# A SIMPLE NON-PARAMETRIC GOODNESS-OF-FIT TEST FOR ELLIPTICAL COPULAS

MIRIAM JASER, STEPHAN HAUG AND ALEKSEY MIN

*Technical University of Munich*

*Abstract:* In this paper, we propose a simple non-parametric goodness-of-fit test for elliptical copulas of any dimension. It is based on the equality of Kendall's tau and Blomqvist's beta for all bivariate margins. Nominal level and power of the proposed test are investigated in a Monte Carlo study. An empirical application illustrates our goodness-of-fit test at work.

*Key words and phrases:* Blomqvist's beta, elliptical copulas, goodness-of-fit test, Kendall's tau

## 1 Introduction

Nowadays, copulas are a standard tool for modeling multivariate dependence. There exist many copula classes such as Archimedean, elliptical and Marshall-Olkin copulas (see e.g. Scherer (2012)) and the choice of the right copula class is crucial for an accurate multivariate data analysis. Therefore, goodness-of-fit tests for copulas have been an objective of active research in recent years, see e.g. Genest et al. (2009), Berg (2009) and the overview Fermanian (2013). In financial applications, elliptical copulas are commonly used to capture the dependence structure.

This paper is concerned with the construction of a simple non-parametric goodness-of-fit test to examine whether the underlying dependence structure follows some elliptical copula of any dimension $d$. Therefore, the null hypothesis that the unknown copula $C$ of the given data belongs to the class of elliptical copulas $\mathcal{C}^{ellipt}$, i.e.

$$H_0 : C \in \mathcal{C}^{ellipt} ,$$

is tested against the alternative

$$H_1 : C \notin \mathcal{C}^{ellipt} .$$

In case of bivariate elliptical copulas, which are symmetric and radially symmetric, one could first use the tests of Genest et al. (2012) and Genest and Nešlehová (2014) to statistically confirm both symmetry properties. If at least one of these statistical tests is rejected, then the bivariate copula of the underlying data cannot be elliptical. Otherwise, a new statistical test is needed to identify bivariate elliptical copulas within symmetric and radially symmetric copulas. In case of multivariate elliptical copulas, one could test only for radial symmetry.

Li and Peng (2009) construct a goodness-of-fit test for the tail copula of a $d$-dimensional distribution, whose dependence structure is expressed by an elliptical copula. Klüppelberg et al. (2008) derive, in Lemma 1, the parametric form of the tail copula of elliptical distributions and argue, in Section 2, that it depends only on the underlying elliptical copula and is independent of the marginal distributions. Hence, the test in Li and Peng (2009) can also be seen as a goodness-of-fit test for elliptical copulas. To the best of our knowledge, this is the only goodness-of-fit test for elliptical copulas of any dimension $d$. However, this test utilizes the tail dependence concept and therefore, the class of copulas for the null hypothesis has to be restricted to elliptical copulas with positive tail dependence. This tail dependence assumption discards for example the Gaussian copula from the null hypothesis and consequently shrinks the class of elliptical copulas under consideration. Furthermore, the test is based on the upper order statistics of the data and therefore has to deal with the difficulties of extreme value statistics. We propose a new simple non-parametric goodness-of-fit test, which takes into account the dependence structure of the whole data set. In particular, it is based on the equality of Kendall's tau and Blomqvist's beta for all bivariate margins of meta-elliptical distributions resulting from Fang et al. (2002) and Schmid and Schmidt (2007).

Elliptical copulas are specified by their generator function and parameters. If the choice of the generator function is fixed, many general goodness-of-fit tests can be used to test whether an underlying copula belongs to this specified subclass of elliptical copulas. However, the choice of the generator function is not an obvious and simple task. Our goodness-of-fit test does not require the knowledge of the generator function and in this sense, it is general. Moreover, it is simple since its critical values are directly computed from an asymptotic $\chi^2$-distribution of a test statistic.

The rest of the paper is organized as follows. Section 2 briefly discusses elliptical copulas as well as Kendall's tau and Blomqvist's beta. Section 3 presents our test statistic and its limiting $\chi^2$-distribution under the assumption that the copula data comes from an elliptical family. The description of the simulation study and numerical results on the nominal level of the test as well as on its power are given in Section 4. Section 5 deals with an application of our goodness-of-fit test to real data. Section 6 concludes and Appendix A contains the proof of the asymptotic $\chi^2$-distribution of our test statistic.

# 2 Basics

## 2.1 Elliptical copulas

Here and in the sequel, we will just consider distribution functions with continuous margins. A copula is a cumulative distribution function over the unit square $[0,1]^d$ with uniform margins. One of the most prominent parametric classes of copulas are elliptical copulas. They are implicit copulas, which do not possess a simple closed-form analytic expression. More precisely, elliptical copulas are derived from multivariate elliptical distribution functions with the help of Sklar's theorem from Sklar (1959).

**Theorem 2.1.** ***(Sklar's theorem)***
*Let $H$ be a cumulative distribution function on $\mathbb{R}^d$ with continuous margins $F_1, \ldots, F_d$. Then there exists a unique copula $C : [0,1]^d \to [0,1]$ such that for all $\mathbf{x} \in \mathbb{R}^d$, it holds that*

$$H(\mathbf{x}) = C\left(F_1(x_1), \ldots, F_d(x_d)\right) .$$

In particular, Sklar's theorem allows to treat margins and the copula separately resulting in two independent and simpler problems. Further, Sklar's theorem provides an universal construction framework for copulas. Without loss of generality, let $F_i^-$ be the generalized inverses of $F_i$, $i \in \{1, \ldots, d\}$. Then, the copula $C(\mathbf{u})$ of $H$ for any $\mathbf{u} \in [0,1]^d$ is given by

$$C(\mathbf{u}) = H\left(F_1^-(u_1), \ldots, F_d^-(u_d)\right) .$$

Further, we denote by $C_{k\ell}$ the marginal copula of the $k$-th and $\ell$-th component with $k, \ell \in \{1, \ldots, d\}$ and $k \neq \ell$.

We first introduce elliptical distributions, from which elliptical copulas are derived. Our exposition follows Chapter 2 in Fang et al. (1990) and is based on spherical distributions that stay invariant under orthogonal transformations of the underlying random vectors. Spherical distributions are an important sub-class of elliptical distributions.

**Definition 2.2.** ***(Elliptical distribution)***
*Let $\mathbb{S}_d$ denote the space of all symmetric $d \times d$ matrices. A random vector $\boldsymbol{X} \in \mathbb{R}^d$ is said to have an (non-degenerate) elliptical distribution with parameters $\boldsymbol{\mu} \in \mathbb{R}^d$ and $\boldsymbol{\Sigma} = (\sigma_{k\ell})_{k,\ell \in \{1,\dots,d\}} \in \mathbb{S}_d$, if*

$$\boldsymbol{X} = \boldsymbol{\mu} + \mathcal{A}\boldsymbol{Y},$$

*where $\boldsymbol{Y}$ has a $m$-dimensional spherical distribution and $\mathcal{A}$ is a $d \times m$ matrix such that $\mathcal{A}\mathcal{A}^\top = \boldsymbol{\Sigma}$ with $\operatorname{rank}(\boldsymbol{\Sigma}) = m$.*

Thus, elliptical distributions are defined as the class of affine transformations of spherical distributions. A bivariate elliptically distributed random vector $\boldsymbol{X}$ resulting from the application of the linear transformation $\mathcal{A}$ to the spherically distributed random vector $\boldsymbol{Y}$ has elliptically contoured density level surfaces. This explains the name of elliptical distributions. Definition 2.2 is the stochastic representation of elliptical distributions. Note that elliptical distributions can alternatively be defined through their generator function. For further details about elliptical distributions and the definition of spherical distributions we refer to Fang et al. (1990).

Since Sklar's theorem determines the copula of multivariate distributions with continuous margins in an unique way, elliptical copulas are defined as follows.

**Definition 2.3.** ***(Elliptical copula)***
*Elliptical copulas are the copulas of elliptical distributions.*

Consequently, an elliptical copula $C$ is defined as the copula of the underlying elliptical distribution $H$ and is typically not available in closed form. The two most popular elliptical copulas are the Gaussian and the $t$-copula. Distributions with an elliptical copula are called (meta)-elliptical distributions (see Fang et al. (2002)). These distributions are fully specified through the matrix $\mathcal{R} = (\rho_{k\ell})_{k,\ell \in \{1,\dots,d\}}$ $:= \left(\sigma_{k\ell}/\sqrt{\sigma_{kk}\sigma_{\ell\ell}}\right)_{k,\ell \in \{1,\dots,d\}}$, the generator function and the marginal distributions.

## 2.2 Ordinal measures of dependence

In this section, we will consider ordinal or concordance measures of dependence, which are invariant with respect to monotone increasing, not necessarily linear transformations and can also be expressed in terms of the underlying copula. More precisely, we introduce Kendall's tau and Blomqvist's beta, which are fundamental for our test statistic. The test will be based on the dependence between all bivariate pairs of the components of the random vector $\mathbf{X} \in \mathbb{R}^d$. Therefore, we will introduce these measures in a bivariate setting. For multivariate extensions of Kendall's tau, we refer to Kendall and Babington Smith (1940) and Joe (1990). A multivariate extension of Blomqvist's beta was introduced in Schmid and Schmidt (2007).

### 2.2.1 Kendall's tau

We start with the concordance measure Kendall's tau, which belongs to the most popular dependence measures and is defined as follows.

**Definition 2.4.** ***(Kendall's tau)***
*Let $(X_{11}, X_{21})^\top$ and $(X_{12}, X_{22})^\top$ be independent copies of the random vector $(X_1, X_2)^\top$ of continuous random variables $X_1$ and $X_2$. Then, Kendall's tau is defined by*

$$\begin{aligned}\tau_{12} :&= \mathbb{E}[\operatorname{sgn}(X_{11} - X_{12})\operatorname{sgn}(X_{21} - X_{22})] \\ &= \mathbb{P}((X_{11} - X_{12})(X_{21} - X_{22}) > 0) - \mathbb{P}((X_{11} - X_{12})(X_{21} - X_{22}) < 0),\end{aligned}$$

*where* sgn *denotes the sign function.*

Hence, Kendall's tau equals the probability of concordance minus the probability of discordance. Furthermore, for continuous random variables $X_1$ and $X_2$ with copula $C_{12}$, Kendall's tau is completely determined by their copula $C_{12}$ (see Theorem 5.1.3 in Nelsen (1999)) and can be expressed as

$$\tau_{12} = \tau_{C_{12}} = 4 \int_0^1 \int_0^1 C_{12}(u_1, u_2)\, dC_{12}(u_1, u_2) - 1\,. \tag{1}$$

For the sample version of Kendall's tau, we look at a random sample of $n$ observations $(X_{11}, X_{21})^\top, \ldots,$ $(X_{1n}, X_{2n})^\top$ from the random vector $(X_1, X_2)^\top$. In total, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ different pairs of observations $(X_{1i}, X_{2i})^\top$ and $(X_{1j}, X_{2j})^\top$ and we get

$$\widehat{\tau}_{12,n} := \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} \operatorname{sgn}(X_{1i} - X_{1j})\operatorname{sgn}(X_{2i} - X_{2j}) \tag{2}$$

as the minimum variance unbiased estimator for Kendall's tau (see Denker (1985)).

#### 2.2.2 Blomqvist's beta

The second concordance measure, we want to consider, is Blomqvist's beta, also referred to as the medial correlation coefficient. The intention of Blomqvist (1950) was to design a simple rank correlation coefficient which can be easily applied in practice. Blomqvist's beta is defined as follows.

**Definition 2.5.** ***(Blomqvist's beta)***
*Let $X_1$ and $X_2$ be continuous random variables. Then, Blomqvist's beta is defined by*

$$\begin{aligned}\beta_{12} &:= \mathbb{E}[\operatorname{sgn}(X_1 - \tilde{x}_1)\operatorname{sgn}(X_2 - \tilde{x}_2)] \\ &= \mathbb{P}((X_1 - \tilde{x}_1)(X_2 - \tilde{x}_2) > 0) - \mathbb{P}((X_1 - \tilde{x}_1)(X_2 - \tilde{x}_2) < 0)\,,\end{aligned}$$

*where $\tilde{x}_1$ and $\tilde{x}_2$ denote the population medians of $X_1$ and $X_2$, respectively.*

Hence, Blomqvist's beta equals the probability of $X_1$ and $X_2$ being both either smaller or greater than their respective medians minus the probability of one being smaller and the other one being greater than its median. Blomqvist's beta can easily be expressed in terms of the copula $C_{12}$ of the distribution of $(X_1, X_2)^\top$ and is given by

$$\beta_{12} = \beta_{C_{12}} = 4C_{12}\left(\frac{1}{2}, \frac{1}{2}\right) - 1\,. \tag{3}$$

Consequently, for copulas with a closed-form analytical expression, Blomqvist's beta can be explicitly derived. This displays one advantage of Blomqvist's beta over other more complicated dependence measures.

Now, let $(X_{11}, X_{21})^\top, \ldots, (X_{1n}, X_{2n})^\top$ be again a random sample of $n$ observations from the random vector $(X_1, X_2)^\top$ and let $\tilde{X}_{1,n}$ and $\tilde{X}_{2,n}$ be the sample medians of the components of the sample. Definition 2.5 trivially leads to the following sample version of Blomqvist's beta given by

$$\widehat{\beta}_{12,n} := \frac{1}{n} \sum_{i=1}^{n} \operatorname{sgn}\left(X_{1i} - \tilde{X}_{1,n}\right) \operatorname{sgn}\left(X_{2i} - \tilde{X}_{2,n}\right)\,. \tag{4}$$

### 2.3 Relation between Kendall's tau and Blomqvist's beta for elliptical distributions and copulas

In Theorem 3.1 of Fang et al. (2002), it is proven that the classical relation between Kendall's tau and the linear correlation coefficient known for bivariate normal distributions is valid within the more general class of meta-elliptical distributions. In particular, let $(X_1, X_2)^\top$ be a meta-elliptically distributed random vector with association $\rho_{12}$, which coincides with the correlation between $X_1$ and $X_2$ in case of finite second moments of the latter two. Then, the following relation between Kendall's tau $\tau_{12}$ and $\rho_{12}$ holds:

$$\tau_{12} = \frac{2}{\pi}\arcsin(\rho_{12})\,. \tag{5}$$

Further, Proposition 8 in Schmid and Schmidt (2007) implies a similar result for Blomqvist's beta $\beta_{12}$ and $\rho_{12}$:

$$\beta_{12} = \frac{2}{\pi}\arcsin(\rho_{12})\,. \tag{6}$$

Equations (5) and (6) show that Kendall's tau $\tau_{12}$ and Blomqvist's beta $\beta_{12}$ are uniquely determined by the association $\rho_{12}$ for bivariate meta-elliptical distributions. Second, they coincide. The equality of Kendall's tau and Blomqvist's beta is an intrinsic property of meta-elliptical distributions and therefore of elliptical copulas. Hence, we build our goodness-of-fit test on this characteristic of elliptical copulas. To the best of our knowledge, such a simple goodness-of-fit test has not been considered in the literature so far.

# 3 Goodness-of-fit test for elliptical copulas

In financial applications, it is often assumed that a copula $C$ belongs to the class of elliptical copulas. Therefore, our aim is to provide a statistical test to verify this assumption. From now on, we assume that we are given a copula sample and neglect unknown marginal distribution functions and their estimation. In practice, marginal distribution functions can be estimated parametrically and non-parametrically, which will affect the statistical inference of the test statistic. This is a subject of our future research.

Let $\mathbf{U}_1, \ldots, \mathbf{U_n} \in [0,1]^d$ be a sample from the statistical model $\Big(([0,1]^d)^n, \mathcal{B}([0,1]^d)^{\otimes n}, P^{\otimes n}\Big)$, where $P$ is a distribution with copula $C$ and uniform margins. Under the hypothesis of an elliptical copula $C$, also all marginal copulas have to be elliptical. We construct our test on the equality of Kendall's tau $\tau_{C_{k\ell}}$ and Blomqvist's beta $\beta_{C_{k\ell}}$ given by

$$\tau_{C_{k\ell}} = \beta_{C_{k\ell}}\,, \tag{7}$$

for all pairs $k, \ell \in \{1, \ldots, d\}$ with $k < \ell$. By virtue of (7), our test statistic will be constructed using the difference $\widehat{\beta}_{k\ell,n} - \widehat{\tau}_{k\ell,n}$ between the empirically estimated Blomqvist's beta $\widehat{\beta}_{k\ell,n}$ and Kendall's tau $\widehat{\tau}_{k\ell,n}$. Asymptotic distributions of the empirical estimators for Kendall's tau and Blomqvist's beta are well known and reviewed below.

First, we outline the derivation of the asymptotic distribution of the Kendall's tau estimator. According to (2), an unbiased estimator of $\tau_{k\ell}$ is given by

$$\widehat{\tau}_{k\ell,n} = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \operatorname{sgn}(U_{ki} - U_{kj})\operatorname{sgn}(U_{\ell i} - U_{\ell j})\,. \tag{8}$$

The estimator $\widehat{\tau}_{k\ell,n}$ is a U-statistic and Hoeffding (1948) showed that $\sqrt{n}(\widehat{\tau}_{k\ell,n} - \tau_{k\ell})$ converges weakly to a centred Gaussian random variable with variance $\sigma^2_{\tau_{k\ell}} := \mathbb{V}\mathrm{ar}(2\tilde{h}_{k\ell,1}((U_{k1}, U_{\ell 1})^\top))$, where

$$\tilde{h}_{k\ell,1}((U_{k1}, U_{\ell 1})^\top) := \mathbb{E}\left[\operatorname{sgn}(U_{k1} - U_{k2})\operatorname{sgn}(U_{\ell 1} - U_{\ell 2}) | U_{k1}, U_{\ell 1}\right]\,.$$

If the copula $C_{k\ell}$ is assumed to be known, then $\tilde{h}_{k\ell,1}$ has the following representation

$$\tilde{h}_{k\ell,1}((U_{k1}, U_{\ell 1})^\top) = 1 - 2U_{k1} - 2U_{\ell 1} + 4C_{k\ell}(U_{k1}, U_{\ell 1}) \tag{9}$$

and $\sigma^2_{\tau_{k\ell}}$ can be represented through the copula $C_{k\ell}$ (see Theorem 4.3 in Dengler (2010)) as

$$\begin{aligned}\sigma^2_{\tau_{k\ell}} &= 64\mathbb{E}\left[C^2_{k\ell}(U_{k1},U_{\ell 1})\right] - 64\mathbb{E}\left[U_{k1}C_{k\ell}(U_{k1},U_{\ell 1})\right] - 64\mathbb{E}\left[U_{\ell 1}C_{k\ell}(U_{k1},U_{\ell 1})\right] + 32\mathbb{E}\left[C_{k\ell}(U_{k1},U_{\ell 1})\right] \\ &\quad + 16\mathbb{E}\left[U^2_{k1}\right] + 16\mathbb{E}\left[U^2_{\ell 1}\right] - 16\mathbb{E}\left[U_{k1}\right] - 16\mathbb{E}\left[U_{\ell 1}\right] + 32\mathbb{E}\left[U_{k1}U_{\ell 1}\right] + 1 - 4\tau^2_{k\ell}\,.\end{aligned} \tag{10}$$

The variance $\sigma^2_{\tau_{k\ell}}$ can be further simplified using the theoretical moments of uniformly distributed random variables and Equation (1) for Kendall's tau. We get

$$\begin{aligned}\sigma^2_{\tau_{k\ell}} &= 64\mathbb{E}\left[C^2_{k\ell}(U_{k1},U_{\ell 1})\right] - 64\mathbb{E}\left[U_{k1}C_{k\ell}(U_{k1},U_{\ell 1})\right] - 64\mathbb{E}\left[U_{\ell 1}C_{k\ell}(U_{k1},U_{\ell 1})\right] \\ &\quad + 32\mathbb{E}\left[U_{k1}U_{\ell 1}\right] + \frac{20}{3} + 8\tau_{k\ell} - 4\tau^2_{k\ell}\,.\end{aligned} \tag{11}$$

If we do not impose any parametric assumption on the copula $C_{k\ell}$, the asymptotic variance from (11) needs to be estimated non-parametrically. For this, each expectation involving $C_{k\ell}$ can be consistently estimated with the corresponding $V$-statistic (see Denker (1985) or Mises (1947)) by employing the empirical copula $C_{k\ell,n}$ given by

$$C_{k\ell,n}(u,v) = \frac{1}{n}\sum_{i=1}^{n} I\left\{U_{ki} \le u, U_{\ell i} \le v\right\}\,,$$

where $I\{\cdot,\cdot\}$ denotes the indicator function. The remaining mixed moment can be consistently estimated by the corresponding empirical moment and $\tau_{k\ell}$ can be estimated by $\widehat{\tau}_{k\ell,n}$ from (8). However, this framework cannot ensure a positive variance estimate, since $\sigma^2_{\tau_{k\ell}}$ from (11) has been computed using theoretical moments of the uniform distribution as well as Equation (1). If we additionally estimate the moments of the uniform distribution in Equation (10) empirically, then the resulting variance estimate can still be negative due to the direct estimation of $\tau_{k\ell}$.

Below we describe our estimation framework for $\sigma^2_{\tau_{k\ell}}$, which is the variance of $2\tilde{h}_{k\ell,1}((U_{k1},U_{\ell 1})^\top)$. For a sample $(U_{k1},U_{\ell 1})^\top, \ldots, (U_{kn},U_{\ell n})^\top$, we propose to estimate $\tilde{h}_{k\ell,1}((U_{ki},U_{\ell i})^\top)$ non-parametrically by

$$\hat{h}_{k\ell,1}((U_{ki},U_{\ell i})^\top) = 1 - 2U_{ki} - 2U_{\ell i} + 4C_{k\ell,n}(U_{ki},U_{\ell i})\,, \quad i \in \{1,\ldots,n\}\,. \tag{12}$$

Now, $\sigma^2_{\tau_{k\ell}}$ is estimated by the sample variance of

$$2\hat{h}_{k\ell,1}((U_{k1},U_{\ell 1})^\top)\,,\ldots,2\hat{h}_{k\ell,1}((U_{kn},U_{\ell n})^\top)\,.$$

This leads to a consistent and positive estimation of $\sigma^2_{\tau_{k\ell}}$. Consistency follows again from the consistency of the corresponding $V$-statistics resulting from the empirical copula $C_{k\ell,n}$ combined with the estimation of moments. Note that our variance estimate is equivalent to the estimate based on Equation (10), when $\tau_{k\ell}$ is estimated using the empirical copula $C_{k\ell,n}$.

For copula data $(U_{k1},U_{\ell 1})^\top,\ldots,(U_{kn},U_{\ell n})^\top$, the empirical estimator for Blomqvist's beta $\beta_{k\ell}$ is given by

$$\widehat{\beta}_{k\ell,n} = \frac{1}{n}\sum_{i=1}^{n} \operatorname{sgn}\left(U_{ki} - 0.5\right)\operatorname{sgn}\left(U_{\ell i} - 0.5\right)\,.$$

The asymptotic normality of the estimator $\widehat{\beta}_{k\ell,n}$ of Blomqvist's beta follows in the case of known marginal distributions trivially from the central limit theorem and was already stated in Blomqvist (1950). Thus, we have the following result

$$\sqrt{n}\left(\widehat{\beta}_{k\ell,n} - \beta_{k\ell}\right) \rightsquigarrow N(0,\sigma^2_{\beta_{k\ell}})\,,$$

where

$$\sigma^2_{\beta_{k\ell}} = \mathbb{V}\mathrm{ar}\left[\operatorname{sgn}\left(U_{k1} - 0.5\right)\operatorname{sgn}\left(U_{\ell 1} - 0.5\right)\right] = 1 - \beta^2_{k\ell}$$

and $\rightsquigarrow$ denotes convergence in distribution.

Now, we know how to estimate Kendall's tau and Blomqvist's beta for each pair $(k, \ell)$ of coordinates. The test statistic will be based on all $d(d-1)/2$ differences between the corresponding estimators for Kendall's tau and Blomqvist's beta. Hence, we define the statistic $\mathbf{D}_n$

$$\mathbf{D}_n := \mathrm{vec}_u(\widehat{\boldsymbol{\beta}}) - \mathrm{vec}_u(\widehat{\boldsymbol{\tau}}), \tag{13}$$

in terms of the matrices $\widehat{\boldsymbol{\beta}} := (\widehat{\beta}_{k\ell,n})_{k,\ell\in\{1,\dots,d\}}$ and $\widehat{\boldsymbol{\tau}} := (\widehat{\tau}_{k\ell,n})_{k,\ell\in\{1,\dots,d\}}$, where $\widehat{\beta}_{kk,n} = \widehat{\tau}_{kk,n} := 1$ and $\mathrm{vec}_u(\mathcal{A})$ is the vectorization operator that extracts the elements strictly above the main diagonal of a matrix $\mathcal{A} \in \mathbb{S}_d$ in a row-wise manner, i.e.

$$\mathrm{vec}_u(\mathcal{A}) := (a_{12}, a_{13}, \dots, a_{1d}, a_{23}, a_{24}, \dots, a_{2d}, \dots, a_{d-1,d})^\top.$$

The following theorem contains the asymptotic distribution of $\mathbf{D}_n$ for a sample from an elliptical copula. Moreover, it states our test statistic $T_n$ and its limiting distribution under the null hypothesis $C \in \mathcal{C}^{ellipt}$.

**Theorem 3.1.**
*Let $\mathbf{U}_1, \dots, \mathbf{U_n} \in [0,1]^d$ be a sample from the statistical model $\left(([0,1]^d)^n, \mathcal{B}([0,1]^d)^{\otimes n}, P^{\otimes n}\right)$, where $P$ is a distribution with elliptical copula $C$ and uniform margins. Then, the statistic $\mathbf{D}_n$ defined in* (13) *has the following asymptotic distribution*

$$\sqrt{n} \cdot \mathbf{D}_n \rightsquigarrow N(0, \mathcal{V})$$

*with*

$$\mathcal{V} = \begin{pmatrix} \mathcal{I}_{d(d-1)/2} & -\mathcal{I}_{d(d-1)/2} \end{pmatrix} \boldsymbol{\Sigma} \begin{pmatrix} \mathcal{I}_{d(d-1)/2} \\ -\mathcal{I}_{d(d-1)/2} \end{pmatrix},$$

*where $\boldsymbol{\Sigma}$ is defined in Equation* (18) *and $\mathcal{I}_{d(d-1)/2}$ is the unit matrix of dimension $d(d-1)/2$.*

*Now, let $\widehat{\mathcal{V}}_n$ be a consistent estimator of $\mathcal{V}$ and consider the Wald-type statistic*

$$T_n := n\mathbf{D}_n^\top \widehat{\mathcal{V}}_n^{-1} \mathbf{D}_n. \tag{14}$$

*Then, it holds that*

$$T_n \rightsquigarrow \chi^2_{d(d-1)/2},$$

*where $\chi^2_m$ denotes the $\chi^2$-distribution with $m$ degrees of freedom.*

The proof of Theorem 3.1 is given in the Appendix. The second result of Theorem 3.1 depends on a consistent estimator of the covariance matrix $\mathcal{V}$ since $\boldsymbol{\Sigma}$ is unknown. In the following remark, we indicate the construction of such a consistent estimator $\widehat{\mathcal{V}}_n$.

**Remark 3.2.** *The asymptotic covariance matrix $\boldsymbol{\Sigma}$ depends on the unobserved $\tilde{h}_{k\ell,1}((U_{k1}, U_{\ell 1})^\top)$, for $k, \ell \in \{1, \dots, d\}$ and $k \neq \ell$. However, $\boldsymbol{\Sigma}$ can be consistently estimated using $\hat{h}_{k\ell,1}((U_{ki}, U_{\ell i})^\top)$, $i = 1, \dots, n$, defined in* (12)*. This results in the consistent estimator $\widehat{\mathcal{V}}_n$ of the covariance matrix $\mathcal{V}$.*

Based on Theorem 3.1, we propose the test function

$$\delta(\mathbf{U}_1, \dots, \mathbf{U}_n) = I\{T_n > \chi^2_{d(d-1)/2, 1-\alpha}\}$$

to test

$$H_0 : C \in \mathcal{C}^{ellipt} \qquad \text{against} \qquad H_1 : C \notin \mathcal{C}^{ellipt},$$

where $\chi^2_{m,\alpha}$ denotes the $\alpha$-quantile of the $\chi^2$-distribution with $m$ degrees of freedom.

# 4 Simulation study

In order to assess the finite-sample performance of the proposed test for ellipticity based on the test statistic $T_n$, a Monte Carlo study was conducted. We are interested in the ability of the test to hold its nominal level as well as the power of the test to detect alternatives. For ease of notation we skip all indices in the bivariate examples and just use them when they are needed.

## 4.1 Setup

First of all, we fixed a significance level of $\alpha = 0.05$ for the test throughout the study. Furthermore, the number of Monte Carlo replications was set to $N = 1000$. The simulation study was then carried out for different dimensions $d$, copula families, levels of dependence (measured in terms of Kendall's tau) and sample sizes. In particular, we have considered samples of dimension $d = 2, 3$ and 6.

To investigate the level of the test, random samples from two elliptical copula families were considered, namely the Gaussian copula and the $t$-copula (with 5 and 10 degrees of freedom). To study the power of the test, random samples from non-elliptical copula families were examined (see Section 4.2). Here, we looked at random samples from the Frank, Clayton and Gumbel family as well as from a mixture of two elliptical copulas and a copula derived from the mixture of two elliptical distributions with different association parameters, respectively. For the mixtures, we chose a Gaussian and a $t$-copula as well as a Gaussian and a $t$-distribution, respectively.

In order to assess the effect of the strength of dependence, five different levels of dependence were chosen, according to $\tau \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$. Each value of $\tau$ was converted to a unique association or dependence parameter of a multivariate copula. As a consequence, all bivariate marginal copulas of the resulting multivariate copula are then identical. For the copulas based on mixtures, four different levels of dependence were considered. The different levels are given by a combination of $\tau_G$ for the Gaussian copula/distribution and $\tau_t$ for the $t$-copula/distribution. These parameters $(\tau_G, \tau_t)$ had values in $\{(0.25, 0.75), (0.75, 0.25), (0.5, 0.25), (0.5, 0.75)\}$. Finally, for every choice of copula family and fixed level of dependence, random samples of size $n \in \{100, 250, 500, 1000, 5000\}$ were considered.

To get an impression of the common copula families used in the simulation study, Figure 1 displays scatter plots of bivariate random samples of size $n = 1000$ for the levels of dependence corresponding to $\tau \in \{0.25, 0.5, 0.75\}$. Further, scatter plots of the bivariate mixture copula and of the copula derived from the mixture of bivariate elliptical distributions are illustrated for the different combinations of $\tau_G$ and $\tau_t$ in Figure 2 and Figure 3, respectively. First, we would like to point out that the scatter plots for the Gaussian and the Frank copula in Figure 1 are quite difficult to distinguish. Moreover, the scatter plots for the mixtures in Figures 2 and 3 could easily be assigned erroneously to data from elliptical copulas.

## 4.2 Non-elliptical copula classes for the power study

In the following, we briefly overview Archimedean copulas and copulas based on special mixtures of elliptical copulas or elliptical distributions, which constitute three non-elliptical copula classes used for the power study.

### 4.2.1 Archimedean copulas

Here, we outline bivariate Archimedean copulas and follow Nelsen (1999). For $d$-dimensional Archimedean copulas with $d > 2$, we refer to Chapter 2 of Scherer (2012). Thus, we consider the simplest construction of multivariate Archimedean copulas, which are exchangeable and have only one parameter.

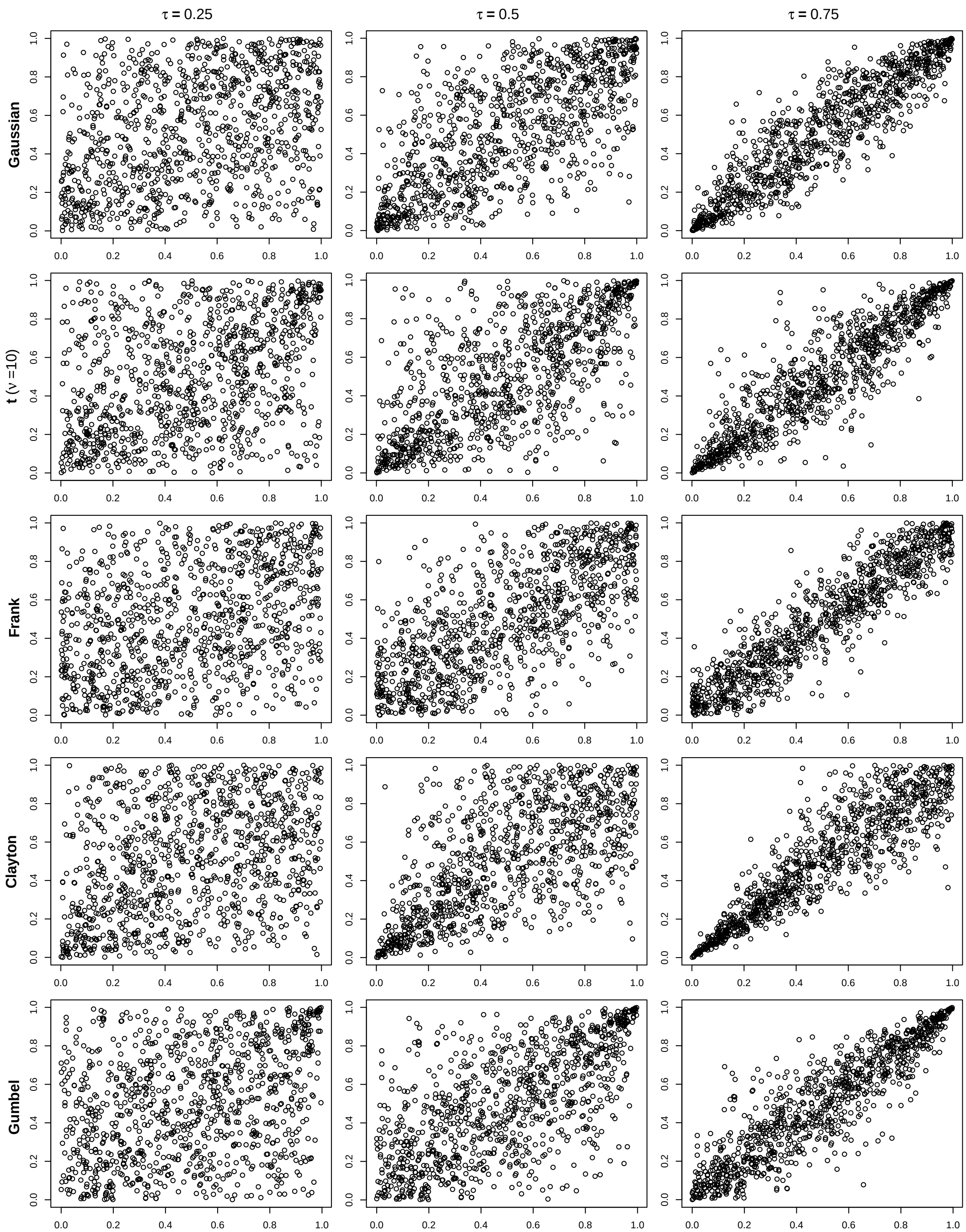


Figure 1: Scatter plots of random samples of size 1000 from the bivariate Gaussian, t (with $\nu = 10$), Frank, Clayton and Gumbel copula (from top to bottom) with $\tau = 0.25$ (left), 0.5 (middle), 0.75 (right).

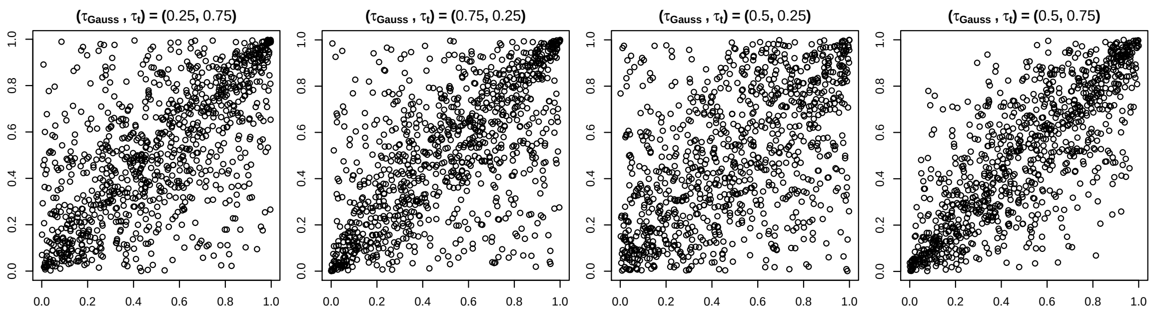


Figure 2: Scatter plots of random samples of size 1000 from the copula based on the mixture of two bivariate elliptical copulas with $(\tau_G, \tau_t) = (0.25, 0.75)$, $(0.75, 0.25)$, $(0.25, 0.5)$ and $(0.75, 0.5)$ (from left to right).

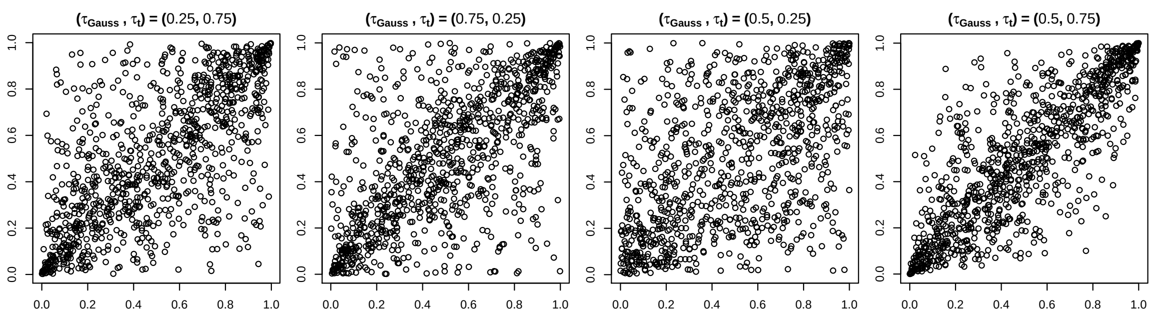


Figure 3: Scatter plots of random samples of size 1000 from the copula based on the mixture of two bivariate elliptical distributions with $(\tau_G, \tau_t) = (0.25, 0.75)$, $(0.75, 0.25)$, $(0.25, 0.5)$ and $(0.75, 0.5)$ (from left to right).

**Definition 4.1.** ***(Bivariate Archimedean copula)***
*Let $\varphi : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing, convex function with $\varphi(1) = 0$. Then, the function $C_\varphi : [0,1]^2 \to [0,1]$ given by*

$$C_\varphi(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \tag{15}$$

*is a copula, where $\varphi^{[-1]}$ is a pseudo-inverse of $\varphi$ . Copulas of this form are called Archimedean copulas and $\varphi$ is called a generator. If $\varphi(0) = \infty$, the generator is called strict, $\varphi^{[-1]} = \varphi^{-1}$ and $C_\varphi(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v))$ is said to be a strict Archimedean copula.*

Table 1 summarizes generators with parameter ranges and the resulting explicit expression for the bivariate Archimedean copulas from the Frank, Clayton and Gumbel family.

For a bivariate Archimedean copula $C$, one can compute Kendall's tau using its generator $\varphi$. More precisely, the following relation (see Genest and MacKay (1986)) holds:

$$\tau_C = 1 + 4 \int_0^1 \frac{\varphi(t)}{\varphi'(t)}\, dt\,.$$

Further, Equation (3) and Definition 4.1 imply for Blomqvist's beta:

$$\beta_C = 4\, \varphi^{[-1]} \left( 2\, \varphi \left( \frac{1}{2} \right) \right) - 1\,.$$

| Copula family | $\varphi_\theta(t)$ | $\theta \in$ | $C_\theta(u,v)$ |
|---|---|---|---|
| Frank | $-\ln \dfrac{e^{-\theta t}-1}{e^{-\theta}-1}$ | $\mathbb{R}\setminus\{0\}$ | $-\dfrac{1}{\theta}\ln\left(1+\dfrac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right)$ |
| Clayton | $\dfrac{1}{\theta}\left(t^{-\theta}-1\right)$ | $(0,\infty)$ | $\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$ |
| Gumbel | $(-\ln t)^\theta$ | $[1,\infty)$ | $\exp\left(-\left[(-\ln u)^\theta+(-\ln v)^\theta\right]^{1/\theta}\right)$ |

Table 1: Summary of generators, parameter ranges and explicit expressions for the bivariate Frank, Clayton and Gumbel copula.

Archimedean copulas are exchangeable by construction. Moreover, the bivariate Frank copula is even radially symmetric, i.e. the survival copula coincides with the copula itself. Being exchangeable and radially symmetric, bivariate Frank copulas possess the same symmetry properties as bivariate elliptical copulas. Therefore, it is very important to distinguish between them when modeling the dependence of bivariate data. Table 2 reports Kendall's tau, Blomqvist's beta and the symmetry properties for the bivariate Frank, Clayton and Gumbel copula.

| Copula family | $\tau_\theta$ | $\beta_\theta$ | exch. | rad. sym. |
|---|---|---|---|---|
| Frank | $1+\dfrac{4}{\theta}\left(D_1(\theta)-1\right)$ | $1+\dfrac{4}{\theta}\ln\left(\dfrac{1}{2}\left(e^{-\theta/2}+1\right)\right)$ | ✓ | ✓ |
| Clayton | $\dfrac{\theta}{\theta+2}$ | $4\left(2^{\theta+1}-1\right)^{-1/\theta}-1$ | ✓ | ✗ |
| Gumbel | $\dfrac{\theta-1}{\theta}$ | $2^{2-2^{1/\theta}}-1$ | ✓ | ✗ |

Table 2: Summary of Kendall's tau, Blomqvist's beta, and the symmetry properties for the bivariate Frank, Clayton and Gumbel copula. Note: $D_k(x)$ is the Debye function, which is defined for any $k \in \mathbb{N}$ by $D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t-1}\,dt$ .

### 4.2.2 Mixture of bivariate elliptical copulas

The aim of this section is to consider another class of bivariate non-elliptical copulas, which are symmetric (i.e. exchangeable) and radially symmetric. For this, we mix two bivariate elliptical copulas with different parameters. More precisely, a bivariate Gaussian copula with correlation $\rho_G$ and a bivariate $t$-copula with $\nu$ degrees of freedom and association parameter $\rho_t$, where $\rho_t \neq \rho_G$, are mixed with probabilities $p \in [0,1]$ and $1-p$, respectively. The resulting bivariate mixture copula is given by

$$C^{mixt,cop}(u,v) = p\, C^{Gauss}_{\rho_G}(u,v) + (1-p)\, C^t_{\nu,\rho_t}(u,v), \quad (u,v) \in [0,1]^2 .$$

By choosing $\rho_G \neq \rho_t$, we expect this bivariate mixture copula to be non-elliptical. However, this is not trivial to show since elliptical copulas are only implicitly defined as the copulas of elliptical distributions. To the best of our knowledge, such mixtures of elliptical copulas have not been investigated so far.

It should be noted that the proposed construction of such mixture copulas is general and can be extended to any dimension. Further, it is very easy to draw a random sample from the mixture copula. For this, the random sample is drawn from the Gaussian copula $C^{Gauss}_{\rho_G}$ with probability $p$ and with probability $(1-p)$ from the $t$-copula $C^t_{\nu,\rho_t}$. In our simulation study, we set $p = 0.5$, $\nu = 5$ and varied the association parameters $\rho_G$ and $\rho_t$. By virtue of the one-to-one correspondence between Kendall's tau and the association parameter $\rho$ (correlation coefficient for $\nu \geq 2$) given in (5), this is equivalent to varying Kendall's tau.

#### 4.2.3 Copulas derived from the mixture of bivariate elliptical distributions

Here, we have a closer look on bivariate copulas derived from the mixture of bivariate elliptical distributions. Again, the framework presented below is general and can be extended to any dimension. The idea is to mix two bivariate elliptical distributions in such a way that the resulting bivariate distribution is not elliptical any more. We expect that its copula is then also non-elliptical, but we have no theoretical justification. Without loss of generality, we set $\boldsymbol{\mu} = \mathbf{0}$ in Definition 2.2. Now, one can easily argue that the mixture of two bivariate elliptical distributions with different parameters $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ is not elliptical.

In the following, a bivariate Gaussian distribution $N_2(\mathbf{0}, \mathcal{P}_G)$ with correlation $\rho_G$ and a bivariate $t$-distribution $t_2(\nu, \mathbf{0}, \mathcal{P}_t)$ with correlation $\rho_t$, where $\rho_t \neq \rho_G$, are mixed with probabilities $p \in [0,1]$ and $1-p$, respectively. The cumulative distribution function $H^{mixt}$ of the resulting bivariate mixture distribution is given by

$$H^{mixt}(x,y) = p\,\Phi_{\rho_G}(x,y) + (1-p)\,t_{\nu,\rho_t}(x,y), \quad (x,y) \in \mathbb{R}^2,$$

where $\Phi_{\rho_G}$ and $t_{\nu,\rho_t}$ are the cumulative distribution functions of $N_2(\mathbf{0}, \mathcal{P}_G)$ and $t_2(\nu, \mathbf{0}, \mathcal{P}_t)$, respectively. The margins $F^{mixt}$ and $G^{mixt}$ of this mixture distribution can be determined using the margins of the underlying Gaussian and $t$-distribution. Then, according to Sklar's theorem (see Theorem 2.1), the bivariate copula $C^{mixt,distr}(u,v)$ of the mixture distribution $H^{mixt}$, for any $u, v \in [0,1]$, is given by

$$C^{mixt,distr}(u,v) = H^{mixt}((F^{mixt})^-(u), (G^{mixt})^-(v)),$$

where $(F^{mixt})^-$ and $(G^{mixt})^-$ denote the generalized inverses of $F^{mixt}$ and $G^{mixt}$, respectively. Since we chose $\rho_G \neq \rho_t$, the resulting bivariate mixture distribution $H^{mixt}$ is non-elliptical.

Just like for the mixture of elliptical copulas, it is easy to draw a random sample from the mixture distribution. First, the random sample is drawn with probability $p$ from the bivariate Gaussian distribution $N_2(\mathbf{0}, \mathcal{P}_G)$ and with probability $(1-p)$ from the bivariate $t$-distribution $t_2(\nu, \mathbf{0}, \mathcal{P}_t)$.Then, the random sample is transformed using the margins $F^{mixt}$ and $G^{mixt}$ to get copula data. For the simulation study, we set again $p = 0.5$, $\nu = 5$ and varied the association parameters $\rho_G$ and $\rho_t$. With the same argument as before, this is equivalent to varying Kendall's tau.

### 4.3 Level

Tables 3, 4 and 5 display the empirical level of the test for ellipticity with significance level $\alpha = 0.05$ as observed in 1000 random samples for dimension $d = 2, 3$ and 6, respectively, and all possible scenarios from the simulation setup. Note that for $d = 3$ and $d = 6$, all off-diagonal elements of the correlation matrix $\mathcal{R}$ of the Gaussian and $t$-copula are identical and related to the level of dependence $\tau_C$.

For dimensions $d = 2$ and $d = 3$, the test seems to hold its nominal level (see Tables 3 and 4). Only for large values of Kendall's tau in combination with a small sample size of $n = 100$ or $n = 250$, the test turns out to be too liberal. As the distributional result for the test statistic holds only asymptotically, this explains why there might occur some problems especially for small sample sizes.

Table 5 shows that the proposed test requires large sample sizes to hold its level for higher dimensions. For $d = 6$ and medium level of dependence $\tau_C$, a sample size of at least $n = 1000$ is required. This can be explained by the asymptotic nature of our test. The accuracy of the distributional approximation with the limiting $\chi^2$-distribution is very poor for small sample sizes and gets improved significantly for larger sample sizes. This is illustrated by the QQ-plots for the $t$-copula with 5 degrees of freedom in Figure 4.

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Gauss | 0.10 | 0.046 | 0.052 | 0.046 | 0.061 | 0.044 |
| | 0.25 | 0.044 | 0.045 | 0.060 | 0.053 | 0.053 |
| | 0.50 | 0.063 | 0.044 | 0.044 | 0.055 | 0.048 |
| | 0.75 | 0.054 | 0.040 | 0.049 | 0.054 | 0.051 |
| | 0.90 | 0.090 | 0.047 | 0.053 | 0.053 | 0.064 |
| $t$ ($\nu = 5$) | 0.10 | 0.048 | 0.048 | 0.047 | 0.049 | 0.053 |
| | 0.25 | 0.049 | 0.051 | 0.042 | 0.043 | 0.053 |
| | 0.50 | 0.047 | 0.042 | 0.034 | 0.051 | 0.047 |
| | 0.75 | 0.072 | 0.052 | 0.049 | 0.050 | 0.046 |
| | 0.90 | 0.077 | 0.046 | 0.035 | 0.054 | 0.059 |
| $t$ ($\nu = 10$) | 0.10 | 0.063 | 0.051 | 0.058 | 0.050 | 0.051 |
| | 0.25 | 0.041 | 0.043 | 0.044 | 0.052 | 0.052 |
| | 0.50 | 0.052 | 0.046 | 0.050 | 0.053 | 0.038 |
| | 0.75 | 0.056 | 0.056 | 0.060 | 0.057 | 0.065 |
| | 0.90 | 0.073 | 0.051 | 0.047 | 0.062 | 0.050 |

Table 3: $d = 2$: Empirical level of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Gauss | 0.10 | 0.054 | 0.046 | 0.045 | 0.049 | 0.041 |
| | 0.25 | 0.054 | 0.044 | 0.062 | 0.055 | 0.045 |
| | 0.50 | 0.048 | 0.066 | 0.062 | 0.062 | 0.059 |
| | 0.75 | 0.081 | 0.057 | 0.055 | 0.053 | 0.045 |
| | 0.90 | 0.213 | 0.111 | 0.077 | 0.063 | 0.043 |
| $t$ ($\nu = 5$) | 0.10 | 0.047 | 0.048 | 0.042 | 0.039 | 0.048 |
| | 0.25 | 0.067 | 0.039 | 0.051 | 0.058 | 0.061 |
| | 0.50 | 0.055 | 0.052 | 0.054 | 0.038 | 0.053 |
| | 0.75 | 0.071 | 0.058 | 0.045 | 0.054 | 0.042 |
| | 0.90 | 0.173 | 0.080 | 0.066 | 0.072 | 0.048 |
| $t$ ($\nu = 10$) | 0.10 | 0.059 | 0.052 | 0.051 | 0.067 | 0.044 |
| | 0.25 | 0.051 | 0.045 | 0.054 | 0.045 | 0.044 |
| | 0.50 | 0.060 | 0.050 | 0.053 | 0.062 | 0.045 |
| | 0.75 | 0.073 | 0.059 | 0.049 | 0.062 | 0.045 |
| | 0.90 | 0.169 | 0.096 | 0.062 | 0.056 | 0.048 |

Table 4: $d = 3$: Empirical level of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

Hence, the results of our simulation study for dimension $d = 6$ are reliable only for large sample sizes. Therefore, we consider only samples of size $n = 1000$ and $n = 5000$ in the power study for dimension $d = 6$.

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Gauss | 0.10 | 0.155 | 0.092 | 0.069 | 0.063 | 0.043 |
| | 0.25 | 0.195 | 0.091 | 0.075 | 0.044 | 0.059 |
| | 0.50 | 0.266 | 0.126 | 0.074 | 0.060 | 0.067 |
| | 0.75 | 0.484 | 0.243 | 0.153 | 0.104 | 0.057 |
| | 0.90 | 0.642 | 0.591 | 0.403 | 0.193 | 0.075 |
| $t$ ($\nu = 5$) | 0.10 | 0.167 | 0.069 | 0.070 | 0.054 | 0.059 |
| | 0.25 | 0.196 | 0.092 | 0.090 | 0.054 | 0.046 |
| | 0.50 | 0.244 | 0.122 | 0.067 | 0.054 | 0.037 |
| | 0.75 | 0.479 | 0.220 | 0.127 | 0.069 | 0.040 |
| | 0.90 | 0.501 | 0.528 | 0.285 | 0.154 | 0.056 |
| $t$ ($\nu = 10$) | 0.10 | 0.160 | 0.074 | 0.058 | 0.048 | 0.052 |
| | 0.25 | 0.197 | 0.080 | 0.065 | 0.053 | 0.054 |
| | 0.50 | 0.275 | 0.113 | 0.079 | 0.065 | 0.047 |
| | 0.75 | 0.469 | 0.225 | 0.128 | 0.086 | 0.046 |
| | 0.90 | 0.605 | 0.572 | 0.332 | 0.182 | 0.074 |

Table 5: $d = 6$: Empirical level of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

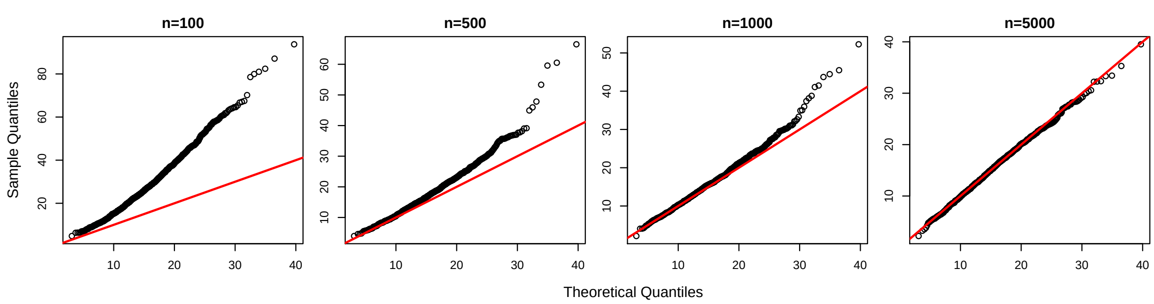


Figure 4: QQ-plots of $T_n$ for $t$-copula with 5 degrees of freedom, $d = 6$, $\tau_C = 0.75$ and $n = 100, 500, 1000$ and 5000.

## 4.4 Power

The results for the empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on 1000 random samples from the Frank, Clayton and Gumbel family are presented in Tables 6, 7 and 8 for the different dimensions $d = 2, 3$ and 6. For the random samples from the mixture copula and the copula derived from elliptical distributions, we report the results only for $d = 2$ in Tables 9 and 10, respectively. This is due to the fact that huge sample sizes are generally needed to achieve satisfactory empirical power for the bivariate mixture copula constructions. This lacks in practical relevance and, therefore, we do not consider these mixture copulas in higher dimensions.

First of all, when we look at Tables 6 - 10, we notice that the observed power varies enormously across the level of dependence and the sample size as well as across the copula families. In general, the rejection rate increases with the sample size, as expected. In addition to that, the rejection rate increases with the level of dependence. Since the non-ellipticity becomes more apparent for higher values of Kendall's tau,

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Frank | 0.10 | 0.063 | 0.072 | 0.063 | 0.074 | 0.243 |
| | 0.25 | 0.074 | 0.097 | 0.179 | 0.298 | 0.903 |
| | 0.50 | 0.145 | 0.256 | 0.492 | 0.743 | 1.000 |
| | 0.75 | 0.157 | 0.341 | 0.567 | 0.854 | 1.000 |
| | 0.90 | 0.181 | 0.232 | 0.379 | 0.620 | 1.000 |
| Clayton | 0.10 | 0.049 | 0.052 | 0.052 | 0.056 | 0.058 |
| | 0.25 | 0.062 | 0.061 | 0.047 | 0.053 | 0.066 |
| | 0.50 | 0.053 | 0.050 | 0.075 | 0.102 | 0.228 |
| | 0.75 | 0.121 | 0.136 | 0.266 | 0.452 | 0.981 |
| | 0.90 | 0.169 | 0.194 | 0.288 | 0.514 | 0.986 |
| Gumbel | 0.10 | 0.050 | 0.050 | 0.051 | 0.062 | 0.049 |
| | 0.25 | 0.047 | 0.052 | 0.037 | 0.067 | 0.067 |
| | 0.50 | 0.056 | 0.053 | 0.044 | 0.029 | 0.055 |
| | 0.75 | 0.059 | 0.044 | 0.073 | 0.053 | 0.080 |
| | 0.90 | 0.088 | 0.046 | 0.065 | 0.074 | 0.086 |

Table 6: $d = 2$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Frank | 0.10 | 0.070 | 0.065 | 0.077 | 0.114 | 0.376 |
| | 0.25 | 0.096 | 0.145 | 0.234 | 0.430 | 0.992 |
| | 0.50 | 0.175 | 0.323 | 0.603 | 0.883 | 1.000 |
| | 0.75 | 0.232 | 0.379 | 0.651 | 0.918 | 1.000 |
| | 0.90 | 0.411 | 0.348 | 0.457 | 0.710 | 1.000 |
| Clayton | 0.10 | 0.056 | 0.056 | 0.052 | 0.054 | 0.059 |
| | 0.25 | 0.059 | 0.044 | 0.057 | 0.049 | 0.065 |
| | 0.50 | 0.075 | 0.065 | 0.083 | 0.091 | 0.272 |
| | 0.75 | 0.142 | 0.177 | 0.272 | 0.504 | 1.000 |
| | 0.90 | 0.336 | 0.284 | 0.365 | 0.549 | 0.997 |
| Gumbel | 0.10 | 0.065 | 0.050 | 0.053 | 0.048 | 0.068 |
| | 0.25 | 0.044 | 0.047 | 0.060 | 0.050 | 0.076 |
| | 0.50 | 0.064 | 0.050 | 0.063 | 0.056 | 0.051 |
| | 0.75 | 0.099 | 0.057 | 0.073 | 0.068 | 0.082 |
| | 0.90 | 0.217 | 0.120 | 0.105 | 0.076 | 0.104 |

Table 7: $d = 3$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

which can also be observed in Figure 1, this is also expected. The empirical power also increases with increasing dimension as soon as the distributional approximation with the $\chi^2$ distribution is sufficiently accurate. Some exceptions occur in connection with the Gumbel family, which we discuss later on.

| C | $\tau_C$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|
| Frank | 0.10 | 0.176 | 0.794 |
| | 0.25 | 0.678 | 1.000 |
| | 0.50 | 0.974 | 1.000 |
| | 0.75 | 0.959 | 1.000 |
| | 0.90 | 0.862 | 1.000 |
| Clayton | 0.10 | 0.059 | 0.077 |
| | 0.25 | 0.056 | 0.061 |
| | 0.50 | 0.122 | 0.260 |
| | 0.75 | 0.532 | 0.996 |
| | 0.90 | 0.745 | 1.000 |
| Gumbel | 0.10 | 0.062 | 0.080 |
| | 0.25 | 0.055 | 0.057 |
| | 0.50 | 0.074 | 0.062 |
| | 0.75 | 0.101 | 0.104 |
| | 0.90 | 0.238 | 0.139 |

Table 8: $d = 6$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

### 4.4.1 Power for Archimedean copula families

For the Frank copula, the test appears to perform well for all considered dimensions. If Kendall's tau has a value of at least 0.5, a sample size of $n = 1000$ suffices to achieve a good power. For dimension $d = 2$ and $d = 3$ and small levels of dependence, a larger sample size is needed. Table 8 shows that the empirical power for $d = 6$ is larger than the corresponding power for lower dimensions, such that here a sample size of $n = 1000$ is sufficient for small levels of dependence.

From Section 4.2, we know that the bivariate Frank copula is the only Archimedean copula which is not only exchangeable but also radially symmetric. Since radial symmetry is an important necessary condition for a copula to be elliptical, the fact that the test performs quite well for this family is a very promising feature. Note that elliptical copulas of $d > 3$ can but do not have to be exchangeable.

In case of the Clayton family, quite similar observations can be made, though with slightly lower rejection rates. Still, we can say that the test seems to be good in detecting the lack of ellipticity if the level of dependence is not too close to independence.

In contrast to the previous results, the rejection rates for the Gumbel family appear to be very low. Since the test statistic $T_n$ is based on the difference between Kendall's tau and Blomqvist's beta, we have to take a closer look at those two measures in order to find some explanation. Figure 5 illustrates Kendall's tau and Blomqvist's beta as functions of the copula family parameter $\theta$ for the bivariate Frank, Clayton and Gumbel copulas. Here, the reason for the low rejection rates becomes apparent: Kendall's tau and Blomqvist's beta are very close and almost not distinguishable for the Gumbel family. Nevertheless, even in this case, the test is able to provide some indication against the null hypothesis for huge sample sizes if the level of dependence is high enough, meaning Kendall's tau being equal to 0.75 or higher. To confirm this presumption, we carried out the simulation study for the bivariate Gumbel copula with a Kendall's tau of 0.75 and chose a sample size of $n = 10^5$, which delivered a quite acceptable rejection rate of 0.648.

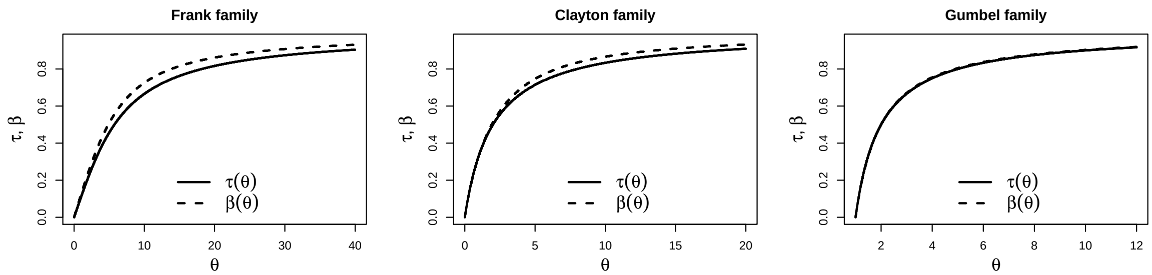


Figure 5: Comparison of Kendall's tau and Blomqvist's beta as functions of the copula family parameter $\theta$ for different copula families.

### 4.4.2 Power for the bivariate mixture copula constructions

For the mixture of bivariate elliptical copulas, the test generally achieves good power only for huge sample sizes ($n = 10^5$), which we do not consider in our simulation study. If the absolute difference of the values of Kendall's tau for the Gaussian and the $t$-copula is large enough then an acceptable empirical power can be observed already for a sample size of $n = 5000$.

| C | $\tau_G$ | $\tau_t$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|---|
| Mixture ($p = 0.5$) | 0.25 | 0.75 | 0.077 | 0.080 | 0.134 | 0.249 | 0.776 |
| | 0.75 | 0.25 | 0.079 | 0.070 | 0.118 | 0.176 | 0.604 |
| | 0.50 | 0.25 | 0.041 | 0.043 | 0.061 | 0.046 | 0.054 |
| | 0.50 | 0.75 | 0.048 | 0.062 | 0.069 | 0.075 | 0.202 |

Table 9: $d = 2$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from a mixture $C$ of bivariate elliptical copulas with Kendall's tau combinations $(\tau_G, \tau_t)$.

| C | $\tau_G$ | $\tau_t$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|---|
| Mixture ($p = 0.5$) | 0.25 | 0.75 | 0.053 | 0.064 | 0.068 | 0.099 | 0.232 |
| | 0.75 | 0.25 | 0.092 | 0.128 | 0.236 | 0.366 | 0.948 |
| | 0.50 | 0.25 | 0.047 | 0.042 | 0.056 | 0.061 | 0.150 |
| | 0.50 | 0.75 | 0.041 | 0.061 | 0.039 | 0.051 | 0.065 |

Table 10: $d = 2$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of size $n$ from a copula $C$ derived from the mixture of bivariate elliptical distributions with Kendall's tau combinations $(\tau_G, \tau_t)$.

Similar observations on the empirical power can be made for the copula derived from the mixture of bivariate elliptical distributions. There is only one exception. It turns out that the empirical power depends not only on the absolute difference but also on the sign of the difference. Thus, the empirical power of 0.948 for the combination of $\tau_G = 0.75$ and $\tau_t = 0.25$ is observed. Whereas, we get the empirical power of 0.232 if we switch the values of the dependence levels.

Since it is not easy to graphically detect the non-ellipticity for the samples of the mixture copulas used in the simulation study, our test is still useful.

## 4.5 Level and power for pseudo-observations

As it was suggested by one of the referees, we have investigated the empirical level and power of the proposed test for the more realistic situation with unknown marginal distributions. For this, we have simulated copula data from the considered copula families and transformed the uniform marginal distributions to exponential distributions with unit rate. Thus, we deal now with observations $\mathbf{X}_i \in \mathbb{R}_+^d$, $i = 1, \ldots, n$. We have applied our test to pseudo-observations $\widehat{U}_{ki} = nF_{k,n}(X_{ki})/(n+1)$, where $F_{k,n}$ is the empirical cumulative distribution function of the $k$-th component, $k \in \{1, \ldots, d\}$. So, we do not make any assumptions on the marginal distributions, which corresponds to practical applications. Below, we present our results for the bivariate case.

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Gauss | 0.10 | 0.063 | 0.049 | 0.054 | 0.042 | 0.043 |
| | 0.25 | 0.053 | 0.048 | 0.063 | 0.048 | 0.049 |
| | 0.50 | 0.046 | 0.060 | 0.058 | 0.044 | 0.041 |
| | 0.75 | 0.069 | 0.055 | 0.036 | 0.053 | 0.045 |
| | 0.90 | 0.082 | 0.060 | 0.051 | 0.050 | 0.049 |
| $t$ ($\nu = 5$) | 0.10 | 0.063 | 0.049 | 0.059 | 0.056 | 0.053 |
| | 0.25 | 0.055 | 0.047 | 0.050 | 0.043 | 0.046 |
| | 0.50 | 0.058 | 0.055 | 0.043 | 0.060 | 0.048 |
| | 0.75 | 0.058 | 0.053 | 0.044 | 0.058 | 0.057 |
| | 0.90 | 0.075 | 0.067 | 0.048 | 0.045 | 0.050 |
| $t$ ($\nu = 10$) | 0.10 | 0.044 | 0.052 | 0.057 | 0.046 | 0.054 |
| | 0.25 | 0.053 | 0.041 | 0.053 | 0.047 | 0.059 |
| | 0.50 | 0.049 | 0.047 | 0.043 | 0.048 | 0.055 |
| | 0.75 | 0.051 | 0.072 | 0.054 | 0.061 | 0.046 |
| | 0.90 | 0.080 | 0.059 | 0.054 | 0.048 | 0.054 |

Table 11: $d = 2$: Empirical level of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of pseudo-observations of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

Table 11 shows the empirical level of our test for $d = 2$ and unknown margins. As one can observe, the test keeps its nominal level across all sample sizes and dependence levels for the considered copula families. Compared to Table 3, the empirical levels are similar for both situations: known and unknown margins. This supports our testing procedure for copula data in real applications.

Table 12 now shows the empirical power of our test for $d = 2$ and unknown margins. We do not observe any significant differences in comparison to the empirical power results from Table 6. Thus, it seems that the test is equally powerful for known as well as unknown marginal distributions. Summarizing the empirical findings of this section, we can recommend our test also in the case of unknown marginal distributions, although the observations are now dependent and therefore the limit results do not hold as stated in Theorem 3.1.

## 4.6 Power under the local alternatives

As indicated by one referee, the simple functional form of the test statistic allows to investigate the power of the proposed goodness-of-fit test under local alternatives. Since the accuracy of the distributional approximation for our test statistic $T_n$ is not satisfactory for small sample sizes and large dimensions, we

| C | $\tau_C$ | $n = 100$ | $n = 250$ | $n = 500$ | $n = 1000$ | $n = 5000$ |
|---|---|---|---|---|---|---|
| Frank | 0.10 | 0.052 | 0.082 | 0.060 | 0.093 | 0.251 |
| | 0.25 | 0.092 | 0.101 | 0.176 | 0.310 | 0.910 |
| | 0.50 | 0.125 | 0.258 | 0.475 | 0.753 | 1.000 |
| | 0.75 | 0.169 | 0.336 | 0.534 | 0.851 | 1.000 |
| | 0.90 | 0.166 | 0.224 | 0.372 | 0.609 | 1.000 |
| Clayton | 0.10 | 0.047 | 0.055 | 0.051 | 0.063 | 0.049 |
| | 0.25 | 0.041 | 0.050 | 0.038 | 0.072 | 0.065 |
| | 0.50 | 0.054 | 0.043 | 0.061 | 0.087 | 0.218 |
| | 0.75 | 0.098 | 0.138 | 0.272 | 0.445 | 0.982 |
| | 0.90 | 0.168 | 0.176 | 0.292 | 0.506 | 0.990 |
| Gumbel | 0.10 | 0.050 | 0.055 | 0.057 | 0.052 | 0.056 |
| | 0.25 | 0.049 | 0.040 | 0.051 | 0.049 | 0.061 |
| | 0.50 | 0.039 | 0.056 | 0.057 | 0.033 | 0.046 |
| | 0.75 | 0.046 | 0.047 | 0.047 | 0.057 | 0.077 |
| | 0.90 | 0.078 | 0.071 | 0.058 | 0.070 | 0.086 |

Table 12: $d = 2$: Empirical power of the test for ellipticity with significance level $\alpha = 0.05$ based on the test statistic $T_n$: rate of rejecting $H_0$ as observed in 1000 random samples of pseudo-observations of size $n$ from copula family $C$ with Kendall's tau $\tau_C$.

restrict ourselves to dimension $d = 2$. For the null hypothesis $H_0 : \beta = \tau$, local alternatives of the form $\beta = \tau + \Delta/\sqrt{n}$ are considered for varying $\Delta$. It follows in lines of the proof of Theorem 3.1 that the asymptotic distribution of the test statistic under the local alternatives is the non-central $\chi^2$-distribution with one degree of freedom and non-centrality parameter $\Delta^2/v^2$, where $v^2$ is the asymptotic variance of $\mathbf{D}_n$ for $d = 2$. In applications, the asymptotic variance $v^2$ should be consistently estimated and depends on the underlying data.

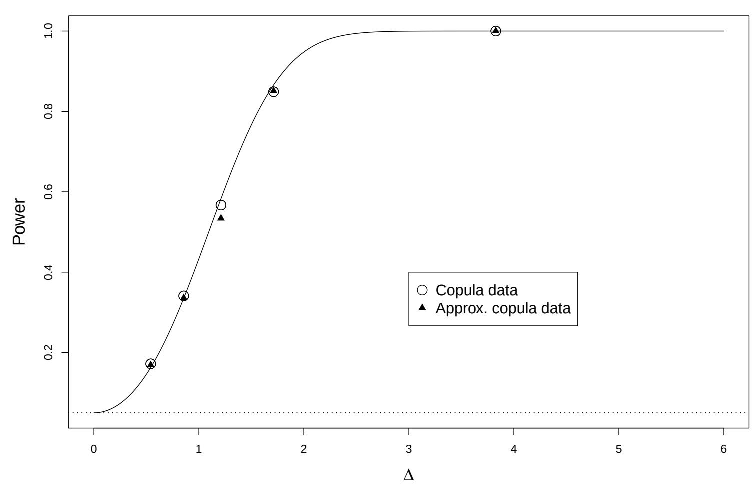


Figure 6: Asymptotic local power curve for the bivariate Frank copula with $\tau_C = 0.75$ and $\beta_C = 0.804$. Circles and triangles correspond to the empirical power for copula data and pseudo-observations, respectively, of sample sizes $n = 100, 250, 500, 1000, 5000$.

For varying $\Delta$, Figure 6 shows the theoretical asymptotic power of our test under the sequence of local alternatives when the data comes from a Frank copula with $\tau_C = 0.75$ and $\beta_C = 0.804$. The asymptotic variance is estimated using a sample of size 10000. This estimate is then used instead of the unknown asymptotic variance $v^2$. Further, the five circles in Figure 6 indicate the empirical power of our test from Table 6 for the Frank copula and the five sample sizes $n = 100, 250, 500, 1000$ and 5000. For each sample size $n$, the position of the circles on the $x$-axis is computed by $\sqrt{n}\,(\beta_C - \tau_C)$. Thus, the circles are located further to the right with increasing sample size $n$. We see that the asymptotic local power is in good agreement with our empirical results. Moreover, the five triangles in Figure 6 similarly display the empirical power of our test applied to pseudo-observations from Section 4.5. For the considered simulation scenario, Figure 6 shows that the empirical power of our asymptotic test does not significantly fall in quality and agrees well with the asymptotic local power even if marginal distributions are unknown.

# 5 Empirical analysis

We consider the daily log-returns of the DAX, the Dow Jones Industrial Average and the Euro Stoxx 50 indices for 10 years starting from January 1, 2006 till December 31, 2016. For our test, we need i.i.d. data. Therefore, we fit a time series model to each series of log-returns and then use the standardized residuals of these models. More precisely, we choose ARMA$(1, 1)$ - GARCH$(1, 1)$ models with Student's $t$ innovations to capture autocorrelation and volatility clustering in the daily log-returns. The model fits have been validated with QQ-plots of the standardized residuals.

To get the copula data, the standardized residuals have to be transformed to achieve approximate i.i.d. uniform margins. This can be done non-parametrically by using the empirical cumulative distribution functions. Apart from that, one can use a Student's $t$ distribution to parametrically transform the residuals, which is due to the fact that the considered ARMA$(1, 1)$ - GARCH$(1, 1)$ models have Student's $t$ innovations. Figure 7 displays the scatter plots of the standardized residuals after the non-parametric (above the diagonal) as well as the parametric transformation with the fitted $t$-distribution (below the diagonal). Here, we can visually observe a high dependence between the margins as well as symmetry and radial symmetry of the underlying data. Therefore, an elliptical copula would be a natural choice to model the dependence structure of the standardized residuals of the three indices.

Now, we apply our goodness-of-fit test to the underlying copula data. We get $p$-values of 0.030 and 0.048 for the non-parametrically and the parametrically transformed residuals, respectively. Hence, our test rejects the null hypothesis that the dependence structure of the considered data can be captured by a three dimensional elliptical copula at the significance level of 5%. This is a surprising statistical result and indicates that one should be careful when choosing elliptical copulas in financial applications.

Further, we get $p$-values between 0.018 and 0.056, when we apply our test to the bivariate margins of the non-parametrically and the parametrically transformed residuals. Even if we cannot reject the null hypothesis of ellipticity for some bivariate margins, we would not favour elliptical copulas for modeling the two dimensional dependence structures of the given data.

# 6 Conclusion

In this paper, we derive a simple non-parametric goodness-of-fit test for elliptical copulas of any dimension. It is based on the equality of Kendall's tau and Blomqvist's beta for all bivariate margins. However, to our best knowledge, it is an open problem whether this equality does completely characterize elliptical copulas. The distinguishing property of our test is its ability to differentiate between elliptical and non-elliptical copulas of any dimension even if the underlying copulas are radially symmetric. In the bivariate case, our test can even detect symmetric non-elliptical copulas.

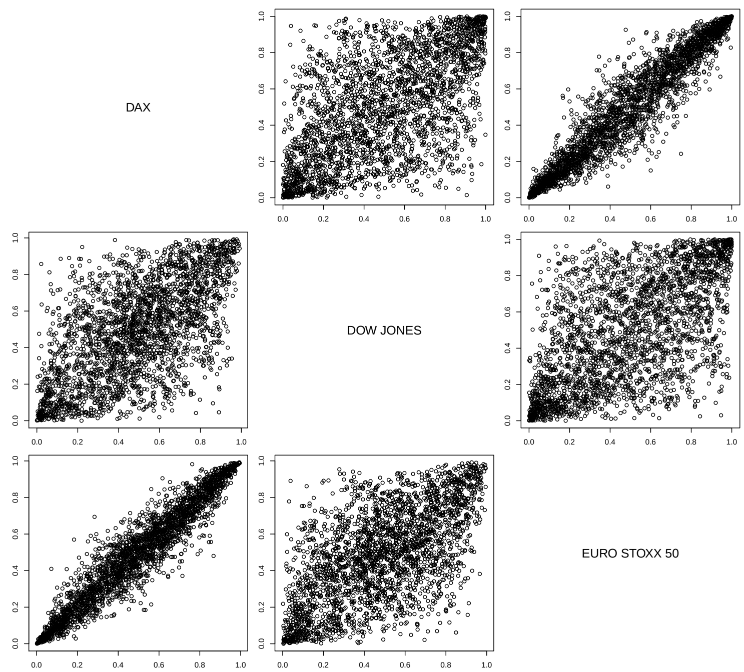


Figure 7: Pairwise scatter plots of the non-parametrically (above the diagonal) and parametrically (below the diagonal) transformed residuals of the ARMA-GARCH models for the DAX, Dow Jones and Euro Stoxx 50 indices.

In an intensive Monte Carlo study, we investigate the nominal level and the power of the proposed test. Unfortunately, our test is not powerful enough to reject samples of moderate and large sizes from the Gumbel copula. In the bivariate case, we propose to use it in combination with tests for symmetry and radial symmetry by Genest et al. (2012) and Genest and Nešlehová (2014), respectively. For moderate dependent data, our test has sufficient power starting from sample size 1000 for the considered Archimedean copulas, except the Gumbel family. When considering bivariate copulas derived from mixture constructions, the power depends on the values of the association parameters and the distance between them. In some cases, sufficient power can already be achieved using samples of size 5000. In future research, we aim to develop and design more powerful goodness-of-fit tests for elliptical copulas using small and moderate sample sizes.

Our test requires copula data, which is usually not available in empirical applications due to unknown marginal distributions. It seems that the performance of our asymptotic test is not significantly influenced by non-parametric estimation of unknown marginal distributions. The referees pointed out that the limiting distribution of the test statistic in case of unknown margins and bivariate data can be derived by considering the empirical copula process and applying the functional Delta method (see Theorem 3.9.4 in van der Vaart and Wellner (1996)). In the following, we outline this derivation.

Under non restrictive smoothness assumptions on the copula $C$, Segers (2012) showed that the empirical copula process converges weakly towards the Gaussian field $\mathbb{G}_C$, whose covariance structure is determined by the unknown copula $C$. Formulated for the bivariate case, it holds that

$$\sqrt{n}(C_n - C) \rightsquigarrow \mathbb{G}_C\,, \qquad in \quad \ell^\infty([0,1]^2),$$

where
$$C_n(u_1,u_2) = \frac{1}{n}\sum_{i=1}^{n} I\{F_{1,n}(X_{1i}) \le u_1, F_{2,n}(X_{2i}) \le u_2\}$$
is the empirical copula for the bivariate random sample $(X_{11}, X_{21})^\top, \ldots, (X_{1n}, X_{2n})^\top$ and $\ell^\infty([0,1]^2)$ denotes the metric space of all uniformly bounded functions on the unit square $[0,1]^2$ equipped with the metric induced by the supremum norm.

Using the representation of empirical Blomqvist's beta (see Schmid and Schmidt (2007) or Genest et al. (2013)) and Kendall's tau (see Gaenssler and Stute (1987)) through the empirical copula, our test statistic can be rewritten as
$$\widehat{\beta}_n - \widehat{\tau}_n = 4\, C_n\left(\frac{1}{2}, \frac{1}{2}\right) - 4\int_{[0,1]^2} C_n(v_1,v_2)dC_n(v_1,v_2)\,.$$
Generalizing Blomqvist's beta for any cutting point $(u_1,u_2)$ instead of $\left(\frac{1}{2}, \frac{1}{2}\right)$ in (3), one obtains $\beta(u_1,u_2)$ (see (5) in Schmid and Schmidt (2007)) and its empirical counterpart is then given by $\widehat{\beta}_n(u_1,u_2)$. By the functional Delta method, it follows that
$$\sqrt{n}\left(\widehat{\beta}_n(u_1,u_2) - \widehat{\tau}_n - (\beta(u_1,u_2)-\tau)\right) = \sqrt{n}\cdot(\phi(C_n)-\phi(C)) \rightsquigarrow \phi'(\mathbb{G}_C)\,,$$
where $\phi : \ell^\infty([0,1]^2) \to \ell^\infty([0,1]^2)$, $h \mapsto 4h - 4\int_{[0,1]^2} h(v_1,v_2)dh(v_1,v_2)$. Note that the mapping $\phi$ is Hadamard-differentiable at $\phi(C)$ due to Lemma 3.9.17 in van der Vaart and Wellner (1996) and this leads to the limiting Gaussian field
$$\tilde{\mathbb{G}}_C(u_1,u_2) := \phi'(\mathbb{G}_C) = 4\mathbb{G}_C(u_1,u_2) - 4\int_{[0,1]^2} C(v_1,v_2)d\mathbb{G}_C(v_1,v_2) - 4\int_{[0,1]^2} \mathbb{G}_C(v_1,v_2)dC(v_1,v_2)\,.$$
Hence, we get the following weak convergence result for our test statistic under the null hypothesis
$$\sqrt{n}\left(\widehat{\beta}_n\left(\frac{1}{2},\frac{1}{2}\right) - \widehat{\tau}_n\right) \rightsquigarrow N(0, \sigma^2_{\tilde{\mathbb{G}}_C})\,,$$
where $\sigma^2_{\tilde{\mathbb{G}}_C} = \mathbb{E}\left[\tilde{\mathbb{G}}_C\left(\frac{1}{2},\frac{1}{2}\right)^2\right]$.

The limiting Gaussian field $\tilde{\mathbb{G}}_C(u_1,u_2)$ and hence $\sigma^2_{\tilde{\mathbb{G}}_C}$ depend on the unknown copula $C$. Therefore, bootstrap procedures should be used to approximate the limiting distribution (see e.g. Chapter 2 in Bücher (2011)), which is possible due to Theorem 3.9.11 in van der Vaart and Wellner (1996). In future research, we consider bootstrap approximations for the limiting distribution $N(0,\sigma^2_{\tilde{\mathbb{G}}_C})$ and compare them to our naive approach from Section 4.5. Moreover, we intend to extend the described approach to the $d$-dimensional case.

### Acknowledgement

The authors would like to thank the Editor, the Associate Editor and two anonymous referees for constructive comments and helpful remarks, which lead to a significantly improved presentation of the manuscript. This work was supported by the German Research Foundation (DFG) and the Technical University of Munich within the funding programme Open Access Publishing.

## A Proof

In this section, we give the proof of Theorem 3.1 about the asymptotic normality of the difference statistic $\mathbf{D}_n$ and the limit distribution of the test statistic $T_n$.

*Proof of Theorem 3.1.* Let $d \geq 2$ be the dimension of the sample $\mathbf{U}_1, \ldots, \mathbf{U_n} \in [0,1]^d$ from the statistical model

$$\left(([0,1]^d)^n, \mathcal{B}([0,1]^d)^{\otimes n}, P^{\otimes n}\right),$$

where $P$ is a distribution with copula $C$ and uniform marginals. Next, we define the matrices

$$\mathcal{U}_i := (\mathrm{sgn}(U_{ki} - 0.5)\mathrm{sgn}(U_{\ell i} - 0.5))_{k,\ell \in \{1,\ldots,d\}} \in \mathbb{S}_d \quad \text{and} \quad \mathcal{H}_i := \left(2\tilde{h}_{k\ell,1}((U_{ki}, U_{\ell i})^\top)\right)_{k,\ell \in \{1,\ldots,d\}} \in \mathbb{S}_d$$

as well as

$$\boldsymbol{\beta} := (\beta_{k\ell})_{k,\ell \in \{1,\ldots,d\}} \in \mathbb{S}_d \qquad \text{and} \qquad \boldsymbol{\tau} := (\tau_{k\ell})_{k,\ell \in \{1,\ldots,d\}} \in \mathbb{S}_d\,,$$

where $\beta_{kk} := 1$ and $\tau_{kk} := 1$. Using the matrices $\mathcal{U}_i$ and $\mathcal{H}_i$, we define the vectors $\mathbf{Z}_i^\beta := \mathrm{vec}_u(\mathcal{U}_i)$ and $\mathbf{Z}_i^\tau := \mathrm{vec}_u(\mathcal{H}_i)$.

Now, we consider the empirical estimator

$$\widehat{\tau}_{k\ell,n} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \mathrm{sgn}(U_{ki} - U_{kj})\mathrm{sgn}(U_{\ell i} - U_{\ell j}) \tag{16}$$

of $\tau_{k\ell}$, which is a $U$-statistic of degree two with kernel function $h(\mathbf{w}_1, \mathbf{w}_2) := \mathrm{sgn}(w_{11} - w_{12})\mathrm{sgn}(w_{21} - w_{22})$, where $\mathbf{w}_i = (w_{1i}, w_{2i})^\top$, for $i = 1, 2$. Hoeffding's decomposition for $U$-statistics implies (see Theorem 1.2.1 in Denker (1985)) that $\widehat{\tau}_{k\ell,n} - \tau_{k\ell}$ can be represented as $2U_{k\ell,n1} + U_{k\ell,n2}$, where

$$U_{k\ell,n1} := \frac{1}{n} \sum_{i=1}^{n} \left(\tilde{h}_{k\ell,1}((U_{ki}, U_{\ell i})^\top) - \tau_{k\ell}\right)$$

and $U_{k\ell,n2} := (\widehat{\tau}_{k\ell,n} - \tau_{k\ell}) - 2U_{k\ell,n1}$. Note that $U_{k\ell,n2}$ is a $U$-statistic of degree two with the degenerate kernel $h_{k\ell,2}(\mathbf{w}_1, \mathbf{w}_2) = h(\mathbf{w}_1, \mathbf{w}_2) - \tilde{h}_{k\ell,1}(\mathbf{w}_1) - \tilde{h}_{k\ell,1}(\mathbf{w}_2) + \tau_{k\ell}$, i.e.

$$\mathbb{E}\left[h_{k\ell,2}\left((U_{k1}, U_{\ell 1})^\top, (U_{k2}, U_{\ell 2})^\top\right) |(U_{k1}, U_{\ell 1})^\top\right] = \mathbb{E}\left[h_{k\ell,2}(\mathbf{W}_1, \mathbf{W}_2) |\mathbf{W}_1\right] = 0$$

almost surely with $\mathbf{W}_1, \mathbf{W}_2 \overset{i.i.d.}{\sim} C_{k\ell}$. From Theorem 1.2.4 in Denker (1985) it follows that

$$\mathbb{E}\left[(\sqrt{n} \cdot U_{k\ell,n2})^2\right] \leq \frac{A_{k\ell,h}}{n}, \tag{17}$$

where $A_{k\ell,h}$ is a constant depending only on the kernel $h(\cdot,\cdot)$. Therefore, $\sqrt{n} \cdot U_{k\ell,n2} \xrightarrow{L_2} 0$ as $n \longrightarrow \infty$, and

$$\sqrt{n} \cdot (\widehat{\tau}_{k\ell,n} - \tau_{k\ell}) \quad \text{and} \quad \sqrt{n} \cdot 2U_{k\ell,n1} = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} 2\tilde{h}_{k\ell,1}((U_{ki}, U_{\ell i})^\top) - 2\tau_{k\ell}\right)$$

have the same limiting normal distribution. The multivariate central limit theorem implies

$$\sqrt{n}\left(\overline{\mathbf{Z}}_n - \begin{pmatrix} \mathrm{vec}_u(\boldsymbol{\beta}) \\ \mathrm{vec}_u(\boldsymbol{\tau}) \end{pmatrix}\right) \rightsquigarrow N(\mathbf{0}, \boldsymbol{\Sigma}),$$

where $\mathbf{Z}_i := \begin{pmatrix} \mathbf{Z}_i^\beta \\ \mathbf{Z}_i^\tau \end{pmatrix}$, $\overline{\mathbf{Z}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{Z}_i$ and

$$\boldsymbol{\Sigma} = \mathbb{C}\mathrm{ov}(\mathbf{Z}_1)\,. \tag{18}$$

For example, the covariance matrix $\boldsymbol{\Sigma}$ for $d = 2$, $k = 1$ and $l = 2$ has the following form

$$\begin{pmatrix} \mathbb{V}\mathrm{ar}\left(\mathrm{sgn}(U_{11} - 0.5)\mathrm{sgn}(U_{21} - 0.5)\right) & \mathbb{C}\mathrm{ov}\left(\mathrm{sgn}(U_{11} - 0.5)\mathrm{sgn}(U_{21} - 0.5), 2\tilde{h}_1((U_{11}, U_{21})^\top)\right) \\ \mathbb{C}\mathrm{ov}\left(\mathrm{sgn}(U_{11} - 0.5)\mathrm{sgn}(U_{21} - 0.5), 2\tilde{h}_1((U_{11}, U_{21})^\top)\right) & \mathbb{V}\mathrm{ar}\left(2\tilde{h}_1((U_{11}, U_{21})^\top)\right) \end{pmatrix}$$

For an elliptical copula $C$, the multivariate statistic $\mathbf{D}_n$ is then equal to $\overline{\mathbf{Z}^{\beta}}_n - \mathrm{vec}_u\left((\widehat{\tau}_{k\ell,n})_{k,\ell\in\{1,\ldots,d\}}\right)$, which has the same limiting distribution as

$$\overline{\mathbf{Z}^{\beta}}_n - \overline{\mathbf{Z}^{\tau}}_n\,.$$

By applying the Delta method (see Proposition 6.2 in Bildeau and Brenner (1999)) with

$$\phi : \mathbb{R}^{d(d-1)} \to \mathbb{R}^{d(d-1)/2}, \mathbf{x} \mapsto \left(x_1, \ldots, x_{d(d-1)/2}\right)^{\top} - \left(x_{d(d-1)/2+1}, \ldots, x_{d(d-1)}\right)^{\top},$$

we obtain

$$\sqrt{n}\left(\overline{\mathbf{Z}^{\beta}}_n - \overline{\mathbf{Z}^{\tau}}_n\right) \rightsquigarrow N\left(\mathbf{0}, \mathcal{V}\right)$$

under the null hypothesis $C \in \mathcal{C}^{ellipt}$, where

$$\mathcal{V} = \phi'\left(\begin{pmatrix}\mathrm{vec}_u(\boldsymbol{\beta})\\ \mathrm{vec}_u(\boldsymbol{\tau})\end{pmatrix}\right)\boldsymbol{\Sigma}\phi'\left(\begin{pmatrix}\mathrm{vec}_u(\boldsymbol{\beta})\\ \mathrm{vec}_u(\boldsymbol{\tau})\end{pmatrix}\right)^{\top}$$

and $\phi'$ denotes the Jacobian matrix of $\phi$. Since $\phi$ is a linear map, $\phi'$ is independent of $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$. Moreover, it is given by

$$\begin{pmatrix}\mathcal{I}_{d(d-1)/2} & -\mathcal{I}_{d(d-1)/2}\end{pmatrix},$$

where $\mathcal{I}_{d(d-1)/2}$ is the unit matrix of dimension $d(d-1)/2$.

The second statement of the theorem is obvious. The asymptotic distribution of $T_n$ defined in (14) follows from the asymptotic normality of $\mathbf{D}_n$, the multivariate Slutsky Theorem (see Lemma 6.3 in Bildeau and Brenner (1999)) and the continuous mapping theorem.

□

# References

Berg, D. (2009). Copula goodness-of-fit testing: an overview and power comparison. *Eur. J. Finance 15(7-8)*, 675-701.

Bilodeau, M. and D. Brenner (1999). *Theory of Multivariate Statistics*. New York: Springer.

Blomqvist, N. (1950). On a measure of dependence between two random variables. *Ann. Math. Stat. 21(4)*, 593-600.

Bücher, A. (2011). *Statistical inference for copulas and extremes*. PhD thesis, Ruhr-Universität Bochum, Germany.

Dengler, B. (2010). *On the asymptotic behaviour of the estimator of Kendall's Tau*. PhD thesis, Vienna University of Technology, Austria.

Denker, M. (1985). *Asymptotic Distribution Theory in Nonparametric Statistics*. Braunschweig: Vieweg.

Fang, K.-T., S. Kotz, and K. Ng (1990). *Symmetric Multivariate and Related Distributions*. London: Chapman and Hall.

Fang, H.-B., K.-T. Fang, and S. Kotz (2002). The meta-elliptical distributions with given marginals. *J. Multivariate Anal. 82(1)*, 1-16.

Fermanian, J. D. (2013). An overview of the goodness-of-fit test problem for copulas. In P. Jaworski, F. Durante and W. K. Härdle (Eds.), *Copulae in Mathematical and Quantitative Finance*, pp. 61-89. Springer, Berlin.

Gaenssler, P. and W. Stute (1987). *Seminar on Empirical Processes*. DMV Seminar Band 9. Basel: Birkhäuser.

Genest, C., A. Carabarín-Aguirre, and F. Harvey (2013). Copula parameter estimation using Blomqvist's beta. *J. SFdS 154(1)*, 5-24.

Genest, C. and J. MacKay (1986). The joy of copulas: bivariate distributions with uniform marginals. *Amer. Statist. 40(4)*, 280-283.

Genest, C. and J. G. Nešlehová (2014). On tests of radial symmetry for bivariate copulas. *Statist. Papers 55(4)*, 1107-1119.

Genest, C., J. Nešlehová, and J. F. Quessy (2012). Tests of symmetry for bivariate copulas. *Ann. Inst. Statist. Math. 64(4)*, 811-834.

Genest, C., B. Rémillard, and D. Beaudoin (2009). Goodness-of-fit tests for copulas: A review and a power study. *Insurance Math. Econom. 44(2)*, 199-213.

Hoeffding, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Stat. 19(3)*, 293-325.

Joe, H. (1990). Multivariate concordance. *J. Multivariate Anal. 35*, 12-30.

Kendall, M. G. and B. Babington Smith (1940). On the method of paired comparisons. *Biometrika 31(3/4)*, 324-345.

Klüppelberg, C., G. Kuhn, and L. Peng (2008). Semi-parametric models for the multivariate tail dependence function - the asymptotically dependent case. *Scand. J. Stat. 35(4)*, 701-718.

Kotz, S. and S. Nadarajah (2004). *Multivariate t-Distributions and Their Applications*. Cambridge: Cambridge University Press.

Li, D. and L. Peng (2009). Goodness-of-fit test for tail copulas modeled by elliptical copulas. *Statist. Probab. Lett. 79(8)*, 1097-1104.

Mises, R. v. (1947). On the asymptotic distribution of differentiable statistical functions. *Ann. Math. Stat. 18(3)*, 309-348.

Nelsen, R. B. (1999). *An Introduction to copulas*. Volume 139 of Lecture Notes in Statistics. New York: Springer Science and Business Media.

Mai, J.-F. and M. Scherer (2012). *Simulating Copulas: Stochastic Models, Sampling Algorithms and Applications*. Series in Quantitative Finance: Volume 4. Singapore: World Scientific.

Schmid, F. and R. Schmidt (2007). Nonparametric inference on multivariate versions of Blomqvist's beta and related measures of tail dependence. *Metrika 66(3)*, 323-354.

Segers, J. (2012). Asymptotics of empirical copula processes under non-restrictive smoothness assumptions. *Bernoulli 18(3)*, 764-782.

Sklar, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Université de Paris 8*, 229-231.

van der Vaart, A. W. and J. A. Wellner (1996). *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer Series in Statistics. New York: Springer

Miriam Jaser
Technical University of Munich
Boltzmannstraße 3
D-85748 Garching, Germany
miriam.jaser@tum.de

Stephan Haug
Technical University of Munich
Boltzmannstraße 3
D-85748 Garching, Germany
haug@tum.de

Aleksey Min
Technical University of Munich
Boltzmannstraße 3
D-85748 Garching, Germany
min@tum.de