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Random walks on oriented percolation and in recurrent environments

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Contents

1 Introduction

In this thesis we take a look at two models from the field of random processes in random environments. A random process is a sequence of variables where each variable depends on the previous ones in a random fashion. The basic random process for this thesis is the random walk (RW). To visualize a random walk, imagine it as a disoriented group of tourists in a city with a modern grid street plan that stretches endlessly in all directions. As the tourists are notoriously bad with their maps and GPS does not work well in deep canyons of high-rise buildings, at each junction they discuss where to go. Because they cannot decide on the right way, they determine the new direction by coin tossing. It is lucky for the tourists that streets are predominantly two-dimensional structures and they are sure to reach their destination eventually. It is unlucky for them that it will probably take a very long time and an extensive detour to get there. Of course, this scenario is completely unrealistic. In reality, the tourists would never toss coins, but rather look at each of the four directions and then decide based on what they see. It is far more likely that they choose a street that promises to have the best shops and food-stores over a bleak and dirty one. If we want to describe realistic tourist behaviour mathematically, we need to take the attraction of each street on a group of tourists into account. We do this by defining an environment, which rates the attractiveness of each street in numbers. The type of shops in each street and how much our tourists like them is usually not very deterministic. Thus we may choose the environment randomly. The resulting tourist behaviour is a random walk in random environment. If in addition there is construction work in the city and some streets are completely blocked, then the model of choice for the environment is a percolation cluster. Since in a percolation certain passages are blocked, a city with major construction work looks like a maze to a tourist. We will talk about these models in the first part of the thesis. To make matters worse, in real life there is usually more than one group of tourists in the same city. Tourist groups often try to avoid each other to get the most authentic experience of the place. We may assume that a tourist group will never pick a street that leads to a junction which is already occupied by other tourists. Such a system is mathematically an interacting particle system. We will deal with interacting particles that avoid each other in the second part of the thesis. Returning from stories about tourists to mathematics, we will start with the general aim of this thesis.

Since the field of random processes in random environments is rather young, there are two approaches to achieve progress. One can try to push general results and techniques further to establish a universal theory. Unfortunately, there has only been good progress, if the random environment is nice and these methods cannot be used to tackle any other cases. The second approach is to search for special models that show interesting behaviour under scaling to explore all the possible results that a universal theory needs to cover. This thesis falls into the second category. We will explore and describe two models and their scaling behaviour. The first model is a variation of random walks on oriented percolation clusters, where we explore how adding random weights onto the percolation structure affects the scaling limits. In the second model a finite number of particles perform a random walk simultaneously in the same environment with some simple rules for the interaction. The random environment is a recurrent environment on the integers. The first important result on the localization of random walks in recurrent one-dimensional i.i.d. environments was published by Sinaĭ (1982) and we refer to them as Sinaĭ environments in this thesis.

In the first part of the thesis we discuss random walks on random environments to establish the context for the main result on random walks on oriented percolation clusters. We first give the most general definition and then establish methods for the case of reversible random walks, as there are many results known in this specific case. We will make use of the concept of resistor networks that are applicable in

this case. Then, we use the rather simple example of random walks on the integers to explore the different possible scaling behaviours. While the different behaviours are completely understood in one-dimensional lattices, this is far from true for any higher dimension. Random walks on the strip live in between both worlds and are a generalization of random walks on the integers. We discuss them briefly in the first part of the thesis as they are relevant to interacting particle systems with finitely many particles, which we discuss in the second part. On higher dimensional lattices, we will focus on known results about percolation and then give scaling limits for random walks on weighted, oriented percolation clusters. Random walks on oriented graphs are a model for ancestral lineages in a population model where the amount of offspring depends locally on the occupation of neighbouring sites. In this way, it is a simple population model that incorporates competition, since each site can only be occupied by one particle at a time. Random walks on oriented percolation clusters correspond to a population model, where some sites are not habitable. We go a step further and introduce a random field that represents the maximal occupation number of each site. Thus, some sites might have enough resources to support many individuals, while others can feed one individual at a time. The random field acts like weights on the random walk in this environment. In this work, we demonstrate how the independent percolation structure can be used to show scaling limits even in the presence of mixing weights. The proofs for these results can be found in Section 4 at the end of the thesis.

During the first part of the thesis, we deal with ordinary Markov chains. For the second part, we need more general theory on Markov processes, as they provide the main language to describe interacting particle systems. After a basic introduction of definitions and notation we focus on the exclusion process. The big remaining open question of the thesis is to establish a limit theorem for the tagged particle of an exclusion process in Sinaĭ environments. The hope is, to see how the localization behaviour of the Sinaĭ environment affects the scaling in addition to the exclusion dynamics, which already cause a change of scale in the limit theorem compared to a single particle. To date there are few results for the exclusion process, let alone the tagged particle, in any inhomogeneous environment. The known results on exclusion processes that are relevant to this question are presented. Then we focus on an interacting particle system with exclusion dynamics but with finitely many particles. These models have been studied before and are known as spiders. We add our own results to what is known and do some tentative steps towards tackling the full exclusion process with infinitely many particles. All proofs for these results can be found in Section 5 at the end of the thesis.

The two rather different models presented in the two parts of the thesis aim to show how different known mechanisms in random processes in random environment combine to produce new behaviour. We examine whether one mechanism dominates over the other as in the first model, or whether both mechanisms act together to a maximized effect as in the second model. Any general theory in the field must aim to incorporate these results. But until then and possibly also after that, we need concrete examples to help our understanding along.

2 Random walks in random environments

Random walks are Markov chains on graphs, which can be used to represent many different real world phenomena. In many models the transition probabilities of the Markov chain are not homogeneous on the graph, but depend on local properties. Whenever these local properties can be described by a random field or even a random process, the model falls into the class of random walks in random environments. Classical examples can be found in abundance all over natural sciences. One of them is the diffusion of a particle in a fluid that contains obstacles, which could be an emulsion or a porous stone. Random walks in random environments can also be used to model ancestral lineages in populations in an environment with non-homogeneous resources or it may be used to model the propagation of information in a randomly chosen graph.

The formal definition of *random walks in random environments* (RWRE) is as follows. Let $\mathcal{G} = (V, E)$ be an infinite, oriented graph with countable vertex set V and edge set E. If two vertices $v, w \in V$ are adjacent, i.e. connected by an edge $(v, w) \in E$, we write $v \sim w$. The set of all adjacent vertices $N_v = \{w \in V : w \sim v\}$ is called the *neighbourhood* of v. For each vertex we denote the family of all probability measures on V with support on N_v by $\mathcal{M}_1(N_v)$ and equip it with the weak topology to get a Polish space. Denote by $\mathcal{F}$ the Borel-σ-algebra on $\Omega = \prod_{v \in V} \mathcal{M}_1(N_v)$, which is the σ-algebra generated by cylinder events. We fix a probability measure P on $(\Omega, \mathcal{F})$, which we call the environment law. A random environment $\omega \in \Omega$ with law P is an element of the probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

A random walk in random environment is a Markov chain $(X_n)_{n \in \mathbb{N}}$ with state space V and transition law P_ω that depends locally on the environment ω. For any $x \in V$ and $y \in N_v$ it has transition probabilities

$$P_\omega \left(X_{n+1} = y | X_n = x \right) = \omega_x(y),$$

where the random walk starts at the origin, $P_\omega(X_0 = 0) = 1$. We call the law P_ω of the random walk for a fixed environment the *quenched law*. The quenched law is a probability measure on $(V^{\mathbb{N}}, \mathcal{F}')$, where $\mathcal{F}'$ is again the σ-algebra generated by cylinder events. This also defines a measure $\mathbb{P} = \mathsf{P} \otimes P_\omega$ on the product space $(\Omega \times V^{\mathbb{N}}, \mathcal{F} \times \mathcal{F}')$. Its marginal on $V^{\mathbb{N}}$ is also denoted by $\mathbb{P}$ and called the *annealed law*. The annealed law describes the law of the random walk, when we average over different realizations of the random environment.

In many situations, especially if the environment is nice, the behaviour of the random walk is very similar under the quenched and annealed law. For example take the integer lattice $V = \mathbb{Z}^d$ and an i.i.d. environment law. If P is uniform on the neighbourhood up to a small perturbation, the random walk in random environment under the quenched law should behave very similar to a simple random walk. This is indeed the case, but the proof is not easy. It was first done for dimensions $d \geq 4$ by Bolthausen and Sznitman (2002). They also present an example, where the quenched and annealed behaviour are very different. Take the lattice $V = \mathbb{Z}^d$ as the vertex set with nearest-neighbour edges $E = \{(x, y) : x, y \in V \text{ and } ||x - y||_2 = 1\}$, where $|| \cdot ||_2$ is the usual Euclidean norm on $\mathbb{Z}^d$. For each $v \in V$ choose one vector $\mathbf{e}$ from the Euclidean basis uniformly and independent of all other vertices and set $\omega_{(v, v+\mathbf{e})} = 1$ and $\omega_{(v,w)} = 0$ for all $w \neq v + \mathbf{e}$. Thus, in $d = 2$ dimensions the transitions from each vertex are only allowed either to the north or to the east neighbour uniformly at random. Consequently, under the quenched law, when ω is fixed, then the random walk is deterministic. On the other hand, since the random walk can never return to a vertex that it has visited before, under the annealed law the random walk performs a normal north & east random walk, which has a non-degenerate central limit theorem, since it has i.i.d. increments that are not constant.

Technically, both the quenched law and the annealed law have their own difficulties when we want to prove scaling limits. The random walk in random environment is a

Markov chain under the quenched law. However, in $d \geq 2$ it is not reversible without further assumptions. Reversibility allows us to use methods from harmonic analysis and homogenization theory and we will discuss reversible models in Section 2.1. The random walk under the annealed law, on the other hand, suffers from the problem that it is not Markovian, since it uncovers the environment as it moves and when it returns some parts of the environment are already known.

The general aim for every model is to prove a law of large numbers and annealed and quenched central limit theorems. Usually, if a central limit theorem holds, then a functional version of the theorem can be proved with only little more effort. A more difficult question is to establish local central limit theorems and large deviation results. However, these are not the topic of this thesis. We begin with the definition of the limit theorems that we are interested in. We write E_ω and $\mathbb{E}$ for the expectation under the quenched and annealed law, respectively.

Definition 2.1. Let $(X_n)_{n\in\mathbb{N}}$ be a random walk in random environment $(\omega_{(x,y)})_{x,y\in\mathbb{Z}^d}$ with annealed measure $\mathbb{P}$ and quenched measure P_ω and associated expectations $\mathbb{E}$ and E_ω respectively.

(i) A *law of large numbers* (LLN) holds, if there is a constant $\vec{\mu} \in \mathbb{R}^d$ such that

$$P_\omega\left(\frac{X_n}{n} \xrightarrow{n\to\infty} \vec{\mu}\right) = 1 \quad \text{for } \mathbb{P}\text{ -almost every } \omega.$$

Let $X \sim \mathcal{N}(0,\Sigma)$ be a normal random variable on $\mathbb{R}^d$ with covariance matrix Σ. Denote by $C_b(\mathbb{R}^d)$ the set of all continuous, bounded functions $f : \mathbb{R}^d \to \mathbb{R}$. We say that:

(ii) An *annealed central limit theorem* (aCLT) holds, if for all $f \in C_b(\mathbb{R}^d)$

$$\mathbb{E}\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] \xrightarrow{n\to\infty} \mathbb{E}\left[f(X)\right].$$

(iii) A *quenched central limit theorem* (qCLT) holds, if for all $f \in C_b(\mathbb{R}^d)$

$$E_\omega\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] \xrightarrow{n\to\infty} \mathbb{E}\left[f(X)\right] \quad \text{for } \mathbb{P}\text{ -almost every } \omega.$$

For any $t \geq 0$, denote the scaled linear interpolation of the discrete random walk by

$$X_t^{(n)} = \frac{1}{\sqrt{n}}\left(X_{\lfloor nt\rfloor} + (tn - \lfloor tn\rfloor)\left(X_{\lfloor tn\rfloor+1} - X_{\lfloor tn\rfloor}\right) - nt\vec{\mu}\right).$$

Fix $T > 0$ and write $(C[0,T], \mathcal{W}_T)$ for the space of continuous functions from the interval $[0,T]$ to $\mathbb{R}^d$ equipped with the σ-algebra $\mathcal{W}_T$ of Borel sets relative to the supremum topology. We say that:

(iv) An *annealed functional central limit theorem* (aFCLT) holds, if for all $T > 0$, the law of $\left(X_t^{(n)} : 0 \leq t \leq T\right)$ under $\mathbb{P}$ converges as $n \to \infty$ on $C([0,T], \mathcal{W}_T)$ to the law of Brownian motion $(B_t : 0 \leq t \leq T)$ on $\mathbb{R}^d$ with covariance matrix Σ.

(v) A *quenched functional central limit theorem* (qFCLT) holds, if for all $T > 0$ and for $\mathbb{P}$-almost every ω, the law of $\left(X_t^{(n)} : 0 \leq t \leq T\right)$ under P_ω converges as $n \to \infty$ on $C([0,T], \mathcal{W}_T)$ to the law of Brownian motion $(B_t : 0 \leq t \leq T)$ on $\mathbb{R}^d$ with covariance matrix Σ.

Note that from this definition it is not guaranteed that the limits are non-degenerate, i.e. that Σ is of full rank. Proving non-degeneracy of the limit is often more difficult than proving the central limit theorem itself. In the previous definition of the qCLT the drift is deterministic, which is not always the case. We will see a quenched limit theorem on the integers with random centring, Theorem 2.13, where the quenched expectation replaces the annealed expectation $n\vec{\mu}$. In this case, the limit laws of the quenched and annealed limit theorems do not coincide.

2.1 Conductance model

For the random conductance model we build the transition probabilities such that the random walk is reversible. Let $(\omega_e)_{e\in E}$ be a family of positive random numbers $\omega_e \geq 0$ with the symmetry condition $\omega_{(x,y)} = \omega_{(y,x)}$ for any $(x, y) \in E$. The random variable $\omega_{(x,y)}$ is called the *conductance* between vertices x and y in a reference to its role in the representation of the model as a resistor network. We give a short introduction to resistor networks in Section 2.1.1. For a more comprehensive treatment, see for example the book of Lyons and Peres (2016).

Definition 2.2. A random conductance model with conductances $(\omega_e)_{e\in E}$ is called *uniformly elliptic*, if there is a constant $\kappa \in (0, 1)$ such that

$$\kappa < \omega_e < \frac{1}{\kappa} \quad \text{for all } e \in E.$$

It is called *elliptic*, if

$$0 < \omega_e < \infty \quad \text{for all } e \in E.$$

We call κ the *ellipticity constant*. Ellipticity is important, since it ensures that all parts of the graph are connected. Otherwise, one has to check that there is an infinite connected component in the graph to get non-trivial behaviour of the random walk. The problem to decide, whether there is an infinite connected component, is known as percolation and discussed in Section 2.4. The random walk $(X_n)_{n\geq 0}$ with transition probabilities

$$p(x, y) = P_\omega\left(X_{n+1} = y | X_n = x\right) = \frac{\omega_{(x,y)}}{\sum_{z\sim x} \omega_{(x,z)}}$$

is reversible with stationary measure

$$\pi(x) = \sum_{z\sim x} \omega_{(x,z)}.$$

The reversibility follows from the symmetry of the conductances,

$$\pi(x)p(x, y) = \omega_{(x,y)} = \omega_{(y,x)} = \pi(y)p(y, x).$$

Note that the random walk is well defined, if $\pi(x) < \infty$ for all $x \in V$. This is automatically satisfied, if the environment is elliptic and the graph is locally finite.

2.1.1 Resistor networks

We can use an analogue with resistor or electrical networks to express events for the random conductance model in a convenient way. We interpret the values $\omega_{(x,y)}$ as conductances $c(x, y)$ in the resistor network. In other words, we place a resistor with resistance $1/\omega_{(x,y)}$ onto any edge of the graph and connect them at the vertices, as depicted in Figure 1. When we create a voltage difference between two different parts

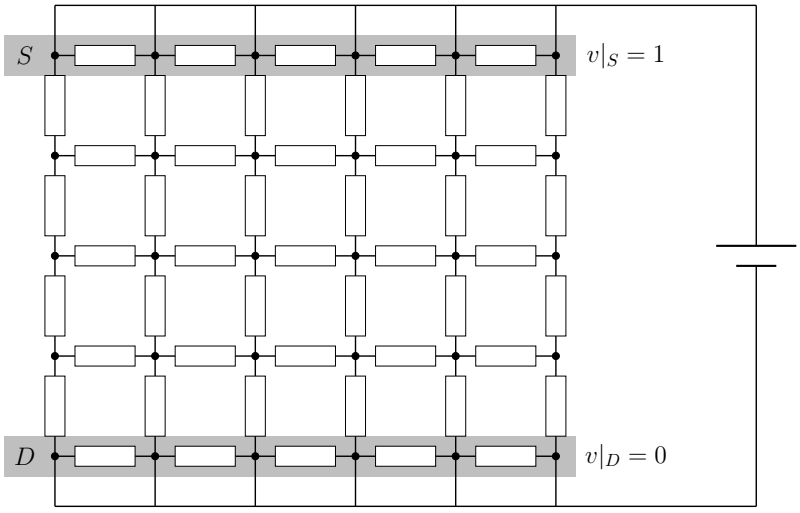

Figure 1: Simple resistor network for a two dimensional lattice graph that has a battery with 1 Volt hooked up between sets S and D. These two sets are highlighted by grey boxes. Each edge has a resistor assigned, where the resistance is determined by the environment. The resistors are connected at the vertices, which are represented by small black circles.

of the network, an electrical current flows through the resistor network. Following the book of Lyons and Peres (2016), for any finite graph (V, E) with conductances $c(x, y) = \omega_{(x,y)}$ we can define the *voltage* or *potential* v between two disjoint subsets S and D of the vertex set V. We set $v(x) = 1$ for $x \in S$ and $v(x) = 0$ for $x \in D$. The voltage is the unique function $v : V \to \mathbb{R}$ such that for every $x \in V \setminus (S \cup D)$

$$v(x) = \frac{1}{\pi(x)} \sum_{y \sim x} c(x, y)v(y).$$

By definition v is harmonic on $V \setminus (S \cup D)$. We can imagine a battery with 1 Volt hooked up between the two sets S and D, which are the source and drain for the current. The set-up is shown in Figure 1. The current $i(x, y)$ between two adjacent vertices $x \sim y$ has to obey Ohm's law, so

$$v(x) - v(y) = i(x, y)r(x, y),$$

where $r(x, y) = 1/c(x, y)$ is the resistance. Furthermore, by Kirchoff's law all currents that flow in and out of any vertex have to sum up to zero.

The most important use of the representation as resistor networks is the ability to express probabilities of hitting times. Let τ_A be the first hitting time of a set $A \subset V$ of the random walk. That is

$$\tau_A = \inf \{n \geq 0 : X_n \in A\}.$$

For convenience, we want to denote the law of the random walk starting in $z \in V$ by $P^z_\omega(\cdot) = P_\omega(\cdot | X_0 = z)$. Then, by the Markov property of the random walk for every $z \notin A \cup D$

$$P^z_\omega(\tau_A < \tau_D) = \sum_{z \sim y} p(z, y)P^y_\omega(\tau_A < \tau_D) = \frac{1}{\pi(z)} \sum_{z \sim y} c(z, y)P^y_\omega(\tau_A < \tau_D)$$

and we see that $P^z_\omega(\tau_A < \tau_D)$ is a harmonic function in z on $V \setminus (A \cup D)$. Since v is both harmonic and linear in $v(a)$ by the superposition principle, we get for a singleton

set $A = \{a\}$

$$P^z_\omega(\tau_{\{a\}} < \tau_D) = \frac{v(z)}{v(a)}.$$

Furthermore, we can set the first return time of the random walk that is starting in a vertex a to the same vertex a,

$$\tau^+_a = \inf\{n > 0 : X_n = a \text{ when } X_0 = a\},$$

in relation with the voltage

$$v(a) = \frac{\sum_{x\sim a} i(a,x)}{\pi(a)P^a_\omega(\tau_D < \tau^+_a)} =: \sum_{x\sim a} i(a,x)\mathcal{R}(a \leftrightarrow D). \tag{2.1}$$

In the last equation we have implicitly defined the *effective resistance* $\mathcal{R}(a \leftrightarrow D)$ between a and D. Its reciprocal is called the *effective conductance* $\mathcal{C}(a \leftrightarrow D) = \mathcal{R}(a \leftrightarrow D)^{-1}$. The effective resistance tells us, whether the random walk is recurrent, see e.g. Grimmett (2010). Let $d(x,y)$ be the graph distance between two vertices $x, y \in V$. Then, we can exhaustively cover any graph with increasingly large boxes around the origin. Define their boundaries as

$$\Lambda_n = \{x \in V : d(x,0) = n\}.$$

The effective resistance between the origin and infinity is defined as

$$\mathcal{R}(0 \leftrightarrow \infty) = \lim_{n\to\infty} \mathcal{R}(0 \leftrightarrow \Lambda_n).$$

By Rayleigh's principle, the effective resistance is a non-decreasing function of the edge-resistances, which implies that the previous limit exists. We can now conclude from Equation (2.1) that recurrence of the random walk can be determined by this effective resistance, since it implies

$$P^0_\omega\left(\tau_{\Lambda_n} < \tau^+_0\right) = \frac{1}{\pi(0)\mathcal{R}(0 \leftrightarrow \Lambda_n)}.$$

By taking the limit $n \to \infty$, we get the probability for the random walk to escape towards infinity.

Corollary 2.3. *The Markov chain $(X_n)_{n\in\mathbb{N}}$ in the random environment with conductances $(c(x,y))_{x,y\in V}$ is recurrent if and only if $\mathcal{R}(0 \leftrightarrow \infty) = \infty$. Otherwise it is transient.*

If the Markov chain is recurrent and its stationary measure is finite, then

$$\sum_{x\in V} \pi(x) = \sum_{x,y\in V} \omega_{(x,y)} < \infty$$

and the Markov chain is positive recurrent. Otherwise it is null-recurrent. The following lemma relates the hitting times of two sets with effective resistances.

Lemma 2.4. *Let $\mathcal{G} = (V,E)$ be any finite graph such that $|V|, |E| < \infty$. Take two subsets of vertices $A, B \subset V$ such that $A \cap B = \emptyset$ and a starting point for the random walk $z \notin A \cup B$. Denote by P^z_ω the law of the random walk $(X_n)_{n\in\mathbb{N}}$ on the graph $\mathcal{G}$ with conductances $(\omega_{(x,y)})_{x,y\in V}$ and such that $X_0 = z$. Then we have*

$$P^z_\omega(\tau_A < \tau_B) \le \frac{\mathcal{R}(z \leftrightarrow B)}{\mathcal{R}(z \leftrightarrow A)}. \tag{2.2}$$

Proof. Define the first return time to the start vertex by

$$\tau_z^+ = \inf\{n > 0 \colon X(n) = z\}.$$

We consider the events

$$R = \left\{\tau_z^+ < \tau_{A\cup B}\right\} \text{ and}$$
$$S = \{\tau_A < \tau_B\} \cap R^c,$$

where $R^c = \Omega \setminus R$ denotes the complement of R. We can decompose the time to hit either the set A or the set B into successive excursions from z. By the Markov property these excursions are independent. On the event R we have neither hit A nor B before returning to z and thus we may try again. On the event S the random walk successfully hits A before hitting B or returning to z and

$$S = \left\{\tau_A < \tau_B < \tau_z^+\right\} \cup \left\{\tau_A < \tau_z^+ < \tau_B\right\} \subset \left\{\tau_A < \tau_z^+\right\}.$$

Thus, we can rewrite the event $\{\tau_A < \tau_B\}$ in terms of events R and S and get the upper bound

$$P_\omega^z(\tau_A < \tau_B) = \sum_{r=0}^{\infty} P_\omega^z(S)\left(P_\omega^z(R)\right)^r = \frac{P_\omega^z(S)}{1 - P_\omega^z(R)} \leq \frac{P_\omega^z(\tau_A < \tau_z^+)}{P_\omega^z(\tau_{A\cup B} < \tau_z^+)}.$$

According to Section 2.2 of Lyons and Peres (2016), we have on any finite graph

$$\frac{P_\omega^z(\tau_A < \tau_z^+)}{P_\omega^z(\tau_{A\cup B} < \tau_z^+)} = \frac{\mathcal{R}(z \leftrightarrow A \cup B)}{\mathcal{R}(z \leftrightarrow A)} \leq \frac{\mathcal{R}(z \leftrightarrow B)}{\mathcal{R}(z \leftrightarrow A)}, \tag{2.3}$$

which proves the lemma. □

One may observe that whenever $\mathcal{R}(z \leftrightarrow B) > \mathcal{R}(z \leftrightarrow A)$, then the upper bound provided by the previous lemma is trivial. However, in this case we get a non-trivial lower bound, since

$$P_\omega^z(\tau_A < \tau_B) = 1 - P_\omega^z(\tau_B < \tau_A) \geq 1 - \frac{\mathcal{R}(z \leftrightarrow A)}{\mathcal{R}(z \leftrightarrow B)} > 0.$$

2.2 RWRE on the integers

A random walk on the integers can easily get trapped by the random environment as a short sequence of exceptionally small conductance is enough to form a strong barrier for the random walk. Therefore, the behaviour of random walks on the integers is different from random walks on integer lattices in higher dimensions. However, the one-dimensional case lets us demonstrate which different limit laws can occur. It is very well understood and in this section we present what is known about them. Random walks in higher dimensional environments are covered in Section 2.4.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, where $\Omega = [0,1]^{\mathbb{Z}}$ and P is a product measure on Ω. Let $\omega = (\omega_x)_{x\in\mathbb{Z}}$ be a random environment with values in Ω. Let the random variables ω_x be mutually independent and identically distributed for all $x \in \mathbb{Z}$. The random environment is best described using a sequence $(\rho_x)_{x\in\mathbb{Z}}$ of random variables defined as

$$\rho_x = \frac{1 - \omega_x}{\omega_x}.$$

To exclude the trivial cases we assume that the random environment is non-degenerate and elliptic, namely

$$\begin{aligned}&\mathsf{Var}(\log \rho_0) > 0 \qquad \text{and}\\ &\exists \kappa \in (0, 1/2) \text{ such that } \mathsf{P}\left(\kappa \leq \omega_0 \leq 1-\kappa\right) = 1.\end{aligned}$$

We assume that these conditions hold for all results in this section. Most results are known to hold under more general assumptions then i.i.d. conductances. The most general known assumptions for each claim are mentioned after the theorem and proof. We consider a nearest-neighbour random walk $(X_n)_{n\in\mathbb{N}}$ in the environment $(\omega_x)_{x\in\mathbb{Z}}$. At each step the random walk at site x will move either to the right with probability ω_x or to the left with probability $1-\omega_x$. Thus, for any start point $z \in \mathbb{Z}$ and vertices $x, y \in \mathbb{Z}$ it has transition probabilities

$$\begin{aligned}p(x,y) &:= P_\omega^z\left(X_{n+1} = y \middle| X_n = x\right)\\ &= \begin{cases}\omega_x & \text{if } y = x+1\\ 1-\omega_x & \text{if } y = x-1\\ 0 & \text{if } y \nsim x\end{cases},\end{aligned}$$

where we write $x \sim y$ if and only if x and y are neighbours, $|x-y| = 1$. Using the detailed balance equation we can calculate a reversible measure $(\pi(x))_{x\in\mathbb{Z}}$ through the recurrence relation

$$\pi(x+1) = \pi(x)\frac{p(x,x+1)}{p(x+1,x)} = \pi(x)\frac{\omega_x}{1-\omega_{x+1}}.$$

By choosing $\pi(0) = 1/\omega_0$, we find that a reversible measure for all sites $x > 0$ is

$$\pi(x) = \pi(0)\frac{\omega_0}{1-\omega_1}\cdot\frac{\omega_1}{1-\omega_2}\cdot\ldots\cdot\frac{\omega_{x-1}}{1-\omega_x} = \frac{1}{\omega_x}\prod_{i=1}^{x}\rho_i^{-1}.$$

The reversible measure for all sites $x < 0$ is accordingly

$$\pi(x) = \frac{1}{\omega_x}\prod_{i=x+1}^{0}\rho_i.$$

We define a function $V : \mathbb{Z} \to \mathbb{R}$ by

$$V(x) = \begin{cases}\sum_{i=1}^{x}\log\rho_i & \text{if } x > 0\\ 0 & \text{if } x = 0\\ \sum_{i=x+1}^{0} -\log\rho_i & \text{if } x < 0\end{cases}. \tag{2.4}$$

Note that

$$\frac{1}{\omega_x} = \frac{\omega_x + (1-\omega_x)}{\omega_x} = 1 + \rho_x.$$

Thus, we can conveniently express the reversible measure in terms of V,

$$\pi(x) = (1+\rho_x)\,e^{-V(x)} = e^{-V(x)} + e^{-V(x-1)}. \tag{2.5}$$

If the random walk in random environment is viewed as a resistor network, we can write the random conductances $(c(x,y))_{x,y\in\mathbb{Z}}$ as

$$c(x,x+1) = \pi(x+1)p(x+1,x) = \frac{1-\omega_{x+1}}{\omega_{x+1}}\prod_{i=1}^{x+1}\rho_i^{-1}$$

for any $x > 0$. A similar expression holds for $x < 0$. This implies

$$c(x, x+1) = e^{-V(x)}.$$

We can also express the effective resistance $\mathcal{R}_x^y = \mathcal{R}(x \leftrightarrow y)$ between two sites $x < y$ by

$$\mathcal{R}_y^x = \mathcal{R}_x^y = \sum_{i=x}^{y-1} \frac{1}{c(i, i+1)} = \sum_{i=x}^{y-1} e^{V(i)}$$

using the function V defined in Equation (2.4). Therefore, we may view $(X_n)_{n\in\mathbb{N}}$ as a random walk in the random potential V. The random potential is itself a random walk with drift

$$\lim_{x\to\infty} \frac{V(x)}{x} = \mathsf{E}\left[\log \rho_0\right].$$

We can now determine whether the random walk in random environment is recurrent or transient depending on the value of $\mathsf{E}[\log \rho_0]$, which was first done by Solomon (1975).

Lemma 2.5 (Solomon (1975))**.** *The random walk $(X_n)_{n\in\mathbb{N}}$ is recurrent if and only if $\mathsf{E}[\log \rho_0] = 0$. In particular, we have the following.*

(i) If $\mathsf{E}[\log \rho_0] < 0$, then $X_n \to +\infty$ as $n \to \infty$ P-almost surely.

(ii) If $\mathsf{E}[\log \rho_0] > 0$, then $X_n \to -\infty$ as $n \to \infty$ P-almost surely.

(iii) If $\mathsf{E}[\log \rho_0] = 0$, then $\liminf X_n = -\infty$ and $\limsup X_n = +\infty$ P-almost surely.

We prove this lemma by evaluating hitting times, which we define as

$$\tau_x = \inf\{k \geq 0 \colon X_k = x\}$$

for the hitting time of a site $x \in \mathbb{Z}$. Then, we have the following lemma.

Lemma 2.6. *Let* $\operatorname{sign}(n) \neq \operatorname{sign}(m)$. *Then*

$$P_\omega^0(\tau_n < \tau_m) = \left(1 + \frac{\mathcal{R}_n^0}{\mathcal{R}_0^m}\right)^{-1}.$$

Proof. The law of the event $\{\tau_n < \tau_m\}$ for the random walk on $\mathbb{Z}$ is the same as for the random walk on the finite set $[n, m] \cap \mathbb{Z}$. Thus, the lemma is a direct consequence of a variation of Lemma 2.4, which applies to any finite one-dimensional graph. On the integers we can turn all inequalities in the proof of the Lemma into equalities, since every excursion from the start point $z = 0$ can only hit the set $\{n\}$ or the set $\{m\}$ before returning to the origin and never both. This implies $S = \{\tau_n < \tau_0^+\}$ and consequently

$$P_\omega^0(\tau_n < \tau_m) = \frac{P_\omega^0(\tau_n < \tau_0^+)}{P_\omega^0(\tau_{n\wedge m} < \tau_0^+)}.$$

We can also use the fact that resistances are always in series to express them explicitly as

$$\mathcal{R}(0 \leftrightarrow \{n, m\}) = \left(\frac{1}{\mathcal{R}_n^0} + \frac{1}{\mathcal{R}_0^m}\right)^{-1}.$$

Using both equalities in Equation (2.3), we prove the claim. □

Proof of Lemma 2.5. Having Lemma 2.6 in mind, we want to consider three different cases.

(i) If $\mathcal{R}^0_{-\infty} = \infty$ and $\mathcal{R}^\infty_0 < \infty$, then for any $k > 0$

$$\lim_{n\to\infty} P^0_\omega(\tau_n < \tau_{-k}) = \lim_{n\to\infty} \left(1 + \frac{\mathcal{R}^n_0}{\mathcal{R}^0_{-k}}\right)^{-1} > 0$$

and

$$\lim_{k\to\infty}\lim_{n\to\infty} P^0_\omega(\tau_n < \tau_{-k}) = \lim_{k\to\infty}\lim_{n\to\infty} \left(1 + \frac{\mathcal{R}^n_0}{\mathcal{R}^0_{-k}}\right)^{-1} = 1.$$

Thus

$$P^0_\omega\left(\lim_{n\to\infty} X(n) = \infty\right) = 1,$$

and the random walk is transient to the right.

(ii) If $\mathcal{R}^0_{-\infty} < \infty$ and $\mathcal{R}^\infty_0 = \infty$, then we get similarly

$$P^0_\omega\left(\lim_{n\to\infty} X(n) = -\infty\right) = 1$$

and the random walk is transient to the left.

(iii) If $\mathcal{R}^0_{-\infty} = \infty$ and $\mathcal{R}^\infty_0 = \infty$, then for any $k > 0$

$$\lim_{n\to\infty} P^0_\omega(\tau_k < \tau_{-n}) = \lim_{n\to\infty} \left(1 + \frac{\mathcal{R}^k_0}{\mathcal{R}^0_{-n}}\right)^{-1} = 1$$

and

$$\lim_{n\to\infty} P^0_\omega(\tau_{-k} < \tau_n) = \lim_{n\to\infty} \left(1 + \frac{\mathcal{R}^0_{-k}}{\mathcal{R}^n_0}\right)^{-1} = 1.$$

Thus

$$P^0_\omega\left(\liminf_{n\to\infty} X(n) = -\infty \quad \text{and} \quad \limsup_{n\to\infty} X(n) = \infty\right) = 1$$

and the random walk is recurrent. Alternatively, the total effective resistance in this electrical network is

$$\mathcal{R} = \left(\frac{1}{\mathcal{R}^\infty_0} + \frac{1}{\mathcal{R}^0_{-\infty}}\right)^{-1}.$$

Thus, by applying Corollary 2.3, we get recurrence directly. Together with irreducibility of the random walk, the claim follows.

The rest of the proof is taken from Zeitouni (2004). Note that $\{\mathcal{R}^\infty_0 < \infty\}$ and $\{\mathcal{R}^0_{-\infty} < \infty\}$ are 0-1-events and by definition of the potential V in Equation (2.4) and stationarity of ω we have

$$\begin{aligned} \mathcal{R}^\infty_0 < \infty \quad &\Longrightarrow \quad \mathcal{R}^0_{-\infty} = \infty \quad \text{and} \\ \mathcal{R}^0_{-\infty} < \infty \quad &\Longrightarrow \quad \mathcal{R}^\infty_0 = \infty. \end{aligned}$$

Therefore, we need to show that

$$\begin{aligned} P_\omega^0\left(\mathcal{R}_0^\infty < \infty\right) \quad &\iff \quad \mathsf{E}\left[\log \rho_0\right] < 0 \quad \text{and} \\ P_\omega^0\left(\mathcal{R}_{-\infty}^0 < \infty\right) \quad &\iff \quad \mathsf{E}\left[\log \rho_0\right] > 0. \end{aligned}$$

Here, we only prove the first claim, as the second claim can be proven analogously. First, assume $\mathsf{E}\left[\log \rho_0\right] < 0$. Then, we can find constants $\epsilon(\omega) > 0$ and $N(\omega) \in \mathbb{N}$ such that

$$\frac{V(n)}{n} = \frac{1}{n} \sum_{k=1}^{n-1} \log \rho_k < -\epsilon$$

for $n > N(\omega)$ chosen large enough by the ergodic theorem. Consequently, for almost every $\omega \in \Omega$ exists some constant $C(\omega)$ such that

$$\mathcal{R}_0^\infty = C(\omega) + \sum_{k=N}^{\infty} e^{V(k)} \leq C(\omega) + \sum_{k=N}^{\infty} e^{-k\epsilon} < \infty.$$

On the other hand, if $\mathcal{R}_0^\infty = \sum_{k=0}^{\infty} e^{V(k)} < \infty$, then $V(n) \to -\infty$ as $n \to \infty$. Since ω is stationary, we can apply a theorem from Kesten (1975) to get

$$\lim_{n\to\infty} \frac{V(n)}{n} < 0.$$

This implies directly the desired result, $\mathsf{E}\left[\log \rho_0\right] < 0$. □

The lemma is true for a much wider range of environment laws P. In particular it holds true, whenever Birkhoff's ergodic theorem applies to the averaged potential, i.e. if P is stationary and ergodic and $\log \rho_0$ is integrable on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$. In fact, it is even enough that $\mathsf{E}\left[\log \rho_0\right]$ is well defined, including values $\pm\infty$, see Theorem 2.1.2 in Zeitouni (2004) for a proof.

Corollary 2.7. *If the random walk $(X_n)_{n\in\mathbb{N}}$ is recurrent, then it is null-recurrent.*

Proof. If the random walk is recurrent, then $\mathcal{R}_0^\infty = \infty$. By the definition of the potential and shift invariance of the environment, we have almost surely

$$\mathcal{R}_0^\infty = \infty \quad \iff \quad \sum_{x<0} \pi(x) = \infty.$$

Thus, the stationary measure π is not finite and the random walk is null-recurrent. □

The behaviour of the random walk is very different in the recurrent versus the transient regime. In the recurrent regime the potential traps the random walk and causes localization behaviour, while in the transient regime we can get limit theorems, if the potential is not too rough.

2.2.1 Recurrent Regime

In the recurrent regime the potential V is a random walk with zero drift. Consequently, it returns to the origin infinitely often and performs increasingly large excursions in between,

$$\liminf_{|x|\to\infty} V(x) = -\infty \quad \text{and} \quad \limsup_{|x|\to\infty} V(x) = +\infty.$$

A random walk in such a potential discovers increasingly large valleys as it explores the environment. Denote the location of the largest discovered valley up to time n by b_n. These valleys trap the random walk for a sufficient amount of time to reduce its fluctuations around b_n to a squared logarithm.

Theorem 2.8 (Sinaĭ (1982)). *There exists a random process $(b(n))_{n\in\mathbb{N}}$ that depends only on ω such that for all $\eta > 0$*

$$\mathbb{P}\left(\left|\frac{X(n) - b(n)}{\log^2 n}\right| > \eta\right) \to 0 \quad \text{as } n \to \infty.$$

The probability distributions of $b(n)/\log^2 n$ converge weakly to some limit distribution as $n \to \infty$ under the environment law P.

The theorem also holds under more general assumptions. The main requirement on the law of the environment P is that there exists some variance $\sigma_{\mathsf{P}}^2 > 0$ such that a functional invariance principle holds for $(V(\pm n)/\sqrt{n\sigma_{\mathsf{P}}^2})_{n\in\mathbb{N}}$. A very neat version of the proof can be found in the lecture notes of Zeitouni (2004), of which we give a short summary here. As we will see in the sketch of the proof, the random process $(b(n))_{n\in\mathbb{N}}$ is in fact equal to the sequence of locations of valleys $(b_n)_{n\in\mathbb{N}}$. We formally define a valley of the potential as a triple (a, b, c), $a < b < c$, such that

$$\begin{aligned} V(a) &= \max_{a\le x\le b} V(x), \\ V(b) &= \min_{a\le x\le c} V(x) \quad \text{and} \\ V(c) &= \max_{b\le x\le c} V(x). \end{aligned}$$

The depth of the valley is

$$d(a, b, c) = \min\{V(a) - V(b), V(c) - V(b)\}.$$

Now, we can define (a_n, b_n, c_n) to be the smallest valley with $a_n < 0 < c_n$ and with depth $d(a_n, b_n, c_n) \ge \log n$. We will typically find such a valley on a scale of $\log^2 n$, i.e. such that $|a_n| + |c_n| \le J \log^2 n$ for some large constant J. We will also most likely find a valley such that it does not contain another valley with height larger than $(1 - \delta)\log n$, neither in the interval $[a_n, b_n]$ nor in $[b_n, c_n]$ for any small $\delta > 0$. The notation is illustrated in Figure 2, which shows an example potential together with the logarithmic lower bound on the depth of the valley and a selected smallest valley.

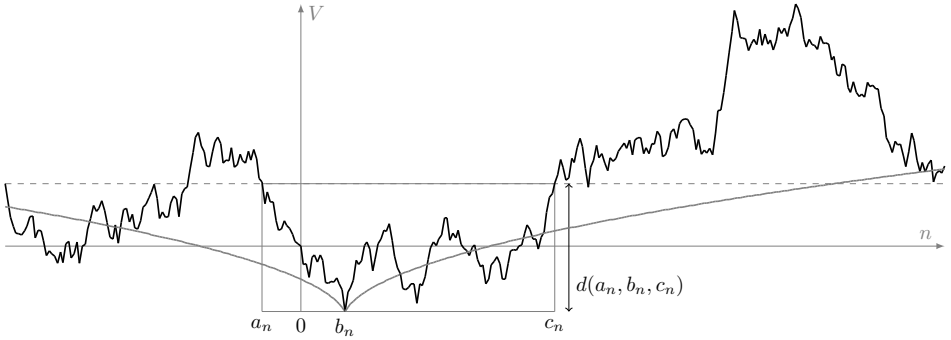

Figure 2: The potential (thick black line) has a smallest valley (a_n, b_n, c_n) of height $d(a_n, b_n, c_n)$. The logarithmic lower bound for the height of the valley relative to the bottom b_n is shown as a thick grey line.

In such a situation the random walk reaches the site b_n before time n with high probability,

$$P_\omega^0(\tau_{b_n} \le n) \to 1 \quad \text{as } n \to \infty. \tag{2.6}$$

Once the random walk has reached the bottom of the valley b_n, it stays there for n time steps,

$$\max_{k \le n} P_\omega^{b^n} \left(\left| \frac{X(k) - b_n}{\log^2 n} \right| > \delta \right) \to 0 \quad \text{as } n \to \infty. \tag{2.7}$$

Therefore, a valley (a_n, b_n, c_n) will trap the random walk up to time n. The random process $(b(n))_{n\in\mathbb{N}}$ can be identified as the sequence $(b_n)_{n\in\mathbb{N}}$ of locations of smallest valleys. The random walk is forced to stay close to $b(n) = b_n$ until it can escape to the next deeper valley around b_{n+1}. Since a valley has a width of $\log^2 n$, this is the scale on which the random walk shows its localization behaviour.

Furthermore, Theorem 2.8 implies that the distribution of $X(n)/\log^2 n$ converges weakly for $n \to \infty$ to the same limit distribution as $b(n)/\log^2 n$ under the annealed law $\mathbb{P}$. This limit distribution was identified later by Golosov (1983) and Kesten (1986) independently.

Theorem 2.9 (Golosov (1983), Kesten (1986)). *Denote by $\sigma^2 := \mathsf{E}\left[\log^2 \rho_0\right]$ the variance of the increments of V. The distribution of*

$$\frac{\sigma^2 b(n;\omega)}{\log^2 n}$$

converges weakly under the environment measure P to the distribution of L, where L is a functional of Brownian motion with density

$$\frac{d\mathsf{P}(L \le x)}{dx} = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{\pi^2 (2k+1)^2}{8} |x| \right) \, dx.$$

We have seen previously that the random walk is null-recurrent in the original environment. However, the particle visits increasingly deep valleys as it performs its excursions and explores the environment. As a consequence the environment seen from the particle converges to an infinitely large valley. More precisely, the stationary limit of the environment seen from the particle has the same law as the original environment, conditioned to be non-negative on the positive integers and strictly positive on the negative integers. The environment seen from the particle is positive recurrent. As a consequence, the random walk centred around the sequence of valley locations $(b_n)_{n\in\mathbb{N}}$ converges without any scaling to some limit distribution.

Theorem 2.10 (Golosov (1984)). *(i) The distribution of the random process $(X(n) - b(n;\omega))_{n\in\mathbb{N}}$ converges to a limit function F_σ for any $\sigma > 0$.*

(ii) There exists a distribution function G such that

$$\lim_{\sigma \to 0} F_\sigma \left(\frac{x}{\sigma^2} \right) = G(x).$$

The last result implies that with high probability and for large times n the random walk stays in a finite neighbourhood of $b(n)$, which is of size $1/\sigma^2$.

2.2.2 Transient Regime

In the transient regime there is a phase transition in the speed of the random walk. In one phase the speed of the walk is zero, while in the other it is strictly positive. We also have a second phase transition for the existence of a diffusive central limit theorem. Without loss of generality take $\mathsf{E}[\log \rho_0] < 0$ throughout this section, so that almost surely the random walk is transient to the right, $X(n) \to +\infty$ for $n \to \infty$. The

case where the random walk is transient to the left follows by reflection. Then, the two phase transitions can be characterized by the number

$$s := \sup\{r \ : \ \mathsf{E}[\rho_0^r] < 1\}, \tag{2.8}$$

which is the largest moment of ρ_0 that exists and is bounded by one. Note that s may take values in $[0,\infty)\cup\{\infty\}$. The two phase transitions occur at values $s=1$ and $s=2$ respectively. Since we look at random walks that are transient to the right, we need to be concerned with moments of hitting times

$$\tau_1 = \inf\{n>0 : X(n)=1\}.$$

In particular it has been shown that for all $\gamma < s$ the following moments exist,

$$\mathbb{E}^0\left[(E_\omega\tau_1)^\gamma\right] < \infty \quad \text{and} \quad \mathbb{E}^0\left[\tau_1^\gamma\right] < \infty,$$

where the superscript indicates that $X(0)=0$ almost surely. For the first estimate, see for example Peterson (2008). The second estimate is shown in Dembo et al. (1996). Thus, we see that the phase transitions occur exactly at those points at which we gain a first and second moment of the hitting times.

The transient regime has been intensely studied and by now we have an almost complete characterization, which is mainly due to the works of Solomon (1975) for the results on the speed, Kesten et al. (1975) for the annealed results and Goldsheid (2007) and the works of Peterson and Zeitouni for the quenched results. All these results are summarized in Table 1 at the end of this section. For i.i.d. random environments the limit theorems are known, except for quenched limit laws for critical values $s=1$ and $s=2$. The first phase transition at $s=1$ concerns the speed of the random walk, which is

$$v := \lim_{n\to\infty}\frac{X(n)}{n}.$$

Theorem 2.11 (LLN, Solomon (1975)). *The speed of the random walk is zero* P*-almost surely if and only if* $s \le 1$*. The speed is positive, if* $s>1$*. Then*

$$v = \frac{1-\mathsf{E}[\rho_0]}{1+\mathsf{E}[\rho_0]} > 0 \quad \mathbb{P}\text{-almost surely.}$$

Note that the definition of s implies that $\mathsf{E}[\rho_0]<1$ if and only if $s>1$. Annealed scaling limits are known for all $s>0$. The two limit laws that can appear are Gaussian laws and stable laws. Let Φ be the distribution function of a Gaussian random variable,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy. \tag{2.9}$$

Let $F_{\alpha,\beta}$, $\alpha\neq 0$, be the distribution function of a stable law with characteristic function

$$\hat{F}_{\alpha,\beta}(t) = \exp\left(-\beta|t|^\alpha\left(1 - i\cdot\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sign}(t)\right)\right). \tag{2.10}$$

Theorem 2.12 (Annealed scaling limits, Kesten et al. (1975)). *Assume that the environment is such that* $s>0$*,* $\mathsf{E}\log\rho_0 < 0$ *and* $\mathsf{E}[\rho_0^s\log\rho] < \infty$*. Assume furthermore that the distribution of* $\log\rho_0$ *is non-lattice, i.e. it is not concentrated on a set of points* $\{ax+b\}_{x\in\mathbb{Z}}$*, for any* $a,b\in\mathbb{R}$*.*

(i) If $s\in(0,1)$*, then an annealed stable limit law holds with*

$$\mathbb{P}\left(\frac{X(n)}{n^s}\le x\right) \to 1 - F_{s,\beta}(x^{-1/s}) \quad \text{as } n\to\infty,$$

where β *has been identified explicitly in Enriquez et al. (2009).*

(ii) *If $s = 1$, then there exist a constant $0 < C_1 < \infty$ and function $\delta(n) \sim n/(C_1 \log n)$ such that an annealed stable limit law holds with*

$$\mathbb{P}\left(\frac{X(n) - \delta(n)}{n^{-1}\log^2 n} \leq x\right) \to 1 - F_{1,\beta}(-C_1^2 x) \quad \text{as } n \to \infty,$$

where β is not explicitly known.

(iii) *If $s \in (1,2)$, then an annealed stable limit law holds with*

$$\mathbb{P}\left(\frac{X(n) - nv}{v^{1+1/s} n^{1/s}} \leq x\right) \to 1 - F_{s,\beta}(-x) \quad \text{as } n \to \infty,$$

where β is not explicitly known.

(iv) *If $s = 2$, then a super-diffusive central limit theorem holds with deterministic variance $\sigma_a > 0$,*

$$\mathbb{P}\left(\frac{X(n) - nv}{\sigma_a \sqrt{n \log n}} \leq x\right) \to \Phi(x) \quad \text{as } n \to \infty. \tag{2.11}$$

(v) *If $s > 2$, then an annealed CLT holds with deterministic variance $\sigma_a^2 > 0$,*

$$\mathbb{P}\left(\frac{X(n) - nv}{\sigma_a \sqrt{n}} \leq x\right) \to \Phi(x) \quad \text{as } n \to \infty. \tag{2.12}$$

Under the quenched law, there are no limit laws for certain values of s. A further speciality to random walks in one-dimensional environments compared to higher dimensions is that the quenched variance of the limit theorem is smaller than the annealed one. The reason is that the quenched expectation $\mathsf{E}_\omega[X(n)]$ fluctuates around the annealed expectation nv. Thus, a part of the total variance is contained in those fluctuations.

Theorem 2.13 (Quenched scaling limits). *(i) (Goldsheid (2007); Peterson (2008)) For $s > 2$, a quenched central limit theorem holds with deterministic variance $\sigma_q^2 < \sigma_a^2$ and quenched expectation. That is, for $\mathbb{P}$-almost every ω,*

$$P_\omega\left(\frac{X(n) - E_\omega[X(n)]}{\sigma_q \sqrt{n}} \leq x\right) \to \Phi(x) \quad \text{as } n \to \infty. \tag{2.13}$$

(ii) (Peterson and Zeitouni (2009); Peterson (2009)) For $s \in (0,1)$ and $s \in (1,2)$ no quenched limit laws exist. In fact, for almost every ω there exist two different random sub-sequences along which we can observe different limit laws.

The first statement of the theorem holds for a much wider class than i.i.d. environments. Independently of each other, Peterson (2008) proved the statement for α-mixing environments, while Goldsheid (2007) could show the result even for ergodic environments. The previous results about transient one-dimensional random walks in random environment are summarized in Table 1.

Eventually, a local limit theorem for the annealed and quenched law in the diffusive regime $s > 2$ was proven by Dolgopyat and Goldsheid (2013). These results allowed them to show that the environment seen from the particle converges for almost every environment.

Theorem 2.14 (Dolgopyat and Goldsheid (2013)). *For every continuous function $f : \Omega \to \mathbb{R}$*

$$E_\omega\left[f\left(\theta_{X(n)}\omega\right)\right] \xrightarrow{n\to\infty} \frac{\mathsf{E}[\rho_0 f]}{\mathsf{E}[\rho_0]} \quad \text{for } \mathbb{P}\text{-almost every } \omega$$

where $(\theta_n \omega)(x) = \omega_{x+n}$ is the usual shift operator.

	$s \in (0,1)$	$s=1$	$s \in (1,2)$	$s=2$	$s>2$
LLN	$v=0$		$v=(1-\mathsf{E}\rho_0)/(1+\mathsf{E}\rho_0)$		
annealed LLs	s-stable limit law			aCLT with $\sigma_a^2>0$	
quenched LLs	~~		~~		qCLT, $0<\sigma_q^2<\sigma_a^2$

Table 1: Characterization of limit laws in the transient regime. The results in the crossed out cells have been disproven, while the results for the empty cells are still open.

2.3 RWRE on the strip

Random walks in random environments on the strip $\mathbb{Z} \times \{1, \dots, M\}$ are studied as a generalization of one-dimensional models. This class includes in particular random walks in one-dimensional random environments with bounded jumps. It can be also applied to certain interacting particle systems with a finite number of particles, which we will discuss later in Section 3.4.3. We call the set $\{k\} \times \{1, \dots, M\}$ a layer of the strip and consider a family of transition matrices $(\omega_k)_{k\in\mathbb{Z}} = \{(P_k, Q_k, R_k)\}_{k\in\mathbb{Z}}$ with three matrices $P_k, Q_k, R_k \in \mathbb{R}^{M\times M}$ for each layer. Each triplet (P_k, Q_k, R_k) is chosen such that all matrices are positive definite and $(P_k + Q_k + R_k)\mathbf{1} = \mathbf{1}$. The matrix P_k contains the transition probabilities from layer k to layer $k+1$, the matrix Q_k from layer k to layer $k-1$ and the matrix R_k are the transition probabilities within layer k. The random walk in this set-up can only jump within layers or to neighbouring layers. Let $(X_n)_{n\in\mathbb{N}}$ be a discrete time Markov chain with state space $\mathbb{Z} \times \{1, \dots, M\}$, where $X_n = (\eta_n, \xi_n)$. Thus, η_n denotes the layer and ξ_n denotes the vertex within the layer for the position of the the random walk at time n. Then, the transition probabilities relate to the matrices $\{(P_k, Q_k, R_k)\}_{k\in\mathbb{Z}}$ according to

$$\begin{aligned} P_k(i,j) &= \mathbb{P}\left(X_{n+1} = (k+1, j) \middle| X_n = (k,i)\right), \\ Q_k(i,j) &= \mathbb{P}\left(X_{n+1} = (k-1, j) \middle| X_n = (k,i)\right) \text{ and} \\ R_k(i,j) &= \mathbb{P}\left(X_{n+1} = (k, j) \middle| X_n = (k,i)\right), \end{aligned}$$

for each $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i, j \in \{1, \dots, M\}$. The matrices form the random environment of the random walk on the strip. They are chosen such that the sequence

$$\{(P_n, Q_n, R_n)\}_{n\in\mathbb{Z}} \text{ is stationary and ergodic .} \tag{C1}$$

Define the matrix norm

$$||A|| = \max_{1\le i\le M} \sum_{j=1}^{M} |A(i,j)|$$

for any $A \in \mathbb{R}^{M\times M}$. We require further that first inverse logarithmic moments exists,

$$\begin{aligned} &\mathsf{E}\left[\log\left(1 - \|P_k + R_k\|\right)^{-1}\right] < \infty \text{ and} \\ &\mathsf{E}\left[\log\left(1 - \|Q_k + R_k\|\right)^{-1}\right] < \infty \end{aligned} \tag{C2}$$

for all $k \in \mathbb{Z}$. This condition implies that for all $1 \le i \le M$

$$\sum_{j=1}^{M} P_k(i,j) > 0 \quad \text{and} \quad \sum_{j=1}^{M} Q_k(i,j) > 0.$$

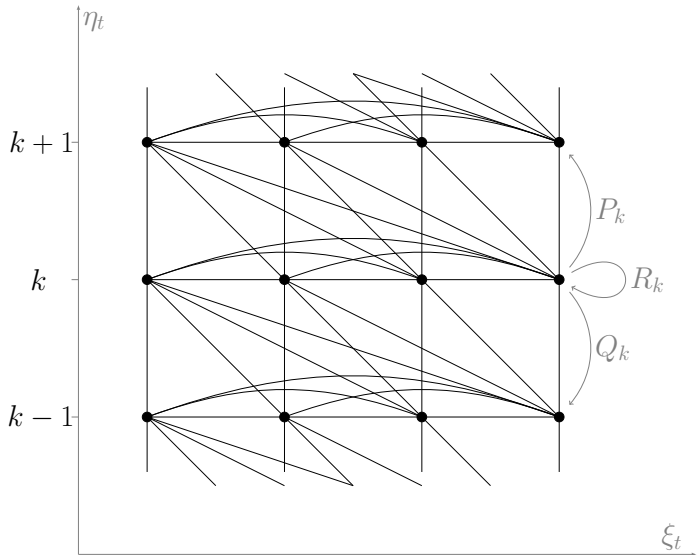

Figure 3: One-dimensional random walk $X_n = (\eta_n, \xi_n)$ with jumps bounded by maximal distance $M = 4$ represented as a random walk on the strip $\mathbb{Z} \times \{1, \ldots, 4\}$. All allowed transitions are marked by an edge. All missing edges are for transitions which have probability zero.

We may also assume that for all $1 \leq j \leq M$

$$\sum_{i=1}^{M} P_k(i,j) > 0 \quad \text{and} \quad \sum_{i=1}^{M} Q_k(i,j) > 0. \tag{C3}$$

We furthermore need some form of irreducibility. The minimal irreducibility assumption would require the whole strip to be in one communicating class. Usually, stronger assumptions need to be made. Note that it is not reasonable to assume all matrix elements to be positive, since such an assumption would be violated whenever the model is derived from a one-dimensional random walk. As an example consider the one-dimensional random walk $(Y_n)_{n\in\mathbb{N}}$ in random environment with bounded jumps and let M be the maximal distance for a jump. Assume furthermore that all jump probabilities for the one-dimensional random walk are strictly positive,

$$p(i,j) = \mathbb{P}(Y_{n+1} = j | Y_n = i) > 0$$

for all $i, j \in \mathbb{Z}$ such that $1 \leq |i - j| \leq M$ and $n \in \mathbb{N}$. By chopping the integers into pieces $\{1, \ldots, M\}$, $\{M+1, \ldots, 2M\}$, ... and so on, we can represent the state space of this random walk as the strip $\mathbb{Z} \times \{1, \ldots, M\}$. The corresponding random walk on the strip is the Markov chain $(X_n)_{n\in\mathbb{N}}$ such that

$$X_n = (\eta_n, \xi_n) \quad \text{with} \quad \eta_n = \lfloor Y_n / M \rfloor \quad \text{and} \quad \xi_n = Y_n \mod M.$$

Here, $\lfloor \cdot \rfloor$ denotes the largest integer that bounds the argument from below. While all transitions within layers have positive probability, this is not the case for transitions to neighbouring layers. This is illustrated in Figure 3, where all transitions with positive probability are marked by an edge between states. For the definition of a strong irreducibility assumption which is not too restrictive, set the first hitting time of neighbouring layers $+1$ and -1 to be

$$\tau^+ = \inf\{n \geq 0 : \eta_n = 1\} \quad \text{and} \quad \tau^- = \inf\{n \geq 0 : \eta_n = -1\}.$$

We want a positive probability for the event that a walker starting at site (n, i) reaches the neighbouring layers upon first entry of the layer at the sites $(n+1, j)$ and $(n-1, j)$ respectively,

$$P_{\theta_n\omega}(\xi_{\tau^+} = j | \xi_0 = i) > \epsilon \quad \mathbb{P} - \text{a.s. for all } 1 \leq i, j \leq M \tag{C4.1(ϵ)}$$
$$P_{\theta_n\omega}(\xi_{\tau^-} = j | \xi_0 = i) > \epsilon \quad \mathbb{P} - \text{a.s. for all } 1 \leq i, j \leq M. \tag{C4.2(ϵ)}$$

These two assumptions imply Condition (C2). It is a result of Bolthausen and Goldsheid (2000) that there exists a unique sequence of $M \times M$ matrices $(\psi_n)_{n\in\mathbb{Z}}$ that satisfies the equation

$$\psi_{k+1} = P_k + R_k\psi_{k+1} + Q_k\psi_k\psi_{k+1}.$$

Also, define the matrices

$$A_k = (I - R_k - Q_k\psi_k)^{-1} Q_k.$$

Then, by Kingman's subadditive ergodic theorem,

$$\lambda^+ = \lim_{k\to\infty} \frac{1}{k} \log \left\| \prod_{i=1}^{k} A_i \right\|$$

exists almost surely and is constant. Using this constant, we can determine recurrence and transience just as in the one-dimensional case.

Theorem 2.15 (Recurrence and Transience, Bolthausen and Goldsheid (2000))**.** *Assume that conditions (C1), (C2) and (C3) hold and that the zeroth layer (and consequently every layer) is in the same communicating class.*

(i) If $\lambda^+ > 0$, then $\lim_{n\to\infty} \eta_n = -\infty$ almost surely,

(ii) If $\lambda^+ < 0$, then $\lim_{n\to\infty} \eta_n = +\infty$ almost surely and,

(iii) If $\lambda^+ = 0$, then $\liminf_{n\to\infty} \eta_n = -\infty$ and $\limsup_{n\to\infty} \eta_n = +\infty$ almost surely.

As one would expect, if the distribution of (P_k, Q_k, R_k) is equal to the distribution of (Q_k, P_k, R_k), then $\lambda^+ = 0$ and the random walk on the strip is recurrent. There is also a law of large numbers, which was proven independently in two papers. For the random walk on $\mathbb{Z}$, the positivity of the speed was determined by the constant s as defined in Equation (2.8). This role is now taken over by

$$s = \sup \left\{ r : \limsup_{n\to\infty} \frac{1}{n} \log |\mathsf{E} \|A_1 \dots A_n\|^r| < 0 \right\}.$$

Theorem 2.16 (LLN, Goldsheid (2008); Roitershtein (2008))**.** *Assume that conditions (C1), (C3) and (C4.1(ϵ)), (C4.2(ϵ)) hold for some $\epsilon > 0$ and that the zeroth layer is in the same communicating class. Then, the limit*

$$v = \lim_{n\to\infty} \frac{\eta_n}{n}$$

exists and is constant $\mathbb{P}$-almost surely. Furthermore, if $\lambda^+ < 0$, then $\mathbb{P}$-almost surely

$$v > 0 \quad \text{if} \quad s > 1 \text{ and}$$
$$v = 0 \quad \text{if} \quad s < 1.$$

In the transient case, we get annealed and quenched central limit theorems, where we have a quenched correction of the speed as for the random walk in one-dimensional environments, Theorems 2.12 and 2.13.

Theorem 2.17 (Annealed and quenched CLT, Goldsheid (2008); Roitershtein (2008)). *Assume that conditions* (C1), (C3) *and* (C4.1(ϵ)), (C4.2(ϵ)) *hold for some* $\epsilon > 0$ *and that the zeroth layer is in the same communicating class. Assume furthermore that* $s > 2$. *Then, there is an annealed and quenched central limit theorem with quenched correction of the speed.*

While Conditions (C3) and (C2) or (C4.1(ϵ)), (C4.2(ϵ)) in Theorems 2.15, 2.16 and 2.17 might not be satisfied if the random walk on a strip is derived from a one-dimensional model, Goldsheid (2008) claims that it may be proven under milder assumptions. He requires that the whole strip is one communicating class and that there is some $\epsilon > 0$ such that either $(I - R)^{-1}P(i, j) \geq \epsilon$ or $(I - R)^{-1}Q(i, j) \geq \epsilon$ for all $1 \leq i, j \leq M$ instead.

If the random walk on the strip is recurrent, we get the same behaviour as on $\mathbb{Z}$ and the random walk concentrates around some random sequence $(b(n))_{n\in\mathbb{N}}$.

Theorem 2.18 (Bolthausen and Goldsheid (2008)). *Let the random walk on the strip* $(X_n)_{n\in\mathbb{N}}$ *be recurrent and set* $X_n = (\eta_n, \xi_n)$. *Assume that*

(i) the sequence $(P_k, Q_k, R_k)_{k\in\mathbb{Z}}$ *is i.i.d. with law* μ,

(ii) there exists $\epsilon > 0$ *and* $l < \infty$ *such that Conditions* (C4.1(ϵ)) *and* (C4.2(ϵ)) *hold and such that* $||R^l|| < 1 - \epsilon$,

(iii) $\mathrm{supp}(\mu) \not\subseteq \{\pi(P_0 - Q_0)\mathbf{1} = 0 \in \mathbb{Z}\}$, *where* π *is the unique row vector such that* $\pi(P_0 + Q_0 + R_0) = \pi$ *and* $\sum_{i=1}^{M} \pi_i = 1$.

Then, there exists a sequence of random variables $(b_n) = (b(n; \omega))$ *which converges weakly as* $n \to \infty$ *and for every constant* $\delta > 0$,

$$\mathbb{P}\left(\left|\frac{\eta_n - b_n}{\log^2 n}\right| > \delta\right) \to 0 \text{ as } n \to \infty.$$

In their paper Bolthausen and Goldsheid (2008) remark that the theorem could also be proved under less strict conditions, by replacing Condition *(ii)* of the theorem with the assumption that the strip is the only communicating class of the random walk and that there is an $\epsilon > 0$ and a triple $(P, Q, R) \in \mathrm{supp}(\mu)$ such that either Assumption (C4.1(ϵ)) or Assumption (C4.2(ϵ)) hold. They also remark that Condition *(iii)* of the theorem is necessary by giving an example where the violation of this condition leads to Gaussian behaviour. Finally, Bolthausen (2008) remarks in his lecture notes that for constant transition matrices (P, Q, R) the third condition is equivalent to $\lambda^+ = 0$, i.e. to the recurrence of the random walk.

2.4 RWRE on integer lattices

The picture for random walks in random environments on integer lattices in more than two dimensions is much less clear than on the integers. First, while random walks on the integers are always reversible and can be expressed as a random conductance model, this is not the case in higher dimensions. Thus, we lack much of the machinery presented so far. Even if we restrict ourselves to random conductance models on $\mathbb{Z}^d$, there are still many difficulties. This is partially due to the fact that traps can take much more complicated shapes. In very high dimensions, traps are less likely to form, since many conductances have to work together to form a trap. For an illustration of this effect consider the simplest possible trap, which is a single large conductance between to sites x and y, while all other conductances adjacent to x and y are small. In $d = 2$ this requires six conductances to be small. In general we need to control $2(2d - 1)$ conductances to get a trap. Thus, the probability for such an

event in an i.i.d. conductance model decreases as p^{4d-2}, where p is the probability for a conductance to be small enough. In addition to that, we also have to find an estimate on the probability for the random walk to enter a trap. On the integers, this is much easier, since the random walker has to enter and leave every trap that lies on its way at least once.

Another complication is that we may allow non-elliptic environments. On the integers $\mathbb{Z}$, whenever the probability for a zero conductance is positive, the random walk becomes confined to finite intervals of $\mathbb{Z}$. For $d \geq 2$, one can obtain an infinite connected component, if the probability for conductances to be zero is small. This effect is known as percolation. If there is an infinite connected component, the random walk has non-trivial behaviour on this sub-graph. We will first briefly describe the most recent results for random conductance models with no intention to cover the topic. Then we summarize the results for random walks on percolation clusters. We conclude the chapter with our own results on random walks on oriented percolation clusters. Thus, some of the material presented here has already appeared in Miller (2016).

2.4.1 In elliptic environments

Let $(X_n)_{n\geq 0}$ be a random walk in a random elliptic i.i.d. environment on $\mathbb{Z}^d$. As the situation is already complicated in $d = 1$, one might expect it to be even more so in $d \geq 2$. Indeed, in general cases it is not even known what the precise conditions for transience or ballisticity of a random walk should be. As these results are not of direct importance for the main result of this section, we refer for example to Zeitouni (2004) for an overview. Most results are known for the random conductance model. Here, the strict ellipticity condition on the conductances was removed in recent works on the topic and replaced by moment conditions, for example Andres et al. (2015). However, in contrast to percolating environments, the conductances have to be strictly between zero and infinity. One may also allow conductances to change in time. These models are known as dynamic random conductance models. For the previously mentioned model the dynamic version has been treated in Andres et al. (2016). We can reduce a dynamic random conductance model on $\mathbb{Z}^d$ to a static one, by adding another dimension for the time such that the state space is $\mathbb{Z}^d \times \mathbb{Z}$ or $\mathbb{Z}^d \times \mathbb{R}$. Then, we force the random walk to be transient with speed one in the time direction. The same effect is caused by choosing an oriented graph as we do for random walks on oriented percolation clusters.

2.4.2 On percolation clusters

In this section, we will only consider percolation on integer lattices, although percolation on more general graphs is a big field of research in itself. Denote by $V = \mathbb{Z}^d$ the vertex set and let E be the set of all nearest-neighbour edges of V,

$$E = \{(x, y) \ : \ x, y \in V \text{ and } ||x - y||_2 = 1\}.$$

Then, the pair $\mathcal{G} = (V, E)$ is a graph. We say two vertices x, y are adjacent in the graph $\mathcal{G}$ if $(x, y) \in E$. We say two edges (x, y) and (x', y') are adjacent, if they share a vertex, i.e. either $x \in \{x', y'\}$ or $y \in \{x', y'\}$. We may now assign a Bernoulli random variable either to each vertex $v \in V$ or each edge $e \in E$. We write I for the index set of the family of random variables and choose $I = V$ or $I = E$. Then, a percolation is a family $(\omega_x)_{x\in I}$ of i.i.d. Bernoulli random variables with parameter $p \in [0, 1]$ such that $\omega_x \in \{0, 1\}$ for all $x \in I$. If $I = V$, we get site percolation, if $I = E$ we get bond percolation. We say that a vertex or edge x is open, if $\omega_x = 1$ and closed if $\omega_x = 0$. Thus, a vertex or edge is open with probability p and closed with probability $1 - p$ independent of all other vertices or edges. We say two vertices x and y are connected by an open path, if two things happen. First, there is a sequence of vertices $z_1, \ldots, z_N$

such that $z_1 = x$, $z_N = y$ and z_i and z_{i+1} are adjacent. Second, we require for site percolation that all vertices $z_1, \dots, z_N$ are open and for edge percolation that all edges $(z_1, z_2), \dots, (z_{N-1}, z_N)$ are open. We write $x \to y$ to denote that there is an open path from x to y.

In this thesis, we generally work with site percolation. Thus, the following notation is for $I = V$. We define the connected open component $\mathcal{C}_x$ of a site x as the set of all vertices that can be reached from x by an open path. If x is closed, then $\mathcal{C}_x = \emptyset$. If x is open, then

$$\mathcal{C}_x = \{y \in V : x \to y\}.$$

We say that $(\omega_x)_{x \in V}$ percolates if there is some $x \in V$ such that $|\mathcal{C}_x| = \infty$. We define the critical probability as

$$p_c = \sup \{p \geq 0 : \mathsf{P}(|\mathcal{C}_0| = \infty) = 0\}.$$

We say that the percolation is *supercritical*, if the probability for an edge or vertex to be open is bigger than the critical probability, $p > p_c$. The main theorem on percolation states that this phase transition happens at a non-trivial critical probability p_c.

Theorem 2.19. *For percolation on integer lattices with $d \geq 2$, we have that $0 < p_c < 1$. Provided that $p > p_c$, the infinite connected component is unique.*

For a proof of this theorem, see for example Grimmett (2010), Theorem 3.2 and Theorem 5.22, which was originally shown in Aizenman et al. (1987). The random walk on a percolation cluster is the random walk that chooses its next step uniformly from all adjacent open sites. Assume that the origin is in the infinite cluster, $0 \in \mathcal{C}_\infty$. The random walk on $\mathcal{C}_\infty$ is a Markov chain $(X_n)_{n \in \mathbb{N}}$ with state space $\mathbb{Z}^d$ starting at $X_0 = 0$ with transition probabilities

$$P^0_\omega (X_{n+1} = y | X_n = x) = \frac{\mathbb{1}_{\{y \in \mathcal{C}_\infty\}}}{\sum_{z \sim x} \mathbb{1}_{\{z \in \mathcal{C}_\infty\}}}$$

for every environment ω such that $0 \in \mathcal{C}_\infty$.

The next two theorems were proven for bond percolation only. With some extra work one can transfer the proofs to site percolation as well. The annealed central limit theorem on a percolation cluster on integer lattices was proven for $d = 2$ by De Masi et al. (1989), while the result for general dimensions follows from Gaussian bounds on the transition densities proven in Barlow (2004).

Theorem 2.20 (Annealed central limit theorem for random walk on percolation clusters, De Masi et al. (1989); Barlow (2004))**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a random walk on supercritical bond percolation on $\mathbb{Z}^d$ with $d \geq 2$ and such that $X_0 = 0$ almost surely. Then, an annealed invariance principle holds for $(X_n)_{n \in \mathbb{N}}$.*

The quenched central limit theorem was first shown by Sidoravicius and Sznitman (2004) for dimensions $d \geq 4$ and then by Berger and Biskup (2007) and independently by Mathieu and Piatnitski (2007) for dimensions $d = 2, 3$.

Theorem 2.21 (Quenched central limit theorem for random walk on percolation clusters, Sidoravicius and Sznitman (2004); Berger and Biskup (2007); Mathieu and Piatnitski (2007))**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a random walk on supercritical bond percolation on $\mathbb{Z}^d$ with $d \geq 2$ and such that $X_0 = 0$ almost surely. Then, a quenched functional invariance principle holds for $(X_n)_{n \in \mathbb{N}}$.*

Going back to the most general definition of random walks in random environments, we can add orientation to the edges of the graph and only allow the random walk to pass

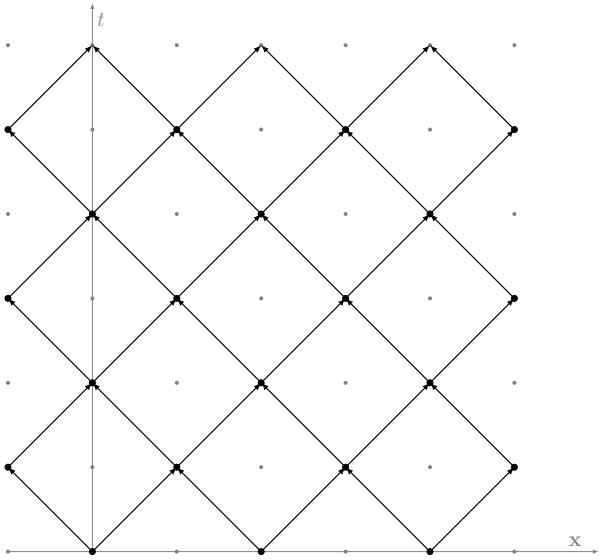

Figure 4: Oriented lattice graph for $d = 1$. The vertices of the underlying lattice $\mathbb{Z}^d \times \mathbb{Z}$ are shown in grey. We see that there is in fact two disjoint copies of this graph on $\mathbb{Z}^d \times \mathbb{Z}$. One of them is shown in here, the second can be obtained by translation along the x-axis. We choose the connected component that contains the origin.

an edge following its orientation. Such a model is not a random conductance model. In oriented percolation, the orientation of the edges distinguishes one dimension and breaks the symmetry. Thus, we work on the discrete space $V := \mathbb{Z}^d \times \mathbb{Z}$, which we will refer to as the *full lattice*. The first $d \geq 1$ dimensions in V are *space dimensions* and the last dimension is the *time dimension*. We turn the lattice V into an oriented graph (V, E) with vertices V by adding edges

$$E := \{|(x, n), (y, k)\rangle : (y, k) \in U^+(x, n))\},$$

where $|(x, n), (y, k)\rangle$ denotes an oriented edge from (x, n) to (y, k) and

$$U^+(x, n) := \{(y, k) \in V : ||x - y||_\infty = 1, k = n + 1\} \tag{2.14}$$

is the set of consecutive vertices of (x, n). The specific choice of the set of consecutive vertices U^+ is not important as long as it is finite and symmetric. The corresponding oriented graph for our choice of U^+ and $d = 1$ is shown in Figure 4. There is always more than one disjoint connected component of this graph and only one of them is shown. We will always work on the component that contains the origin, without further mentioning.

Let $(\omega(x, n))_{(x,n)\in V}$ be a family of independent and identically distributed Bernoulli random variables with parameter $p \in (p_c, 1]$ that represents a supercritical site percolation on the vertex set V. The constant $0 < p_c < 1$ is the critical probability of oriented site percolation on (V, E). Existence and non-triviality of p_c was proven in Grimmett and Hiemer (2002), similar to Theorem 2.19. As before, we say a site $(x, n) \in V$ is *open*, if $\omega(x, n) = 1$. Otherwise, we call it *closed*. With the notion of open sites we can define open paths. A directed path on the oriented graph (V, E) from vertex (x, n) to vertex (y, m) is called open, if all vertices on that path are open. For an open directed path from (x, n) to (y, m) we write $(x, n) \to (y, m)$. Analogously, we write $(x, n) \to \infty$ if there is an infinite, directed open path on (V, E) starting in (x, n). The oriented percolation cluster is not a good graph for a random walk, as it

would get trapped in finite time. We define the percolation process $\xi^P := (\xi_n^P)_{n\in\mathbb{Z}}$ by

$$\xi_n^P(x) := \begin{cases} 1 & \text{if } (x,n) \to \infty \\ 0 & \text{otherwise.} \end{cases}$$

Any random walk on the oriented percolation cluster has to be confined to sites that are open in the percolation process to avoid traps. Since a site (x,n) can only ever be visited, if $\xi_n^P(x) = 1$, we refer to ξ^P as the environment. The place of the oriented percolation cluster as the sub-graph for the random walk is thus taken by its *backbone*, which is denoted by

$$\mathcal{C} := \{(x,n) \in V : (x,n) \to \infty\}$$

and is a proper subset of the oriented percolation cluster. It describes all sites that lie on an infinite directed open path on (V,E). Since the random walk is only non-trivial, if it starts on the backbone, we define a new measure conditioned on the origin being in the backbone. We can do this, since this event has a positive probability in supercritical percolation. We therefore write $\tilde{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|0 \in \mathcal{C})$ for the annealed measure conditioned on the backbone containing the origin. The random walk $(X_n)_{n\in\mathbb{N}}$ on the environment ξ^P such that $0 \in \mathcal{C}$ has transition probabilities

$$P_\xi^0(X_{n+1} = y | X_n = x) = \frac{\mathbb{1}_{\{(y,n+1)\in U^+(x,n)\cap\mathcal{C}\}}}{\sum_{z\sim x} \mathbb{1}_{\{(z,n+1)\in U^+(x,n)\cap\mathcal{C}\}}}. \tag{2.15}$$

The usual limit theorems for this model were shown recently.

Theorem 2.22 (Scaling limits for random walk on oriented percolation clusters, Birkner et al. (2013))**.** *Let $(X_n)_{n\geq 0}$ be a random walk on the backbone of an oriented bond percolation on $\mathbb{Z}^d \times \mathbb{Z}$ for $d \geq 1$. Then*

(i) a law of large numbers holds with $\vec{\mu} = 0$,

(ii) an annealed central limit theorem holds with non-degenerate covariance matrix Σ and

(iii) a quenched central limit theorem holds with the same limit law as for the annealed case.

The random walk on the backbone of an oriented percolation cluster is a simple model for ancestral lineages in populations with local competition. The connection to a population model is shown in Figure 5 on the example of a finite section of V. Each open site in V can be inhabited by at most one individual. However, an open site cannot be inhabited if there is no individual in the previous generation that can become its parent. The local construction is as follows. First, we put individuals at all open sites at some initial time, here $s = 1$. In the figure this first generation is highlighted by the grey box. Each of these individuals places offspring to neighbouring sites in the next time step $s+1$ in case the new site is open. The backbone is the set of all sites that are inhabited eventually by progeny of the first generation. It is shown in black in the top panel of the figure. For the global construction of the backbone $\mathcal{C}$, we take the first generation back to time $-\infty$. We can see that in many occasions a site can receive offspring from two parents. In this case, only one of the potential parents can place their offspring, which is a simple form of local competition. An ancestral line is the path of its ancestors backwards through the generations. It is shown in black in the bottom panel. To construct the ancestral line, first take an individual from the backbone that lives in the last generation. Successively, we choose one of the possible parents from generation $s-1$ uniformly at random. If there is no choice, we pick the only parent available. This makes the ancestral line a simple random walk on the backbone. By construction, when we follow the path of the random walk with time t, we go backwards through the generations s.

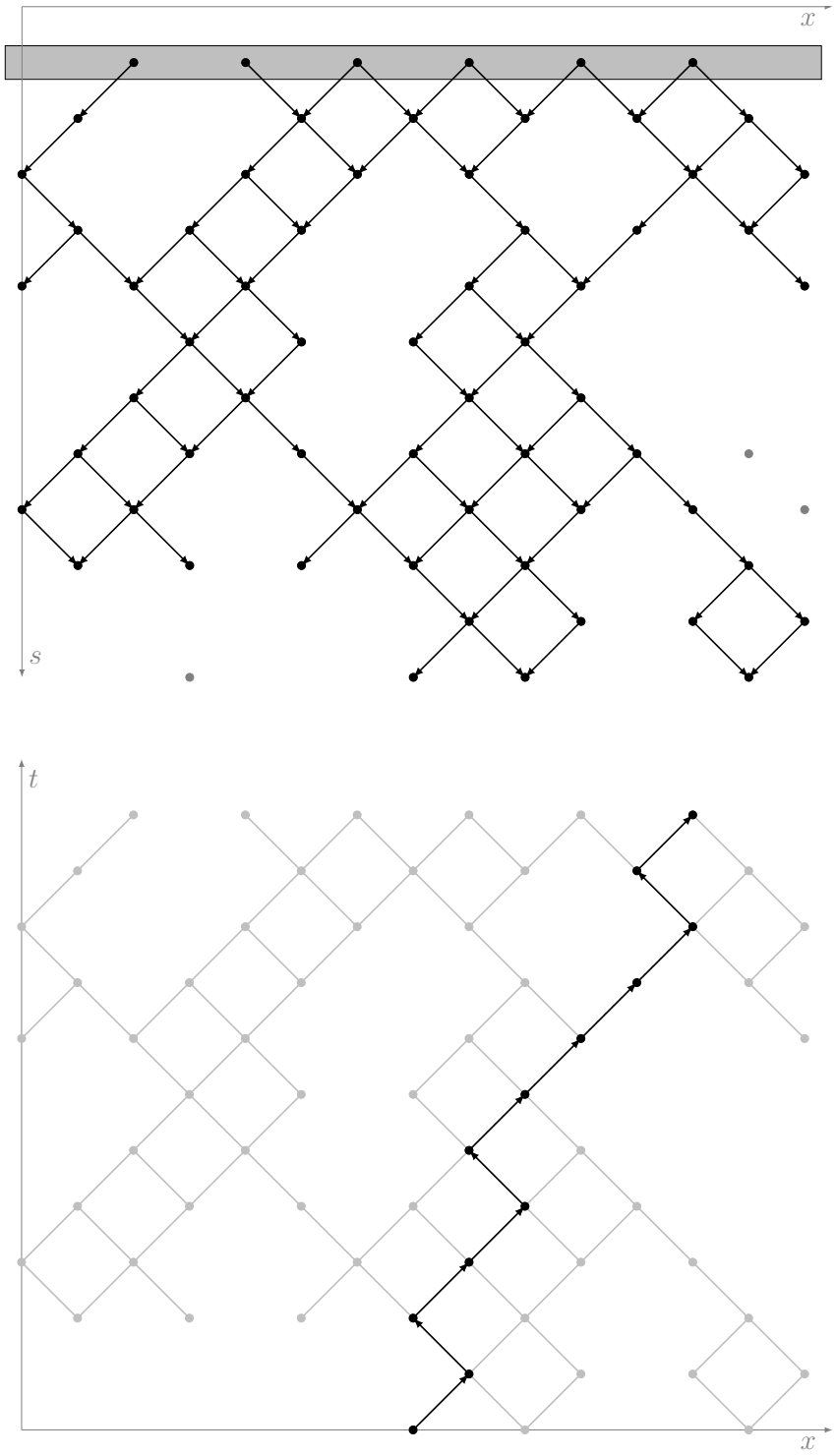

Figure 5: The construction of the backbone is shown in the top panel. Open sites in V are denoted by dots and individuals are placed initially at all open sites in the grey box. The locally constructed backbone is shown in black. The bottom panel shows an ancestral line in the backbone.

2.4.3 On weighted, oriented percolation clusters

Random walks in non-elliptic environments are always random walks on the sub-graph that is the open cluster of the origin, since every conductance that is equal to zero is equivalent to a closed edge in a percolation. Starting from Bernoulli percolation as random environment, we observe that changing all non-zero conductances to i.i.d. uniformly elliptic ones does not lead to different behaviour of the random walk. Thus, for most of the results on random walks on percolation clusters this fact was already mentioned by the authors as in Birkner et al. (2013) or the result was directly proven for bounded conductances, as in De Masi et al. (1989).

We generalize the pure percolation model introduced in the previous section by adding a random field $(K_x)_{x\in V}$. The percolation process denoted by $\xi^K = (\xi_n^K)_{n\in\mathbb{Z}}$ becomes

$$\xi_n^K(x) := \begin{cases} K(x,n) & \text{if } (x,n) \to \infty \\ 0 & \text{otherwise.} \end{cases}$$

We modify the transition kernel to be

$$P_\xi^0(X_{n+1} = y | X_n = x) = \frac{K(y,n+1)\mathbb{1}_{\{(y,n+1)\in U^+(x,n)\cap\mathcal{C}\}}}{\sum_{z\sim x} K(z,n+1)\mathbb{1}_{\{(z,n+1)\in U^+(x,n)\cap\mathcal{C}\}}}, \tag{2.16}$$

for every environment ξ^K such that $0 \in \mathcal{C}$. The random field K acts as weights for the random walk on the backbone. The orientation of the percolation cluster implies that the environment seen from the particle changes dynamically in time, since the particle can never return to a previously visited site. The percolation process with additional weights ξ^K has exactly the same non-zero sites as ξ^P. Thus, the backbone $\mathcal{C}$ is the same. This independence between the percolation $(\omega_x)_{x\in V}$ and the weights $(K_x)_{x\in V}$ makes it possible to transfer methods from Birkner et al. (2013) to this model. The weights allow us to have a slightly more realistic population model. Each site can still be habitable or not habitable depending on ω. In addition, if a site is habitable, it can now support K individuals instead of only one. We choose one of the possible parents to put its offspring at a new site and populate it to its capacity K. In order to construct the ancestral line, we simply pick one of those potential parents uniformly at random, which makes the ancestral line the same as a weighted random walk on $\mathcal{C}$ with weights K.

Before stating the results, we first have to be more specific on the random field K. We introduce some notion of mixing for random fields. For a brief overview on mixing conditions we refer the reader to the survey paper of Bradley (2005).

Definition 2.23 (Mixing conditions and mixing coefficients)**.** Let K be a random field on V. Denote by

$$\sigma(K) := \sigma\{K(v) : v \in V\}$$

the σ-algebra of the weights and by

$$\operatorname{supp}(A) := \bigcap \{U \subset V : A \in \sigma(K(v) : v \in U)\}$$

the support of an event $A \in \sigma(K)$. Furthermore, we say that a set C is a cone with apex in $(x,l) \in V$, if

$$C = \{(y,k) \in V : k \geq l \text{ and } ||x-y||_\infty \leq |k-l|\}.$$

This defines a cone with aperture $\pi/2$.

(i) We say that K is *α-mixing (or strongly mixing) in space* w.r.t. the law $\mathbb{P}$ if the mixing coefficients $(\alpha_n)_{n\in\mathbb{N}}$ satisfy $\alpha_n \xrightarrow{n\to\infty} 0$, where

$$\begin{aligned}\alpha_n := \sup\{&|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| : \\ &A, B \in \sigma(K), \operatorname{dist}^{\mathrm{s}}(\operatorname{supp}(A), \operatorname{supp}(B)) > n\}\end{aligned} \tag{2.17}$$

and we take the distance in the first d coordinates (space coordinates), i.e.

$$\operatorname{dist}^{\mathrm{s}}(U, W) = \inf\{||x-y||_\infty : (x,n)\in U, (y,m)\in W\}. \tag{2.18}$$

(ii) We say that K is *ϕ-mixing (or uniformly mixing) in time* w.r.t. the law $\mathbb{P}$ if the mixing coefficients $(\phi_n)_{n\in\mathbb{N}}$ satisfy $\phi_n \xrightarrow{n\to\infty} 0$, where

$$\begin{aligned}\phi_n := \sup\{&|\mathbb{P}(B|A) - \mathbb{P}(B)| : \\ &A, B \in \sigma(K), \mathbb{P}(A) > 0, \operatorname{supp}(B) \subseteq C, C \text{ is a cone and} \\ &\operatorname{dist}^{\mathrm{t}}(\operatorname{supp}(A), C) > n\}\end{aligned} \tag{2.19}$$

and we take the distance in the last coordinate of V (time coordinate),

$$\operatorname{dist}^{\mathrm{t}}(U, W) = \inf\{|n-m| : (x,n)\in U, (y,m)\in W\}. \tag{2.20}$$

This definition of ϕ-mixing is in fact a cone mixing condition since the usual definition of ϕ-mixing for random sequences does not generalize well to random fields. While the notion of α-mixing makes sense on its own for random fields, the usual definition of ϕ-mixing applied to random fields is a form of finite dependence. Bradley (1989) shows that there is a constant $r > 0$ such that $\phi_n \in \{0, 1\}$ for all $n \geq r$. However, since the neighbourhood U^+ is finite, we use a ϕ-mixing condition on cone-shaped subsets. With this definition of mixing, we obtain the following scaling limits similar to oriented percolation without weights. We choose K stationary, mixing and independent of ω for all our results. Note that for the results, ω denotes only the percolation. The full environment including the weights K is represented by the environment process ξ. The following results were published in Miller (2016) and a proof can be found in Section 4.

Lemma 2.24 (LLN for polynomially time-mixing weights)**.** *Let $d \geq 1$ and $p \in (p_c, 1]$. If K is independent of ω, strictly positive, stationary and ϕ-mixing in the time coordinate with mixing coefficients $\phi_n \in \mathcal{O}(n^{-(1+\delta)})$ for some $\delta > 0$, then a LLN holds, i.e. there is a constant $\vec{\mu} \in \mathbb{R}^d$ such that $||\vec{\mu}||_\infty < 1$ and*

$$P_\xi^0\left(\frac{X_n}{n} \xrightarrow{n\to\infty} \vec{\mu}\right) = 1 \quad \text{for } \tilde{\mathbb{P}}\text{- almost every } \xi^K. \tag{2.21}$$

Theorem 2.25 (Annealed CLT for polynomially time-mixing weights)**.** *Let $d \geq 1$ and $p \in (p_c, 1)$. If K is independent of ω, strictly positive, stationary and ϕ-mixing in the time coordinate with mixing coefficients $\phi_n \in \mathcal{O}(n^{-(2+\delta)})$ for some $\delta > 0$, then an annealed CLT holds, i.e. for all continuous and bounded functions $f \in C_b(\mathbb{R}^d)$*

$$\tilde{\mathbb{E}}\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] \xrightarrow{n\to\infty} \Phi(f), \tag{2.22}$$

where $\vec{\mu}$ is the same drift vector as in Lemma 2.24, $\Phi(f) := \int f(x)\Phi(\mathrm{d}x)$ and Φ is a non-trivial centred d-dimensional Gaussian law with full rank covariance matrix Σ.

While the LLN, Lemma 2.24, holds for $p = 1$, we can prove the central limit theorems under the given mixing conditions only for $p < 1$. For the aCLT on the full lattice, $p = 1$, the main difficulty is to show non-degeneracy of the limit, since we have no assumptions on moments of the random field K.

Theorem 2.26 (Quenched CLT for exponentially space-time-mixing weights)**.** *Let $d \geq 2$ and $p \in (p_c, 1)$. If K is independent of ω, strictly positive, stationary, ϕ-mixing in the time coordinate with mixing coefficients $\phi_n \in \mathcal{O}(e^{-c_1 n})$ and α-mixing in space with mixing coefficients $\alpha_n \in \mathcal{O}(e^{-c_2 n})$, $0 < c_1, c_2 < \infty$, then a quenched CLT holds with the same limit as in Theorem 2.25, i.e. for all continuous and bounded functions $f \in C_b(\mathbb{R}^d)$*

$$E_\xi^0 \left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}} \right) \right] \xrightarrow{n\to\infty} \Phi(f) \quad \text{for } \tilde{\mathbb{P}}\text{-almost every } \xi^K, \tag{2.23}$$

where $\vec{\mu}$ is the same drift vector as in Lemma 2.24 and Φ is the same law as in Theorem 2.25.

3 Interacting particle systems with exclusion

The standard reference for interacting particle systems is Liggett (1985). He describes them in his introduction as follows:

> *"A typical interacting particle system consists of finitely or infinitely many particles which, in the absence of the interaction, would evolve according to independent finite or countable state Markov chains. Superimposed on this underlying motion is some type of interaction. As a result of the interaction, the evolution of an individual particle is no longer Markovian."*

Typical examples of interacting particle systems are the Ising model, the voter model, the contact process and the exclusion process. The latter will be described in the Section 3.3. For that we need some general theory for Markov processes.

3.1 Markov processes

Interacting particle systems are usually described as Markov processes. The state space of a Markov process is a compact metric space endowed with the σ-algebra of Borel sets. The canonical path space Ω is the set of all cad-lag functions $\omega : [0, \infty) \to X$. Let $\mathcal{F}$ be the smallest σ-algebra on Ω such that the mapping $\omega \mapsto \omega(t)$ is measurable for all $t \geq 0$ and $\omega \in \Omega$.

Definition 3.1 (Liggett (1985))**.** A Markov process on the state space X is a collection of probability measures $(\mathbb{P}^\zeta)_{\zeta \in X}$ on Ω together with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ on Ω to which the random variables $\eta(t, \omega) = \omega(t)$ are adapted. Furthermore, for every $\zeta \in X$ it satisfies

(i) $\mathbb{P}^\zeta(\eta(0) = \zeta) = 1$,

(ii) $\zeta \mapsto \mathbb{P}^\zeta(A)$ is measurable for every $A \in \mathcal{F}$ and

(iii) for every $\zeta \in X$ and $A \in \mathcal{F}$

$$\mathbb{P}^\zeta(\eta_{s+\cdot} \in A | \mathcal{F}_s) = \mathbb{P}^{\eta_s}(A) \qquad \mathbb{P}^\zeta\text{-almost surely.}$$

Let $\mathcal{P}$ be the set of probability measures on X with the topology of weak convergence. If we start a Markov process with law $(\mathbb{P}^\zeta)_{\zeta \in X}$ from some initial distribution $\mu \in \mathcal{P}$ we may define the measure of this process by

$$\mathbb{P}^\mu = \int_X \mathbb{P}^\zeta \, \mu(d\zeta),$$

which turns the Markov process into a stochastic process. Let $C(X)$ be the collection of all continuous functions on the state space X, which is a Banach space with norm $||f|| = \sup |f(\eta)|$. The Markov semi-group of a Markov process $(\mathbb{P}^\zeta)_{\zeta \in X}$ is a collection of linear operators $(S(t))_{t \geq 0}$ such that

$$\mathbb{E}^\zeta f(\eta_t) = S(t) f(\zeta)$$

for all $f \in C(X)$, $\zeta \in X$ and $t \geq 0$. We prefer to work with Feller processes instead of Markov processes, since they have two convenient representations.

Definition 3.2 (Liggett (1985))**.** A Markov process is a Feller process, if for every $f \in C(X)$ and for all $t \geq 0$ we have $S(t)f \in C(X)$.

There is a one-to-one correspondence between Feller processes, their Markov semigroup $(S(t))_{t\geq 0}$ and their Markov generators L, which is explained in detail in Liggett (1985). We need a few definitions, before we can state the ergodic theorem for Markov processes and discuss interacting particle systems.

Definition 3.3 (Liggett (1985))**.** A measure $\mu \in \mathcal{P}$ is said to be invariant for the process, if $\mu S(t) = \mu$ for all $t \geq 0$ and we write

$$\mathcal{I} = \{\mu \in \mathcal{P} : \mu S(t) = \mu \text{ for all } t \geq 0\}$$

for the set of all invariant measures. Let $\mathcal{I}_e$ be the set of extreme points of $\mathcal{I}$.

We are interested in the set of all invariant measures for the process, because they help us to identify the limit of $\mu S(t)$ as $t \to \infty$.

Theorem 3.4 (Liggett (1985))**.**

(i) *$\mu \in \mathcal{P}$ is invariant for the Feller process if and only if*

$$\int S(t)f \, d\mu = \int f \, d\mu$$

for all $f \in C(X)$ and $t \geq 0$.

(ii) *$\mathcal{I}$ is not empty.*

(iii) *$\mathcal{I}$ is the closed convex hull of $\mathcal{I}_e$.*

(iv) *If $\nu = \lim \mu S(t)$ as $t \to \infty$ exists for some $\mu \in \mathcal{P}$, then $\nu \in \mathcal{I}$.*

Definition 3.5 (Liggett (1985))**.** A Markov process $(\eta_t)_{t\geq 0}$ is called

(i) *stationary*, if the joint distributions of $(\eta_{s_1+t}, \ldots, \eta_{s_N+t})$ are independent of t for all $N \in \mathbb{N}$ and $(s_1, \ldots, s_N)$,

(ii) *ergodic*, if $\mathcal{I} = \{\nu\}$ is a singleton and $\lim \mu S(t) = \nu$ as $t \to \infty$ for all $\mu \in \mathcal{P}$.

If we have a Markov process $(\mathbb{P}^\zeta)_{\zeta \in X}$ and take a measure $\mu \in \mathcal{I}$ from the set of invariant measures for this process, we can construct a new stationary Markov process $\mathbb{P}^\mu$ by taking the invariant measure μ as initial distribution. For such a stationary Markov process we have a connection between μ being extremal and ergodicity of the process.

Theorem 3.6 (Liggett (1999))**.** *Let $(\eta_t)_{t\geq 0}$ be a stationary Markov process such that the distribution of η_t is equal to $\mu \in \mathcal{I}$ for all $t \geq 0$. Then, the process is ergodic in the sense of Definition 3.5 (iii), if and only if $\mu \in \mathcal{I}_e$.*

Finally, we conclude this section with the ergodic theorem for Markov processes as it is stated in Liggett (1999).

Theorem 3.7 (Birkhoff's Ergodic Theorem. Birkhoff (1931))**.** *Let $(\eta_t)_{t\geq 0}$ be a Markov process.*

(i) *If $(\eta_t)_{t\geq 0}$ is stationary, then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\eta_t) \, ds$$

exists for any bounded, measurable function f, but may not be constant.

(ii) *If $(\eta_t)_{t\geq 0}$ is stationary and ergodic, then*

$$\frac{1}{t} \int_0^t f(\eta_t) \, ds \xrightarrow{t\to\infty} \mathbb{E}\left[f(\eta_0)\right] \quad \text{almost surely}$$

for any bounded, measurable function f.

3.2 Interacting particle systems

Using the notation we have introduced for Feller processes, we can now proceed to interacting particle systems. For the general set-up as described in Liggett (1985), denote a countable set of sites by S and by W a compact metric space. The state space of the interacting particle system is $X = W^S$. We describe the dynamics of the system using transition measures for any finite subset $T \subset S$. Let the transition rates $q_T(\eta, \mathrm{d}\zeta)$ be a finite positive measure for any $\eta \in X$ such that $\eta \mapsto q_T(\eta, \mathrm{d}\zeta)$ is a continuous mapping with the topology of weak convergence. Define

$$\eta^\zeta(x) = \begin{cases} \zeta(x) & \text{if } x \in T \\ \eta(x) & \text{if } x \notin T \end{cases}$$

for any $\eta \in X$ and $\zeta \in W^T$. These transition rates provide us with a pre-generator on the Lipschitz functions on X.

Lemma 3.8 (Liggett (1985))**.** *If*

$$\sup_{x \in S} \sum_{T \ni x} \sup \left\{ q_T(\eta, W^T) : \eta \in X \right\} < \infty,$$

then for any Lipschitz function f on X the operator L defined by

$$Lf(\eta) = \sum_T \int_{W^T} q_T(\eta, \mathrm{d}\zeta) \left[f(\eta^\zeta) - f(\eta) \right]$$

is a Markov pre-generator.

We obtain a Markov generator with associated semi-group $(S(t))_{t \geq 0}$ by taking the closure of the pre-generator, if the transition rates depend not too much on a single coordinate in the configuration. A measure of this dependence is defined by

$$q_T(y) = \sup \left\{ \| q_T(\eta_1, \mathrm{d}\zeta) - q_T(\eta_2, \mathrm{d}\zeta) \|_{TV, W^T} : \eta_1(z) = \eta_2(z) \ \forall z \neq y \right\}, \tag{3.1}$$

where $|| \cdot ||_{TV, W^T}$ is the total variation norm on the subspace W^T.

Theorem 3.9 (Liggett (1985))**.** *Assume that*

$$\sup_{x \in S} \sum_{T \ni x} \sum_{y \neq x} q_T(y) < \infty.$$

Then the closure of L is a Markov generator of a Markov semi-group $S(t)$.

3.3 Exclusion process

The exclusion process is an infinite interacting particle system, where each site can be occupied by at most one particle. A particle at site x tries to jump to site y, whenever its exponential clock with rate $q(x, y)$ rings. It performs the jump, if the target site y is empty. If the target site is already occupied, it does not move and waits for its exponential clock to ring again for another try. Thus, the exclusion process is a continuous time Markov process $(\eta_t)_{t \geq 0}$ on the state space $X = \{0, 1\}^S$, where we consider only the lattice and take $S = \mathbb{Z}^d$. We say the site x is occupied at time t if $\eta_t(x) = 1$ and it is empty if $\eta_t(x) = 0$. Let $(p(x, y))_{x, y \in S}$ be the transition probabilities of the single particle associated with the transition rates $(q(x, y))_{x, y \in S}$, namely for all $x, y \in S$

$$p(x, y) = \frac{q(x, y)}{\sum_{z \in S} q(x, z)}.$$

Denote by $\eta_{x,y}$ the same configuration as η but with coordinates x and y interchanged. Now, we can define the pre-generator L that acts on any Lipschitz function $f : \{0,1\}^S \to \mathbb{R}$ by

$$Lf(\eta) = \sum_{x,y \in S} \eta(x)(1-\eta(y))p(x,y)\left[f(\eta_{x,y}) - f(\eta)\right].$$

The generator says that the configuration changes from η to $\eta_{x,y}$, if the site x is occupied, the site y is not occupied and we choose this jump with probability $p(x,y)$. The closure of L is the generator of the semi-group $(S(t))_{t\geq 0}$ of a Markov process on $\{0,1\}^S$, if

$$\sup_{y \in S} \sum_{x \in S} p(x,y) < \infty. \tag{3.2}$$

We may check that both conditions in Lemma 3.8 and Theorem 3.9 are satisfied, if Equation (3.2) holds. We can explicitly calculate the measure of dependence of transition rates q_T for finite subsets $T \subset \mathbb{Z}^d$ as defined in Equation (3.1). Firstly, $q_T = 0$, whenever $|T| \neq 2$, since there are exactly two particles involved in every transition. Thus, take $T = \{x,y\}$ for some $x, y \in S$. If $\eta \in X$ is such that $\eta(x) = \eta(y)$, then $q_{\{x,y\}}(\eta, \mathrm{d}\zeta) = 0$, since no transition is possible. If, however, $\eta(x) = 1$ and $\eta(y) = 0$, then the particle at x can jump to y and the measure $q_T(\eta, \mathrm{d}\zeta)$ puts mass $p(x,y)$ on $\eta_{x,y}$ and mass $1 - p(x,y)$ on η. Since $q_T = 0$ whenever $|T| \neq 2$, and $q_T(y) = \max\{p(x,y), p(y,x)\}$ for any $T = \{x,y\}$, both assumptions follow from Equation (3.2).

If the transition probabilities of the single particle $(p(x,y))_{x,y\in S}$ are either doubly stochastic or if there is a reversible measure $(\alpha(x))_{x\in S}$ associated to them, then the product measures with marginals

$$\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x) \tag{3.3}$$

are invariant for the dynamics of the Markov process, which is the following result.

Theorem 3.10 (Liggett (1985))**.**

(i) If the transition probabilities $(p(x,y))_{x,y\in S}$ are doubly stochastic, then $\nu_\alpha \in \mathcal{I}$ for every constant $\alpha \in [0,1]$.

(ii) If the motion of a single particle is reversible under $(p(x,y))_{x,y\in S}$, i.e. there exists a function $(\pi(x))_{x\in S}$ such that

$$\pi(x)p(x,y) = \pi(y)p(y,x) \quad \text{for all } x, y \in S, \tag{3.4}$$

then $\nu_\alpha \in \mathcal{I}$, where

$$\alpha(x) = \frac{\pi(x)}{1+\pi(x)}.$$

Note here that any reversible measure may be multiplied by a constant and is still reversible. Thus, the measures ν_α form a one-parameter family. Let the reversible measure be scaled such that $\pi(0) = 1$. Then, every constant $c \in [0,\infty]$ corresponds to a reversible measure $c\pi$ and function α such that

$$\alpha(x) = \frac{c\pi(x)}{1+c\pi(x)}$$

for every $x \in S$. If $\alpha \equiv \rho \in [0,1]$ is constant, then ρ corresponds to the density of the particles on S. In fact, α is constant if and only if $(p(x,y))_{x,y\in S}$ is symmetric, since

$$\pi(x) = \pi(y)\frac{p(y,x)}{p(x,y)} = \pi(y) \qquad \Longleftrightarrow \qquad \frac{p(y,x)}{p(x,y)} = 1.$$

If α is not constant, then $\alpha(x)$ is the local density and we write $\nu_{\alpha\equiv\rho}$, where ν_α is defined in Equation (3.3). Thus, we can parametrize the family of measures by $\alpha(0) \in [0,1]$ and the whole family is $\{\nu_\alpha : 0 \leq \alpha(0) \leq 1\}$. However, not every α corresponds to a global particle density, since the limit

$$\lim_{x\to\infty} \frac{1}{x}\sum_{y=1}^{x} \alpha(y)$$

may not exist. For example, consider the case when the transition probabilities $(p(x,y))_{x,y\in\mathbb{Z}}$ are nearest-neighbour and i.i.d. such that

$$\mathbb{E}\left[\log \frac{p(x,x-1)}{p(x,x+1)}\right] = 0,$$

which corresponds to an exclusion process in a Sinai environment. In this case $\log \pi(x)$ is a symmetric simple random walk, which performs increasingly large excursions as $x \to \infty$ and thus there are regions of all sizes which have very high or very low particle density.

In order to find all invariant measures, we try to describe the extremal invariant measures, since the set of all invariant measures $\mathcal{I}$ is the convex hull of the set of extremal invariant measures $\mathcal{I}_e$, see Definition 3.3. Once we know that a measure is extremal for a class of measures, we get ergodicity with respect to time shifts by applying Theorem 3.6 for the process started from any distribution $\mu \in \mathcal{I}_e$. For the remainder of this section, we first describe results for spatially homogeneous systems, which are well understood. Afterwards, we summarize what is known for spatially inhomogeneous systems. We then consider the exclusion process on the full lattice $\mathbb{Z}^d$ although there are some results known for $S = \mathbb{N}$, see for example Chapter III.3 in Liggett (1999). For $S = \mathbb{N}$ very little is known about the tagged particle process, as the system is not translation invariant.

3.3.1 Homogeneous environments

In this section we consider exclusion processes on $\mathbb{Z}^d$ with translation invariant transition probabilities $(p(x,y))_{x,y\in S}$, i.e. $p(x,y) = p(0,y-x)$ for all $x,y \in S$. The first goal is to determine the invariant measures for translation invariant systems. Write $\mathcal{T}$ for the set of all translation invariant measures for the exclusion process,

$$\mathcal{T} = \{\mu \in \mathcal{P} : \theta_x\mu = \mu \text{ for all } x \in S\},$$

where $(\theta_x)_{x\in S}$ is the usual shift operator on S such that $(\theta_x\eta)(y) = \eta(x+y)$ for any $\eta \in X$ and $x,y \in S$. We do not know the complete set of invariant extremal measures, but only those which are also translation invariant. Define for any $t \geq 0$ and $x,y \in S$

$$p_t(x,y) = e^{-t}\sum_{n=0}^{\infty} \frac{t^n}{n!} p^n(x,y),$$

where $(p^n(x,y))_{x,y\in S}$ are the n-step transition probabilities of the Markov chain with transition law $(p(x,y))_{x,y\in S}$.

Theorem 3.11 (Liggett (1976); Andjel (1981)). *Take $S = \mathbb{Z}^d$ and assume that for every $x, y \in S$ there is some $t \geq 0$ such that*

$$p_t(x, y) + p_t(y, x) > 0. \tag{3.5}$$

(i) If the transition probabilities $(p(x, y))_{x,y\in S}$ are translation invariant, then the product measures ν_α are extremal in the class of all stationary and translation invariant distributions,

$$(\mathcal{I} \cap \mathcal{T})_e = \{\nu_\alpha : 0 \leq \alpha(0) \leq 1\}.$$

(ii) Let the starting distribution μ be translation invariant with density $\rho = \mu\{\eta : \eta(x) = 1\}$. Then the limit

$$\bar{\mu} = \lim_{t\to\infty} \mu S(t)$$

exists and $\bar{\mu} = \nu_{\alpha\equiv\rho}$.

In the special case of the nearest-neighbour one-dimensional homogeneous system, $p(x, x+1) = p$, $p(x, x-1) = q$ and $p+q = 1$ for all $x \in \mathbb{Z}$, we know the set of all extremal invariant measures explicitly. This includes the totally asymmetric exclusion process, which we get be setting $p = 1$. In the previous theorem, the condition of irreducibility has been replaced by Condition (3.5), which includes the totally asymmetric exclusion process. However, for the asymmetric exclusion process, $p \in (0, 1)$ and $p \neq 1/2$, the extremal invariant measures are known anyway.

Define another one-parameter family of measures (ν_n) that we obtain by conditioning ν_α defined as in Equation (3.3) on the events

$$A_n = \begin{cases} \{\eta : \sum_x \eta(x) = n\} & \text{if } \sum_x \alpha(x) < \infty \\ \{\eta : \sum_x (1 - \eta(x)) = n\} & \text{if } \sum_x (1 - \alpha(x)) < \infty \\ \{\eta : \sum_{x\in T} \eta(x) - \sum_{x\in T^c} (1 - \eta(x)) = n\} & \text{otherwise} \end{cases},$$

where $T \subset S$ is chosen such that $\sum_{x\in T} \alpha(x) < \infty$ and $\sum_{x\in T^c}(1 - \alpha(x)) < \infty$. Define the measures by

$$\nu_n(\cdot) = \begin{cases} \nu_\alpha(\cdot|A_n) & \text{if } n \in \mathbb{Z} \\ \text{the point mass on } \eta \equiv 1 & \text{if } n = \infty \\ \text{the point mass on } \eta \equiv 0 & \text{if } n = -\infty \end{cases}. \tag{3.6}$$

These measures are independent of the choice of T and the choice of $\alpha(0) \in [0, 1]$.

Theorem 3.12 (Liggett (1976)). *Let $(p(x, y))_{x,y\in\mathbb{Z}}$ be homogeneous and nearest neighbour with $p \neq 1/2$. Then*

$$\mathcal{I}_e = \{\nu_{\alpha\equiv\rho} : 0 \leq \rho \leq 1\} \cup \{\nu_n : -\infty < n < \infty\}.$$

A detailed study of the asymmetric exclusion process can be found in Chapter III.2 in Liggett (1999).

3.3.2 Inhomogeneous environments

We can state the set of extremal invariant measures explicitly, when the translation probabilities have asymptotically zero mean and there is a reversible measure as defined in Equation (3.4). The following result does not require translation invariance, but we want the transition probabilities to be irreducible, such that all states in the state space are in the same communicating class. In fact, for the next result it is enough if the system is asymptotically translation invariant.

Theorem 3.13 (Jung (2003)). *Let $S = \mathbb{Z}$ and assume $(p(x,y))_{x,y\in S}$ is irreducible and reversible with ν_α and π defined as in Theorem 3.10. Assume further that there exist two transition kernels $(q_i(x))_{x\in\mathbb{Z}}$, $i \in \{1,2\}$ such that*

(i) $\sum_{x\in\mathbb{Z}} x q_i(x) = 0$ *for* $i \in \{1,2\}$,

(ii) $\sum_{x\in\mathbb{Z}} |x| q_i(x) < \infty$ *for* $i \in \{1,2\}$,

(iii) $\lim_{K\to\infty} \sum_{x\geq 0} \sum_{|y|\geq|x-K|} |p(x,x+y) - q_1(y)| = 0$ *and*

(iv) $\lim_{K\to\infty} \sum_{x\leq 0} \sum_{|y|\geq|x+K|} |p(x,x+y) + q_2(y)| = 0$.

Then, the extremal invariant measures are explicitly known and the three cases are as follows.

(i) If $\sum_i \alpha_i(1-\alpha_i) = \infty$, *then*

$$\mathcal{I}_e = \{\nu_\alpha : 0 \leq \alpha(0) \leq 1\}.$$

(ii) If $\sum_i \alpha_i(1-\alpha_i) < \infty$ *and either* $\sum_i \pi(x) < \infty$ *or* $\sum_i 1/\pi(i) < \infty$, *then*

$$\mathcal{I}_e = \{\nu_n : 0 \leq n \leq \infty\}.$$

(iii) If $\sum_i \alpha_i(1-\alpha_i) < \infty$, *but* $\sum_i \pi(x) = \infty = \sum_i 1/\pi(i)$, *then*

$$\mathcal{I}_e = \{\nu_n : -\infty \leq n \leq \infty\}.$$

A simpler version of the following theorem has been first proven by Liggett (1976). Note that the condition of asymptotically zero mean is a necessary condition for the reversible measures (ν_n) to be the only invariant measures. The existence of non-reversible invariant measure has been proven in Chayes and Liggett (2007).

Unfortunately, very little is known if the transition probabilities are not translation invariant and the environment of the exclusion process is inhomogeneous except under the previous assumption of asymptotically zero mean. The invariant extremal measures are known only in the special case that the transition probabilities are positive recurrent. Then, the system converges in law to total occupancy of all sites. The measures (ν_n) are defined as in Equation (3.6).

Theorem 3.14 (Liggett (1974)). *Let $(p(x,y))_{x,y\in S}$ be positive recurrent, reversible and irreducible. Then*

(i) $\mathcal{I}_e = \{\nu_n : 0 \leq n \leq \infty\}$.

(ii) If $\mu\{\eta : \sum_x \eta(x) = \infty\} = 1$, *then*

$$\nu_\infty = \lim_{t\to\infty} \mu S(t),$$

where ν_∞ is the point mass on $\eta \equiv 1$.

We can see that for positive recurrent Markov chains the invariant measures are independent of the initial particle density whenever we start with infinitely many particles. In fact, part (ii) of this theorem says that the exclusion process converges weakly to the fully occupied state and we can imagine the particles to fill the valley of the potential over time. It is still an open problem to show almost sure convergence, i.e. that under the same assumptions $\mathbb{P}^\eta(\eta_t(x) = 1) \to 1$ as $t \to \infty$ for all $x \in S$ and $\eta \in X$.

3.3.3 The tagged particle

Much work has been dedicated into understanding the motion of a selected particle, which is traced while it moves as part of the exclusion process. Up to now, results are only known in the case where the transition probabilities are translation invariant as in Section 3.3.1. To start with, the motion of the so called *tagged particle* is not Markovian since it does not keep track of the positions of any other particles. We can make the system Markovian by observing the tagged particle z_t together with the full process η_t. The generator of this Markov process is

$$\begin{aligned} Lf(z,\eta) = &\sum_{x,y\in S\setminus\{z\}} \eta(x)(1-\eta(y))p(x,y)\left[f(z,\eta_{x,y}) - f(z,\eta)\right] \\ &+ \sum_{x\in S} p(x)(1-\eta(z+x))\left[f(z+x,\eta_{z,z+x}) - f(z,\eta)\right]. \end{aligned}$$

If $(p(x,y))_{x,y\in S}$ is translation invariant, then we can rewrite this generator in the reference frame of the tagged particle and get a Markovian description of the configuration seen from the tagged particle. Denote the configuration seen from the particle by $\xi_t = \theta_{z_t}\eta_t$, where θ is the usual shift operator such that $\xi_t(x) = \eta_t(x+z_t)$. Now, the new process ξ_t is itself a Markov process with an explicit generator. We get the following result about invariant measures, which tells us that the process seen from the tagged particle is stationary. It can be found for example as Propositions 4.3 and 4.8 in Liggett (1999), where ergodicity is due to Saada (1987).

Lemma 3.15 (Liggett (1999); Saada (1987))**.** *Let $(p(x,y))_{x,y\in S}$ be translation invariant, irreducible and have finite range.*

(i) The Bernoulli product measures conditioned to have the origin occupied $\nu'_\rho(\cdot) = \nu_{\alpha\equiv\rho}\{\cdot|\eta(0)=1$ and $0\le\rho\le 1\}$ are all invariant for the process $(\xi_t)_{t\ge 0}$.

(ii) The process $(\xi_t)_{t\ge 0}$ is stationary and ergodic.

Using the stationary distribution of the process seen from the tagged particle, we get a law of large numbers.

Lemma 3.16 (Spitzer (1970); Kipnis (1986); Saada (1987))**.** *Let $\sum_y ||y||p(0,y) < \infty$ and start the exclusion process from a stationary initial distribution ν_α as described in the previous Lemma 3.15. Then*

$$\lim_{t\to\infty}\frac{z_t}{t} = (1-\alpha)\sum_{y\in S} yp(0,y)$$

almost surely and in L_1.

The expectation $\mathbb{E}[z_t]$ was first computed by Spitzer (1970). He also showed the existence of the limit almost surely and in L_1. However, he could not show that the limit was constant almost surely. Later, Kipnis (1986) proved the result for the one-dimensional asymmetric nearest-neighbour case and Saada (1987) proved the remaining cases.

It is generally believed, that the tagged particle exhibits exceptional behaviour only, if the dynamics are nearest neighbour, $d=1$ and $p(0,1)=p(0,-1)=1/2$ but is diffusive in all other cases. This conjecture can be found for example in Sethuraman (2006), which contains also a recent summary of the topic.

Conjecture 3.17 (Sethuraman (2006))**.** *Let $(p(x,y))_{x,y\in S}$ be translation invariant and start the exclusion process from a stationary initial distribution ν_α with $0<\alpha<1$.*

(i) For any continuous bounded function $f \in C_b(\mathbb{R}^d)$

$$\mathbb{E}\left[f\left(\frac{z_t - \mu t}{\sqrt{t}}\right)\right] \to \mathbb{E}\left[f(Z)\right] \quad \text{as } t \to \infty,$$

where $\mu \in \mathbb{R}^d$ *is the deterministic drift and* $Z \sim \mathcal{N}(0, \Sigma)$ *is a centred Gaussian random variable with covariance matrix* Σ*.*

(ii) The covariance matrix Σ *is non-degenerate for all cases but the exceptional case, i.e.* $d = 1$ *and* $p(0,1) = p(0,-1) = 1/2$*.*

The covariance matrix is not explicit except for the one-dimensional nearest-neighbour case. Let $p = p(x, x+1)$ and $q = p(x, x-1)$ such that $p + q = 1$ and $p > 1/2$. Then $Z \sim \mathcal{N}(0, \sigma^2)$ and the limiting variance of the tagged particle is

$$\sigma^2 = \frac{1-\alpha}{p-q},$$

which was first calculated by De Masi and Ferrari (1985). This result includes the totally asymmetric case, $p = 1$, which was attributed to H. Kesten in Spitzer (1970).

The conjecture has been proven for a number of cases starting with the totally asymmetric simple exclusion process on $\mathbb{Z}$, where the particles can only jump to the right by Spitzer (1970). It was followed by the proof for all nearest-neighbour asymmetric cases in dimension $d = 1$ by Kipnis (1986). In the same year Kipnis and Varadhan (1986) obtained a proof for general symmetric translation probabilities excluding the exceptional case. The result for asymmetric translation probabilities with zero mean was obtained by Varadhan (1995) and in general for dimensions $d \geq 3$ by Sethuraman et al. (2000).

Theorem 3.18 (The central limit theorem except for the exceptional case)**.** *Let* $S = \mathbb{Z}^d$ *and* $(p(x,y))_{x,y\in S}$ *be translation invariant. The conjecture holds in each of the following cases except for the exceptional case* ($d = 1$ *and* $p(x, x+1) = p(x, x-1) = 1/2$)*.*

(i) If $d = 1$ *and translation probabilities are nearest-neighbour* $p(x, x+1) = 1 - p(x, x-1) \neq 1/2$*.*

(ii) If $d = 1, 2$ *and the translation probabilities are symmetric,* $p(x,y) = p(x,y)$ *for all* $x, y \in S$*.*

(iii) If $d = 1, 2$ *and the translation probabilities have zero mean,*

$$\sum_{y\in S} yp(0,y) = 0.$$

(iv) If $d \geq 3$*.*

All the remaining cases, where the translation probabilities are asymmetric with non-zero mean in dimensions one and two are open, but Sethuraman (2006) made some progress towards a proof.

The tagged particle in the exceptional case shows significant reduction in fluctuations and the central limit theorem is sub-diffusive.

Theorem 3.19 (The exceptional case, Arratia (1985))**.** *Let* $d = 1$ *and* $p(0,1) = p(0,-1) = 1/2$*. Start the exclusion process from a stationary initial distribution* ν'_ρ *with density* ρ *and the tagged particle at zero. Then, for any function* $f \in C_b(\mathbb{R})$

$$\mathbb{E}\left[f\left(\frac{z_t}{t^{1/4}}\right)\right] \to \mathbb{E}\left[f(Z)\right] \quad \text{as } t \to \infty,$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is a centred Gaussian random variable with variance

$$\sigma^2 = \sqrt{\frac{2}{\pi}} \left[\frac{1-\rho}{\rho}\right].$$

A central limit theorem is also known for the tagged particle of an exclusion process in a random environment with bond disorder. Jara and Landim (2008) show a central limit theorem for the tagged particle starting from non-equilibrium distributions. The generator for the exclusion process on $\mathbb{Z}$ is

$$\begin{aligned} Lf(\eta) &= \sum_{x\in\mathbb{Z}} \omega_x \left[f(\eta_{x,x+1}) - f(\eta)\right] \\ &= \sum_{x,y\in\mathbb{Z}} \eta(x)(1-\eta(y))q(x,y)\left[f(\eta_{x,y}) - f(\eta)\right]. \end{aligned} \tag{3.7}$$

where $(\omega_x)_{x\in\mathbb{Z}}$ is the random environment and transition rates are $q(x,y) = 0$ if $|x-y| \neq 1$ and

$$q(x, x+1) = \omega_x \quad \text{and} \quad q(x, x-1) = \omega_{x-1} \qquad \text{if } |x-y| = 1.$$

Note that this environment is not of the site disorder type described in Section 2.2. Here, the jump rates are not constant and equal to one and the potential of the environment is not a random walk, but independent for each $x \in S$ up to scaling by ω_0,

$$\begin{aligned} V(x) &= \sum_{i=1}^{x} \log\left(\frac{p(i,i-1)}{p(i,i+1)}\right) \\ &= \sum_{i=1}^{x} \log\left(\frac{\omega_{i-1}}{\omega_i}\right) \\ &= \log\left(\frac{\omega_0}{\omega_x}\right). \end{aligned}$$

The quenched non-equilibrium central limit theorem of Jara and Landim (2008) implies in particular the quenched equilibrium central limit theorem with the same scaling as in the exceptional case studied by Arratia (1985), Theorem 3.19.

Theorem 3.20 (Quenched central limit theorem in inhomogeneous environments, Jara and Landim (2008))**.** *Let $(\omega_x)_{x\in\mathbb{Z}}$ be i.i.d. and uniformly elliptic. Let $(z_t)_{t\geq 0}$ be the tagged particle of the exclusion process starting in zero with generator as in Equation (3.7). Let the initial distribution ν'_ρ be a Bernoulli distribution with density ρ and particle at the origin. Then, for any continuous bounded function $f \in C_b(\mathbb{R})$*

$$E_\omega\left[f\left(\frac{z_t}{t^{1/4}}\right)\right] \xrightarrow{t\to\infty} \mathbb{E}\left[f(Z)\right] \quad \textit{for } \mathbb{P}-\textit{almost every } \omega,$$

where $Z \sim \mathcal{N}(0, \sigma^2)$ is a centred Gaussian random variable with variance

$$\sigma^2 = \frac{2}{\sqrt{\pi \mathsf{E}[\omega_0^{-1}]}} \left[\frac{1-\rho}{\rho}\right]. \tag{3.8}$$

Note that this is consistent with the result of Arratia (1985), Theorem 3.19, where the exclusion process runs in a constant environment such that $\omega \equiv 1/2$, which implies $\mathsf{E}[\omega_0^{-1}] = 2$. We retrieve the equilibrium variance σ^2 as a corollary of Theorem 2.3

in Jara and Landim (2008) by calculating $E_\omega[W_t^2]$, where $W_t = z_{N^2 t}/\sqrt{N}$. Set $\gamma = \mathsf{E}[\omega_0^{-1}]$. Then, using their formula with constant density ρ we get

$$\begin{aligned} E_\omega[W_t^2] &= \frac{2}{\gamma}\frac{1-\rho}{\rho}\int_0^t \int_{-\infty}^{\infty} \left(\sqrt{\frac{\gamma}{2\pi(t-r)}}\exp\left(-\frac{\gamma v^2}{2(t-r)}\right)\right)^2 dv\, dr \\ &= \frac{1}{\sqrt{\pi\gamma}}\frac{1-\rho}{\rho}\int_0^t \frac{1}{\sqrt{t-r}}\, dr \\ &= \frac{2}{\sqrt{\pi\gamma}}\frac{1-\rho}{\rho}\sqrt{t}. \end{aligned}$$

From this, the variance in Equation (3.8) follows by identifying σ^2 in $E_\omega[W_t^2] = \sigma^2\sqrt{t}$.

There is no equivalent result for exclusion processes with site disorder and constant jump rate. The random walk in such environments was discussed in Section 2.2. For these environments the potential V performs a random walk, which has large excursions. In this model one would expect that fluctuations of a tagged particle are reduced due to the combined effect of the exclusion process and trapping in valleys of the potential. A first step towards an answer is made in Section 5. Before that we discuss results for finite interacting particles systems with exclusion dynamics to understand the effect of the potential better.

3.4 Spider random walks

Spider random walks were introduced in the papers by Gallesco et al. (2011b,a). Spiders are finite systems of nearest-neighbour random walks with exclusion dynamics, so that no two particles can be at the same site at the same time. Furthermore, the particles can have at most some distance L from one another. Usually, spiders have finite span L, but we will also consider the case $L = \infty$. Since we have a finite number of particles, we take discrete-time random walks and only one particle can move at the same time-step. All results hold for spiders in continuous time with constant speed random walks as well. While Gallesco et al. (2011a) consider spiders on general graphs, we will restrict our attention to $\mathbb{Z}$, as it was done in Gallesco et al. (2011b). A spider on $\mathbb{Z}$ with finite number of legs N and span $L \geq N$ or $L = \infty$ is defined by a finite set of admissible leg configurations,

$$C \subseteq \{(x_1, \ldots, x_N) : x_1 = 0,\ x_N \leq L,\ x_1 < \ldots < x_N \in \mathbb{Z}\} \subset \mathbb{Z}^N,$$

where the first leg is at the origin. All other admissible configurations are obtained by translation of the set C by the shift operator θ_x,

$$C_x = \theta_x C = \left\{(x, x_2 \ldots, x_N) \in \mathbb{Z}^N : (0, x_2 - x \ldots, x_N - x) \in C\right\}.$$

The set of all admissible leg configurations is

$$\mathcal{L} = \bigcup_{x\in\mathbb{Z}} C_x.$$

Spiders obtained through translation of a finite configuration are called transitive spiders. We allow the spider to move one leg at a time to an adjacent unoccupied vertex. Thus, a spider may move from configuration $\mathbf{x} \in \mathcal{L}$ to configuration $\mathbf{y} \in \mathcal{L}$ if $||\mathbf{x} - \mathbf{y}||_2 = 1$. In this case we write $\mathbf{x} \overset{\mathcal{L}}{\sim} \mathbf{y}$, otherwise $\mathbf{x} \overset{\mathcal{L}}{\nsim} \mathbf{y}$. We can represent all allowed transitions for the spider as edges in a graph. Let

$$E = \left\{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \overset{\mathcal{L}}{\sim} \mathbf{y}\right\}$$

be the set of all nearest-neighbour edges in $\mathcal{L}$. We call the graph $\mathcal{G} = (\mathcal{L}, E)$ the *spider graph*. By construction, this graph is invariant under translations by integer multiples of the vector $(1, 1, \ldots, 1)$. We define an environment for the spider as a collection of random variables $(\omega_x)_{x\in\mathbb{Z}}$, with $\rho_x = (1-\omega_x)/\omega_x$, which are elliptic and not constant,

$$\exists \kappa \in (0, 1/2) \text{ such that } \mathsf{P}(\kappa \leq \omega_x \leq 1-\kappa) = 1 \quad \text{and} \tag{A1}$$

$$\mathsf{Var}(\log \rho_x) > 0. \tag{A2}$$

for all $x \in \mathbb{Z}$. The spider walk in a random environment is a discrete time Markov chain $(\mathbf{S}(n))_{n\in\mathbb{N}}$, where $\mathbf{S}(n) = (S_1(n), \ldots, S_N(n))$ denotes the positions of the N legs at time n. A leg at site x can move to site $x+1$ with probability ω_x, if the new configuration is in $\mathcal{L}$. It can also move to site $x-1$ with probability $1-\omega_x$, if the new configuration is in $\mathcal{L}$. In particular, since no two legs can be at the same site at the same time in the set of configurations $\mathcal{L}$, a particle can only move if the target site is empty. Thus, a spider random walk is a N-particle random walk simultaneously in the same environment with exclusion dynamics. Furthermore, if the span L is finite, then a further restriction on the spider walk is that the first and the last leg cannot be further apart than distance L, $S_N(n) - S_1(n) \leq L$ for all $n \in \mathbb{N}$.

Let $z \in \mathcal{L}$ be the start configuration for the spider random walk. The transition probabilities given the environment are

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}) :=& P_\omega^z \left(\mathbf{S}(n+1) = \mathbf{y} \middle| \, \mathbf{S}(n) = \mathbf{x} \right) \\ =& \frac{1}{N} \cdot \begin{cases} \omega_{x_i} & \text{if } \mathbf{y} \overset{\mathcal{L}}{\sim} \mathbf{x} \text{ and } \exists i \ : \ y_i = x_i + 1, \\ 1 - \omega_{x_i} & \text{if } \mathbf{y} \overset{\mathcal{L}}{\sim} \mathbf{x} \text{ and } \exists i \ : \ y_i = x_i - 1, \\ p(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x}, \\ 0 & \text{if } \mathbf{y} \overset{\mathcal{L}}{\nsim} \mathbf{x} \text{ and } \mathbf{y} \neq \mathbf{x} \end{cases} \end{aligned} \tag{3.9}$$

for any $\mathbf{x}, \mathbf{y} \in \mathcal{L}$ and any $\omega \in \Omega$. Any leg stays at its position, if its chosen target site is not admissible. Therefore, the spider random walk stays in configuration $\mathbf{x}$ with probability

$$p(\mathbf{x}) = \sum_{1\leq i\leq N} \omega_x \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_i \notin \mathcal{L}\}} + (1-\omega_x) \mathbb{1}_{\{\mathbf{x}-\mathbf{e}_i \notin \mathcal{L}\}}, \tag{3.10}$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_N)$ is the canonical basis of $\mathbb{Z}^N$. The spider graph for a simple two-legged spider with corresponding transition rates is shown in Figure 6.

Since spiders are interacting particle systems with a finite number of particles, we can describe them as discrete time Markov chains. We could have defined the spider random walk as a continuous time Markov chain as well by interpreting the ω_x and $1-\omega_x$ as transition rates to the right and left respectively. However, since both models are equivalent up to a relabelling of time, we will work here with the discrete time Markov chain, which allows for slightly easier notation.

Not all choices of C yield a connected graph $\mathcal{G}$ for the spider random walk. We can obtain a connected spider graph, if we require it to be of bounded span as it was defined in Gallesco et al. (2011a).

Definition 3.21. A transitive spider on the graph $\mathcal{G}$ has bounded span, if

$$C = \{(x_1, \ldots, x_N) : x_1 = 0, \ x_N \leq L, \ x_1 < \ldots < x_N \in \mathbb{Z}\}.$$

The spider graph for a spider with two legs and with bounded span is shown in Figure 7, left panel. All spider graphs for spiders with bounded span are connected. As an example for a spider that is not of bounded span, consider a two-legged spider with $C = \{(x, x+1), (x, x+2), (x, x+4), (x, x+5)\}$, as shown in Figure 7, right panel. In

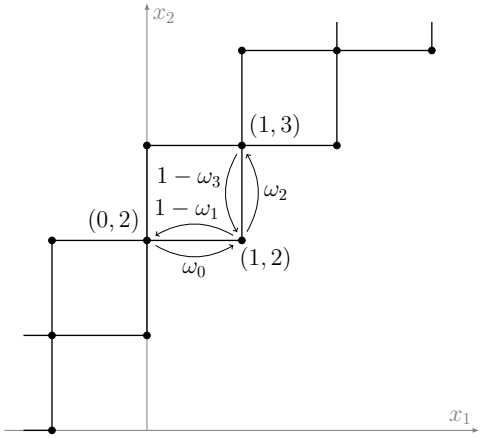

Figure 6: Configuration graph for a two-legged spider with admissible configurations $C = \{(x, x+1), (x, x+2)\}$, i.e. $N = 2$ and $L = 3$. Transition rates q from or to configuration $(1, 2)$ are marked on their respective arrows. Transition probabilities are calculated from the rates as $p(\cdot, \cdot) = q(\cdot, \cdot)/N$. The random walk also has a chance to stay at site $(1, 2)$. This happens with probability $(\omega_1 + (1 - \omega_2))/N$.

this case, the configuration graph has two connected components, each corresponding to the graph for a spider with $C = \{(x, x+1), (x, x+2)\}$. For a spider to be non-trivial, we only need to have that

$$\mathcal{G} \text{ has an infinite connected component } G \subset \mathcal{G} \text{ and initial configuration } z \in G. \tag{B1}$$

However, as the example shows, if the spider graph has more than one infinite connected component, then we may as well reduce the spider graph to one of its connected components. Thus, we can simplify our notation and require

$$\mathcal{G} \text{ is connected.} \tag{B2}$$

While for two legged spiders, $N = 2$, the spider graph is connected if and only if the spider has bounded span, this is not the case for more legs. Observe the example

$$C = \{(x, x+1, x+2), (x, x+1, x+3), (x, x+2, x+3), (x, x+2, x+4)\},$$

which is shown in Figure 8. The configurations $(x, x+1, x+4)$ and $(x, x+3, x+4)$ are missing and the spider is not of bounded span. However, the spider graph is connected nonetheless. While most arguments for spider random walks need only Condition (B2) to work, it might be more convenient to ask for a little more and require

$$\mathcal{G} \text{ has bounded span.} \tag{B3}$$

3.4.1 Transient spiders on $\mathbb{Z}$

Spiders with bounded span and $L < \infty$ were studied in Gallesco et al. (2011b) for environments $(\omega_x)_{x\in\mathbb{Z}}$ that are i.i.d. and transient to the right for the random walk of a single particle,

$$\mathsf{E}[\log \rho_0] < 0, \tag{A3}$$

where $\rho_x = (1 - \omega_x)/\omega_x$ as shown in Lemma 2.5. They work with continuous-time spiders, where each leg performs a constant-speed random walk. Under the above assumptions on the environment together with (A1) and (A2) there is a unique constant

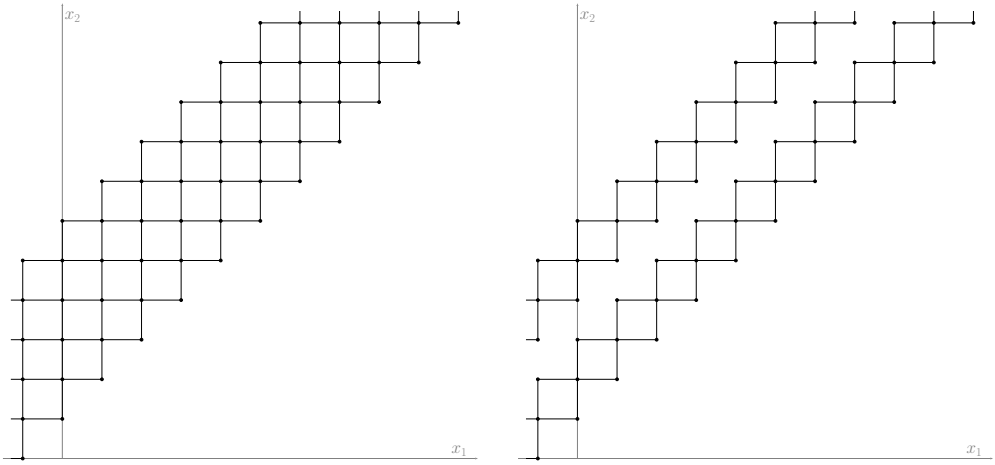

Figure 7: The left panel shows the configuration graph for a two-legged spider of bounded span with $L = 5$. Consequently, the spider graph has admissible configurations $C = \{(x, x+1), (x, x+2), (x, x+3), (x, x+4), (x, x+5)\}$. The right panel shows the configuration graph for a two-legged spider, which is not of bounded span and the graph is not connected. This spider graph has admissible configurations $C = \{(x, x+1), (x, x+2), (x, x+4), (x, x+5)\}$.

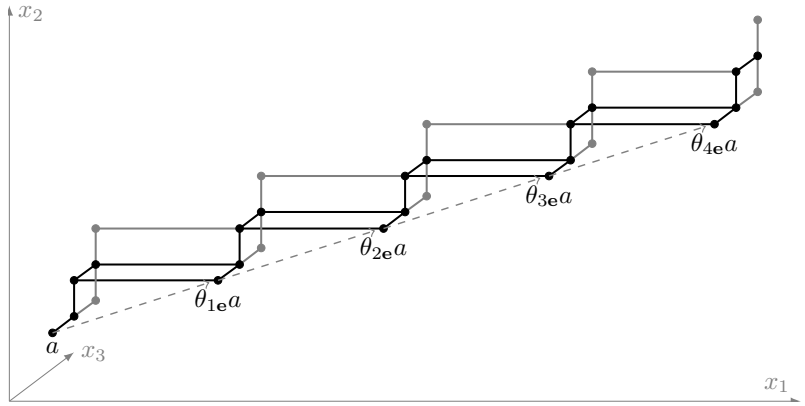

Figure 8: Spider graph for three-legged spider with configuration set $C = \{(x, x+1, x+2), (x, x+1, x+3), (x, x+2, x+3), (x, x+2, x+4)$ is drawn in black. The vertex $a = (1, 2, 3)$ and its translations by multiples of the vector $\mathbf{e} = (1, 1, 1)$ are labelled. The missing vertices and edges to turn the spider into a spider with bounded span with $L = 5$ are shown in grey.

$s > 0$ such that

$$s = \sup\{r > 0 : \mathbb{E}[\rho_0^r] < 1\}. \tag{3.11}$$

Consequently, we have $\mathbb{E}[\rho_0^s] = 1$. We have already seen this definition in the context of normal random walks in transient one-dimensional transient environments, Section 2.2.2. Then, not only a single particle but also the spider is transient to the right, see Proposition 2.1 in Gallesco et al. (2011b) and we define the speed of the spider v as the speed of the first leg

$$v = \lim_{t\to\infty} \frac{S_1(t)}{t}.$$

Define furthermore the first return time to the initial configuration $z \in \mathcal{L}$ or to any shifted-version $\theta_x z$ by

$$\tau_z^\theta = \inf\left\{t > 0 : S_1(t) > 0 \text{ and } S(t) = \theta_{S_1(t)} z\right\}.$$

There is a phase transition in the speed of the transient spider.

Theorem 3.22 (Gallesco et al. (2011b))**.** *Let Assumptions (A1) and (A3) hold for the environment and let Assumption (B2) hold for the spider graph $\mathcal{G}$ with bounded span $L < \infty$. Assume furthermore that*

$$\mathsf{P}(\omega_0 > 1/2) > 0 \quad \text{and} \quad \mathsf{P}(\omega_0 \leq 1/2) > 0.$$

The speed of a spider with N legs and constant $s > 0$ defined in Equation (3.11) is well defined.

(i) If $s/N < 1$, then $v = 0$.

(ii) If $s/N > 1$, then

$$v = \frac{\mathbb{E}\left[S_1(\tau_z^\theta)\right]}{\mathbb{E}[\tau_z^\theta]} > 0.$$

The speed of the spider in the critical case $s/N = 1$ is so far not known.

3.4.2 Recurrent spiders on $\mathbb{Z}$

Recurrent spiders on $\mathbb{Z}$ are spiders in Sinaĭ environments, which means we require them to be recurrent,

$$\mathsf{E}[\log \rho_0] = 0, \tag{A3'}$$

where $\rho_x = (1-\omega_x)/\omega_x$. In such environments we can show that for any finite number of particles the results for random walks in recurrent random environment from Section 2.2 hold. We work with discrete-time spiders. They are of bounded span and $L \geq N$ or $L = \infty$. Write $S_j(n)$ for the j-th leg of $\mathbf{S}(n)$, where $j \in \{1, \ldots, N\}$.

Theorem 3.23. *Fix $L \geq N$ or $L = \infty$. For any $\mathbf{z} \in \mathcal{L}$ and small constant $\eta > 0$ there exists a random process $(b(n))_{n\in\mathbb{N}} = (b(n;\omega))_{n\in\mathbb{N}}$ such that*

$$\mathbb{P}^z\left(\left|\frac{S_j(n) - b(n)}{\log^2 n}\right| > \eta\right) \to 0 \qquad \text{as } n \to \infty. \tag{3.12}$$

This theorem implies that all N particles stay close to the same process $b(n)$ and therefore all particles stay together as well.

Corollary 3.24. *If $L = \infty$, then for any $\mathbf{z} \in \mathcal{L}$ and $\eta > 0$*

$$\mathbb{P}^z \left(\left| \frac{S_N(n) - S_1(n)}{\log^2 n} \right| > \eta \right) \to 0 \qquad \text{as } n \to \infty. \tag{3.13}$$

Lemma 3.25. *The Markov chain $(\mathbf{S}(n))_{n\in\mathbb{N}}$ is recurrent for all $L \geq N$ and $L = \infty$. In particular*

$$\liminf_{n\to\infty} |S_N(n) - S_1(n)| = N - 1 \quad \text{and}$$
$$\limsup_{n\to\infty} |S_N(n) - S_1(n)| = L.$$

So far we have dealt with the law of single particles. However, the positions of all N particles of the process converge simultaneously to some limit process under the annealed law. We denote the mean vector of the process by $\mathbf{b}(n;\omega) = b(n;\omega)(1,1,\ldots,1)$ for any $n \in \mathbb{N}$. This vector is used in the following theorem to shift the spider or N-particle exclusion process (i.e. every single leg) by $b(n;\omega)$, so that we can observe its fluctuations around $b(n;\omega)$. The shifted N-particle process converges to a stationary limit similar to the result for one particle shown in Golosov (1984).

Theorem 3.26. *Let $L \geq N$ or $L = \infty$. The finite dimensional marginals of the random process $(\mathbf{S}(k+n) - \mathbf{b}(k;\omega))_{n\in\mathbb{N}}$ converge under the annealed law $\mathbb{P}$ to those of a stationary random process $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ as $k \to \infty$. The process $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ is positive recurrent.*

We call $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ the N-particle exclusion process in the infinite well. For the exclusion process with infinitely many particles in positive recurrent environments, it was shown that the process converges in law to full occupancy, which is the configuration where each site is occupied by a particle, see Theorem 3.14. However, so far the convergence was not shown to hold almost surely. In a configuration where each site is occupied, any particle motion is forbidden. Thus, naturally the tagged particle becomes localized at some point and its fluctuations around the point of localization vanish. Thus, if we study the exclusion process in recurrent environments, we have to expect particles to get trapped in the large excursions of the potential. It is a natural extension of these results to show limit theorems for the tagged particle in an exclusion process in recurrent environment. In light of the localization effect of both the potential as well as the exclusion dynamics it is an interesting question whether the combination of both leads to even stronger localization.

The proofs for the results in this section can be found in Section 5. We conclude this chapter with a discussion of the relationship between spiders and random walks on a strip, which were presented in Section 2.3. This connection could also be used to prove results for spiders in random environments. However, all results presented here are proven with direct methods.

3.4.3 Spiders as Random Walk on a Strip

Spiders with bounded span and $L < \infty$ can be described as a random walk on a strip, since the set of leg configurations C is finite. Choose any mapping of leg configurations to integers $C \to \{1,\ldots,|C|\}$ and let the position of the first leg S_1 determine the layer in the strip, while the second coordinate is given by the integer assigned to the leg configuration. Then, the movement of the spider is a random walk on $\mathbb{Z} \times \{1,\ldots,|C|\}$. This set-up is described briefly in Section 2.3. The transition matrices do not obey the usual ellipticity conditions for random walks on strips, as many transitions between layers are forbidden and the corresponding matrix elements are zero. A depiction of the strip is shown in Figure 9. All transitions with strictly positive probability are marked by an edge. We can compare this embedding in the strip to the normal

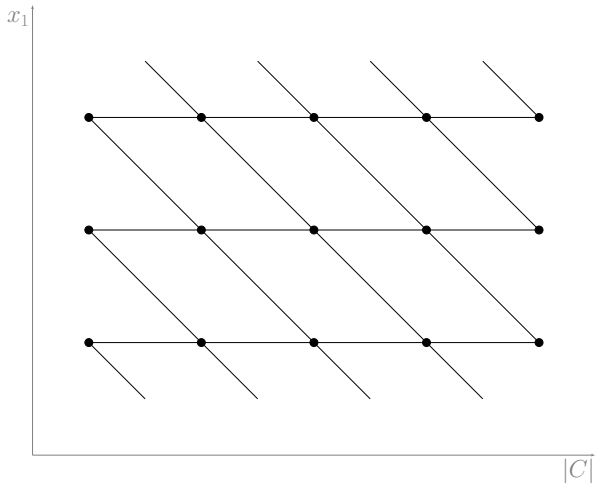

Figure 9: Two-legged spider with bounded span and $L = 5$ represented as a random walk on the strip $\mathbb{Z} \times \{1, \ldots, 5\}$. All possible transitions of the spider are marked by an edge in the graph with the strip as the vertex set. We see that most transitions are forbidden.

parametrization as it is shown in Figure 6, left panel. In the following paragraph, we will see whether the conditions for random walks on a strip presented in Section 2.3 hold true for spiders.

Let us consider a spider with $N = 2$ legs and maximal span $L > N$. The transition matrices for the random walk on a strip are identically distributed and thus we may set $(P, Q, R) = (P_0, Q_0, R_0)$ for $P, Q, R \in \mathbb{R}^{L \times L}$. We have for transitions from layer 0 to layer 1

$$P = \frac{1}{2} \begin{pmatrix} 0 & & & 0 \\ \omega_0 & 0 & & \\ & \ddots & \ddots & \\ 0 & & \omega_0 & 0 \end{pmatrix},$$

for transitions from layer 0 to layer -1

$$Q = \frac{1}{2} \begin{pmatrix} 0 & 1-\omega_0 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1-\omega_0 \\ 0 & & & 0 \end{pmatrix},$$

and for transitions within layer 0 we have a tridiagonal matrix

$$R = \frac{1}{2}\begin{pmatrix} \omega_0 + 1 - \omega_1 & \omega_1 & & & & 0 \\ 1-\omega_2 & 0 & \omega_2 & & & \\ & 1-\omega_3 & 0 & \omega_3 & & \\ & & 1-\omega_4 & 0 & \ddots & \\ & & & \ddots & \ddots & \omega_{L-1} \\ 0 & & & & 1-\omega_L & 1-\omega_0+\omega_L \end{pmatrix}.$$

Naturally, Condition (C1) is always valid, the triples (P_k, Q_k, R_k) are identically distributed and thus stationary and ergodic. However, the matrices $(R_k)_{k\in\mathbb{N}}$ are clearly not mutually independent. For Condition (C2) we calculate

$$\|P+R\| = \frac{1}{2}\max\{1+\omega_0, 1, 2\} = 1 \quad \text{and}$$
$$\|Q+R\| = \frac{1}{2}\max\{2, 1, 2-\omega_0\} = 1.$$

Thus, the second condition cannot hold. Also, Condition (C3) does not hold, since the column sums of the first column of Q and the last column of P are zero. We can see from the graph in Figure 9, that it is impossible to enter layer 1 from layer 0 at the site $(1, L)$ and from layer 0 to layer -1 at site $(-1, 1)$. Therefore, Conditions (C4.1(ϵ)) and (C4.2(ϵ)) cannot hold. Thus, spiders do not fit into the usual set of conditions for random walks in random environment on a strip. However, they do fulfil alternative conditions given in Goldsheid (2008), which imply the result on recurrence and transience, Theorem 2.15, the law of large numbers, Theorem 2.16, and the central limit theorem, Theorem 2.17. The alternative condition that we have to check is that $(I-R)^{-1}(i,j) > 0$ for all $1 \le i, j \le L$, where I is the identity matrix. In our case, we want to show that the matrix $T = 2(I-R)$ with

$$I - R = \frac{1}{2}\begin{pmatrix} 1-\omega_0+\omega_1 & -\omega_1 & & & & 0 \\ -(1-\omega_2) & 2 & -\omega_2 & & & \\ & -(1-\omega_3) & 2 & \ddots & & \\ & & \ddots & \ddots & -\omega_{L-1} & \\ 0 & & & -(1-\omega_L) & 1+\omega_0-\omega_L \end{pmatrix}$$

has a strictly positive inverse. A matrix is positive inverse if and only if it is monotone, i.e. $Tv \ge 0$ implies $v \ge 0$ for all $v \in \mathbb{R}^L$. The inequalities have to hold element-wise. We need to strengthen the result in one direction to get a condition for a strictly positive inverse.

Lemma 3.27. *Let $T \in \mathbb{R}^{L\times L}$ and $v \in \mathbb{R}^L$. If $Tv \ge 0$ implies $v > 0$ for all $v \ne 0$, then $T^{-1} > 0$.*

Proof. For a proof by contradiction we assume that T^{-1} contains a non-positive entry and denote the column containing this entry by c. Then, Tc is a column of the identity matrix and thus $Tc \ge 0$. Since c contains a non-positive entry we have $c \not> 0$, which is a contradiction. □

Thus, we only need to check that T is strictly monotone in the sense of Lemma 3.27, which can be easily done using the tridiagonal matrix algorithm (TDMA) for

the solution of a tridiagonal matrix equation $Tv = x$, where the column vector $x \neq 0$ has non-negative entries $x \geq 0$. The algorithm can be found for example in Conte and Boor (1980) and is a simplified version of Gaussian elimination. Denote the entries of the tridiagonal matrix T by three sequences $(a_n)_{n=1}^L$, $(b_n)_{n=1}^L$ and $(c_n)_{n=1}^{L-1}$,

$$T = \begin{pmatrix} b_1 & c_1 & & 0 \\ a_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{L-1} \\ 0 & & a_L & b_L \end{pmatrix}.$$

We set $a_1 = 0$. We calculate two sequences $(\gamma_n)_{n=1}^{L+1}$ and $(\beta_n)_{n=1}^{L+1}$ recursively starting from $\gamma_1 = \beta_1 = 0$ by

$$\gamma_{n+1} = \frac{-c_n}{a_n\gamma_n + b_n} \quad \text{and} \quad \beta_{n+1} = \frac{x_n - a_n\beta_n}{a_n\gamma_n + b_n}.$$

Then, we can solve the system by backwards substitution starting from $v_{L+1} = 0$ using

$$v_{n-1} = \gamma_n v_n + \beta_n. \tag{3.14}$$

We will now show that $\gamma_n > \kappa/2$ for all $2 \leq n \leq L$ using uniform ellipticity of the environment. We start by bounding γ_2 from below,

$$\gamma_2 = \frac{\omega_1}{1 - \omega_0 + \omega_1} \geq \frac{\kappa}{1 - \kappa + (1 - \kappa)} = \frac{\kappa}{2(1 - \kappa)} > \frac{\kappa}{2}.$$

From here, we can show by induction that the bound holds for the remaining elements of the sequence. For all $2 \leq n \leq L - 1$

$$\gamma_{n+1} = \frac{\omega_n}{2 - (1 - \omega_n)\gamma_n} \geq \frac{\kappa}{2 - (1 - \kappa)\kappa/2} > \frac{\kappa}{2}.$$

We need to take special care of the last element of the sequence. Since $b_L = 1 - \omega_0 + \omega_L \leq 2 - 2\kappa < 2$, we get the same lower bound as before and $\gamma_{L+1} > \kappa/2$. From that follows $\beta_n \geq 0$ for all $2 \leq n \leq L$ again by induction starting from $\beta_1 = 0$ and using

$$\beta_{n+1} = \gamma_{n+1}\frac{x_n - a_n\beta_n}{-c_n} \geq 0,$$

since $x_n \geq 0$, $-a_n > 0$ and $-c_n > 0$. Furthermore, $x \neq 0$ implies that there is some $1 \leq i \leq L$ such that $x_i > 0$ and we get strict positivity of β_n for all $i < n \leq L + 1$. In particular, this implies $\beta_{L+1} > 0$. Using the backwards substitution formula, Equation (3.14), it follows that $v > 0$ element-wise. Thus, by Lemma 3.27, the alternative condition from Goldsheid (2008) includes spider random walks. Consequently, we can apply Theorems 2.16 and 2.17 to transient spiders. The result on the speed of the spider random walk in transient environments by Gallesco et al. (2011b) is of course more explicit than the results for random walks on strips.

For recurrent spiders we cannot apply the results from random walks on a strip. For the localization theorem for recurrent random walks on a strip, Theorem 2.18, there are two conditions that do not hold true for spiders. First, the sequence of transition matrices is not i.i.d., though it is finitely dependent and usually with some work proofs can be extended to cover that case. Since neither (C4.1(ϵ)) nor (C4.2(ϵ)) hold, the proposed weaker condition for cases, where the walk is derived from a one-dimensional model, does not help. It should still be possible to find a suitable condition that covers spiders and prove results using the methods used in Bolthausen and Goldsheid (2008).

To conclude this section, we establish the implications of Condition *(iii)* in Theorem 2.18 for spiders. For simplicity, take $L = 2$ for a two-legged spider. We want to solve $\pi(P + Q + R) = \pi$ for a normalized row vector $\pi = (\pi_1, \pi_2)$,

$$\pi \begin{pmatrix} \omega_0 + 1 - \omega_1 & 1 - \omega_0 + \omega_1 \\ \omega_0 + 1 - \omega_2 & 1 - \omega_0 + \omega_2 \end{pmatrix} = 2\pi.$$

Then we get from the equation for the first coordinate and from $\pi_1 + \pi_2 = 1$ that

$$\pi_1(\omega_0 + 1 - \omega_1) + (1 - \pi_1)(\omega_0 + 1 - \omega_2) = 2\pi_1,$$

which we solve to

$$\pi_1 = \frac{1 - \omega_2 + \omega_0}{2 - \omega_2 + \omega_1} \quad \text{and} \quad \pi_2 = \frac{1 - \omega_0 + \omega_1}{2 - \omega_2 + \omega_1}.$$

Now, we ask which law the environment needs to have such that

$$\pi(P - Q)1 = -\pi_1(1 - \omega_0) + \pi_2\omega_0 = 0. \tag{3.15}$$

A quick calculation shows that this is equivalent to

$$\omega_0 - \omega_2\omega_0 + \omega_1\omega_0 + \omega_2 = 1. \tag{3.16}$$

Since the environment $(\omega_x)_{x\in\mathbb{Z}}$ is i.i.d., we can set $\mu = \mathsf{E}[\omega_0]$ and $\sigma^2 = \mathsf{Var}(\omega_0)$ and determine these moments from Equation (3.16). This gives us that Condition *(iii)* in Theorem 2.18 is violated, if $\mu = 1/2$ and $\sigma^2 = 0$. Thus, the only law such that all environments in $\mathrm{supp}(\mu)$ solve (3.16), is by setting $\omega_n \equiv 1/2$ for all $n \in \mathbb{Z}$. This case is exactly what is excluded by Assumptions (A2) and (A3') for normal spiders.

4 Proofs for RWs on weighted, oriented percolation clusters

The key ingredient of our proofs is the mixing property of the percolation structure and environment. We will use it to show that we can define a regeneration structure, which is mixing itself, such that standard results for stationary, mixing sequences of random variables apply.

Lemma 4.1 (The environment is mixing)**.** *Let $d \geq 1$ and K be stationary and independent of ω.*

(i) The processes ξ^P and ξ^K are stationary under the law $\mathbb{P}$.

(ii) The processes ξ^P is mixing in space-time under the law $\mathbb{P}$ in the following sense: Fix $n \in \mathbb{N}$. Let $V_B \subset V$ be any cone shaped subset of V, i.e. there is a site $(x,l) \in V$ and angle $\beta \in [\pi/4, \pi/2]$ such that

$$V_B := \{(y,k) \in V : k \geq l \text{ and } ||x-y||_\infty \leq |k-l| \tan(\beta)\}. \tag{4.1}$$

Let $V_A \subset V$ be such that $L := |V_A| < \infty$ and $\operatorname{dist}(V_A, V_B) \geq n$. Then there exist constants $0 < c, C < \infty$ such that for any two events $A, B \in \sigma(\xi_k^P(y) : (y,k) \in V)$ with $\operatorname{supp}(A) \subseteq V_A$ and $\operatorname{supp}(B) \subseteq V_B$ we have

$$\alpha_n^P(A,B) := |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq C 2^L L^2 e^{-cn}. \tag{4.2}$$

Remark 4.2. The environment is not stationary under the conditional law $\tilde{\mathbb{P}}$. However, the environment is mixing under the conditional law $\tilde{\mathbb{P}}$, since the event B_0 can either be included in the event A or in the event B in Lemma 4.1. Therefore Equation (4.2) holds with constants $L' = L + 1$ and $c' = c/2$ also for $\tilde{\mathbb{P}}$.

Proof of Lemma 4.1. (i) The process ξ^P is stationary with respect to $\mathbb{P}$, which follows from the fact that the time-reversed process is a stationary discrete time contact process as explained in Birkner et al. (2013). The environment ξ^K is stationary with respect to $\mathbb{P}$, since it is the product of two independent stationary processes.

(ii) First, define the length of the longest path on the oriented percolation cluster given by ξ^P and starting in some point $(y,k) \in V$ by

$$l(y,k) := \sup\{n \geq 1 : \exists (y', k+n) \in V : (y,k) \to (y', k+n)\}. \tag{4.3}$$

Note that $l(y,k) = \infty$ if $(y,k) \in \mathcal{C}$. Define a subset $V_B \subset V$ and event B as in Lemma 4.1. We will successively consider more complicated events for A. To begin with, let the second event be $A_1 := \{\xi_{k_1}^P(x_1) = 0\}$ for some $(x_1, k_1) \in V$ such that $\operatorname{dist}(\{(x_1,k_1)\}, V_B) \geq n$. By Lemma A.1 in Birkner et al. (2013) we know that

$$\mathbb{P}\left(A_1 \cap \{l(x_1,k_1) \geq n\}\right) \leq C e^{-cn}. \tag{4.4}$$

The event $\{l(x_1,k_1) < n\} \cap A_1$ is measurable with respect to $\sigma(\omega(v) : v \in V \setminus V_B)$ and therefore independent of B. We can write

$$\begin{aligned}
\mathbb{P}(A_1 \cap B) &= \mathbb{P}(A_1 \cap B \cap \{l(x_1,k_1) < n\}) + \mathbb{P}(A_1 \cap B \cap \{l(x_1,k_1) \geq n\}) \\
&\leq \mathbb{P}(B)\mathbb{P}(A_1 \cap \{l(x_1,k_1) < n\}) + \mathbb{P}(A_1 \cap \{l(x_1,k_1) \geq n\}) \\
&\leq \mathbb{P}(B)\mathbb{P}(A_1) + \mathbb{P}(A_1 \cap \{l(x_1,k_1) \geq n\})
\end{aligned}$$

and similarly

$$\begin{aligned}
\mathbb{P}(A_1 \cap B) &\geq \mathbb{P}(B)\mathbb{P}(A_1 \cap \{l(x_1,k_1) < n\}) \\
&= \mathbb{P}(B)\left(\mathbb{P}(A_1) - \mathbb{P}(A_1 \cap \{l(x_1,k_1) \geq n\})\right) \\
&\geq \mathbb{P}(B)\mathbb{P}(A_1) - \mathbb{P}(A_1 \cap \{l(x_1,k_1) \geq n\}).
\end{aligned}$$

We conclude, using Equation (4.4), that

$$\alpha_n^P(A_1, B) = |\mathbb{P}(A_1 \cap B) - \mathbb{P}(A_1)\mathbb{P}(B)| \leq Ce^{-cn}. \tag{4.5}$$

The same upper bound follows for $\alpha_n^P(A_1^c, B)$ with $A_1^c := \{\xi_{k_1}^P(x_1) = 1\}$, if we use that

$$\mathbb{P}(A_1^c) = 1 - \mathbb{P}(A_1) \text{ and}$$
$$\mathbb{P}(B \cap A_1^c) = \mathbb{P}(B) - \mathbb{P}(B \cap A_1).$$

We want to generalize this result to events that have support of more than one point. Consider events of the form

$$A_L^0 := \left\{\xi_{k_1}^P(x_1) = 0\right\} \cap \ldots \cap \left\{\xi_{k_L}^P(x_L) = 0\right\}$$

for L points $(x_1, k_1), \ldots, (x_L, k_L) \in V$ such that

$$\operatorname{dist}\left(\{(x_1, k_1), \ldots, (x_L, k_L)\}, V_B\right) \geq n$$

By subadditivity, using the same steps as before, we get

$$\alpha_n^P(A_L^0, B) \leq CLe^{-cn}. \tag{4.6}$$

Observe that an arbitrary event of the form

$$A_L^s := \left\{\xi_{k_1}^P(x_1) = s_1\right\} \cap \ldots \cap \left\{\xi_{k_L}^P(x_L) = s_L\right\}$$

for any $s := (s_1, \ldots, s_L) \in \{0, 1\}^L$ can be written as the disjoint union of two events of the form A_{L+1}. For example $A_1^0 = A_2^{(0,0)} \uplus A_2^{(0,1)}$. Since we have already established the mixing property for events A_1^0 and $A_2^{(0,0)}$ in Equation (4.6), we can use the triangle inequality to get the mixing property for $A_2^{(0,1)}$. The same argument allows us to derive the bounds for arbitrary sets A_L^s, where we have to pay a price on the upper bound for each time we apply the triangle inequality. After adding all the upper bounds of the appearing terms, we get

$$\alpha_n^P(A_L, B) \leq CL^2e^{-cn}. \tag{4.7}$$

Finally, it remains to observe that any event A, with $L = |V_A|$, can be written as a disjoint union of at most 2^L events of the type A_l^s, $1 \leq l \leq L$ and the claim follows. □

4.1 The Law of Large Numbers

The process ξ^K is not stationary with respect to $\tilde{\mathbb{P}}$, so we need to use a regeneration structure that has stationary increments. The definition of the appropriate regeneration structure is similar to the case of i.i.d. weights K in Birkner et al. (2013). It uses additional random permutations to achieve a local construction of the random walk. For every $(x, n) \in V$ we let $\tilde{\omega}(x, n)$ be a random permutation of sites in $U^+(x, n)$, which is chosen from the set of all permutations according to the law

$$\mathbb{P}\left(\tilde{\omega}(x, n) = (y_1, \ldots, y_N) \middle| K\right) = \prod_{l=1}^{2^d} \frac{K(y_l, n+1)}{\sum_{k=l}^N K(y_k, n+1)}. \tag{4.8}$$

The sum runs over all consecutive vertices of (x, n). The number of consecutive vertices $|U^+(x, n)| = 2^d$ is the number of corners in a d-dimensional hypercube. Our construction of the local path will be measurable with respect to the σ-algebra of all weights and permutations in the time interval of interest,

$$\mathcal{G}_n^m := \sigma\left(\omega(y, k), \tilde{\omega}(y, k), y \in \mathbb{Z}^d, n \leq k < m\right). \tag{4.9}$$

We need to know the length of the longest open path $l(x,n)$ starting at (x,n). Then $l_k(x,n) := l(x,n) \wedge k$ is measurable with respect to $\mathcal{G}_n^{n+k+1}$. For the local construction of the path we furthermore need the set of possible next steps if we want to stay on paths, which have at least length k. For any $k \geq -1$, we define this set as

$$M_k(x,n) := \begin{cases} U^+(x,n) & \text{if } k = -1, \\ \left\{ v \in U^+(x,n) : l_k(v) = \max_{z \in U^+(x,n)} l_k(z) \right\} & \text{otherwise.} \end{cases} \tag{4.10}$$

Finally, we complete our auxiliary notation by choosing $m_k(x,n) \in M_k(x,n)$ to be the first element in the permutation $\tilde{\omega}(x,n)$. Given a percolation ω, a permutation $\tilde{\omega}$ and a starting point $(x,n) \in V$ we finally define the local path $\gamma_k = \gamma_k^{(x,n)}$ by

$$\gamma_k(j) := \begin{cases} (x,n) & \text{if } j = 0, \\ m_{k-j-1}(\gamma_k(j-1)) & \text{if } j = 1, 2, \ldots, k. \end{cases} \tag{4.11}$$

The law of the local path $(\gamma_\infty^{(x,n)}(j))_{j \geq 0}$ is the same as the law of the random walk $(X_j, n+j)_{j \geq 0}$ by Lemma 2.1 in Birkner et al. (2013). A more detailed description of this construction and a picture can be found in their paper as well.

For $p < 1$ we want the set S_{2m} to contain all sites $(x,n) \in V$ for which every directed open path returns to the space coordinate x after $2m$ steps,

$$S_{2m} := \{(x,n) \in V : (x,n) \to (x, n+2m),\ \mathbb{P}(X_{n+2m} = x | X_n = x) = 1\}. \tag{4.12}$$

Note that we get a strictly positive lower bound on the probability that any site $(x,n) \in V$ is in this set conditioned that it is on the backbone $\mathcal{C}$ by considering a single path. Since it is already on the backbone we only need to make sure that all sites that are adjacent to the single path are closed, i.e. if $p < 1$

$$\tilde{\mathbb{P}}\left(v \in S_{2m} \middle| \, v \in \mathcal{C} \right) \geq (1-p)^{2m(2d-1)} > 0. \tag{4.13}$$

Our definition of the regeneration times differs from the paper of Birkner et al. (2013) in the additional requirement that a regeneration can only happen at points in S_{2m}. Define the *regeneration times* recursively by $T_0 = 0$ and

$$T_n := \inf \left\{ k \geq T_{n-1} + 2m : \gamma_{k-2m}(k-2m) \in \mathcal{C} \cap S_{2m} \right\}. \tag{4.14}$$

The corresponding regeneration increments are

$$\tau_n := T_n - T_{n-1} \quad \text{and} \quad Y_n := X_{T_n} - X_{T_{n-1}}.$$

The regeneration times are those times at which the local construction discovers a point that is in the backbone and is followed by an episode in the percolation cluster that forces the random walk to return after $2m$ steps independent of the weights K. Since behaviour of the random walk during these episodes does not depend on the weights K it can be used to decrease dependency between regeneration increments by increasing m. Later in the proof we will choose m large, see Equation (4.32), to show that the covariance matrix has full rank.

The regeneration times are not measurable with respect to the past of the environment $(\mathcal{G}_0^n)_n$. The local construction allows us to define *potential regeneration times* $(\sigma_k)_{k \geq 0}$ for the $(i+1)$th regeneration by $\sigma_0 = T_i$ and

$$\sigma_{k+1} = \sigma_k + l(\gamma_{\sigma_k}(\sigma_k)) + 2. \tag{4.15}$$

The potential regeneration times are those times at which the local construction discovers that a local path was finite and jumps to another branch, see Figure 10. The

construction ends when the new branch is part of the backbone and we have discovered an infinite path. In this case, we have found the next regeneration time as these times are $(\mathcal{G}_0^n)_n$-measurable and therefore stopping times. We only need to check at potential regeneration times (i.e. jumps to the next branch) whether all conditions for a regeneration are met. With this procedure we achieve minimal dependence on the future.

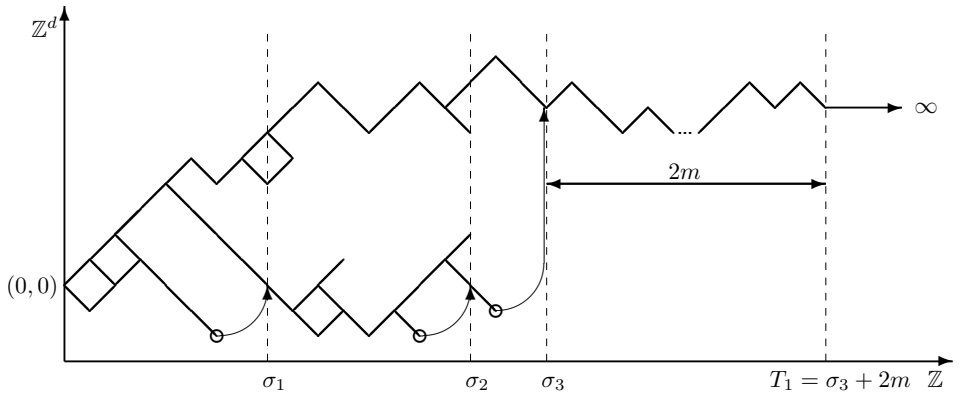

Figure 10: Example for the regeneration structure in dimension $d = 1$. The vertex set V is not shown. The visible edges are those that can be reached from the origin $(0, 0)$ by visiting open sites only. These edges are in the oriented percolation cluster of the origin. For a better visualization the permutations $\tilde{\omega}$ are chosen non-randomly and such that sites with smaller space coordinates are visited first. The local construction discovers three finite branches of the cluster before finding a regeneration time T_1. The end of each of these branches is marked by a circle. Afterwards the local discovery of the cluster is continued at the sites marked by the thin arrows. In this example only the topmost branch is connected by an open path to infinity and thus is in the backbone $\mathcal{C}$.

Lemma 4.3 (Increments of the random walk are ergodic)**.** *Let $d \geq 1$, K be independent of ω, stationary and ϕ-mixing in the time coordinate with mixing coefficients $(\phi_n)_{n\in\mathbb{N}}$. Then the process $(Y_n, \tau_n)_{n\in\mathbb{N}}$ is stationary and ϕ-mixing with respect to $\tilde{\mathbb{P}}$ with mixing coefficients*

$$(\phi_n^X)_{n\in\mathbb{N}} = (\phi_{2mn} + 2\alpha_{2mn}^P)_{n\in\mathbb{N}}, \tag{4.16}$$

where $\alpha_n^P = Ce^{-cn}$, $n \in \mathbb{N}$ are the mixing coefficients for ξ^P from Lemma 4.1, Equation (4.2).

Proof. Fix a site $(x, l) \in V$ such that $||x||_\infty \leq l$. Then $\tilde{\mathbb{P}}(\gamma_{T_n}(T_n) = (x, l)) > 0$. We observe that for all $n \in \mathbb{N}$ by the local construction of the random walk there exists an event

$$A' \in \sigma\left(\omega(y, k), \tilde{\omega}(y, k) : (y, k) \in V, 0 \leq k < l\right)$$

such that

$$\{\gamma_{T_n}(T_n) = (x, l)\} = A' \cap \{(x, l) \to \infty\} \subset B_0. \tag{4.17}$$

Let $\theta_z : \Omega \mapsto \Omega$, $z \in V$ be the standard shift operator such that $(\theta_z\omega)(z') = \omega(z + z')$ for any $\omega \in \Omega$, $z, z' \in V$. Then we can write

$$\theta_{(x,l)}^{-1}(\{\gamma_{T_n}(T_n) = (x, l)\}) = \theta_{(x,l)}^{-1}(A') \cap B_0. \tag{4.18}$$

Thus, for every event $A \in \sigma(\xi_k^K(y) : (y,k) \in V)$ and (x,l) we have

$$\begin{aligned}\mathbb{P}\left(\theta_{(x,l)}(A) \cap B_0 \middle| \gamma_{T_n}(T_n) = (x,l)\right) &\overset{(4.17)}{=} \mathbb{P}\left(\theta_{(x,l)}(A) \middle| \gamma_{T_n}(T_n) = (x,l)\right) \\ &= \mathbb{P}\left(A \middle| \theta_{(x,l)}^{-1}(\{\gamma_{T_n}(T_n) = (x,l)\})\right) \\ &\overset{(4.18)}{=} \mathbb{P}\left(A \cap B_0 \middle| \theta_{(x,l)}^{-1}(\{\gamma_{T_n}(T_n) = (x,l)\})\right),\end{aligned}$$

which implies

$$\begin{aligned}&\mathbb{P}(\theta_{\gamma_{T_n}(T_n)}(A) \cap B_0) \\ &\quad = \sum_{(x,l)} \mathbb{P}\left(\theta_{(x,l)}(A) \cap B_0 \middle| \gamma_{T_n}(T_n) = (x,l)\right) \mathbb{P}\left(\gamma_{T_n}(T_n) = (x,l)\right) \\ &\quad = \mathbb{P}(A \cap B_0).\end{aligned}$$

Consequently $\tilde{\mathbb{P}}(\theta_{\gamma_{T_n}(T_n)}(A)) = \tilde{\mathbb{P}}(A)$ and both processes are stationary with respect to $\tilde{\mathbb{P}}$.

Denote by $\mathcal{W}$ the σ-algebra that contains all possible paths of the random walk between regeneration times, namely

$$\mathcal{W}_k^l := \sigma\left(\{(Y_i(\omega), \tau_i(\omega))\}_{i=k}^l : \omega \in \Omega\right)$$

and $\mathcal{W} = \mathcal{W}_0^\infty$. Then, the mixing coefficients for the process $(Y_n, \tau_n)_{n\in\mathbb{N}}$ are given by

$$\phi_n^X = \sup_{N\in\mathbb{N}} \sup_{\substack{W\in\mathcal{W}, \\ A^N := W\cap\mathcal{W}_0^N, \\ B^N := W\cap\mathcal{W}_{N+n}^\infty}} \left| \frac{\tilde{\mathbb{P}}(A^N \cap B^N)}{\tilde{\mathbb{P}}(A^N)} - \tilde{\mathbb{P}}(B^N) \right|. \tag{4.19}$$

Note that by definition of the random walk and since $K > 0$ we have $\tilde{\mathbb{P}}(A^N) > 0$ for all $A^N \in W \cap \mathcal{W}_0^N$. We will from now on leave out the subscripts of the suprema. Furthermore, note that for every $(x,l) \in V$ exists an event

$$A_{(x,l)}^N \in \sigma(\omega(y,k), \tilde{\omega}(y,k) : y \in \mathbb{Z}^d, k < l)$$

such that

$$A^N \cap \{(X_N, T_N) = (x,l)\} = A_{(x,l)}^N \cap \{\xi_l^P(x) = 1\}.$$

This allows us to split up the events into disjoint subsets depending on where the path

ends. We rewrite the mixing coefficients as

$$\begin{aligned}
\phi_n^X &= \sup\sup \frac{1}{\tilde{\mathbb{P}}(A^N)} \Bigg| \sum_{(x,l)\in\mathbb{Z}^{d+1}} \tilde{\mathbb{P}}\left(A^N \cap B^N \cap \{(X_N, T_N) = (x,l)\}\right) \\
&\qquad\qquad - \tilde{\mathbb{P}}\left(A^N \cap \{(X_N, T_N) = (x,l)\}\right) \tilde{\mathbb{P}}\left(B^N\right) \Bigg| \\
&= \sup\sup \frac{1}{\tilde{\mathbb{P}}(A^N)} \Bigg| \sum_{(x,l)} \tilde{\mathbb{P}}\left(A^N_{(x,l)} \cap B^N \cap \{\xi_l^P(x) = 1\}\right) \\
&\qquad\qquad - \tilde{\mathbb{P}}\left(A^N_{(x,l)} \cap \{\xi_l^P(x) = 1\}\right) \tilde{\mathbb{P}}\left(B^N\right) \Bigg| \\
&= \sup\sup \frac{1}{\mathbb{P}(B_0)\tilde{\mathbb{P}}(A^N)} \Bigg| \sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)} \cap B^N \cap \{\xi_l^P(x) = 1\}\right) \\
&\qquad\qquad - \mathbb{P}\left(A^N_{(x,l)} \cap \{\xi_l^P(x) = 1\}\right) \tilde{\mathbb{P}}\left(B^N\right) \Bigg|.
\end{aligned}$$

The last equation follows from the fact that $B_0 \subset A^N_{(x,l)} \cap \{\xi_l^P(x) = 1\}$. We can use independence of $A^N_{(x,l)}$ and $\{\xi_l^P(x) = 1\}$ and the mixing property of the weights K to get

$$\begin{aligned}
\phi_n^X \leq \sup\sup \frac{1}{\mathbb{P}(B_0)\tilde{\mathbb{P}}(A^N)} \Bigg| \sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)}\right) \mathbb{P}\left(B^N \cap \{\xi_l^P(x) = 1\}\right) \\
- \mathbb{P}\left(A^N_{(x,l)}\right) \mathbb{P}\left(\{\xi_l^P(x) = 1\}\right) \tilde{\mathbb{P}}\left(B^N\right) \Bigg| + \mathcal{E}_1(n),
\end{aligned} \tag{4.20}$$

where

$$\mathcal{E}_1(n) := \operatorname{ess\,sup} \frac{1}{\mathbb{P}(B_0)} \frac{\sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)}\right)}{\tilde{\mathbb{P}}(A^N)} \phi_{T_{N+n} - T_N}. \tag{4.21}$$

This error term has a deterministic upper bound. We use stationarity of the environment and $\mathbb{P}(B_0) > 0$ to see that in fact

$$\tilde{\mathbb{P}}(A^N) = \sum_{(x,l)} \frac{\mathbb{P}\left(A^N_{(x,l)} \cap \{\xi_l^P(x) = 1\}\right)}{\mathbb{P}(B_0)} = \sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)}\right). \tag{4.22}$$

Since regeneration increments include a deterministic path, they have a length of at least $2m$ and thus we get

$$\mathcal{E}_1(n) \leq \frac{1}{\mathbb{P}(B_0)} \phi_{2mn}. \tag{4.23}$$

Note that since K is ϕ-mixing instead of α-mixing, the factor $\sum_{(x,l)} \mathbb{P}(A^N_{(x,l)})$ appears in the upper bound in Equation (4.23), which cancels with the denominator and makes the error term finite.

We can use the mixing property of ξ^P from Lemma 4.1 to factor $\mathbb{P}(B^N \cap \{\xi_l^P(x) = 1\})$ and $\tilde{\mathbb{P}}(B^N) = \mathbb{P}(B^N \cap B_0)/\mathbb{P}(B_0)$. This leads to the upper bound

$$\phi_n^X \leq \sup\sup \frac{1}{\mathbb{P}(B_0)\tilde{\mathbb{P}}(A^N)} \Bigg| \sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)}\right) \mathbb{P}\left(B^N\right) \mathbb{P}\left(\xi_l^P(x) = 1\right) \tag{4.24}$$

$$- \mathbb{P}\left(A^N_{(x,l)}\right) \mathbb{P}\left(\xi_l^P(x) = 1\right) \mathbb{P}\left(B^N\right) \Bigg| + \mathcal{E}_1(n) + 2\mathcal{E}_2(n)$$

$$= \mathcal{E}_1(n) + 2\mathcal{E}_2(n), \tag{4.25}$$

with

$$\mathcal{E}_2(n) := \operatorname{ess\,sup} \frac{1}{\mathbb{P}(B_0)} \frac{\sum_{(x,l)} \mathbb{P}\left(A^N_{(x,l)}\right)}{\tilde{\mathbb{P}}(A^N)} \alpha^P_{T_{N+n}-T_N} \leq \frac{1}{\mathbb{P}(B_0)} \alpha^P_{2mn}. \tag{4.26}$$

Combining Equations (4.23), (4.22) and (4.26) tells us that overall the sequence of regeneration increments is ϕ-mixing and the mixing coefficients are bounded above by

$$\phi_n^X \leq \frac{1}{\mathbb{P}(B_0)} \left(\phi_{2mn} + 2\alpha^P_{2mn}\right). \tag{4.27}$$

□

With this preparation the LLN, Lemma 2.24, follows directly from the previous results.

Proof of Lemma 2.24. By Lemma 4.3, the sequence $(Y_n)_{n\in\mathbb{N}}$ is stationary and mixing and therefore ergodic. The law of large numbers follows from the ergodic theorem (Birkhoff, 1931) together with standard arguments from renewal theory and the drift vector takes the usual form,

$$\vec{\mu} = \frac{\tilde{\mathbb{E}}[X_{T_1}]}{\tilde{\mathbb{E}}[T_1]}. \tag{4.28}$$

□

The next example shows that the average $\vec{\mu}$ can indeed be non-zero on the full lattice, even if the weights K are independent in time. The example was provided in private communication by Noam Berger.

Example 4.4. Let $d = 1$ and take $p = 1$. We construct an environment from bounded weights that are independent in time such that the random walk is ballistic in the space coordinate, i.e. $\mu \neq 0$. Let $(\beta(n))_{n\in\mathbb{N}}$ be a family of independent random variables, each of them uniformly distributed on the set $\{0, 1, 2\}$. Choose weights for all $x \in \mathbb{Z}$ according to

$$K(x, n) = ((\beta(n) + 3|x| + x) \mod 3) + 1 \in \{1, 2, 3\}.$$

Then the average speed is

$$\mu = \mathbb{E}\left[\frac{K(1, n) - K(-1, n)}{K(1, n) + K(-1, n)}\right] = -1/90 < 0$$

for any $n \in \mathbb{N}$.

4.2 The Annealed Central Limit Theorem

The aCLT follows without much additional work from the results we already established for the LLN.

Proof of Theorem 2.25. We begin with the proof for $p < 1$, where we have to consider the percolation cluster. As defined in Equation (4.3) the random variable $\tau^{(x,n)}$ denotes the length of the longest open path starting at the site (x, n). The proof is similar to the proof of Lemma 2.5 in Birkner et al. (2013), since K is independent of ω and the bounds derive from the structure of the open cluster. In particular, the increments $(\sigma_{k+1} - \sigma_k) \leq l(\gamma_{\sigma_k}(\sigma_k))$ are dominated by a random variable, which is independent of the weights K. Furthermore the number of trials to find a regeneration time is dominated by a geometric random variable with success probability $\mathbb{P}(B_0)(1-p)^{2m(2d-1)} > 0$ by Equation (4.13). Consequently, the first regeneration time has exponential tails,

$$\tilde{\mathbb{P}}\left(T_1 > n\right) \leq Ce^{-cn}. \tag{4.29}$$

The same bound holds for the space increment $||Y_1||_\infty$, since $||Y_1||_\infty \leq T_1$ for all $n \in \mathbb{N}$. We have shown in Lemma 4.3 that the sequence of regeneration increments $(Y_n, \tau_n)_{n\in\mathbb{N}}$ is stationary and ϕ-mixing with coefficients ϕ_n^X for n large enough. Therefore, all increments have exponential tail bounds. Under the mixing conditions of Theorem 2.25, $\phi_n \in \mathcal{O}(n^{-(2+\delta)})$ for some $\delta > 0$, we get that

$$\sum_{k=1}^{\infty} (\phi_k^X)^{\frac{1}{2}} \leq \frac{1}{\mathbb{P}(B_0)^{1/2}} \sum_{k=1}^{\infty} (\phi_{2mk} + 2\alpha_{2mk}^P)^{\frac{1}{2}} < \infty. \tag{4.30}$$

This is the condition of Ibragimov and Linnik (1971) for the CLT for ϕ-mixing sequences, Theorem 18.5.2. We prove the aCLT first in the case $d = 1$. Define centred random variables $Z_n = Y_n - \tilde{\mathbb{E}}[Y_n] = Y_n - \tilde{\mathbb{E}}[Y_1]$ for all $n \in \mathbb{N}$. By Equation (4.29) we know that $\tilde{\mathbb{E}}\left[|Z_n|^{D+2}\right] < \infty$ and $\tilde{\mathbb{E}}[\tau_n^{D+2}] < \infty$. Since Z_n is centred, Theorem 17.2.3 in Ibragimov and Linnik (1971) and Equation (4.30) imply that

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \tilde{\mathbb{E}}[Z_0 Z_n]\right| &\leq 2\sum_{n=1}^{\infty} \tilde{\mathbb{E}}[Z_0^2]^{1/2} \tilde{\mathbb{E}}[Z_n^2]^{1/2} (\phi_n^X)^{1/2} \\ &= 2\tilde{\mathbb{E}}[Z_0^2] \sum_{n=1}^{\infty} (\phi_n^X)^{1/2} < \infty. \end{aligned} \tag{4.31}$$

Furthermore, the sum can be made arbitrary small if we choose m large enough. Choose m so that $|\sum_{n=1}^{\infty} (\phi_n^X)^{1/2}| < 1/4$. Then the variance is strictly positive,

$$\sigma^2 = \tilde{\mathbb{E}}[Z_0^2] + 2\sum_{n=1}^{\infty} \tilde{\mathbb{E}}[Z_0 Z_n] \geq \tilde{\mathbb{E}}[Z_0^2] \left(1 - 4\sum_{n=1}^{\infty} (\phi_n^X)^{1/2}\right) > 0. \tag{4.32}$$

This choice of the distance $2m$ between two pieces of the regeneration increments allows us to conclude that the variance is strictly positive and the central limit theorem has a non-degenerate limit. Here, we use the percolation cluster explicitly to bound the variance away from zero. Using a central limit theorem for stationary and ϕ-mixing sequences, e.g. Theorem 18.5.2 in Ibragimov and Linnik (1971), and renewal arguments (Kuczek, 1989) we get a non-degenerate central limit theorem for the sequence $(Y_n, \tau_n)_{n\in\mathbb{N}}$.

Furthermore, we can generalize this result to the multivariate case using e.g. Lévy's continuity theorem as in Rio (2013), Corollary 4.1. In this case, we have to choose m

large enough, such that the covariance matrix Σ has full rank. The covariance matrix $\Sigma := (\Sigma_{ij})_{1\le i,j\le d}$ is given by

$$\Sigma_{ij} = \tilde{\mathbb{E}}\left[\langle Z_0, e_i\rangle\langle Z_0, e_j\rangle\right] + 2\sum_{k=1}^{\infty}\tilde{\mathbb{E}}[\langle Z_0, e_i\rangle\langle Z_k, e_j\rangle], \tag{4.33}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean scalar product and $\{e_1,\dots,e_d\}$ is the canonical basis of $\mathbb{Z}^d$. □

4.3 The Quenched Central Limit Theorem

The main idea for the proof is to study a pair of random walks on the same environment and show that their behaviour is close enough to the behaviour of two random walks on independent copies of the environment. As we did for the regeneration structure for a single random walk we define the sequence of regeneration times for two random walks starting at times $T_0 = T'_0 = 0$ for $j \ge 1$ by

$$\begin{aligned} T_j &:= \inf\left\{k > T_{j-1} + 2m : \gamma^{(x,0)}_{k-2m}(k-2m) \in \mathcal{C} \cap S_{2m}\right\}, \\ T'_j &:= \inf\left\{k > T'_{j-1} + 2m : \gamma'^{(x',0)}_{k-2m}(k-2m) \in \mathcal{C} \cap S_{2m}\right\}. \end{aligned} \tag{4.34}$$

Set $J_0 = J'_0 = 0$ and for $m \in \mathbb{N}$ and define auxiliary times

$$\begin{aligned} J_j &:= \inf\{k > T_{j-1} : T_k = T'_{k'} \text{ for some } k' > J'_j\} \quad \text{and} \\ J'_j &:= \inf\{k > T'_{j-1} : T'_{k'} = T_k \text{ for some } k > J_j\}. \end{aligned} \tag{4.35}$$

Define the sequence of simultaneous regeneration times by

$$T^{\text{sim}}_m := T_{J_m} = T'_{J'_m}, \quad m \ge 0 \tag{4.36}$$

or recursively $T^{\text{sim}}_0 = 0$ and

$$T^{\text{sim}}_m = \min\left(\{T_j : T_j > T^{\text{sim}}_{m-1}\} \cap \{T'_j : T'_j > T^{\text{sim}}_{m-1}\}\right). \tag{4.37}$$

The increments Y_k, Y'_k, τ_k and τ'_k are defined as in the single walk case and we set for $m, l \in \mathbb{N}$

$$\begin{aligned} \tilde{X}_m &:= X_{T_m}, \quad \tilde{X}'_m := X'_{T'_m} \\ \hat{X}_l &:= X_{T^{\text{sim}}_l}, \quad \hat{X}'_l := X'_{T^{\text{sim}}_l}. \end{aligned} \tag{4.38}$$

Finally denote the pieces between simultaneous regenerations $\Xi_m \in \mathcal{W} := \mathbb{F}\times\mathbb{F}\times\mathbb{Z}^d\times\mathbb{Z}^d$ by

$$\Xi_m := \left((Y_k,\tau_k)_{k=J_{m-1}+1}^{J_m}, (Y'_k,\tau'_k)_{k=J'_{m-1}+1}^{J'_m}, X_{T_{J_m}}, X'_{T'_{J'_m}}\right), \tag{4.39}$$

where $\mathbb{F} := \bigcup_{n=1}^{\infty}(\mathbb{Z}^d\times\mathbb{N})^n$. We need some more notation to indicate when we are considering two random walks simultaneously on the same percolation cluster. Take two starting points for the random walks $x, x' \in \mathbb{Z}^d$. Let

$$B_{x,x'} := \{\xi^P_0(x) = 1\} \cap \{\xi^P_0(x') = 1\}$$

be the event that both starting points are in the backbone $\mathcal{C}$. Conditioned on $B_{x,x'}$ let $X := (X_n)_n$ and $X' := (X'_n)_n$ be two independent random walks started at $(x,0)$

and $(x', 0)$ respectively and both with transition probabilities as in Equation (2.15). Write for the law of the two walkers conditioned on $B_{x,x'}$

$$\tilde{\mathbb{P}}^{\text{joint}}_{x,x'}(\cdot) = \mathbb{P}^{\text{joint}}_{x,x'}(\,\cdot \mid B_{x,x'}) = \mathbb{P}^{\text{joint}}(\,\cdot \mid X_0 = x, X'_0 = x', B_{x,x'})\,, \tag{4.40}$$

where the superscript indicates that the two walks run on the same realization of the environment. We will describe the joint law by comparing it to the law of two independent random walks $\tilde{\mathbb{P}}^{\text{ind}}_{x,x'}$, which is the product measure of two independent copies of single random walks with laws $\tilde{\mathbb{P}}^1_x$ and $\tilde{\mathbb{P}}^2_{x'}$ on two independent copies Ω_1, Ω_2 of the environment,

$$\tilde{\mathbb{P}}^{\text{ind}}_{x,x'}(\cdot) = \tilde{\mathbb{P}}^1_x(\cdot)\tilde{\mathbb{P}}^2_{x'}(\cdot).$$

To describe the second random walk, let $\tilde{\omega}'$ be another, independent random permutation distributed like $\tilde{\omega}$ and define paths $\gamma'^{(x,n)}_k$ analogously to $\gamma^{(x,n)}_k$ using $\tilde{\omega}'$ instead of $\tilde{\omega}$. For given n, k the construction of both paths are measurable w.r.t

$$\hat{\mathcal{G}}^k_n := \sigma\left(\omega(y,i), \tilde{\omega}(y,i), \tilde{\omega}'(y,i) \;:\; y \in \mathbb{Z}^d, n \le i < k\right). \tag{4.41}$$

Conditioned on $B_{x,x'}$ we may couple the random walks by

$$(X_k, k) = \lim_{n\to\infty} \gamma^{(x,0)}_n(k), \quad (X'_k, k) = \lim_{n\to\infty} \gamma'^{(x',0)}_n(k).$$

The proof of Theorem 2.26 is analogous to the proof of Theorem 2 in the paper of Birkner et al. (2013). However, some of the lemmas along the way have to be modified. Most of the proofs in this paper are kept rather short, if a similar and more detailed version can be found in the original paper. Here, we will list the essential adaptations needed to make it suitable for our problem. We get the exponential bounds on the joint regeneration times with the same argument as in the proof of Theorem 2.25, Equation (4.29), i.e. for all $x, x' \in \mathbb{Z}^d$

$$\tilde{\mathbb{P}}^{\text{joint}}_{x,x'}\left(T^{\text{sim}}_1 > n\right) \le Ce^{-cn}. \tag{4.42}$$

Also, the sequence of joint regeneration increments is again stationary and α-mixing by a similar reasoning as is used in Lemma 4.3.

Lemma 4.5 (Total variation distance of joint and independent law, cf. Lemma 3.4 in Birkner et al. (2013))**.** *There exist constants $0 < c, C < \infty$ such that for all $x, x' \in \mathbb{Z}^d$*

$$\left\|\mathbb{P}^{\text{joint}}_{x,x'}(\Xi_1 = \cdot) - \mathbb{P}^{\text{ind}}_{x,x'}(\Xi_1 = \cdot)\right\|_{TV} \le Ce^{-c||x-x'||},$$

where $||\cdot||_{TV}$ is the total variation norm.

Proof. As in the original paper, without loss of generality, we prove the lemma for two start points $x = 0$ and $x'e_1$, where e_1 is the first coordinate vector in $\mathbb{Z}^d$ and $x' > 2m$. Let Ω_1, Ω_2 and Ω_3 be three independent copies of environment and permutations Ω_i, i.e. for all $i \in \{1, 2, 3\}$ we define

$$\Omega_i := \{\omega_i(v), K_i(v), \tilde{\omega}_i(v) : v \in V\}.$$

Throughout this proof, we will add Ω_i as an argument to our random variables to indicate which realization of percolation and permutation is used in the construction. Detailed definitions can be found in the proof of Lemma 3.4 in Birkner et al. (2013). For example, we write

$$B_{x,x'}(\Omega_i, \Omega_j) := \{\xi^P_0(x; \Omega_i) = \xi^P_0(x'; \Omega_j) = 1\}$$

for the condition to start the walks on the backbones of Ω_i and Ω_j respectively,

$$\begin{aligned} T^{\text{sim}}_{i,j} &:= T^{\text{sim}}(\Omega_i, \Omega_j) \\ &:= \inf\left\{ n \geq 1 : \xi^P_n(\gamma^{(x,n)}_n(n;\Omega_i);\Omega_i) = \xi^P_n(\gamma^{(x,n)}_n(n;\Omega_j);\Omega_j) = 1 \right\} \end{aligned}$$

for the simultaneous regeneration times and

$$\begin{aligned} &\Xi_1(\Omega_i, \Omega_j) \\ &:= \left((Y_k(\Omega_i), \tau_k(\Omega_i))_{k=1}^{J_1(\Omega_i,\Omega_j)}, (Y'_k(\Omega_j), \tau'_k(\Omega_j))_{k=1}^{J'_1(\Omega_i,\Omega_j)}, X_{T^{\text{sim}}_{i,j}}(\Omega_i), X'_{T'^{\text{sim}}_{i,j}}(\Omega_j) \right). \end{aligned}$$

for the simultaneous regeneration increments. To construct a simultaneous regeneration increment of two independent walks $\Xi^{\text{ind}}_{x,x'}$, we will start one random walk at $x = 0$ on Ω_1 and another random walk at x' on Ω_2. Similarly we construct the simultaneous regeneration increment of two walks on the same cluster $\Xi^{\text{joint}}_{x,x'}$ by starting two random walks in x and x' respectively both on Ω_3. It is convenient to write

$$\begin{aligned} \Xi^{\text{joint}}_{x,x'} &:= \begin{cases} \Xi_1(\Omega_3, \Omega_3), & \text{if } B_{x,x'}(\Omega_3, \Omega_3) \text{ occurs,} \\ \Delta & \text{otherwise,} \end{cases} \\ \Xi^{\text{ind}}_{x,x'} &:= \begin{cases} \Xi_1(\Omega_1, \Omega_2), & \text{if } B_{x,x'}(\Omega_1, \Omega_2) \text{ occurs,} \\ \Delta & \text{otherwise,} \end{cases} \end{aligned}$$

with some cemetery state Δ. If we start the random walks far enough apart, then with high probability the regeneration event will happen in two disjoint subsets of V that have a distance of $x'/2$. This allows us to use the mixing properties of the environment, Lemma 4.1. Define the two disjoint subsets of $S_1, S_2 \subseteq V$ by

$$\begin{aligned} S_1 &:= \{(y,k) \in V : |y| \leq k \text{ for all } 0 \leq k \leq |x - x'|/4\}, \\ S_2 &:= \{(y,k) \in V : |y - x'| \leq k \text{ for all } 0 \leq k \leq |x - x'|/4\}. \end{aligned}$$

Then $\text{dist}(S_1, S_2) = x'/2$ as shown in Figure 11. Finally, define the events

$$\begin{aligned} L_1 &:= \{l(x,0;\Omega_1) \vee l(x',0;\Omega_2) \vee l(x,0;\Omega_3) \vee l(x',0;\Omega_3) \leq |x - x'|/4\}, \\ L_2 &:= \{\xi^P_0(x;\Omega_1) = \xi^P_0(x';\Omega_2) = \xi^P_0(x;\Omega_3) = \xi^P_0(x';\Omega_3) = 1\} \\ &\quad \cap \{T^{\text{sim}}_{1,2} \leq |x - x'|/4\} \cap \{T^{\text{sim}}_{3,3} \leq |x - x'|/4\}. \end{aligned}$$

Conditioned on these events the two random walks stay far enough apart. Note that the events L_1 and L_2 are disjoint. Since the probabilities of the complements of both sets have exponential bounds in x' by Lemma A.1 in Birkner et al. (2013) and Equation (4.29), we know that $\mathbb{P}(L_1^c \cap L_2^c)$ has an exponential tail bound in $x'/4$. So,

$$\begin{aligned} &\left| \mathbb{P}(\Xi^{\text{joint}}_{x,x'} = w) - \mathbb{P}(\Xi^{\text{ind}}_{x,x'} = \omega) \right| \\ &\quad \leq \left| \mathbb{P}(\{\Xi^{\text{joint}}_{x,x'} = w\} \cap L_1) - \mathbb{P}(\{\Xi^{\text{ind}}_{x,x'} = \omega\} \cap L_1) \right| \\ &\qquad + \left| \mathbb{P}(\{\Xi^{\text{joint}}_{x,x'} = w\} \cap L_2) - \mathbb{P}(\{\Xi^{\text{ind}}_{x,x'} = \omega\} \cap L_2) \right| \\ &\qquad + Ce^{-cx'/4}. \end{aligned} \tag{4.43}$$

On the event L_1 the regeneration increments are supported on the sets S_1 and S_2. Thus, since on L_1 both increments are equal to the cemetery state Δ, we have

$$\left| \mathbb{P}(\{\Xi^{\text{joint}}_{x,x'} = w\} \cap L_1) - \mathbb{P}(\{\Xi^{\text{ind}}_{x,x'} = w\} \cap L_1) \right| = 0. \tag{4.44}$$

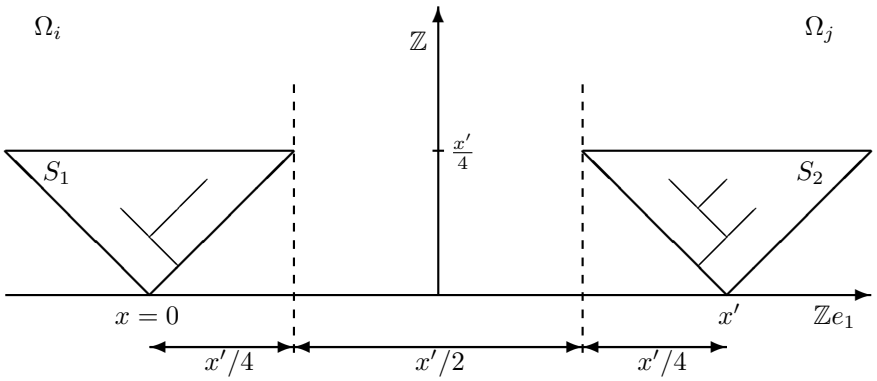

Figure 11: Conditioned on either of the events L_1 and L_2 both regeneration increments depend on percolation cluster and permutations only in the sets S_1 and S_2 respectively. Since both sets have at least distance $x'/2$, we can use the mixing property of the environment to bound the difference of probabilities of a joint versus an independent pair of regeneration increments.

Very similar to what we did in the proof of Lemma 4.3 we obtain the bounds on our second term by summing over all possible endpoints for the increment. For every site $(y,k) \in S_r$, $r \in \{1,2\}$ there exist events

$$A_r^{(y,k)}(\Omega_i) \in \sigma(\omega(z,l);\Omega_i), \tilde{\omega}(z,l;\Omega_i) : (z,l) \in S_r)$$

such that

$$\begin{aligned}\{\Xi^{\text{joint}}_{x,x'} = w\} \cap \left\{\left(X_{T^{\text{sim}}_{3,3}}(\Omega_3), X'_{T^{\text{sim}}_{3,3}}(\Omega_3), T^{\text{sim}}_{3,3}\right) = (y,y',k)\right\} \cap L_2 \\ = A_1^{(y,k)}(\Omega_3) \cap A_2^{(y',k)}(\Omega_3) \cap \{\xi_k^P(y;\Omega_3) = 1\} \cap \{\xi_k^P(y';\Omega_3) = 1\}\end{aligned}$$

and

$$\begin{aligned}\{\Xi^{\text{ind}}_{x,x'} = w\} \cap \left\{\left(X_{T^{\text{ind}}_{1,2}}(\Omega_1), X'_{T^{\text{ind}}_{1,2}}(\Omega_2), T^{\text{ind}}_{1,2}\right) = (y,y',k)\right\} \cap L_2 \\ = A_1^{(y,k)}(\Omega_1) \cap A_2^{(y,k)}(\Omega_2) \cap \{\xi_k^P(y;\Omega_1) = 1\} \cap \{\xi_k^P(y';\Omega_2) = 1\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\mathbb{P}(\{\Xi^{\text{joint}}_{x,x'} = w\} \cap L_2) - \mathbb{P}(\{\Xi^{\text{ind}}_{x,x'} = w\} \cap L_2)\right| \\ &\quad\leq \sum_k \sum_{\substack{y:(y,k)\in S_1 \\ y':(y',k)\in S_2}} \qquad (4.45) \\ &\qquad \Big|\mathbb{P}\left(A_1^{(y,k)}(\Omega_3) \cap \{\xi_k^P(y;\Omega_3) = 1\} \cap A_2^{(y',k)}(\Omega_3) \cap \{\xi_k^P(y';\Omega_3) = 1\}\right) \\ &\qquad - \mathbb{P}\left(A_1^{(y,k)}(\Omega_1) \cap \{\xi_k^P(y;\Omega_1) = 1\}\right) \mathbb{P}\left(A_2^{(y',k)}(\Omega_2) \cap \{\xi_k^P(y';\Omega_2) = 1\}\right)\Big| \\ &\quad\leq \sum_{(y,k)\in S_1} \phi_{x'/4} + \alpha^X_{x'/4} \leq |S_1| \left(\phi_{x'/4} + \alpha^X_{x'/4}\right). \qquad (4.46)\end{aligned}$$

Using $|S_1| = x'^{d+1}/8$, we can combine the three previous estimates in Equations (4.43),

(4.44) and (4.45) to obtain

$$\begin{aligned}
&\left\|\mathbb{P}^{\text{joint}}_{x,x'}(\Xi_1 = \cdot) - \mathbb{P}^{\text{ind}}_{x,x'}(\Xi_1 = \cdot)\right\|_{\text{TV}} \\
&\quad = \sup_{w \in \mathcal{W} \cup \{\Delta\}} \left|\mathbb{P}(\Xi^{\text{joint}}_{x,x'} = w) - \mathbb{P}(\Xi^{\text{ind}}_{x,x'} = w)\right| \\
&\quad \leq Ce^{-cx'/4} + \alpha_{x'/2} + \frac{x'^{d+1}}{8}\left(\phi_{x'/4} + \alpha^X_{x'/4}\right).
\end{aligned}$$

The conclusion of the lemma follows since all mixing coefficients are exponentially decreasing. □

We have established that the total variation distance between the laws of two independent walks and two walks on the same cluster becomes small if the walks start far apart. Now, we need estimates on the probabilities to find two independent walks closer together or further apart after some time. For this, we compare it with standard Brownian motion and estimate exit probabilities from an annulus.

Lemma 4.6 (Escape time from an annulus, cf. Lemma 3.6 in Birkner et al. (2013))**.** *Let U be the linear, bijective map that decomposes the inverse covariance matrix from Theorem 2.25, $\Sigma^{-1} = U^T U$. Write for $r > 0$*

$$\begin{aligned}
h(r) &:= \inf\{k \in \mathbb{N} : ||U(\hat{X}_k - \hat{X}'_k)||_\infty \leq r\} \\
H(r) &:= \inf\{k \in \mathbb{N} : ||U(\hat{X}_k - \hat{X}'_k)||_\infty \geq r\}
\end{aligned}$$

and set for $r_1 < r < r_2$

$$f_d(r; r_1, r_2) = \begin{cases} \frac{\log(r) - \log(r_1)}{\log(r_2) - \log(r_1)} & \text{for } d = 2 \\ \frac{r_1^{2-d} - r^{2-d}}{r_1^{2-d} - r_2^{2-d}} & \text{for } d \geq 3. \end{cases}$$

Then for every $\epsilon > 0$ there are large constants R and $\tilde{R}$ such that for all $r_2 > r_1 > R$ and $r_2 - r_1 > \tilde{R}$ and for all starting points $x, y \in \mathbb{Z}^d$ such that $r = ||U(x - y)||_\infty$, $r_1 < r < r_2$,

$$(1 - \epsilon) f_d(r; r_1, r_2) \leq \tilde{\mathbb{P}}^{\text{ind}}_{x,y}(H(r_2) < h(r_1)) \leq (1 + \epsilon) f_d(r; r_1, r_2).$$

Proof. Under the law $\tilde{\mathbb{P}}^{\text{ind}}$ the two copies of the random walk $(\hat{X}_k)_{k \in \mathbb{N}}$ and $(\hat{X}'_k)_{k \in \mathbb{N}}$ are independent and their difference is again a random walk with finite variance and zero mean. By Theorem 2.2 in Merlevède (2003) or Theorem 4.3 in Rio (2013) we get a functional central limit theorem for $(\hat{X}_k - \hat{X}'_k)_{k \in \mathbb{N}}$ under the same assumptions as in Theorem 2.25. The limit law is Brownian motion with some covariance operator Σ. Since the covariance matrix Σ is symmetric and positive semi-definite it has a Cholesky decomposition. It has full rank and so the inverse has a decomposition $\Sigma^{-1} = U^T U$, where U has full rank as well. Then the limit law of the random walk (X_n) under the map U has identity covariance matrix. If we define h and H as above, we can compare the random walks to the standard estimates of the exit probability from annuli for Brownian motion, which gives the conclusion. □

Lemma 4.7 (Separation lemma, cf. Lemma 3.8 in Birkner et al. (2013))**.** *For dimension $d \geq 2$ there are constants $b_1, b_2 \in (0, 1/2)$, $b_3 > 0$, $b_4 \in (0, 1)$, $C > 0$ such that for large n*

$$\tilde{\mathbb{P}}^{\text{joint}}_{0,0}\left(H(n^{b_1}) \geq n^{b_2}\right) \leq Ce^{-b_3 n^{b_4/2}}. \tag{4.47}$$

Proof. For the proof, we have to be a little bit more careful as our environment ξ^K does not have the Markov property and the regeneration increments are not independent. In the first step of the proof of Equation (4.47) we observe that instead of forcing two paths on the same cluster to move in opposite directions by choosing $\tilde{\omega}$, we can as well choose the percolation ω to our needs. In this case, we get the required bounds, Equation (3.29) in Birkner et al. (2013), with the further advantage that the construction depends on the percolation only. This allows us to rely on the Markov property of ξ^P. Furthermore, define the event

$$A_n := \{(\hat{X}_n, \hat{X}'_n) \text{ has reached distance } n^{b_1} \text{ in at most } n^{3b_1} + n^{b_6} \text{ steps}\}.$$

Following the proof in Birkner et al. (2013) we know from their Equations (3.34) and (3.35) that there exist $b_1 \in (0, 1/6)$ and $b_6 \in (0, 1/2)$ such that $\tilde{\mathbb{P}}^{\text{joint}}_{x,y}(A_n) > \delta$ for some $\delta > 0$ and uniformly in x, y. This bound is based on the escape time estimates of Lemma 4.6. We can pick $b_2 \in (3b_1 \vee b_6, 1/2)$ such that $n^{b_2} \geq n^{3b_1} + n^{b_6}$. We get the required upper bound for the probability to fail to reach the distance n^{b_1} at least n^{b_4} times by looking only at every second regeneration increment and then using mixing properties to bound dependencies between them. This way we get

$$\begin{aligned}
\tilde{\mathbb{P}}^{\text{joint}}_{0,0}\left(\bigcap_{k=1}^{n^{b_4}} A_k^c\right) &\leq \tilde{\mathbb{P}}^{\text{joint}}_{0,0}\left(\bigcap_{k=1,k \text{ odd}}^{n^{b_4}} A_k^c\right) \\
&\leq \tilde{\mathbb{P}}^{\text{joint}}_{0,0}(A_1^c)\tilde{\mathbb{P}}^{\text{joint}}_{0,0}\left(\bigcap_{k=3,k \text{ odd}}^{n^{b_4}} A_k^c\right) + \alpha_{n^{3b_1}+n^{b_6}} \\
&\leq \ldots \leq \prod_{k=1,k \text{ odd}}^{n^{b_4}} \tilde{\mathbb{P}}^{\text{joint}}_{0,0}(A_k^c) + \frac{n^{b_4}}{2}\alpha_{n^{3b_1}+n^{b_6}} \\
&\leq (1-\delta)^{n^{b_4}/2} + Cn^{b_4}e^{-cn^{3b_1}-cn^{b_6}} \\
&\leq Cn^{b_4}e^{-b_3 n^{b_4}/2} \\
&\leq Ce^{-b_3 n^{b_4}/2}
\end{aligned}$$

for n large enough, $b_3 := \min\{-\log(1-\delta), c\}$ and $b_4 < \min\{6b_1 \vee 2b_6, b_2 - 3b_1 \vee b_6\}$. By construction these attempts take at most

$$n^{b_4}(n^{3b_1} + n^{b_6}) \leq n^{b_2 - 3b_1 \vee b_6} n^{3b_1 \vee b_6} = n^{b_2}$$

steps. This proves Equation (4.47) for $d \geq 3$ and similarly in $d = 2$, see Birkner et al. (2013). □

Lemma 4.8 (cf. Lemma 3.10 in Birkner et al. (2013))**.** *For $d \geq 2$ there exist constants $b, C > 0$ such that for every pair of bounded Lipschitz functions $f, g : \mathbb{R}^d \to \mathbb{R}$ with Lipschitz constants L_f and L_g respectively*

$$\begin{aligned}
&\left|\tilde{\mathbb{E}}^{\text{joint}}_{0,0}\left[f\left(\frac{\tilde{X}_n - n\tilde{\mu}}{\sqrt{n}}\right) g\left(\frac{\tilde{X}'_n - n\tilde{\mu}}{\sqrt{n}}\right)\right] - \tilde{\mathbb{E}}^{\text{ind}}_{0,0}\left[f\left(\frac{\tilde{X}_n - n\tilde{\mu}}{\sqrt{n}}\right) g\left(\frac{\tilde{X}'_n - n\tilde{\mu}}{\sqrt{n}}\right)\right]\right| \\
&\quad \leq C\left(1 + ||f||_\infty + L_f\right)\left(1 + ||g||_\infty + L_g\right) n^{-b},
\end{aligned}$$

where $\tilde{\mu} := \mathbb{E}[\tau_1]\vec{\mu}$ and $\vec{\mu}$ is as in Theorem 2.25.

Proof. The proof remains almost the same as in Birkner et al. (2013), using the separation lemma, Lemma 4.7, and a coupling of dependent and independent Ξ-chains, introduced in their Lemma 3.9. Furthermore, we do not have standard large deviation

estimates. Instead, we need use the Markov inequality together with an estimate on the expectation of the product of mixing random variables, e.g. Theorem 17.2.2 in Ibragimov and Linnik (1971) to get

$$\begin{aligned}\tilde{\mathbb{P}}_{0,0}^{\text{joint}}\left(T_n^{\text{sim}} \geq Kn\right) &\leq e^{-Kn}\tilde{\mathbb{E}}_{0,0}^{\text{joint}}\left[\prod_{k=1}^{n} e^{\tau_k^{\text{joint}}}\right] \\ &\leq e^{-Kn}\sum_{k=1}^{n}\alpha_{2m}^{\frac{D}{D+2}}\tilde{\mathbb{E}}_{0,0}^{\text{joint}}\left[e^{\tau_k^{\text{joint}}}\right]^k \\ &\leq n\alpha_{2m}^{\frac{D}{D+2}}\exp\left(\log\tilde{\mathbb{E}}_{0,0}^{\text{joint}}\left[e^{\tau_k^{\text{joint}}}\right]n - Kn\right).\end{aligned}$$

Since the time increments τ_n^{sim} have exponential moments under the joint law by Equation (4.29), we can choose the constant $K > \log\tilde{\mathbb{E}}_{0,0}^{\text{joint}}[e^{\tau_1^{\text{sim}}}]$ to get exponential tail bounds

$$\tilde{\mathbb{P}}_{0,0}^{\text{joint}}(T_n^{\text{sim}} \geq Kn) \leq Ce^{-cn}. \tag{4.48}$$

This shows that Equation (3.47) in Birkner et al. (2013) holds in our case and completes the proof by following their steps. □

Proof of Theorem 2.26. As in Birkner et al. (2013) we show that for any bounded Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$

$$\left|E_\xi\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] - \Phi(f)\right| \xrightarrow{n\to\infty} 0 \quad \text{for } \tilde{\mathbb{P}}\text{-a.e. } \xi^K, \tag{4.49}$$

where $\Phi(f) = \int f(x)\Phi(\mathrm{d}x)$ and Φ is a non-trivial d-dimensional normal law and $\vec{\mu}$ is as in Theorem 2.25. We do this by finding an upper bound of different terms and show that each of these terms converge individually as $n \to \infty$. Let L_f be the Lipschitz constant of f and write

$$\begin{aligned}&\left|E_\xi\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] - \Phi(f)\right| \\ &\quad\leq \left|E_\xi\left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}}\right)\right] - E_\xi\left[f\left(\frac{\tilde{X}_{[n/\mathbb{E}\tau_1]} - n\vec{\mu}}{\sqrt{n/\mathbb{E}\tau_1}}\frac{1}{\sqrt{\mathbb{E}\tau_1}}\right)\right]\right| &(4.50)\\ &\qquad+ \left|E_\xi\left[f\left(\frac{\tilde{X}_{[n/\mathbb{E}\tau_1]} - n\vec{\mu}}{\sqrt{n/\mathbb{E}\tau_1}}\frac{1}{\sqrt{\mathbb{E}\tau_1}}\right)\right] - \Phi(f)\right|, &(4.51)\end{aligned}$$

where we write $[n]$ for the integer part of an index n. In Term (4.50) we split the position of the random walk according to

$$\frac{X_n - n\vec{\mu}}{\sqrt{n}} = \frac{X_n - \tilde{X}_{V_n}}{\sqrt{n}} + \frac{\tilde{X}_{V_n} - \tilde{X}_{[n/\mathbb{E}[\tau_1]]}}{\sqrt{n}} + \frac{\tilde{X}_{[n/\mathbb{E}[\tau_1]} - n\vec{\mu}}{\sqrt{n/\mathbb{E}[\tau_1]}}\frac{1}{\sqrt{\mathbb{E}[\tau_1]}}, \tag{4.52}$$

where

$$V_n := \max\{k > 0 : T_k \leq n\}.$$

Conditioning on three suitable events (see terms (i)-(iii) in Equation (4.53)) and using the properties of Lipschitz functions we get for constants $0 < \gamma' < 1/2 < \beta < 1$ and

$\gamma \in (\beta/2, 1/2)$

$$\begin{aligned}
&\left| E_\xi \left[f\left(\frac{X_n - n\vec{\mu}}{\sqrt{n}} \right) \right] - E_\xi \left[f\left(\frac{\tilde{X}_{[n/\mathbb{E}\tau_1]} - n\vec{\mu}}{\sqrt{n/\mathbb{E}\tau_1}} \frac{1}{\sqrt{\mathbb{E}\tau_1}} \right) \right] \right| \qquad (4.53) \\
&\leq \; L_f \frac{n^{\gamma'} + n^{\gamma}}{\sqrt{n}} + 2||f||_\infty \left(\underbrace{P_\xi \left(\left\| X_n - \tilde{X}_{V_n} \right\| \geq n^{\gamma'} \right)}_{(i)} + \underbrace{P_\xi \left(\left| V_n - \frac{n}{\mathbb{E}\tau_1} \right| \geq n^{\beta} \right)}_{(ii)} \right) \\
&\quad + 2||f||_\infty \underbrace{P_\xi \left(\sup_{|k - n/\mathbb{E}\tau_1| < n^\beta} \left\| \tilde{X}_k - \tilde{X}_{[n/\mathbb{E}\tau_1]} \right\| \geq n^{\gamma} \right)}_{(iii)}.
\end{aligned}$$

For term (i) in Equation (4.53) we need to look at polynomial tails instead of logarithmic tails as in the original paper. For every $\epsilon > 0$ we get from the Markov inequality and stationarity of the joint regeneration increments under the annealed law that

$$\tilde{\mathbb{P}} \left(P_\xi \left(\tau_n \geq n^{\gamma'} \right) > \epsilon \right) \leq \frac{1}{\epsilon} \tilde{\mathbb{E}} \left(P_\xi \left(\tau_n \geq n^{\gamma'} \right) \right) = \frac{1}{\epsilon} \tilde{\mathbb{P}} \left(\tau_n \geq n^{\gamma'} \right) \leq \frac{C}{\epsilon} e^{-cn^{\gamma'}}$$

is summable in n for every $\epsilon > 0$. We conclude by Borel-Cantelli that

$$P_\xi \left(\left\| X_n - \tilde{X}_{V_n} \right\| \geq n^{\gamma'} \right) \leq P_\xi \left(\tau_n \geq n^{\gamma'} \right) \xrightarrow{n \to \infty} 0 \text{ for } \tilde{\mathbb{P}} - \text{a.e. } \xi^K, \qquad (4.54)$$

since $||X_n - \tilde{X}_{V_n}|| \leq \tau_n$. For term (ii) in Equation (4.53) we proceed as in Birkner et al. (2013) and use their equations (3.64) and (3.67) to show almost sure convergence. Here we only need to remark that their Equation (3.64) holds by a law of the iterated logarithm for stationary mixing sequences, e.g. Theorem 6.4 in Rio (2013). For term (iii) in Equation (4.53) we use Equations (3.68) and (3.69) from Birkner et al. (2013), where Equation (3.69) holds by Inequality (I.6) in Rio (2013). Therefore, Term (4.50) converges for $\tilde{\mathbb{P}}$-a.e. ξ^K to 0 as $n \to \infty$.

For Term (4.51), we choose Φ to be a rescaled normal law $\Phi(f(\cdot)) := \tilde{\Phi}(f(\cdot/\sqrt{\mathbb{E}\tau_1}))$ and the almost sure convergence follows from Lemma 4.8 together with Lemma 3.12 in Birkner et al. (2013). Lemma 4.8 is used to control the covariance of two walks under $\mathbb{P}^{\text{joint}}$ by comparing it to the variance of a single walker under the annealed law. Then Lemma 3.12 turns this into a quenched CLT for $\tilde{X}_n$. The proof of their Lemma 3.12 holds true in our case as there is a moderate deviation principle for stationary, strongly mixing sequences, Theorem 4 in Merlevède et al. (2009). We complete the proof of Theorem 2.26 with the remark that any continuous bounded function can be approximated by bounded Lipschitz functions in a locally uniform way and Equation (4.49) holds for any continuous bounded function. □

5 Proofs for exclusion dynamics in recurrent environments

A survey of random walks in recurrent environments is given in the lecture notes of Zeitouni (2004). We mostly use his notation in this paper. Define the *potential* of the environment ω as the random process $(V(x))_{x\in\mathbb{Z}}$ such that for all $x \in \mathbb{Z}$

$$V(x) = \begin{cases} \sum_{i=1}^{x} \log \rho_i & \text{if } x \geq 1 \\ 0 & \text{if } x = 0 \\ \sum_{i=x+1}^{0} -\log \rho_i & \text{if } x \leq -1 \end{cases}, \tag{5.1}$$

where $\rho_x = (1-\omega_x)/\omega_x$.

Lemma 5.1. *The Markov chain $(\mathbf{S}(n))_{n\in\mathbb{N}}$ on the state space $\mathcal{L}$ with transition probabilities as in Equation (3.9) has an invariant reversible measure π which is given by*

$$\pi(\mathbf{x}) = \prod_{j=1}^{N} \left[e^{-V(x_j)} + e^{-V(x_j-1)} \right]$$

for any $\mathbf{x} \in \mathcal{L}$.

Proof. The reversible measure $\tilde{\pi}$ for the 1-particle system is

$$\tilde{\pi}(x) = \frac{1}{\omega_x} e^{-V(x)} = e^{-V(x)} + e^{-V(x-1)}$$

for any $x \in \mathbb{Z}$. The reversible measure for N-particles on the same environment with state space $\mathbb{Z}^N$ is the product measure

$$\pi(\mathbf{x}) = \prod_{j=1}^{N} \tilde{\pi}(x_j),$$

since the detailed balance condition still holds in every coordinate and the remaining terms are a constant factor on both sides. Restricting the state space from $\mathbb{Z}^N$ to $\mathcal{L}$ does not change the reversible measure, since the restriction only affects diagonal terms $p(\mathbf{x},\mathbf{x}) = p(\mathbf{x})$ for $\mathbf{x} \in \mathbb{Z}^N$, which do not appear in the detailed balance equation. □

5.1 Valleys

We define a *valley* of the potential V as a triple (a,b,c), with $a < b < c$ and such that V attains a local maximum at a and c and a local minimum at b,

$$\begin{aligned} V(a) &= \max_{a\leq x\leq b} V(x), \\ V(b) &= \min_{a\leq x\leq c} V(x) \text{ and} \\ V(c) &= \max_{b\leq x\leq c} V(x). \end{aligned}$$

A valley (a,b,c) has *depth*

$$d(a,b,c) := (V(a) - V(b)) \wedge (V(c) - V(b)), \tag{5.2}$$

where we denote the minimum of two numbers a, b by $a \wedge b := \min\{a,b\}$. We can obtain a smaller valley (a,b',c') such that $a < b' < c' < b$ by a *left refinement*, if the new valley is such that

$$V(c') - V(b') = \max_{a\leq b''<c''\leq b} V(c'') - V(b'').$$

The same procedure also allows us to define a *right refinement*. By a finite sequence of refinements of any valley (a_0^n, b_0^n, c_0^n), with $a_0^n < 0 < c_0^n$ and depth $d(a_0^n, b_0^n, c_0^n) \geq \log n$ for any $n \in \mathbb{N}$, we can find the smallest valley (a^n, b^n, c^n) with depth $d(a^n, b^n, c^n) \geq \log n$ such that $a^n < 0 < c^n$. Denote by $(a_\delta^n, b_\delta^n, c_\delta^n)$ the smallest valley with $a_\delta^n < 0 < c_\delta^n$, while $d_{(a_\delta^n, b_\delta^n, c_\delta^n)} \geq (1+\delta)\log n$. Fix $\delta > 0$, $J > 0$ and define the event

$$A_n^{J,\delta} := \left\{ \begin{array}{ll} \omega \in \Omega : & 1)\ b^n = b_\delta^n, \\ & 2)\ \text{any refinement } (a,b,c) \text{ of } (a_\delta^n, b_\delta^n, c_\delta^n) \text{ s.t. } b \neq b^n \\ & \qquad \text{has depth } d_{(a,b,c)} < (1-\delta)\log n, \\ & 3)\ |a_\delta^n| + |c_\delta^n| \leq J \log^2 n, \\ & 4)\ \min_{k \in [a^n, c^n] \setminus [b^n - \delta \log^2 n, b^n + \delta \log^2 n]} \{V(k) - V(b^n)\} \\ & \qquad > \delta^3 \log n \end{array} \right\}.$$

By the invariance principle and the properties of Brownian motion

$$\lim_{\delta \to 0} \lim_{J \to \infty} \lim_{n \to \infty} \mathsf{P}\left(A_n^{J,\delta}\right) = 1. \tag{5.3}$$

Note that all claims for S_j, $1 \leq j \leq N$, with $b^n > 0$ and $\mathbf{S}(0) = \mathbf{z}$ hold also for S_{N-j+1} with $b^n < 0$ and $\mathbf{S}(0) = (-z_N, \ldots, -z_1)$, since the reflected environment ω such that $\omega_x = \omega_{-x}$ for all $x \in \mathbb{Z}$ is recurrent and strictly elliptic as well. Therefore we can assume $b^n > 0$ without loss of generality.

5.2 Localization for N-particle exclusion processes

After defining hitting times for single particles of the process

$$\tau_{j,x} = \inf\{n \geq 0 : S_j(n) = x\}$$

for some $j \in \{1, \ldots, N\}$ and $x \in \mathbb{Z}$, we are ready to state the next lemma.

Lemma 5.2. *Let the Markov chain $(\mathbf{S}(n))_{n \in \mathbb{N}}$ start at vertex $\mathbf{z} = \mathbf{S}(0)$ and choose $z_1 = 0$. Fix $\delta > 0$ small enough and J, k large enough such that $\omega \in A_n^{J,\delta}$ for all $n \geq k$, then*

$$P_\omega^z\left(\tau_{j,b^n} < \tau_{1,a_\delta^n} \wedge \tau_{N,c_\delta^n}\right) \xrightarrow{n \to \infty} 1, \tag{5.4}$$

where $a \wedge b := \min\{a, b\}$.

Proof. Assume without loss of generality $a^n < 0 < b^n$. Take n large enough such that $\mathbf{z} \in (a^n, b^n)^N$. Since the minimum b^n is to the right of all particles in their initial configuration $\mathbf{z}$, $b^n > z_N$, the leftmost particle S_1 will be the last of all particles to reach the minimum at b^n, i.e. $\tau_{j,b^n} \leq \tau_{1,b^n}$ for all $j \in \{1, \ldots, N\}$. Thus, we only work with the first particle S_1, and Equation (5.4) follows directly from that for all other particles $j > 1$. Define the time of the first return to the starting position $\mathbf{z}$ as $\tau_z^+ = \inf\{n > 0 : \mathbf{S}(n) = \mathbf{z}\}$. Let $\tilde{\tau}_A$ and $\tilde{\tau}_z^+$ have the same laws as τ_A and τ_z^+, except that the Markov chain $(\mathbf{S}(n))_{n \in \mathbb{N}}$ is reflected at the sets $\{\mathbf{x} \in \mathcal{L} : x_1 = a_\delta^n\}$ and $\{\mathbf{x} \in \mathcal{L} : x_N = c_\delta^n\}$. We now consider the Markov chain on the finite graph $\tilde{\mathcal{G}}$ with vertices $\tilde{\mathcal{L}} = \mathcal{L} \cap [a_\delta^n, c_\delta^n]^N$. Note that the events $\{\tilde{\tau}_{1,b^n} < \tilde{\tau}_{1,a_\delta^n} \wedge \tilde{\tau}_{N,c_\delta^n}\}$ and $\{\tau_{1,b^n} < \tau_{1,a_\delta^n} \wedge \tau_{N,c_\delta^n}\}$ have the same probability. If we can show

$$\frac{P_\omega^z\left(\tilde{\tau}_{1,a_\delta^n} \wedge \tilde{\tau}_{N,c_\delta^n} < \tilde{\tau}_z^+\right)}{P_\omega^z\left(\tilde{\tau}_{1,b^n} < \tilde{\tau}_z^+\right)} \xrightarrow{n \to \infty} 0, \tag{5.5}$$

then by Equation (2.2) the statement of the lemma, Equation (5.4), follows.

Let $\mathcal{R}_\omega(\mathbf{a} \leftrightarrow B)$ be the effective resistance between the point $\mathbf{a} \in \tilde{\mathcal{L}}$ and the set $B \subset \tilde{\mathcal{L}}$. Since we are on a finite graph $\tilde{\mathcal{G}}$, we can apply Equation (2.3) to get

$$\begin{aligned}
&\frac{P_\omega^z\left(\tilde{\tau}_{1,a_\delta^n} \wedge \tilde{\tau}_{N,c_\delta^n} < \tilde{\tau}_z^+\right)}{P_\omega^z\left(\tilde{\tau}_{1,b^n} < \tilde{\tau}_z^+\right)} \\
&\quad \leq \frac{P_\omega^z\left(\tilde{\tau}_{1,a_\delta^n} < \tilde{\tau}_z^+\right) + P_\omega^z\left(\tilde{\tau}_{N,c_\delta^n} < \tilde{\tau}_z^+\right)}{P_\omega^z\left(\tilde{\tau}_{1,b^n} < \tilde{\tau}_z^+\right)} \\
&\quad = \frac{\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\})}{\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = a_\delta^n\})} + \frac{\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\})}{\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_N = c_\delta^n\})}.
\end{aligned} \tag{5.6}$$

By Rayleigh's principle, e.g. Theorem 1.29 in Grimmett (2010), the effective resistance is non-increasing in edge resistances of the electrical network. Therefore, we may remove edges until only a single path γ remains to connect $\{\mathbf{a}\}$ and B and thus get an upper bound on the effective resistance between them. We then write $\mathbf{a} \overset{\gamma}{\leftrightarrow} B$, if $\mathbf{a}$ is connected to B by the path γ. Also, we may add edges until the vertex set is the whole lattice $\mathbb{Z}^N$ to get a lower bound.

Let $\gamma = \{\mathbf{y}^0, \dots, \mathbf{y}^M\}$ be the unique path connecting $\mathbf{y}^0$ with some end point $\mathbf{y}^M$, $M \in \mathbb{N}$. Then the effective resistance along the path is the sum of the edge resistances along the path,

$$\begin{aligned}
\mathcal{R}_\omega^{\text{single path}}(\gamma) &= \sum_{i=0}^{M-1} \left(\pi(\mathbf{y}^i) p(\mathbf{y}^i, \mathbf{y}^{i+1})\right)^{-1} \\
&= \sum_{i=0}^{M-1} \frac{1}{p(\mathbf{y}^i, \mathbf{y}^{i+1})} \prod_{j=1}^{N} \left[e^{-V(y_j^i)} + e^{-V(y_j^i - 1)}\right]^{-1}.
\end{aligned} \tag{5.7}$$

Note that by Assumption (A1) all transition probabilities are bounded

$$\kappa/N < p(\mathbf{x}, \mathbf{y}) < 1 - \kappa/N \qquad \text{for all } \mathbf{x} \overset{\mathcal{L}}{\sim} \mathbf{y}.$$

Thus, we may write for any path γ such that $\mathbf{y}^M \in \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\}$

$$\begin{aligned}
\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\}) &\leq \mathcal{R}_\omega^{\text{single path}}\left(\mathbf{z} \overset{\gamma}{\leftrightarrow} \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\}\right) \\
&\leq \sum_{i=0}^{M-1} \exp\left(\sum_{j=1}^{N} V\left(y_j^i\right)\right) \frac{N(1-\kappa)^N}{\kappa}.
\end{aligned} \tag{5.8}$$

From here on, we need to treat the case $L = \infty$ and the spider random walk, $L < \infty$, separately. In the first case any two particles can cross high parts of the potential separately, so the particle system behaves similar to N independent particles in the same potential. In the second case, all particles have to cross obstacles together and the particle system behaves similar to a single particle with a slowdown in time by a factor of N, as we will see in Lemma 5.3. This is equivalent to an increase of the height of the potential by a factor of N, see for example Equation (5.15) versus Equation (5.19).

Case $L = \infty$: For the estimate of the resistance along a single path we need to choose the path γ connecting $\mathbf{z}$ with $\{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = b^n\}$ such that the effective resistance does not grow too fast and behaves similar to the resistance of N independent particles in the same potential. A good choice is to send the N-th particle to the minimum b^n first, then send the $(N-1)$-th particle and continue until the first particle reaches the bottom of the valley. This ensures that at most one particle has to cross

any local maximum of V on the interval $[z_1, b^n]$ at the same time. In particular, choose $\gamma = \{\mathbf{y}^0, \dots, \mathbf{y}^M\}$, such that $\mathbf{y}^0 = \mathbf{z}$ and $\mathbf{y}^M = \beta^n$, where the final configuration is

$$\beta^n := (b^n, b^n + 1, \dots, b^n + (N-1)). \tag{5.9}$$

Denote by $l_j := |b^n + (j-1) - z_{N-j+1}|$ the distance that the $(N-j+1)$-th particle has to travel to reach y^M_{N-j+1} and set

$$\mathbf{y}^i := \mathbf{y}^{i-1} + \sum_{k=1}^{N} \mathbf{e}_k \mathbb{1}_{\{\sum_{m=1}^{k-1} l_m < i \le \sum_{m=1}^{k} l_m\}}, \tag{5.10}$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_N)$ is the canonical basis of $\mathbb{Z}^N$. Then the number of steps in γ is bounded by

$$M \le N|b^n + N - z_1| \le N(J \log^2 n + N) \le 2N^2 J \log^2 n.$$

Using Equation (5.8) we have

$$\begin{aligned}
&\mathcal{R}_\omega^{\text{single path}}\left(\mathbf{z} \overset{\gamma}{\leftrightarrow} \{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = b^n\}\right) \\
&\quad \le \frac{2N^3 J(1-\kappa)^N \log^2 n}{\kappa} \exp\left(\max_{\mathbf{y}\in\gamma} \sum_{j=1}^{N} V(y_j)\right) \\
&\quad \le \frac{2N^3 J(1-\kappa)^N \log^2 n}{\kappa} \exp\left(\max_{x\in[0,b^n+N]} V(x) + \sum_{j=1}^{N-1} V(z_j) \vee V(b^n + j)\right).
\end{aligned} \tag{5.11}$$

The last inequality holds, since the maximum of the sum of the potential over all particles is attained when one particle is at the local maximum while all other particles are still at their respective starting positions z_j or already at their respective end positions β^n_j. Finally, by ellipticity of ω we can bound

$$\max_{x\in(b^n,b^n+N]} V(x) + \sum_{j=1}^{N-1} V(z_j) \vee V(b^n + j) \le 2(N-1) z_N \log\left(\frac{1-\kappa}{\kappa}\right).$$

Defining the constant

$$C_1(N, \kappa, \mathbf{z}) := \frac{2N^3 J(1-\kappa)^N}{\kappa} \left(\frac{1-\kappa}{\kappa}\right)^{2(N-1)z_N},$$

we obtain the desired upper bound on the resistance

$$\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}} : x_1 = b^n\}) \le C_1(N, \kappa, \mathbf{z}) \log^2 n \exp\left(\max_{x\in[0,b^n]} V(x)\right). \tag{5.12}$$

For the corresponding lower bound we work on the integer lattice graph with vertex set $\mathbb{Z}^N$. On the lattice graph we can calculate the voltage function $u : \mathbb{Z}^N \to \mathbb{R}$ explicitly, which gives us a direct bound on the effective resistance between $\mathbf{z}$ and $\{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = a^n_\delta\}$ by

$$\mathcal{R}_\omega^{\text{full lattice}}(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = a^n_\delta\}) = \frac{u(\mathbf{z})}{\sum_{\mathbf{y}\sim\mathbf{z}} i(\mathbf{z}, \mathbf{y})}. \tag{5.13}$$

Here $i(\mathbf{z}, \mathbf{y})$ is the unit electrical current flowing from $\mathbf{z}$ to $\mathbf{y}$ and the voltage u is such that $u \equiv 0$ on $\{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = a_\delta^n\}$. The same trick works for the effective resistance between $\mathbf{z}$ and $\{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_N = c_\delta^n\}$. The voltage u is a harmonic function on the electrical network. A harmonic function $f : \mathbb{Z}^N \to \mathbb{R}$ on a graph solves the equation

$$f(\mathbf{x}) = \sum_{\mathbf{y} \sim \mathbf{x}} f(\mathbf{y}) p(\mathbf{x}, \mathbf{y}). \tag{5.14}$$

A solution to Equation (5.14) on the integer lattice satisfying the boundary condition $u \equiv 0$ on $\{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = a_\delta^n\}$ is

$$u(x_1, \ldots, x_N) = \sum_{i=a_\delta^n}^{x_1 - 1} e^{V(i)}.$$

To check that u solves Equation (5.14), we calculate

$$\begin{aligned}
&\sum_{\mathbf{y} \sim \mathbf{x}} u(\mathbf{y}) p(\mathbf{x}, \mathbf{y}) \\
&\quad = \frac{1}{N} \sum_{i=1}^{N} u(\ldots, x_i + 1, \ldots) \omega_{x_i} + u(\ldots, x_i - 1, \ldots)(1 - \omega_{x_i}) \\
&\quad = \frac{1}{N} \left(N \sum_{i=a_\delta^n}^{x_1 - 1} e^{V(i)} - e^{V(x_1 - 1)} \omega_{x_1} + e^{V(x_1 - 2)} (1 - \omega_{x_1}) \right) \\
&\quad = u(\mathbf{x}).
\end{aligned}$$

With our expression for the voltage we can also bound the outgoing current of the start vertex $\mathbf{z}$ by a constant

$$\begin{aligned}
\sum_{\mathbf{y} \sim \mathbf{z}} i(\mathbf{z}, \mathbf{y}) &= \sum_{\mathbf{y} \sim \mathbf{z}} c(\mathbf{z}, \mathbf{y}) [u(\mathbf{z}) - u(\mathbf{y})] \\
&\leq 2N \sup_{\mathbf{y} \sim \mathbf{z}} c(\mathbf{z}, \mathbf{y}) [u(\mathbf{z}) - u(\mathbf{y})] \\
&\leq 2(1 - \kappa) \pi(\mathbf{z}) \max_{1 \leq k \leq N} e^{V(z_k - 1)} \\
&\leq 2(1 - \kappa) \pi(\mathbf{z}) \left(\frac{1 - \kappa}{\kappa} \right)^{z_N} \\
&=: C_2(N, \kappa, \mathbf{z}).
\end{aligned}$$

Consequently, it follows that

$$\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = a_\delta^n\}) \geq \frac{e^{V(a_\delta^n)}}{C_2(N, \kappa, \mathbf{z})}. \tag{5.15}$$

We get the same bound for the effective resistance $\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}}_\infty : x_1 = c_\delta^n\})$. By Equations (2.2) and (5.6) and both bounds from Equations (5.12) and (5.15) we find that

$$\begin{aligned}
\frac{P_\omega^z \left(\tilde{\tau}_{1, a_\delta^n} \wedge \tilde{\tau}_{N, c_\delta^n} < \tilde{\tau}_z^+ \right)}{P_\omega^z \left(\tilde{\tau}_{1, b^n} < \tilde{\tau}_z^+ \right)} &\leq C_3 \log^2 n \cdot \frac{\max_{x \in [0, b^n]} e^{V(x)} e^{-V(b^n)} e^{V(b^n)}}{e^{V(a_\delta^n) \wedge V(c_\delta^n)}} \\
&\leq C_3 \log^2 n \cdot \frac{\max_{x \in [0, b^n]} e^{V(x) - V(b^n)}}{e^{V(a_\delta^n) \wedge V(c_\delta^n) - V(b^n)}} \\
&\leq C_3 \log^2 n \cdot \frac{n}{n^{1 + \delta}} \xrightarrow{n \to \infty} 0,
\end{aligned} \tag{5.16}$$

where $C_3(N,\kappa,\mathbf{z}) := 2C_1(N,\kappa,\mathbf{z})C_2(N,\kappa,\mathbf{z})$. This proves the lemma for pure exclusion dynamics.

Case $L < \infty$: For the spider Markov chain, we proceed in the same way as before, now trying to get estimates on the resistances that are similar to the behaviour of a single particle in the potential with some scaling by a factor of N. For the lower bound on the resistance we take a path $\gamma = \{\mathbf{y}^0, \dots, \mathbf{y}^M\}$ such that $\mathbf{y}^0 = \mathbf{z}$, $\mathbf{y}^M = \beta^n$, where β^n is defined as in Equation (5.9). The intermediate points $0 < i < M$ are chosen such that all particles move with minimal possible distance between them,

$$\mathbf{y}^i = \mathbf{y}^{i-1} + \sum_{k=1}^{N} \mathbf{e}_k \mathbb{1}_{\{k=i-\lfloor i/N \rfloor + 1\}}, \tag{5.17}$$

where $\lfloor x \rfloor := \max\{i \in \mathbb{N} : i \le x\}$. Then by Equation (5.11)

$$\begin{aligned}
&\mathcal{R}_\omega^{\text{single path}}\left(\mathbf{z} \overset{\gamma}{\leftrightarrow} \{\mathbf{x} \in \tilde{\mathcal{L}}_L : x_1 = b^n\}\right) \\
&\quad\le \frac{2N^3 J(1-\kappa)^N \log^2 n}{\kappa} \exp\left(\max_{\mathbf{y}\in\gamma} \sum_{j=1}^{N} V(y_j)\right) \\
&\quad\le \frac{2N^3 J(1-\kappa)^N \log^2 n}{\kappa} \exp\left(\max_{x\in[0,b^n+N]} NV(x) + NL\log\left(\frac{1-\kappa}{\kappa}\right)\right) \\
&\quad\le C_1'(N,\kappa,L)(\log^2 n)\exp\left(\max_{x\in[0,b^n+N]} NV(x)\right),
\end{aligned} \tag{5.18}$$

where

$$C_1'(N,\kappa,L) := \frac{2N^3 J(1-\kappa)^N}{\kappa}\left(\frac{1-\kappa}{\kappa}\right)^{LN}.$$

For the estimate on the lower bound we identify all vertices in the electrical network with the same first coordinate x_1. There are $|\{\mathbf{x} \in \mathcal{L}_L : x_1 = k\}| \le (L-N+2)^N$ such vertices for all $k \in \mathbb{N}$. Then the new conductances c^{ident} between neighbouring identified vertices are bounded around $e^{-NV(x_1)}$, since all other particles stay close to the first one in the chosen path γ. In particular

$$\begin{aligned}
c^{\text{ident}}(k,k+1) &:= c\left(\{\mathbf{x} \in \mathcal{L}_L : x_1 = k\}, \{\mathbf{x} \in \mathcal{L}_L : x_1 = k+1\}\right) \\
&\le C_2'(N,\kappa,L)e^{-NV(k)},
\end{aligned}$$

where

$$C_2'(N,\kappa,L) := \frac{\kappa(L-N+2)^N}{(1-\kappa)^N}\left(\frac{1-\kappa}{\kappa}\right)^{LN}.$$

Then

$$\begin{aligned}
\mathcal{R}_\omega(\mathbf{z} \leftrightarrow \{\mathbf{x} \in \tilde{\mathcal{L}}_L : x_1 = a_\delta^n\}) &\ge \sum_{k=0}^{a_\delta^n} \left(c^{\text{ident}}(k,k+1)\right)^{-1} \\
&\ge \frac{1}{C_2'(N,\kappa,L)} e^{NV(a_\delta^n)}.
\end{aligned} \tag{5.19}$$

Overall, by the same calculation as in Equation (5.16)

$$\frac{P_\omega^z\left(\tilde{\tau}_{1,a_\delta^n} \wedge \tilde{\tau}_{N,c_\delta^n} < \tilde{\tau}_z^+\right)}{P_\omega^z\left(\tilde{\tau}_{1,b^n} < \tilde{\tau}_z^+\right)} \le C_3' \log^2 n \frac{Nn}{(Nn)^{1+\delta}} \xrightarrow{n\to\infty} 0,$$

where $C_3'(N,\kappa) := 2C_1'(N,\kappa,L)C_2'(N,\kappa,L)$. This proves the lemma for spider dynamics. □

Lemma 5.3. *Fix $\delta > 0$ small enough and J, k large enough such that $\omega \in A_n^{J,\delta}$ for all $n \geq k$. For any $\mathbf{z} \in \mathcal{L}$ such that $z_1 = 0$ and $j \in \{1, \dots, N\}$*

$$P_\omega^z\left(\tau_{j,b^n} > n\right) \xrightarrow{n\to\infty} 0 \text{ for } L = \infty \text{ and}$$
$$P_\omega^z\left(\tau_{j,b^n} > Nn\right) \xrightarrow{n\to\infty} 0 \text{ for } L < \infty.$$

Proof. Define the set $B_n = [a_\delta^n, c_\delta^n]^N$ and the hitting time $\tilde{\tau}_{j,b^n}$ of the reflected Markov chain on $\tilde{\mathcal{L}} = \mathcal{L} \cap B_n$ as in Lemma 5.2. Also, define the first return time on $\tilde{\mathcal{L}}$ as

$$\tilde{\tau}_x^+ = \left\{n > 0 : \tilde{\mathbf{S}}(n) = \mathbf{x}\right\}.$$

Since $P_\omega^z\left(\tilde{\tau}_{j,b^n} > n\right) \leq P_\omega^z\left(\tilde{\tau}_{1,b^n} > n\right)$, we may only consider the case $j = 1$. We use a technique due to Golosov (1984). Take n large enough. Define the number of times the Markov chain visits a site $\mathbf{x}$ before arriving at the bottom of the largest valley as $Z_x = \{k < \tilde{\tau}_{1,b^n} : \mathbf{S}(k) = \mathbf{x}\}$. Set $\lambda_x = P_\omega^x\left(\tilde{\tau}_{1,b^n} < \tilde{\tau}_x^+\right)$. Then the random variable Z_x is dominated by a geometric random variable, since for all $l \in \mathbb{N}$

$$P_\omega^z\left(Z_x = l\right) = P_\omega^z\left(\tilde{\tau}_x^+ < \tilde{\tau}_{1,b^n}\right)(1-\lambda_x)^{l-1}\lambda_x \leq (1-\lambda_x)^l\lambda_x.$$

Consequently $E_\omega^z[Z_x] \leq \lambda_x^{-1}$. On the finite graph $\tilde{\mathcal{G}}$ we have (e.g. by Proposition 9.5 in Levin et al. (2009)) that

$$\lambda_x^{-1} = \pi(\mathbf{x})\mathcal{R}_\omega(\mathbf{x} \leftrightarrow \{\mathbf{y} \in \tilde{\mathcal{L}} : y_1 = b^n\}).$$

Case $L = \infty$: Choose γ as in Equation (5.10), i.e. we take one particle at a time from x_j to the minimum of the valley at b^n. Then we get

$$\begin{aligned}
E_\omega^z[\tilde{\tau}_{1,b^n}] &= \sum_{\mathbf{x}\in B_n} P_\omega^z(\tilde{\tau}_x^+ < \tilde{\tau}_{1,b^n})E_\omega^x[Z_x] \\
&\leq \sum_{\mathbf{x}\in B_n} \pi(\mathbf{x})\mathcal{R}_\omega^{\text{single path}}\left(\mathbf{x} \overset{\gamma}{\leftrightarrow} \{\mathbf{y} \in \tilde{\mathcal{L}} : y_1 = b^n\}\right) \\
&\leq C_1|\gamma|\log^2 n \sum_{\mathbf{x}\in B_n} \exp\left(-\sum_{j=1}^N V(x_j) + \max_{x\in\gamma}\sum_{j=1}^N V(x_j)\right).
\end{aligned}$$

As before, $|\gamma| \leq 2N^2 J \log^2 n$ and $|B_n| \leq J^N(\log n)^{2N}$. Define a new constant $C_4 := C_4(N, \kappa, \mathbf{z}) := 2C_1(N, \kappa, \mathbf{z})N^2J^{N+1}$. The term in the exponent is only positive, if the maximum of the potential along the path γ is bigger than the potential $V(x_j)$ at the respective starting positions x_j. If this is the case, then on $\omega \in A_n^{J,\delta}$ the potential at the maximum can be at most $(1-\delta)\log n$ bigger than $V(x_j)$ and we get the bound

$$\begin{aligned}
E_\omega^z[\tilde{\tau}_{1,b^n}] &\leq C_1|\gamma|\log^2 n \sum_{x\in B_n} \exp\left(\max_{1\leq j\leq N}\left(-V(x_j) + \max_{x_j < y \leq b^n} V(y)\right)\right) \\
&\leq C_4(\log n)^{2(N+2)} e^{(1-\delta)\log n}.
\end{aligned}$$

By Markov's inequality we conclude that

$$P_\omega^z\left(\tilde{\tau}_{1,b^n} > n\right) \leq \frac{E_\omega^z[\tilde{\tau}_{1,b^n}]}{n} \leq C_4(\log n)^{2(N+2)} n^{-\delta} \xrightarrow{n\to\infty} 0. \tag{5.20}$$

Case $L < \infty$: For the spider case, choose the path γ as in Equation (5.17) and use the upper bound for the resistance from Equation (5.18). Then

$$E_\omega^z[\tilde{\tau}_{1,b^n}] \leq C_1'|\gamma||B_n|\log^2 n(Nn)^{1-\delta},$$

and again by Markov's inequality

$$P_\omega^z\left(\tilde{\tau}_{1,b^n} > Nn\right) \leq \frac{E_\omega^z\left[\tilde{\tau}_{1,b^n}\right]}{Nn} \leq C_4'(\log n)^{2(N+2)} N^{-\delta} n^{-\delta} \xrightarrow{n\to\infty} 0, \tag{5.21}$$

where $C_4' = 2C_1' N^2 J^{N+1}$. □

Now, we know that all N particles move to the bottom of the valley before time n and without hitting the boundary $\partial B_n := \{\mathbf{x} \in B_n : \exists \mathbf{y} \sim \mathbf{x} \text{ s.t. } \mathbf{y} \notin B_n\}$. We still need to show that once the N particles arrive at the bottom of the valley, they will not leave before time n.

Lemma 5.4. *Define $\beta^n = (b^n, b^n+1, \ldots, b^n+N-1)$. Then for any $j \in \{1,\ldots,N\}$*

$$\sup_{\omega\in A_n^{J,\delta}} \max_{k\leq n} P_\omega^{\beta^n}\left(\left|\frac{S_j(k)-b^n}{\log^2 n}\right| > \delta\right) \xrightarrow{n\to\infty} 0 \text{ if } L=\infty \text{ and}$$

$$\sup_{\omega\in A_n^{J,\delta}} \max_{k\leq Nn} P_\omega^{\beta^n}\left(\left|\frac{S_j(k)-b^n}{\log^2 n}\right| > \delta\right) \xrightarrow{n\to\infty} 0 \text{ if } L<\infty.$$

Proof. Let again $\tau_{\beta^n}^+ := \inf\{n>0 : \mathbf{S}(n) = \beta^n\}$ be the first return time to the starting vertex, but now we start the walk at $\beta^n = (b^n, b^n+1, \ldots, b^n+N-1)$. If we do not leave the set $B_n = [a_\delta^n, c_\delta^n]^N$ before time n, we may again restrict our attention to the finite sub-graph with vertices in B_n. By the detailed balance equation for any $\mathbf{x} \in B_n \cap \mathcal{L}$ such that $x_1 = a_\delta^n$ or $x_N = c_\delta^n$ we get the following estimate

$$\begin{aligned} P_\omega^{\beta^n}(\tau_x \leq n) &\leq \sum_{i=1}^n P_\omega^{\beta^n}(\mathbf{S}(i) = \mathbf{x}) \\ &= \sum_{i=1}^n \frac{\pi(\mathbf{x})}{\pi(\beta^n)} P_\omega^x(\mathbf{S}(i) = \beta^n) \\ &\leq n\frac{\pi(\mathbf{x})}{\pi(\beta^n)}. \end{aligned} \tag{5.22}$$

From this we get a bound on the time the Markov chain stays in the valley by

$$\begin{aligned} &P_\omega^{\beta^n}\left(\tau_{1,a_\delta^n} \wedge \tau_{N,c_\delta^n} \leq n\right) \\ &\quad\leq |B_n| \sup_{\{\mathbf{x}\in B_n : x_1 = a_\delta^n \text{ or } x_N = c_\delta^n\}} P_\omega^{\beta^n}(\tau_x \leq n) \\ &\quad\leq |B_n| \sup_{\{\mathbf{x}\in B_n : x_1 = a_\delta^n \text{ or } x_N = c_\delta^n\}} n\frac{\pi(\mathbf{x})}{\pi(\beta^n)}. \end{aligned} \tag{5.23}$$

Case $L=\infty$: The supremum in Equation (5.23) for $L=\infty$ is attained either at approximately $\mathbf{x} = (a_\delta^n, b^n+1, \ldots, b^n+N-1)$ or otherwise at approximately $\mathbf{x} = (b^n, b^n+1, \ldots, b^n+N-2, c_\delta^n)$,

$$\begin{aligned} &P_\omega^{\beta^n}\left(\tau_{1,a_\delta^n} \wedge \tau_{N,c_\delta^n} \leq n\right) \\ &\quad\leq |B_n|\left(\frac{1-\kappa}{\kappa}\right)^N n\exp\left(-V(a_\delta^n) \vee V(c_\delta^n) - (N-1)V(b^n) + NV(b^n)\right. \\ &\qquad\qquad \left. + \sum_{i=1}^{N-1}\left(V(b^n+i) - V(b^n)\right)\right) \\ &\quad\leq |B_n|\left(\frac{1-\kappa}{\kappa}\right)^N n\exp\left(-(1+\delta)\log n + N^2\log\left(\frac{1-\kappa}{\kappa}\right)\right) \\ &\quad\leq C(\log n)^N n^{-\delta} \xrightarrow{n\to\infty} 0. \end{aligned} \tag{5.24}$$

Thus, we may again consider the Markov chain reflected at $\{\mathbf{x} \in \mathcal{L} : x_1 = a_\delta^n \text{ or } x_N = c_\delta^n\}$ and denote it by $\tilde{\mathbf{S}}$. By the Carne–Varopoulos bound, Varopoulos (1985), we get

$$P_\omega^{\beta^n}\left(\tilde{S}(k) = \mathbf{x}\right) \leq \sqrt{\frac{\pi(\mathbf{x})}{\pi(\beta^n)}} \qquad \text{for any } k \in \mathbb{N}$$

and therefore

$$\begin{aligned}
&P_\omega^{\beta^n}\left(\left|\tilde{S}_j(k) - b^n\right| > \delta \log^2 n\right) \\
&\qquad \leq \sum_{\mathbf{x} \in B_n : |x_j - b^n| > \delta \log^2 n} P_\omega^{\beta^n}\left(\tilde{S}(k) = \mathbf{x}\right) \\
&\qquad \leq |B_n| \left(\frac{1-\kappa}{\kappa}\right)^N \max_{\mathbf{x} \in B_n : |x_j - b^n| > \delta \log^2 n} \exp\left(\frac{1}{2}\sum_{i=1}^N -V(x_i) + V(\beta_i^n)\right) \\
&\qquad \leq \left(J \log^2 n\right)^N \left(\frac{1-\kappa}{\kappa}\right)^{N^2} n^{-\delta^3/2}, \qquad (5.25)
\end{aligned}$$

which is uniform in k for all $k \leq n$. This proves the claim for $\tilde{S}_j$ and by Equation (5.24) also for S_j.

Case $L < \infty$: The bound in Equation (5.25) is in fact also uniform for all times $k \leq Nn$, provided that the Markov chain is confined to $B_n \cap \mathcal{L}_L$ until time Nn with high probability. Indeed, similar to Equation (5.24), the supremum is now attained for some $\mathbf{x} \in \mathcal{L}_L$ such that either $x_1 = a_\delta^n$ or $x_N = c_\delta^n$, while all other particles can be at most distance L away, thus

$$\begin{aligned}
&P_\omega^{\beta^n}\left(\tau_{1,a_\delta^n} \wedge \tau_{N,c_\delta^n} \leq Nn\right) \\
&\qquad \leq |B_n| \left(\frac{1-\kappa}{\kappa}\right)^N Nn \exp\left(-N\left(V(a_\delta^n) \vee V(c_\delta^n)\right) + NL \log\left(\frac{1-\kappa}{\kappa}\right)\right. \\
&\qquad\qquad \left. + NV(b^n) + \sum_{i=1}^{N-1}\left(V(b^n + i) - V(b^n)\right)\right) \\
&\qquad \leq C(\log n)^N (Nn)^{-\delta} \xrightarrow{n \to \infty} 0 \qquad (5.26)
\end{aligned}$$

which completes the proof for $L < \infty$. □

Proof of Theorem 3.23. We choose the starting position of the Markov chain $\mathbf{z} \in \mathcal{L}$ without loss of generality such that $z_1 = 0$. By translation invariance of the environment the results hold for all other choices of $\mathbf{z}$, but the definition of b^n would be for valleys centred around z_1 instead of 0. Fix $\delta < \eta/2$ and take J and k large enough such that $\omega \in A_n^{J,\delta}$ for all $n \geq k$. By putting Lemma 5.3 and Lemma 5.4 together we get for $L = \infty$

$$P_\omega^z\left(\left|\frac{S_j(n) - b^n}{\log^2 n}\right| > \eta\right) \leq P_\omega^z\left(\tau_{j,b^n} > n\right) + \max_{k \leq n} P_\omega^{\beta^n}\left(\left|\frac{S_j(k) - b^n}{\log^2 n}\right| > \delta\right)$$

and for $L < \infty$

$$P_\omega^z\left(\left|\frac{S_j(n) - b^n}{\log^2 n}\right| > \eta\right) \leq P_\omega^z\left(\tau_{j,b^n} > Nn\right) + \max_{k \leq Nn} P_\omega^{\beta^n}\left(\left|\frac{S_j(k) - b^n}{\log^2 n}\right| > \delta\right).$$

Therefore, choosing $b(n;\omega) = b^n$, we prove the claim by using Equation (5.3) and taking the limits. □

Remark 5.5. The result in Theorem 3.23 holds for $L = \infty$ in fact not only if the number of particles N is finite, but also if it grows slowly enough. For example, choose $N = \rho\sqrt{\log\log n}$ for some constant $\rho > 0$ and the initial configuration $\mathbf{z}(N) = (0, 1, \dots, N-1)$ such that $z_N = N - 1$. Then for large n

$$\begin{aligned} C_2(N(n), \kappa, \mathbf{z}(N(n))) &< C_1(N(n), \kappa, \mathbf{z}(N(n))) \\ &\leq \frac{2N^3(n)J}{\kappa^2} \exp\left(2N^2(n)\log\left(\frac{1-\kappa}{\kappa}\right)\right) \\ &= \frac{2J\rho^3}{\kappa^2}(\log n)^{2\rho(\log(1-\kappa)-\log\kappa)}(\log\log n)^{3/2}. \end{aligned}$$

Consequently, the constants $C_3 = 2C_1C_2$ and $C_4 = 2C_1N^2J^{N+1}$ grow slower than polynomial and both Equations (5.16) and (5.21) still hold. By the same argument Equation (5.25) and thus also Theorem 3.23 hold. For $L < \infty$ we can choose both $N, L \in \mathcal{O}(\sqrt{\log\log n})$ and get the same results.

5.3 The stationary limit law

Proof of Lemma 3.25. Part (i): The joint random walk of N independent particles in the same realization of the environment ω is recurrent by Corollary 1.4 in Gantert et al. (2014). The Markov chain $(\mathbf{S}(n))_{n\in\mathbb{N}}$ lives on a sub-graph of $\mathbb{Z}^N$ and is therefore recurrent as well.

Part (ii): Since $(\mathbf{S}(n))_{n\in\mathbb{N}}$ is recurrent and irreducible, every state is visited eventually. In particular for any $M \leq L$ we visit a state $\mathbf{y} \in \mathbb{Z}^N$ such that $|y_1 - y_N| \geq M$ in finite time and return infinitely often.

□

Proof of Theorem 3.26. We define the *infinitely deep well* (Golosov (1984)) or *infinite valley* (Gantert et al. (2010)) as the random process $(\bar{V}(x))_{x\in\mathbb{Z}}$, which has the law of the environment $(V(x))_{x\in\mathbb{Z}}$ conditioned to stay non-negative for $x > 0$ and conditioned to stay strictly positive for $x < 0$. This process is defined by a limit procedure. It can be found together with a proof of existence in Golosov (1984) or Bertoin (1993). Denote by $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ the Markov chain on the environment $\bar{\omega}$, which corresponds to the potential $(\bar{V}(x))_{x\in\mathbb{Z}}$, where the relation between the environment and the potential is defined as in Equation (5.1).

The Markov chain on the original environment ω is null-recurrent, which is also true for N independent particles on the same copy of the environment, see Gantert et al. (2014). However, the Markov chain in the infinite well is positive recurrent, i.e.

$$\sum_{x\in\mathbb{Z}} e^{-\bar{V}(x)} < \infty,$$

which is due to Golosov (1984). Denote by $\bar{p}$, $\bar{c}$ and $\bar{\pi}$ the transition probabilities, conductances and invariant measure corresponding to the environment $\bar{\omega}$. We have the same result for our N-particle Markov chain

$$\begin{aligned} \sum_{(\mathbf{x},\mathbf{y})\in\mathcal{L}\times\mathcal{L}} \bar{c}(\mathbf{x},\mathbf{y}) &\leq \sum_{(\mathbf{x},\mathbf{y})\in\mathbb{Z}^N\times\mathbb{Z}^N} \bar{\pi}(\mathbf{x})\bar{p}(\mathbf{x},\mathbf{y}) \\ &\leq (1-\kappa)\left(\frac{1-\kappa}{\kappa}\right)^N \sum_{\mathbf{x}\in\mathbb{Z}^N}\prod_{i=1}^N e^{-\bar{V}(x_i)} \\ &= (1-\kappa)\left(\frac{1-\kappa}{\kappa}\right)^N \left(\sum_{x\in\mathbb{Z}} e^{-\bar{V}(x)}\right)^N < \infty. \end{aligned}$$

This implies that the N-particle process $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ is positive recurrent in the infinitely deep well potential $(\bar{V}(x))_{x\in\mathbb{Z}}$ for both pure exclusion and spider dynamics. The process is also irreducible and aperiodic and consequently the law of $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ converges for fixed $\bar{\omega}$ to some limit law $\varphi(\bar{\omega})$. Denote by $(\theta_y)_{y\in\mathbb{Z}}$ the shift operator, i.e. for any $\omega\in\Omega$ and $x\in\mathbb{Z}$ we have $\theta_y\omega_x=\omega_{x+y}$. It was shown by Golosov (1984), Lemma 4, that the finite dimensional distributions of the environment seen from the bottom of the deepest valley $\theta_{b^n}\omega$ converge to those of the infinitely deep well potential $\bar{\omega}$. In the proof of his theorem, part (ii), he combines these facts and thus shows convergence of finite dimensional distributions of $(\mathbf{S}(k+n)-\mathbf{b}(k;\omega))_{n\in\mathbb{N}}$ to those of $(\mathbf{Y}(n))_{n\in\mathbb{N}}$ as $k\to\infty$ under the annealed measure. □

References

M. Aizenman, H. Kesten, and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation. *Comm. Math. Phys.*, 111(4):505–531, 1987.

E. D. Andjel. The asymmetric simple exclusion process on $\mathbb{Z}^d$. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 58(3):423–432, 1981.

S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.*, 43(4):1866–1891, 07 2015.

S. Andres, A. Chiarini, J.-D. Deuschel, and M. Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *ArXiv e-prints*, Feb. 2016.

R. Arratia. Symmetric exclusion processes: A comparison inequality and a large deviation result. *Ann. Probab.*, 13(1):53–61, 02 1985.

M. T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.*, 32 (4):3024–3084, 10 2004.

N. Berger and M. Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probability Theory and Related Fields*, 137(1):83–120, 2007.

J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and lévy processes. *Stochastic Processes and their Applications*, 47(1):17 – 35, 1993.

G. D. Birkhoff. Proof of the ergodic theorem. *Proceedings of the National Academy of Sciences*, 17(12):656–660, 1931.

M. Birkner, J. Černý, A. Depperschmidt, and N. Gantert. Directed random walk on the backbone of an oriented percolation cluster. *Electron. J. Probab.*, 18:no. 80, 35, 2013.

E. Bolthausen. Random walks in random environment. Lecture notes for workshop "Interacting particle systems, statistical mechanics and probability theory" at Institut Henri Poincaré, UPMC, 2008.

E. Bolthausen and I. Goldsheid. Recurrence and transience of random walks in random environments on a strip. *Communications in Mathematical Physics*, 214(2):429–447, 2000.

E. Bolthausen and I. Goldsheid. Lingering random walks in random environment on a strip. *Communications in Mathematical Physics*, 278(1):253–288, 2008.

E. Bolthausen and A.-S. Sznitman. On the static and dynamic points of view for certain random walks in random environment. *Methods Appl. Anal.*, 9(3):345–376, 09 2002.

R. C. Bradley. A caution on mixing conditions for random fields. *Statistics & Probability Letters*, 8(5):489–491, 1989.

R. C. Bradley. Basic properties of strong mixing conditions. A survey and some open questions. *Probab. Surv.*, 2:107–144, 2005. Update of, and a supplement to, the 1986 original.

L. Chayes and T. M. Liggett. One dimensional nearest neighbor exclusion processes in inhomogeneous and random environments. *Journal of Statistical Physics*, 129(2): 193–203, 2007.

S. D. Conte and C. W. D. Boor. *Elementary Numerical Analysis: An Algorithmic Approach*. McGraw-Hill Higher Education, 3rd edition, 1980. ISBN 0070124477.

A. De Masi and P. A. Ferrari. Self-diffusion in one-dimensional lattice gases in the presence of an external field. *Journal of Statistical Physics*, 38(3):603–613, 1985.

A. De Masi, P. A. Ferrari, S. Goldstein, and W. D. Wick. An invariance principle for reversible Markov processes. Applications to random motions in random environments. *Journal of Statistical Physics*, 55(3):787–855, 1989.

A. Dembo, Y. Peres, and O. Zeitouni. Tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.*, 181(3):667–683, 1996.

D. Dolgopyat and I. Goldsheid. Local limit theorems for random walks in a 1D random environment. *Archiv der Mathematik*, 101(2):191–200, 2013.

N. Enriquez, C. Sabot, and O. Zindy. Limit laws for transient random walks in random environment on $\mathbb{Z}$. *Annales de l'institut Fourier*, 59(6):2469–2508, 2009.

C. Gallesco, S. Müller, and S. Popov. A note on spider walks. *ESAIM: PS*, 15:390–401, 2011a.

C. Gallesco, S. Müller, S. Popov, and M. Vachkovskaia. Spiders in random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 8:129–147, 2011b.

N. Gantert, Y. Peres, and Z. Shi. The infinite valley for a recurrent random walk in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, 46(2):525–536, 05 2010.

N. Gantert, M. Kochler, and F. Pène. On the recurrence of some random walks in random environment. *ALEA Lat. Am. J. Probab. Math. Stat.*, 11(1):483–502, 2014.

I. Y. Goldsheid. Simple transient random walks in one-dimensional random environment: the central limit theorem. *Probability Theory and Related Fields*, 139(1): 41–64, 2007.

I. Y. Goldsheid. Linear and sub-linear growth and the CLT for hitting times of a random walk in random environment on a strip. *Probability Theory and Related Fields*, 141(3):471–511, 2008.

A. O. Golosov. Limit distributions for random walks in random environments. *Soviet Mathematics: Doklady*, 28:18–22, 1983.

A. O. Golosov. Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.*, 92(4):491–506, 1984.

G. Grimmett. *Probability on graphs*, volume 1 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-14735-4. Random processes on graphs and lattices.

G. Grimmett and P. Hiemer. Directed percolation and random walk. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 273–297. Birkhäuser Boston, Boston, MA, 2002.

I. A. Ibragimov and Y. V. Linnik. *Independent and stationary sequences of random variables*. Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman.

M. D. Jara and C. Landim. Quenched non-equilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder. *Ann. Inst. H. Poincaré Probab. Statist.*, 44(2):341–361, 04 2008.

P. Jung. Extremal reversible measures for the exclusion process. *Journal of Statistical Physics*, 112(1):165–191, 2003.

H. Kesten. Sums of stationary sequences cannot grow slower than linearly. *Proceedings of the American Mathematical Society*, 49:205–211, 1975.

H. Kesten. The limit distribution of Sinai's random walk in random environment. *Physica A: Statistical Mechanics and its Applications*, 138(1):299 – 309, 1986.

H. Kesten, M. V. Kozlov, and F. Spitzer. A limit law for random walk in a random environment. *Compositio Mathematica*, 30(2):145–168, 1975.

C. Kipnis. Central limit theorems for infinite series of queues and applications to simple exclusion. *Ann. Probab.*, 14(2):397–408, 04 1986.

C. Kipnis and S. R. S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Communications in Mathematical Physics*, 104(1):1–19, 1986.

T. Kuczek. The central limit theorem for the right edge of supercritical oriented percolation. *Ann. Probab.*, 17(4):1322–1332, 1989.

D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. ISBN 978-0-8218-4739-8. With a chapter by James G. Propp and David B. Wilson.

T. Liggett. *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer, 1999. ISBN 9783540659952.

T. M. Liggett. Convergence to total occupancy in an infinite particle system with interactions. *Ann. Probab.*, 2(6):989–998, 12 1974.

T. M. Liggett. Coupling the simple exclusion process. *Ann. Probab.*, 4(3):339–356, 06 1976.

T. M. Liggett. *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985. ISBN 0-387-96069-4.

R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, 2016.

P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 463(2085):2287–2307, 2007.

F. Merlevède. On the central limit theorem and its weak invariance principle for strongly mixing sequences with values in a Hilbert space via martingale approximation. *J. Theoret. Probab.*, 16(3):625–653, 2003.

F. Merlevède, M. Peligrad, and E. Rio. Bernstein inequality and moderate deviations under strong mixing conditions. In *High dimensional probability V: the Luminy volume*, volume 5 of *Inst. Math. Stat. Collect.*, pages 273–292. Inst. Math. Statist., Beachwood, OH, 2009.

K. Miller. Random walks on weighted, oriented percolation clusters. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13:53–77, 2016. erratum-ibid. 14:173–175, 2017.

J. Peterson. *Limiting distributions and large deviations for random walks in random environments.* PhD thesis, 2008.

J. Peterson. Quenched limits for transient, ballistic, sub-Gaussian one-dimensional random walk in random environment. *Ann. Inst. H. Poincaré Probab. Statist.*, 45 (3):685–709, 08 2009.

J. Peterson and O. Zeitouni. Quenched limits for transient, zero speed one-dimensional random walk in random environment. *The Annals of Probability*, 37(1):143–188, 2009.

E. Rio. *Inequalities and limit theorems for weakly dependent sequences.* Laboratoire de Mathmatiques de Versailles (LM-Versailles), 2013.

A. Roitershtein. Transient random walks on a strip in a random environment. *Ann. Probab.*, 36(6):2354–2387, 11 2008.

E. Saada. A limit theorem for the position of a tagged particle in a simple exclusion process. *The Annals of Probability*, 15(1):375–381, 1987.

S. Sethuraman. Diffusive variance for a tagged particle in $d \leq 2$ asymmetric simple exclusion. *ALEA: Latin American Journal of Probability and Mathematical Statistics*, 1:305–332, 2006.

S. Sethuraman, S. Varadhan, and H. Yau. Diffusive limit of a tagged particle in asymmetric simple exclusion processes. *Communications on Pure and Applied Mathematics*, 53(8):972–1006, 8 2000.

V. Sidoravicius and A.-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probability Theory and Related Fields*, 129(2):219–244, 2004.

Y. G. Sinaĭ. The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatnost. i Primenen.*, 27(2):247–258, 1982.

F. Solomon. Random walks in a random environment. *Ann. Probab.*, 3(1):1–31, 02 1975.

F. Spitzer. Interaction of Markov processes. *Advances in Mathematics*, 5(2):246 – 290, 1970.

S. Varadhan. Self diffusion of a tagged particle in equilibrium for asymmetric mean zero random walk with simple exclusion. *Annales de l'I.H.P. Probabilités et statistiques*, 31(1):273–285, 1995.

N. T. Varopoulos. Long range estimates for Markov chains. *Bull. Sci. Math. (2)*, 109 (3):225–252, 1985.

O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.