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POST CORRECTING FOR GYROSCOPIC EFFECTS VIA DYNAMIC SUBSTRUCTURING.

M. Haeussler, R. Reichart, A. Bartl and D. J. Rixen

Institute of Applied Mechanics, Faculty of Mechanical Engineering, Technical University of Munich, Boltzmannstr. 15, 85748 Garching, Germany
email: michael.haeussler@tum.de, rixen@tum.de

Frequency response functions (FRFs) often serve as a basis for predicting sound and vibration levels at a receiver position, with a known excitation at a source position. Within the dynamic substructuring framework it is possible to build the FRFs of a complex assembly from the known FRFs of its subcomponents. However, in the case of subcomponents with revolving parts the task is further complicated due to gyroscopic effects. These components are changing their dynamic behavior depending on the operating speed. A correct approach would require measuring the FRFs of the rotating machinery at each operating speed, which is a difficult and tedious task. Thus, the unmeasured gyroscopic effects are often neglected (but not always negligible) in practice. We propose a dynamic substructuring based approach, for analytically coupling the gyroscopic reaction moments to an FRF matrix, measured on the idling subcomponent. Gyroscopic terms only influence subcomponent motions that are tilting the rotation axis. The proposed method will thus be interpreted and derived as a coupling in the subspace of this tilting motion. An analytical testcase is used to exemplify and validate the proposed method. We show how the tilting angles can be determined from an overdetermined set of measured sensor motions, based on a kinematic assumption. The validity of this kinematic assumption certainly influences the solution, which will also be shown on the example.

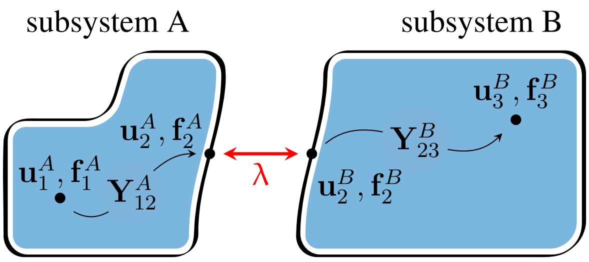
Keywords: *gyroscopic effects, experimental substructuring, transfer path analysis*

Nomenclature:

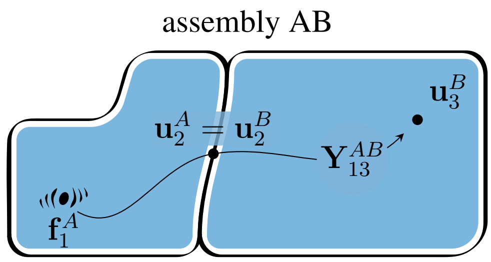
$\mathbf{u}, \mathbf{f}$	measured displacements / forces	$\mathbf{q}, \mathbf{m}$	generalized displacements / forces
$\mathbf{Y}, \mathbf{Z}$	admittance / impedance matrix	$\mathbf{B}$	signed Boolean matrix
λ	Lagrange multiplier vector	$\mathbf{R}$	reduction mode matrix
$\mathbf{T}$	transformation matrix	$\mathbf{\Theta}$	rotational inertia tensor
Ω	angular velocity of rotor	$\mathbf{Y}_0$	idling component admittance
$\mathbf{Y}_{op}$	operational component admittance	$\mathbf{Y}_{gy}$	gyroscopic admittance
$_T(\star)$	measured in frame of reference T	$(\star)_i$	pertaining to set of dofs i
$(\star)_{u/f}$	pertaining to displacements/forces	$(\star)^A$	quantity pertaining to substructure A
$(\star)^{A\|B}$	uncoupled block notation of A and B	$(\star)^{AB}$	coupled quantity of A and B
FBS	frequency based substructuring	OSI	operational system identification
FRF	frequency response function	IDM	interface displacement mode
TPA	transfer path analysis		

1. Introduction

Modal testing of rotating structures aims at extracting the modal properties of a rotating structure from a set of measurements. See e.g. [1] for a comprehensive review on the topic. However, the measured



(a) Admittances: uncoupled systems A and B.



(b) Admittance: coupled assembly AB.

Figure 1: The substructuring problem and its application for TPA [5].

FRFs of the structure will change depending on the operating speed of the rotating components, due to gyroscopic effects. Even a change of rotation direction (forward or backward) will alter the measured FRFs. An approach solely based on testing is thus very time consuming. To make matters worse, it is non-trivial to get a clean FRF measurement on a machine while it is in operation. All sorts of operational excitations, coming from the internals of the machine, are masking the actual sensor response to e.g. a shaker input. Some techniques are available for performing these measurements, like e.g. the operating system identification (OSI) method [2], but the involved signal processing and averaging further complicates the FRF determination. The idea in this paper is to use frequency based substructuring (FBS) [3, 4], for analytically coupling the gyroscopic effects to a set of FRF measurements performed on the idling structure. To the authors best knowledge, no one has published this approach yet. This technique might prove particularly valuable in component based transfer path analysis, where a correct FRF of the source component is vital for the determination of blocked forces [5].

2. Frequency Based Substructuring

In FBS the admittance of an assembled system $\mathbf{Y}^{AB}$ is derived from the separate admittances of two subsystems $\mathbf{Y}^A$ and $\mathbf{Y}^B$ (see figure 1). The admittances of both subsystems $\mathbf{Y}^A$ and $\mathbf{Y}^B$ are known and their degrees of freedom (dofs) are grouped into some internal dofs ($(\star)_1^A$ and $(\star)_3^B$) and some common dofs on the interface ($(\star)_2^A$ and $(\star)_2^B$). Displacements are denoted by $\mathbf{u}$, external forces in the respective dofs are denoted by $\mathbf{f}$. The admittance of the *uncoupled* substructures can be written in block diagonal form $\mathbf{Y}^{A|B}$. The following equations are the starting point for coupling $\mathbf{Y}^A$ and $\mathbf{Y}^B$, but also aim at clarifying the notation in verbose and compact form:

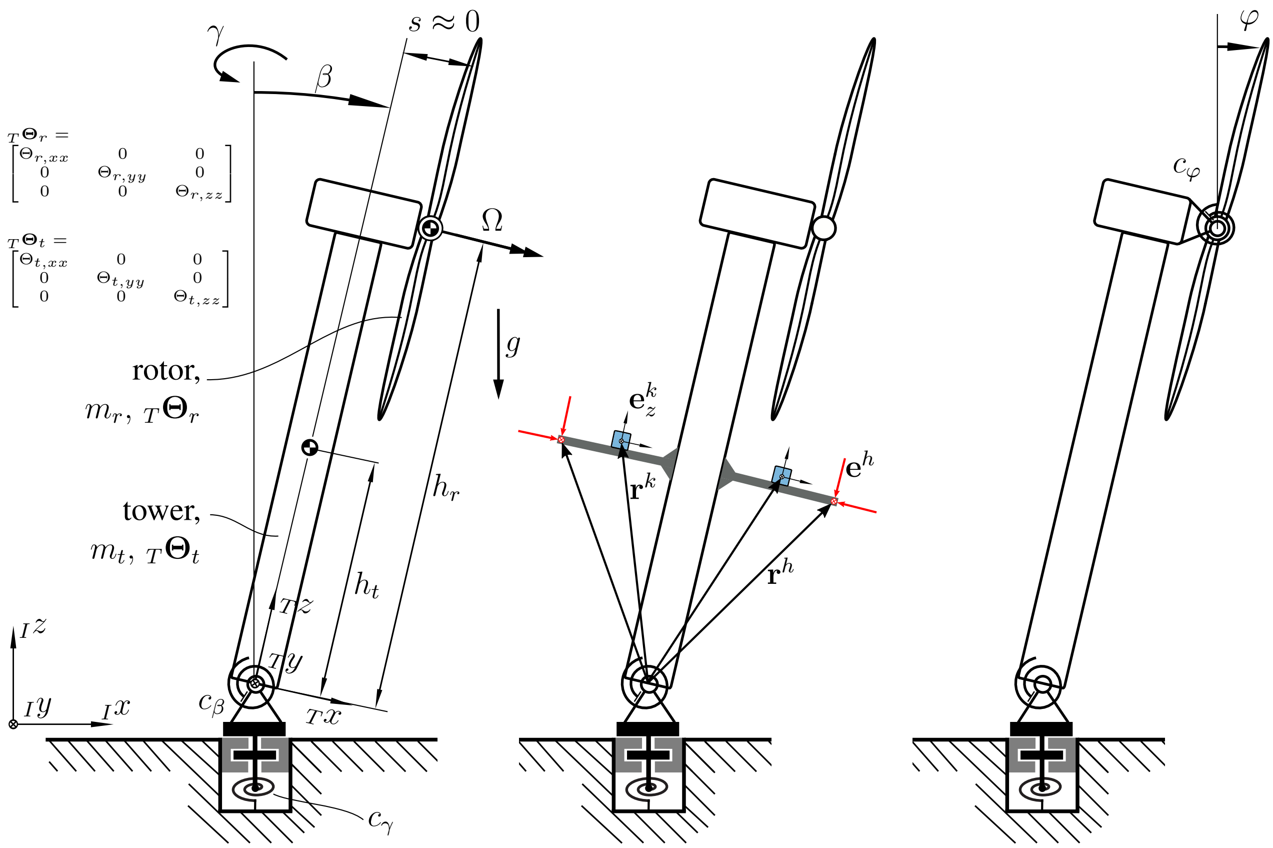
$$\mathbf{Y}^{A|B}\left(\mathbf{f}+\mathbf{B}^T\lambda\right) = \begin{bmatrix} \mathbf{Y}_{11}^A & \mathbf{Y}_{12}^A & \mathbf{0} & \mathbf{0} \\ \mathbf{Y}_{21}^A & \mathbf{Y}_{22}^A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Y}_{22}^B & \mathbf{Y}_{23}^A \\ \mathbf{0} & \mathbf{0} & \mathbf{Y}_{32}^B & \mathbf{Y}_{33}^A \end{bmatrix} \left(\begin{bmatrix} \mathbf{f}_1^A \\ \mathbf{f}_2^A \\ \mathbf{f}_2^B \\ \mathbf{f}_3^B \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \lambda \\ -\lambda \\ \mathbf{0} \end{bmatrix} \right) = \begin{bmatrix} \mathbf{u}_1^A \\ \mathbf{u}_2^A \\ \mathbf{u}_2^B \\ \mathbf{u}_3^B \end{bmatrix} = \mathbf{u}, \tag{1}$$

$$\mathbf{B}\mathbf{u} = \mathbf{0}, \qquad \text{where} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} \end{bmatrix}. \tag{2}$$

The internal interface forces, necessary for coupling the two substructures, are denoted by λ. The matrix $\mathbf{B}$ is commonly called a 'signed Boolean matrix'. When coupled, the two substrucutures have to fulfill *compatibility* on the interface ($\mathbf{u}_2^A = \mathbf{u}_2^B$) which is expressed in eq. (2). Inserting eq. (1) in eq. (2) and solving for λ we find:

$$\lambda = -\underbrace{\left(\mathbf{B}\mathbf{Y}^{A|B}\mathbf{B}^T\right)}_{\mathbf{F}_{int}}{}^{-1}\underbrace{\mathbf{B}\mathbf{Y}^{A|B}\mathbf{f}}_{\Delta\mathbf{u}_2}, \tag{3}$$

where $\Delta\mathbf{u}_2$ is the interface gap which would result between both structures if they where uncoupled (due to external forces $\mathbf{f}$). The term $\mathbf{F}_{int}$ is the 'interface flexibility' whose inverse relates the interface



(a) Dimensions and symbols. (b) Applied acceleration sensors and force inputs. (c) Flexibility c_φ between rotor and tower.

Figure 2: Example of windturbine for coupling of gyroscopic effects.

gap $\Delta\mathbf{u}_2$ to the reaction forces λ needed for 'closing the gap'. Reinserting eq. (3) in eq. (1) yields the admittance matrix of the assembled system $\mathbf{Y}^{AB}$:

$$\underbrace{\left(\mathbf{Y}^{A|B} - \mathbf{Y}^{A|B}\mathbf{B}^T\left(\mathbf{B}\mathbf{Y}^{A|B}\mathbf{B}^T\right)^{-1}\mathbf{B}\mathbf{Y}^{A|B}\right)}_{=\mathbf{Y}^{AB}}\mathbf{f} = \mathbf{u} \tag{4}$$

Remark 1: *Note that the assembled matrix $\mathbf{Y}^{AB}$ in eq.* (4) *still has the same size as the unassembled block matrix $\mathbf{Y}^{A|B}$. When writing down eq.* (4) *in the verbose form indicated in eq.* (1)*, we would notice that the second and third block row of $\mathbf{Y}^{AB}$ are equal (since compatibility: $\mathbf{u}_2^A = \mathbf{u}_2^B$) and that the second and third block column in $\mathbf{Y}^{AB}$ are equal (external interface forces $\mathbf{f}_2^A$ and $\mathbf{f}_2^B$ create the same response). It is thus common practice to remove these redundant rows and columns from the final matrix $\mathbf{Y}^{AB}$, thereby treating the interface dofs as the common quantities they are:* $\mathbf{u}_2^A = \mathbf{u}_2^B =: \mathbf{u}_2^{AB}$ *and* $\mathbf{f}_2^A + \mathbf{f}_2^B =: \mathbf{f}_2^{AB}$.

3. Coupling Gyroscopic Effects

The FBS approach shown above has been successfully used to couple the admittances of two separate structures. This approach can equivalently be used for coupling non-measured gyroscopic effects to a set of measured FRFs. To make things more tangible we will start considering the example of a windturbine like structure, shown in figure 2a.
The windturbine is mounted on a flexible support, allowing it to rotate around the longitudinal axis (spring stiffness c_γ, angle γ) and to tilt forward (spring stiffness c_β, angle β). The vector of uncon-

strained coordinates is $\mathbf{q} = \begin{bmatrix}\gamma & \beta\end{bmatrix}^T$. The system consists of two bodies, the tower (mass m_t, inertia tensor ${}_T\boldsymbol{\Theta}_t$) and the rotor (mass m_r, inertia tensor ${}_T\boldsymbol{\Theta}_r$) with their centers of mass being at height h_t and h_r respectively. The lower left subscript ${}_T(\star)$ indicates that the quantity is described in frame of reference T which is fixed to the tower. The rotor can rotate at varying operational speeds Ω. For simplicity, the distance of the rotor center of mass to the vertical axis is assumed to be zero. Hence, we have an equilibrium position for $\mathbf{q}_0 = \begin{bmatrix}0 & 0\end{bmatrix}^T$, where the springs are undeformed. For studying the vibrations around this equilibrium position, the linearized equations of motion can be written as:

$$\underbrace{\left(-\omega^2\mathbf{M} + i\omega\mathbf{G} + \mathbf{K}\right)}_{=:\mathbf{Z}_{op}(\omega)}\mathbf{q} = \mathbf{m}, \qquad \text{in this example:}$$

$$\mathbf{M} = \begin{bmatrix}\Theta_{t,zz} + \Theta_{r,zz} & 0 \\ 0 & \Theta_{t,yy} + \Theta_{r,yy} + m_t h_t^2 + m_r h_r^2\end{bmatrix}, \tag{5}$$

$$\mathbf{G} = \begin{bmatrix}0 & -\Omega\Theta_{r,xx} \\ \Omega\Theta_{r,xx} & 0\end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix}c_\gamma & 0 \\ 0 & c_\beta - (m_t h_t + m_r h_r)g\end{bmatrix}.$$

External moments in the respective dofs are denoted by $\mathbf{m}$. Matrices $\mathbf{M}, \mathbf{G}, \mathbf{K}$ denote the mass, gyroscopic and stiffness matrix respectively. They are often combined in the dynamic stiffness $\mathbf{Z}_{op}(\omega)$, where the subscript $(\star)_{op}$ indicates that the matrix is dependent on the operating conditions, i.e. the speed of the rotor Ω. The excitation frequency is denoted by ω.[1] The admittance matrix which could be measured on the idling system is $\mathbf{Y}_0$, where the subscript $(\star)_0$ denotes the idling component. However, we are interested in obtaining the admittance matrix of the operating system $\mathbf{Y}_{op}$:

$$\mathbf{Y}_0 = \left(-\omega^2\mathbf{M} + \mathbf{K}\right)^{-1}, \qquad \mathbf{Y}_{gy} = (i\omega\mathbf{G})^{-1}, \qquad \mathbf{Y}_{op} = \left(-\omega^2\mathbf{M} + i\omega\mathbf{G} + \mathbf{K}\right)^{-1}. \tag{6}$$

The unmeasured gyroscopic effect can be seen as an additional substructure admittance $\mathbf{Y}_{gy}$ which can be coupled to the idle component $\mathbf{Y}_0$ in a post processing step via dynamic substructuring. When using the same ideas as in FBS eq. (4) (in this case only interface dofs, $\mathbf{B} = [\mathbf{I} \quad -\mathbf{I}]$) we obtain:

$$\mathbf{Y}_{op} = \mathbf{Y}_0 - \mathbf{Y}_0\left(\mathbf{Y}_{gy} + \mathbf{Y}_0\right)^{-1}\mathbf{Y}_0. \tag{7}$$

Remark 2: *One can show that the FBS result in eq.* (7) *is equivalent to directly assembling the operational dynamic stiffness* $\mathbf{Z}_{op} = \mathbf{Y}_0^{-1} + \mathbf{Y}_{gy}^{-1}$ *and inverting it. We need to show that:*

$$\begin{aligned}\underbrace{\left(\mathbf{Y}_0^{-1} + \mathbf{Y}_{gy}^{-1}\right)}_{\mathbf{Z}_{op}}{}^{-1} &= \mathbf{Y}_0 - \underbrace{\mathbf{Y}_0\left(\mathbf{Y}_{gy} + \mathbf{Y}_0\right)^{-1}}_{=\left(\mathbf{Y}_{gy}\mathbf{Y}_0^{-1} + \mathbf{I}\right)^{-1}}\mathbf{Y}_0 \\ &= \left(\mathbf{I} - \left(\mathbf{Y}_{gy}\mathbf{Y}_0^{-1} + \mathbf{I}\right)^{-1}\right)\mathbf{Y}_0 \\ &= \left(\mathbf{Y}_{gy}\mathbf{Y}_0^{-1} + \mathbf{I}\right)^{-1}\left(\mathbf{Y}_{gy}\mathbf{Y}_0^{-1} + \mathbf{I} - \mathbf{I}\right)\mathbf{Y}_0 \\ &= \left(\mathbf{Y}_{gy}\mathbf{Y}_0^{-1} + \mathbf{I}\right)^{-1}\mathbf{Y}_{gy} = \left(\mathbf{Y}_0^{-1} + \mathbf{Y}_{gy}^{-1}\right)^{-1} \qquad \square\end{aligned}$$

A proof that was already used in [6]. Note that a similar result for the coupling of gyroscopic terms was found in [1, appendix C], though not derived from FBS but from pure linear algebra.

[1] For simplicity the dependence of dynamic stiffness $\mathbf{Z}$ and admittance $\mathbf{Y}$ on ω will be omitted in the rest of the text.

Remark 3: *Note that above it was implicitly assumed that the rotor's inertia tensor ${}_T\boldsymbol{\Theta}_r$ is independent of the actual rotation angle Ωt (i.e. $\Theta_{r,zz} = \Theta_{r,yy} = const.$) and that frame of reference T is a principal axes system for the inertia tensor.*
This is often referred to as an 'isotropic rotating component' [1]. The gyroscopic matrix $\mathbf{G}$ of the above example is representative for these cases, as can be seen from the Euler equations for the rotor:

$$ {}_T\boldsymbol{\Theta}_r\, {}_T\dot{\omega}_{op} + {}_T\omega_T \times {}_T\boldsymbol{\Theta}_r\, {}_T\omega_{op} = \mathbf{m}, $$

where ${}_T\omega_{op}$ is the angular velocity of the rotor measured in coordinate system T, ${}_T\omega_T$ is the angular velocity of coordinate system T and ${}_T\boldsymbol{\Theta}_r$ is the inertia of the rotor. If we assume only small rotations and a constant rotor operating speed (in the example above $\gamma \ll 1, \beta \ll 1, \Omega = const.$) this means ${}_T\omega_{op} \approx [\Omega \;\; \dot{\beta} \;\; \dot{\gamma}]^T$ and ${}_T\omega_T \approx [0 \;\; \dot{\beta} \;\; \dot{\gamma}]^T$. Inserting in the Euler equations for the rotor yields:

$$ \begin{bmatrix} 0 \\ \Theta_{r,yy}\ddot{\beta} \\ \Theta_{r,zz}\ddot{\gamma} \end{bmatrix} + \begin{bmatrix} (\Theta_{r,zz} - \Theta_{r,yy})\dot{\beta}\dot{\gamma} \\ \Omega\Theta_{r,xx}\dot{\gamma} \\ -\Omega\Theta_{r,xx}\dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 \\ m_\beta \\ m_\gamma \end{bmatrix}. $$

With the degrees of freedom $\mathbf{q} = \begin{bmatrix}\gamma & \beta\end{bmatrix}^T$ and linearizing for only small pertubations (i.e. assuming small $\dot{\beta}$ and $\dot{\gamma}$ and neglecting terms of higher order), we get the same gyroscopic matrix $\mathbf{G}$ as in eq. (5).

4. Projecting Gyroscopic Effects on Measured FRFs

Consider the situation depicted in figure 2b. The windturbine is equipped with a rigid fixture on which some sensors (indexed with k) are mounted and some force inputs (indexed with h) can be applied. Usually the set of measurements is performed on the idling component and the single measurement channels, grouped in the vector $\mathbf{u}$, do not directly correspond to the tilting angles of the rotor axis $\mathbf{q}$. Likewise, the applied forces, grouped in the vector $\mathbf{f}$, do not directly correspond to the tilting moments of the rotor axis $\mathbf{m}$. Consider the *kinematic assumption* of rigidity for the windturbine being valid. Then the linearized response in e.g. the z-channel of sensor k due to a small γ and β is:

$$ u_z^k = \left(\mathbf{e}_z^k\right)^T \left(\begin{bmatrix}0 & \beta & \gamma\end{bmatrix}^T \times \mathbf{r}^k\right) = \left(\mathbf{e}_z^k\right)^T \begin{bmatrix} r_z^k & -r_y^k \\ 0 & r_x^k \\ -r_x^k & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \end{bmatrix}. \tag{8} $$

The resulting moments $\mathbf{m}^h$ in the axis tilting directions, due to one force input f^h are:

$$ \mathbf{m}^h = \begin{bmatrix} m_\beta^h \\ m_\gamma^h \end{bmatrix} = \begin{bmatrix} r_z^h & 0 & -r_x^h \\ -r_y^h & r_x^h & 0 \end{bmatrix} \mathbf{e}^h f^h, \tag{9} $$

where $\mathbf{r}$ denotes the position vectors of sensors / forces and $\mathbf{e}$ denotes their unit direction vectors (see figure 2b). Similarly to eq. (8) and eq. (9), one finds general expressions between all sensor channels / force inputs ($\mathbf{u}$ and $\mathbf{f}$) and the tilting angles / tilting moments ($\mathbf{q}$ and $\mathbf{m}$).[2] Thus we can find the

[2]Methodically all these derivations are equivalent to projecting measurements on interface displacement modes (IDMs) first described in [2] and extended in [7] which we refer to, for a thorough explanation.

measured FRF matrix $\mathbf{Y}_{uf}$ from the FRF matrix in the subspace of the tilting angles $\mathbf{Y}_{qm}$:

$$\mathbf{u} = \mathbf{R}_u \mathbf{q}, \qquad \mathbf{m} = \mathbf{R}_f^T \mathbf{f}, \qquad \mathbf{Y}_{uf} = \mathbf{R}_u \mathbf{Y}_{qm} \mathbf{R}_f^T, \tag{10}$$

$$\mathbf{q} = \underbrace{\left(\mathbf{R}_u^T \mathbf{R}_u\right)^{-1} \mathbf{R}_u^T}_{=:\mathbf{T}_u} \mathbf{u}, \qquad \mathbf{f} = \underbrace{\mathbf{R}_f \left(\mathbf{R}_f^T \mathbf{R}_f\right)^{-1}}_{=:\mathbf{T}_f^T} \mathbf{m}, \tag{11}$$

where $\mathbf{R}_u$ and $\mathbf{R_f}$ contain the kinematic assumption for relating the tilting angles and tilting moments to the set of measured channels. The matrices $\mathbf{T}_u$ and $\mathbf{T}_f^T$ are basically the pseudo inverses of $\mathbf{R}_u$ and $\mathbf{R_f}$ (i.e. $\mathbf{T}_u \mathbf{R}_u = \mathbf{T}_f \mathbf{R}_f = \mathbf{I}$).
$\mathbf{T}_u$ transforms measured signals $\mathbf{u}$ to the tilting angles $\mathbf{q}$ in a least squares sense. $\mathbf{T}_f^T$ determines a minimal set of forces $\mathbf{f}$ for producing a specific tilting moment $\mathbf{m}$, also in a least squares sense. The subscript $(\star)_{uf}$ refers to a FRF matrix being measured between force inputs and sensor channels. The subscript $(\star)_{qm}$ denotes a FRF matrix being measured in the subspace of the tilting angles (between tilting moments and the tilting angles).
Our goal is to predict the FRF matrix of the operating system $\mathbf{Y}_{uf,op}$ from the measured FRF matrix of the idling component $\mathbf{Y}_{uf,0}$ and the known gyroscopic admittance $\mathbf{Y}_{qm,gy}$ (rotational intertia and operating speed of the rotor must thus be known). The FBS approach thus needs to ensure compatibility between the tilting angles of the idling component (inferred from the measured sensor channels $\mathbf{u}$ via eq. (11)) and the tilting angles of the subsequently coupled 'gyroscopic substructure'. This compatibility is stated in eq. (13). The gyroscopic reaction moments needed for ensuring compatibility are denoted as λ. They have to be transformed to an equivalent set of measured forces for applying them to the measured idling component (via $\mathbf{T}_f^T$, which is stated in eq. (12)). The formulation of the coupling can be put as follows:

$$\mathbf{Y}^{0|gy} \left(\tilde{\mathbf{f}} + \mathbf{B}_f^T \lambda\right) = \begin{bmatrix} \mathbf{Y}_{uf,0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{qm,gy} \end{bmatrix} \left(\begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{T}_f^T \lambda \\ -\lambda \end{bmatrix} \right) = \begin{bmatrix} \mathbf{u} \\ \mathbf{q} \end{bmatrix} = \tilde{\mathbf{u}}, \tag{12}$$

$$\mathbf{B}_u \tilde{\mathbf{u}} = \mathbf{0}, \qquad \text{where} \quad \mathbf{B}_u = \begin{bmatrix} \mathbf{T}_u & -\mathbf{I} \end{bmatrix}, \qquad \text{and} \quad \mathbf{B}_f = \begin{bmatrix} \mathbf{T}_f & -\mathbf{I} \end{bmatrix}. \tag{13}$$

Inserting eq. (12) in eq. (13) results in:

$$\mathbf{Y}_{uf,op} = \mathbf{Y}_{uf,0} - \mathbf{Y}_{uf,0} \mathbf{T}_f^T \left(\mathbf{Y}_{qm,gy} + \mathbf{T}_u \mathbf{Y}_{uf,0} \mathbf{T}_f^T\right)^{-1} \mathbf{T}_u \mathbf{Y}_{uf,0}. \tag{14}$$

Note that the above result corresponds to the upper left block of the FBS result when using the block matrix notation indicated in eq. (12). This is perfectly fine considering the argument of remark 1. Knowing that there are no other external moments on the 'gyroscopic substructure', apart from the compatibility moments λ (see the 'zero' entry in the external forces $\tilde{\mathbf{f}}$ in eq. (12)), one concludes that the result corresponds to the FRF matrix one would measure in operation.[3]

Remark 4: *The above result is equal to just expanding the matrix of the operational system from eq.* (7)*, which is restated here:*

$$\mathbf{Y}_{qm,op} = \mathbf{Y}_{qm,0} - \mathbf{Y}_{qm,0} \left(\mathbf{Y}_{qm,gy} + \mathbf{Y}_{qm,0}\right)^{-1} \mathbf{Y}_{qm,0}$$

The expansion of a matrix to the sensor channels and force inputs can be done via:

$$\mathbf{Y}_{uf,op} = \mathbf{R}_u \mathbf{Y}_{qm,op} \mathbf{R}_f^T \qquad \mathbf{Y}_{uf,0} = \mathbf{R}_u \mathbf{Y}_{qm,0} \mathbf{R}_f^T$$

Starting from eq. (14) *with the definitions of the transformation matrices in eq.* (11) *it is easy*

[3]This can also be seen as a weak coupling of the gyroscopic effects to the rest of the tower dynamics, which might even be non-rigid in directions not tilting the rotation axis (see e.g. [8] for further discussion).

to show:

$$\mathbf{Y}_{uf,op} = \mathbf{Y}_{uf,0} - \mathbf{Y}_{uf,0}\mathbf{T}_f^T\big(\mathbf{Y}_{gy} + \underbrace{\mathbf{T}_u\mathbf{Y}_{uf,0}\mathbf{T}_f^T}_{=\mathbf{Y}_{qm,0}}\big)^{-1}\mathbf{T}_u\mathbf{Y}_{uf,0}$$

$$= \mathbf{Y}_{uf,0} - \underbrace{\mathbf{Y}_{uf,0}\mathbf{T}_f^T}_{=\mathbf{R}_u\mathbf{Y}_{qm,0}}\left(\mathbf{Y}_{gy} + \mathbf{Y}_{qm,0}\right)^{-1}\underbrace{\mathbf{T}_u\mathbf{Y}_{uf,0}}_{=\mathbf{Y}_{qm,0}\mathbf{R}_f^T}$$

$$= \mathbf{R}_u\underbrace{\left(\mathbf{Y}_{qm,0} - \mathbf{Y}_{qm,0}\left(\mathbf{Y}_{qm,gy} + \mathbf{Y}_{qm,0}\right)^{-1}\mathbf{Y}_{qm,0}\right)}_{\mathbf{Y}_{qm,op}}\mathbf{R}_f^T \qquad \square$$

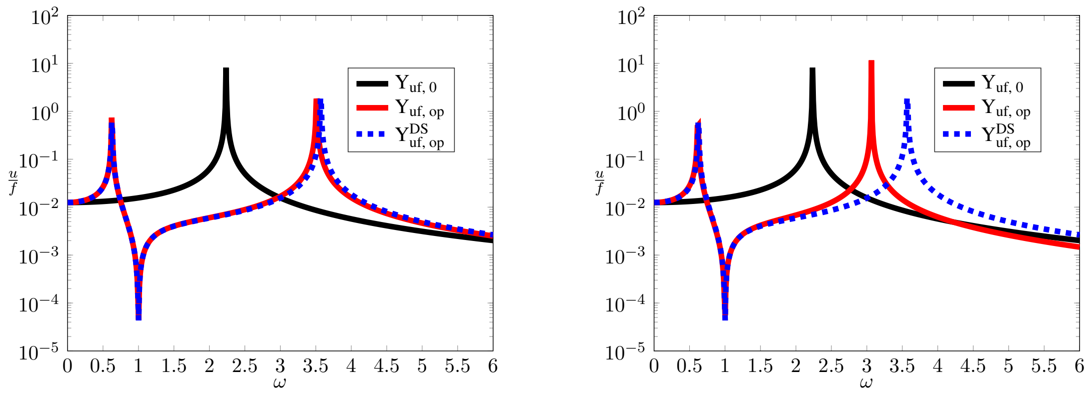
5. Analysis of kinematic assumptions

In the previous paragraph we have shown, that the suggested FBS approach yields the excact solution, provided the kinematic assumption (relating the axis tilting motion to the sensor motion) is valid. We will now show the importance of this assumption for the quality of the results. Consider the situation in figure 2c, with a spring c_φ between the tower and the rotor. Assume we still want to use FRF measurements between the sensors and impacts on the tower as shown in figure 2b. The kinematic assumption of a rigid connection between the sensors and the rotor axis will be deteriorated as c_φ is reduced. In fact we have three coordinates now $\mathbf{q} = [\gamma\ \beta\ \varphi]^T$, where φ describes the absolute angle of the rotor as it tilts over. For a very stiff c_φ, the coordinates β and φ will be almost identical, which approves our kinematic assumption. However, for a reduced c_φ the kinematic assumption will further deteriorate. The result for an arbitrary choice of parameters[4] is shown in figure 3. It can be seen that the response of the idle system shows only one resonance peak, since in the case shown, with an excitation in the y-direction, one is only exciting a rotation around the z-axis (i.e. the coordinate γ). The rotations of the two free coordinates (β and φ) are decoupled from this motion. The gyroscopic reaction moments introduce the coupling with these coordinates and their resonances start to show up in this FRF as we introduce a rotor velocity $\Omega \neq 0$. Due to the significant difference between idle and operational FRF, it can be argued that a consideration of the gyroscopic effects in this case is essential for an accurate estimate of the FRF at different operating speeds. The results for differing stiffnesses c_φ (cf. figure 3a and figure 3b) also show that an accurate consideration of the structure kinematics is vital for good results.

6. Conclusion

We have shown how gyroscopic effects of rotating components in a substructure can significantly alter its dynamic behavior. The proposed method can be used as a comparatively easy way for considering these gyroscopic effects in a dynamic model, without the need for performing a new set of measurements for each operating speed of the rotor. It is also well suited for providing the uncertain engineer with a first estimate for the importance of gyroscopic effects in a specific design. Though not shown in this paper, the kinematic assumptions, for inferring the the rotor tilting angles from a set of measurement channels, doesn't have to be rigidity. If e.g. a finite element model of the component is available, one might determine the modal participation of important modes from a set of measurements and get the tilting angles from those modal participations (reduction matrices $\mathbf{R}_u$ and $\mathbf{R}_f$ would change). Note, that in component based TPA [5] it is important to consider these effects, when determining the blocked forces, since otherwise they are not transferable to a different design.

[4] All parameter appearing in the system matrices of eq. (5) are set to 1 apart from: $c_\gamma = c_\beta = 10$; $h_r = 2, \Omega = 10$.



(a) $c_\varphi = 1000$ (b) $c_\varphi = 100$

Figure 3: FRF between impact point h in y-direction and sensor k in y-direction (cf. figure 2b). Comparison of idle FRF $Y_{uf,0}$, reference solution $Y_{uf,op}$, and dynamic substructuring result $Y^{DS}_{uf,op}$, obtained with the violated kinematic assumption.

The method could prove particularly valuable for components that are relatively compact, e.g. electric motors or small compressors in stiff housings (when compared with their usually soft support, for decoupling them from the rest of the structure, e.g. a car). Also the general idea of this paper, namely to use concepts from dynamic substructuring for coupling a non-measurable physical effect to a measured substructure, may also be transferable to other problems.

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References

[1] I. Bucher and D. J. Ewins. "Modal analysis and testing of rotating structures". In: *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 359.1778 (2001), pp. 61–96.

[2] D. de Klerk. "Dynamic Response Characterization of Complex Systems through Operational Identification and Dynamic Substructuring". PhD thesis. TU Delft, 2009.

[3] B. Jetmundsen, R. L. Bielawa, and W. G. Flannelly. "Generalized Frequency Domain Substructure Synthesis". In: *Journal of the American Helicopter Society* 33.1 (Oct. 1988), pp. 55–64.

[4] D. de Klerk et al. "Solving the RDoF Problem in Experimental Dynamic Substructuring". In: *Proceedings of the 26th International Modal Analysis Conference - IMAC*. 2008.

[5] M. V. van der Seijs, D. de Klerk, and D. J. Rixen. "General framework for transfer path analysis: History, theory and classification of techniques". In: *Mechanical Systems and Signal Processing* 68-69 (Feb. 2016), pp. 217–244.

[6] D. Rixen. "How measurement inaccuracies induce spurious peaks in frequency based substructuring". In: *IMAC XXVI - International Modal Analysis Conference*. 2008.

[7] M. V. van der Seijs et al. "An improved methodology for the virtual point transformation of measured frequency response functions in dynamic substructuring". In: *4th ECCOMAS Thematic Conference on Computational Methods in Structural Dynamics and Earthquake Engineering* June (2013).

[8] M. Häußler and D. J. Rixen. "Optimal transformation of frequency response functions on interface deformation modes". In: *IMAC XXXV - International Modal Analysis Conference*. 2017.