Technische Universität München

Faculty of Mathematics
Chair for Applied Geometry and Discrete Mathematics

Optimal Containment – Geometric Inequalities, Extremal Sets, and Applications

Cumulative Habilitation Thesis by René Brandenberg

Mentors: Prof. Dr. Peter Gritzmann
Prof. Dr. Martin Henk
Prof. Dr. Andreas Schulz

Thesis summary

1 Introduction

The thesis [11] collects several results on radii of convex bodies in Euclidean spaces. Most of them concern geometric inequalities between them, but also the closely related matter of bodies of constant width is tackled and generalized.
Most of the papers in this collection take a wider view of the subject. Leaving the concentration on Euclidean space behind, we will see that one should do two steps in one. The typical generalization in the literature to *only* allow general *0-symmetric* gauge bodies (thus going over to general normed spaces) often is an unnecessary restriction. It turns out that allowing arbitrary gauge bodies simplifies and unifies the matter in many cases, allowing new results even for the symmetric case. The idea of allowing even non-symmetric gauge bodies stems from the interpretation of the involved problems as optimal containment problems under homothety: cover a given geometric object with a minimally dilated translate of another geometric object. The object used to cover is usually called the container and corresponds to the gauge body, when translating back to radii.
All papers collected deal with aspects of geometric optimal containment problems and therefore (as can be seen below) with the radii of convex bodies. For this purpose optimal containment under homothety serves somehow as the base problem of the whole subject. This is why it is so important to understand it in depth. With this problem as the central building block we will handle the four basic radii (in- and circumradius, width and diameter), extensions like radii of j-dimensional intersections, coverings with several copies of the container, and also contaiment problems under wider classes of transformations (e. g., allowing also rotations or affine transformations). There exist quite a few direct applications to "real life"-problems, from which some are described in [33, 35] and others in the according sections. However, most important are the results in the development of the area itself and in applying them to closely related fields.
Let us list a few of the most important subjects and key words dealt with in the cumulated papers:

- A fundamental subject in convex geometry and far beyond are *geometric inequalities*, a field that includes such famous inequalities as the isoperimetric inequality or the Brunn-Minkowski inequality (see [14] for an extensive summary on geometric inequalities).

  There also exist many inequalities only relating radii and among them those concerning the basic radii have a long history, too. The most prominent may be *Jung's inequality* [48], upper bounding the circumradius-diameter ratio in

Euclidean space. Jung's inequality "is intimitely connected with the well-known *Helly theorem*" [14, p. 11.1.1] as we will see below. It is cited and used as a tool in important publications of many other subjects (see e. g. [21], [59], and [69]) and has been generalized in several ways before (see e. g. [6, Proposition 4], [19], and [40]).

- The Jung constant of an arbitrary normed space measures the upper bound for the circumradius-diameter ratio within this space. *Bohnenblust's inequality* [9] gives an upper bound for the Jung constants over all normed spaces.

  The well-known *Hadamard matrices* are closely related to Bohnenblust's inequality. In [20] it is shown that the Jung constant of an n-dimensional ℓ_1-space attains the bound of Bohnenblust's inequality, if and only if there exists a Hadamard matrix of order $n+1$.

- The *bodies of constant width* are an active part of research in convex geometry and the existence of such bodies besides the ball was already known to Euler [25]. The best known examples are the Reuleaux triangle in the plane (described in [63]) and the Meißner bodies in 3-space (see [10] for a description). There are quite a few challenging open questions concerning these bodies, e. g., "among all bodies of the same constant width, which minimizes the volume"? While in 2-space it is well known that the Reuleaux triangle is the corresponding minimizer, this question is open even in 3-space. The Meißner bodies are often conjectured to be the minimizers. However, Danzer suspected that a body of constant width having the same symmetry group as the regular simplex will attain the minimum (see [36, p. 261] and [50]). The Minkowski sum of the two Meißner bodies is an example with the same symmetry group as a tetrahedron, but it has a bigger volume as the two Meißner bodies. This fact easily follows from the Brunn-Minkowski inequality, which is pointed out already in [50].

  Bodies of constant width serve as extremal bodies for many geometric inequalities, especially as in Euclidean space constant width and *completeness* of a convex body fall together. Even though it was believed for (at least) sixteen years in the 20th century that this is true in general normed spaces [51], it was shown to be false in [24]. One speaks about a *perfect norm* if in the corresponding space every complete body is of constant width. Characterizing spaces equipped with a perfect norm is an outstanding open problem [58].

- Another fundamental issue in convex geometry and in applications like pattern recognition (see e. g. [77]) are *asymmetry measures*.

  The paper of Grünbaum [37] is an important source for the theory of measuring central symmetry. Since then almost all results obtained still cite [37], regardless if concerning the description of asymmetry measures or involving asymmetry measures in geometric inequalities or their stability analysis. However, the matter is still vivid; just recently Toth published the book "Measures of Symmetry for Convex Sets and Stability" [72].

Possibly the asymmetry measure most commonly used is the *Minkowski asymmetry*, which measures the smallest dilatation factor needed to cover a set K by its origin reflection $-K$. It can also be phrased as the circumradius of K with respect to $-K$. From the many (partly recent) results concerning the Minkowski asymmetry an important one for applications should be mentioned already: the Minkowski asymmetry of any polytope given by its vertices or facets can be obtained via linear programming [7].

- *John's theorem* [47] giving optimality conditions for containment under affinities and the corresponding inequality play an important role in this work, too. On the one hand a similar characterization for containment under homothety is used frequently. On the other hand a simplified proof is given for a sharpening of John's inequality. Using an asymmetry measure quite similar to the one of Minkowski, this sharpening has originaly been derived in [7],

- Closely related to John's theorem is the *Banach-Mazur distance* (see [32, Theorem 3.5 and Theorem 3.8] for optimality conditions for containment under affinities similar to that of John). Determining a tight upper bound for the Banach-Mazur distance of two general convex sets is a famous challenging open problem on the edge between functional analysis and convex geometry. Surely, this problem falls into the topic of geometric inequalities. A bit surprising at first sight may be the fact that it is also very closely related to the Minkowski asymmetry. It is already stated in [37] that the Minkowski asymmetry of any body K measures its minimal Banach-Mazur distance to any symmetric set. This fact is used in quite a few recent results tightening the upper bound on the maximal Banach-Mazur distance in terms of the asymmetry of the involved bodies (see, e. g., [32, Corollary 5.10]).

- The possibly most directly applied problem tackled in this habilitation is the so called *k-center problem*. When considering the circumradius one wants to cover a set of points (customers) with a single circle/ball (a facility and its serving range). In the k-center problem, we are allowed to place k copies of the same ball (serving range) at different centers (locations) for covering all data points (customers). In this sense the circumradius is just the special case $k = 1$.

  This problem is somehow the prototype of a geometric facility location problem. Facility location problems are an important class of problems in discrete optimization and have many applications (see e. g. [74, Chapters 5, 24, 25] and below).

  Whenever it is not the accumulated distance to the facilities to be placed what matters, but the maximal distance of an individual, then one becomes a sort of k-center problem. This is quite obvious, e. g. for emergency helicopter bases or fire alarm systems [17], but also applies for the placement of transmission towers with a limited range. Another example is the placing of charging stations for electric cars by a car-sharing company. The stations should be placed such that customers are willing to bring the cars back to one of them at the end of their trip. Not putting a high charge for not bringing the car to a loading unit, customers will only do it, if the walking distance afterwards is short enough. This leads to (a variant of) the k-center problem.

- The notion *core-set* is extremely young in comparision to the other notions above. It is introduced as a dimension reduction tool in computational geometry in general (see [2] for an overview) and especially for 1- and k-center problems [3, 4]. Within a decade these papers have been cited quite often and the results found applications in fields like machine learning [73] or linear programming [54] via Khachiyan's ellipsoid method [52].

  It will be shown below that there is a close relation to the generalized radii of convex sets as e. g. considered in [40]. And the bridge is built via Helly's theorem, which also brings the *Helly dimension* into the matter.

Connections to the topics above and to other important research fields in convex geometry will be frequently outlined below. However, before going into details we need some notation.

This is a cumulative habilitation encompassing papers of mine on variants of (geometric) optimal containment problems. The notation used in this summary unifies the presentation used in these papers and therefore may differ from the original ones. However, it should always be easy to translate between the newer and the older notation.

If $k \in \mathbb{N}$ we abbreviate $[k] := \{1, \ldots, k\}$.
For any $A, B \subset \mathbb{R}^n$ we denote by $\mathrm{bd}(A)$ the *boundary* of A and by $\mathrm{lin}(A), \mathrm{aff}(A)$, and $\mathrm{conv}(A)$ the *linear hull*, the *affine hull*, and *convex hull* of A, respectively. For short, we use $\langle s \rangle := \mathrm{lin}(\{s\})$, $s \in \mathbb{R}^n \setminus \{0\}$ and $[x, y]$ to denote the *line segment* $\mathrm{conv}(\{x, y\})$ between x and y in $\mathbb{R}^n$. The *Minkowski sum* of A and B is defined by $A + B := \{a + b \ : \ a \in A, b \in B\}$. We write $\rho A := \{\rho x : x \in A\}$, $\rho \in \mathbb{R}$, for the *ρ-dilation* of A (where a negative ρ means that the $|\rho|$-dilation of $-A$, i. e. A mirrored at the origin, is considered).
A polytope P is given in *$\mathcal{V}$-representation*, if we know $p^1, \ldots, p^k \in \mathbb{R}^n$ such that $P = \mathrm{conv}\left\{p^1, \ldots, p^k\right\}$ and it is given in *$\mathcal{H}$-representation*, if we know $a^1, \ldots, a^m \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ such that $P = \left\{x \in \mathbb{R}^n : (a^i)^T x \leq b_i, \ i \in [m]\right\}$.
Throughout this summary $S, T, P \subset \mathbb{R}^n$ denote a simplex, a regular simplex, and a (general) polytope, respectively. while $K, C \subset \mathbb{R}^n$ denote *(convex) bodies* , i. e. compact

convex sets (not neccessarily full-dimensional), and $\mathcal{K}^n$ the family of all bodies in n-space. We often call C a *container*. However, if C contains 0 in its interior (thus being full-dimensional), we also call it a *gauge body*. If C is a gauge body it induces a *gauge function* $\|x\|_C := \min\{\rho > 0 : x \in \rho C\}$, which defines a *norm* if C is 0-symmetric, i. e. if $C = -C$. If the gauge function induced by C is a p-norm, $p \in [1, \infty]$, we write $\mathbb{B}_p$ instead of C for its unit ball and $\mathbb{S}_p$ for the corresponding unit sphere.

We write $K \subset_t C$ to express that there exists a translation vector $c \in \mathbb{R}^n$, such that $K \subset C + c$. The *outer radius* (or *circumradius*) of K with respect to C is defined by

$$R(K, C) := \min\{\rho : K \subset_t \rho C\}.$$

While the definition of the outer radius would easily allow general sets for the first and the second argument, we usually restrict to K and C as defined above. This is important in certain results on the behaviour of the outer radius. One should recognize that for general $A \subset \mathbb{R}^n$ we obtain $R(A, C) := R(\mathrm{conv}(A), C)$ due to the convexity of C. Moreover, it is shown in [H7] that unboundedness of the arguments in general may cause problems. However, this is not the case when only allowing lineality spaces to cause the unboudedness, i. e. when the argument can be written as $K + F$ with K as above and F a linear subspace. Finally, we write $K \subset^{\mathrm{opt}} C$ and say K is *optimally contained in* C to denote that $K \subset C$ and $R(K, C) = 1$ (thus $K \not\subset_t \rho C$ for any $\rho < 1$).

The well-known *Minkowski asymmetry* $s(C)$ measures the asymmetry of C by the smallest dilation factor $\rho \geq 0$ needed to cover $-C$ by a translate of ρC. Using our outer radius definition we may simply define $s(C) := R(-C, C)$. Moreover, any $c \in \mathbb{R}^n$ such that $-(C - c) \subset R(-C, C)(C - c)$ is called a *Minkowski center* of C and we say C is *Minkowski centered*, if 0 is a Minkowski center of C.

It should be mentioned at this point that $1 \leq s(C) \leq n$ with equality on the left if and only if C is symmetric and equality on the right if and only if C is an n-dimensional simplex (an *n-simplex*) (see, e. g., [37]). Moreover, important for computational issues is the fact that $s(C)$ can be determined via linear programming whenever C is a polytope in $\mathcal{V}$-representation or $\mathcal{H}$-representation [7].

If C is Minkowski centered, we call any translation vector c such that $K \subset c + R(K, C)C$ a *circumcenter* of K (with respect to C). Using the gauge function $\|\cdot\|_C$ one may rewrite $R(K, C) = \min\{\rho \geq 0 : \|x - c\|_C \leq \rho \text{ for some } c \in \mathbb{R}^n\}$.

The optimization task given in the definition of the outer radius states one of the most basic containment problems, the containment under homothety: over all translates of C find one that needs the minimal dilation to contain K.

There are many applications in which containment problems have to be solved. Basic radii play a role in a huge variety of mathematical problems and direct applications for some of the problems lie on hand. Many kinds of facility location problems, e. g., are based on k-center, some applications concerning radii are listed in [33], others will be motivated below, e. g., in Section 12.

The main aim of the papers summarized below is to achieve a better understanding of optimal containments – both from the purely theoretical point of view, but also focusing on computations and applications. Since the recent papers are those dealing with more basic and fundamental results, they are presented first.

2 Sharpening geometric inequalities using computable symmetry measures

Besides the outer radius defined above, there are three more well-known basic "radii" usually considered with respect to symmetric C: the inradius, the diameter, and the width. For our purposes we give more general definitions that also allow asymmetric C. In the case of symmetric C they coincide with the known definitions.
The *inradius* of K with respect to C is defined as

$$r(K,C) := \max\left\{\rho : \rho C \subset_t K\right\}.$$

Note that $r(K,C) = R(C,K)^{-1}$ (with the convention that $\infty^{-1} = 0$ if $C \not\subset \operatorname{aff}(K)$).
For any $s \in \mathbb{R}^n \setminus \{0\}$ we define the *s-length* of K with respect to C as

$$l_s(K,C) := 2 \max_{c \in \mathbb{R}^n} R(K \cap (c + \langle s \rangle), C) = \max_{\substack{x,y \in K \\ x-y \in \langle s \rangle}} 2R([x,y],C).$$

Hence $\frac{1}{2} l_s(K,C)$ is the maximal radius of a segment in K parallel to s. The *s-breadth* of K with respect to C is defined by

$$b_s(K,C) := 2R(K, C + \langle s \rangle^\perp) = 2 \cdot \frac{\max_{x,y \in K} s^T(x-y)}{\max_{x,y \in C} s^T(x-y)}.$$

It measures the distance of two hyperplanes orthogonal to s supporting K relative to the distance of two such hyperplanes with respect to C. The latter is often also called the *s-width* or the *directional width orthogonal to s*. Note that it depends only on the direction, but not on the length of s. We use the notation C° to denote the *polar set* $\left\{a \in \mathbb{R}^n : a^T x \le 1 \text{ for all } x \in C\right\}$ of C. In case that $0 \in \operatorname{int}(C)$ the definition of the s-breadth can now also be written as $\max_{x,y \in K} s^T(x-y)$ for $s \in \operatorname{bd}(C^\circ)$ (being aware of the fact that $\operatorname{bd}(C^\circ)$ is the dual unit sphere, if C is the gauge body of a normed space).
Now the *width* of K with respect to C (sometimes also called the *minimal width*) is defined as

$$w(K,C) := \min_{s \in \mathbb{R}^n} b_s(K,C).$$

Further the *diameter* of K with respect to C is given by

$$D(K,C) := \max_{s\in\mathbb{R}^n} l_s(K,C) = \max_{x,y\in K} 2R([x,y],C).$$

If C is symmetric, $D(K,C)$ equals $\max_{x,y\in K} \|x-y\|_C$. It may, however, differ from the latter expression for asymmetric C.
It is well-known for symmetric C (and can also be shown for non-symmetric C) that $D(K,C) = \max_{s\in\mathbb{R}^n\setminus\{0\}} b_s(K,C)$ and $w(K,C) = \min_{s\in\mathbb{R}^n} l_s(K,C)$ (even though $l_s(K,C) < b_s(K,C)$ is possible for some s).
The first result from "Sharpening Geometric Inequalities using Computable Symmetry Measures" which should be mentioned in this summary is [H7, Lemma 2.8]. It generalizes well-known results when restricting to symmetric C (Parts (a,b)) and comprises a consequence only possible when allowing asymmetric C (Part (c)):

Lemma 1. a) $\frac{1}{2}D(K,C) = R(K-K, C-C)$,

b) $\frac{1}{2}w(K,C) = r(K-K, C-C)$, and

c) $\frac{1}{2}w(K,C) = (\frac{1}{2}D(C,K))^{-1}$.

Besides their direct geometrical meaning, the above lemma allows us to understand the diameter and the width as the outer and inner radii of the *central symmetrization* $\frac{1}{2}(K-K)$ of K with respect to the central symmetrization $\frac{1}{2}(C-C)$ of C. (One may keep this in mind when considering the inequalities below.)
In [9] Bohnenblust stated the following inequality between the outer radius and the diameter for all K and all symmetric C:

$$\frac{R(K,C)}{D(K,C)} \le \frac{n}{n+1}. \tag{1}$$

Already in the papers [H6, H9] we derived a sharpened version of (1) involving the asymmetry of the container C, while [66] involved the asymmetry of K. In [H7, Theorem 4.1] we stated a version involving both:

Theorem 2. For all K, C it holds

$$\frac{R(K,C)}{D(K,C)} \le \frac{(s(C)+1)s(K)}{2(s(K)+1)}$$

and this bound is tight, whichever values of $s(K)$ and $s(C)$ are prescribed.

In [57] Leichtweiß reconsidered the inequality of Bohnenblust and derived an analogous inequality for the inradius and the width for all K and all symmetric C:

$$\frac{w(K,C)}{r(K,C)} \le n+1.$$

Allowing asymmetric C one can now just interchange the arguments and use Lemma 1 ([H7, Lemma 2.8]) to obtain [H7, Corollary 4.3] directly from Theorem 2 ([H7, Theorem 4.1]), a sharpened and generalized version of the inequality of Leichtweiß:

Corollary 3. For all K, C it holds

$$\frac{w(K,C)}{r(K,C)} \le \frac{2(s(K)+1)s(C)}{s(C)+1}$$

and this bound is tight, whichever values of $s(K)$ and $s(C)$ are prescribed.

One should recognize that Leichtweiß [57] already considered generalizations of the radii for non-symmetric gauge bodies and geometric inequalities between them. However, his definition of the diameter and the width differs from ours for non-symmetric C. Hence they do not match the identities given in Lemma 1 ([H7, Lemma 2.8]) and therefore do not allow to transfer one result into the other.

Our generalized definitions of the basic radii for non-symmetric C show that the two results above are somehow two sides of the same story. However as soon as one wants to restrict to symmetric C, e. g. to obtain results about normed spaces, interchanging arguments would be prohibited again. To do so we need these results phrased independently for the outer radius/diameter-ratio and the width/inradius-ratio. The specialized versions of the inequalities of Bohnenblust and Leichtweiß for the Euclidean case are the well-known inequalities of Jung [48].

$$\frac{R(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)} \le \sqrt{\frac{n}{2(n+1)}} \tag{2}$$

and Steinhagen [70]

$$\frac{w(K,\mathbb{B}_2)}{r(K,\mathbb{B}_2)} \le \begin{cases} 2\sqrt{n} & \text{if } n \text{ is odd,} \\ \frac{2(n+1)}{\sqrt{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

Surely, they are much tighter for general K than those inequalities of Bohnenblust and Leichtweiß. However, it turns out in [H7, Corollaries 5.1 and 5.2] that from involving the Minkowski asymmetry of K no better bounds can be derived than the ones obtained from combining the original inequalities with Theorem 2 ([H7, Theorem 4.1]) and Corollary 3 ([H7, Corollary 4.3]):

Corollary 4. For all K it holds

$$\frac{R(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)} \leq \min\left\{\sqrt{\frac{n}{2(n+1)}}, \frac{s(K)}{s(K)+1}\right\},$$

$$\frac{w(K,\mathbb{B}_2)}{r(K,\mathbb{B}_2)} \leq \begin{cases} \min\left\{\, 2\sqrt{n}, \quad 2(s(K)+1)\right\} & \text{if } n \text{ is odd,} \\ \min\left\{\frac{2(n+1)}{\sqrt{n+2}}, 2(s(K)+1)\right\} & \text{if } n \text{ is even,} \end{cases}$$

and these bounds are tight, whichever values of $s(K)$ and $s(C)$ are prescribed.

A classical inequality (see, e. g., [27, p. 28]) states that the outer radius of a simplex in Euclidean space is at least n times larger than its inradius, while for general K one cannot give a better constant lower bound for this ratio than 1. Involving the Minkowski asymmetry [H7, Theorem 6.1] for the first time allows a meaningful inequality for general K and C:

Theorem 5. For all K, C it holds

$$\frac{R(K,C)}{r(K,C)} \geq \max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\},$$

and this bound is tight, whichever values of $s(K)$ and $s(C)$ are prescribed.

When C is symmetric the inequality in the above theorem simplifies to

$$\frac{R(K,C)}{r(K,C)} \geq s(K).$$

This allows to arrange the inequalities in Theorem 2 ([H7, Theorem 4.1]), Corollary 3 ([H7, Corollary 4.3]), and Theorem 5 ([H7, Theorem 6.1]) in one chain of inequalities (cf. [H7, Remark 6.3]):

Theorem 6. For all K and all symmetric C it holds

$$w(K,C) \leq (s(K)+1)r(K,C) \leq r(K,C)+R(K,C) \leq \frac{s(K)+1}{s(K)}R(K,C) \leq D(K,C).$$

The development of this chain allows several corollary inequalities, which are collected in [H7, Corollary 6.4]. However, its relevance will show up even more in Section 3 below.

Let us finally remark that the inequality in Theorem 5 ([H7, Theorem 6.1]) can also be reinterpreted as a lower bound for containment under affinities: If for general K and C we ask for the optimal ratio between the volume of an affine transformation

C' of C to contain K and the volume of a dilation of C' being contained in K, then this ratio must be at least $\max\left\{\frac{s(K)}{s(C)}, \frac{s(C)}{s(K)}\right\}$. In the special case $C = \mathbb{B}_2$ this gives a lower bound on the volume ratio of a pair of ellipsoids containing and being contained in K, respectively. The corresponding upper bounds n and $\sqrt{n}$ in case of K symmetric are due to John [47] (see also [6, 5]). Defining the *John asymmetry* $s_0 := \min\{\rho \geq 0 : -(K - c_K) \subset \rho(K - c_K)\}$, where c_K denotes the (unique) center of the ellipsoid of maximal volume contained in K, we obtain a very similar asymmetry measure as the Minkowski asymmetry, having the same bounds as $s(K)$ and being polynomially approximable to any accuracy for $\mathcal{H}$-represented polytopes. Using it one can sharpen John's inequality, too (cf. [H7, Theorem 7.1]):

Theorem 7. For all K, denoting the volume maximal ellipsoid contained in K by $\mathcal{E}(K)$ and its center by c_K, there exists ρ with $s(K) \leq \rho \leq \sqrt{s_o(K)n}$ such that $K - c_K \subset \rho(\mathcal{E}(K) - c_K)$.

Even though we realized that the upper bound is already shown in [7], using a technique of Ball [5, 6] our proof is significantly simpler.

3 The asymmetry of complete and constant width bodies in general normed spaces and the Jung constant

For any K and any C we say that K is of *constant width* with respect to C, if $b_s(K, C)$ is constant over all choices of s. The term *constant breadth*, used e. g. in [H1], would be more adequate in combination with the usage of s-breadth, but the former is more common. For symmetric C it is well-known that K is of constant width if and only if $K - K = \rho(C - C)$ for some dilation factor ρ and that this is again equivalent to $w(K, C) = D(K, C)$. From the relations between width, diameter, s-breadth, and s-length, the latter also implies that $b_s(K, C) = l_s(K, C)$ are constant over all choices of s (see, e. g., [15]). With our definitions of the basic radii all this extends easily to non-symmetric C. Moreover, $w(K, C) = D(K, C)$ directly implies equality for the full inequality chain in Theorem 6 ([H7, Remark 6.3]).
A set K is called *complete* with respect to C, if $D(K, C) < D(K \cup \{x\}, C)$ for any $x \notin K$. Furthermore, any complete set $K^* \supset K$ with $D(K, C) = D(K^*, C)$ is called a *completion* of K and *Scott completion* if in addition $R(K, C) = R(K^*, C)$.
Both terms, constant width and completeness, are deeply studied subjects in convex geometry, especially in connection with geometric inequalities. However, many major problems are still unsolved (see, e. g., [16]). It is well-known, for example, that both terms are equivalent if $C = \mathbb{B}_2$. In general, norms for which completeness and constant width coincide are called perfect. A characterization of perfect norms is still open.

Recently, in [46], an inequality bounding the asymmetry of constant width bodies K in Euclidean spaces is presented, stating that

$$s(K) \leq \frac{n + \sqrt{2n(n+1)}}{n+2}, \tag{3}$$

which is shown in [45] to be sharp for completions of the regular n-simplex.
The impression that [45, 46] could easily be simplified with our results from [H7], is a major motivation for "The asymmetry of complete and constant width bodies in general normed spaces and the Jung constant" [H4]. In the following it turns out that for complete bodies equality can be kept in Theorem 6 ([H7, Remark 6.3]) for all but the leftmost inequality. Moreover, a characterization of the corresponding equality case allows several nice and easy conclusions. One of these is a new proof of (3), which is simpler than the rather technical original proof and generalizes the result to arbitrary Minkowski spaces.

First we need some additional notation: If C is symmetric, the maximal ratio

$$j_C := \max \left\{ \frac{R(K,C)}{D(K,C)} \ : \ K \text{ a convex body} \right\}$$

is usually called the *Jung constant* of the normed space $(\mathbb{R}^n, \|\cdot\|_C)$ induced by C. Since it easily extends allowing also general C, we will simply talk about the Jung constant with respect to C or plainly the Jung constant, if C is known from the context. One should recognize that it is a simple corollary of Helly's theorem, that there always exists a simplex S (depending on C) such that $R(S,C)/D(S,C) = j_C$.
Using (2) we have $j_{\mathbb{B}_2} = \sqrt{\frac{n}{2(n+1)}}$ and thus the inequality in (3) may be rewritten as

$$s(K) \leq \frac{j_{\mathbb{B}_2}}{1 - j_{\mathbb{B}_2}}.$$

[H4, Theorem 1.1] therefore generalizes (3) as follows:

Theorem 8. For any symmetric C and any K complete with respect to C it holds

$$s(K) \leq \frac{j_C}{1 - j_C},$$

with equality if and only if K is the completion of an n-simplex S with circumradius-diameter ratio j_C.

Defining the *asymmetry constant* with respect to C by $s_C := \max\{s(K) : K \text{ complete}\}$, the above theorem gives a direct relation between the two constants s_C and j_C:

$$s_C = \frac{j_C}{1 - j_C} \quad \text{or} \quad j_C = \frac{s_C}{s_C + 1}.$$

To prove [H4, Theorem 1.1], we combined Theorem 6 ([H7, Remark 6.3]) with the fact that in particular all complete bodies satisfy the equation $R(K,C)+r(K,C) = D(K,C)$, as shown in [62]. Investigating the equality case of the inequality $R(K,C)+r(K,C) \leq D(K,C)$ further, which is part of the chain, we first state in [H4, Lemma 2.4] that this already forces concentricity of the in- and circumcenters (which do not have to be unique):

Lemma 9. For any symmetric C and any K it holds $R(K,C) + r(K,C) \leq D(K,C)$. Moreover, if $R(K,C)+r(K,C) = D(K,C)$ then any incenter of K is also a circumcenter.

Now, we say that a set K is *pseudo-complete* (with respect to a symmetric C) if there exists $c \in \mathbb{R}^n$ such that $c + (D(K,C) - R(K,C))C \subset K \subset c + R(K,C)C$ and for any circumcenter c of K, the set $K^+ := \operatorname{conv}\{K \cup (c + (D(K,C) - R(K,C))C)\}$ is called a *pseudo-completion* of K. As shown below all complete bodies are pseudo-complete and for C being a simplex both terms coincide. Moreover, even the non-complete pseudo-complete bodies share quite a few properties of complete bodies.
Using Lemma 9 ([H4, Lemma 2.4]) we are able to characterize the equality case of $R(K,C) + r(K,C) \leq D(K,C)$ in the following way [H4, Lemma 2.5]:

Lemma 10. Let C be symmetric. Then the following are equivalent:

(i) $D(K,C) = r(K,C) + R(K,C)$,

(ii) K is pseudo-complete with respect to C,

(iii) $D(K,C) = (s(K)+1)r(K,C)$, and

(iv) for every incenter c of K it holds that

$$K - K \subset D(K,C)C \subset (s(K)+1)((K-c) \cap (-(K-c))).$$

Recall that every complete K fulfills $D(K,C) = r(K,C) + R(K,C)$, which means that the lemma shows that they are also pseudo-complete.
The above lemma is the central technical result of the paper on which everything else is based. The first is an observation on the control of the radii, when (pseudo-)completing [H4, Lemma 2.7]:

Lemma 11. a) If $K \in \mathcal{K}^n$ and K^+ is a pseudo-completion of K, then $R(K^+) = R(K)$ and $D(K^+) = D(K)$.

b) If K is pseudo-complete and K^* is a completion of K, then $R(K^*) = R(K)$, $r(K^*) = r(K)$ and $s(K^*) = s(K)$.

Simply combining Parts (a) and (b) of the above lemma we obtain as a direct corollary a short proof for the existence of Scott completions in all Minkowskis spaces (cf. [68] for the Euclidean case and [75] for its generalization):

Corollary 12. Let C be symmetric. Then there exists a Scott completion K^* of K.

[H4, Corollary 2.10] shows that for simplices pseudo-completeness and completeness coincide.

Corollary 13. Let C be symmetric and let S be an n-simplex. Then the following are equivalent:

(i) S is complete with respect to C.

(ii) $D(S,C) = r(S,C) + R(S,C)$.

(iii) $S-S \subset D(S,C)C \subset (n+1)((S-c)\cap(-(S-c)))$, where c is the (unique) incenter of S.

(iv) $R(S,C)/D(S,C) = n/(n+1)$ (which means equality in (1)).

Taking Lemma 10 ([H4, Lemma 2.5]) and the above corollary together we have everything needed to prove [H4, Theorem 1.3]. It is the main interpretation of the results as a condition on the gauge body C of a normed space, when K is known to be complete within that space:

Theorem 14. If K is Minkowski centered and complete with respect to a 0-symmetric C, then C must fulfill the following property

$$K - K \subset D(K,C)C \subset (s(K)+1)(K \cap (-K)).$$

If K is an n-simplex and complete then either the homothetic copies of C are the only bodies of constant width or $K - K = D(K,C)C$.

One should remark that when an n-simplex is complete the second fact in this theorem shows somehow a discrete behaviour of the maximal asymmetry s_C: it can only have the values 1 (no other shapes of constant width than that of the gauge body) or n (there exists a simplex of constant width).
From [H4, Lemma 2.12] we know that there always exists a complete K such that $R(K,C)/D(K,C) = j_C$. Using this suffices to prove Theorem 8 ([H4, Theorem 1.1]) as well as Theorem 15 below ([H4, Theorem 1.2]). The latter tightens two geometric inequalities of [22] (Euclidean case) and [62] (general Minkowski spaces) on the inradius-diameter ratio of complete sets:

Theorem 15. Let C be symmetric and K complete with respect to C. Then

$$\frac{r(K,C)}{D(K,C)} = 1 - \frac{R(K,C)}{D(K,C)} \geq 1 - j_C,$$

with equality, if and only if $R(K,C)/D(K,C) = j_C$.

After proving the main results the paper collects several applications: In [H4, Corollary 2.15 and Remark 2.16] it is shown that the results in [44] on the maximal asymmetry of constant width bodies of revolution in Euclidean spaces, can easily be obtained and generalized from the results above.
A further result concerns the *Helly dimension* $\mathrm{him}(C)$ of C. It is the smallest positive integer k, such that for every family of indices $I \neq \emptyset$ with $\bigcap_{i \in J}(x_i + C) \neq \emptyset$, for any $J \subset I$ with $|J| \leq k+1$ and $x_i \in \mathbb{R}^n$, for $i \in I$, we have $\bigcap_{i \in I}(x_i + C) \neq \emptyset$.
In [H4, Corollary 2.19], using Theorem 8 ([H4, Theorem 1.1]), we obtain a direct inequality between the Helly dimension and s_C:

Corollary 16. If C is symmetric then

$$\lceil s_C \rceil \leq \mathrm{him}(C)$$

and $s_C = \mathrm{him}(C)$ if and only if there exists a $\mathrm{him}(C)$-dimensional simplex S such that $s(S^+) = s(S^*) = \mathrm{him}(C)$ for all of its completions S^*. Moreover, in that case it holds $S = S^+ \cap \mathrm{aff}(S)$.

The paper is finished by a section stating several implications on the *Banach-Mazur distance*

$$d_{BM}(K, C) := \min\{\rho > 0 : K \subset_t AC \subset_t \rho K, \text{ for some } A \in \mathbb{R}^{n \times n} \text{ regular}\}.$$

It is well-known (see, e. g., [38]) that

$$s(K) = \min\{d_{BM}(K, C) \ : \ C \text{ symmetric}\} = d_{BM}(K, K - K),$$

saying that the asymmetry of K measures the minimal Banach-Mazur distance of K to any symmetric body C and this minimum is attained when K is of constant width. [H4, Corollary 3.3] of this section extends the above to:

Corollary 17. For all symmetric C and all K (pseudo-) complete with respect to C it holds $s(K) = d_{BM}(K, C)$.

In [38] Grünbaum suggested two properties to be fulfilled by an asymmetry measure: For any two convex sets K, K' and any asymmetry measure $\bar{s}$ it should be true that

(i) $\bar{s}(K + K') \leq \max\{\bar{s}(K), \bar{s}(K')\}$ (*supermaximality condition*) and

(ii) if $\bar{s}(K + K') = \max\{\bar{s}(K), \bar{s}(K')\}$ then either K, K' are symmetric or K, K' are homothetics of each other.

While the first property obviously holds true for the Minkowski asymmetry (as well as for many other known asymmetry measures) in [H4] an example is given, showing that (ii) is false for the Minkowski asymmetry. To do so it suffices to consider two non-similar completions of the regular simplex in Euclidean n-space, $n \geq 3$, e. g., the two Meißner bodies for $n = 3$.

4 Is a complete, reduced set necessarily of constant width?

As learned from the preceding section, not all spaces are perfect, i. e. there exist Minkowski spaces in which a complete body does not need to be of constant width. A quite similar concept to completion and completeness is the following:
A set K is called *reduced* with respect to C, if $w(K,C) > w(K \cap H^-, C)$ for any half-space H^- intersecting the interior of K. Furthermore, any reduced set $K_* \subset K$ with $w(K,C) = w(K_*,C)$ is called a *reduction* of K.
It is well-known that even in Euclidean spaces there exist reduced sets, which are not of constant width. However, it is quite obvious that any constant width body is complete and reduced. Hence, it seems a natural question to ask, if the opposite is true, i. e. if a body being reduced and complete must be of constant width. This question is formulated and tackled in “Is a complete, reduced set necessarily of constant width?” [H5].
As a direct consequence of the defintion of the Minkowski asymmetry and [H6, Theorem 2.3] (see Theorem 37 below) first a chain of optimal containment is derived in [H5, Corollary 3.4].

Corollary 18. Let $K \in \mathcal{K}^n$ be Minkowski centered. Then

$$\left(1+\frac{1}{s(K)}\right)\operatorname{conv}(K \cup (-K)) \subset^{\mathrm{opt}} K - K \subset^{\mathrm{opt}} (s(K)+1)(K \cap (-K)).$$

Moreover, in case that K is a polytope, there exist vertices p^i and facet normals a^i of K, with $i \in [m]$ for some $2 \leq m \leq n+1$, such that $0 \in \operatorname{conv}(\{a^1,\dots,a^m\})$ and $\pm(1+1/s(K))p^i$ is a vertex of $(1+1/s(K))\operatorname{conv}(K \cup (-K))$ contained in a facet of $K-K$, which itself is completely contained in a facet of $(s(K)+1)(K \cap (-K))$, both with outer normal $\mp a^i$.

As an example let $T \subset \mathbb{R}^3$ be the regular tetrahedron with centroid at the origin. Then the above corollary describes the (well-known) fact that the cube $\operatorname{conv}(T \cup (-T))$, the cuboctahedron $T-T$, and the octahedron $T \cap (-T)$ can be placed such that the cube is optimally contained in the octahedron, but still the cuboctahedron fits in between (cf. Figure 1) the other two.

In [H5, Lemma 3.2] it is shown (besides others) that we may concentrate on 0-symmetric C:

Lemma 19. Let $K, C \in \mathcal{K}^n$. Then the following statements hold true: K is complete, reduced, or of constant width with respect to C if and only if K is complete, reduced, of constant width, respectively, with respect to $C-C$.

In Corollary 13 ([H4, Corollary 2.10]) the situation when a simplex is complete is characterized. A similar characterization for a simplex to be reduced is given in [56, Corollary 7]:

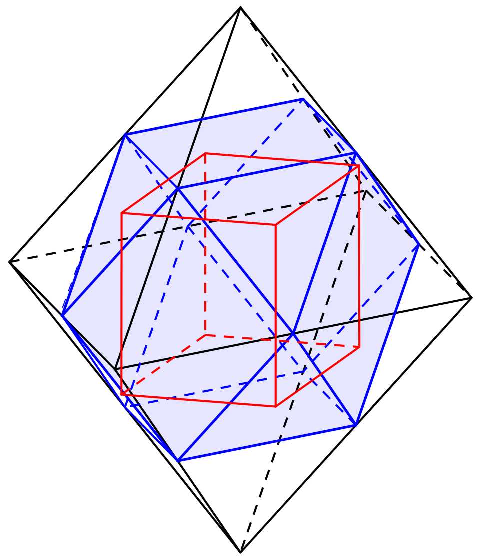


Figure 1: A cube optimally contained in an octahedron, and a cuboctahedron fitting in between.

Proposition 20. Let $S, C \in \mathcal{K}^n$ with S being an n-simplex and $C = -C$. Then the following are equivalent:

(i) S is reduced with respect to C.

(ii) $w(S,C)C \subset S - S$ touches all facets of $S - S$ with outer normals parallel to outer normals of facets of $\pm S$.

From joining the optimal containment chain in Corollary 13 ([H5, Corollary 3.4]) with the two characterizations of complete and reduced simplices, Corollary 18 ([H4, Corollary 2.10]) and Proposition 20 ([56, Corollary 7]), respectively, one obtains the first of the two main results of the paper [H5, Theorem 3.8]:

Theorem 21. Let $S, C \in \mathcal{K}^n$ be such that S is a complete and reduced simplex with respect to C. Then S is of constant width with respect to C.

The second main result, [H5, Theorem 3.18], gives a positive answer to the question in the title of the paper for a much bigger class of bodies. It is based on well-known statements about diametrical chords of complete bodies and width slabs of reduced bodies.

Theorem 22. If K is complete and reduced with respect to C and there exists a smooth extreme point of K, then K is of constant width.

The remainder of the paper concerns the perfectness of normed spaces (or, in more general the perfectness of gauge bodies C, after extending the notion).
There is a close relation between the perfectness of gauge bodies and the main question of the paper, which is expressed in [H5, Remark 4.1]:

Remark 23. If for a given gauge body C any complete and reduced K is of constant width then C is perfect if and only if completeness implies reducedness.

An n-dimensional polytope is called *simple*, if all its vertices are contained in exactly n edges. In [24] Eggleston showed that in 3-space any perfect polytopal 0-symmetric gauge body must be simple. This statement cannot be generalized saying that a perfect polytopal 0-symmetric gauge body must be simple in higher dimensions. However, it is generalized to arbitrary dimensions $n \geq 3$ in [H5, Lemma 4.3] in the following way:

Lemma 24. Let $n \geq 3$ and $C \in \mathcal{K}^n$ be a 0-symmetric polytope. If C is perfect, then every pair of non-disjoint facets F_1, F_2 of C intersects in at least an edge of C.

Combining the above lemma and Corollary 13 ([H4, Corollary 2.10]) we obtain in [H5, Corollary 4.6] that all equality cases of the Bohnenblust inequality (see [57] and cf. (1)) belong to non-perfect gauge-bodies:

Corollary 25. Let $C \in \mathcal{K}^n$ be 0-symmetric, $n \geq 3$, and let S be an n-simplex such that

$$S - S \subset D(S, C)C \subset (n+1)(S \cap (-S)).$$

Then C is not perfect.

[H3, Lemma 2.2] (see Lemma 27 below) states a linearity property of the width function for the Minkowski sum of a body and any of its completions in Euclidean space. [H5, Theorem 4.7] shows that this property is characteristic for perfect spaces in general:

Theorem 26. Let $C \in \mathcal{K}^n$. The following are equivalent:

(i) C is perfect.

(ii) For all $K \in \mathcal{K}^n$ and any completion K^* of K it holds that

$$w(\lambda K + (1-\lambda)K^*, C) = \lambda w(K, C) + (1-\lambda) w(K^*, C) \text{ for all } \lambda \in [0, 1].$$

5 A complete 3-dimensional Blaschke-Santaló-diagram

Instead of developing a single inequality between certain geometric functionals $g_0, \ldots, g_m$ one may aim for a complete system of inequalities describing their relations and dominating all other possible inequalities between these functionals. To do so one usually tries to describe the boundaries of the image of $g(\mathcal{K}^n)$, where

$$g(K) = (g_1(K)/g_0(K), \ldots, g_m(K)/g_0(K)),\ K \in \mathcal{K}^n, g_0(K) \neq 0.$$

The most prominent such mapping is the so called *Blaschke-diagram*, which investigates the functionals volume, surface area, and integral mean curvature of 3-dimensional convex sets (in Euclidean space). Proposed by Blaschke in 1916 [8], completing the description of the diagram is still one of the most challenging open problems in the field of geometric inequalities (see, e. g., [64]).

Santaló [65] and later Hernández-Cifre et al. [43, 41, 42] investigated several of these images for planar convex sets and 3-tuples of the functionals area, perimeter, diameter, width, in- and outer radius (all Euclidean). They derived complete systems of inequalities for 13 of the 20 possible tuples. Since Santaló was the first to do so, all kinds of these images $g(\mathcal{K}^n)$ were subsumed under the notion *Blaschke-Santaló-diagram* later on.

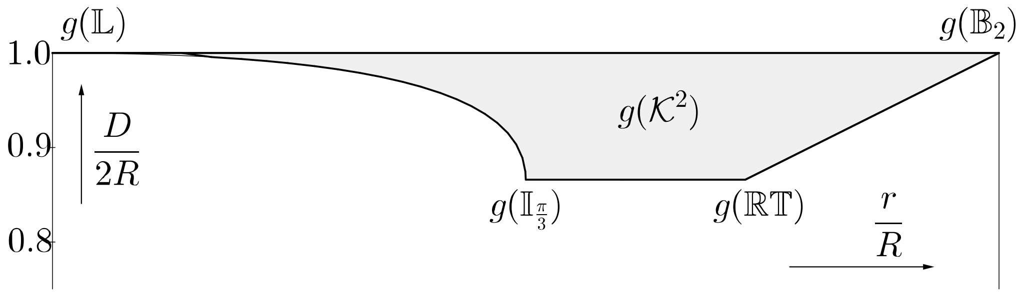


Figure 2: The diagram $g(\mathcal{K}^2)$ with $g_0 = R$, $g_1 = r$, and $g_2 = D/2$. The boundaries are given via the inequalities $D \leq 2R$, $r + R \leq D$, $\sqrt{3}R \leq D$ (Jung's inequality (2)), and $2R\left(2R + \sqrt{4R^2 - D^2}\right) r \geq D^2\sqrt{4R^2 - D^2}$ (see [65]). The vertices are the (images of the) Euclidean ball $\mathbb{B}_2$, the line segment $\mathbb{L}$, the equilateral triangle $\mathbb{I}_{\pi/3}$ and the Reuleaux triangle $\mathbb{RT}$ (cf. Figure 3).

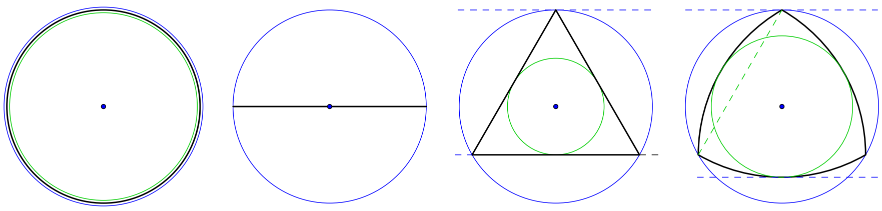


Figure 3: From left to right: the Euclidean ball $\mathbb{B}_2$, the line $\mathbb{L}$, the equilateral triangle $\mathbb{T}_{\pi/3}$, and the Reuleaux triangle $\mathbb{RT}$. The inballs are drawn in green and the circumballs in blue. The diameters are given in dashed green and the widths are indicated in dashed blue.

Blaschke-Santaló-diagrams for 4-tuples of the collected measures above were first investigated in [11] and in [71]. In both cases the descriptions are incomplete. Since not even all included 3-tuples of the diagram considered in [71] have been completed so far, there is little hope to complete the description of that one in near future. However, all four 3-tuples only involving the fundamental radii (width, diameter, in- and circumradius) belong to the 13 completed diagrams mentioned above.

In “A complete 3-dimensional Blaschke-Santaló-diagram” [H3] a full description of the diagram

$$f : \mathcal{K}^n \to [0,1]^3, \quad f(K) = \left(\frac{r(K,\mathbb{B}_2)}{R(K,\mathbb{B}_2)}, \frac{w(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}, \frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)} \right)$$

is provided.
To do so we first collect some general lemmas concerning the diagram.
A first lemma is [H3, Lemma 2.1], which was already shown in [11]. It states the star-shapedness of such diagrams only involving radii with respect to the image point of the unit ball. Even though not explicitly mentioned, it remains true even when replacing the Euclidean unit ball $\mathbb{B}_2$ in the definition of f above by any other symmetric container C.
Both, [H3, Lemma 2.2 and Lemma 2.3] show further linearity properties with respect to completions:

Lemma 27. Let $K, K^* \in \mathcal{K}^n$ be such that K^* is a completion of K and $\lambda \in [0,1]$. Then

$$D(\lambda K + (1-\lambda K^*),\mathbb{B}_2) = D(K,\mathbb{B}_2) \quad \text{and}$$
$$w(\lambda K + (1-\lambda K^*),\mathbb{B}_2) = \lambda w(K,\mathbb{B}_2) + (1-\lambda) w(K^*,\mathbb{B}_2).$$

Lemma 28. Let $K, K^* \in \mathcal{K}^n$ be such that $w(K,\mathbb{B}_2) = r(K,\mathbb{B}_2) + R(K,\mathbb{B}_2)$ and K^* is a Scott completion of K. Then

$$f(\lambda K + (1-\lambda K^*)) = \lambda f(K) + (1-\lambda) f(K^*)$$

for all $\lambda \in [0,1]$.

Since the proof of Lemma 27 ([H3, Lemma 2.2]) uses the fact that the completion K^* is of constant width it cannot be generalized to arbitrary symmetric containers C. However, it is generalized in Theorem 26 (see [H5, Theorem 4.7]) to show that the property is characteristic for perfect spaces. The proof of Lemma 28 ([H3, Lemma 2.3]) makes strong use of Euclidean properties, which makes it much harder to generalize.

In the following we present the inequalities necessary to describe the diagram; first from [H3, Proposition 3.1] the previously known ones (mostly mentioned before):

Proposition 29.

$$\begin{array}{rcl}
2r(K,\mathbb{B}_2) & \leq & w(K,\mathbb{B}_2) \\
D(K,\mathbb{B}_2) & \leq & 2R(K,\mathbb{B}_2) \\
w(K,\mathbb{B}_2) & \leq & R(K,\mathbb{B}_2) + r(K,\mathbb{B}_2) \\
R(K,\mathbb{B}_2) + r(K,\mathbb{B}_2) & \leq & D(K,\mathbb{B}_2) \\
\sqrt{3}R(K,\mathbb{B}_2) & \leq & D(K,\mathbb{B}_2) \qquad \text{(Jung's inequality, cf. (2))} \\
(4R(K,\mathbb{B}_2)^2 - D(K,\mathbb{B}_2)^2)D(K,\mathbb{B}_2)^4 & \leq & 4w(K,\mathbb{B}_2)^2 R(K,\mathbb{B}_2)^4 \\
 & & \qquad \text{(Hernández-Gomis' inequality, see [43]).}
\end{array}$$

All the above inequalities only involve three of the four radii. This is not surprising, as they are all already describing boundaries of the 2-dimensional diagrams. In [H3, Theorems 3.2 – 3.4] the new inequalities involving all four radii are given:

Theorem 30.

$$\begin{aligned}
w(K,\mathbb{B}_2) \geq 2D(K,\mathbb{B}_2)\sqrt{1-\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right)^2} \cos\Bigg[&\arccos\left(\frac{D(K,\mathbb{B}_2)}{2(D(K,\mathbb{B}_2)-r(K,\mathbb{B}_2))}\right) \\
&+\arccos\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right) - \arcsin\left(\frac{r(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)-r(K,\mathbb{B}_2)}\right)\Bigg],
\end{aligned}$$

$$w(K,\mathbb{B}_2) \leq r(K,\mathbb{B}_2)\left(1+\frac{2\sqrt{2}R(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)}\sqrt{1+\sqrt{1-\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right)^2}}\right),$$

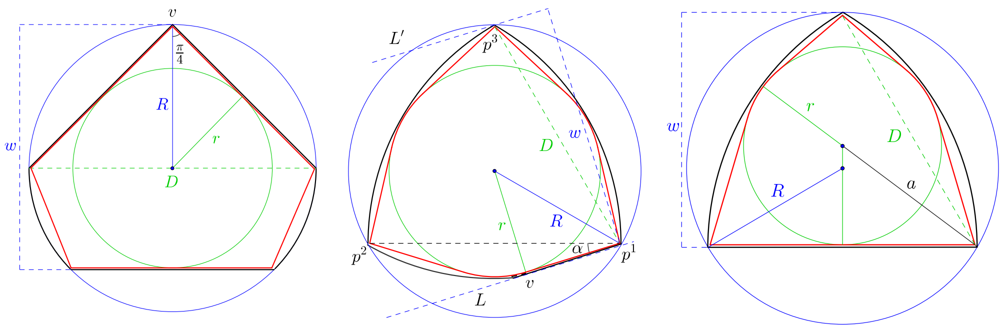
$$w(K,\mathbb{B}_2) \leq 2r(K,\mathbb{B}_2)\left(1+\frac{2r(K,\mathbb{B}_2)R(K,\mathbb{B}_2)}{D(K,\mathbb{B}_2)^2}\left(1+\sqrt{1-\left(\frac{D(K,\mathbb{B}_2)}{2R(K,\mathbb{B}_2)}\right)^2}\right)\right).$$

One should emphasis the following two facts:

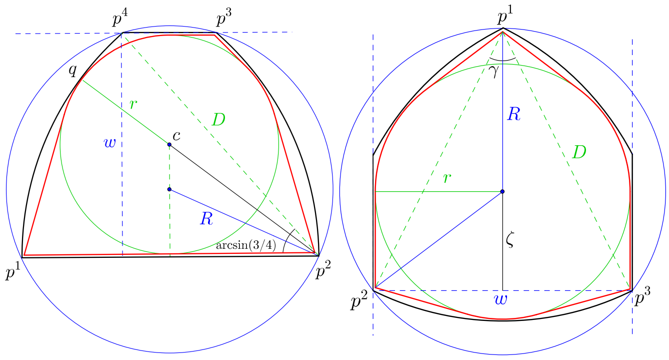
(i) The first inequality can as well as all the others be stated as an algebraic equation (see [H3, Remark 1]).

(ii) The second inequality sharpens the 2-dimensional version of Steinhagen's inequality (see [H3, Remark 2]).

To prepare the proofs of Theorem 30 ([H3, Theorems 3.2 – 3.4]) we collect extreme convex sets describing the *skeleton* of the diagram. The skeleton consists of what we call the *vertices*, *edges*, and *facets* of the diagram (in analogy to polytopal structures).



(a) The sailing boat $\mathbb{SB}$. **(b)** The sliced Reuleaux triangle $\mathbb{SR}$. **(c)** The flattened Reuleaux triangle $\mathbb{FR}$.



(d) The bent trapezoid $\mathbb{BT}$. **(e)** The hood $\mathbb{H}$.

Figure 4: The five new vertices: displayed in black (red) maximal (minimal) shapes – with respect to set inclusion – mapped to the same coordinates in the diagram.

The vertices of the diagram are realized by these convex sets fulfilling three or more of the inequalities listed above with equality. In total there exist ten vertices, besides well-known sets, which are the *Euclidean ball (disc)*, the *equilateral triangle*, the *(isosceles) right-angled triangle* , the *line segment*, and the *Reuleaux triangle*, also five more, which we call the *sailing boat*, the *sliced Reuleaux triangle*, the *flattened Reuleaux triangle*, the *bent trapezoid*, and the *hood*. For detailed constructions we refer to [H3, Section 4.1]. However, it should be pointed out that the minimal sets mapped to the coordinates of the sliced Reuleaux triangle and the hood are in both cases pseudo-completions of the triangles $\mathrm{conv}\{p^1, p^2, p^3\}$ of points on the circumcircle (see the sets drawn in red in Figures 4b and 4e).

Each edge is realized by a one-parametric family of convex sets (transforming one endpoint of the edge as given above into another). Each set within such a family fulfills the same two of the inequalities with equality. Let us mention three examples: the first are isosceles triangles I_γ with an angle $\gamma \in [\pi/2, \pi/3]$ being the parameter. All of them fulfill the second and the third inequality in Theorem 30 ([H3, Theorems 3.3 and 3.4]) with equality.
The second are the bodies of constant width. Since $w(K) = D(K)$ for every body of constant width, they fulfill the third and the forth inequality in Proposition 29 with equality. To reduce to a one-parametric family realizing every point of the edge, one may consider the outer parallel bodies of a Reuleux triangle (cf. [H3, Lemma 2.1]).
As a third example of an edge consider the (general) bent trapezoids BT_γ with $\gamma \in [\arcsin(3/4), \pi/3]$. The construction starts with an isosceles $\mathrm{I}_\gamma = \mathrm{conv}\{p^1, p^2, p^3\}$ such that γ is the angle at p^1. Moreover, let p^4 be the point on the circumsphere of I_γ different from p^3, such that $\mathrm{conv}\{p^1, p^2, p^4\}$ is congruent with $\mathrm{conv}\{p^1, p^2, p^3\}$ and possesses its angle γ at p^2. Finishing the construction, we replace the two edges $[p^1, p^4]$ and $[p^2, p^3]$ of $\mathrm{conv}\{p^1, p^2, p^3, p^4\}$ by two arcs of radius $D(\mathrm{I}_\gamma)$ with centers in p^1 and p^2, respectively (cf. Figure 5). The result is what we call a (general) bent trapezoid BT_γ. The bent trapezoids build a one-parametric family of convex sets joining the vertices $\mathbb{BT}$ and $\mathbb{FR}$ for $\gamma = \arcsin(3/4)$ and $\gamma = \pi/3$, respectively. All bent trapezoids BT_γ with $\gamma \in [\arcsin(3/4), \pi/3]$ fulfill the last inequality in Proposition 29 and the the first inequality in Theorem 30 ([H3, Theorem 3.2]) with equality.
See [H3, Section 4.2] for more details and the constructions and proofs for the remaining edges.

Finally, in [H3, Section 4.3] the facets are described. Each of them is realized by a two-parametric family of convex sets, where each set within a family fulfills a single of the inequalities with equality. Let us again consider an example: for any $D \in [\sqrt{3}, 2]$ consider the two angles γ_i, $i = 1, 2$ with $\gamma_1 \in [0, \pi/3]$ and $\gamma_2 \in [\pi/3, \pi/2]$ such that I_{γ_1} and I_{γ_2} are both of circumradius 1 and diameter D. Obviously, for every $r \in [r(\mathrm{I}_{\gamma_1}), r(\mathrm{I}_{\gamma_2})]$ there exists an acute angled triangle $\mathrm{T}_{r,D}$ with inradius r, diameter D and circumradius 1. Let p^3 denote the vertex opposing the diametral edge of $\mathrm{T}_{r,D}$,

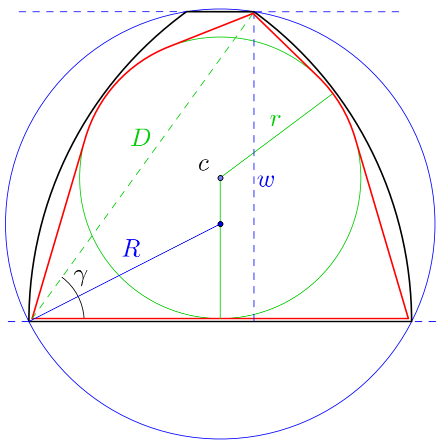


Figure 5: A general bent trapezoid (black), which is a maximal set (with respect to set inclusion) mapped to this coordinates in the diagram. In red a minimal set mapped to the same coordinates.

and s the distance between this vertex and the touching points of the inball to the edges meeting in p^3. Then the following identities (formulated using the radii) are well known for acute triangles:

- $w(\mathrm{T}_{r,D})D(\mathrm{T}_{r,D}) = 2r(\mathrm{T}_{r,D})(s + D(\mathrm{T}_{r,D}))$,
- $D(\mathrm{T}_{r,D}) = 2R(\mathrm{T}_{r,D})\sin(\gamma)$, and
- $r = s\tan(\gamma/2)$.

Using these three identities it is shown in [H3, Section 4.4] that all acute triangles $\mathrm{T}_{r,D}$ fulfill the third inequality in Theorem 30 ([H3, Theorem 3.4]) with equality.
Since the facets include all edge-sets, they completely describe the boundary of the diagram.

Knowing the skeleton then suffices to provide the proofs for the validity of the inequalities in Theorem 30 ([H3, Theorems 3.2 – 3.4]) and the completeness of the given collection for the description of the diagram.

Theorem 31. The inequalities presented in Proposition 29 and in Theorem 30 ([H3, Theorems 3.2 – 3.4] together completely describe $f(\mathcal{K}^2)$.

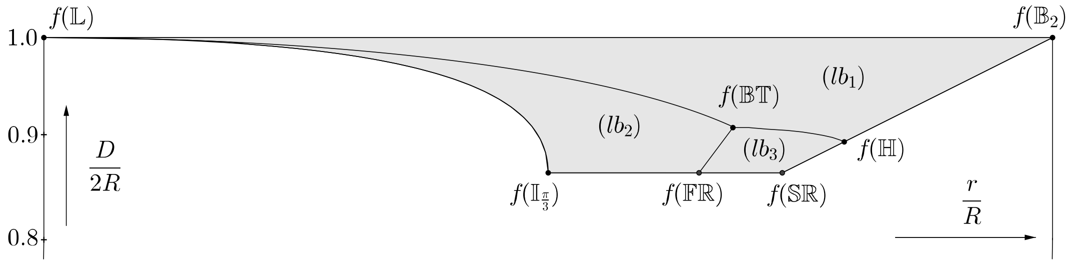


Figure 6: Bottom view of the diagram $f(\mathcal{K}^2)$ with the three facets lb_i, $i = 1, 2, 3$, obtained from inequalities lower bounding the width for fixed inradius, diameter, and circumradius ($lb_1 : 2r \leq w$; lb_2 : Hernandez-Gomis' inequality; lb_3 : inequality in [H3, Theorem 3.2]).

In difference to the more analytic proofs used in [41, 42, 43] our proofs are based on geometric transformations of a general starting set resulting in one of the extremal sets belonging to one of the facets.

6 Minimal containment under homothetics. A simple cutting plane approach

Recall that Lemma 1 ([H7, Lemma 2.8]) shows that the computational problems for each of the four basic radii introduced above are minimal containment problems under homothetics. The same holds true for the Minkowski asymmetry $s(C) = R(-C, C)$. This fact may be taken as a first motivation for the computational study presented in

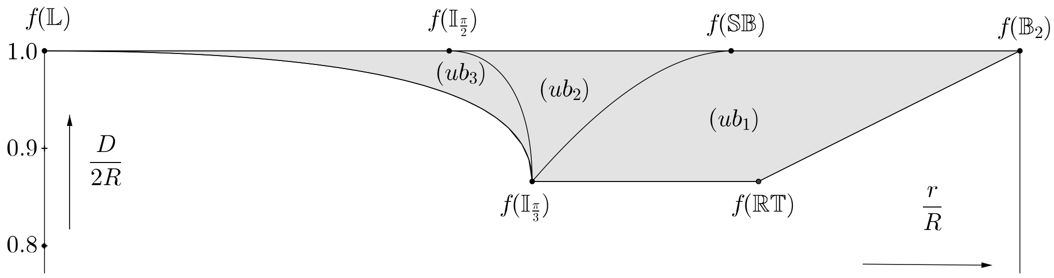


Figure 7: Top view of the diagram $f(\mathcal{K}^2)$ with the three facets ub_i, $i = 1, 2, 3$, obtained from inequalities upper bounding the width for fixed inradius, diameter, and circumradius ($ub_1 : w \leq r + R$; lb_2, lb_3 : second and third inequality in [H3, Theorems 3.3 and 3.4]).

"Minimal containment under homothetics. A simple cutting plane approach" [H8].

Many optimal containment problems are hard to be tackled algorithmically (not only but also from the complexity point of view). Hence often simpler variants have to be solved very efficiently, such that they may be used in a (possibly non-polynomial) framework for (approximatively) solving the actual problem.

Let us consider two examples, first the *minimal containment under similarities*: given a set K (possibly finite) and a container C, find a rotation matrix A to minimize $\rho \geq 0$, such that $K \subset_t \rho AC$. Surely, if $C = \mathbb{B}_2$ the choice of A is irrelevant and the problem is equivalent to computing $R(K, \mathbb{B}_2)$. In general, however, the set of rotation matrices is non-convex and thus there may exist many local optima. In fact, there is only little known how to solve this problem nicely in general.
[28] presents a quadratic model for the optimal containment of a $\mathcal{V}$-represented polytope in an $\mathcal{H}$-represented polytope. However, the focus of the paper is more on the theoretical side as it does not really aim for the computational aspects (numerical solution, complexity) of the problem. Usually brute force methods are used, which all are based somehow on discretizations of the unit sphere to replace the contaiment problem under similarities by many copies of the corresponding problems under homothety.

A second example are the *k-containment problems*, which are the object of the papers being summarized in Sections 7 and 8 below. In k-containment problems we are given a set P (typically finite) and k containers $C_1, \dots, C_k$. Then the task is to find suitable translation vectors (centers) $c^i \in \mathbb{R}^n$, $i \in [k]$ and a minimal dilation factor $\rho \geq 0$, such that $P \subset \bigcup_{i=1}^k \rho(c^i + C_i)$. In analogy to the notation we used for the case of one container, we write $R(P, C_1, \dots, C_k)$ to denote the optimal ρ. Note that $R(P, C) = R(\mathrm{conv}(P), C)$ since C is convex, but possibly $R(P, C_1, \dots, C_k) \neq R(\mathrm{conv}(P), C_1, \dots, C_k)$ since $\bigcup_{i=1}^k \rho(c^i + C_i)$ does not have to be convex anymore.
Again the problem is hard to solve, even if all of the C_i are equal, which is called the (general) *k-center problem*. However the problem simplifies to k separate 1-center problems, i. e. to k minimal containment problems under homothety, if we somehow are able to decide which points in P are covered by which container C_i. In fact, most of the algorithms given for k-center and k-containment problems decide how to split the points in P onto the C_i by branch-and-bound like methods (see [H9, 26] and the references therein).

Both examples show the necessity of fast solution methods for the computation of $R(K, C)$. To provide this at least for $\mathcal{V}$-represented polytopes K (or, equivalently, for finite point sets) and a big class of possible containers C is the aim of [H8]. In [30, 33] it is shown that the problem can be solved in polynomial time for $\mathcal{H}$-represented polyhedra K and C, while it gets $\mathbb{NP}$-complete for $\mathcal{H}$-represented polyhedra K and C being a Euclidean ball or a $\mathcal{V}$-represented polytope.

In [H8, Section 2] we start with a collection of approaches for special types of containers.

This includes the well-known LP formulations for $\mathcal{V}$- or $\mathcal{H}$-represented polytopes C. An overview on the state of the art for the Euclidean 1-center problem is given, including the LP-type approach [29], the second order cone problem (SOCP) formulation (see, e. g., [78]), and the core-set approach (see, e. g., [55] and Section 8 for more details). Next, LP and SOCP formulations for combined containers are presented [H8, Subsection 2.3]:

Lemma 32. Let $k, l \in \mathbb{N}_0$, $k + l \geq 1$, $m \in \mathbb{N}$, $r_i > 0$, $c^i \in \mathbb{R}^n$, $i \in [k]$, $K = \operatorname{conv}\{v^1, \ldots, v^m\}$ be a $\mathcal{V}$-represented polytope, and Q_j be $\mathcal{V}$- or $\mathcal{H}$-represented polytopes, $j \in [l]$.

a) If $C = \bigcap_{i=1}^{k}(c^i + r_i\mathbb{B}_2) \cap \bigcap_{j=1}^{l} Q_j$ then $R(K, C)$ is the solution of the following SOCP (which becomes an LP if $k = 0$):

$$\begin{array}{lrcll} \min & \rho & & & \\ & \left\|v^j - c - \rho c^i\right\|_2 & \leq & \rho r_i & i \in [k], j \in [m], \\ & v^j - c & \in & \rho Q_i & i \in [l], j \in [m], \end{array}$$

where the constraints $v^j - c \in \rho Q_i$ can be expressed as linear constraints depending on the representation of Q_i.

b) If $l \in \{0, 1\}$ and $C = \sum_{i=1}^{k} Q_i + l\mathbb{B}$ then $R(K, C)^{-1}$ is the solution of the following SOCP (which becomes an LP if $l = 0$):

$$\begin{array}{lrcll} \max & \rho' & & & \\ & \rho' v^j - c' - \sum_{i=1}^{k+l} x^{ij} & = & 0 & j \in [m], \\ & x^{ij} & \in & Q_i & i \in [k+l], j \in [m], \end{array}$$

where the constraints $x^{ij} \in Q_i$ can be expressed as linear constraints depending on the representation of Q_i.

One should recognize that in case we have $Q_i = [\alpha_i, \beta_i]z^i$ in Part (b) for some $\alpha_i < \beta_i$ and $z^i \in \mathbb{R}^n$, $i \in [k]$, then $C = \sum_{i=1}^{k} Q_i$ becomes a zonotope. Moreover, the above lemma covers also the important symmetrizations $C - C$ and $C \cap (c - C)$ (for some $c \in \mathbb{R}^n$). For the case that C is just the outer parallel body of an $\mathcal{H}$-represented polytope a spezialized SOCP is also given.

From the complexity point of view the problem of covering a finite set of points with a homothetic copy of a convex set C can be solved in polynomial time via the ellipsoid algorithm whenever we can provide a separation oracle for C (see [H8] for details). However, in practical computations, e. g. for Linear Programming, the ellipsoid algorithm performs much worse than the simplex algorithm – even though all known pivot rules for the latter lead to an exponential theoretical worst case running time. Hence for practical purposes the simplex algorithm is still in use.

This motivated us to investigate a cutting plane approach for the situation as explained above [H8, Subsection 2.4]: Cover a finite point set $\{v^1, \ldots, v^m\}$ with a homothetic copy of a convex set C for which a separation oracle is provided.
The idea of the cutting plane approach is as follows: we start with some $\mathcal{H}$-polytope $H \supset C$. As better this polytope approximates C as faster the algorithm will terminate. Then the optimal containment with H replacing C is solved. The corresponding centers c and radius $\bar{\rho}$ of the cover of $\{v^1, \ldots, v^m\}$ with $c + \bar{\rho}_* H$ is then used to compare with the radius $\bar{\rho}$ needed for covering with a dilate of C translated by c. If the ratio is less than the tolerance $1 + \varepsilon$ the algorithm terminates, otherwise the separation oracle for C is used with $1/\rho_*(v^j - c)$ to obtain a new half-space to refine the $\mathcal{H}$-polytopal approximation of C by H. This is iterated until $\bar{\rho}/\rho_* \leq 1 + \varepsilon$.
The algorithm essentially assumes that a strong separation oracle is provided: For any input $x \in \mathbb{R}^n$ a *strong separation oracle* asserts that $x \in C$ or produces a halfspace $\{x : a^T x \leq 1\}$ supporting C in x/ρ, where $\rho > 1$ is chosen such that $x \in \mathrm{bd}(\rho C)$.
However, one can easily obtain from a general separation oracle an approximate version of the strong variant.

Algorithm 1 (Cutting-Plane Algorithm).

INPUT: $K = \mathrm{conv}\{v^1, \ldots, v^m\}$, C via (strong) separation oracle,
$H = \bigcap_{i=1}^k \{x : (a^i)^T x \leq 1\}$ a bounding polytope of C, and $\varepsilon > 0$.

Set $\bar{\rho} = \infty$, loop = TRUE.
WHILE(loop)
solve the LP: $\rho_* := \min \{\rho : \ \rho + (a^i)^T c \geq \max_j (a^i)^T v^j \ \forall\ i\}$
IF $v^j - c \in \rho_* C\ \forall j$, set $\bar{\rho} = \rho_*$, $\bar{c} = c$, loop = FALSE.
ELSE compute $\bar{\rho} = \min\{\bar{\rho}, \max_{j \in [m]} \left\| v^j - c \right\|_C\}$,
set $\bar{c}$ to the corresponding c.
IF $\bar{\rho}/\rho_* \leq 1 + \varepsilon$, set loop = FALSE.
ELSE get a^{k+1} from the strong separation oracle with input $(v^j - c)$,
set $H = H \cap \{x : (a^{k+1})^T x \leq 1\}$.
END
OUTPUT: ε-approximation $\bar{\rho}$ of ρ^* and center $\bar{c}$.

While the idea is quite simple, the algorithmic success is impressive. This can be seen from the tests provided in [H8, Section 3]. Assuming, e. g., that $C = \mathbb{B}_1$ – the regular cross-polytope possessing only $2n$ vertices but 2^n facets – one would usually use its $\mathcal{V}$-representation and solve the problem using the LP-formulation provided in Section 2 of the paper (called "$\mathcal{V}$in $\mathcal{V}$"-LP in Table 1). However approximating it via an $\mathcal{H}$-polytope and then using the cutting plane method, yields much better running times (see Table [H8, Table 2] or Table 1 below)–e. g., only 200 of the 2^{30} facets of the

30-dimensional cross-polytope suffice to approximate the problem in all tests made up to the tolerance of the used LP-solver.

Input		10			30		
	n						
	m	100	1000	10000	100	1000	10000
"$\mathcal{V}$ in $\mathcal{V}$"-LP	time (s)	0.37	15.20	2143.60	2.11	107.42	5594.63
Cutting plane	iterations	24	24	27	194	196	182
	time (s)	0.04	0.04	0.16	0.67	0.85	4.12

Table 1: Running times of the cutting plane and the "$\mathcal{V}$ in $\mathcal{V}$"-LP. Here $C = \mathbb{B}_1$ is the unit cross polytope and K are samples of $(0,1)$-normally distributed data points. In both cases, the problem is solved up to an accuracy of $\epsilon < 10^{-14}$.

7 New algorithms for k-center and extensions

In "New algorithms for k-center and extensions" [H9] we deal with the k-containment problem as defined in the preceding section – with one difference: P now denotes a finite pointset and not a polytope anymore. For the outer radius there is no difference between the radius of a polytope and the radius of its vertex set. However, for k-center this is not the case anymore.
Until the beginning of the 21$^{\text{st}}$ century it was believed that even for the Euclidean k-center problem (i. e. $C_1 = \ldots = C_k = \mathbb{B}_2$) there would be little hope for algorithmic solutions of the problem in acceptable periods of time for $n \geq 3$ and $k \geq 2$ (and even in the planar case for slightly bigger k). The situtation changed with the introduction of the so called core-sets [3, 4]: Let $S \subset P$ such that $R(S,C) \leq R(P,C) \leq (1+\varepsilon)R(S,C)$. Then S is called an *ε-core-set* of P (with respect to C).

In [4] it is shown that for any given P and $\varepsilon > 0$ it is possible to find an ε-core-set S of P with $|S| \leq \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$ in time linearly depending on n and $|P|$. Thus the bound on the size of S does neither depend on the dimension nor on the size of P. The latter is not really surprising considering Helly's theorem (or simply the corollary that $\mathrm{him}(C) \leq \dim(C)$), but the dimension independence is to be recognized. Moreover in [3] this bound is even improved to $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$. Actually these results do not substantially improve 1-center solution routines in practical computations (which is the reason why we did not explain them in detail in the preceding section). However, they have a significant impact on k-center.
On the theoretical side it is shown in [4] that (assuming k to be constant) one obtains a *polynomial time approximation scheme* (*PTAS*) for Euclidean k-center using these

core-sets (see [31] for an introduction into complexity theory). Since $|S|$ neither depends on n nor $|P|$ this is done by simply using the fact, that checking all possible partitions of S is polynomial in the input data (and only exponential in k and $1/\varepsilon$).

For practical purposes enumerating all possible partitions of $|S|$ is not recommendable even for moderate choices of k and ε. However it turns out that the core-set approach works very well in combination with a branch-and-bound (B&B) scheme and this even for other variants of k-containment.
A first branch-and-bound approach is described in [53], but concludes that solving Euclidean k-center instances in 2- or 3-space with $k \leq 4$ and $\varepsilon \geq 0.01$ is the best one can expect. Our paper expands the possibilities of using branch-and-bound methods in combination with core-sets to higher dimensions and moderately higher values of k. Moreover, the method is capable for solving general k-containment problems of almost any kind.

To do so, we first generalize the notion of core-sets to better fit to k-containment:
Let $C_1, \ldots, C_k$ be containers and $S \subset P$ such that

$$R(S, C_1, \ldots, C_k) \leq R(P, C_1, \ldots, C_k) \leq (1+\varepsilon) R(S, C_1, \ldots, C_k).$$

Then S is called an *ε-core-set* of P (with respect to $C_1, \ldots, C_k$).
When determining such core-sets S one searches for the following: a partition $(S_1, \ldots, S_k)$ of S into k clusters, a value ρ such that

$$\rho = \max_{i \in [k]} R(S_i, C_i) \leq R(P, C_1, \ldots, C_k) \leq (1+\varepsilon)\rho,$$

and translations $c^1, \ldots, c^k$, such that for all $p \in P$ there exists an index i such that $p \in c^i + (1+\varepsilon)\rho C^i$.
Our main core-set based branch-and-bound scheme aims for finding an S and a corresponding partition $(S_1, \ldots, S_k)$ with the above property [H9, Algorithm 1]:

Algorithm 2 (*k*-containment B&B-scheme).

INPUT: $P, C_1, \ldots, C_k$, ε, and possibly $\bar{\rho}$ an upper bound for $R(P, C_1, \ldots, C_k)$ (if known beforehand).

Set $S_i = \emptyset$, $\rho_i = 0$, c^i arbitrarily for all i,
k-Containment(S_i, ρ_i, c^i):
Compute $\delta = \max_{p \in P \setminus \bigcup S_i} \min_i (\|p - c^i\|_{C_i})$
Let p^* be the point where the maximum is attained
IF $\delta < \bar{\rho}$ THEN set $\bar{\rho} = \delta$
IF $(1+\varepsilon)\max_i \rho_i \geq \delta$ THEN RETURN
ELSE sort cluster indices in an descending order with respect to $\|p^* - c^i\|_{C_i}$

FOR $j = i_1, \ldots, i_k$:
  Recompute c^j and ρ_j for $S_j = S_j \cup p^*$
  IF $(1+\varepsilon)\max_i \rho_i < \bar{\rho}$:
    k-Containment(S_i, ρ_i, c^i)
RETURN the best S_i, ρ_i, and c^i found.

Since it matches a well-known greedy-type approximation algorithm, a guarantee for the approximation factor of 2 for identical, symmetric containers after the first k steps is obtained (and a weaker bound for identical, but possibly asymmetric containers, too).

Data set	m	n	k	Algorithm in [53]	Algorithm 2		
Cat	352	3	2	3.4	2.5	1.0	0.9
Cat	352	3	3	62.2	29.3	10.0	9.0
Cat	352	3	4	*	1952.1	552.7	505.6
Shark	1744	3	2	3.3	2.4	1.0	0.6
Shark	1744	3	3	248.8	15.8	6.7	3.8
Seashell	18033	3	2	29.8	11.4	8.2	1.3
Seashell	18033	3	3	169.9	81.5	65.4	11.5
Dragon	437645	3	2	132.9	70.7	69.1	3.6
Dragon	437645	3	3	5536.2	2468.1	2196.3	154.9
Norm. dist.	1000	5	3	929.8	460.0	104.0	86.3
Norm. dist.	10000	5	3	10843.5	7074.8	2840.1	1085.1

Table 2: Comparison of the branch-and-bound scheme proposed in [53] and Algorithm 2 ([H9, Algorithm 1]). The rule for picking core-set points, the 1-center computation routine, and Euclidean distance computation are exchanged to improve the performance. The 3D geometric model data sets are chosen comparable to the ones used in [53]. The "norm. dist." data sets refer to examples of (0,1) normally distributed points. Running times are given in seconds for $\varepsilon = 0.01$. Concerning the entry "*", the calculation was unfinished after 24 hours.

Table 2 ([H9, Table 1]) illustrates that after adding some slight changes to the calculations of the intermediate steps, our "pure B&B scheme" Algorithm 2 ([H9, Algorithm 1]) already improves the running times of the scheme proposed in [53] for Euclidean k-center substantially.

Further improvement is derived from modelling the k-containment problem as a mixed integer convex program (MICP) [H9, Subsection 4.1]. The model requires that for every $i \in [k]$ at least one point in K is already assigned to S_i (e. g., from the B&B algorithm).

$$
\begin{array}{rcll}
\min\ \rho & & & \\
\left\|p^j - c^i\right\|_{C_i} & \leq & \rho & \forall p^j \in S_i,\ i \in [k] \\
\left\|\lambda_{ij} p^j - c^i + (1-\lambda_{ij}) q^{ij}\right\|_{C_i} & \leq & \rho & p^j \in S_0,\ i \in [k] \\
\sum_{i=1}^{k} \lambda_{ij} & = & 1 & p^j \in S_0 \\
\lambda_{ij} & \in & \{0,1\} & p^j \in S_0,\ i \in [k]
\end{array}
\tag{4}
$$

where S_0 denotes a subset of the yet unassigned points.
The MICP (4) becomes a mixed integer linear program (MILP), if all containers C_i are polytopes. If in addition to polytopes intersections of Euclidean balls or intersections of balls and polytopes are allowed (4) becomes a mixed integer second order cone program.
The convex/linear relaxation of (4) has a simple geometric interpretation (see [H9, Figure 1]) and allows to reduce the running time of the B&B scheme.

				Algorithm 2			Algorithm 2 using (4)		
Data set	m	n	k	Nodes	Leaves	time	Nodes	Leaves	time
Cat	352	3	4	10353	2138	505.6	2380	144	207.7
Shark	1744	3	4	649	126	26.1	225	27	13.6
Seashell	18033	3	4	12718	2365	925.6	3266	479	371.0
Dragon	437645	3	3	341	96	154.9	161	43	89.2
Rand. box	1000	5	3	889.3	57.8	44.6	623.9	35.7	45.6
Rand. box	1000	5	4	20919.9	3249.8	1272.6	6544.0	238.4	611.2
Rand. box	10000	5	3	2595.1	167.6	166.6	1577.7	84.3	139.0
Rand. box	10000	5	4	32611.9	3021.6	2273.7	13768.3	808.1	1459.4
Trunc. cube	1000	5	3	1146.9	96.8	111.7	690.9	43.7	60.5
Trunc. cube	1000	5	4	17407.5	689.8	1783.3	9313.0	165.8	942.8
Trunc. cube	10000	5	3	2282.8	148.3	258.5	1380.2	70.2	145.3
Trunc. cube	10000	5	4	62343.6	2771.5	8087.4	32311.5	965.0	4096.1

Table 3: The table displays running times of the branch-and-bound algorithm with and without SOCP bounds for Euclidean k-center and $\varepsilon = 0.01$. The 3D geometric model data sets are the same as in Table 2. The 5D “rand. box” data sets refer to equally distributed points within boxes with randomly scaled axes. The 5D “trunc. cube” data sets refer to equally distributed points within the unit cube, truncated by $n+1$ randomly generated hyperplanes. Numbers of nodes and leaves of the branch-and-bound tree and running times in seconds are listed (in case of the random data sets, the mean over samples of 20).

The progress achieved by (4) (for Euclidean k-center) in comparison with the results in Table 2 ([H9, Table 1]) is summarized in Table 3 ([H9, Table 2]). For all considered Euclidean 4-center instances the running times are approximatively halved and the potential for improvement is even higher for polytopal containers as the LP-techniques are much more developed. Since Table 3 ([H9, Table 2]) shows that the reduction in the number of leaves in the B&B tree is even more substantial than that in the running time, the potential improvement even increases with growing k.

[H9, Section 5] is devoted to testing the potential of diameter partitioning. The idea of the diameter partitioning approach is to obtain good approximations for k-center problems by pairwise comparisons of point distances in K. Here we can strongly use the bounds for the Jung constant j_C for general containers C derived in Theorem 2 ([H7, Theorem 4.1]) or special containers like $\mathbb{B}_2$ or $\mathbb{B}_\infty$.
For all $\rho > 0$ the graph $G(\rho) = (P, E)$ with edges for all pairs $\{p, q\} \subset P$ with $R([p, q], C) > \rho$ is called the *ρ-distance graph* of (P, C). Using the ρ-distance graphs the k-center problem is transformed into a k-coloring problem [H9, Algorithm 2]:

Algorithm 3 (Diameter partitioning).

INPUT: P, C, and ε.

FOR ALL l pairs $\{p, q\}$ of points in P:
  Compute $\rho_j = R([p, q], C)$, $1 \leq j \leq l$
Label such that $\rho_1 \geq \ldots \geq \rho_l$
FOR $j = 1, \ldots, l$:
  IF $G(\rho_j)$ is not k-colorable THEN BREAK
  ELSE set $\rho = \rho_j$
RETURN ρ

Surely, k-coloring for general k is an extremely hard algorithmic problem itself, but easy to solve at least for $k = 2$.

Even if the containers are allowed to be different, we are able to apply a similar transformation:

a) Let $G = (V, E_1, \ldots, E_k)$ be an (edge-colored) multigraph with vertex set V and edge sets $E_1, \ldots, E_k$. A *generalized k-coloring* of G is a partition $V_1, \ldots, V_k$ of the vertices V such that for any $\{v, w\} \in E_i$, $i \in [k]$, it follows $\{v, w\} \not\subset V_i$.

b) Let $\rho > 0$. Then the *ρ-distance graph* of $(P, C_1, \ldots, C_k)$ is the edge-colored multigraph $G(\rho) = (P, E_1, \ldots, E_k)$ with edges in E_i for every pair $\{p, q\} \subset P$ with $R([p, q], C_i) > \rho$.

The existence of a solution of the generalized k-coloring problem for the ρ-distance graph $G(\rho)$ implies again that ρ is a lower bound for $R(P, C_1, \ldots, C_k)$. Determining an optimal ρ is (theoretically) done in [H9, Algorithm 3]:

Algorithm 4 (Diameter partitioning for k-containment).

INPUT: $P, C_1, \dots, C_k$, and ε.

FOR ALL l combinations of pairs $\{p, q\} \subset P$ and $i \in [k]$:
    Compute $\rho_j = R([p, q], C_i)$, $j \in [l]$
Label such that $\rho_1 \geq \ldots \geq \rho_l$
FOR $j \in [l]$:
    IF $G(\rho_j)$ has no valid generalized k-coloring THEN BREAK
    Set $\rho = \rho_j$
RETURN ρ

The following lemma summarizes the approximation errors attainable, assuming that we can solve the (generalized) k-coloring problem [H9, Lemma 2 and Lemma 4]:

Lemma 33. Algorithms 3 and 4 ([H9, Algorithms 2 and 3]) compute a $\max_{i \in [k]} j_{C_i}$-approximation for the general k-containment problem under homothety.

Respecting the edge colors seems to make generalized k-coloring more difficult than usual k-coloring. However, the problem is still polynomially solvable for $k = 2$ [H9, Lemma 3], since it may be reduced to the polynomially solvable *2*-SAT problem, a variant of SATISFIABILITY, where all clauses do only consist of two literals (see [31] for details).

Lemma 34. The generalized 2-coloring problem can be reduced to 2-Sat.

In [H9, Table 3] computational results are summarized for 2-center problems using diameter partitioning. The theoretical core-set results imply that the number of points in P have no significant impact on the running time and this is almost true in the practical B&B-routine. In contrast, diameter partitioning is quadratic in $|P|$ and thus, for larger $|P|$, it is not suitable to be combined with the B&B-algorithm. However, for moderate sizes of P diameter partitioning may reduce running times (especially, if both containers are parallelotopes, as then it is exact and no B&B-algorithm is needed).

In [H9, Section 6] we show that for planar containers and small k our B&B-routine even allows to approximate solutions for k-center problems under similarities, i. e. when allowing to (separately) rotate the containers in addition to translation and dilation. This is especially important for the medical application studied in Section 12.

8 No dimension independent core-sets for containment under homothetics

When studying the positive results about Euclidean k-center for the preceding paper and generalizing the approach to general k-containment, the following question automatically arises: is it possible to prove comparably good theoretical results in more generality than the Euclidean setting – e. g., for general k-center or when restricting to symmetric containers? The answer we derived for the two mentioned cases is negative as the title of the paper *No dimension independent core-sets for containment under homothetics* [H6] suggests.

For the explanation we need the following well-known series of successive radii:
Let $j \in [n]$. Then

a) the *maximal outer cylindric j-radius* is defined as

$$R_j^\pi(K,C) := \max\{R(K, C+F) \ : \ F \text{ a linear } (n-j)\text{-space}\}$$

and

b) the *maximal outer intersection j-radius* is defined as

$$R_j^\sigma(K,C) := \max\{R(K \cap E, C) \ : \ E \text{ an affine } j\text{-space}\}.$$

In case of $C = \mathbb{B}_2$ (but only in that case), i. e., defining the radii with respect to the Euclidean unit ball $\mathbb{B}_2$, we may also understand $R_j^\pi(K,\mathbb{B}_2)$ as the outer radius of an orthogonal projection of K maximized over all j-subspaces (orthogonal to F above).
In [40] two series of geometric inequalites are derived:

$$\frac{R_{j_2}^\pi(K,\mathbb{B}_2)}{R_{j_1}^\pi(K,\mathbb{B}_2)} \leq \sqrt{\frac{(j_1+1)j_2}{j_1(j_2+1)}}, \tag{5}$$

$$\frac{R_{j_2}^\sigma(K,\mathbb{B}_2)}{R_{j_1}^\sigma(K,\mathbb{B}_2)} \leq \sqrt{\frac{(j_1+1)j_2}{j_1(j_2+1)}}, \tag{6}$$

for $1 \leq j_1 \leq j_2 \leq n$.
Both inequalities are tight for a regular n-simplex (independently of j_1, j_2) and since $2R_1^\pi(K,C) = 2R_1^\sigma(K,C) = D(K,C)$ (even for general C), both generalize Jung's inequality (2).
In [H6], searching for lower bounds on core-set sizes, we defined a new series of successive radii: For $j \in [n]$ the *j-th core-radius* of K with respect to C is defined as $R_j(K,C) := \max\{R(S,C) : S \subset K, \ |S| = j+1\}$.

As a corollary from Helly's theorem one obtains [H6, Lemma 2.2] (which essentially states nothing else than the fact that the Helly dimension is bounded by n):

Lemma 35. For all K, C it holds $R_n(K,C) = R(K,C)$, which means there always exists a (full-dimensional) simplex $S \subset K$ such that $R(S,C) = R(K,C)$ and thus S is an $(n+1)$-point 0-core-set of K.

[H6, Theorem 3.3] shows that indeed all three series of sucessive radii are identical:

Theorem 36. Let $j \in [n]$. Then $R_j(K,C) = R_j^\sigma(K,C) = R_j^\pi(K,C)$.

An important ingredient in the proof of the above theorem is the optimality characterization for containment under homothetics derived in [H6, Theorem 2.3]:

Theorem 37. Let $K \subset C$. Then $R(K,C) = 1$, if and only if for $l \in \{2,\dots,n+1\}$ there exist $p^1,\dots,p^l \in K$ and half-spaces $\{x : (a^i)^T x \le 1\}$ supporting K and C in p^i, $i \in [l]$ such that $0 \in \operatorname{conv}\{a^1,\dots,a^l\}$.

This theorem is in a certain sense a simple version of the optimality characterization for minimal-volume enclosing ellipoids provided in [47], but as far as we know, nowhere explicitely stated before.
Using the identity of the three series of succesive radii the tightness of the inequalities in (5) and (6) for $j_2 = n$ can be interpreted as a lower bound on core-set sizes for $C = \mathbb{B}_2$, saying that in Euclidean spaces an ε-core-set S of K must have at least $\lceil \frac{1}{2\varepsilon+\varepsilon^2} \rceil = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$ points.
This motivated us to develop similar tight inequalities for general C and for general symmetric C.

First, we could solve the general case [H6, Theorem 4.1]:

Theorem 38. Let $1 \le j_1 \le j_2 \le n$. Then

$$\frac{R_{j_2}(K,C)}{R_{j_1}(K,C)} \le \frac{j_2}{j_1}, \tag{7}$$

with equality if $K = -C$ is a regular n-simplex.

For general symmetric C it follows from the inequality of Bohnenblust (1) that

$$\frac{R_{j_2}(K,C)}{R_{j_1}(K,C)} \le \frac{2j_2}{j_2+1}, \tag{8}$$

for all $j_2 \in [n]$ and $j_1 = 1$, but also for all $j_1 \in [n]$, since $R_{j_1}(K,C) \ge R_1(K,C)$.
However, it turned out from considering K to be a regular simplex centered in 0 and $C = K \cap (-K)$ (see [H6, Lemma 4.5]) that simply the combination of the two inequalities (7) and (8) is the best one can derive for general symmetric C [H6, Theorem 4.6]:

Theorem 39. Let $1 \le j_1 \le j_2 \le n$ and C be symmetric. Then

$$\frac{R_{j_2}(K,C)}{R_{j_1}(K,C)} \le \min\left\{\frac{j_2}{j_1}, \frac{2j_2}{j_2+1}\right\}. \tag{9}$$

Now taking $j_2 = n$, we may derive directly from Theorems 38 and 39 ([H6, Theorems 4.1 and 4.6]) the followong theorem on the sizes of core-sets ([H6, Corollary 4.2 and Theorem 1.2]):

Theorem 40. For all $\varepsilon \ge 0$ there exists an ε-core-set of K with respect to C of size at most $\left\lceil \frac{n}{1+\varepsilon} \right\rceil + 1$ and this bound is tight for general C and stays tight for general symmetric C if $\varepsilon < 1$.

One should remark that in case C is symmetric, a diametral pair of points is already a 1-core-set.
The above theorem proves that in general there do not exist dimension independent core-sets, not even sublinear ones, for general (symmetric) containers. In light of the PTAS results derived for Euclidean k-center in [4] using core-sets, the above blasts all hope to find a similar result for general (symmetric) containers.
Surely, better bounds can be derived for special container classes, e. g., when considering containers of small Helly dimensions.

9 Isoradial bodies

The paper on "Isoradial bodies" [H1] is motivated by the question, whether for every $j \in [n-1]$ there exists a convex body K different from $\mathbb{B}_2$, such that $R(K, \mathbb{B}_2 + F)$ is constant over all choices of linear $(n-j)$-spaces F. First we need the following definitions: Let $j \in [n]$. Then

a) the *minimal outer cylindric j-radius* is defined as

$$R_\pi^j(K,C) := \min\{R(K, C+F) \ : \ F \text{ a linear } (n-j)\text{-space}\}$$

and

b) the *minimal outer intersection j-radius* is defined as

$$R_\sigma^j(K,C) := \min\{\max_{t\in\mathbb{R}^n} R(K\cap(E+t), C) \ : \ E \text{ a linear } j\text{-space}\}.$$

In contrast to the equality of the corresponding maximal radii Theorem 36 ([H6, Theorem 3.3]) it holds $R_\sigma^j(K,C) \le R_\pi^j(K,C)$, but for $j \ge 2$ equality is generally not true (e. g., for the regular simplex T it holds $R^j_\sigma(T, \mathbb{B}_2) < R^j_\pi(T, \mathbb{B}_2)$ for all $1 < j < n$). The only exception (besides the trivial case $j = n$) is $j = 1$, since then

$$2R^1_\pi(K, C) = \min_{s\in\mathbb{R}^n} b_s(K, C) = w(K, C) = \min_{s\in\mathbb{R}^n} l_s(K, C) = 2R^1_\sigma(K, C)$$

(as already stated in the beginning of Section 2).
However, $\max_{t\in\mathbb{R}^n} R(K \cap (E + t), C)$ being constant over all choices of linear j-spaces E is equivalent to $R^j_\sigma(K, C) = R^\sigma_j(K, C)$. Now, using Theorem 36 ([H6, Theorem 3.3]), we have $R^j_\sigma(K, C) \leq R^j_\pi(K, C) \leq R^\pi_j(K, C) = R^\sigma_j(K, C)$. Hence $R^j_\sigma(K, C) = R^\sigma_j(K, C)$ implies equality in the whole chain.
The main purpose formulated in [H1] is to show that there exist K fulfilling $R^j_\pi(K, \mathbb{B}_2) = R^\pi_j(K, \mathbb{B}_2)$. However, essentially it is shown that $R^j_\sigma(K, \mathbb{B}_2) = R^\sigma_j(K, \mathbb{B}_2)$ can be achieved for any $j \in [n - 1]$. Any such K is called *(outer) j-isoradial*.

Since $2R(K, C + F) = b_s(K, C)$ for any $(n - 1)$-subspace F with outer normal s, it holds that K is 1-isoradial with respect to C, if and only if K is of constant width with respect to C. In this light isoradiality is just a generalization of constant width.

The proof of the existence of j-isoradial bodies for arbitrary j employs the following two results, the first [H1, Lemma 3.2] a simple observation:

Lemma 41. Let $j_1 \in [n - 1]$. Then every K fulfilling $R^{j_1}_\sigma(K, \mathbb{B}_2) = R^{j_1+1}_\sigma(K, \mathbb{B}_2) = \ldots = R^{n-1}_\sigma(K, \mathbb{B}_2) = R(K, \mathbb{B}_2)$ is j-isoradial for all $j \geq j_1$.

According to this lemma every constant width body K fulfilling $R^2_\sigma(K, \mathbb{B}_2) = R(K, \mathbb{B}_2)$ is j-isoradial for all $j \in [n - 1]$.
The second result ([H1, Proof of Theorem 5.1]) describes some sufficient conditions on K such that $R^{j_1}_\sigma(K, \mathbb{B}_2) = R^{j_1+1}_\pi(K, \mathbb{B}_2) = \ldots = R^{n-1}_\pi(K, \mathbb{B}_2) = R(K, \mathbb{B}_2)$ holds. It is a direct corollary of Theorem 37 ([H6, Theorem 2.3]).

Lemma 42. If K is the convex hull of a family of regions $\mathcal{T}$ on the unit sphere $\mathbb{S}_2$ possessing the properties

- for any $x \in \mathcal{T}$ it holds $-x \notin \mathcal{T}$ and
- for each intersection of $\mathbb{S}_2$ with a linear 2-space there exist at least 3 points in $\mathcal{T}$ which belong to this intersection, and which have 0 in their convex hull

then $K \neq \mathbb{B}_2$ fulfills $D(K, \mathbb{B}_2) < 2R^2_\sigma(K, \mathbb{B}_2) = 2R(K, \mathbb{B}_2)$, thus being j-isoradial for all $j \geq 2$.

The existence of such a family $\mathcal{T}$ is proven with the help of [H1, Theorem 4.1] adapting the concept of dark cloudes (see [18] and cf. [12, Section 2]) and transfering it onto the unit sphere to form a so called *dark orbit*.

The paper also asks for *(inner) j-isoradial* bodies using inner intersection radii (see [H1] for technical details).
Finally, if a body K is inner and outer j-isoradial for all j it is called *totally isoradial*. [H1, Lemma 3.4] shows that we can easily obtain a totally isoradial body from any body fulfilling $R^2_\sigma(K,C) = R(K,C)$:

Lemma 43. Let K be such that $R^2_\sigma(K,\mathbb{B}_2) = R(K,\mathbb{B}_2)$, then any Scott completion K^* of K is totally isoradial.

Taking everything together we are able to state the desired result [H1, Theorem 5.1]:

Theorem 44. There exist non-spherical totally-isoradial bodies for any dimension $n \geq 2$.

Proving the existence of totally isoradial bodies K different from the unit ball implies that there exist bodies K, such that $D(K,\mathbb{B}_2) < 2R^2_\pi(K,\mathbb{B}_2)$. This extends an old result of Eggleston [23], showing that $D(K,\mathbb{B}_2) < 2R^{n-1}_\pi(K,\mathbb{B}_2)$ is possible, to the extreme.
At the end of [H1] it is stated that the construction of dark orbits easily extends to strictly symmetric C and, if wanted, it should be possible to generalize even further. It is also stated that this would only result in a construction of bodies which are outer j-isoradial for all $j \geq 2$. Neither the existence of a Scott completion is clear for $C \neq \mathbb{B}_2$, nor that a completion would be of constant width.
We know now that Scott completions exist with respect to arbitrary symmetric C [75] (cf. Corollary 12). We also learned in Section 3 that besides the fact that not all norms are perfect, it still holds that K fulfills $r(K) + R(K) = D(K)$ whenever K is complete, which is the most important ingredient in the proof of Lemma 43 ([H1, Lemma 3.4]). This suffices to generalize [H1, Lemma 3.4] to arbitrary symmetric containers (and thus to arbitrary normed spaces):

Lemma 45. Let C be symmetric and K such that $R^2_\sigma(K,C) = R(K,C)$, then any Scott completion K^* of K is inner and outer j-isoradial for all $j \geq 2$ and if C induces a perfect norm then K^* is totally isoradial.

10 Algebraic methods for computing smallest enclosing and circumscribing cylinders of simplices

The paper "Algebraic methods for computing smallest enclosing and circumscribing cylinders of simplices" [H10] deals especially with the minimal outer cylindric $(n-1)$-radius of a convex set $K \subset \mathbb{R}^n$ within Euclidean spaces. It is mainly motivated by the papers [33, 34] on general minimal outer cylindric j-radii (and the many applications given in these papers).

Usually authors considering computational issues focus on the complexity of the problem or, since computing these radii is most often $\mathbb{NP}$-hard (see, e. g., [33]), try to find efficient approximation algorithms (see, e. g., [39]).

In [H10] we derive a framework for computing these radii for simplices within a computer algebra tool. A major motivation to do this is the fact that in many approaches for computing or approximating outer cylindric j-radii the calculation of these radii of simplices appears as a subproblem (see, e. g., [1, 67] and cf. Lemma 35 – [H6, Lemma 2.2]).

Calling an enclosing cylinder of a simplex S *circumscribing*, if all the vertices of S lie on the boundary of the cylinder, the problem is first considered for $n = 3$ and $j = 2$ and solved in two steps:

First we reduce the problem of finding the minimal enclosing cylinder of a simplex, to the problem of finding a minimal circumscribing cylinder. This is covered by [H10, Theorem 1] (cf. [H10, Figures 1 and 2]):

Theorem 46. Let $K = \operatorname{conv}\{p^1, \dots, p^m\}$, $m \geq 4$ be an at least 2-dimensional polytope in Euclidean 3-space. Then the following holds true: If there exists a 1-dimensional linear subspace F (the axis of the cylinder) such that $K \subset \rho\mathbb{B}_2 + F$, then there exists a 1-dimensional linear subspace F' such that $K \subset \rho\mathbb{B}_2 + F'$ and $\rho\mathbb{S}_2$ passes through

a) at least four vertices of K, or

b) three vertices of K, and F is contained in

  (i) the boundary of the cylinder of radius ρ and axis through two of these points,

  (ii) the boundary of the double cone with apex in the middle of the segment connecting two of these points, optimally containing balls of radius ρ around the two points (and these balls are disjoint), or

  (iii) the set of lines which are tangent to the two balls of radius ρ centered at two of these points and which are contained in the plane equidistant from these points (and the balls intersect).

Second we characterize axis directions of locally optimal solutions for radius minimal circumscribing cylinders. To do so, the problem of finding the direction vector v of an axis of such cylinders is transformed into a polynomial optimization problem (see [H10, Section 3.1] for details), assumming w. l. o. g. that one vertex of the simplex is the origin and the other are p^1, p^2, p^3. Let $M := (p^1, p^2, p^3)^T$, then M is invertible. Now, abbreviating

$$z := \frac{1}{2} M^{-1} \begin{pmatrix} \|v\|_2^2 \, \|p^1\|_2^2 - (v^T p^1)^2 \\ \|v\|_2^2 \, \|p^2\|_2^2 - (v^T p^2)^2 \\ \|v\|_2^2 \, \|p^3\|_2^2 - (v^T p^3)^2 \end{pmatrix}$$

the optimization problem derived consists of the objective $f(v) = \|z\|_2^2$ and the constraints $g_1(v) = v^T z = 0$ and $g_2(v) = \|v\|_2^2 = 1$.
It follows that for any locally optimal v it must hold

$$\det \begin{pmatrix} -\frac{\partial f}{\partial v_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_2}{\partial v_1} \\ -\frac{\partial f}{\partial v_2} & \frac{\partial g_1}{\partial v_2} & \frac{\partial g_2}{\partial v_2} \\ -\frac{\partial f}{\partial v_3} & \frac{\partial g_1}{\partial v_3} & \frac{\partial g_2}{\partial v_3} \end{pmatrix} = 0 \tag{10}$$

from which [H10, Lemma 3] is derived:

Lemma 47. a) For any normalized direction vector $v \in \mathbb{R}^3$ of an axis of a locally minimal circumscribing cylinder, the determinant (10) vanishes. If there is a finite number of such locally extreme, normalized direction vectors, then that number is bounded by 36.

b) For a generic simplex the above number is indeed finite, and all solutions have multiplicity one.

In the following subsections tighter results for special classes of simplices are derived: The first is [H10, Lemma 5]:

Lemma 48. If the four faces of the simplex can be partitioned into two pairs of equal area faces, then there are at most 28 isolated local extrema for the minimal circumscribing cylinder. They can be computed from two polynomial systems with Bézout numbers 20 and 8, respectively.

The second [H10, Lemma 6] applies to simplices of which all four facets have the same area (and thus covers the regular case too):

Lemma 49. If all four faces of the simplex T have the same area then the axis of a minimum circumscribing cylinder is perpendicular to two opposite edges.

Hence for equifacial simplices we do not need to solve a system of polynomial equations at all.

Generalizing the approach to general dimensions, the following summarizes the results derived in [H10, Lemma 8 and Lemma 9]:

Lemma 50. For $2 \leq n \leq 7$, the number of isolated local extrema for the minimal circumscribing cylinder is bounded by $6\left\{{n+1 \atop 3}\right\}$, where $\left\{{n \atop k}\right\}$ denotes the *Stirling-number of the second kind*, and for $n \geq 8$, the number of isolated local extrema for the minimal circumscribing cylinder is bounded by $2 \cdot 3^{n+1}$.

Finally, concerning regular n-simplices embedded with their vertices into the unit vectors in $\mathbb{R}^{n+1}$, we could provide the following important structural result for the coordinates of any axis direction of a (locally) optimal circumscribing cylinder ([H10, Theorem 12]):

Theorem 51. Let $u^1, \ldots, u^{n+1}$ denote the unit vectors in $\mathbb{R}^{n+1}$. Then the components of the direction vector $v = (v_1, \ldots, v_{n+1})^T$ of any locally extreme circumscribing cylinder (with $\sum_{i=1}^{n+1} v_i = 0$) of the regular n-simplex $\operatorname{conv}\{u^1, \ldots, u^{n+1}\}$ take at most three distinct values.

Applying our results for the regular simplex T, we found a mistake in the calculations (not in the result) of $R_\pi^{n-1}(T, \mathbb{B}_2)$ for odd dimensions n in [76]. This observation was a major motivation for starting the following paper.

11 Radii minimal projections of polytopes and constrained optimization of symmetric polynomials

In "Radii minimal projections of polytopes and constrained optimization of symmetric polynomials" [H11] the wrongly proven result given in [76] for $R_\pi^{n-1}(T, \mathbb{B}_2)$ for odd dimensions n is revised. To do so, we first derive a general characterization of the linear $(n-j)$-flats F fulfilling $R_\pi^j(P, \mathbb{B}_2) = R(P, \mathbb{B}_2 + F)$ for polytopes P [H11, Theorem 1], which generalizes [34, Theorem 1.9] from $j = 1$ to arbitrary $j \in [n]$:

Theorem 52. Let $1 \leq j \leq n < m$ and $P = \operatorname{conv}\{v^1, \ldots, v^m\} \subset \mathbb{R}^n$ be an n-dimensional polytope and F an $(n-j)$-flat such that $R_\pi^j(P, \mathbb{B}_2) = R(P, \mathbb{B}_2 + F)$. Then one of the following is true:

a) there exist $n+1$ affinely independent vertices v^i, $i \in I$, $|I| = n+1$ of P such that $v^i \in R(P, \mathbb{B}_2 + F)(\mathbb{S}_2 + F)$, or

b) $j \geq 2$ and $R_j(P) = R_{j-1}(P \cap H)$ for some hyperplane H spanned by vertices of P.

If $j = 1$ or if P is a regular simplex then case (a) always holds. Moreover, the number k of affinely independent vertices v^i of P such that $v^i \in R(P, \mathbb{B}_2 + F)(\mathbb{S}_2 + F)$ is at least $n - j + 2$ and there exists a $(k-1)$-flat F such that $R_j(P) = R_{j-(n-k+1)}(P \cap F)$. The bound $n - j + 2$ is best possible.

Applying this theorem to simplices, one obtains that the radius of a simplex is either already achieved as the usual outer radius of one of its facets or the optimal enclosing cylinder circumscribes the simplex. In both cases this is a crucial step towards a solution, which is obvious for the first case and pointed out in Section 10 for the second.

Moreover, as stated in Theorem 52 ([H11, Theorem 1]) for a regular simplex a minimal enclosing cylinder is always circumscribing.
Using the latter fact one can already guess an axis of the optimal cylinder for $R_\pi^{n-1}(T,\mathbb{B}_2)$, n odd, and use it to obtain an upper bound.
In the final section of [H11] we formulate the problem to determine $R_\pi^j(P,\mathbb{B}_2)$ for general j as an algebraic optimization problem (embedding the whole situation into $\mathbb{R}^{n+1}$ again). Simplifying for the regular simplex T we obtain [H11, Theorem 9]:

Theorem 53. Let $j \in [n]$. A set of vectors $s^1,\dots,s^{n-j} \in \mathbb{S}_2$ spans the $(n-j)$-dimensional subspace of a minimal enclosing j-cylinder of a regular n-simplex T, embedded with its vertices into the unit vectors of $\mathbb{R}^{n+1}$, if and only if it is an optimal solution of the problem

$$\begin{array}{rrcl} \min & \sum\limits_{i=1}^{n+1}\left(\sum\limits_{k=1}^{n-j}(s_i^k)^2\right)^2 & & \\ \text{s.t.} & \sum\limits_{i=1}^{n+1}\sum\limits_{k'=1}^{n-j}(s_i^{k'})^2 s_i^k & = & 0, \quad k \in [n-j]\,, \\ & s^1,\dots,s^{n-j} & \in & \mathbb{S}_2 \quad \text{pairwise orthogonal.} \end{array} \tag{11}$$

Now, it is easy to see that in case $j = n-1$ the program (11) reduces to a program in symmetric polynomials:

$$\begin{array}{rrcl} \min & \sum\limits_{i=1}^{n+1} s_i^4 & & \\ \text{s.t.} & \sum\limits_{i=1}^{n+1} s_i^3 & = & 0\,, \\ & \sum\limits_{i=1}^{n+1} s_i^2 & = & 1\,, \\ & \sum\limits_{i=1}^{n+1} s_i & = & 0\,. \end{array} \tag{12}$$

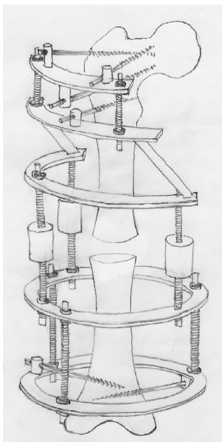
Based on exploiting symmetries and Theorem 51 ([H10, Theorem 12]), we reduce (12) for arbitrary n to an optimization problem in six variables with additional integer constraints:

$$\begin{array}{rrcl} \min & k_1s_1^4+k_2s_2^4+k_3s_3^4 & & \\ \text{s.t.} & k_1s_1^3+k_2s_2^3+k_3s_3^3 & = & 0\,, \\ & k_1s_1^2+k_2s_2^2+k_3s_3^2 & = & 1\,, \\ & k_1s_1+k_2s_2+k_3s_3 & = & 0\,, \\ & k_1+k_2+k_3 & = & n+1\,, \\ & \multicolumn{3}{l}{s_1,s_2,s_3 \in \mathbb{R}, \quad k_1,k_2,k_3 \in \mathbb{N}_0\,.} \end{array} \tag{13}$$

Finally, we show that the optimal solution of (13) is lower bounded by $1/n$, which suffices to show that the upper bound we derived before for $R_\pi^{n-1}(T,\mathbb{B}_2)$ is sharp.

12 Modeling and optimization of correction measures for human extremities

The paper on “Modeling and optimization of correction measures for human extremities” [H2] is a study on integrating tools from computational geometry related to the j-radii and k-center problems described above into a 3D-planning device for semi-automated operation planning for fully implantable intramedullary limb lengtheners (see the paper for explanations of the medical terms).
Deformities of the lower extremeties (congenital or post-traumatic) are treated by callus distraction, which was done for a long time via the external Ilizarov apparatus (see Figure 8, left). On the one hand this treatment was extremely unpleasant for the patients and always carried the danger of bacterial infections. On the other hand, since it was always possible to adjust the apparatus during the distraction process, the planning of the operation could easily be performed.



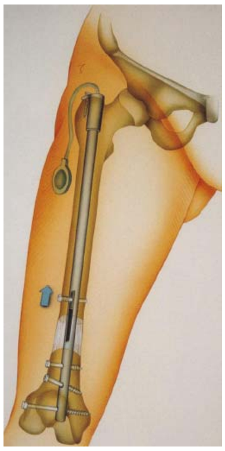


Figure 8: The Ilizarov apparatus, classical operation planning in 2D, and an implanted intramedullary nail.

Our collaborators from the Limb Lengthening Center Munich designed a new method for callus distraction, replacing the Ilizarov apparatus by the fully implantable intramedullary limb lengtheners (see Figure 8, right). This method is by far more gentle for the patients and also immensely reduces the threat of infections. However, the new method relies heavily on an exact operation planning, as post operation corrections, besides an additional surgical intervention, are impossible. This was the reason for attempting a new planning scheme based on 3-dimensional computer tomography data.

Skipping the details about the treatment of deformities and the medical tasks arrising (see [H2, Section 2]), we immediately jump into the description of the underlying mathematical problems and the discussion of algorithmic solutions. For the description

of the developed software-tool we again refer to the paper itself [H2, Section 5].
The section on the mathematical problems resulting from the medical tasks is organized in two parts: first a general framework is given by the notion of optimal containment:

Definition 54 (Optimal Containment). Let $\mathcal{B}, \mathcal{C}$ be two families of closed sets in $\mathbb{R}^n$ and $\omega : \mathcal{B} \times \mathcal{C} \to \mathbb{R}_0^+ \cup \{\infty\}$ a functional. The task is to find $B^* \in \mathcal{B}$ and $C^* \in \mathcal{C}$ such that $B^* \subset C^*$ and $\omega(B^*, C^*)$ is minimal or to decide that no such pair exists.

Second, the mathematical problems arising from the medical tasks are phrased as specified Optimal Containment problems in the above sense, such that an algorithmical treatment is possible.

The first such specification is the *k-containment problem under similarities* (see [H2, Problem 1] and cf. [H9, Section 6]):

Problem 55. For a finite point set $P \subset \mathbb{R}^n$ and closed convex sets $C_1, \ldots, C_k \subset \mathbb{R}^n$, find a rotation map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $c^i \in \mathbb{R}^n$, $i \in [k]$, and $\rho > 0$, such that $P \subset C^* = \bigcup_i \rho \Phi_i(C_i) + c^i$, while minimizing ρ.

Obviously the above problem generalizes two of the major problems we discused in the preceding subsections: finding a minimal enclosing cylinder (which is a very special 1-containment problem under similarities) and the k-containment problem under homothety, not allowing rotations.
In [61], it is shown that the k-line center problem, i. e. the problem of covering a given point set with k cylinders of equal radius, is already $\mathbb{NP}$-complete in the plane, if k is part of the input. If the dimension is part of the input the problem becomes $\mathbb{NP}$-complete even for $k = 1$ [60]. However, for our purposes we only need to solve them for 3-dimensional input-data and small values of k. And, moreover, we have good a-priori knowledge of the desired result (e. g., the orientation of the axes we are looking for).
In the practical treatment of the problem we use this a-priori knowledge of the overall situation in combination with the core-set based branch-and-bound scheme described in Section 7. To do so, in each leaf of the branch-and-bound tree k 1-containment problems have to be solved. While one can use standard methods for fast approximation of smallest enclosing balls, the fast solution of the occuring cylindrical problems strongly relies on good estimates of the axis direction based on our a-priori knowledge in combination with good lower bounds from ellipsoidal or semidefinite approximation.

[H2, Problem 2] is called *(restricted) double-ray center* :

Problem 56. For a finite set of points $P \subset \mathbb{R}^n$, an approximate center $c' \in \mathbb{R}^n$, and a maximal distance δ, determine two rays $r_1 = \{c + \lambda v^1 : \lambda \geq 0\}$ and $r_2 = \{c + \lambda v^2 : \lambda \geq 0\}$, with $v^1, v^2 \in \mathbb{R}^n \setminus \{0\}$, emanating from the same center $c \in \mathbb{R}^n$ such that $P \subset (r_1 \cup r_2) + \rho\mathbb{B}_2$ and $\|c - c'\| \leq \delta$, minimizing ρ.

Problem 56 is closely related to the 2-line center problem mentioned above, differing in the additional constraint that the two rays should originat from a common source point. The joint origin adds further difficulties for computations as guessing the correct partition – which points have to be covered by which half-cylinder – does not reduce the problem to two unrelated 1-ray center problems (different from the k-center or the k-line center problems). Thus (again differently from the k-line center problem) even well-shaped input data may yield unwanted solutions, if we do not respect the common source restriction. However, in the practical treatment for our purposes in operation planning, we can again add good knowledge about the possible regions for the center and the directions for the rays, thus permitting good approximations of the problem again.

The notion *optimal traversing cylinder* [H2, Problem 3] stems from the task to find a proper position for the nail in the medulla of the bone:

Problem 57. Let $\{E_1, \ldots, E_m\}$ be a finite set of $(n-1)$-dimensional ellipsoids in $\mathbb{R}^n$. Find a line l and a maximal ρ, such that $(l + \rho\mathbb{B}_2) \cap \mathrm{aff}(E_i) \subset E_i$ for all $i \in [m]$, or decide that no such line exists.

Differently from the preceding problems we now have a maximal in-containment problem. But the raw bone-data, obtained from computer tomography, is a finite point set and therefore not immediately suitable for the description of the containing set. Instead of first trying to reconstruct the complete surface of the bone, we take advantage of the 2-dimensional layers the 3-dimesional tomography data was aquired from and approximate the bone by ellipses in some of these layers.
While this problem in general is again very hard to tackle, it is at least known that the solution space for the feasibility version – a stabbing problem – consists of up to at most m connected components in $\mathbb{R}^3$ [13], which means that if two lines belong to the same component one can transform one into the other without leaving the component. Moreover all lines belonging to the same component define the same unique order of traversal (up to reversal) through the ellipses E_i, $i \in [m]$.
However, the order of the ellipses is in principle known from the layer-structure of the tomographic data. Knowing this order insures that there exists at most one permutation of the ellipses allowing a line transversal (again up to reversal) and therefore at most one of the connected components in the solution space of the corresponding stabbing problem[49]. This means we may find an approximately optimal traversal via discretization of the remaining space of directions. Our practical solution algorithm therefore solves the problem by projecting onto the space orthogonal to any of these finitely many traversal lines and then computes a maximal ball contained within the projected ellipses.

Papers belonging to the cumulative habilitation

[H1] R. Brandenberg, A. Dattasharma, P. Gritzmann, and D. Larman. "Isoradial bodies". In: *Discrete Comput. Geom.* 4 (2004), pp. 581–594.

[H2] R. Brandenberg, T. Gerken, P. Gritzmann, and L. Roth. "Modeling and optimization of correction measures for human extremities". In: *Mathematics–Key Technology for the Future*. Springer Berlin Heidelberg, 2008, pp. 131–148.

[H3] R. Brandenberg and B. González Merino. "A complete 3-dimensional Blaschke-Santaló diagram". In: *Math. Inequal. Appl.* 20.2 (2017), pp. 301–348.

[H4] R. Brandenberg and B. González Merino. "The asymmetry of complete and constant width bodies in general normed spaces and the Jung constant". Accepted for publication in Israel J. Math.

[H5] R. Brandenberg, B. González Merino, T. Jahn, and H. Martini. "Is a complete, reduced set necessarily of constant width?" Submitted.

[H6] R. Brandenberg and S. König. "No dimension-independent core-sets for containment under homothetics". In: *Discrete Comput. Geom.* 49.1 (2013), pp. 3–21.

[H7] R. Brandenberg and S. König. "Sharpening geometric inequalities using computable symmetry measures". In: *Mathematika* 61 (2015), pp. 559–580.

[H8] R. Brandenberg and L. Roth. "Minimal containment under homothetics: a simple cutting plane approach". In: *Comp. Optim. Appl.* 48.2 (2011), pp. 325–340.

[H9] R. Brandenberg and L. Roth. "New algorithms for k-center and extensions". In: *J. Comb. Optim.* 18.4 (2009), pp. 376–392.

[H10] R. Brandenberg and T. Theobald. "Algebraic methods for computing smallest enclosing and circumscribing cylinders of simplices". In: *Appl. Algebra Eng. Commun. Comput.* 14 (2004), pp. 439–460.

[H11] R. Brandenberg and T. Theobald. "Radii minimal projections of polytopes and constrained optimization of symmetric polynomials". In: *Adv. Geom.* 6 (2005), pp. 71–83.

Other references

[1] P. Agarwal, B. Aronov, and M. Sharir. "Line transversals of balls and smallest enclosing cylinders in three dimensions". In: *Discrete Comput. Geom.* 21 (1999), pp. 373–388.

[2] P. Agarwal, S. Har-Peled, and K. Varadarajan. "Geometric approximation via coresets". In: *Combinatorial and computational geometry* 52 (2005), pp. 1–30.

[3] M. Bădoiu and K. Clarkson. "Smaller core-sets for balls". In: *Proc. 14th Annu. ACM-SIAM Sympos. Discrete Algorithms.* 2003, pp. 801–802.

[4] M. Bădoiu, S. Har-Peled, and P. Indyk. "Approximate clustering via core-sets". In: *Proc. 34th Annu. ACM Sympos. Theory Comput.* Montreal, Quebec, Canada: ACM Press, 2002, pp. 250–257.

[5] K. Ball. "An elementary introduction to modern convex geometry". In: *Flavors of geometry.* Ed. by S. Levy. MSRI Publications, 1997, pp. 1–55.

[6] K. Ball. "Ellipsoids of maximal volume in convex bodies". In: *Geom. Dedicata* 41.2 (1992), pp. 241–250.

[7] A. Belloni and R. Freund. "On the symmetry function of a convex set". In: *Math. Prog.* 111.1-2 (2008), pp. 57–93.

[8] W. Blaschke. "Eine Frage über konvexe Körper". In: *Jahresber. Deutsch. Math.-Verein.* 25 (1916), pp. 121–125.

[9] H. Bohnenblust. "Convex regions and projections in Minkowski spaces". In: *Ann. Math.* 39 (1938), pp. 301–308.

[10] T. Bonnesen and W. Fenchel. *Theorie der konvexen Körper.* Translation: *Theory of convex bodies,* BCS Associates, Moscow, Idaho (USA), 1987. Springer, Berlin, 1974.

[11] R. Brandenberg. "Radii of convex bodies". PhD thesis. Technische Universität München, Universitätsbibliothek, 2002.

[12] R. Brandenberg and D. Larman. "Dark clouds on spheres and totally non-spherical bodies of constant breadth". In: *Beitr. Algebra Geom.* 44 (2003), pp. 531–538.

[13] H. Brönnimann, H. Everett, S. Lazard, F. Sottile, and S. Whitesides. "Transversals to line segments in three-dimensional space". In: *Discrete Comput. Geom.* 34.3 (2005), pp. 381–390.

[14] Y. D. Burago and V. A. Zalgaller. *Geometric inequalities.* Vol. 285. Springer Science & Business Media, 2013.

[15] G. Chakerian. "Sets of constant relative width and constant relative brightness". In: *Trans. Am. Math. Soc.* 129 (1967), pp. 26–37.

[16] G. Chakerian and H. Groemer. “Convex bodies of constant width”. In: *Convexity and its applications*. Ed. by P. Gruber and J. Wills. Birkhäuser, 1983, pp. 49–96.

[17] J. Current and M. O’Kelly. “Locating emergency warning sirens”. In: *Decision Sciences* 23.1 (1992), pp. 221–234.

[18] L. Danzer. “Packungs- und Überdeckungsprobleme”. Manuscript. 1976.

[19] B. Dekster. “An extension of Jung’s theorem”. In: *Israel Journal of Mathematics* 50.3 (1985), pp. 169–180.

[20] V. L. Dol’nikov. “Jung constant in l_1^n”. In: *Matematicheskie Zametki* 42.4 (1987), pp. 519–526.

[21] H. Edelsbrunner, D. G. Kirkpatrick, and R. Seidel. “On the shape of a set of points in the plane”. In: *Information Theory, IEEE Transactions on* 29.4 (1983), pp. 551–559.

[22] H. Eggleston. “Measure of asymmetry of convex curves of constant width and restricted radii of curvature”. In: *Quart. J. Math.* 3.1 (1952), pp. 63–72.

[23] H. Eggleston. “Minimal universal covers in E^n”. In: *Israel J. Math.* 16 (1963), pp. 149–155.

[24] H. Eggleston. “Sets of constant width in finite dimensional Banach spaces”. In: *Israel Journal of Mathematics* 3.3 (1965), pp. 163–172.

[25] L. Euler. “De curvis triangularibus”. In: *Acta Acad. Sci. Imp. Petropolitanae* II (1778), pp. 3–30.

[26] H. A. Fayed and A. F. Atiya. “A mixed breadth-depth first strategy for the branch and bound tree of Euclidean k-center problems”. In: *Computational Optimization and Applications* 54.3 (2013), pp. 675–703.

[27] L. Fejes Tóth. *Lagerungen in der Ebene auf der Kugel und im Raum*. Springer, 1953.

[28] M. Firsching. “Computing Maximal Copies of Polyhedra Contained in a Polyhedron”. In: *Experimental Mathematics* 24.1 (2015).

[29] K. Fischer, B. Gärtner, and M. Kutz. “Fast Smallest-Enclosing-Ball Computation in High Dimensions”. In: *Proc. 11th Annu. European Sympos. Algorithms*. 2003, pp. 630–641.

[30] R. Freund and J. Orlin. “On the complexity of four polyhedral set containment problems”. In: *Mathematical Programming* 33.2 (1985), pp. 139–145.

[31] M. R. Garey and D. S. Johnson. *Computers and intractability*. Vol. 29. W. H. Freeman and Company, New York, 2002.

[32] Y. Gordon, A. Litvak, M. Meyer, A. Pajor, et al. “John’s decomposition in the general case and applications”. In: *Journal of Differential Geometry* 68.1 (2004), pp. 99–119.

[33] P. Gritzmann and V. Klee. “Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces”. In: *Math. Program.* 59 (1993), pp. 163–213.

[34] P. Gritzmann and V. Klee. “Inner and outer j-radii of convex bodies in finite-dimensional normed spaces”. In: *Discrete Comput. Geom.* 7 (1992), pp. 255–280.

[35] P. Gritzmann and V. Klee. “On the complexity of some basic problems in computational convexity. I. Containment problems”. In: *Discrete Math.* 136 (1994), pp. 129–174.

[36] P. Gruber and R. Schneider. “Problems in geometric convexity”. In: *Contributions to Geometry.* Springer, 1979, pp. 255–278.

[37] B. Grünbaum. “Measures of symmetry for convex sets”. In: *Proc. Symp. Pure Math., 1 (Convexity)* (1963), pp. 271–284.

[38] B. Grünbaum. “Measures of symmetry for convex sets 1”. In: *Convexity: Proc. 7th Symp. Pure Math.* Vol. 7. American Mathematical Society. 1963, pp. 233–270.

[39] S. Har-Peled and K. Varadarajan. “High-Dimensional Shape Fitting in Linear Time”. In: *Discrete Comput. Geom.* 32.2 (2004), pp. 269–288.

[40] M. Henk. “A generalization of Jung’s theorem”. In: *Geom. Dedicata* 42.2 (1992), pp. 235–240.

[41] M. Hernández Cifre. “Is there a planar convex set with given width, diameter, and inradius?” In: *Amer. Math. Monthly* 107 (2000), pp. 893–900.

[42] M. Hernández Cifre, G. Salinas, and S. Segura Gomis. “Complete Systems Of Inequalities”. In: *J. Ineq. Pure Appl. Math* 2.Article 10 (2001). Online Journal; http://jipam.vu.edu.au/.

[43] M. Hernández Cifre and S. Segura Gomis. “The missing boundaries of the Santalo diagrams for the cases (d, ω, R) and (ω, R, r)”. In: *Discrete Comput. Geom.* 23 (2000), pp. 381–388.

[44] H. Jin. “Asymmetry for convex body of revolution”. In: *Wuhan University Journal of Natural Sciences* 20.2 (2015), pp. 97–100.

[45] H. Jin and Q. Guo. “A note on the extremal bodies of constant width for the Minkowski measure”. In: *Geom. Dedicata* 164.1 (2013), pp. 227–229.

[46] H. Jin and Q. Guo. “Asymmetry of convex bodies of constant width”. In: *Discrete Comput. Geom.* 47.2 (2012), pp. 415–423.

[47] F. John. “Extremum problems with inequalities as subsidiary conditions”. In: *Courant Anniversary Volume*. Interscience, 1948, pp. 187–204.

[48] H. Jung. “Über die kleinste Kugel, die eine räumliche Figur einschließt”. In: *J. Reine Angew. Math.* 123 (1901), pp. 241–257.

[49] M. Katchalski. “Thin sets and common transversals”. In: *J. Geom.* 14.2 (1980), pp. 103–107.

[50] B. Kawohl and C. Weber. “Meissner’s mysterious bodies”. In: *The Mathematical Intelligencer* 33.3 (2011), pp. 94–101.

[51] P. J. Kelly. “On Minkowski bodies of constant width”. In: *Bulletin of the American Mathematical Society* 55.12 (1949), pp. 1147–1150.

[52] L. G. Khachiyan. “Polynomial algorithms in linear programming”. In: *USSR Computational Mathematics and Mathematical Physics* 20.1 (1980), pp. 53–72.

[53] P. Kumar. “Clustering and reconstructing large data sets”. PhD thesis. Department of Computer Science, Stony Brook University, 2004.

[54] P. Kumar and A. Yildirim. “Approximate Minimum Volume Enclosing Ellipsoids Using Core Sets.” To appear in *J. Optimization Theory Appl.* 2004.

[55] P. Kumar, J. Mitchell, and E. Yıldırım. “Approximate minimum enclosing balls in high dimensions using core-sets”. In: *J. Exp. Algorithmics* 8 (2003), p. 1.1.

[56] M. Lassak and H. Martini. “Reduced convex bodies in finite dimensional normed spaces: a survey”. In: *Results Math.* 66 (2014), pp. 405–426.

[57] K. Leichtweiss. “Zwei Extremalprobleme der Minkowski-Geometrie”. In: *Math. Zeitschr.* 62 (1955), pp. 37–49.

[58] H. Martini, C. Richter, and M. Spirova. “Intersections of balls and sets of constant width in finite-dimensional normed spaces”. In: *Mathematika* 59.02 (2013), pp. 477–492.

[59] J. Matoušek, M. Sharir, and E. Welzl. “A subexponential bound for linear programming”. In: *Algorithmica* 16.4-5 (1996), pp. 498–516.

[60] N. Megiddo. “On the complexity of some geometric problems in unbounded dimension”. In: *J. Symbolic Comp.* 10.3 (1990), pp. 327–334.

[61] N. Megiddo and A. Tamir. “On the complexity of locating linear facilities in the plane”. In: *Oper. Res. Lett.* 1.5 (1982), pp. 194–197.

[62] J. Moreno and R. Schneider. “Diametrically complete sets in Minkowski spaces”. In: *Israel J. Math.* 191.2 (2012), pp. 701–720.

[63] F. Reuleaux. *Theoretische Kinematik, Grundzüge einer Theorie des Maschinenwesens.* Vieweg, 1875.

[64] J. R. Sangwine-Yager. "The missing boundary of the Blaschke diagram". In: *Amer. Math. Monthly* 96 (1989), pp. 233–237.

[65] L. A. Santaló. "Sobre los sistemas completos de desigualdades entre tres elementos de una figura convexa plana". In: *Math. Notae* 17 (1961), pp. 82–104.

[66] R. Schneider. "Stability for some extremal properties of the simplex". In: *J. Geom.* 96.1 (2009), pp. 135–148.

[67] E. Schömer, J. Sellen, M. Teichmann, and C. Yap. "Smallest enclosing cylinders". In: *Algorithmica* 27 (2000), pp. 170–186.

[68] P. Scott. "Sets of constant width and inequalities". In: *Q. J. Math. Oxf. II. Ser.* 32 (1981), pp. 345–348.

[69] C. Sohl, M. Gustafsson, and G. Kristensson. "Physical limitations on broadband scattering by heterogeneous obstacles". In: *Journal of Physics A: Mathematical and Theoretical* 40.36 (2007), p. 11165.

[70] P. Steinhagen. "Über die größte Kugel in einer konvexen Punktmenge". In: *Abh. Math. Sem. Univ. Hamburg* 1 (1921), pp. 15–26.

[71] L. Ting and J. Keller. "Extremal convex planar sets". In: *Discrete Comput. Geom.* 33.3 (2005), pp. 369–393.

[72] G. Toth. *Measures of Symmetry for Convex Sets and Stability.* Universitext. Springer, 2015.

[73] I. W. Tsang, J. T. Kwok, and P.-M. Cheung. "Core vector machines: Fast SVM training on very large data sets". In: *Journal of Machine Learning Research.* 2005, pp. 363–392.

[74] V. V. Vazirani. *Approximation algorithms.* Springer Science & Business Media, 2013.

[75] S. Vrecica. "A note on sets of constant width". In: *Publ. Inst. Math.* 29 (1981), pp. 289–291.

[76] B. Weissbach. "Über Umkugeln von Projektionen regulärer Simplexe". In: *Beitr. Algebra Geom.* 16 (1983), pp. 127–137.

[77] H. Zabrodsky, S. Peleg, and D. Avnir. "Symmetry as a continuous feature". In: *Pattern Analysis and Machine Intelligence, IEEE Transactions on* 17.12 (1995), pp. 1154–1166.

[78] G. Zhou, K. Toh, and J. Sun. "Efficient Algorithms for the Smallest Enclosing Ball Problem". In: *Comput. Optim. Appl.* 30.2 (2005), pp. 147–160.