

The smallest sets of points not determined by their X-rays*

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Abstract

Let F be an n -point set in \mathbb{K}^d with $\mathbb{K} \in \{\mathbb{R}, \mathbb{Z}\}$ and $d \geq 2$. A (discrete) X-ray of F in direction s gives the number of points of F on each line parallel to s . We define $\psi_{\mathbb{K}^d}(m)$ as the minimum number n for which there exist m directions s_1, \dots, s_m (pairwise linearly independent and spanning \mathbb{R}^d) such that two n -point sets in \mathbb{K}^d exist that have the same X-rays in these directions. The bound $\psi_{\mathbb{Z}^d}(m) \leq 2^{m-1}$ has been observed many times in the literature. In this note we show $\psi_{\mathbb{K}^d}(m) = O(m^{d+1+\varepsilon})$ for $\varepsilon > 0$. For the cases $\mathbb{K}^d = \mathbb{Z}^d$ and $\mathbb{K}^d = \mathbb{R}^d$, $d > 2$, this represents the first upper bound on $\psi_{\mathbb{K}^d}(m)$ that is polynomial in m . As a corollary we derive bounds on the sizes of solutions to both the classical and two-dimensional Prouhet-Tarry-Escott problem. Additionally, we establish lower bounds on $\psi_{\mathbb{K}^d}$ that enable us to prove a strengthened version of Rényi's theorem for points in \mathbb{Z}^2 .

1 Introduction

The problem of reconstructing point sets from their X-rays has a long history; perhaps the 1952 paper [20] by Rényi represents one of the first works in this field. Of special interest are questions of uniqueness. Two sets with the same X-rays are said to be *tomographically equivalent* [8], [9]; the sets are also commonly referred to as *switching components* [13], [22] or *ghosts* [12, Sect. 15.4]. In [15] Matoušek, Přívětivý, and Škovroň show that almost all sets of m directions (in the sense of measure) allow for a unique reconstruction of $2^{Cm/\log(m)}$ -point sets in the real plane (here $C > 0$ is a constant and the result holds for large m). For almost all choices of m directions there thus exist only superpolynomial size switching components. By a careful selection of directions, however, we can reduce them to a polynomial size.

To make this precise, let F be an n -point set in \mathbb{K}^d with $\mathbb{K} \in \{\mathbb{R}, \mathbb{Z}\}$ and $d \geq 2$. A (discrete) X-ray of F in direction s gives the number of points of F on each line parallel to s . We define $\psi_{\mathbb{K}^d}(m)$ as the minimum number n for which there exist m directions s_1, \dots, s_m (pairwise linearly independent and spanning \mathbb{R}^d) such that two different n -point sets in \mathbb{K}^d exist that have the same X-rays in these directions. We derive lower and upper bounds on $\psi_{\mathbb{K}^d}$.

Two constructions are known to yield upper bounds on $\psi_{\mathbb{K}^d}$. The first construction is based on regular polygons. The two disjoint m -point sets of alternate vertices of a regular $2m$ -gon in \mathbb{R}^2 yield $\psi_{\mathbb{R}^2}(m) \leq m$. This cannot be transferred to \mathbb{Z}^d as any (planar) regular polygon with integer vertices must have 3, 4 or 6 vertices [21, 3]. The functions $\psi_{\mathbb{R}^2}$ and $\psi_{\mathbb{Z}^2}$ are, in fact, different functions as we show $\psi_{\mathbb{Z}^2}(m) \geq m + 1$ if $m = 5$ or $m > 6$ (see Thm. 2.2). From this we derive a strengthened version of Rényi's theorem (see Thm. 2.1 and Cor. 2.3 in Sect. 2).

The second well-known construction for upper bounds on $\psi_{\mathbb{K}^d}$ is based on two-colorings of the unit cube $[0, 1]^m$ in \mathbb{Z}^m . More precisely, two different sets with equal X-rays in coordinate directions are obtained as the two disjoint sets of 2^{m-1} alternate vertices of $[0, 1]^m$. By projecting into \mathbb{Z}^d , the bound $\psi_{\mathbb{Z}^d}(m) \leq 2^{m-1}$ is obtained. This construction seems to be due to Lorentz [14]; see also [2], [5, Lem. 2.3.2], and [7, Thm. 4.3.1]. As $\mathbb{Z}^d \subseteq \mathbb{R}^d$, this, of course, yields also $\psi_{\mathbb{R}^d}(m) \leq 2^{m-1}$.

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Our main observation is contained in the statement of Thm. 3.3, where we prove $\psi_{\mathbb{Z}^d}(m) = O(m^{d+1+\varepsilon})$ for $\varepsilon > 0$. This is, to our knowledge, the first upper bound on $\psi_{\mathbb{Z}^d}(m)$ that is polynomial in m . Our proof is non-constructive.

We conclude in Sect. 4 by stating some remarks and consequences that relate our bounds to the *Prouhet-Tarry-Escott* problem from number theory (see, e.g., [10, Sect. 21.9]).

Throughout the paper, ζ is the Riemann zeta function, m and n denote natural numbers, and \mathbb{Z} , \mathbb{R} , $\mathbb{N} = \{1, 2, \dots\}$ are, respectively, the sets of integers, reals, and natural numbers. We use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[n] = \{1, \dots, n\}$, and $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$. With $G_n^d = [n]^d$ we denote the set of d -tuples of positive integers less than or equal to n . If $\xi \in \mathbb{R}$, then $\lceil \xi \rceil$ denotes the smallest integer greater than or equal to ξ . The symbol O has the usual meaning: $f(m) = O(g(m))$ means that $f(m)/g(m)$ is bounded as $m \rightarrow \infty$. A property is said to hold for large m if that property holds for all m larger than some m_0 .

2 Lower bounds

In this section we derive lower bounds on $\psi_{\mathbb{K}^d}$. The key ideas are not new, but appear scattered and isolated in different contexts in the literature (see [20] and the proof of Thm. 2.2 in [1]).

Theorem 2.1. *For every $d \geq 2$ we have $\psi_{\mathbb{K}^d}(m) \geq m$.*

Proof. This is a reformulation of Rényi's theorem (proved in [20] and generalized to arbitrary dimensions by Heppes [11]), which states that any n -point set in \mathbb{K}^d is uniquely determined by its X-rays from $n + 1$ different directions. For completeness, we reproduce a short proof. Suppose there are two sets F, F' with equal X-rays in $m + 1$ directions, each set containing at most m points. Without loss of generality there exists a point $p \in F \setminus F'$. Since F and F' have equal X-rays, there needs to be a point of F' on each of the $m + 1$ lines through p . This implies that F' contains at least $m + 1$ points, a contradiction. \square

The bound is tight for $m \in \{1, 2, 3, 4, 6\}$ and $\mathbb{K}^d = \mathbb{Z}^2$; examples showing this for $m = 1, 2, 3, 4, 6$ are respectively provided by any two 1-point sets in \mathbb{Z}^2 , two-colorings of the unit cube in \mathbb{Z}^2 , the sets $F = \{(0, 0), (1, 2), (2, 1)\}$, $F' = \{(1, 0), (0, 1), (2, 2)\}$, and the examples shown in Fig. 4.3 and Fig. 4.5 of [7]. For the remaining cases, however, we can improve the bound as stated in the following result.

Theorem 2.2. *If $m = 5$ or $m > 6$ then $\psi_{\mathbb{Z}^2}(m) \geq m + 1$.*

Proof. Let $m = 5$ or $m > 6$, and suppose there exist different n -point sets $F, F' \subseteq \mathbb{Z}^2$ with equal X-rays in $m \geq n$ directions. Without loss of generality we can assume that $F \cap F' = \emptyset$. The convex hull P of $F \cup F'$ is a non-degenerate polygon with at most $2n$ vertices. Parallel to each of the m directions there are two lines that support P with each line containing a single point from F and F' , respectively (since otherwise one of F and F' contains more than n points). Since this implies that P has at least $2m$ edges, we conclude that at least $2m$ of the elements of $F \cup F'$ are vertices of P (i.e. $n = m$), proving that $F \cup F'$ is the set of vertices of the non-degenerate convex $2m$ -gon P . Since F and F' have the same X-rays, P has the property that any line through a vertex of P in any of the m directions meets another vertex of P . Such polygons are known as lattice U -gons with U denoting the set of m directions. They, however, do not exist for $m > 6$ (see Thm. 4.5 in [6]). As is shown in the proof of Thm. 4.5 of [6] or (more simply) in Thm. 6 of [1], there are also no lattice U -gons for exactly 5 directions. In other words, we have $\psi_{\mathbb{Z}^2}(m) > m$ for $m = 5$ or $m > 6$. \square

The bound is tight for $m = 5$. For this consider the 6-point sets

$$F = \{(0, 2), (1, 4), (2, 2), (3, 0), (4, 3), (5, 1)\} \text{ and } F' = \{(0, 3), (1, 1), (2, 4), (3, 2), (4, 0), (5, 2)\}.$$

It is easily verified that F and F' have the same X-rays in the 5 directions

$$S = \{(1, 0), (0, 1), (1, 1), (1, -1), (-2, 1)\}.$$

A reformulation of Thm. 2.2 provides a strengthened version of Rényi's theorem for \mathbb{Z}^2 .

Corollary 2.3. *Any n -point set in \mathbb{Z}^2 with $n = 5$ or $n > 6$ is uniquely determined by its X-rays taken from at least n different directions.*

3 Upper bounds

In this section we prove a polynomial upper bound on $\psi_{\mathbb{K}^d}$. As a prelude, we prove an upper bound on the number of lines parallel to a given direction that intersect points of G_n^d . This is followed by a lemma that asserts the existence of certain coverings of a specified finite part of the integer lattice by m families of parallel lines.

Lemma 3.1. For any relatively prime d -tuple $s = (\sigma_1, \dots, \sigma_d) \in \mathbb{N}_0^d \setminus \{0\}$ with $d \geq 2$ there are at most $dn^{d-1} \cdot \max\{\sigma_1, \dots, \sigma_d\}$ lines parallel to s that intersect G_n^d .

Proof. For each line ℓ parallel to $s = (\sigma_1, \dots, \sigma_d)$ that intersects G_n^d , there is a unique point $p \in \ell \cap G_n^d$ for which $p - s \notin G_n^d$. The point $p - s$ needs to have a non-positive component, i.e.,

$$p \in V_i = \{(\xi_1, \dots, \xi_d) \in G_n^d : 1 \leq \xi_i \leq \sigma_i\}$$

for an $i \in [d]$. As the number of points in $\bigcup_{i=1}^d V_i$ is clearly bounded by $dn^{d-1} \cdot \max\{\sigma_1, \dots, \sigma_d\}$, we obtain the claimed result. (Tight bounds can be obtained similarly via the inclusion-exclusion principle, but they are not needed in the present context.) \square

Lemma 3.2. Let $\varepsilon > 0$, $m \in \mathbb{N}$, $d \geq 2$, and $n \in \{ \lceil m^{1+(1+\varepsilon)/d} \rceil, \lceil m^{1+(1+\varepsilon)/d} \rceil + 1 \}$. Then, for large m there is a set $S = \{s_1, \dots, s_m\} \subseteq \mathbb{Z}^d$ with the property that

- (i) the elements of S are pairwise linearly independent spanning \mathbb{R}^d ;
- (ii) the total number l of lines that are parallel to a direction in S and intersect G_n^d is bounded from above by $2^{1+1/d} dn^{d-1} m^{1+1/d}$.

Proof. For the number $R(p, d)$ of relatively prime d -tuples in G_p^d , $p \in \mathbb{N}$, it holds by [17] that

$$\lim_{p \rightarrow \infty} \frac{R(p, d)}{p^d} = \frac{1}{\zeta(d)}.$$

As ζ decreases for values larger than 1 and since $\zeta(2) = \pi^2/6 < 2$, we have

$$R(p, d) > p^d/2$$

for large p .

Setting $q = \lceil (2m)^{1/d} \rceil$, we note that $q \leq 2(2m)^{1/d}$ and $q \leq n$ for $m \geq 2$. For large m we have

$$R(q, d) > q^d/2 \geq m,$$

so for our set S we can select m elements from $G_q^d \subseteq G_n^d$. We can assume that the elements of S span \mathbb{R}^d since otherwise we replace d of the directions by the standard unit vectors. Property (i) is thus fulfilled (note that the elements of S are relatively prime d -tuples).

The entries of the elements in S are bounded by q , so by Lem. 3.1 we have at most

$$mdn^{d-1}q \leq 2^{1+1/d} dn^{d-1} m^{1+1/d}$$

lines parallel to a direction in S that intersect G_n^d . \square

Theorem 3.3. For every $\varepsilon > 0$ and $d \geq 2$ it holds that $\psi_{\mathbb{Z}^d}(m) = O(m^{d+1+\varepsilon})$.

Proof. We assume that m is large enough that the set S from Lem. 3.2 exists. We set

$$n = \left\{ \left\lceil m^{1+(1+\varepsilon)/d} \right\rceil, \left\lceil m^{1+(1+\varepsilon)/d} \right\rceil + 1 \right\} \cap 2\mathbb{Z}$$

and $k = \frac{1}{2}n^d$. Note that $k \in \mathbb{N}$, $k = O(m^{d+1+\varepsilon})$, and that we can assume that $n \geq 4$.

Let l_i , $i \in [m]$, denote the number of lines parallel to s_i that intersect G_n^d . The X-ray in direction s_i of a set in G_n^d with cardinality k gives a *weak k -composition* of l_i , i.e., a solution to $\xi_1 + \dots + \xi_{l_i} = k$ in nonnegative integers [23, p. 15]. (The converse is generally false, because the corresponding X-ray lines may intersect G_n^d in fewer points than provided by a weak k -composition of l_i .) The number of weak k -compositions of l_i is given by

$$N(k, l_i) = \binom{k + l_i - 1}{l_i - 1}$$

and thus represents an upper bound for the number of different X-rays of k -point subsets of G_n^d in the direction s_i .

With $l = l_1 + \dots + l_m$ we thus obtain the following upper bound on the number of different X-rays (for the directions in S) that can originate from a subset of G_n^d with cardinality k :

$$\prod_{i=1}^m N(k, l_i) \leq \prod_{i=1}^m \binom{n^d/2 + l_i}{l_i} \leq \prod_{i=1}^m \left(\frac{(n^d/2 + l_i)^e}{l_i} \right)^{l_i} = \prod_{i=1}^m \left(\frac{n^d e}{2l_i} + e \right)^{l_i} \leq (ne + e)^l \leq n^{2l};$$

here the inequalities (from left to right) follow from $N(k, l_i) \leq N(k, l_i + 1)$, a standard inequality for binomial coefficients (see, e.g., [18, Eq. (4.9)]), $l_i \geq n^{d-1}$, and $n \geq 4$, respectively.

There are

$$\binom{n^d}{n^d/2} \geq 2^{n^d/2}$$

subsets of cardinality k in G_n^d . We claim that

$$n^{2l} < 2^{n^d/2}$$

holds for large m , which, by the pigeonhole principle, concludes the proof as it implies the existence of two sets in G_n^d with cardinality k and equal X-rays in the directions in S .

For the claim we first note that

$$m^{1+(1+\varepsilon)/d} \leq n \leq 3m^{1+(1+\varepsilon)/d} \quad (1)$$

holds as $m^{1+(1+\varepsilon)/d} \geq 1$. It is easy to see that $\lim_{x \rightarrow \infty} x^a/2^{x^b} = 0$ for $a, b > 0$. Thus for large m and $C = 2^{3+1/d}$ we have

$$3^C m^{C(1+(1+\varepsilon)/d)} < 2^{m^{\varepsilon/d}},$$

which, by (1) and Property (ii) of Lem. 3.2, gives

$$n^C < 2^{m^{\varepsilon/d}} \Rightarrow n^{Cm^{1+1/d}} < 2^n \Rightarrow n^{Cn^{d-1}m^{1+1/d}} < 2^{n^d} \Rightarrow n^{4l} < 2^{n^d},$$

proving the claim. \square

4 Remarks and consequences

The previously mentioned regular $2m$ -gon construction in \mathbb{R}^2 , together with the inequality $\psi_{\mathbb{R}^d}(m) \leq \psi_{\mathbb{Z}^d}(m)$ for $d \geq 2$, yields the following corollary to Thm. 3.3.

Corollary 4.1. *For every $\varepsilon > 0$ and $d \in \mathbb{N}$, it holds that*

$$\psi_{\mathbb{R}^d}(m) = \begin{cases} m & \text{if } d = 2, \\ O(m^{d+1+\varepsilon}) & \text{if } d > 2. \end{cases}$$

In [1] the *general Prouhet-Tarry-Escott problem* (PTE_r) was introduced: Given $k, n, r \in \mathbb{N}$, find two different multi-sets $\{x_1, \dots, x_n\}, \{y_1, \dots, y_n\} \subseteq \mathbb{Z}^r$ where $x_i = (\xi_{i1}, \dots, \xi_{ir})$, $y_i = (\eta_{i1}, \dots, \eta_{ir})$ for $i \in [n]$ such that

$$\sum_{i=1}^n \xi_{i1}^{j_1} \xi_{i2}^{j_2} \cdots \xi_{ir}^{j_r} = \sum_{i=1}^n \eta_{i1}^{j_1} \eta_{i2}^{j_2} \cdots \eta_{ir}^{j_r}$$

for all nonnegative integers j_1, \dots, j_r with $j_1 + \dots + j_r \leq k$. The parameter k is called the *degree* and n the *size* of the solution. Tracing back to works of Euler and Goldbach [4, p. 705], the Prouhet-Tarry-Escott problem (PTE₁) is an old and largely unsolved problem in Diophantine analysis. The following corollary sharpens the bound of [1, Thm. 12] on the size of solutions, which for (PTE₁) is due to Prouhet [19].

Corollary 4.2. *For every $\varepsilon > 0$ there exists a constant $C > 0$ such that there are solutions of (PTE₂) of degree k and size bounded by $Ck^{3+\varepsilon}$.*

Proof. In Thm. 8 of [1] it was shown that tomographically equivalent sets in \mathbb{Z}^2 for m directions yield (PTE₂) solutions of degree $m - 1$. This and Thm. 3.3 for $d = 2$ imply the statement of this corollary. \square

Remark 1. *As the products cancel, it is evident that solutions of (PTE₁) can be obtained by applying to (PTE₂) solutions a suitable linear functional that maps $(\xi_1, \xi_2) \in \mathbb{Z}^2$ to $\alpha_1 \xi_1 + \alpha_2 \xi_2$ where $\alpha_1, \alpha_2 \in \mathbb{Z}$ are suitably chosen. The current best bounds for (PTE₁) are quadratic in k (see [16], [24]); the bound from Thm. 3.3 is in this case weaker.*

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