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# Inequalities for Matrix Powers and Absolute Values: A Generalization of London's Conjecture

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**Abstract**

We provide an inequality for absolute row and column sums of the powers of a complex matrix. This inequality generalizes several other inequalities. As a result, it provides an inequality that compares the absolute entry sum of the matrix powers to the sum of the powers of the absolute row/column sums. This provides a proof for a conjecture of London, which states that for all complex matrices $A$ such that $|A|$ is symmetric, we have $\operatorname{sum}(|A^p|) \leq \sum_{i=1}^{n} r_i(|A|)^p$.

## 1 Introduction

We consider $n \times n$ matrices, denoted by $A$, with complex entries $a_{ij}$. In particular, we look at the row and column sums of $A$, which are denoted by $r_i(A)$ and $c_j(A)$, respectively. If $A$ is clear from the context, we abbreviate this by $r_i$ and $c_j$. We use $|c|$ for the absolute value (modulus) of $c \in \mathbb{C}$ and $|A|$ for the matrix where every single entry $a_{ij}$ of $A$ is replaced by its modulus $|a_{ij}|$. For the matrix power $A^p$, $p \in \mathbb{N}$, we define the following abbreviations: $a_{ij}^{[p]} := (A^p)_{ij}$, $r_i^{[p]} := r_i(A^p)$, and $c_j^{[p]} := c_j(A^p)$. We assume that $A^0 = I$ is the identity matrix.

As a special case, we consider directed and undirected (multi-)graphs $G = (V, E)$ with $n := |V|$ vertices and $m := |E|$ edges (and their adjacency matrices). The in-degree and the out-degree of a vertex $v \in V$ are denoted by $d_{\text{in}}(v)$ and $d_{\text{out}}(v)$, respectively. In undirected graphs, the degree of a vertex $v \in V$ is denoted by $d(v)$. A *walk* in a multigraph $G = (V, E)$ is an alternating sequence $(v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k)$ of vertices $v_i \in V$ and edges $e_i \in E$ where each edge $e_i$ of the walk must connect vertex $v_{i-1}$ to vertex $v_i$ in $G$, that is, $e_i = (v_{i-1}, v_i)$ for all $i \in \{1, \ldots, k\}$. Vertices and edges can be used repeatedly in the same walk. If the multigraph has no parallel edges, then the walks could also be specified by the sequence of vertices $(v_0, v_1, \ldots, v_{k-1}, v_k)$ without the edges. The *length* of a walk is the number of edge traversals. That means, the walk $(v_0, \ldots, v_k)$ consisting of $k+1$ vertices and $k$ edges is a walk of length $k$. We call it a *$k$-step walk*. Let $s_k(v)$ denote the number of $k$-step walks starting at vertex $v \in V$ and let $e_k(v)$ denote the number of $k$-step walks ending at $v$. If $G$ is undirected, then we have $w_k(v) := s_k(v) = e_k(v)$. The total number of $k$-step walks is denoted by $w_k$. For walks of length 0, we have $s_0(v) = e_0(v) = 1$ for each vertex $v$ and $w_0 = n$. For walks of length 1, we have $s_1(v) = d_{\text{out}}(v)$ and $e_1(v) = d_{\text{in}}(v)$, i.e., $w_1(v) = d(v)$ for undirected graphs. This implies $w_1 = \sum_{v \in V} d_{\text{out}}(v) = \sum_{v \in V} d_{\text{in}}(v) = m$ for directed graphs. For undirected graphs, we have $w_1 = \sum_{v \in V} d(v) = 2m$ by the handshake lemma.

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## 2 Related Work

### 2.1 Undirected Graphs and Symmetric Nonnegative Matrices

The $k$-th moment of the degree sequence has been discussed in the papers by Füredi and Kündgen [FK06] and Cioabă [Cio06]. Cioabă [Cio06] used the equality

$$\sum_{v \in V} d(v)^{k+1} = \sum_{v \in V} d(v) \cdot m_k(v)$$

(where $m_k(v) = \sum_{\{v,w\} \in E} d(w)^k / d(v)$ is the average of the $k$-th powers of the degrees of the neighbors of $v$) to deduce the following inequality using Chebyshev's inequality:

$$\sum_{v \in V} d(v)^{k+1} \geq \frac{2m}{n} \sum_{v \in V} d(v)^k \quad .$$

Note that this corresponds to the inequality

$$w_0 \sum_{v \in V} d(v)^{k+1} \geq w_1 \sum_{v \in V} d(v)^k \quad .$$

Let us remark here that it is easy to obtain the following inequality using the same argument:

$$\begin{aligned} \frac{1}{n} \sum_{v \in V} d(v)^2 \cdot d(v)^k &\geq \frac{1}{n} \sum_{v \in V} d(v)^2 \cdot \frac{1}{n} \sum_{v \in V} d(v)^k \\ \sum_{v \in V} d(v)^{k+2} &\geq \frac{w_2}{n} \cdot \sum_{v \in V} d(v)^k \quad . \end{aligned}$$

This corresponds to the inequality

$$w_0 \sum_{v \in V} d(v)^{k+2} \geq w_2 \sum_{v \in V} d(v)^k \quad .$$

Both results can be generalized using the following theorem (and also to row sums of symmetric matrices). This inequality for undirected graphs was conjectured by Noy and proven by Fiol and Garriga [FG09].

**Theorem 1** (Fiol and Garriga)**.** *For every undirected graph, the number $w_k$ of walks of length $k$ does not exceed the sum of the $k$-th powers of the vertex degrees, i.e.,*

$$w_k \leq \sum_{v \in V} d(v)^k \quad .$$

Let us emphasize that there are close connections to the graph homomorphism numbers for paths and stars: Suppose that $H$ is an undirected graph with $m_H$ edges. A *graph homomorphism* from a graph $H$ to a graph $G$ is a mapping $f : V(H) \mapsto V(G)$ such that, for each edge $\{u, v\}$ of $H$, $\{f(u), f(v)\}$ is an edge of $G$. Suppose that $h_H(G)$ denotes the number of homomorphisms from $H$ to $G$. If $H$ is a path consisting of $k$ edges, then $h_H(G)$ is the number of $k$-step walks in $G$, i.e., $h_H(G) = w_k(G)$. If $H$ is a star consisting of $k$ edges, then $h_H(G)$ denotes the sum of the $k$-th vertex degree powers of $G$, i.e., $h_H(G) = \sum_{v \in V(G)} d(v)^k$.

Several applications for the sum of vertex degree powers are discussed in the article of Cao, Dehmer, and Shi [CDS14]. In particular, a corresponding graph entropy measure was considered.

Theorem 1 is a special case of an older theorem (for powers of nonnegative symmetric matrices and their row or column sums) which was conjectured by London [Lon66] and proven by Hoffman [Hof67].

**Theorem 2** (London; Hoffman). *For every symmetric nonnegative matrix and $p \in \mathbb{N}$, we have*

$$\operatorname{sum}(A^p) \leq \sum_{i=1}^{n} r_i^p$$

Another proof of this theorem has been published by Sidorenko [Sid85b; Sid85a]. He also showed that for $k > 1$ equality is achieved if and only if $A$ is decomposable into a direct sum of matrices which are proportional to doubly stochastic matrices.

The inequality of Fiol and Garriga has been refined by Täubig [Täu12; Täu15] in the following way.

**Theorem 3.** *For all undirected graphs and $p, q \in \mathbb{N}$ with $p \geq 1$, we have*

$$\sum_{v \in V} d(v)^q w_p(v) \leq \sum_{v \in V} d(v)^{q+1} w_{p-1}(v) \quad .$$

In particular, this implies the following.

**Corollary 4.** *For all undirected graphs and $p, q \in \mathbb{N}$, we have*

$$w_{p+q} \leq \sum_{v \in V} d(v)^q w_p(v) \leq \sum_{v \in V} d(v)^{p+q} \quad .$$

More generally, the inequality of London and Hoffman has been refined as follows (see Täubig [Täu12; Täu15]).

**Theorem 5.** *For every nonnegative symmetric matrix $A$ and $p, q \in \mathbb{N}$ with $p \geq 1$, we have*

$$\sum_{i=1}^{n} r_i(A)^q \cdot r_i\left(A^p\right) \leq \sum_{i=1}^{n} r_i(A)^{q+1} \cdot r_i\left(A^{p-1}\right) \quad .$$

This corresponds to the short form $\sum_{i=1}^{n} r_i^q r_i^{[p]} \leq \sum_{i=1}^{n} r_i^{q+1} r_i^{[p-1]}$.

**Corollary 6.** *For every nonnegative symmetric matrix $A$ and $p, q \in \mathbb{N}$, we have*

$$\operatorname{sum}\left(A^{p+q}\right) \leq \sum_{i=1}^{n} r_i(A)^q \cdot r_i\left(A^p\right) \leq \sum_{i=1}^{n} r_i(A)^{p+q} \quad .$$

This corresponds to the short form $\operatorname{sum}(A^{p+q}) \leq \sum_{i=1}^{n} r_i^q r_i^{[p]} \leq \sum_{i=1}^{n} r_i^{p+q}$. These inequalities were generalized to directed graphs and nonsymmetric matrices, see Theorems 7 and 10.

## 2.2 Directed Graphs and Arbitrary Nonnegative Matrices

### 2.2.1 Directed Graphs

Now we briefly discuss generalizations of Theorem 1 to directed graphs and corresponding generalizations of Theorem 2 to nonsymmetric nonnegative matrices.

We have $w_k \not\leq \sum_{x\in V} d_{\text{in}}(x)^k$ and $w_k \not\leq \sum_{x\in V} d_{\text{out}}(x)^k$ (see Täubig [Täu12; Täu15]). Also, trying to generalize the inequality by using direct products of $d_{\text{in}}(x)$ and $d_{\text{out}}(x)$ is not successful, e.g., $w_k \not\leq \sum_{x\in V} \sqrt{d_{\text{in}}(x) \cdot d_{\text{out}}(x)}^k$.

While the power sum for $d_{\text{in}}(x)$ or $d_{\text{out}}(x)$ alone is not suitable for bounding $w_k$, it was shown that a combination (namely, the geometric mean) of both sums is sufficient. To this end, it was shown before that for the consideration of power sums with exponent $q$ over the set of walks of length $p$, the total cannot decrease if we shorten the walk length while at the same time the exponent is increased by the same amount.

**Theorem 7.** *For all directed graphs $G = (V, E)$ and for all $p, q \in \mathbb{N}$, $p \geq 1$, the following inequality holds*

$$\sum_{x\in V} d_{in}(x)^q s_p(x) \sum_{y\in V} d_{out}(y)^q e_p(y) \;\leq\; \sum_{x\in V} d_{in}(x)^{q+1} s_{p-1}(x) \sum_{y\in V} d_{out}(y)^{q+1} e_{p-1}(y) \;.$$

This was shown in a more general form for nonnegative matrices (see Theorem 10).

Theorem 7 implies a chain of inequalities where the smallest and the largest elements are

$$\sum_{x\in V} s_{p+q}(x) \sum_{y\in V} e_{p+q}(y) = w_{p+q}^2 \qquad \text{and} \qquad \sum_{x\in V} d_{\text{in}}(x)^{p+q} \sum_{y\in V} d_{\text{out}}(y)^{p+q} \;.$$

Hence it directly implies the following corollary (see also the more general form in Corollary 11).

**Corollary 8.** *For every directed graph $G = (V, E)$ and for all $k \in \mathbb{N}$, we have*

$$w_k \leq \sqrt{\left(\sum_{v\in V} d_{in}(v)^k\right)\left(\sum_{v\in V} d_{out}(v)^k\right)} \;.$$

That means, although $w_k \not\leq \sum_{x\in V} d_{\text{in}}(x)^k$ and $w_k \not\leq \sum_{x\in V} d_{\text{out}}(x)^k$, we know for the geometric mean of the two power sums that

$$w_k \leq \mathfrak{G}\left(\sum_{v\in V} d_{\text{in}}(v)^k, \sum_{v\in V} d_{\text{out}}(v)^k\right) = \sqrt{\left(\sum_{x\in V} d_{\text{in}}(x)^k\right)\left(\sum_{x\in V} d_{\text{out}}(x)^k\right)} \;.$$

Corollary 8 implies the following statement by applying the inequality of arithmetic and geometric means.

**Corollary 9.** *For every directed graph $G = (V, E)$ and for all $k \in \mathbb{N}$, we have*

$$w_k \leq \frac{1}{2}\left(\sum_{v\in V} d_{in}(v)^k + d_{out}(v)^k\right).$$

Therefore, at least one of the two power sums must be greater than or equal to $w_k$:

$$w_k \leq \max\left\{\sum_{x \in V} d_{\text{in}}(x)^k, \sum_{x \in V} d_{\text{out}}(x)^k\right\}$$

Note that Corollaries 8 and 9 contain Theorem 1 by Fiol and Garriga as a special case (since $d_{\text{in}}(x) = d_{\text{out}}(x)$ holds for all $x$). Both corollaries could also be derived from corresponding inequalities for nonnegative matrices that were shown by Merikoski and Virtanen [MV95] and Virtanen [Vir90] (see Corollaries 11 and 12).

### 2.2.2 Row and Column Sums in Nonnegative Matrices

Theorems 2 and 7 have been generalized to the case of arbitrary nonnegative matrices (see Täubig [Täu12; Täu15]).

**Theorem 10.** *For every nonnegative $n \times n$-matrix $A$ with row sums $r_i$ and column sums $c_i$, $i \in [n]$, and for all $p, q \in \mathbb{N}$, $p \geq 1$, the following inequality holds:*

$$\sum_{i \in [n]} c_i^q r_i^{[p]} \sum_{j \in [n]} r_j^q c_j^{[p]} \leq \sum_{i \in [n]} c_i^{q+1} r_i^{[p-1]} \sum_{j \in [n]} r_j^{q+1} c_j^{[p-1]} \quad .$$

This theorem is a refinement of the following corollary which was already shown by Merikoski and Virtanen [MV95].

**Corollary 11** (Merikoski and Virtanen)**.** *For every nonnegative $n \times n$-matrix $A$ with row sums $r_i$ and column sums $c_i$, $i \in [n]$, and all $p \in \mathbb{N}$, we have*

$$\text{sum}(A^p) \leq \sqrt{\left(\sum_{i=1}^n c_i^p\right)\left(\sum_{i=1}^n r_i^p\right)} \quad .$$

Corollary 11 generalizes Theorem 2 since $r_i$ equals $c_i$ for symmetric matrices. It also generalizes Corollary 8.

Since the right hand side of Corollary 11 can be interpreted as a geometric mean, the inequality of arithmetic and geometric means directly implies the following corollary, which was already proven by Virtanen [Vir90] using majorization. An alternative proof for the same result has been published by Merikoski and Virtanen [MV91].

**Corollary 12** (Virtanen)**.** *For every nonnegative $n \times n$-matrix $A$ with row sums $r_i$ and column sums $c_i$, $i \in [n]$, and all $p \in \mathbb{N}$, we have*

$$\text{sum}(A^p) \leq \frac{1}{2}\left(\sum_{i=1}^n c_i^p + r_i^p\right) \quad .$$

This is a more general form of Theorems 2 and 9.

## 3 Main Results

### 3.1 Absolute Values and Complex Matrices

Now, we will generalize Theorem 10 and Corollary 11 to the case of arbitrary complex matrices. This provides a generalization of the following conjecture of London [Lon66].

**Conjecture 13.** *For every complex $n \times n$-matrix such that $|A|$ is symmetric and for all $p \in \mathbb{N}$, we have*

$$\operatorname{sum}(|A^p|) \leq \sum_{i=1}^{n} r_i(|A|)^p \ .$$

**Theorem 14.** *For every complex $n \times n$-matrix $A$ with row sums $r_i$ and column sums $c_i$, $i \in [n]$, and for all $p, q \in \mathbb{N}$, $p \geq 1$, the following inequality holds:*

$$\sum_{i\in[n]} c_i(|A|)^q r_i(|A^p|) \sum_{j\in[n]} r_j(|A|)^q c_j(|A^p|) \leq \sum_{i\in[n]} c_i(|A|)^{q+1} r_i(|A^{p-1}|) \sum_{j\in[n]} r_j(|A|)^{q+1} c_j(|A^{p-1}|) \ .$$

*Proof.*

$$\begin{aligned}
&\sum_{i\in[n]} c_i(|A|)^q r_i(|A^p|) \sum_{j\in[n]} r_j(|A|)^q c_j(|A^p|) \\
&= \left(\sum_{i=1}^{n} c_i(|A|)^q \sum_{j=1}^{n} |a_{ij}^{[p]}|\right)\left(\sum_{j=1}^{n} r_j(|A|)^q \sum_{i=1}^{n} |a_{ij}^{[p]}|\right) \\
&= \left(\sum_{i=1}^{n} c_i(|A|)^q \sum_{j=1}^{n} \left|\sum_{k=1}^{n} a_{ik}^{[p-1]} a_{kj}\right|\right)\left(\sum_{j=1}^{n} r_j(|A|)^q \sum_{i=1}^{n} \left|\sum_{\ell=1}^{n} a_{i\ell} a_{\ell j}^{[p-1]}\right|\right) \\
&\leq \left(\sum_{i=1}^{n} c_i(|A|)^q \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}^{[p-1]} a_{kj}|\right)\left(\sum_{j=1}^{n} r_j(|A|)^q \sum_{i=1}^{n}\sum_{\ell=1}^{n} |a_{i\ell} a_{\ell j}^{[p-1]}|\right) \\
&= \left(\sum_{i=1}^{n} c_i(|A|)^q \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}^{[p-1]}||a_{kj}|\right)\left(\sum_{j=1}^{n} r_j(|A|)^q \sum_{i=1}^{n}\sum_{\ell=1}^{n} |a_{i\ell}||a_{\ell j}^{[p-1]}|\right) \\
&= \sum_{i=1}^{n} c_i(|A|)^q \sum_{k=1}^{n} |a_{ik}^{[p-1]}| r_k(|A|) \sum_{j=1}^{n} r_j(|A|)^q \sum_{\ell=1}^{n} |a_{\ell j}^{[p-1]}| c_\ell(|A|) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n} |a_{ik}^{[p-1]}|\ c_i(|A|)^q\ r_k(|A|)\ |a_{\ell j}^{[p-1]}|\ r_j(|A|)^q\ c_\ell(|A|) \\
&= \sum_{(i,k)\in[n]^2} \Bigg( \sum_{\substack{(\ell,j)\in[n]^2 \\ (\ell,j)=(i,k)}} |a_{ik}^{[p-1]}|\ |a_{\ell j}^{[p-1]}|\ c_i(|A|)^q\ r_j(|A|)^q\ r_k(|A|)\ c_\ell(|A|) \\
&\qquad + \sum_{\substack{(\ell,j)\in[n]^2 \\ (\ell,j)\neq(i,k)}} |a_{ik}^{[p-1]}|\ |a_{\ell j}^{[p-1]}|\ c_i(|A|)^q\ r_j(|A|)^q\ r_k(|A|)\ c_\ell(|A|) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
= & \left( \sum_{(i,k)\in[n]^2} |a_{ik}^{[p-1]}|^2 \; c_i(|A|)^{q+1} \; r_k(|A|)^{q+1} \right) \\
& + \sum_{(i,k)\in[n]^2} \sum_{\substack{(\ell,j)\in[n]^2 \\ (\ell,j)\neq(i,k)}} |a_{ik}^{[p-1]}| \; |a_{\ell j}^{[p-1]}| \; c_i(|A|)^q \; r_j(|A|)^q \; r_k(|A|) \; c_\ell(|A|) \\
= & \left( \sum_{(i,k)\in[n]^2} |a_{ik}^{[p-1]}|^2 \; c_i(|A|)^{q+1} \; r_k(|A|)^{q+1} \right) \\
& + \sum_{(i,k)<(\ell,j)\in[n]^4} |a_{ik}^{[p-1]}| \; |a_{\ell j}^{[p-1]}| \left[ c_i(|A|)^q r_j(|A|)^q r_k(|A|) c_\ell(|A|) + c_\ell(|A|)^q r_k(|A|)^q r_j(|A|) c_i(|A|) \right] \\
\leq & \left( \sum_{(i,k)\in[n]^2} |a_{ik}^{[p-1]}|^2 \; c_i(|A|)^{q+1} \; r_k(|A|)^{q+1} \right) \\
& + \sum_{(i,k)<(\ell,j)\in[n]^4} |a_{ik}^{[p-1]}| \; |a_{\ell j}^{[p-1]}| \left[ c_i(|A|)^{q+1} \; r_j(|A|)^{q+1} + c_\ell(|A|)^{q+1} \; r_k(|A|)^{q+1} \right] \\
= & \sum_{i=1}^n \sum_{k=1}^n \sum_{\ell=1}^n \sum_{j=1}^n |a_{ik}^{[p-1]}| \; |a_{\ell j}^{[p-1]}| \; c_i(|A|)^{q+1} \; r_j(|A|)^{q+1} \\
= & \sum_{i=1}^n c_i(|A|)^{q+1} \sum_{k=1}^n |a_{ik}^{[p-1]}| \sum_{j=1}^n r_j(|A|)^{q+1} \sum_{\ell=1}^n |a_{\ell j}^{[p-1]}| \\
= & \sum_{i=1}^n c_i(|A|)^{q+1} \; r_i(|A^{p-1}|) \sum_{j=1}^n r_j(|A|)^{q+1} \; c_j(|A^{p-1}|)
\end{aligned}
$$

□

This theorem implies the following corollary.

**Corollary 15.** *For every complex $n \times n$-matrix $A$ and all $p \in \mathbb{N}$, we have*

$$
\operatorname{sum}(|A^p|) \leq \sqrt{\left( \sum_{i=1}^n c_i(|A|)^p \right) \left( \sum_{i=1}^n r_i(|A|)^p \right)} \; .
$$

*Proof.* We show the squared form of the inequality. First, we notice that

$$
\sum_{i\in[n]} c_i(|A|)^0 r_i(|A^p|) \sum_{j\in[n]} r_j(|A|)^0 c_j(|A^p|) = \sum_{i\in[n]} r_i(|A^p|) \sum_{j\in[n]} c_j(|A^p|) = (\operatorname{sum}(|A^p|))^2 \; .
$$

Hence, we start with the term on the left hand side and we apply Theorem 14 repeatedly until we end up with

$$
\begin{aligned}
\sum_{i\in[n]} c_i(|A|)^0 r_i(|A^p|) \sum_{j\in[n]} r_j(|A|)^0 c_j(|A^p|) & \leq \; \ldots \\
& \leq \sum_{i\in[n]} c_i(|A|)^p r_i(|A^0|) \sum_{j\in[n]} r_j(|A|)^p c_j(|A^0|) \\
& = \sum_{i\in[n]} c_i(|A|)^p \sum_{j\in[n]} r_j(|A|)^p
\end{aligned}
$$

For the last equality, it can be assumed that $A^0$ is the identity matrix. □

Obviously, Corollary 15 generalizes Corollary 11.

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