitrarily small as the theory is UV-divergent, and due to the given hierarchy of scales in the EST, the size of this regulator is restricted by

$$
\epsilon > \Lambda_{\mathrm{QCD}}^{-1}. \tag{A.84}
$$

One might consider regulating the spatial coordinate z instead of time, since the EST power counting dictates that ∂_t and ∂_r are of the same order ($\sim 1/r$). However this “alternative” regularization would contradict the Dirichlet boundary conditions of the theory, in which the two ends of the string are supposed to coincide with the position of the heavy quark-antiquark pair. This was assumed from the onset of the theory construction. Thus, one can only implement the regulator ϵ into the time variable.

As for the calculation of the transversal component of the two-chromomagnetic field insertion, $\langle\!\langle \boldsymbol{B}_1^l(-it)\boldsymbol{B}_2^l(0)\rangle\!\rangle_c^{\mathrm{NLO}}$, divergences arise from

$$\begin{aligned} &\lim_{t'\to t} \partial_z\partial_{z'} G(t,t';z,z')|_{z=z'=r/2} = \infty\,, \\ &\lim_{t'\to t} \partial_t\partial_z\partial_{z'} G(t,t';z,z')|_{z=z'=r/2} = \infty\,. \end{aligned} \tag{A.85}$$

As the first one was already regulated, let us analyze the second expression. Implementation of the ϵ regulator gives the following expression

$$\lim_{\epsilon\to 0} \partial_t\partial_z\partial_{z'} G(t,t';z,z')|_{z=z'=r/2}^{t'=t+\epsilon} = \frac{2}{\pi\sigma^2\epsilon^3} - \frac{\pi^3\epsilon}{120\sigma r^4} + \frac{\pi^5\epsilon^3}{1512\sigma r^6} + \mathcal{O}(\epsilon^4)\,, \tag{A.86}$$

in which the finite part is zero. This coincides with the result from the zeta function regularization, Eq. (5.148).

Lastly, the divergence coming from two time derivatives on the Green’s function

$$\lim_{t'\to t} \partial_t\partial_{t'} G|_{z=z'=r/2} \to \infty\,, \tag{A.87}$$

cannot be regularized by the ϵ regulator. Let us look at the Fourier integral from Eq. (5.27)

$$\lim_{\epsilon\to 0} \partial_t\partial_{t'} G|_{z=z'=r/2}^{t'=t+\epsilon} \propto \int_{-\infty}^{\infty} dk\, \frac{k^2}{k^2+a} e^{-ik\epsilon}\,, \tag{A.88}$$

where $a = (n\pi/r)^2$. If we take the Taylor expansion of the exponential factor of the integrand (since ϵ is small), the expression is rewritten by

$$\int_{-\infty}^{\infty} dk \left(\frac{k^2}{k^2+a} - \frac{ik^3\epsilon}{k^2+a} + \dots \right)\,, \tag{A.89}$$

and the integral of each term will diverge regardless of the size of the ϵ regulator. Thus, it is necessary to use dimensional regularization for this type of divergent expression, which is given in Eq. (5.151).

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