

## Risk Management and Portfolio Selection Using $\alpha$ -Stable Regime Switching Models

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### Abstract

This article tries to enhance traditional distribution paradigms for modelling asset returns by considering an  $\alpha$ -stable regime-switching model. Our approach is to perform an empirical test of the  $\alpha$ -stable regime-switching model against other common methods in two settings: in risk management and in portfolio selection. Our empirical study will show that the model is better suited than Gaussian and Gaussian regime-switching models to measure risk accurately. A portfolio optimization case study for a traditional stocks and bonds investor is pursued. In this study, the model leads to less risky and more diversified portfolios. In particular, the model avoids outsized losses in times of crisis and thus leads to a better (adjusted) Sharpe ratio and Omega.

**Keywords:** Markov switching, regime switching, stable distribution, risk management, portfolio selection

## 1 Introduction

The dependence structure between asset prices plays a central role in the risk management of complex portfolios. Since the introduction of Risk Metrics in the early nineties, a constant covariance structure underlies many risk management theories, both in the academic literature as well as in regulatory-driven bank implementations. The advent of copulas signified a substantial enhancement to our understanding of risk factors and it has given rise to theories and practices where dependence structures of asset prices are unleashed from prior, constraining Gaussian settings. Much of the success of the copula approach is based on the ability of copulas to use well known distributions (such as the gaussian) in settings where they do not normally apply. One such setting which has not attracted much work is the multivariate  $\alpha$ -stable, despite the fact that when it was first introduced by [1], as a univariate distribution, it displayed several interesting features.

This article is also shaped by the circumstances of the financial crisis, which started in 2007, and its remarkable impact on asset prices. Not only did credit default swap (CDS) spreads increase sharply and (later on) stock prices drop enormously. Volatilities and correlations rose at the same time. Therefore one could doubt whether assets follow the same statistical pattern over time. Indeed the crisis boosts the idea that markets and assets have different regimes, e.g. periods of time with statistical properties which are distinct from each other. Thus, for modeling assets and markets, regime-switching models are seen as one way to accommodate the observed shifts in market regimes. As another consequence of the crisis, the assumption of a Gaussian distribution in many financial models has been challenged by both practitioners and academics. Single stock return time series for example are subject to skewness and excess kurtosis and thus are empirically not normally distributed. But when portfolios of stocks are constructed, multivariate correlation effects add to tail events over and beyond the individual tail characteristics of individual stock distributions. Therefore, tail correlations are key. In this article we will try to include both points in one model. On the one hand, we use a multivariate  $\alpha$ -stable distribution model, which has a powerful dependence structure, and on the other hand we will use a regime-switching model. For simplicity we will restrict the model to two regimes.

Regime-switching models explicitly take into account different state process, which describes the market regimes, is an unobservable Markov chain. This model is, in itself, able to capture kurtosis, skewness, autocorrelation, and volatility clustering (cf. [2]). One of the first to use Markov switching in an economic context was [3]. He used the model to explain shifts in the growth rate of the gross national product and thus identified positive and negative growth regimes. [4] focused on the regime shift in correlations be-

tween equity markets. They employed Gaussian Markov-switching models and found empirical evidence for the existence of a regime with high volatility and high correlations and a regime with low volatility and low correlations. [5] extend this finding using a two-state Gaussian Markov-switching model to model volatility and correlation behaviour between different asset classes. In a portfolio optimization case study, they show that the consideration of regime switching in portfolio optimization leads to a better portfolio performance and lower portfolio risk compared to a non-switching standard Normal distribution model. [6] employ a Markov-switching model for the asset allocation of a fund of hedge funds. An application to the credit market, in particular for the pricing of collateralized debt obligations (CDO) can be found in [7]. [8] employ a multivariate regime-switching approach for the estimation of Value-at-Risk. [9] introduces a general regime-switching Lévy process model in continuous time. He applies this univariate model to option pricing.

In this article, the regime switching is done between  $\alpha$ -stable distributions, introduced by [10]. This distribution was first employed in economics and finance by ([1, 11]) in his works about the distribution of incomes respectively cotton prices. Since then, a lot of different articles and books have been written about the  $\alpha$ -stable distribution. See for example [12] and [13] for a study of the model of  $\alpha$ -stable distributions and its properties. [14] show applications in finance. ([15, 16]) give an overview about using the  $\alpha$ -stable distribution to financial data. ([17, 18]) deal with portfolio selection using  $\alpha$ -stable distributions. [19] present dynamic portfolio strategies and compare the forecasting power of different  $\alpha$ -stable models for Value at Risk prediction. Some authors also combine the stable distribution with generalized autoregressive conditional heteroskedasticity (GARCH) or mixture models (cf. [15]). [15] propose a mixed-stable model for describing the stagnation periods in baltic stocks. But what is critical in our approach is that the  $\alpha$ -stable distribution be multivariate; its dependence structure has been shown to offer unique properties when used in a risk management framework in [20]. As a sample of its characteristics, we note that the dependence structure is essentially modelled by the spectral measure, a measure on the unit multidimensional sphere; when one views the gaussian in this perspective, the corresponding measure would be symmetric.  $\alpha$ -stable distributions do not require the spectral measure to be symmetric, which has interesting financial interpretations. When modeling financial data by an  $\alpha$ -stable distribution, it is generally assumed that  $1 < \alpha \leq 2$ . This is in accord with most empirical estimates of financial asset returns ([15, 16]).

Gaussian regime-switching models are not able to describe regime-dependent fat tails. While a Gaussian regime-switching model can distinguish different market regimes like calm and turbulent periods, it is not able to describe crises adequately. Crisis periods can be regarded as the fat tails of the turbulent mar-

ket regime. For example, the period from 2007 until 2009, referred to as the financial crisis, can be regarded as a turbulent market phase with overall negative returns and high volatilities. During that period, the liquidity crisis in 2008 and the Lehman collapse lead to the most severe drops in asset prices. These extreme events lead to stock price movements, which are typically described best by a fat-tailed distribution. Hence, the liquidity crisis can be considered as the fat tail of the financial crisis. This behaviour can be modelled by a regime-switching model which employs fat-tailed distributions. Observations of empirical data point in the same direction. [5] found in their paper that returns in the turbulent regime are still subject to skewness and excess kurtosis and thus are empirically not normally distributed. To sum up, the proposed  $\alpha$ -stable regime-switching model is capable to describe autocorrelation by employing different market regimes and regime-dependent tail behaviour by using the  $\alpha$ -stable distribution. To our best knowledge, no  $\alpha$ -stable regime switching model has been proposed in a discrete-time multivariate setting, in a risk management or asset management context.

This article makes the following contributions: We propose a univariate as well as a multivariate two-state  $\alpha$ -stable regime-switching model. We intend to use the model in the risk management and asset management context. We consider the univariate model to analyze return time series of a major stock index, estimate the parameters of the model and detect the regimes using the BaumWelch and Viterbi algorithms. Furthermore, we apply this model in a risk management context. Using the multivariate model, we study an application in portfolio selection. We compare different regime-switching and non-switching models with and without fat tails. We show that an  $\alpha$ -stable regime-switching model is better suited to measure risk than the standard Gaussian approach. In applications to the portfolio selection of stocks and bonds, we find that the model leads to less risky and more diversified portfolios, compared to the standard Gaussian and Gaussian switching counterpart. This leads to a better (adjusted) Sharpe ratio and Omega. In addition to that, we found that the model avoids huge drawdowns in times of crisis.

The paper is structured as follows. In Section 2 we give some properties of the  $\alpha$ -stable distribution. In Section 3 we present our model. The methods for estimation are described in Section 3.2. Thereafter, we show the estimation results and the detection of regimes for a major stock index. The model is applied to risk measurement and portfolio selection in Sections 4 and 5. Finally, we conclude.

## 2 Alpha-stable distributions

Following [13], an univariate  $\alpha$ -stable distribution could be defined by its characteristic function. Hence, a random variable  $X$  taking values on the

real line (ie,  $X \in \mathbb{R}$ ) has an  $\alpha$ -stable distribution if there are parameters  $0 < \alpha \leq 2, \sigma > 0, -1 \leq \beta \leq 1$  and  $\mu \in \mathbb{R}$  such that its characteristic function has the form<sup>1</sup>:

$$\varphi(t) = \mathbb{E}[\exp(itX)] = \exp\left\{-\sigma^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sgn}(t) \tan \frac{\pi\alpha}{2}\right) + i\mu t\right\} \quad (1)$$

where  $\operatorname{sgn}(t)$  is the usual signum function, ie,  $\operatorname{sgn}(t) = 1$  if  $t > 0$ ,  $\operatorname{sgn}(t) = 0$  if  $t = 0$ , and  $\operatorname{sgn}(t) = -1$  if  $t < 0$ .

The characteristic function in (1) uses four parameters.  $\alpha$  is the index of stability,  $\beta$  is the skewness parameter,  $\mu$  is the location parameter and  $\sigma$  the scale parameter. Thus, we will denote  $\alpha$ -stable distributions by  $S_\alpha(\sigma, \beta, \mu)$  and write

$$X \sim S_\alpha(\sigma, \beta, \mu),$$

to indicate that  $X$  follows the  $\alpha$ -stable distribution  $S_\alpha(\sigma, \beta, \mu)$ .

The probability densities of  $\alpha$ -stable random variables exist and are continuous, but apart from the Gaussian distribution ( $\alpha = 2$ ), the Cauchy distribution ( $\alpha = 1, \beta = 0$ ) and the Lévy distribution ( $\alpha = 1/2, \beta = 1$ ), they are not known in closed form (cf. [12]). However, several approximation methods exist (see [12, 21, 22, 23]).

The most important properties of the  $\alpha$ -stable distribution are the stability of summation and the generalized central limit theorem. The stability of summation gives a statement about the distribution of the sum of an arbitrary number of independent and identically distributed (i.i.d.)  $\alpha$ -stable random variables. If  $X_1, \dots, X_n$  are i.i.d.  $S_\alpha(\sigma, \beta, \mu)$  and  $\alpha \neq 1$ , then<sup>2</sup>

$$X_1 + \dots + X_n \stackrel{d}{=} n^{1/\alpha} X_1 + \mu(n - n^{1/\alpha}) \quad (2)$$

The property above gives an important property of  $\alpha$ -stable random variables: The sum of i.i.d.  $\alpha$ -stable random variables is again  $\alpha$ -stable.  $\sum_{i=1}^n X_i$  is distributed by  $S_\alpha(\sigma \cdot n^{1/\alpha}, \beta, \mu \cdot n)$ . This result is usually called *closure under convolution* or *invariance under convolution*. The class of  $\alpha$ -stable random variables is thus *closed under convolution*. This is an important property for modeling financial data, as asset prices are the result of a sum of random movements. For  $\alpha = 2$ , (2) is the well-known *closure under convolution* property of the Gaussian distribution.

For the  $\alpha$ -stable distribution the generalized central limit theorem applies. The generalized central limit theorem is a generalization of the central limit theorem to the case that the finite variance assumption is dropped. Under

<sup>1</sup>The formula is valid only for  $\alpha \neq 1$ . For our applications to financial data, which exhibit an  $\alpha > 1$ , this assumption is justified. For  $\alpha = 1$ , a slightly different version applies:  $\varphi(t) = \exp\{-\sigma |t| (1 + i\beta \frac{2}{\pi} \operatorname{sgn}(t) \ln |t|) + i\mu t\}$ .

<sup>2</sup>For  $\alpha = 1$ , the formula is a little bit different:  $X_1 + \dots + X_n \stackrel{d}{=} n^{1/\alpha} X_1 + \frac{2}{\pi} \sigma \beta n \ln(n)$ .

this assumption, the  $\alpha$ -stable distribution is the only limiting distribution of a sum of i.i.d. random variables. This is a rather attractive property of the  $\alpha$ -stable distribution, as it gives a justification for modeling various phenomena as  $\alpha$ -stable random variables.

The tails of an  $\alpha$ -stable random variable with  $0 < \alpha < 2$  follow a power-law<sup>3</sup>, ie, the upper tail probability is given by

$$\mathbb{P}(X > x) \sim C_u(\alpha, \beta, \sigma)x^{-\alpha} \quad (3)$$

as  $x \rightarrow \infty$ .  $C_u(\alpha, \beta, \sigma)$  is constant and only depends on  $\alpha$ ,  $\beta$  and  $\sigma$ . Similarly, the lower tail follows a power-law:  $\mathbb{P}(X < -x) \sim C_l(\alpha, \beta, \sigma)x^{-\alpha}$ . In this case, a slightly different constant  $C_l(\alpha, \beta, \sigma)$  is employed. The tail behavior has some influence on the moments, too. For  $\alpha$ -stable random variables with  $1 < \alpha < 2$  the second and all higher moments are not finite anymore. However, the first moment is finite and the shift parameter  $\mu$  equals the first moment, ie,  $\mathbb{E}[X] = \mu$ . For  $\alpha \leq 1$ , even the first moment is not finite anymore.

Like in the univariate case, the characteristic function for an  $\alpha$ -stable random vector can be given explicitly. It defines a representation of  $\alpha$ -stable random vectors and provides the basis for further analysis of properties of the multivariate  $\alpha$ -stable distribution. The following expression of the characteristic function involves an integration over the unit sphere in  $\mathbb{R}^d$ . The unit sphere<sup>4</sup> in  $\mathbb{R}^d$  is defined as  $S_d := \{\mathbf{s} \in \mathbb{R}^d : \|\mathbf{s}\|_2 = 1\}$ . For example,  $S_1$  is the two point set  $\{-1, 1\}$  and  $S_2$  is the unit circle. In the following,  $\langle \cdot, \cdot \rangle$  denotes the usual inner product, ie, for  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ ,  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^d x_k y_k$ .

Let  $0 < \alpha \leq 2$ . Then  $\mathbf{X} = (X_1, X_2, \dots, X_d)$  is an  $\alpha$ -stable random vector in  $\mathbb{R}^d$  if and only if there exists a finite measure  $\Gamma : S_d \rightarrow \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$  on the unit sphere  $S_d$  of  $\mathbb{R}^d$  and a vector  $\boldsymbol{\mu} \in \mathbb{R}^d$  such that the characteristic function looks as follows:

$$\varphi(\mathbf{t}) = \mathbb{E}[\exp(i \langle \mathbf{t}, \mathbf{X} \rangle)] = \exp(-I(\mathbf{t}) + i \langle \boldsymbol{\mu}, \mathbf{t} \rangle) \quad (4)$$

where

$$I(\mathbf{t}) = \int_{S_d} \psi_\alpha(\langle \mathbf{t}, \mathbf{s} \rangle) d\Gamma(\mathbf{s}) \quad (5)$$

<sup>3</sup>In the Gaussian case, ie,  $\alpha = 2$  the tail property has another form (cf. [24]):

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}(x < -\lambda) = \lim_{\lambda \rightarrow \infty} \mathbb{P}(X > \lambda) = \lim_{\lambda \rightarrow \infty} \frac{\sigma}{\sqrt{\pi\lambda}} \exp(-\lambda^2/(4\sigma^2))$$

<sup>4</sup>The norm in this definition is the *Euclidean norm*. This is the standard used in the representation of the characteristic function of  $\alpha$ -stable random vectors. However, it is possible to use the unit sphere relative to any other norm for the representation of the characteristic function (see [13], Proposition 2.3.8).

and<sup>5</sup>

$$\psi_\alpha(u) = |u|^\alpha \left(1 - i \tan\left(\frac{\pi\alpha}{2}\right) \text{sign}(u)\right) \tag{6}$$

The pair  $(\Gamma, \boldsymbol{\mu})$  is unique for  $0 < \alpha < 2$ . For  $\alpha = 2$ , the characteristic function is given by (4), but the representation is not unique (cf. [13], p. 76f).

Similar to the univariate case, a closed-form density is not available for multivariate  $\alpha$ -stable random vectors. For the subclass of elliptically contoured  $\alpha$ -stable random vectors, ([25, 26]) derives a series expansion for density calculation. For the general case, a numerical density calculation has to be pursued. [27] present a method to calculate two-dimensional  $\alpha$ -stable densities by numerically inverting the characteristic function.

The sum of i.i.d.  $\alpha$ -stable random vectors is again an  $\alpha$ -stable random vector. Thus, the closure under convolution holds for the multivariate  $\alpha$ -stable distribution, too. Moreover, any linear combination of the components of an  $\alpha$ -stable random vector is an  $\alpha$ -stable random variable: Let  $\mathbf{X} = (X_1, X_2, \dots, X_d)$  be an  $\alpha$ -stable random vector in  $\mathbb{R}^d$ . Then any linear combination of the components of  $\mathbf{X}$  of the type  $Y = \sum_{k=1}^d b_k X_k = \langle \mathbf{b}, \mathbf{X} \rangle$ ,  $b \in \mathbb{R}^d$ , is an  $\alpha$ -stable random variable with the same  $\alpha$  parameter as  $\mathbf{X}$ . Conversely, let  $\mathbf{X}$  be a random vector in  $\mathbb{R}^d$ . If all linear combinations  $\sum_{k=1}^d b_k X_k$  are  $\alpha$ -stable with  $\alpha \geq 1$ , then  $\mathbf{X}$  is an  $\alpha$ -stable random vector in  $\mathbb{R}^d$ .

As the spectral measure is "essentially an infinite-dimensional data-structure" ([28]), it is not as easy to deal with as a covariance matrix. Thus, an approximation is necessary. [28] represent the spectral measure as a sum of spherical harmonics. Another way is to approximate the spectral measure by a discrete spectral measure  $\tilde{\Gamma}$  with a finite number of point masses:

$$\tilde{\Gamma}(\mathbf{s}) = \sum_{j=1}^L \gamma_j \delta_{\mathbf{s}_j}(\mathbf{s}) \tag{7}$$

where  $\gamma_j \geq 0$  is the weight and  $\delta_{\mathbf{s}_j}$  the point mass (Dirac delta) of point  $\mathbf{s}_j \in S_d, j = 1, \dots, L$ . Thus,  $I(\mathbf{t})$  in Equation (5) gets to

$$I(\mathbf{t}) = \sum_{j=1}^L \psi_\alpha(\langle \mathbf{t}, \mathbf{s}_j \rangle) \gamma_j \tag{8}$$

For bivariate data, a good way to visualize the spectral measure are polar plots. In Figure 1, such polar plots of discrete spectral measures are shown. Three spectral measures are depicted. The first one shows an estimated spectral measure of two bond indices. To get a rough impression about dependence we add two other spectral measures. The first one shows the spectral measure

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<sup>5</sup>For  $\alpha = 1$ ,  $\psi_\alpha(u) = |u| (1 + i\frac{\pi}{2} \text{sign}(u) \ln(|u|))$ .

of two totally independent assets. In the case of independence, the weights are located at the intersection of the unit sphere with the coordinate axis (cf. [13], Example 2.3.5). The third plot shows a perfectly linear dependent pair of assets with equal scale.

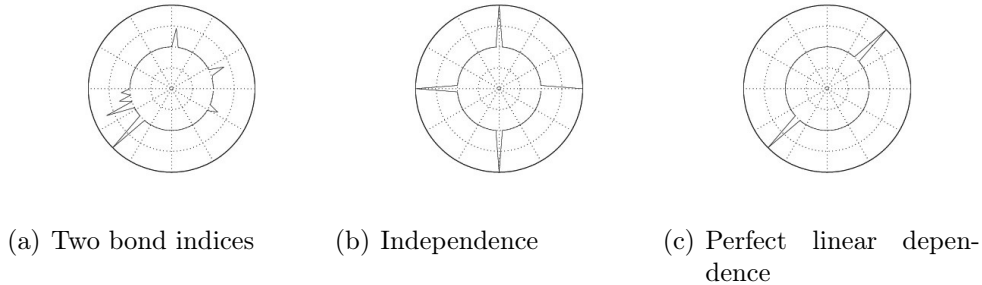


Figure 1: Polar plots of bivariate discrete spectral measures

In general, the points  $\mathbf{s}_j$  can be chosen arbitrarily, for example to form a triangular shape. However the goodness of fit of the approximation depends on the choice of the points (cf. [29], Theorem 2.1). For estimation purposes it seems reasonable to choose  $L$  uniformly spread points  $\mathbf{s}_j$  on the unit sphere  $S_d$ . For  $d = 2$  this would be  $\mathbf{s}_j = \left[ \cos\left(\pi \frac{2(j-1)}{L}\right), \sin\left(\pi \frac{2(j-1)}{L}\right) \right]$ ,  $j = 1, \dots, L$ . [30] show, that for any spectral measure, a discrete spectral measure can be found, such that the corresponding densities are uniformly close. Thus, for simplicity, we restrict ourselves to discrete spectral measures in the remainder of this article.

Several methods for the estimation of the spectral measure have been proposed. For example [28] use harmonical analysis, ([31, 32]) employ the so called Rachev-Xin-Cheng method and [33] employs the discrete spectral measure approximation in (7). We use the latter in our estimation. The method works as follows: One estimates  $I(\mathbf{t})$  in (8), either by using the empirical characteristic function (ECF) or by McCulloch's projection method (cf. [34]), calculates  $\psi_\alpha$  and then uses matrix inversion to obtain the weights  $\gamma_j$ .

A good overview about approximation, estimation and simulation of multivariate spectral measures is given in [29]. Regarding the choice of the point masses, he suggests to use a multiple of four point masses for the bivariate case in order to detect the case of independence. Moreover he states that a number of  $L = 40$  point masses is sufficient for most purposes, except for density calculation.



### 3 Model

#### 3.1 Alpha-stable regime switching

In this article, we will model asset returns by a discrete-time Markov-switching model with  $N = 2$  regimes. A generalization to  $N > 2$  is straightforward. Each regime will be modeled by an  $\alpha$ -stable distribution. In the univariate setting, we model the return as a random variable  $R_t$  which depends on the state of a non-observable homogeneous Markov chain  $Z_t$  with state space  $E = \{1, 2\}$ . The return series  $R_t$  follows an  $\alpha$ -stable distribution, ie,

$$R_t \sim S_{\alpha_{Z_t}}(\sigma_{Z_t}, \beta_{Z_t}, \mu_{Z_t}).$$

For a fixed value of  $Z_t$ ,  $\alpha_{Z_t} \in (0, 2]$  represents the index of stability,  $\beta_{Z_t} \in [-1, 1]$  is the skewness parameter,  $\mu_{Z_t} \in \mathbb{R}$  is the location parameter and  $\sigma_{Z_t} > 0$  the scale parameter. All parameters depend on the state of the Markov chain  $Z_t$ . By restricting  $\alpha$  to 2, one can model one or two regimes with a Gaussian distribution. As both states could be set to  $\alpha = 2$ , our model includes the well-known two-state Gaussian Markov-switching model (as used eg, in [5]) as an instance.

In a multivariate setting, we model the return as a random vector  $\mathbf{R}_t$  which depends on the state of a non-observable Markov chain  $Z_t$  with state space  $E = \{1, 2\}$ . Conditionally to  $Z_t$ ,  $R_t$  follows a multivariate  $\alpha$ -stable distribution,

$$\mathbf{R}_t \sim S_{\alpha_{Z_t}}(\Gamma_{Z_t}, \boldsymbol{\mu}_{Z_t})$$

where  $\alpha_{Z_t}$  is the index of stability,  $\Gamma_{Z_t}$  is the spectral measure and  $\boldsymbol{\mu}_{Z_t}$  is the location vector. By setting  $\alpha$  to 2 in both regimes, the multivariate model includes the multivariate Gaussian regime-switching model, too. However, in that case it is difficult to directly relate the covariance matrix to the spectral measure as the spectral measure is not unique for  $\alpha = 2$ .

In both, the univariate and the multivariate model, the non-observable state process is governed by the homogeneous Markov chain  $Z_t$  with two states, 1 and 2. The transitions of the Markov chain are determined by the transition matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \quad (9)$$

with the transition probabilities  $p = \mathbb{P}(Z_t = 1 | Z_{t-1} = 1)$  and  $q = \mathbb{P}(Z_t = 2 | Z_{t-1} = 2)$ . The initial distribution is denoted by  $\boldsymbol{\pi} = (\eta, 1 - \eta)$  with  $\eta := \mathbb{P}(Z_0 = 1)$ .

The proposed model allows for fat tails and skewness in general as well as

in each regime. Using one regime with  $\alpha = 2$  and one with  $\alpha < 2$  allows for modelling a return distribution with normal tails in a calm regime and fat tails in a turbulent regime. Via distinct spectral measures, the multivariate model enables modelling a regime-dependent dependence structure of assets.

### 3.2 Parameter estimation

For time series estimation we will use discrete returns

$$R_t = \frac{X_t - X_{t-1}}{X_{t-1}}$$

because of their linear additivity property, ie, portfolio returns are the simple weighted sums of the asset returns. This keeps calculations involving the portfolio returns simple.

For both the multivariate and the univariate model, we have to estimate the parameters of the Markov chain  $Z_t$  (the initial distribution  $\boldsymbol{\pi} = (\eta, 1 - \eta)$  and the transition matrix  $P$ ) first. They will be determined using a major stock index, which serves as a market regime indicator for all asset classes. We will use the MSCI World for this purpose, because it has a broad coverage across sectors and geographic regions and thus can be considered as the most representative index for the global capital markets. In addition to that, the data for this major stock market index is available at high frequencies and for long time periods. In Section 3.4, we estimate the parameters using daily data and a long data history. However, for the portfolio selection in Section 5 we employ data for other stock indices and bond indices which is only available weekly. Thus, for the portfolio selection study we use the MSCI World index with weekly datapoints for the detection of the regimes. As the regimes are usually persistent over longer periods, the change of frequency is not a serious issue.

The parameter vector  $\boldsymbol{\theta}_k = (\alpha_k, \sigma_k, \beta_k, \mu_k)$  for each regime  $k \in E$  of the univariate model can be estimated at the same time as the Markov chain parameters. For the multivariate model we first perform the univariate estimation to determine the parameters of the Markov chain. After that, the regimes can be detected and the regime-dependent parameter set  $(\alpha_k, \Gamma_k, \boldsymbol{\mu}_k)_{k \in E}$  can be estimated.

We will use the BaumWelch algorithm (cf. [35]) to estimate the parameters of the Markov chain and the parameter vector  $\boldsymbol{\theta}_k$  for each regime  $k \in E$  of the univariate model. Next, Viterbis algorithm (cf. [36]) will be used to detect the most likely sequence of states. The BaumWelch algorithm is a maximum-likelihood algorithm. Therefore we have to estimate the parameters of the  $\alpha$ -stable distribution by a maximum-likelihood method. For this, we need to evaluate the likelihood function respectively compute the density of

an  $\alpha$ -stable distribution several times. We use a fast Fourier transform (FFT) method similar to the approaches in ([22, 23]) to calculate the density. To keep computation time low, we do not use the Bergström expansion proposed in [23]. In order to do the estimations, we implemented the above-mentioned algorithms and methods in Matlab.

Note, that the implementation of the BaumWelch algorithm is numerically difficult. Multiplying variables between 0 and 1 leads to numerical underflow. Thus, a scaling procedure has to be applied. We use the procedure outlined in [37]. The implementation applies numerical optimization for doing the maximum likelihood estimation which is used in the BaumWelch algorithm. Sequential quadratic programming (SQP), a gradient-based technique, is used for the (corresponding) nonlinear constraint optimization (Matlab function **fmincon**). The routine terminates when the change of the parameters between two iterations falls below a certain tolerance level or a maximum number of iterations is reached. In our implementation we set the tolerance to  $10^{-6}$  and the maximum number of iterations to  $40^6$ . The routine is relatively insensitive to start parameters, except for choosing  $\alpha = 2$ , which results in  $\alpha$  keeping that value.

### 3.3 Simulation

The univariate  $\alpha$ -stable Markov-switching model could be used to simulate a sequence of returns. These simulations could then be used for risk management or portfolio optimization purposes (see Sections 4 and 5). Given the set of states  $E$  and the parameters for the Markov chain  $(P, \boldsymbol{\pi})$ , the distribution parameter vectors  $(\alpha_k, \sigma_k, \beta_k, \mu_k)_{k \in E}$  for each regime  $k \in E$ , and the number of simulation steps (also called simulation time horizon)  $T$ , one can simulate one trajectory of the model according to the following algorithm (cf. [37], p. 261):

1. Draw an initial state  $Z_0$  according to the initial distribution vector  $\boldsymbol{\pi}$ .
2. Set  $t = 0$ .
3. Draw data points, ie, returns, according to the  $\alpha$ -stable distribution  $S_{\alpha_{Z_t}}(\sigma_{Z_t}, \beta_{Z_t}, \mu_{Z_t})$ . A simulation of  $\alpha$ -stable distributions can be done by the Chambers-Mallows-Stuck method (cf. [38, 39]).

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<sup>6</sup>In our test runs of the implementation, the algorithm converges quickly, so with 40 runs the estimation results are usually close to or below the tolerance of  $10^{-6}$ . The computation of the algorithm might take a few minutes on a standard PC. Therefore, computation time is an issue. Thus, limiting the number of iterations is a good instrument to balance convergence and computation time. Note, that for the rebalancing in Section 5, we even use a maximum number of iterations of 30. However, this is no problem, as the algorithm starts with the previous estimates and estimates vary only little from one reallocation time to the next.

4. Draw the new state  $Z_{t+1}$  according to the transition probabilities at state  $Z_t$ . This is determined by the row vector  $(P)_{Z_t}$ , ie, the  $Z_t$ -th row of the transition matrix.
5. Set  $t = t + 1$  and return to step 3 if  $t < T$ . If  $t = T$  terminate.

A similar algorithm can be implemented for the multivariate model. Therefore, one has to use the parameter set  $(\alpha_k, \Gamma_k, \boldsymbol{\mu}_k)_{k \in E}$  instead of  $(\alpha_k, \sigma_k, \beta_k, \mu_k)_{k \in E}$  and use random vectors instead of random variables. A simulation method for the simulation of  $\alpha$ -stable random vectors with discrete spectral measures is given in [29].

### 3.4 Detection of market regimes

We will now apply the univariate  $\alpha$ -stable regime-switching model for the detection of the different market regimes. Therefore we apply the estimation approach to a stock index in order to estimate the parameters of the model and the most likely state sequence. We have used the MSCI World Index. The dataset consists of daily index levels since 1972<sup>7</sup>. We will use the MSCI World Index as reference index for detecting if the world is in a calm or a turbulent state in our applications. The estimation results for the MSCI World are shown below. The transition matrix was estimated as

$$P_{MSCI} = \begin{pmatrix} 0.9893 & 0.0107 \\ 0.0207 & 0.9793 \end{pmatrix}$$

The distribution parameters for daily returns in each regime were estimated as shown in Table 1:

$Z_t$	$\alpha$	$\sigma$	$\beta$	$\mu$
1	1.9731	0.0036	-0.1187	0.0006
2	1.7985	0.0073	-0.0406	-0.0003

Table 1: Distribution parameters of MSCI World for both regimes

The estimate for the initial distribution was  $(\eta, 1 - \eta) = (0, 1)$ . As one can see from the values of  $\alpha$ , the estimation reveals  $\alpha$ -stable behavior in both regimes. Here, the state  $Z_t = 1$  is the calm regime which can be seen by the positive location parameter (corresponds to an annual return of roughly 15 percent) and the moderate value for the scale parameter. In contrast to that, the state  $Z_t = 2$  possesses a negative location parameter (corresponds

<sup>7</sup>We downloaded this dataset from Yahoo Finance. The data ranges from January 3rd, 1972 until June 11th, 2010.

to an annual return of roughly -7 percent) and a scale parameter which is twice the size of the calm regime's scale parameter. Additionally,  $\alpha$  is lower for state  $Z_t = 2$  indicating heavier tails than the calm regime. For this reason, we characterize state  $Z_t = 2$  as the crisis or turbulent regime. The skewness factor is slightly negative for both regimes and the difference there is not too pronounced.

In Figure 2, one can see the MSCI World stock index and the estimated states. Periods where the crisis regime was detected are shaded in grey. One could see, that the identification of crises works quite well. The model detects the crash of 1987, the first Gulf war in 1990/91, the Russian crisis in 1998, the burst of the Dot Com bubble and the recent financial crisis. Apart from that, the model detects some phases in market upswing as crisis. This is mainly due to increased volatility levels in that phases.

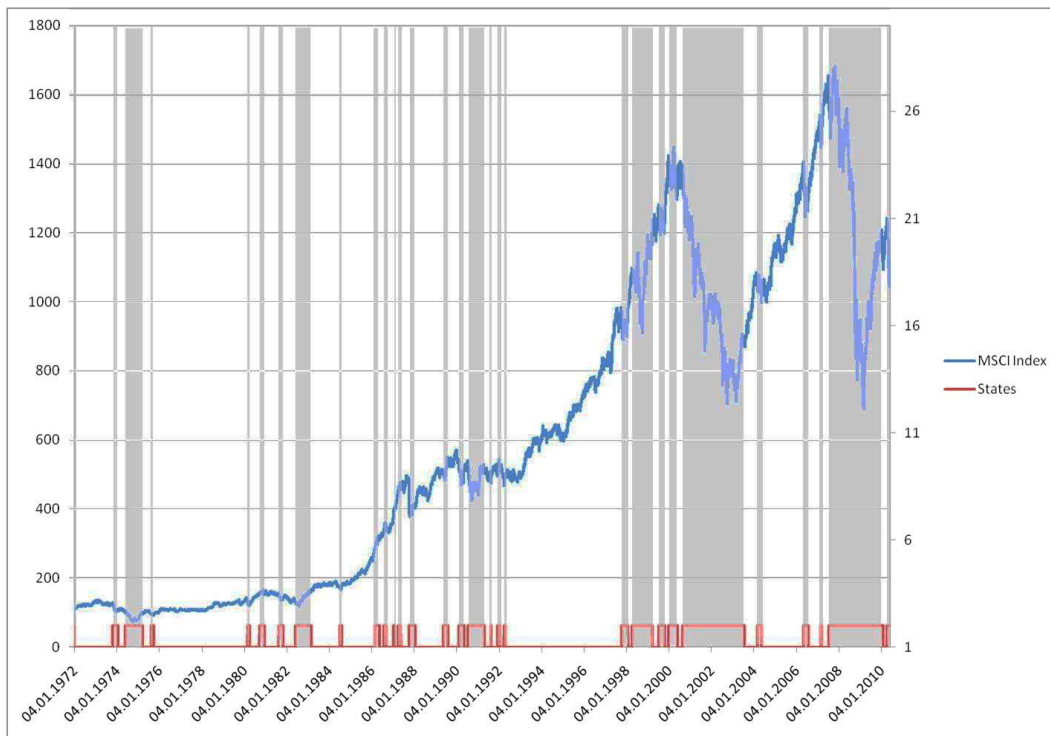


Figure 2: The MSCI World and the estimated crisis regime (shaded in grey)

## 4 Risk measurement

As an application of our univariate model we want to calculate and analyze risk measures like Value at Risk (VaR) and Conditional Value at Risk (CVaR)

for a major stock index. We want to compare the risk measures under different models<sup>8</sup>.

We begin with a definition of Value at Risk and Conditional Value at Risk following [40]. For the continuous random variable  $X \in \mathbb{R}$  and  $q \in (0, 1)$ , the Value at Risk at confidence level  $1 - q$  is defined as

$$VaR_q(X) = -\inf(x \in \mathbb{R} : \mathbb{P}(X \leq x) > q) \quad (10)$$

The Conditional Value at Risk, also called Expected Shortfall (ES), for a continuous random variable<sup>9</sup>  $X$  on  $\mathbb{R}$  and  $q \in (0, 1)$ , is defined as:

$$CVaR_q(X) = -\mathbb{E}[X|X \leq -VaR_q(X)] \quad (11)$$

Note that CVaR is a coherent measure of risk (cf. [40, 41]) while VaR is not. In the remainder of this section, we will give a comparison of risk measures<sup>10</sup> for different models. Our benchmark or proxy for the "true" risk measure is the historical risk measure which is calculated by the realizations of the entire time series, eg, the historical VaR is the empirical quantile of the time series. At first we will compare risk measures under the  $\alpha$ -stable model with those under a Gaussian model. Both models are without taking into account different regimes. After that we will look at Gaussian and  $\alpha$ -stable regime-switching models and compare those to the non-switching models. In addition, we use a slightly modified version of the  $\alpha$ -stable regime-switching model. In this model, we fix the calm regime to  $\alpha = 2$ , thus the calm regime is modelled by a Gaussian distribution while the crisis regime is modelled by an  $\alpha$ -stable distribution without any constraint. As the estimated  $\alpha$  for the calm regime is close to 2 for the MSCI World (see Section 3.4), this modification is reasonable. Thus, the comparison comprises the following models:

---

<sup>8</sup>VaR and CVaR are finite for the  $\alpha$ -stable models we employ. VaR as a quantile function is always finite, and CVaR is finite for  $\alpha > 1$ , which can be easily derived from the finiteness of the first moment in the case  $\alpha > 1$ . In our applications we only use  $\alpha$ -stable distributions with  $\alpha > 1$ , so this restriction causes no problem.

<sup>9</sup>Note, that for general, in particular discrete, distributions a slightly more complicated version applies (see [41]).

<sup>10</sup>In our implementation we calculate the  $\alpha$ -stable VaR with a quantile function. For this we used the Matlab program STBLINV by Mark Veillette, which numerically inverts the cumulative distribution function. The code is available from <http://math.bu.edu/people/mveillette/>. The  $\alpha$ -stable CVaR was calculated using numerical integration.

<b>N</b>	The Normal distribution model. This model assumes a Geometric Brownian motion without switching. Thus returns are normally distributed.
<b>S</b>	The $\alpha$ -stable model. This model assumes that returns follow an $\alpha$ -stable distribution. Thus it takes fat tails explicitly into account.
<b>SSRS</b>	The $\alpha$ -stable regime-switching model. The model consists of a two-state Markov-switching model with $\alpha$ -stable distributions in each regime.
<b>NNRS</b>	The Normal regime-switching model or Gaussian regime-switching model. The model consists of a two-state Markov-switching model with Normal distributions in each regime. This model was used in [5].
<b>NSRS</b>	The modified $\alpha$ -stable regime-switching model. The model consists of a two-state Markov-switching model with one regime governed by a Normal distribution (ie, an $\alpha$ -stable distribution with $\alpha = 2$ ) and the other regime governed by an $\alpha$ -stable distribution.

The calculation of the risk measures for the regime-switching models (the Gaussian Markov-switching model and the  $\alpha$ -stable regime-switching model) could not be done analytically. Therefore a simulation approach, as described in Section 3.3, with 50'000 simulations will be pursued, ie, the risk measures will be determined as the empirical risk measures of these simulations. Finally we will look at the risk measures in the different regimes. Therefore we estimated the regimes under each model as explained in Section 3.2 and split the dataset accordingly in two subsets. Then we computed the risk measures for each subset. We compare the risk measures for the MSCI World, using the same dataset as in Section 3.4. The results can be seen in Table 2.

	VaR			CVaR		
	95%	97.5%	99%	95%	97.5%	99%
<b>Historical</b>	1.24%	1.65%	2.25%	1.96%	2.52%	3.49%
<b>N</b>	1.36%	1.62%	1.93%	1.71%	1.94%	2.21%
<b>S</b>	1.19%	1.62%	2.51%	2.42%	3.47%	5.74%
<b>NNRS</b>	1.29%	1.81%	2.48%	1.99%	2.47%	3.02%
<b>SSRS</b>	1.28%	1.71%	2.42%	2.18%	2.89%	4.22%
<b>NSRS</b>	1.22%	1.63%	2.20%	1.97%	2.55%	3.56%

Table 2: Risk measures of the MSCI World for a 1 day holding period

It can be seen, that modeling returns with an  $\alpha$ -stable distribution has effects on both risk measures, the VaR and the CVaR. The effect however

		VaR			CVaR		
		95%	97.5%	99%	95%	97.50%	99%
Calm	Historical	0.84%	1.02%	1.25%	1.10%	1.28%	1.52%
	Normal	0.82%	0.99%	1.19%	1.05%	1.19%	1.37%
	Stable	0.81%	0.99%	1.23%	1.13%	1.37%	1.77%
Turbulent	Historical	1.85%	2.43%	3.53%	2.91%	3.73%	5.06%
	Normal	2.09%	2.49%	2.95%	2.62%	2.96%	3.37%
	Stable	1.89%	2.41%	3.33%	3.20%	4.28%	6.56%

Table 3: Risk measures of the MSCI World for the calm and turbulent state for a 1 day holding period

depends on the confidence level. For small  $q$  (eg,  $q = 0.01$ ) the  $\alpha$ -stable model **S** gives higher risk measures than the Gaussian model **N**. Thus we get more conservative risk measures. Comparisons to the empirical risk measures show, that the **S** model fits the data better than the **N** model. However, for small and smallest values of  $q$ , the  $\alpha$ -stable model overestimates the risk measures. This is in particular the case for the CVaR.

Looking at the regime-switching models, we observe that the switching models fit the historical risk measures much better than the non-switching models. This is especially the case for the CVaR measure. Firstly, we compare the  $\alpha$ -stable regime-switching **SSRS** model with the well-known Gaussian switching model **NNRS**. While the **SSRS** model fits the historical VaR slightly better than **NNRS** model, it overestimates the CVaR. The **NNRS** model fits the CVaR better, yet underestimates it, especially for the 99% confidence level. Looking at the results for the **NSRS** model, it turns out that this modified model fits both the historical VaR and the historical CVaR rather closely. In particular it has the best fit of all models considered for the 99% confidence level for both VaR and CVaR.

When we split the data into a calm and a turbulent regime (cf. Figure 2), we can detect that the risk measures under the  $\alpha$ -stable distribution fit the empirical risk measures in both regimes very well (see Table 3). The normal distribution would underestimate the risk substantially, especially for the 99% confidence level. In addition to that, the magnitude of this underestimation is much higher in the turbulent regime.

One might ask if the difference in the risk measures calculated above is significant. To answer this question we examined the distribution of the returns. As the VaR and CVaR are functions of the distribution, significantly different distributions will lead to different VaR and CVaR.

We did a Kolmogorov-Smirnov test (cf. [42]) on the historical returns of the MSCI World. We tested the null hypothesis, that the distribution of the



weekly returns of MSCI World are equal to the Gaussian respectively the  $\alpha$ -stable distribution. We did this test on the entire dataset and on the subsets of the two regimes. The results are shown in Table 4.

		$\alpha$ -stable	Gaussian
Entire Dataset	P-Value	0.0521	1.7284E-39
	Statistic	0.0135	0.0669
Calm State	P-Value	0.2762	0.0122
	Statistic	0.0121	0.0195
Turbulent State	P-Value	0.9685	2.1462E-06
	Statistic	0.0085	0.0455

Table 4: Kolmogorov-Smirnov test statistics on the returns of MSCI World

One can see, that Kolmogorov-Smirnov shows that with a confidence level of five percent, the null hypothesis of a Gaussian distribution can be rejected. This even holds when looking at the calm and turbulent state separately. Even for a confidence level of one percent, the only case where the Gaussian distribution cannot be rejected is the calm state. In contrast to that, the null hypothesis of an  $\alpha$ -stable distribution cannot be rejected on a five and one percent level, neither for the entire dataset, nor for the calm and turbulent periods. This emphasizes, that calculating risk measures based on an  $\alpha$ -stable regime switching model is a suitable approach.

## 5 Portfolio selection

Firstly, before we go on to the portfolio selection, we determine the joint distribution of the portfolio returns. Let the asset return vector  $\mathbf{R}_t = (R_{t,1}, R_{t,2}, \dots, R_{t,n})^T$  follow a multivariate  $\alpha$ -stable distribution. The portfolio weights are denoted by  $w_1, w_2, \dots, w_n$  with  $w_i \geq 0, \forall i = 1, \dots, n$  and  $\sum_{i=1}^n w_i = 1$ <sup>11</sup>. Then, the portfolio return  $R_{t,p} = \sum_{i=1}^n w_i R_{t,i}$ , as a linear combination of the components of  $\mathbf{R}_t$ , is an  $\alpha$ -stable random variable. Thus, the portfolio return has the distribution of an univariate  $\alpha$ -stable random variable with the same parameter  $\alpha$  as the asset returns. By incorporating the multivariate dependence structure given through the spectral measure, we could determine the distribution parameters of the portfolio return as shown in [13, Example 2.3.4].

As an application of the model, we will pursue a portfolio optimization case study. Therefore, we will perform a portfolio selection based on risk and return.

<sup>11</sup>In this article we are analyzing long-only portfolios only. However this constraint could be relaxed.

The portfolio size  $n$  has a huge influence on computation time. In particular, in step 3, the computational complexity increases quickly with  $n$ , as one needs to have a sufficient number of point masses for the estimation of the spectral measure. For practical applications, we recommend that the portfolio size should not exceed 5 indices or assets. Computations for the entire portfolio optimization case study took several hours on a standard PC.

We conduct this case study on a *stocks and bonds* dataset comprising of four indices which act as proxies for the respective asset class. This dataset<sup>12</sup> consists of stocks and government bonds from Europe and the USA. The indices EuroStoxx50, S&P500, JPM US Gov., JPM Germany Gov. were used. Data was available weekly ranging from 11/01/1987 until 25/01/2009. Some empirical statistics can be found in Table 5.

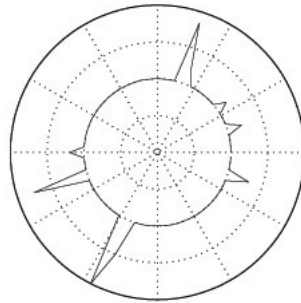
	S&P500	EuroStoxx50	JPM US Govt.	JPM GER Govt.
Mean p.a.	6.81%	6.14%	7.35%	6.28%
Standard Deviation p.a.	17.86%	20.92%	5.09%	3.83%
Skewness	-0.74	-0.56	-0.01	-0.19
Excess Kurtosis	6.37	6.32	3.67	1.80

Table 5: Descriptive statistics for stocks and bonds dataset in the period 11/01/1987 until 25/01/2009

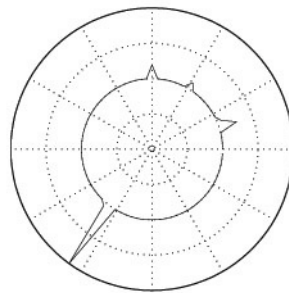
Apparently one can see, that in the chosen time period, the returns of stocks and bonds are of the same magnitude, but the stocks have a much higher standard deviation and have fatter tails. The spectral measures showing the dependence structure in different regimes can be found in Figure 3. One can observe that within one asset class, the spectral measure in turbulent periods is more concentrated than in the calm periods. This indicates a higher dependence in turbulent periods. This is in line with the findings of [5]. An interesting observation can be seen in Figure 3(b). The spectral measure indicates, that in turbulent periods stocks move downwards together, but not upwards at the same time. Between stocks and bonds, the dependence is rather low for both regimes.

To determine the regimes we use the MSCI World Index with the same weekly frequency as the indices mentioned above. The estimated parameters can be found in Table 6. The annualized mean returns corresponding to the location parameter  $\mu$  are roughly 13 percent in regime  $Z_t = 1$  and approximately -12 percent in regime  $Z_t = 2$ . The transition matrix follows below.

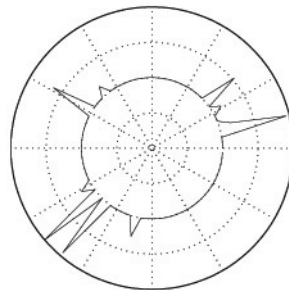
<sup>12</sup>The dataset corresponds to the one used in Bernhart et al. (2009) and was obtained from Reuters (S&P500 and EuroStoxx50), Bloomberg (bond indices) and Datastream (MSCI World Local).



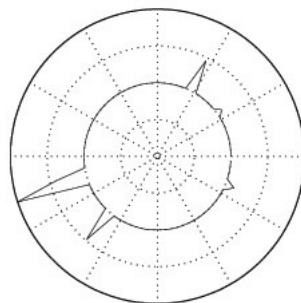
(a) S&P500 vs. Eurostoxx50 - calm regime



(b) S&P500 vs. Eurostoxx50 - turbulent regime



(c) US Govt. Bonds vs. German Govt. Bonds - calm regime



$$P_{MSCI(weekly)} = \begin{pmatrix} 0.9872 & 0.0128 \\ 0.0180 & 0.9820 \end{pmatrix}$$

$Z_t$	$\alpha$	$\sigma$	$\beta$	$\mu$
1	1.8585	0.0084	-1.0000	0.0023
2	1.8572	0.0185	-0.5133	-0.0024

Table 6: Distribution parameters of MSCI World (weekly data, 1987 to 2009) for both regimes

The portfolio case study was based on an initial portfolio value of 100'000 Euros. Cost of trading and taxes were not taken into account. The portfolio optimization problem is given by:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^T \boldsymbol{\mu}_R - \lambda Risk(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{w} \geq 0 \\ & \mathbf{w}^T \mathbf{1} = 1 \end{aligned} \tag{12}$$

In this problem, one maximizes an integrated risk-return function. In (12),  $\boldsymbol{\mu}_R$  denotes the vector of expected returns,  $\mathbf{w}$  denotes the vector of portfolio weights,  $Risk(\mathbf{w})$  denotes a risk measure for the portfolio  $\mathbf{w}$  and  $\lambda \geq 0$  is a parameter, which reflects the risk aversion of the investor. By the constraint  $\mathbf{w} \geq 0$ , we assume short selling is prohibited, thus we show the implications for long-only investors. The risk measure  $Risk(\mathbf{w})$  in these optimization problems can be either the scale parameter  $\sigma(\mathbf{w})$ ,  $VaR_q(\mathbf{w})$  or  $CVaR_q(\mathbf{w})$ . Note that the classical Markowitz' mean-variance framework cannot be applied here as the variance is not finite for  $\alpha$ -stable distributions with  $\alpha < 2$ . In analogy to this classical model, one could do a mean-scale portfolio optimization. This means setting  $Risk(\mathbf{w}) = \sigma(\mathbf{w})$ . Thus the scale parameter of the portfolio is used as risk measure. A detailed discussion of the mean-scale optimization can be found in [18].

However, if one is interested in the risk in the tails, risk measures like VaR or CVaR are more suitable because these risk measures take the (lower) tail into account. However, using VaR leads to a non-linear optimization problem. Moreover, when VaR is calculated from scenarios, eg, when simulations are used, this risk measure might be difficult to optimize. In that case, VaR as a function of the portfolio weights  $\mathbf{w}$  is not convex and non-smooth (cf. [43]). Thus, using VaR leads to problems with local optima.

In contrast to this, CVaR is a convex function of the portfolio weights  $\mathbf{w}$ , thus optimization should find the global optimum. ([44, 43]) show that

using simulated scenarios, problem (12) can be solved by linear optimization methods. Using simulated scenarios comes at the price of a small sampling error. However, as [44] show, this error decreases with increasing sample size and increases with increasing confidence level.

In order to show the implications of the multivariate model on portfolio selection, we will conduct an out-of-sample portfolio optimization case study, using  $Risk(w) = CVaR(w)$  with a confidence level of 95%. In this case study, a portfolio of assets will be reallocated regularly, eg, every 12 weeks, according to the solution of the optimization problem (12). We will use the five models introduced in Section 4 to determine risk and return. The parameters of the underlying Markov chain will be determined using a stock index, which serves as a crisis indicator. As in Section 3.4, we will apply the MSCI world for this purpose using weekly data from 11/01/1987 until 25/01/2009. Based on the parameters of the Markov chain  $Z_t$ , the most likely state sequence will be determined by the Viterbi algorithm. The data will be split according to the detected regimes. For each regime, the distribution parameters will be estimated<sup>13</sup> in dependence of the model considered (ie, assuming either an  $\alpha$ -stable or a Gaussian distribution). Based on these parameters, a simulation study with 4000 runs will be carried out and then the portfolio optimization will be conducted. The investment horizon is 52 weeks, ie, one year. The portfolio optimization started on 05/01/1997 and a reallocation took place every 12 weeks. In summary, the case study will be performed according to the following steps:

1. Estimate regimes of indicator index MSCI World
2. Split data according to regimes
3. Estimate distribution parameters for each regime
4. Perform simulation according to model for the complete investment horizon (52 weeks)
5. Use simulated scenarios for optimization and allocate portfolio accordingly
6. Move forward in time to next rebalancing time (12 weeks)
7. Calculate realized portfolio return over current rebalancing period (12 weeks)

---

<sup>13</sup>For the multivariate  $\alpha$ -stable distribution a discrete spectral measure with 80 point masses, which are uniformly spread around the unit sphere, is estimated.

8. Repeat for every rebalancing time<sup>14</sup>

Note, that the non-switching models will only go through steps 3 to 8 of the algorithm described above.

Now we will focus on the results of the portfolio optimization case study. The optimization was undertaken for a low risk-averse investor, using a risk-aversion factor of  $\lambda = 0.25$ , which is very close to the number proposed in [5] for a low risk averse investor. The development of the portfolio value for each of the models can be seen in Figure 4. The portfolio weights over time for the different models can be seen in Figure 5. To alleviate the interpretation of the results by associating investment periods to dates in history, Table 10 shows some corresponding values. Table 7 contains the final portfolio values and some risk and performance measures calculated from the realized portfolio values respectively the week-on-week portfolio returns.

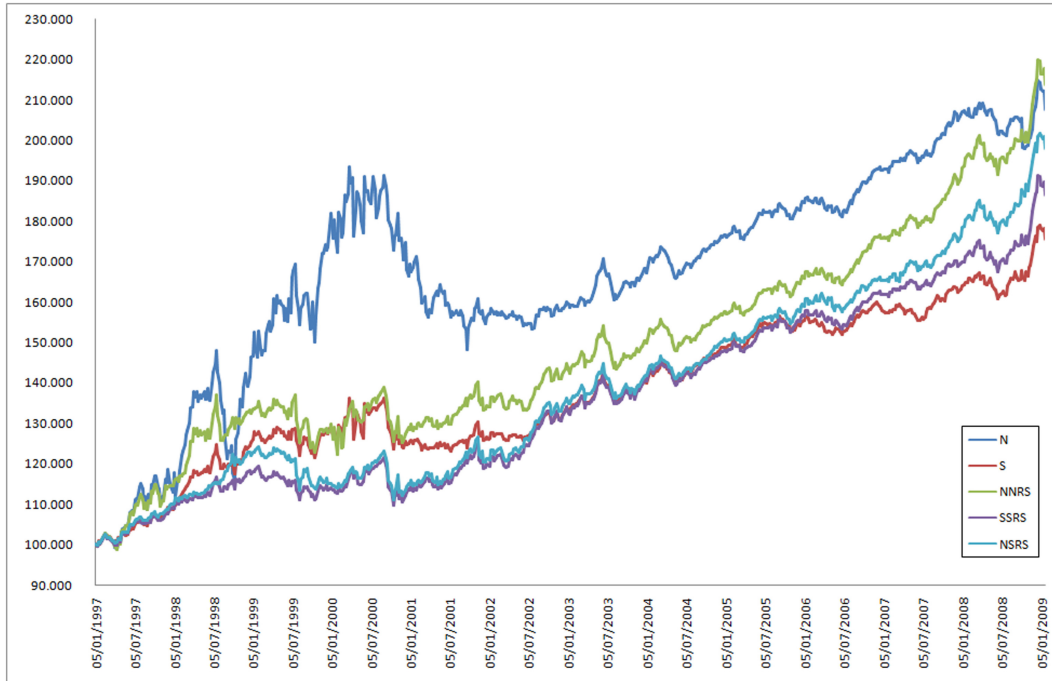


Figure 4: The portfolio values of the different models over time for the Stocks and Bonds dataset.

One can see, that the **NNRS** model has the highest portfolio value at the end of the case study, thus it has the highest mean return. However, looking at the picture in Figure 4, one can see huge differences in the variability of

<sup>14</sup>To be consistent to the previous decisions over time, we keep the classification of regimes until the previous rebalancing time and only add the classification of regimes for the current rebalancing period.

	N	S	NNRS	SSRS	NSRS	S&P500	EuroStoxx50	JPM Gov. US	JPM Gov. GER
Final Portfolio Value	207'598	175'500	213'729	186'369	198'051	108'893	114'590	221'736	189'972
Cumulative Return	107.60%	75.50%	113.73%	86.37%	98.05%	8.89%	14.59%	121.74%	89.97%
Arithm. Mean p.a.	6.85%	4.94%	6.77%	5.42%	5.95%	2.54%	3.92%	6.94%	5.52%
Geo. Mean p.a.	6.22%	4.76%	6.48%	5.28%	5.81%	0.71%	1.13%	6.80%	5.45%
St. Dev. p.a.	11.67%	6.18%	7.83%	5.43%	5.45%	19.56%	24.39%	5.32%	3.84%
Skewness	0.25	-0.12	-0.18	-0.74	-0.95	-0.62	-0.47	-0.46	-0.24
Kurtosis	5.53	14.64	4.43	3.64	4.21	5.59	4.92	1.32	1.17
Sharpe Ratio	0.42	0.48	0.61	0.63	0.73	0.03	0.08	0.93	0.92
Adjusted Sharpe Ratio	0.42	0.42	0.58	0.57	0.62	0.03	0.08	0.92	0.94
Omega(2%)	1.17	1.18	1.26	1.25	1.30	0.99	1.01	1.38	1.38
Omega(0%)	1.27	1.35	1.40	1.45	1.50	1.03	1.04	1.59	1.67
Maximum Drawdown	24.94%	9.74%	10.88%	10.07%	10.77%	61.45%	90.28%	7.07%	5.14%
VaR 99% 1 Week	3.86%	1.87%	2.95%	2.02%	2.04%	6.82%	8.14%	1.75%	1.24%
VaR 95% 1 Week	2.19%	1.15%	1.58%	1.08%	1.11%	4.20%	5.21%	1.00%	0.74%
CVaR 99% 1 Week	5.40%	3.15%	3.67%	2.81%	2.99%	10.42%	11.40%	2.15%	1.48%
CVaR 95% 1 Week	3.53%	1.81%	2.43%	1.70%	1.74%	6.14%	7.27%	1.52%	1.06%

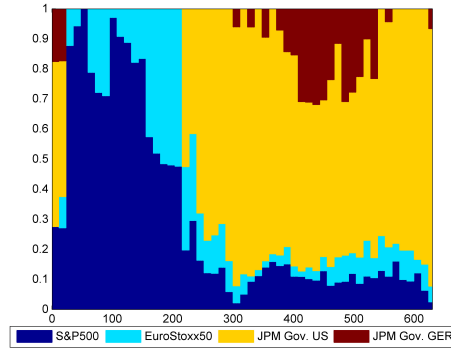
Table 7: Statistics of the different models for the Stocks and Bonds dataset. For comparison the statistics are given for the indices as well.

the portfolio value. Hence one needs to take into account the different risks of the strategies. Comparing the risk measures of the five models, which are given in Table 7, one can see that the Gaussian models (**N** and **NNRS**) have a much higher risk than all  $\alpha$ -stable models, which is reflected in higher values for standard deviation, VaR and CVaR. The numbers for the risk measures of the  $\alpha$ -stable models lie close together, however the **NSRS** model has a slightly higher risk than the **SSRS** model. Looking at the maximum drawdown, one can observe, that the drawdown of the **N** model is more than twice as high than the figures of the other models. The  $\alpha$ -stable model **S** has the least maximum drawdown of all models. Another observation from the week-on-week returns is that the kurtosis of all three switching models is lower than the kurtosis of the non-switching models.

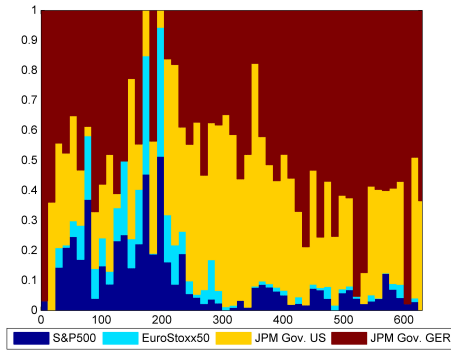
For combining both risk and return, portfolio managers tend to use the Sharpe ratio (cf. [45]) as a measure. The **NSRS** model exhibits the highest Sharpe ratio, having a Sharpe ratio of 0.74. However the data shows skewness and kurtosis for all models. [46] pointed out that for returns with a high skewness, the use of the Sharpe ratio is inappropriate. [47] propose to use an adjusted Sharpe ratio, taking into account skewness and kurtosis. Using this measure the **NSRS** model again dominates all other models. A better performance measure which includes the entire distribution information, is the Omega function, introduced by [48]. The values of the Omega function at the threshold values 0% and 2% (the assumed target rate) confirm the ranking of the Sharpe ratio, with the **NSRS** model leading the rank.

We want to examine, if the distributions of the returns generated by the different models are significantly different from one another. Therefore, we perform a two-sample Kolmogorov-Smirnov test as we did in Chapter 4. The result is displayed in Table 8. On a five percent confidence level, the Kolmogorov-Smirnov test rejects the null hypothesis that **N** is from the same distribution than any other model, except for the case of **N** compared to **NNRS**. It also rejects the null hypothesis, that **S** and **NNRS** are from the same distribution. On a ten percent confidence level, the Kolmogorov-Smirnov test also rejects the null hypothesis that **NNRS** and **SSRS** as well as **NNRS** and **NSRS** are from the same distribution. It thus leads to a separation into the group **N** and **NNRS** on one side and **S**, **NSRS**, and **SSRS** on the other side coming from significantly different distributions. To gain further inside, we additionally undertake a two-sample Anderson-Darling test, see [49]. The result is displayed in Table 9. On a five percent confidence level, the Anderson-Darling test shows that one can reject the null hypothesis that **N** is from the same distribution than any other model. Another finding of the Anderson-Darling test is the fact, that the **NNRS** does not come from the same distribution than any model involving the  $\alpha$ -stable distribution, ie, **S**, **NSRS**, **SSRS**. Additionally, on a one percent confidence level, the Anderson-Darling test cannot reject the

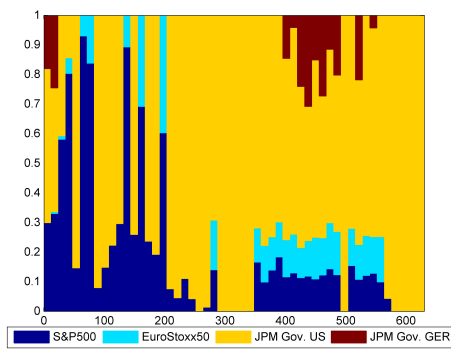




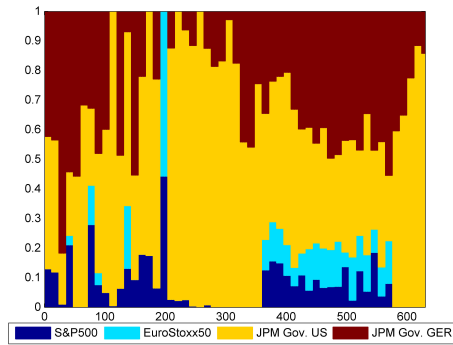
(a) N Model



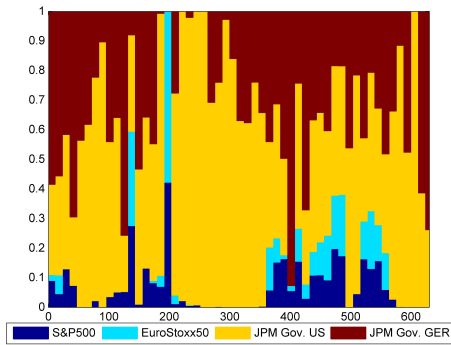
(b) S Model



(c) NNRS Model



(d) SSRS Model



(e) NSRS Model

Figure 5: Portfolio weights over time for the 5 different models for the Stocks and Bonds dataset.

null hypothesis that **N** and **NNRS** are from the same distribution. Thus, the hypothesis tests show that the portfolio returns generated by the models based on the  $\alpha$ -stable distribution (**S**,**NSRS**,**SSRS**) can be distinguished from the models based on the standard **N** as well as on the Gaussian switching model **NNRS**.

Null Hypothesis	P-Value	Statistic
N is from same distribution then S	0.0194	0.0859
N is from same distribution then NNRS	0.1058	0.0684
N is from same distribution then SSRS	0.0065	0.0954
N is from same distribution then NSRS	0.0065	0.0954
S is from same distribution then NNRS	0.0163	0.0874
S is from same distribution then SSRS	0.7944	0.0366
S is from same distribution then NSRS	0.2012	0.0604
NNRS is from same distribution then SSRS	0.0921	0.0700
NNRS is from same distribution then NSRS	0.0597	0.0747
SSRS is from same distribution then NSRS	0.8364	0.0350

Table 8: Kolmogorov-Smirnov test on the portfolio returns

Null Hypothesis	P-Value	Statistic
N is from same distribution then S	0.0001	10.1857
N is from same distribution then NNRS	0.0172	3.1114
N is from same distribution then SSRS	0.0000	11.1556
N is from same distribution then NSRS	0.0000	12.2932
S is from same distribution then NNRS	0.0042	4.8503
S is from same distribution then SSRS	0.4535	-0.4207
S is from same distribution then NSRS	0.2463	0.3298
NNRS is from same distribution then SSRS	0.0086	3.9454
NNRS is from same distribution then NSRS	0.0073	4.1506
SSRS is from same distribution then NSRS	0.5694	-0.8017

Table 9: Anderson-Darling test on the portfolio returns

Regarding the portfolio weights, one can see, that the **N** model puts 100% of the money in shares from December 1997 until November 2000 (investment periods 50 to 200). The other models do not show this bold behaviour. While they invest 100% at some periods, they never keep this allocation long, reverting to more balanced portfolios of stocks and bonds. All the models lower the percentage of shares after the burst of the Dot Com bubble. After that,

the behavior develops differently between switching and non-switching models. While the non-switching models keep a rather constant but low percentage of stocks in their portfolio all the time, the switching models reduce their percentage of stocks to zero in 2002 (investment periods 288 to 347), but then increase their stock percentage to around 20% from 2003 until 2008 (investment periods 348 to 574). In 2008 (from investment period 575 on) they quickly reduce the percentage of stocks to zero again as a reaction on the subprime and the following financial crisis. Thus one could see a distinct switching behaviour in the portfolio weights. Another pattern one could recognize is that the  $\alpha$ -stable model **S** has a higher diversification than the **N** model. Almost any time, the **S** model invests in all four assets, while the Gaussian model often only uses two or three assets. The same effect, but not so pronounced, is observable for the **SSRS** and **NSRS** models compared to the **NNRS** model. Additionally, one could observe that for all models the percentages of stocks in the bull market from 2003 until 2008 are lower than in the Dot Com boom. This could probably be an effect of increasing data history. The models "learn" from the burst of the Dot Com bubble, reflecting the negative returns experienced there. This leads to generally lower portfolio weights of stocks after this bear market.

To further analyze the difference between the different models we look at the behavior of the different strategies in times of crisis. This is particularly interesting because regime-switching models were developed to explicitly take into account times of crisis. In order to answer this question we (heuristically) separated the data into different time periods. The periods are chosen in a way to separate crisis or bear market phases from normal market phases. In the observed time frame from 1997 until 2009, we recognized three different crises:

1. The Russian crisis in 1998
2. The Dot-Com bust from 2000 until 2003
3. The recent financial crisis starting from 2007

Investment Period	1	50	100	150	200
Date in History	05/01/997	14/12/1997	29/11/1998	14/11/1999	29/10/2000
Investment Period	250	300	350	400	450
Date in History	14/10/2001	29/09/2002	14/09/2003	29/08/2004	14/08/2005
Investment Period	500	550	600	630	
Date in History	30/07/2006	15/07/2007	29/06/2008	25/01/2009	

Table 10: Investment periods and the corresponding dates in history

Thus, the dataset was split in six phases, three bull market phases and the three crisis phases above. Then we calculated the period returns during

the different phases as the return between the portfolio value at the start of the period and the portfolio value at period end<sup>15</sup>. The results can be seen in Table 11.

	Bull market	Russian Crisis	Bull market	Dot Com Bust	Bull market	Financial Crisis
Period Start	05/01/1997	26/07/1998	11/10/1998	17/09/2000	30/03/2003	18/11/2007
Period End	26/07/1998	11/10/1998	17/09/2000	30/03/2003	18/11/2007	25/01/2009
<b>N</b>	42.74%	-18.74%	60.50%	-14.10%	27.87%	1.52%
<b>S</b>	22.24%	-3.83%	12.65%	2.11%	20.61%	7.60%
<b>NNRS</b>	32.01%	-2.55%	4.82%	7.95%	30.08%	12.89%
<b>SSRS</b>	15.29%	-1.28%	3.86%	14.28%	25.10%	10.28%
<b>NSRS</b>	15.29%	3.79%	0.10%	14.66%	27.83%	12.80%
<b>MSCI World</b>	45.54%	-20.23%	55.20%	-44.64%	84.55%	-44.59%

Table 11: Analysis of period return for different time periods for the Stocks and Bonds dataset.

One can see that the switching models perform better in crisis periods than the non-switching models. For example, during the Dot Com Bust the **N** strategy lost roughly 14 percent, while the  $\alpha$ -stable switching models **SSRS** and **NSRS** earned roughly 14 percent. Another interesting insight from this analysis is the fact that **S** performs better than **N** in all crises. Analyzing the switching models in more detail, one can see, that the **NSRS** model performed better or equal in crisis times than the **NNRS** model. Note, that our dataset covers the recent financial crisis only until 2009.

The picture looks differently in boom times. In these times, the **N** strategy outperforms the other strategies to a great extent (with the exception of the 2003-2008 boom, where **NNRS** is better). This is due to the full stock investment in the **N** portfolio (and comes as well with a higher drawdown risk). In addition, among the switching models **NNRS** performs better than **SSRS** and **NSRS** in boom times. Especially in bull market period between 11/10/1998 and 17/09/2000 the **NSRS** model has a performance of almost zero. However, this is due to the preceding crisis, in which this model performed best among all models because of the highest percentage of bonds in the portfolio. In the following bull market the model keeps this rather high portion of the portfolio in bonds. However bonds performed poorly in this bull market phase.

In addition to the return figures, we have calculated the maximum drawdown for these six phases. They can be found in Table 12. Looking at the drawdowns, the two-digit percentage drawdowns of the **N** model in the Russia crisis and Dot Com Bust periods catches the eye, while all other models maintain a drawdown of below 10% in these periods. Thus, the **N** model is always worst in crisis periods. The **SSRS** model seems to have the lowest drawdown

<sup>15</sup>Period returns were not annualized. Note, that periods have different length. However, the comparison of models is only undertaken within one period and not between different periods.

	Bull market	Russian Crisis	Bull market	Dot Com Bust	Bull market	Financial Crisis
Period Start	05/01/1997	26/07/1998	11/10/1998	17/09/2000	30/03/2003	18/11/2007
Period End	26/07/1998	11/10/1998	17/09/2000	30/03/2003	18/11/2007	25/01/2009
<b>N</b>	5.70%	18.26%	11.50%	19.96%	6.06%	5.44%
<b>S</b>	1.92%	2.72%	7.44%	5.04%	5.11%	3.83%
<b>NNRS</b>	4.72%	2.96%	10.88%	4.91%	7.05%	4.74%
<b>SSRS</b>	2.49%	1.90%	7.06%	4.91%	5.25%	4.43%
<b>NSRS</b>	2.03%	2.16%	8.78%	4.91%	6.06%	4.41%
<b>MSCI World</b>	9.97%	17.71%	8.01%	62.69%	10.28%	58.34%

Table 12: Analysis of maximum drawdown for different time periods for the Stocks and Bonds dataset.

in most periods. Among the switching models the  $\alpha$ -stable switching models **SSRS** and **NSRS** have a lower or equal maximum drawdown than the **NNRS** model in crisis periods.

## 6 Conclusion

The goal of this paper was to enhance the current Gaussian distribution paradigm in finance by two points: First, the usage of a distribution which captures fat tails and skewness. Second, the consideration of different "states of the world", ie, distinct market phases. Moreover, we wanted to show the implications of these enhancements on risk and asset management.

Regarding the first point, we presented the  $\alpha$ -stable distribution in the univariate and the multivariate case. The second point was taken into account by presenting regime-switching models. We combined both points to an  $\alpha$ -stable regime-switching model and applied this model to risk measurement and portfolio selection.

Using  $\alpha$ -stable models in risk measurement, we observed a better appreciation of risk. However, the  $\alpha$ -stable non-switching model tends to overestimate risk, especially when using the CVaR as risk measure, while the Gaussian non-switching and switching models underestimate the true risk, especially in turbulent market phases. Using  $\alpha$ -stable regime-switching models tempers this overestimation and leads to good approximations of the empirical risk. Thus, employing  $\alpha$ -stable regime-switching models leads to a more accurate risk measurement than using the Gaussian distribution.

A better outcome in terms of risk can also be seen when employing the  $\alpha$ -stable model for portfolio selection. In an out-of-sample portfolio optimization case study we observed that  $\alpha$ -stable models lead to less risky investment strategies. The explicit consideration of tail events leads to more diversified portfolios and the avoidance of big drawdowns. Thus, the model leads to better results in risk-adjusted performance measures like Sharpe ratio or Omega.

The main advantage of  $\alpha$ -stable switching models is the behavior in crisis

phases. In these phases the  $\alpha$ -stable switching models perform better than the  $\alpha$ -stable non-switching model and a Gaussian switching model.

However, the higher focus of the  $\alpha$ -stable regime-switching models on limiting the downside risk comes at the price of a lower return. In boom times, the Gaussian model outperforms both,  $\alpha$ -stable and switching models. In addition, among the switching models the Gaussian regime-switching model performs better than its  $\alpha$ -stable counterparts in boom times. The  $\alpha$ -stable switching models are more "cautious", thus investing less and later in risky assets in a bull market.

Several extensions might be useful and are subject to further research: For a better distinction between turbulent and bear market phases, one could try to split the crisis state into two further regimes, one regime having positive location parameter and the other having a negative one. This results essentially in a three-state MSM. However as the crisis regime is only a subset of the entire set, a further split would require a large dataset for a sound state detection. In addition to that, identifiability problems might occur. Another way of improving the results in the calm regimes could be a regime-dependent risk aversion factor  $\lambda$ , allowing for a more aggressive portfolio allocation in bull markets. Moreover, a calibration which results in the same initial portfolio, instead of the same risk aversion parameters, for each model could be another enhancement.

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**Received: December 22, 2015; Published: February 22, 2016**