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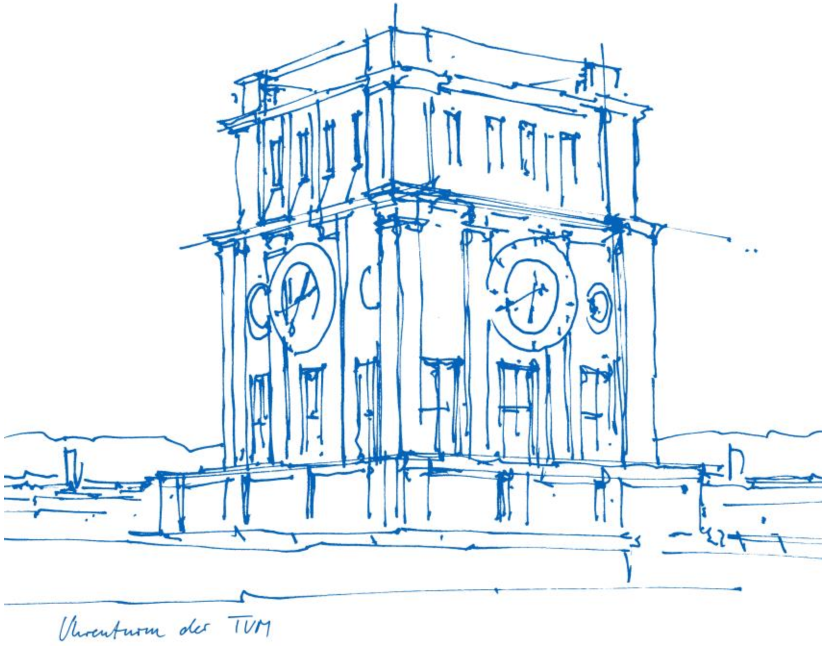
Technical University of Munich

sss & sssMOR - Analysis and Reduction of Large-Scale Dynamic Systems in MATLAB

Alessandro Castagnotto

in collaboration with: Maria Cruz Varona, Boris Lohmann

related publications: “sss & sssMOR: Analysis and reduction of large-scale dynamic systems in MATLAB”, at – Automatisierungstechnik (2017)



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EU-MORNET 2nd Exploratory Workshop | Luxembourg | 10.03.2017

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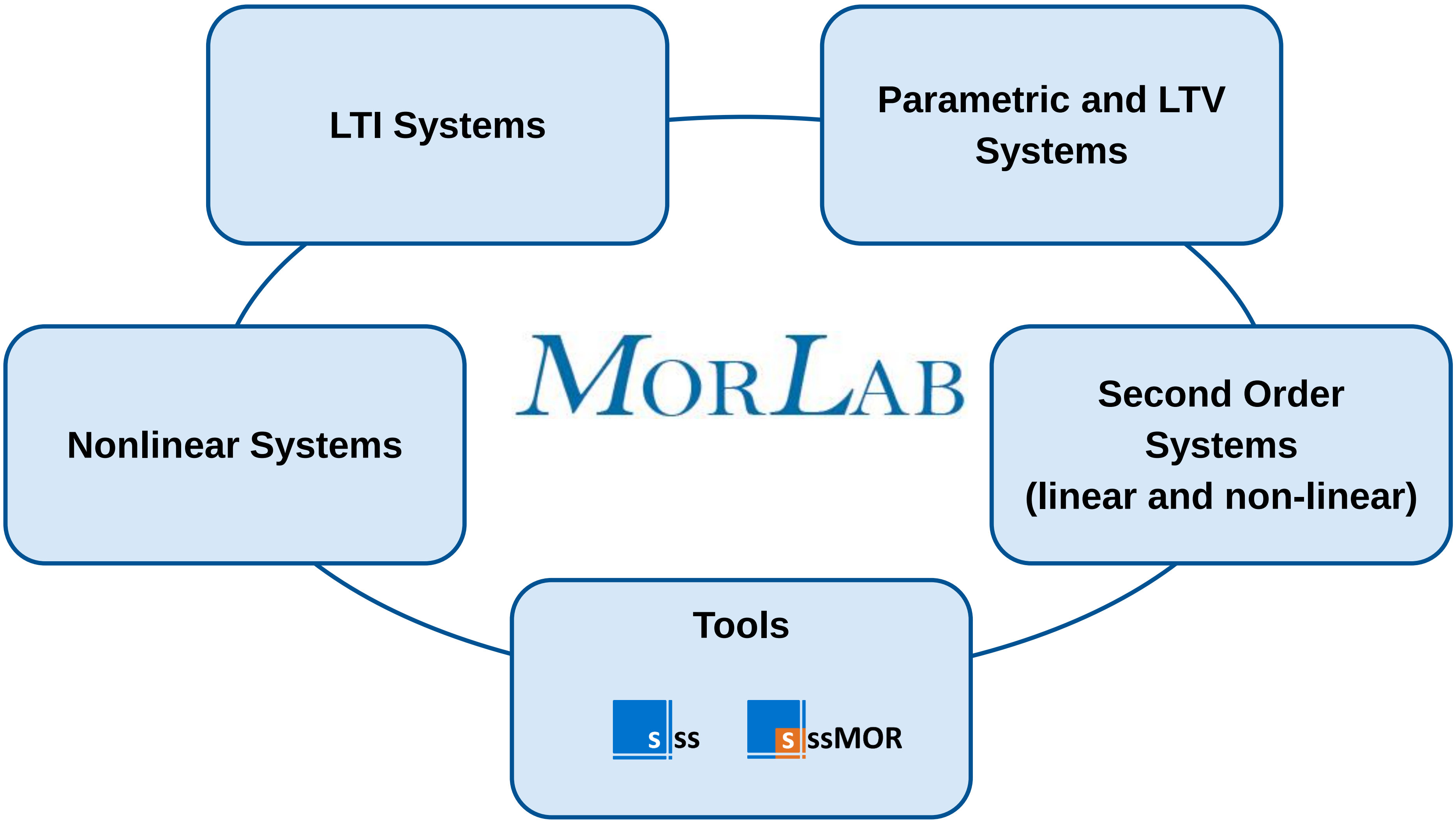
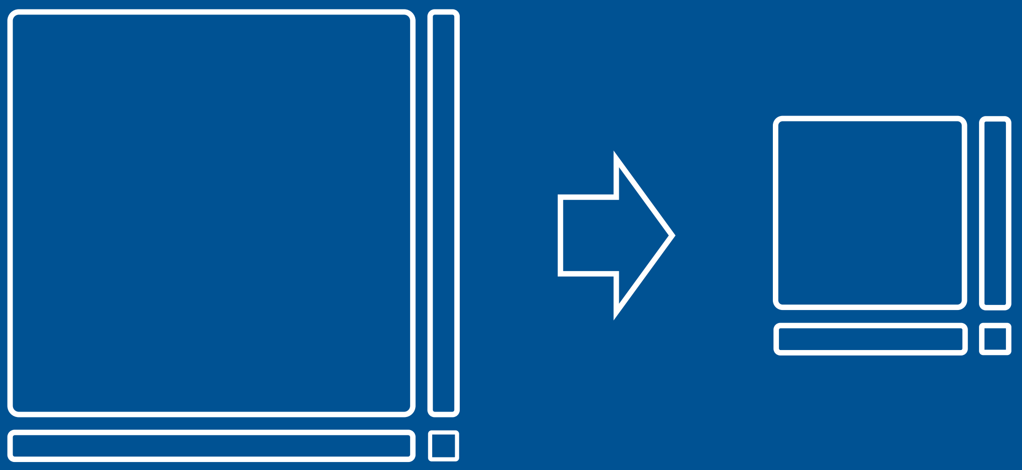
Model Order Reduction @ MORLab



The model order reduction setting



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Model order reduction (MOR)

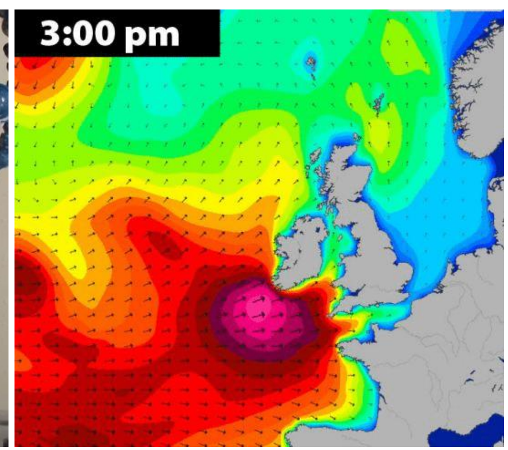
$$\left.\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} x \in \mathbb{R}^N$$

MOR

$$\begin{aligned} E_r\dot{x}_r &= A_r x_r + B_r u \\ y_r &= C_r x_r + D u \end{aligned}$$

$$x_r \in \mathbb{R}^n,\ n \ll N$$

- ✓ high-fidelity approximation
- ✓ preservation of properties
- ✓ numerically efficient



10^3 10^4 10^5 10^6 N

Source(s): nasa.gov, wikimedia.org, dailymail.co.uk

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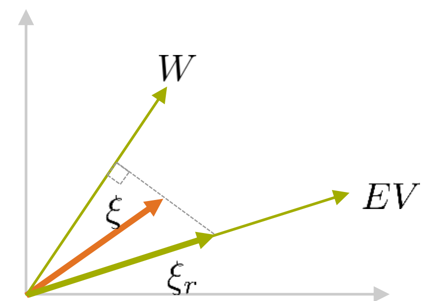
Projective MOR

Approximation in the subspace $\mathcal{V} = \mathrm{Im}\,(V)$

$$x = V x_r + e, \qquad V \in \mathbb{R}^{N \times n}$$



Petrov-Galerkin Projection:

(cf. projection by $\Pi = EV(W^\top EV)^{-1}W^\top$)

$$\underbrace{W^\top E V}_{E_r}\, \dot{x}_r = \underbrace{W^\top A V}_{A_r}\, x_r + \underbrace{W^\top B}_{B_r}\, u$$

$$y \approx y_r = \underbrace{C V}_{C_r}\, x_r + D u$$

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Balanced Truncation

Preserve state-space directions with highest energy transfer

Controllability and Observability

$$A\,\mathcal{W}_\mathcal{C}\,E^\top + E\,\mathcal{W}_\mathcal{C}\,A^\top + BB^\top = 0$$

$$A^\top\,\mathcal{W}_\mathcal{O}\,E + E^\top\,\mathcal{W}_\mathcal{O}\,A + C^\top C = 0$$

…from an „energy“ perspective

$$\min_{s.t.x(t_e)=x_e} \int_0^\infty u^2(\tau)d\tau = x_e^\top\,\mathcal{W}_\mathcal{C}^{-1}\,x_e$$

$$\|y(t)\|_2^2 = x_0^\top\,\mathcal{W}_\mathcal{O}\,x_0$$

Balanced realization

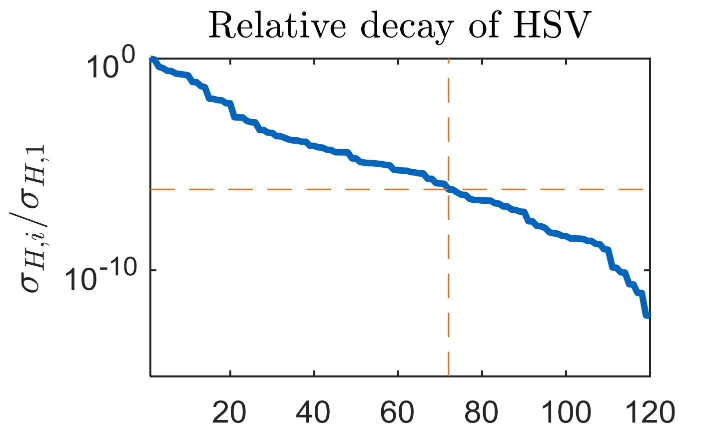
$\mathcal{W}_\mathcal{O}\cdot\mathcal{W}_\mathcal{C}$ is system invariant

$$\Rightarrow\quad \mathcal{W}_\mathcal{O} = \mathcal{W}_\mathcal{C} = \Sigma_H := \mathrm{diag}(\sigma_{H,1},\dots,\sigma_{H,N})$$

$$\mathcal{W}_\mathcal{C} = S^\top S,\quad \mathcal{W}_\mathcal{O} = R^\top R$$

$$SR^\top = [U_1, U_2]\begin{bmatrix}\Sigma_{H,1} & \\ & \Sigma_{H,2}\end{bmatrix}\begin{bmatrix}T_1^\top\\ T_2^\top\end{bmatrix}$$

$$\boxed{W = R^\top T_1 \Sigma_{H,1}^{-1/2} \qquad V = S^\top U_1 \Sigma_{H,1}^{-1/2}}$$



[Moore ’81, Penzl ’00, Li/White ’02, Saak ‘09]

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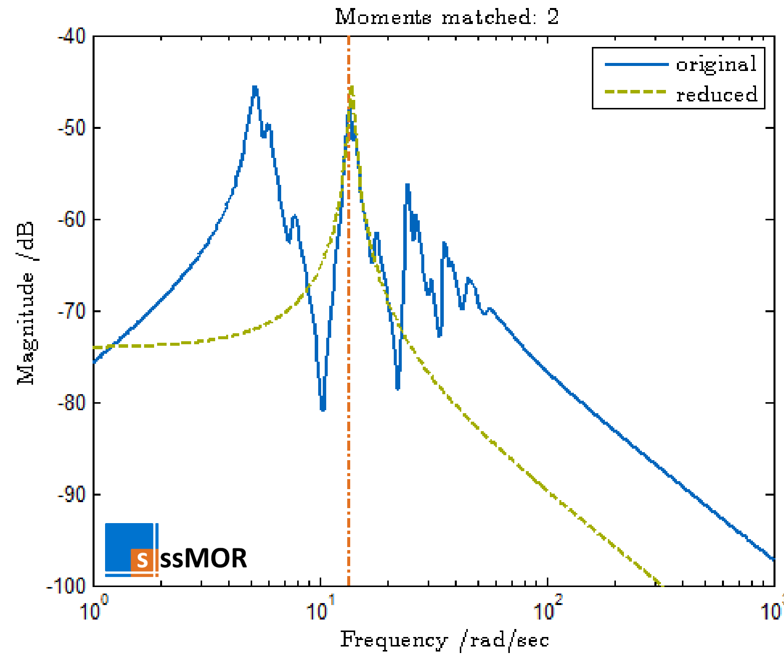
Moment matching (rational interpolation)

Moments of a transfer function

$$G(s) = C(sE - A)^{-1}B$$
$$= G(s_0 + \Delta s) = -\sum_{i=0}^{\infty} M_i(s_0)\Delta s^i$$

s_0 : Interpolation frequency (shift)
$M_i(s_0)$: i-th moment about s_0



Rational Krylov (RK) subspaces

Choose V und W such that:

$$\mathcal{K}_q\left((A - s_0E)^{-1}E, (A - s_0E)^{-1}B\right) \subseteq \mathrm{Im}\,(V)$$
$$\mathcal{K}_r\left((A - s_0E)^{-\top}E^\top, (A - s_0E)^{-\top}C^\top\right) \subseteq \mathrm{Im}\,(W)$$

$\Rightarrow$ $\boxed{M_i(s_0) = M_{r,i}(s_0)}$

for $i = 0, \ldots, q + r - 1$

[Grimme '97, Gallivan et al. '04a]

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$\mathcal{H}_2$-optimal model order reduction

$$\|G - G_r\|_{\mathcal{H}_2} = \min_{\dim(\widetilde{G}_r)=n} \left\| G - \widetilde{G}_r \right\|_{\mathcal{H}_2} \quad \Rightarrow \quad \begin{aligned} G(-\overline{\lambda}_{r,i}) &= G_r(-\overline{\lambda}_{r,i}) \\ G'(-\overline{\lambda}_{r,i}) &= G_r'(-\overline{\lambda}_{r,i}) \end{aligned}$$

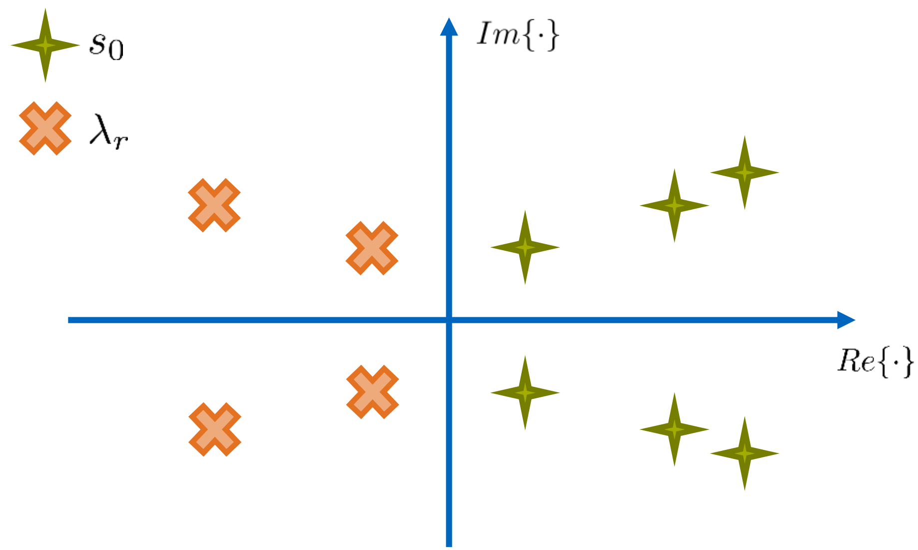
Iterative Rational Krylov Algorithm

$$\Sigma := \{E, A, B, C\} \qquad \Sigma_r := \{E_r, A_r, B_r, C_r\}$$

Algorithm Iterative Rational Krylov Algorithm (IRKA)

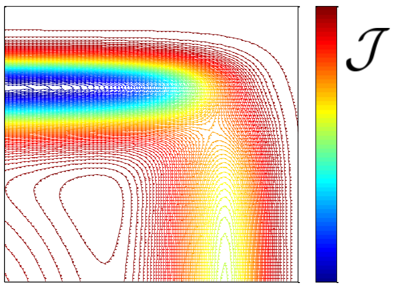
Input: $\Sigma, s_0, \mathrm{tol}$
Output: locally $\mathcal{H}_2$-optimal reduced model Σ_r, s_0^*

while relative change of $s_0 <$ tol **do**
    $\Sigma_r \leftarrow \mathrm{RK}(\Sigma, s_0)$ // Hermite reduction
    $s_0 \leftarrow -\overline{\lambda(\Sigma_r)}$ // mirror reduced eigenvalues
end while
$s_0^* \leftarrow s_0$ // return optimal shifts



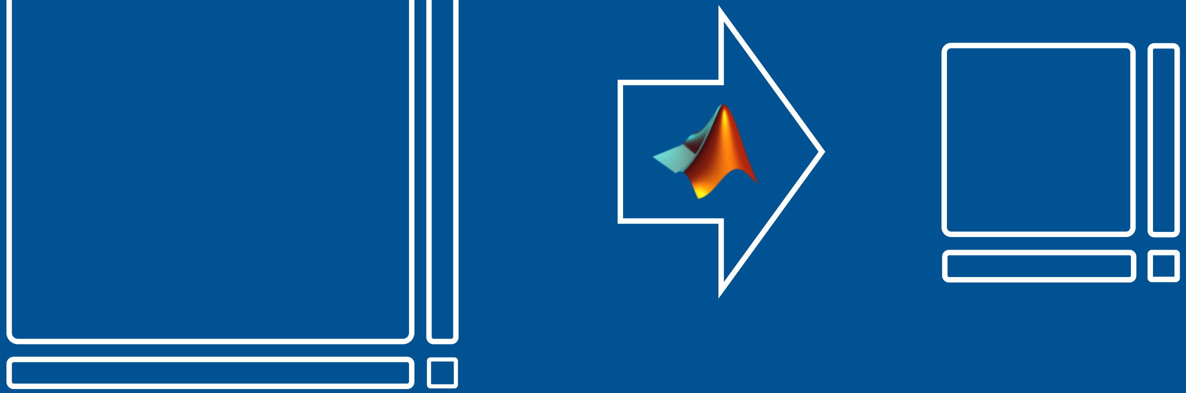
Gradient-based methods

Expressions for gradient and Hessian can be derived and used for trust-region optimization



[Meier/Luenberger '67, Gugercin/Antoulas/Beattie '08, Gugercin/Beattie '12, Panzer et al. '14, C. et al '16]

sss & sssMOR – MATLAB Toolboxes

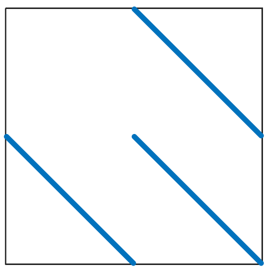


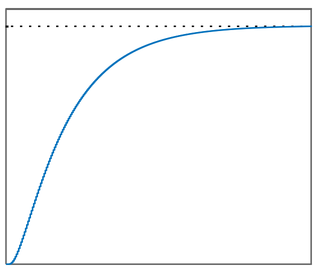
Toolboxes for sparse, large-scale models in

sss

sssMOR

```
sys = sss(A,B,C,D,E);
```

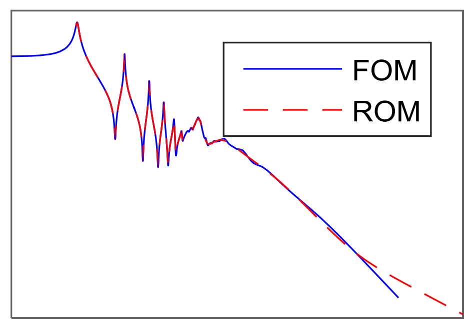


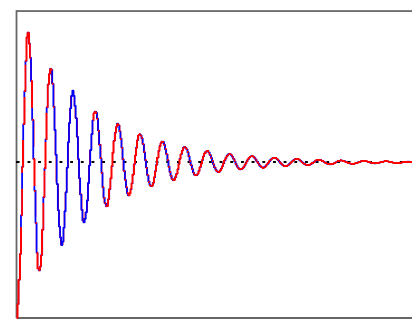


```
bode(sys), sigma(sys)
step(sys), impulse(sys)
norm(sys,2), norm(sys,inf)

c2d, lsim, eigs, connect,…
```

```
sysr = tbr(sys,n)
sysr = rk(sys,s0)
sysr = irka(sys,s0)
sysr = cure(sys)
sysr = cirka(sys,s0)
       ⋮
```





Powered by: **M-M.E.S.S. toolbox** [Saak, Köhler, Benner] for Lyapunov equations
Available at www.rt.mw.tum.de/?sssMOR
[C./Cruz Varona/Jeschek/Lohmann: **„sss & sssMOR: Analysis and Reduction of Large-Scale Dynamic Systems in MATLAB“**, 2017 at-Automatisierungstechnik]

GitLab

Main characteristics

sss

- **State-space models** of very high order on a standard computer $\mathcal{O}\left(\mathbf{10^8}\right)$
- Many Control System Toolbox functions, revisited to **exploit sparsity**
- Allows system analysis in **frequency** (`bode, sigma, ...`) and **time domain** (`step,norm,lsim,...`), as well as **manipulations** (`connect,truncate,...`)
- Is **compatible** with the built-in syntax
- **New functionality:** `eigs, residue, pzmap,...`

sssMOR

- **Classical** (`modalMor, tbr, rk,...`) and **state-of-the-art** (`isrk, irka, cirka, cure,...`) reduction methods
- Both **highly-automatized**

```
sysr = irka(sys,n)
```

  and **highly-customizable**

```
Opts.maxiter = 100
Opts.tol = 1e-6
Opts.stopcrit = 'combAll'
Opts.verbose = true
sysr = irka(sys,n,Opts)
```

- `solveLse` and `lyapchol` as **core functions**

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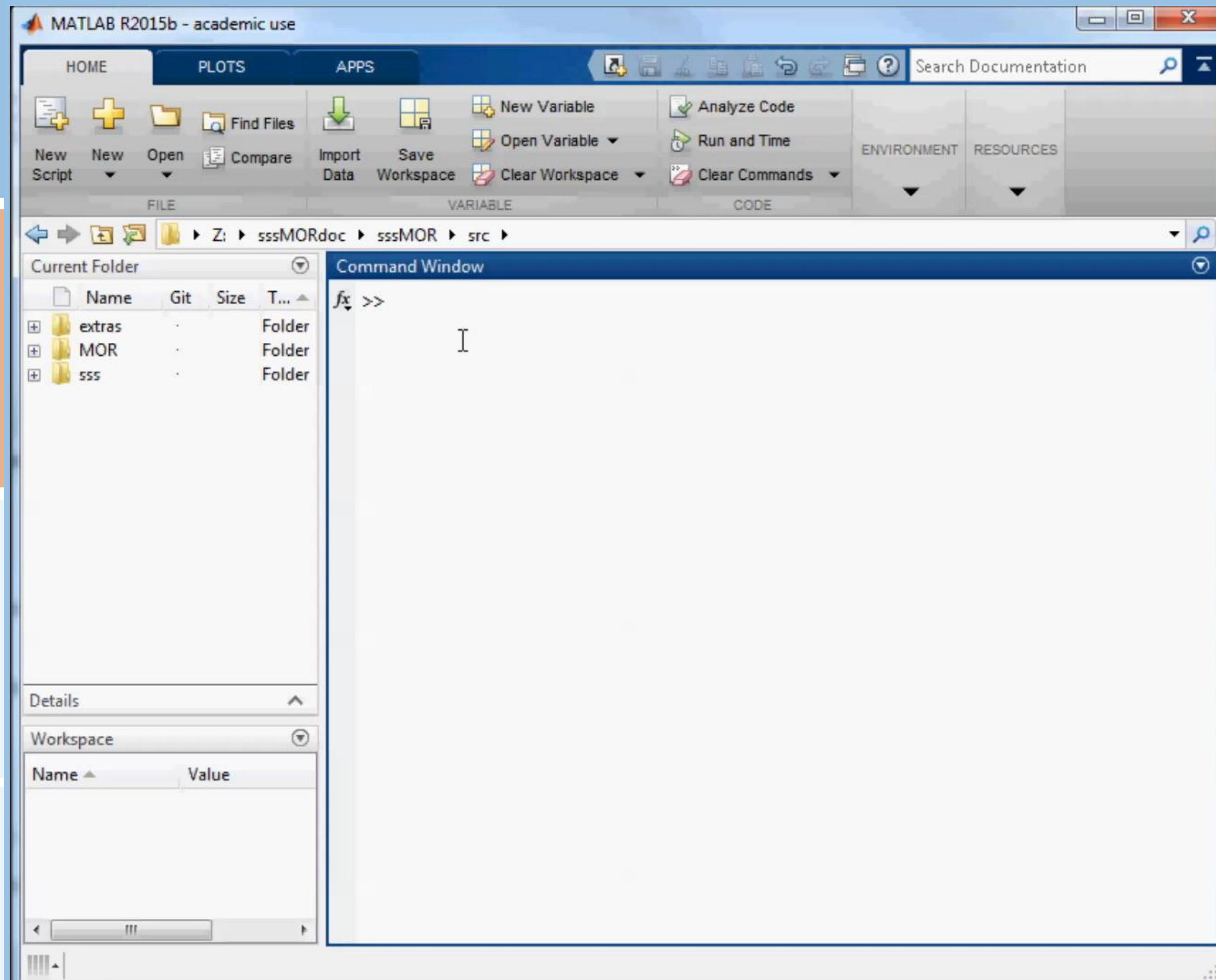
Comprehensive **documentation** with examples and references

sssMOR App
graphical user interface

completely **free**
and **open source**
(contributions welcome)



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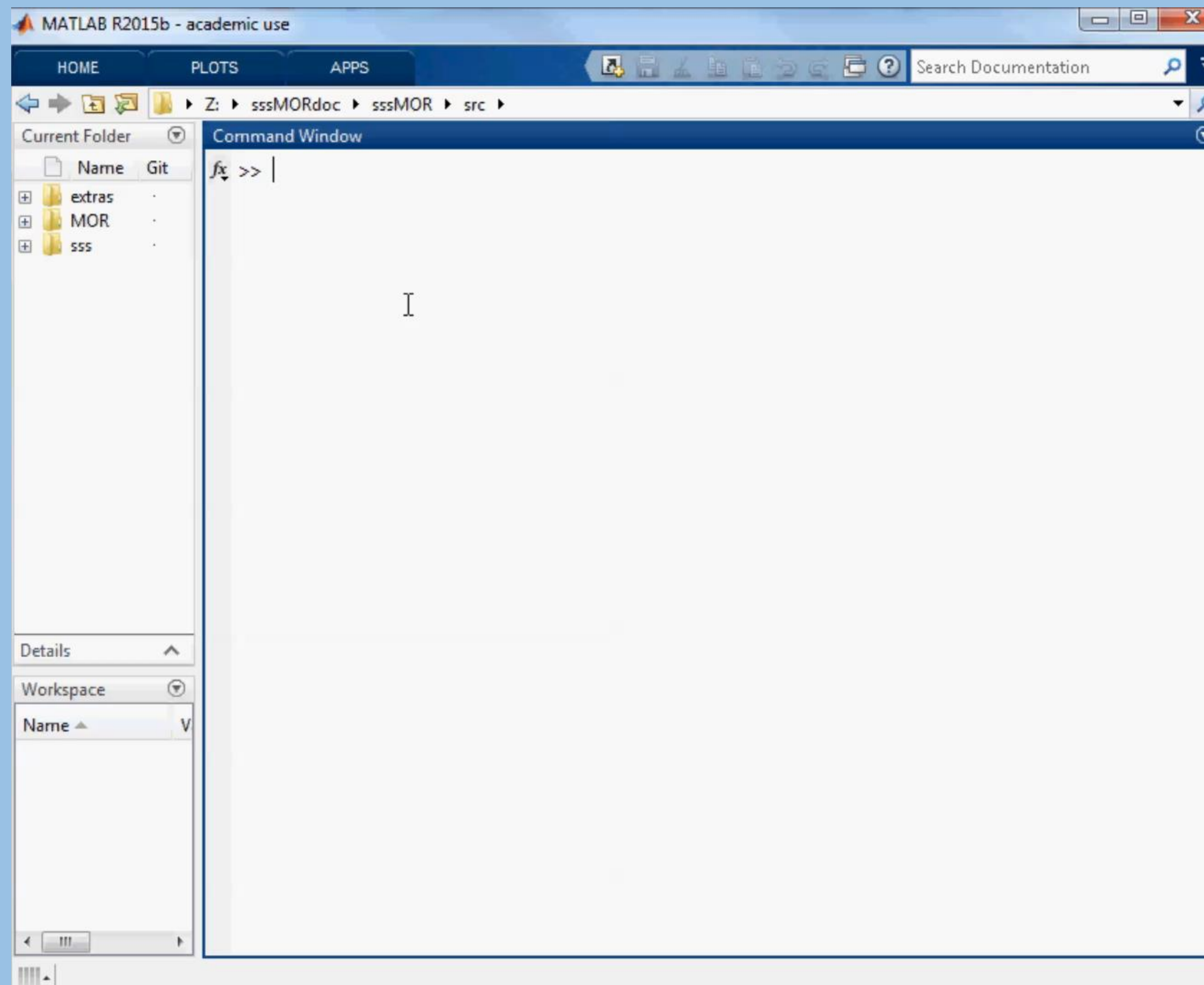
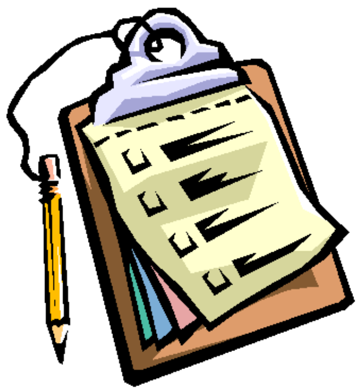


sss & sssMOR – a test case

Live demo available for download here

Summary

sss & **sssMOR**: Analysis and reduction of large-scale dynamic systems in MATLAB



- ☑ Model reduction as projection
- ☑ Balanced truncation, IRKA
- ☑ Live demo
  - ✓ **Control System Toolbox** cannot handle sparsity
  - ✓ **sss** allows analysis of large-scale, sparse state space models
  - ✓ **sssMOR** allows the automated reduction of sss models with a variety of methods
  - ✓ **ssRed** objects can be used in MATLAB tools to efficiently design reduced order controllers

- ☐ **psssMOR** – analysis and reduction of parametric models

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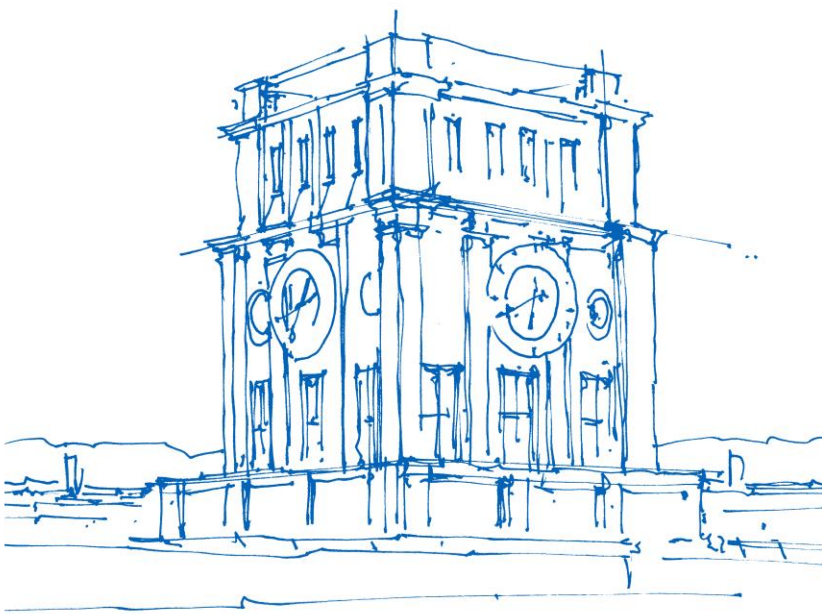

Uhrenturm der TUM

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References

[C. et al. '17]	sss & sssMOR: Analysis and reduction of large-scale dynamic systems in MATLAB
[C./Panzer/Lohmann '16]	Fast H2-optimal model order reduction exploiting the local nature of Krylov subspace methods
[De Villemagne/Skelton '87]	Model reductions using a projection formulation
[Grimme '97]	Krylov projection methods for model reduction
[Gugercin/Antoulas/Beattie '08]	H2-optimal model reduction for large-scale linear dynamical systems
[Gugercin/Beattie '09]	A trust-region method for optimal H2 model reduction
[Meier/Luenberger '67]	Approximation of linear constant systems
[Panzer et al. '14]	Greedy rational Krylov method for H2-pseudooptimal model order reduction with preservation of stability