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On the minimality of the order p^6 chiral Lagrangian

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ABSTRACT

A method to find relations between the operators in the mesonic Lagrangian of Chiral Perturbation Theory at order p^6 is presented. The procedure can be used to establish if the basis of operators in the Lagrangian is minimal. As an example, we apply the method to the two-flavor case in the absence of scalar and pseudo-scalar sources ($s = p = 0$), and conclude that the minimal Lagrangian contains 27 independent operators.

1. Introduction

The global chiral symmetry of the QCD Lagrangian for vanishing quark masses, and its spontaneous breaking to the diagonal group, characterizes the strong interactions among the lightest hadronic degrees of freedom – the pseudoscalar mesons – at low energies. The Nambu–Goldstone nature of these mesons and the mass gap that separates them from the rest of the hadronic spectrum allows one to build an effective field theory (EFT) containing only these modes, with a perturbative expansion in powers of momenta and masses. The framework, called Chiral Perturbation Theory (ChPT), was introduced in its modern form by Weinberg [1], and Gasser and Leutwyler [2,3].

At the lowest order, $\mathcal{O}(p^2)$, the effective ChPT Lagrangian $\mathcal{L}_2$ depends only on two low-energy couplings. One-loop contributions built from the lowest-order vertices generate $\mathcal{O}(p^4)$ divergences that are absorbed by the operators of the next-to-leading-order $\mathcal{L}_4$ Lagrangian [2], introducing seven (ten) additional coupling constants for the two (three) quark flavors case. In the same way, taking the computations to the next-to-next-leading order requires the construction of the effective Lagrangian at $\mathcal{O}(p^6)$. This task was first performed systematically in Ref. [4], and later revisited in [5]. Through the use of partial integration, the equations of motion, Bianchi identities and the Cayley–Hamilton relations for SU(n) matrices, the authors of Ref. [5] managed to write down a basis of operators for $\mathcal{L}_6$ in the even-intrinsic-parity sector for $n = 2$ ($n = 3$) light flavors consisting of 53 (90) terms plus 4 (4) contact terms (*i.e.* terms not containing the pseudo-Goldstone fields, which are only needed for renormalization). In recent years, an additional relation among the operators in the basis of [5] for the $n = 2$ case was proven [6], where no additional manipulations but those already used in [5] were required. This showed that the derivation of an algorithm to exhaust all possible algebraic conditions among the $\mathcal{L}_6$ operators imposed by partial integration, equations of motion, Bianchi identities and, particularly, Cayley–Hamilton relations, is a nontrivial task.

Therefore, the question about the minimality of the $\mathcal{O}(p^6)$ chiral Lagrangian is proper and, to the best of our knowledge, remains unanswered. It is the aim of the present work to describe a method that provides necessary conditions for the existence of additional relations between the operators of the $\mathcal{L}_6$ Lagrangian, and to show its application to the two-flavor case when massless quarks are considered. Our approach does not try to exploit the algebraic conditions mentioned above (and used in [5]), but is rather based on the analysis of Green functions built from arbitrary linear combinations of the operators in the basis. The requirement that the matrix elements built from the latter Green functions must vanish for an arbitrary kinematic configuration is a necessary condition for the linear combination to be true at the operator level. From the method one can conclude that the basis is minimal when the necessary conditions provide no freedom for the existence of new relations. On the other hand, if the method allows for new relations, it cannot immediately answer the question about the minimality of the set, but it has the advantage that it gives the precise form that the (potential) new relations among the operator must have. With the latter information at hand, an algebraic proof of the relation at the operator level shall be greatly simplified.

The method involves the computation of tree-level matrix elements of order p^6. Despite being tree-level, the large number of

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operators in $\mathcal{L}_6$ and their involved Lorentz structure, containing vertices with up to six derivatives, produce rather long expressions. The latter can nevertheless be handled easily with the help of computer tools, and the method lends itself easily to automatization.

The structure of the paper is the following. In Section 2 we provide the basic ingredients of ChPT needed for our analysis. The method that searches for further relations among the $\mathcal{O}(p^6)$ operators is described in Section 3, where details about the calculation of the matrix elements which provide the necessary conditions are given through specific examples. Its application to the two-flavor case in the chiral limit with scalar and pseudo-scalar sources set to zero is then presented in Section 4. Finally, we give our conclusions in Section 5.

2. Chiral perturbation theory

The effective Lagrangian that implements the spontaneous breaking of the chiral symmetry $\mathrm{SU}(n)_L \times \mathrm{SU}(n)_R$ to $\mathrm{SU}(n)_V$ in the meson sector is written as an expansion in powers of derivatives and masses of the pseudo-Goldstone fields [1–3],

$$\mathcal{L} = \sum_{n\geqslant 1} \mathcal{L}_{2n} \,. \tag{2.1}$$

The lowest order reads

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \,, \tag{2.2}$$

where F is the pion decay constant in the chiral limit and $\langle \dots \rangle$ stands for the trace in flavor space. The chiral tensor u_μ,

$$u_\mu = i \left[u^\dagger \left(\partial_\mu - i r_\mu \right) u - u \left(\partial_\mu - i \ell_\mu \right) u^\dagger \right] , \tag{2.3}$$

is built from the Goldstone matrix field

$$u = \exp\left(\frac{i}{\sqrt{2}F} \phi \right) , \tag{2.4}$$

and the left and right $(n \times n)$-dimensional matrix fields in flavor space, $\ell_\mu = v_\mu - a_\mu$, $r_\mu = v_\mu + a_\mu$, with v_μ, a_μ reproducing the couplings of the quarks to the external vector and axial-vector sources, respectively. On the other hand, the tensor χ_+ in (2.2) is built from $\chi = 2B(s+ip)$, with s and p the scalar and pseudo-scalar external matrix fields and B a low-energy parameter. Quark masses are introduced in the ChPT meson amplitudes through the scalar matrix s. Since we restrict ourselves in the specific examples given later to the chiral limit and in addition set p as well as other contributions to s to zero, we can drop all operators containing the χ tensor in what follows.

In the two-flavor case, which will be used for a specific application of our method, the matrix ϕ collects the pion fields,

$$\phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 \end{pmatrix} . \tag{2.5}$$

The vector and axial-vector external fields are general traceless 2×2 matrices,

$$v_\mu = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix}_\mu \qquad \text{and} \qquad a_\mu = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}_\mu , \tag{2.6}$$

since we do not confine ourselves to the Standard Model vector and axial-vector currents, but allow for the parametrization of other possible beyond-the-Standard-Model currents.

The general structure of the $\mathcal{O}(p^6)$ ChPT Lagrangian was studied in [4,5]; adopting the notation of the latter reference, in the $n = 2$ case it reads

$$\mathcal{L}_6^{\mathrm{SU}(2)} = \sum_{i=1}^{53} c_i \mathcal{P}_i + 4 \text{ contact terms} \,, \tag{2.7}$$

where $\mathcal{P}_i$ are the basis elements and c_i are the corresponding low energy constants. In the massless limit with scalar and pseudo-scalar sources set to zero, $27+2$ of the $53+4$ operators in (2.7) remain. For completeness, we give their explicit form in Appendix A.

3. Outline of the method

We describe next the method used to determine the minimal set of monomials of $\mathcal{O}(p^6)$ in the ChPT Lagrangian. It is based on the trivial observation that if a set of operators $\mathcal{P}_i$ from $\mathcal{L}_6$ satisfies a linear relation $\Theta \equiv \sum_i \alpha_i \mathcal{P}_i = 0$, with α_i real or complex numbers, then the matrix elements of (on-shell) pions and currents obtained from the Green functions

$$\begin{aligned} &\langle 0 |\, T\, \phi(x_1) \dots \phi(x_n) \dots j_{f_1}(y_1) \dots j_{f_m}(y_m) \,| 0 \rangle_\Theta \\ &\quad \equiv \frac{(-i)^m}{\mathcal{N}} \frac{\delta^{(m)}}{\delta f_1(y_1) \dots \delta f_m(y_m)} \int [\mathcal{D}\phi]\, \phi(x_1) \dots \phi(x_n) \\ &\qquad \times \left(i \int d^4x\, \Theta(x) \right) \exp i S_{\mathrm{ChPT}}[\phi, f_i] \Big|_{f_i = 0} \,, \end{aligned} \tag{3.8}$$

must also vanish. The Green functions (3.8) are built from $n = 0, 1 \dots$ pion fields (ϕ) and $m = 0, 1 \dots$ vector, axial-vector, scalar or pseudoscalar currents (j_{f_i}), which derive from the ChPT action by functional differentiation with respect the external field sources ($f_i = v,\, a,\, s,\, p$ respectively). The precise definition is given by the path-integral representation provided in the second line of (3.8): the action S_{ChPT}, built from the ChPT Lagrangian (2.1), is a functional of the pion fields and the currents, and the term $\int d^4x\, \Theta(x)$ in the integrand entails that the perturbative expansion of the Green function has Θ in one of the interaction vertices. The normalization $\mathcal{N}$ is fixed such that the Green function with $m = n = 0$ equals one. The corresponding matrix elements involving m currents and n pions are obtained by Fourier transforming (3.8) into momentum space and then amputating the external pion lines and putting them on-shell. Let us note that the vanishing of the matrix elements from (3.8) when the relation $\Theta = 0$ has been obtained using the pion-field equation of motion is only guaranteed if the momenta of the pions are taken on the mass shell. This is because the use of the equations of motion at the operator level can be shown to be equivalent to a redefinition of the pion field in the generating functional [4,5,7], which leaves on-shell S-matrix elements invariant. For off-shell matrix elements, operators that vanish upon use of the equations of motion can give however a non-zero contribution. For our purposes it is sufficient to consider the perturbative computation of the Green function at the leading order in the momentum expansion, which is $\mathcal{O}(p^6)$ because the $\mathcal{P}_i$ operators in the linear combination Θ are already of that order.

The perturbative calculation consists of tree-level diagrams, of the form of a contact interaction, which we shall refer to as "local" in what follows, as well as with intermediate pion exchange ("non-local"); see Fig. 1 for an example. Local contributions contain an $\mathcal{P}_i$ operator at the vertex, whereas non-local contributions have in addition any number of $\mathcal{O}(p^2)$ vertices, which do not change the chiral order of the amplitude. The amplitudes for the matrix elements are rational functions of the momenta, with a pole structure given by the propagators present in the diagrams and a numerator which is a polynomial in the kinematic invariants. If a

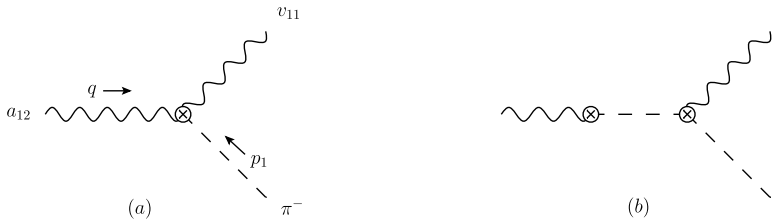

Fig. 1. *a*) Local and *b*) non-local contributions to the $\langle v_{11} a_{12} \pi^- \rangle$ matrix element.

relation between operators holds, the matrix element must vanish for any arbitrary momentum configuration of the fields. This requires that all the coefficients of the terms in the polynomial built from the kinematic invariants are zero, and conditions for the α_i are thus obtained. By requiring that a sufficiently large number of matrix elements computed in this way with increasing number of fields vanish, we obtain a set of conditions for the numerical coefficients α_i in Θ; when these conditions yield non-zero solutions, relations between the operators which are fulfilled for all the processes computed are thus found. One may wish to prove that the relations found hold for matrix elements with an arbitrary number of pions and currents. In that case, the fact that we already know the precise numerical coefficients in the relation between the operators simplifies the task of proving it at the operator level using partial integration, equations of motion, and the Bianchi and Cayley–Hamilton identities. Note also that such a proof may be more a formal matter than one of practical relevance; processes with 6 mesons legs or involving more than two vector or axial-vector currents are rather remote experimentally, so just knowing the relations satisfied among the operators for the phenomenologically relevant processes could be enough.

In order to illustrate how the method works let us consider the computation of the matrix element for two specific cases. The first one involves (3.8) with one external vector (v_{11}), one external axial (a_{12}) and one charged pion field (π^-), which is simple enough to provide explicit formulas. We shall refer to the latter with the abridged notation $\langle v_{11} a_{12} \pi^- \rangle$.

The perturbative computation of this matrix element at $\mathcal{O}(p^6)$ is given by the diagrams in Fig. 1. The operators in Θ contributing to diagram 1a are $\mathcal{P}_{44}$, $\mathcal{P}_{50}$, $\mathcal{P}_{51}$, $\mathcal{P}_{52}$ and $\mathcal{P}_{53}$. For diagram 1b, operators $\mathcal{P}_{51}$, $\mathcal{P}_{52}$ contribute in one of the vertices, whereas the other vertex corresponds to an $\mathcal{O}(p^2)$ interaction. To calculate the amplitude, we take the momenta of the fields incoming and use energy–momentum conservation. We thus have two independent momenta, which we take to be that of the pion, p_1, and that of the axial current, q. In addition we have the "polarization" vectors from the external fields v_{11} and a_{12}, ϵ_v and ϵ_a respectively.[1] Taking into account the on-shell condition for the (massless) pion, $p_1^2 = 0$, the amplitude can then be written in terms of seven different kinematic invariants, $p_1 \cdot q$, $p_1 \cdot \epsilon_v$, $p_1 \cdot \epsilon_a$, $q \cdot \epsilon_v$, $q \cdot \epsilon_a$, $\epsilon_v \cdot \epsilon_a$ and q^2. Adding the result from the diagrams with operators $\mathcal{P}_i$ multiplied by corresponding coefficients α_i, the perturbative amplitude reads

$$\begin{aligned}\mathcal{M} = \frac{1}{q^2}\Big\{ & 4\,(\alpha_{51}-\alpha_{53})\big[\,\epsilon_v\cdot q\,\epsilon_a\cdot q\,(p_1\cdot q)^2 \\ & -\epsilon_v\cdot p_1\,\epsilon_a\cdot q\,(p_1\cdot q)^2 + q^2\,\epsilon_v\cdot p_1\,\epsilon_a\cdot p_1\,p_1\cdot q\,\big] \\ & + (2\alpha_{50}-\alpha_{51}+\alpha_{52}+\alpha_{53})\,q^4\,\epsilon_v\cdot\epsilon_a\,p_1\cdot q \\ & + (\alpha_{51}-\alpha_{52}-\alpha_{53})\,q^2\,\epsilon_v\cdot q\,\epsilon_a\cdot q\,p_1\cdot q \\ & + (2\alpha_{44}-2\alpha_{51}-\alpha_{52}+3\alpha_{53})\,q^2 \\ & \times\big[\,\epsilon_v\cdot p_1\,\epsilon_a\cdot q\,p_1\cdot q - q^2\,\epsilon_v\cdot p_1\,\epsilon_a\cdot p_1\big] \\ & - (2\alpha_{44}+3\alpha_{51}-2\alpha_{52}-2\alpha_{53})\,q^2\,\epsilon_v\cdot\epsilon_a\,(p_1\cdot q)^2 \\ & + (2\alpha_{44}-\alpha_{51}-2\alpha_{52}+2\alpha_{53})\,q^2\,\epsilon_v\cdot q\,\epsilon_a\cdot p_1\,p_1\cdot q \\ & + 2\,\alpha_{50}\,q^4\big[\,q^2\,\epsilon_v\cdot\epsilon_a - \epsilon_v\cdot q\,\epsilon_a\cdot p_1 - \epsilon_v\cdot q\,\epsilon_a\cdot q\big]\Big\}\,, \end{aligned} \tag{3.9}$$

up to a global constant factor, and we have also dropped the Dirac delta function with the momentum conservation. The $1/q^2$ factor arises from the scalar propagator in diagram Fig. 1a; since we have factored out it globally, the resulting polynomial in the numerator is of order z_i^4 in the kinematic invariants $z_i \equiv p_1 \cdot q,\ p_1 \cdot \epsilon_v,\ \ldots$, with the restriction that all monomials must contain both polarization vectors ϵ_v and ϵ_a. The resulting amplitude is therefore of chiral order p^6; recall that the polarization vectors count as $\mathcal{O}(p)$, just like the external fields v_μ, a_μ. The requirement that (3.9) must vanish if a relation between the $\mathcal{O}(p^6)$ operators holds forces the coefficients of all monomials in the numerator to vanish. This translates into the following set of conditions for the α_i:

$$\alpha_{50} = \alpha_{52} = 0 \quad , \quad \alpha_{51} = \alpha_{53} = -2\alpha_{44}\,. \tag{3.10}$$

The first condition in (3.10) implies that no relation involving operators $\mathcal{P}_{50}$ and $\mathcal{P}_{52}$ can be satisfied by the matrix element chosen in this example. Since an operator relation must be true for any process we can already conclude that the operators 50 and 52 belong to the minimal basis of the Lagrangian. The second condition in (3.10) translates into the relation $\mathcal{P}_{44} - 2\mathcal{P}_{51} - 2\mathcal{P}_{53} = 0$ being satisfied for this process. By analyzing other processes we shall conclude in Section 4 that the latter relation is actually part of a larger one involving more terms, that holds exactly for the operators in $\mathcal{L}_6^{\mathrm{SU}(2)}$.

Let us now choose a matrix element with one pion field more, for instance $\langle a_{12} a_{21} \pi^0 \pi^0 \rangle$, which involves two axial-vector currents. This example shall give us an idea of the increasing complexity brought by diagrams with more legs. Fig. 2 shows the diagrammatic contributions to the corresponding matrix element. The pure local term, Fig. 2a, stems from the operators 1–3, 36–44 and 50–53. The non-local contributions include two different type of diagrams: in Figs. 2b, an axial-3π vertex from operators 1–3, 36–38 and 51–53 of the $\mathcal{O}(p^6)$ Lagrangian is combined with the axial-pion vertex from $\mathcal{L}_2^{\mathrm{SU}(2)}$,[2] whereas in Fig. 2c, we need the $\mathcal{O}(p^6)$ 4π vertices from operators $\mathcal{P}_{1\text{-}3}$. The amplitudes for $\langle a_{12} a_{21} \pi^0 \pi^0 \rangle$ depend on 11 independent kinematic invariants, namely $p_1 \cdot p_2$, q^2, $p_1 \cdot q$, $p_2 \cdot q$, $p_1 \cdot \epsilon_{12}$, $p_2 \cdot \epsilon_{12}$, $q \cdot \epsilon_{12}$, $p_1 \cdot \epsilon_{21}$, $p_2 \cdot \epsilon_{21}$, $q \cdot \epsilon_{21}$ and $\epsilon_{12} \cdot \epsilon_{21}$, and we have again considered massless pions. The number of monomials of order p^6 which can be

[1] The introduction of polarization vectors for the external fields is not strictly necessary: we could work with the tensor amplitude with Lorentz indices of the external sources μ, ν left open and require that the coefficients of all tensor structures vanish. The contraction of the tensor amplitude with arbitrary vectors ϵ_v, ϵ_a allows to work with a scalar function, which simplifies handling the long expressions that are obtained for the amplitudes of Green functions with more fields.

[2] We note that there is no axial-π vertex in $\mathcal{L}_6^{\mathrm{SU}(2)}$ with massless pions.

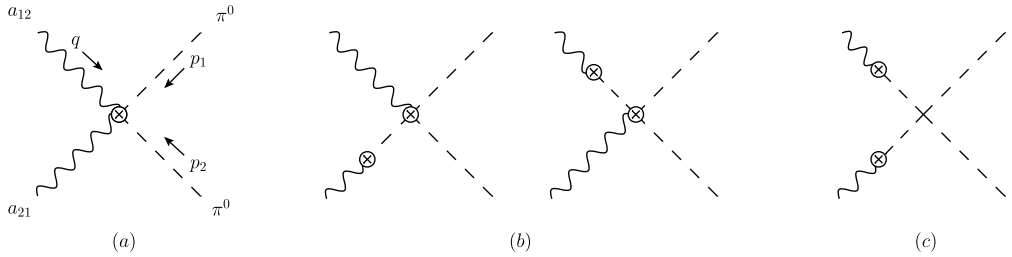

Fig. 2. (*a*) Local and (*b*), (*c*) non-local contributions to the $\langle a_{12}a_{21}\pi^0\pi^0\rangle$ matrix element.

built out of the kinematic invariants is therefore large, and handling the amplitude in order to find out the conditions for the α_i requires automatization. For this task, we have implemented the computation of the tree-level matrix elements at $\mathcal{O}(p^6)$ and the extraction of the relations for the α_i in a MATHEMATICA code. In the case at hand, $\langle a_{12}a_{21}\pi^0\pi^0\rangle$, one obtains an amplitude with 132 independent monomials in the numerator, whose coefficients yield the equations for α_i: 50 of these equations are non-trivially identical, but only 10 turn out to be independent. The solution to this system then provides 10 relations among the coefficients α_i of the 16 operators that contribute to $\langle a_{12}\,a_{21}\pi^0\pi^0\rangle$:

$$\begin{aligned}
&\alpha_{38}=\alpha_{50}=0\,,\quad \alpha_1=-4\alpha_2=\frac{4}{3}\alpha_3=\alpha_{36}=-\alpha_{37}\,,\\
&\alpha_{51}=\alpha_{53}\,,\\
&3\alpha_1-2\alpha_{41}-2\alpha_{42}+4\alpha_{43}-4\alpha_{51}=0\,,\\
&\alpha_1+8\alpha_{39}-8\alpha_{40}+6\alpha_{41}+6\alpha_{42}-12\alpha_{43}-8\alpha_{44}=0\,,\\
&\alpha_1+2\alpha_{39}-2\alpha_{40}+\alpha_{41}+\alpha_{42}-2\alpha_{43}-\alpha_{52}=0\,.
\end{aligned}\tag{3.11}$$

Taking into account these conditions together, one thus finds that the linear combination

$$\begin{aligned}
\Theta=\alpha_1\Big(&\mathcal{P}_1-\frac{1}{4}\mathcal{P}_2+\frac{3}{4}\mathcal{P}_3+\mathcal{P}_{36}-\mathcal{P}_{37}-\frac{3}{4}\mathcal{P}_{40}-\mathcal{P}_{41}-\mathcal{P}_{42}\\
&+\frac{1}{4}\mathcal{P}_{43}-\mathcal{P}_{44}+2\mathcal{P}_{51}+2\mathcal{P}_{53}\Big)\\
&+\alpha_{39}\Big(\mathcal{P}_{39}-\mathcal{P}_{40}-2\,\mathcal{P}_{41}+\mathcal{P}_{43}-\mathcal{P}_{44}+2\,\mathcal{P}_{51}+2\,\mathcal{P}_{53}\Big)
\end{aligned}\tag{3.12}$$

makes the amplitude $\langle a_{12}a_{21}\pi^0\pi^0\rangle$ vanish for arbitrary values of α_1 and α_{39}, implying that the two relations among the $\mathcal{P}_i$ operators between parenthesis in (3.12) are equal to zero for this particular process. We can proceed in the same way for other matrix elements and require a simultaneous vanishing of all of them by solving for the α_i. The latter is a necessary condition for the existence of a relation between the $\mathcal{O}(p^6)$ operators. In the next section we show that the procedure eventually allows for just two relations in the SU(2) case without scalar and pseudo-scalar external fields.

4. SU(2) case with $s=p=0$

As a proof of concept we show in this section how the method described above applies to the two-flavor ChPT Lagrangian in the chiral limit and without additional external scalar or pseudo-scalar sources ($s=p=0$) but $v_\mu, a_\mu\neq 0$. This simplified framework does not lack of phenomenological relevance: it provides a very good approximation to the low-energy interaction of the pions in the presence of electroweak currents, since mass corrections in the u, d quark sector are small and there are no other contributions to the external sources s and p in the Standard Model.[3]

Within this framework, we have computed the matrix elements from (3.8) with the generic field content as listed in the first column of Table 1. The notation $\langle va\,3\pi\rangle$, for instance, stands for all processes involving three pion fields (charged or neutral) and one vector and one axial-vector field component, and similarly for the rest. The second column indicates which $\mathcal{O}(p^6)$ operators contribute to the Green functions. The relations among the operators satisfied for each process, obtained as in the examples of Section 3 by solving a system of equations for the coefficients α_i, are then given in the third column. We have not written the equations for the α_i for each process except for the cases where they require some of the α_i to vanish; the condition $\alpha_i=0$ obtained for a given matrix element already implies that the corresponding operator $\mathcal{P}_i$ cannot be part of any relation, which is an important information. We note that the relations written in Table 1 guarantee that all matrix elements with arbitrary charge (or isospin) configuration of the pion and ChPT currents vanish. For a given charge (isospin) channel additional relations among the operators that contribute can exist, which we do not provide in Table 1.

The relations satisfied for a set of processes can be obtained by combining the equations for the coefficients α_i from each process and looking for a compatible solution. From the table one sees that the combination of matrix elements $\langle vv\rangle$, $\langle vaa\rangle$, $\langle vv\,2\pi\rangle$, $\langle aa\,2\pi\rangle$, $\langle v\,4\pi\rangle$ and $\langle 6\pi\rangle$ already involve all the operators in the SU(2) ChPT Lagrangian with $s=p=0$. The fact that operators $\mathcal{P}_{45}$ and $\mathcal{P}_{55}$ only appear in $\langle vaa\rangle$ requires a further matrix element depending on $\mathcal{P}_{45}$ in order to fix it completely. That is why the matrix element $\langle vva\pi\rangle$ is also computed. The results for the rest of processes in Table 1 is given for completeness; their computation also serves us as a check of the relations found with the minimal set of processes.

Combining the equations for α_i found for the different matrix elements we get that all the latter vanish provided

$$\begin{aligned}
&\alpha_{38}=\alpha_{50}=\alpha_{52}=\alpha_{55}=\alpha_{56}=0\,,\\
&\alpha_1\Big(8\,\mathcal{P}_1-2\,\mathcal{P}_2+6\,\mathcal{P}_3-20\,\mathcal{P}_{24}+8\,\mathcal{P}_{25}+12\,\mathcal{P}_{26}-16\,\mathcal{P}_{28}\\
&\quad-3\,\mathcal{P}_{29}+3\,\mathcal{P}_{30}-6\,\mathcal{P}_{31}+12\,\mathcal{P}_{32}-3\,\mathcal{P}_{33}+8\,\mathcal{P}_{36}-8\,\mathcal{P}_{37}\\
&\quad-11\,\mathcal{P}_{39}+5\,\mathcal{P}_{40}+14\,\mathcal{P}_{41}-8\,\mathcal{P}_{42}-9\,\mathcal{P}_{43}\\
&\quad+3\,\mathcal{P}_{44}-3\,\mathcal{P}_{45}-6\,\mathcal{P}_{51}-6\,\mathcal{P}_{53}\Big)\\
&+\alpha_{27}\Big(8\,\mathcal{P}_{27}+8\,\mathcal{P}_{28}-2\,\mathcal{P}_{29}+2\,\mathcal{P}_{30}-4\,\mathcal{P}_{31}+8\,\mathcal{P}_{32}\\
&\quad-2\,\mathcal{P}_{33}-2\,\mathcal{P}_{39}+2\,\mathcal{P}_{40}+4\,\mathcal{P}_{41}-2\,\mathcal{P}_{43}+2\,\mathcal{P}_{44}\\
&\quad-2\,\mathcal{P}_{45}-4\,\mathcal{P}_{51}-4\,\mathcal{P}_{53}\Big)=0\,,
\end{aligned}\tag{4.13}$$

[3] Let us recall that at low energies the scalar $q\bar{q}$ interaction with the Higgs produces terms in the amplitude suppressed by $1/m_h^2$.

Table 1
Relations among $\mathcal{O}(p^6)$ operators satisfied for each of the matrix elements computed. The second column lists the operators that contribute in each case.

Matrix element	$\mathcal{P}_i$	Operator relations
$\langle v\,v\rangle$	56	$\alpha_{56}=0$
$\langle v\,2\pi\rangle$	51, 53	$\mathcal{P}_{51}+\mathcal{P}_{53}=0$
$\langle va\pi\rangle$	44, 50–53	$\alpha_{50}=\alpha_{52}=0$ $\mathcal{P}_{44}-2\mathcal{P}_{51}-2\mathcal{P}_{53}=0$
$\langle vaa\rangle$	44, 45, 50–53, 55	$\alpha_{50}=\alpha_{52}=0$ $3\mathcal{P}_{45}+8\mathcal{P}_{55}=0$ $\mathcal{P}_{44}-\mathcal{P}_{45}-2\mathcal{P}_{51}-2\mathcal{P}_{53}=0$
$\langle 4\pi\rangle$	1–3	$4\mathcal{P}_1-\mathcal{P}_2+3\mathcal{P}_3=0$
$\langle v\,v\,2\pi\rangle$	29–33, 44, 50–53	$\alpha_{50}=\alpha_{52}=0$ $\mathcal{P}_{29}-\mathcal{P}_{30}+2\mathcal{P}_{31}-4\mathcal{P}_{32}+\mathcal{P}_{33}-\mathcal{P}_{44}+2\mathcal{P}_{51}+2\mathcal{P}_{53}=0$
$\langle aa\,2\pi\rangle$	1–3, 36–44, 50–53	$\alpha_{38}=\alpha_{50}=\alpha_{52}=0$ $\mathcal{P}_{39}-\mathcal{P}_{40}-2\mathcal{P}_{41}+\mathcal{P}_{43}-\mathcal{P}_{44}+2\mathcal{P}_{51}+2\mathcal{P}_{53}=0$ $4\mathcal{P}_1-\mathcal{P}_2+3\mathcal{P}_3+4\mathcal{P}_{36}-4\mathcal{P}_{37}-4\mathcal{P}_{39}+\mathcal{P}_{40}+4\mathcal{P}_{41}-4\mathcal{P}_{42}-3\mathcal{P}_{43}=0$
$\langle a\,3\pi\rangle$	1–3, 36–38, 51, 53	$\alpha_{38}=0$ $4\mathcal{P}_1-\mathcal{P}_2+3\mathcal{P}_3+4\mathcal{P}_{36}-4\mathcal{P}_{37}=0$ $\mathcal{P}_{51}+\mathcal{P}_{53}=0$
$\langle v\,v\,a\,\pi\rangle$	29–33, 44, 45, 50-53	$\alpha_{50}=\alpha_{52}=0$ $4\mathcal{P}_{29}-\mathcal{P}_{30}+2\mathcal{P}_{31}-4\mathcal{P}_{32}+\mathcal{P}_{33}-\mathcal{P}_{44}+\mathcal{P}_{45}+2\mathcal{P}_{51}+2\mathcal{P}_{53}=0$
$\langle v\,4\pi\rangle$	1–3, 27–28, 36–38, 51–53	$\alpha_{38}=0$ $4\mathcal{P}_1-\mathcal{P}_2+3\mathcal{P}_3-6\mathcal{P}_{27}-14\mathcal{P}_{28}+4\mathcal{P}_{36}-4\mathcal{P}_{37}=0$ $2\mathcal{P}_{27}+2\mathcal{P}_{28}-\mathcal{P}_{51}-\mathcal{P}_{53}=0$
$\langle va\,3\pi\rangle$	1–3, 27–44, 50–53	$\alpha_{38}=\alpha_{50}=\alpha_{52}=0$ $\mathcal{P}_{29}+\mathcal{P}_{39}=0$ $\mathcal{P}_{31}+\mathcal{P}_{32}+\mathcal{P}_{41}+\mathcal{P}_{42}=0$ $8\mathcal{P}_1-2\mathcal{P}_2+6\mathcal{P}_3-12\mathcal{P}_{27}-28\mathcal{P}_{28}+8\mathcal{P}_{36}-8\mathcal{P}_{37}-8\mathcal{P}_{39}+2\mathcal{P}_{40}+8\mathcal{P}_{41}-8\mathcal{P}_{42}-6\mathcal{P}_{43}=0$ $4\mathcal{P}_{27}+4\mathcal{P}_{28}+\mathcal{P}_{30}-2\mathcal{P}_{31}+4\mathcal{P}_{32}-\mathcal{P}_{33}+\mathcal{P}_{40}+2\mathcal{P}_{41}-\mathcal{P}_{43}+\mathcal{P}_{44}-2\mathcal{P}_{51}-2\mathcal{P}_{53}=0$
$\langle 6\pi\rangle$	1–3, 24–26	$4\mathcal{P}_1-\mathcal{P}_2+3\mathcal{P}_3-10\mathcal{P}_{24}+4\mathcal{P}_{25}+6\mathcal{P}_{26}=0$

which holds for whatever values of α_1 and α_{27}, meaning that the two linear combinations among the operators $\mathcal{P}_i$ between parenthesis must vanish independently. The relations obtained can be simplified if one uses the second linear combination into the first one. In this way we find:

$$4\,\mathcal{P}_{27}+4\,\mathcal{P}_{28}-\mathcal{P}_{29}+\mathcal{P}_{30}-2\,\mathcal{P}_{31}+4\,\mathcal{P}_{32}-\mathcal{P}_{33}-\mathcal{P}_{39}+\mathcal{P}_{40}+2\,\mathcal{P}_{41}-\mathcal{P}_{43}+\mathcal{P}_{44}-\mathcal{P}_{45}-2\,\mathcal{P}_{51}-2\,\mathcal{P}_{53}=0\,, \tag{4.14}$$

$$8\,\mathcal{P}_1-2\,\mathcal{P}_2+6\,\mathcal{P}_3-20\,\mathcal{P}_{24}+8\,\mathcal{P}_{25}+12\,\mathcal{P}_{26}-12\,\mathcal{P}_{27}-28\,\mathcal{P}_{28}+8\,\mathcal{P}_{36}-8\,\mathcal{P}_{37}-8\,\mathcal{P}_{39}+2\,\mathcal{P}_{40}+8\,\mathcal{P}_{41}-8\,\mathcal{P}_{42}-6\,\mathcal{P}_{43}=0\,. \tag{4.15}$$

The result (4.14) agrees with a relation which is known to hold among the $\mathcal{O}(p^6)$ operators when the scalar and pseudo-scalar sources are set to zero [8].[4] Likewise, (4.15) matches the additional relation found for the SU(2) case in [6], once the operators depending on scalar and pseudo-scalar tensor source χ are neglected in the latter. Since relations (4.14), (4.15) were proven algebraically in these references, they are of course satisfied for all matrix elements with any number of pions and currents. We can moreover state that these are the only two relations between the SU(2) ChPT operators of $\mathcal{O}(p^6)$ in the limit $s=p=0$; otherwise any further relation of the form $\sum\alpha_i'\mathcal{P}_i=0$ would have been obtained from the analysis of the functions of Table 1 with our method (let us recall that the vanishing of any matrix element with an insertion of $\sum\alpha_i'\mathcal{P}_i$ is a necessary condition for the existence of the relation). We therefore conclude that the set of minimal operators of the SU(2) ChPT Lagrangian of $\mathcal{O}(p^6)$ with scalar and pseudo-scalar sources set to zero reduces from the 27+2 operators initially written down in [5] to 25+2 (note that the contact terms do not take part in any of the relations above). Eqs. (4.14), (4.15) can be used to drop two of the 27 basis elements of the set of [5].

The application of our method to the general two- and three-flavor cases is straightforward. For SU(2) including scalar and pseudo-scalar sources, if a similar analysis does not yield additional relations to that of Ref. [6], it would ascertain that the basis of $\mathcal{O}(p^6)$ operators from [5] is minimal up to one term. The case of SU(3) is more involved at the technical level, since we have to consider an octet of pseudo-Goldstone bosons and many more matrix elements can be built. Starting the analysis of processes with less number of fields, one could expect that the space of solutions for the coefficients α_i is either very much constrained, and eventually no solution is allowed after computing a few matrix elements, or that it actually allows for one (or more) relations among the operators. In the former case one could already conclude that the basis of $\mathcal{L}_6^{\mathrm{SU(3)}}$ is minimal. In the latter, one may try to check if the relations found from the analysis of the simpler processes also hold at the level of the operators (*i.e.* for any matrix element with an arbitrary number of fields) by using the same algebraic manipulations as in [5], with the great advantage that one would know beforehand the coefficients that the operators participating in the relation must have. The study of the general two- and three-flavor cases with the automated tools developed in this work will be the subject of future investigation.

5. Summary

The large number of low-energy constants in the mesonic chiral Lagrangian of order p^6 makes their determination by direct com-

[4] Ref. [8] provided relation (4.14) for a number of flavors $n=3$ using the SU(3) operator numbering introduced in [5]. The corresponding relation for SU(2) can be obtained by translating into the SU(2) numbering scheme for the operators, and further using that the operator $\mathcal{P}_{52}$ in the two-flavor case is equal to $-\mathcal{P}_{50}-\mathcal{P}_{51}$ (*i.e.* to $-\mathcal{P}_{27}-\mathcal{P}_{28}$ in the SU(2) numbering scheme).

parison with the experiment rather difficult. To simplify this task, one would like to eliminate possible redundancies by establishing the minimal set of independent operators in $\mathcal{L}_6$, that parametrize the rational part of the $\mathcal{O}(p^6)$ chiral amplitudes.

We have described in this paper a method to search for additional relations among the basis operators that build the $\mathcal{O}(p^6)$ SU(n) chiral Lagrangian. It relies on the computation of tree-level amputated Green functions with insertions of the $\mathcal{L}_6$ operators, which are then required to vanish on-shell for an arbitrary kinematic configuration. The method can be used to establish the minimal basis of operators in the Lagrangian. This has been done in the present work for the two-flavor $\mathcal{O}(p^6)$ Lagrangian without scalar and pseudo-scalar external sources. For this case we have shown that the original basis of 27 measurable terms plus 2 contact terms written in [5] in the even-intrinsic-parity sector has 25+2 independent terms, where the two additional relations between operators that emerge from our method had been already noticed in the literature [6,8].

As a next step, the method shall be applied to determine the minimal basis of operators in the SU(2) case with general scalar and pseudo-scalar sources, as well as in SU(3). Furthermore, one can expect that the method extends naturally to other relevant effective actions containing a large number of operators, and in particular to the linear and non-linear effective theories that describe the breaking of the electroweak symmetry.

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Appendix A

We provide in this appendix the explicit form of the operators in the $\mathcal{O}(p^6)$ ChPT Lagrangian in the SU(2) case (2.7) without scalar and pseudo-scalar sources. The expressions are read off from the list given in the appendix C of Ref. [5] by discarding terms containing the χ tensor.

Besides the chiral tensors already written in Section 2, the following building blocks are needed to construct the operators in Table 2:

$$h_{\mu\nu} = \nabla^\mu u_\nu + \nabla^\nu u_\mu\,,$$
$$f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u\,, \quad \nabla_\rho f_\pm^{\mu\nu}\,, \tag{A.16}$$

with the non-abelian field strength tensor built from the right and left external fields,

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i\left[\ell^\mu, \ell^\nu\right],$$
$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i\left[r^\mu, r^\nu\right] \tag{A.17}$$

Table 2
$\mathcal{O}(p^6)$ operators for SU(2) with $s = p = 0$, in the basis of [5]. The label in the first column refers to the SU(2) numbering scheme used in the latter reference. The last column indicates the simplest process to which the operator contributes.

i	$\mathcal{P}_i$	Matrix element
1	$\langle u\cdot u h_{\mu\nu} h^{\mu\nu}\rangle$	$\langle 4\pi\rangle$
2	$\langle h_{\mu\nu} u_\rho h^{\mu\nu} u^\rho\rangle$	$\langle 4\pi\rangle$
3	$\langle h_{\mu\nu}\left(u_\rho h^{\mu\rho} u^\nu + u^\nu h^{\mu\rho} u_\rho\right)\rangle$	$\langle 4\pi\rangle$
24	$\langle (u\cdot u)^3\rangle$	$\langle 6\pi\rangle$
25	$\langle u\cdot u u_\mu u_\nu u^\mu u^\nu\rangle$	$\langle 6\pi\rangle$
26	$\langle u_\mu u_\nu u_\rho u^\mu u^\nu u^\rho\rangle$	$\langle 6\pi\rangle$
27	$\mathrm{i}\,\langle f_{+\mu\nu} u_\rho u^\mu u^\nu u^\rho\rangle$	$\langle v\,4\pi\rangle$
28	$\mathrm{i}\,\langle f_{+\mu\nu} u^\mu u\cdot u u^\nu\rangle$	$\langle v\,4\pi\rangle$
29	$\langle u\cdot u f_{+\mu\nu} f_+^{\mu\nu}\rangle$	$\langle v\,v\,2\pi\rangle$
30	$\langle f_{+\mu\nu} u_\rho f_+^{\mu\nu} u^\rho\rangle$	$\langle v\,v\,2\pi\rangle$
31	$\langle f_{+\mu\nu} f_+^{\mu\rho} u^\nu u_\rho\rangle$	$\langle v\,v\,2\pi\rangle$
32	$\langle f_{+\mu\nu} f_+^{\mu\rho} u_\rho u^\nu\rangle$	$\langle v\,v\,2\pi\rangle$
33	$\langle f_{+\mu\nu}\left(u_\rho f_+^{\mu\rho} u^\nu + u^\nu f_+^{\mu\rho} u_\rho\right)\rangle$	$\langle v\,v\,2\pi\rangle$
36	$\langle f_{-\mu\nu}\left(h^{\nu\rho} u_\rho u^\mu + u^\mu u_\rho h^{\nu\rho}\right)\rangle$	$\langle a\,3\pi\rangle$
37	$\langle f_{-\mu\nu} h^{\nu\rho}\rangle\langle u^\mu u_\rho\rangle$	$\langle a\,3\pi\rangle$
38	$\langle f_{-\mu\nu}\left(u^\mu h^{\nu\rho} u_\rho + u_\rho h^{\nu\rho} u^\mu\right)\rangle$	$\langle a\,3\pi\rangle$
39	$\langle u\cdot u f_{-\mu\nu} f_-^{\mu\nu}\rangle$	$\langle aa\,2\pi\rangle$
40	$\langle f_{-\mu\nu} u_\rho f_-^{\mu\nu} u^\rho\rangle$	$\langle aa\,2\pi\rangle$
41	$\langle f_{-\mu\nu} f_-^{\mu\rho} u^\nu u_\rho\rangle$	$\langle aa\,2\pi\rangle$
42	$\langle f_{-\mu\nu} f_-^{\mu\rho} u_\rho u^\nu\rangle$	$\langle aa\,2\pi\rangle$
43	$\langle f_{-\mu\nu}\left(u_\rho f_-^{\mu\rho} u^\nu + u^\nu f_-^{\mu\rho} u_\rho\right)\rangle$	$\langle aa\,2\pi\rangle$
44	$\mathrm{i}\,\langle f_{+\mu\nu}[f_-^{\nu\rho}, h_\rho^\mu]\rangle$	$\langle va\,\pi\rangle$
45	$\mathrm{i}\,\langle f_{+\mu\nu}[f_-^{\nu\rho}, f_{-\rho}^\mu]\rangle$	$\langle vaa\rangle$
50	$\langle \nabla_\rho f_{-\mu\nu} \nabla^\rho f_-^{\mu\nu}\rangle$	$\langle aa\rangle$
51	$\mathrm{i}\,\langle \nabla_\rho f_{+\mu\nu}[h^{\mu\rho}, u^\nu]\rangle$	$\langle v\,2\pi\rangle$
52	$\mathrm{i}\,\langle \nabla^\mu f_{+\mu\nu}[f_-^{\nu\rho}, u_\rho]\rangle$	$\langle va\,\pi\rangle$
53	$\mathrm{i}\,\langle \nabla^\mu f_{+\mu\nu}[h^{\nu\rho}, u_\rho]\rangle$	$\langle v\,2\pi\rangle$
Contact terms		
55	$\mathrm{i}\,\langle F_{L\mu\nu} F_L^{\mu\rho} F_{L\rho}^\nu\rangle + L\to R$	$\langle vaa\rangle$
56	$\langle D_\rho F_{L\mu\nu} D^\rho F_L^{\mu\nu}\rangle + L\to R$	$\langle vv\rangle$

and the covariant derivative defined as

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]\,, \tag{A.18}$$

where

$$\Gamma_\mu = \frac{1}{2}\{u^\dagger(\partial_\mu - i r_\mu) u + u(\partial_\mu - i\ell_\mu) u^\dagger\}\,. \tag{A.19}$$

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