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Asymptotically Stable Almost-Periodic Oscillations in Systems with Hysteresis Nonlinearities

M. Brokate, I. Collings, A. V. Pokrovskiĭ and F. Stagnitti

Abstract. We present some sufficient conditions for the asymptotic stability of forced almost-periodic oscillations in nonlinear systems subject to small hysteresis perturbations. The main technical restriction on hysteresis nonlinearity comes to a contraction-type property, which holds for some classical models of hysteresis. Also we require a special stability property of the unperturbed system in the sense of Lyapunov and the bounded input - bounded output.

Keywords: *Nonlinear dynamical systems, almost periodic oscillations, hysteresis nonlinearities, asymptotic stability*

AMS subject classification: Primary 47 H 30, secondary 58 F 10

1. Introduction

The mathematical analysis of dynamical systems arising from applications often consists of two stages. In the first stage, the underlying system is formalized and analyzed within the context of some mathematical setting such as a differential equation. In the simplest case these are ordinary differential equations of the form

$$x' = f(t, x)\,. \tag{1.1}$$

Usually we are especially interested in solutions which are both stable in a reasonable sense and behave rather regularly in time. The simplest example of such a behaviour is, probably, the periodic one. The existence of periodic solutions, however, usually requires the periodicity of the function f in t. It is well known that often the function f is not a periodic function, but some sort of superposition of functions with different, independent, periods. In such situations, almost-periodic solutions are more natural (we refer to [1, 2, 8] and the constructions below).

M. Brokate: Zentrum Math. Techn. Univ., D-80290 München
I. Collings: Deakin Univ., School Comp. & Math., Geelong 3217, Australia
A. V. Pokrovskiĭ: Nat. Univ. Ireland, Inst. Nonlin. Sci., Univ. College, Cork, Ireland; Permanent address: Russ. Acad. Sci., Inst. Inf. Trans. Probl., 19 Ermolovoy St., Moscow 101447, Russia
F. Stagnitti: Deakin Univ., School Ecol. & Envir., Warrnambool 3217, Australia

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The second stage of investigation is often connected with analysis of the influence of possibly complicated, and sometimes not really well known, perturbations on the solution $x^0(\cdot)$. These perturbations can be of deterministic or of stochastic nature. Here, we are interested mainly in hysteresis perturbations which are important in many fields, such as physics, control theory, ecology (e.g., water transport in porous media) and many others (see, for example, [4, 9]).

The theory of functional-differential equations with hysteresis nonlinearities constitutes a new chapter of applied nonlinear functional analysis. In our case a differential-operator perturbation of equation (1.1) can be written, for $t \geq t_0$, in the form

$$x' = f(t,x) + \varepsilon g(x, z(t)), \tag{1.2}$$

$$z(t) = (\Gamma[t_0, z(t_0)]Lx)\,(t). \tag{1.3}$$

Here z takes values $z(t)$ in some subset Z of some Banach space, $L : \mathbb{R}^d \to \mathbb{R}^m$ is a linear mapping, and $\Gamma[t_0, z_0]$ is an operator with initial memory, $z_0 \in Z$, which transforms functions $u : [t_0, \infty) \to \mathbb{R}^m$ to functions $z : [t_0, \infty) \to Z$. As usual, the notation $(\Gamma[t_0, z_0]u)(t)$ refers to the value of the function $z = \Gamma[t_0, z_0]u$ at the time t. We suppose that the operators $\Gamma[t_0, z_0]$ can be understood as describing an autonomous control dynamical system Γ. That is, the family of operators $\Gamma[t_0, z_0]$ satisfies the *Volterra property*

$$u(s) \equiv v(s) \quad (s \in [t_0, t]) \quad \Longrightarrow \quad (\Gamma[t_0, z_0]u)(t) = (\Gamma[t_0, z_0]v)(t)\,, \tag{1.4}$$

the *semi-group property*

$$\big(\Gamma[t_1, (\Gamma[t_0, z_0]u)\,(t_1)]v\big)(t_2) = (\Gamma[t_0, z_0]u)\,(t_2) \qquad (v = u|_{[t_1,\infty)}), \tag{1.5}$$

and is *autonomous*, that is,

$$(\Gamma[t_0, z_0]u)\,(t) = (\Gamma[t_1, z_0]v)\,(t - t_0 + t_1) \qquad (t \geq t_0)\,, \tag{1.6}$$

where $v(t) = u(t - t_1 + t_0)$ $(t \geq t_1)$. Moreover, we assume certain continuity conditions to hold, to be formulated in detail below, which ensure among other things that system (1.2), (1.3) is well-posed (see Corollary 3.2 below). These conditions do hold for the hysteresis nonlinearities which we have in mind; because on the other hand they do not contain explicitly special features of hysteresis like rate independence, they appear to be more general. However, we here do not pursue the question whether there are other interesting nonlinearities besides hysteresis ones to which our results apply.

We ask whether almost-periodic solutions of system (1.2), (1.3) exist for small ε and whether they are asymptotically stable, provided that such solutions exist for the unperturbed equation (1.1). The main result, Theorem 2.1 at the end of the next section, asserts that the answer to this question is 'yes', provided that some natural technical conditions hold. In a general sense, these conditions are summarized in the following two main points:

(i) A specific contraction property of the hysteresis nonlinearity (Definition 2.2 below), which hold for many classical hysteresis systems.

(ii) A certain stability property of the solution of the unperturbed equation (Definition 2.4), which combines essential features of the exponential stability in the sense of Lyapunov and the so-called bounded input, bounded output property from control theory.

In order to prove the results of existence and asymptotic stability, we extend the techniques of [3], where we have already considered the case when f is periodic in t.

2. The main result

At the end of this section, in Subsection 2.5, we formulate the main theorem of this paper. For this purpose, we collect the necessary definitions and introduce the functional setting. These are similar to those presented in [3] but nevertheless will be stated in full. We partition the flow of definitions into four blocks labelled as Subsections 2.1 - 2.4.

2.1 Almost-periodic functions. In this first block we recall some facts from the theory of almost-periodic functions in a form which is convenient for our use. We consider functions

$$\left.\begin{aligned} x &: \mathbb{R} \to \mathbb{R}^d \\ z &: \mathbb{R} \to Z \end{aligned}\right\} \tag{2.1}$$

where Z is a bounded and closed subset of some Banach space Z_0 equipped with a norm $\|\cdot\|_Z$. We set

$$\left.\begin{aligned} Y &= \mathbb{R}^d \times Z \\ Y_0 &= \mathbb{R}^d \times Z_0 \end{aligned}\right\} \tag{2.2}$$

and furnish Y_0 with the product norm

$$\|y\|_Y = |x| + \|z\|_Z \qquad (y = (x,z) \in \mathbb{R}^d \times Z_0)\,.$$

Let $y : \mathbb{R} \to Y$ be continuous. The number $h \in \mathbb{R}$ is called an *ε-almost period* of y, $\varepsilon > 0$, if

$$\sup_{t\in\mathbb{R}} \|y(t+h) - y(t)\|_Y \le \varepsilon\,.$$

A set $H \subset \mathbb{R}$ is called *relatively dense*, if there exists an $l > 0$, called an *inclusion length* for H, such that every interval of length l includes at least one element of H. The function y is called *almost-periodic*, if for all $\varepsilon > 0$, the set $H(\varepsilon)$ of all ε-almost periods of y is relatively dense. An elementary lemma asserts that

$$l(\varepsilon) = \inf\big\{l : l \text{ is an inclusion length for } H(\varepsilon)\big\}$$

is an inclusion length (and thus, the smallest) for $H(\varepsilon)$. We introduce the *translation* or *shift* operator τ^h, which acts on functions u defined on $\mathbb{R}$, by

$$(\tau^h u)(t) = u(t-h)\,.$$

The celebrated *Bochner criterion* states (see [1: p. 8f] or [2: p. 10]) that a continuous function $y : \mathbb{R} \to Y_0$ is almost-periodic if and only if the set $\{\tau^h y\}_{h \in \mathbb{R}}$ of all translates of y is a relatively compact subset of the space of continuous bounded Y_0-valued functions with respect to the maximum norm; or, equivalently, if and only if every sequence $h_n \in \mathbb{R}$ possesses a subsequence $\nu_i = h_{n_i}$ such that $\tau^{\nu_i} y$ is uniformly convergent. Another fundamental theorem – which we will not use – states that y is almost-periodic if and only if it can be represented as the uniform limit of trigonometric polynomials (see [1: p. 14ff] or [2: p. 29]).

2.2 Solution of the perturbed system. Given an almost-periodic solution $x^0 : \mathbb{R} \to \mathbb{R}^d$ of the unperturbed system

$$x' = f(t,x)\,, \tag{2.3}$$

we seek an almost-periodic solution $y^\varepsilon = (x^\varepsilon, z^\varepsilon)$ of the perturbed system, $L : \mathbb{R}^d \to \mathbb{R}^m$ being a linear operator,

$$x' = f(t,x) + \varepsilon g(x, z(t)) \tag{2.4}$$

$$z(t) = (\Gamma[t_0, z(t_0)]Lx)\,(t) \tag{2.5}$$

in the function spaces

$$x^\varepsilon \in W_{\mathbb{R},d} = \Big\{ x : \mathbb{R} \to \mathbb{R}^d \Big| \, x \text{ absolutely continuous on every compact interval} \Big\}\,,$$

$$z^\varepsilon \in C_{\mathbb{R},Z} = \big\{ z : \mathbb{R} \to Z \big| \, z \text{ continuous} \big\}\,.$$

We also require restrictions

$$\left.\begin{aligned} W_{t,d} &= \{x|_{[t,\infty)} :\, x \in W_{\mathbb{R},d}\} \\ C_{t,Z} &= \{z|_{[t,\infty)} : z \in C_{\mathbb{R},Z}\} \end{aligned}\right\}.$$

We consider a family of operators

$$\Gamma[t_0, z_0] :\, W_{t_0,m} \to C_{t_0,Z} \qquad (t_0 \in \mathbb{R}, z_0 \in Z).$$

Definition 2.1 (Notion of Solution). We say that $y = (x,z) \in W_{\mathbb{R},d} \times C_{\mathbb{R},Z}$ is a *solution* of (2.4) - (2.5), if (2.4) holds for almost all $t \in \mathbb{R}$ and if, for every $t_0 \in \mathbb{R}$, (2.5) holds for all $t \geq t_0$, where the restriction to $W_{t_0,m}$ of $Lx \in W_{\mathbb{R},m}$ is again denoted by Lx.

2.3 Normal hysteresis nonlinearities. Besides the Volterra property (1.4), we require the semigroup property which can be written as

$$\big(\Gamma\big[t_0 + h, (\Gamma[t_0, z_0]u)(t_0 + h)\big]v\big)(t) = (\Gamma[t_0, z_0]u)(t) \qquad (t \geq t_0 + h) \tag{2.6}$$

for every $h \geq 0$ and $u \in W_{t_0,m}$, where $v = u|_{[t_0+h,\infty)}$. Property (1.6) of being autonomous becomes, in terms of the shift operator τ^h which maps $W_{t_0,m}$ to $W_{t_0+h,m}$ and $C_{t_0,Z}$ to $C_{t_0+h,Z}$,

$$\tau^h \circ \Gamma[t_0, z_0] = \Gamma[t_0 + h, z_0] \circ \tau^h \qquad (h \in \mathbb{R}). \tag{2.7}$$

Moreover, we assume that Γ satisfies the following two conditions which we state without further explanation and we refer to [3] for a discussion.

(N1) There exists a constant $\gamma_u > 0$ such that for every $t_0 \in \mathbb{R}$, $z_0 \in Z$, every $t \geq s \geq t_0$ and every $u, v \in W_{t_0,m}$ the inequality

$$\left\|(\Gamma[t_0, z_0]u)\,(s) - (\Gamma[t_0, z_0]v)\,(s)\right\|_Z \leq \gamma_u \|u - v\|_{t_0,t}$$

holds, where

$$\|u\|_{t_0,t} = |u(t_0)| + \int_{t_0}^{t} |u'(s)|\, ds\,.$$

(N2) There exists a threshold $\beta > 0$ and a continuous and bounded function $q : \mathbb{R}_+ \to \mathbb{R}_+$ with $q(\alpha) < 1$ for $\alpha > \beta$ such that

$$\left\|(\Gamma[t_0, z_0]u)\,(t) - (\Gamma[t_0, z_1]u)\,(t)\right\|_Z \leq q(\mathrm{osc}_{t_0,t} u)\,\|z_0 - z_1\|_Z \tag{2.8}$$

holds for all $t \geq t_0$, all $z_0, z_1 \in Z$ and all $u \in W_{t_0,m}$. Here,

$$\mathrm{osc}_{t_0,t} u = \sup_{t_0 \leq \tau, \sigma \leq t} |u(\tau) - u(\sigma)| = \sup_{t_0 \leq \tau \leq t} u(\tau) - \inf_{t_0 \leq \tau \leq t} u(\tau)\,.$$

The following definition summarizes the requirements concerning Γ.

Definition 2.2 (Normal Family). The family $\Gamma[t_0, z_0]$ is called *normal with threshold* $\beta > 0$ if it is autonomous (2.7), satisfies the Volterra property (1.4), the semigroup property (2.6) as well as properties (N1) and (N2), the latter with this value of β.

Informally speaking, property (N1) describes rather weak correctness of Γ with respect to perturbations of the input u, whereas property (N2) describes rather strong correctness with respect to perturbation of the initial internal state z_0. In particular, there should be *exponential* convergence of the internal states z for any input $u = u(t)$ of oscillation greater than h. In [3] we have proved that some important hysteresis nonlinearities such as the von Mises yield criterion generate normal families.

2.4 Stability properties. We now discuss again the unperturbed system (2.3). Let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be uniformly continuous in (t, x) and Lipschitz continuous in x. Then the initial value problem

$$\left.\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0 \end{aligned}\right\}$$

has a unique solution $x = x(t; t_0, x_0)$ for any given $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$, which moreover depends continuously on (t_0, x_0). Let $x^0 : \mathbb{R} \to \mathbb{R}^d$ be any solution of (2.3), let $\sigma > 0$. We say that the solution x^0 is *σ-uniformly stable* (cf. [7: p. 103/Definition 1.1]) if

$$\lim_{\tau\to\infty} \sup_{t-t_0>\tau} \sup_{\substack{t_0\in\mathbb{R} \\ |x_0-x^0(t_0)|<\sigma}} |x(t; t_0, x_0) - x^0(t)| = 0\,.$$

We introduce a similar concept concerning the perturbed system

$$x' = f(t, x) + \varepsilon g(x, z(t)) \tag{2.9}$$

$$z(t) = (\Gamma[t_0, z(t_0)]Lx)\,(t). \tag{2.10}$$

Recall that (2.9) - (2.10) is said to be *well posed*, if the corresponding initial value problems with initial conditions

$$\left.\begin{aligned} x(t_0) &= x_0 \\ z(t_0) &= z_0 \end{aligned}\right\} \tag{2.11}$$

have a unique solution

$$y(t) = (x(t), z(t)) = \big(x^\varepsilon(t; t_0, x_0, z_0), z^\varepsilon(t; t_0, x_0, z_0)\big) \qquad (t \geq t_0)$$

for every $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$ and $z_0 \in Z$, which moreover depends continuously on (t_0, x_0, z_0).

Let $y = (x, z) : \mathbb{R}_+ \to \mathbb{R}^d \times Z$ be a solution of the well posed system (2.9) - (2.10), and let $\sigma > 0$.

Definition 2.3 (Uniform Stability of a Hysteresis System). We say that y is *σ-uniformly stable* if

$$\left.\begin{aligned} \lim_{\tau\to\infty} \sup_{t-t_0>\tau} \sup_{\substack{t_0\in\mathbb{R}, z_0\in Z \\ |x_0-x(t_0)|<\sigma}} |x^\varepsilon(t; t_0, x_0, z_0) - x(t)| = 0 \\ \lim_{\tau\to\infty} \sup_{t-t_0>\tau} \sup_{\substack{t_0\in\mathbb{R}, z_0\in Z \\ |x_0-x(t_0)|<\sigma}} \|z^\varepsilon(t; t_0, x_0, z_0) - z(t)\|_Z = 0. \end{aligned}\right\}$$

Note that this stability is global with respect to the unknown initial state $z_0 \in Z$ of the perturbation. Moreover, we say that $y = (x, z)$ is *globally asymptotically stable*, if it is σ-uniformly stable for each $\sigma > 0$.

2.5 The main theorem. Let $x^0 : \mathbb{R} \to \mathbb{R}^d$ be a solution of the unperturbed system (2.3). We define a notion of stability of x^0 related to the perturbed system

$$\left.\begin{aligned} x' &= f(t, x) + \xi(t) \\ x(t_0) &= x_0 \end{aligned}\right\}$$

whose unique solution we denote by $x(t; t_0, x_0, \xi(\cdot))$, and where $\xi : \mathbb{R} \to \mathbb{R}^d$ represents some general continuous perturbation.

Definition 2.4 (Convergence Property). The function f is called *convergent near* x^0, if there exist positive numbers ε_c, δ_c, γ_c, q_c, T_c and a bounded function $\rho_c : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho_c(T) \le q_c < 1$ for all $T \ge T_c$ such that, for all $t_0 \in \mathbb{R}$, the conditions

$$|x_0 - x^0(t_0)|,\ |y_0 - x^0(t_0)| < \delta_c \qquad \text{and} \qquad |\xi(t)|, |\eta(t)| \le \varepsilon_c \quad (t \ge t_0)$$

imply that for all $t \ge t_0$

$$\big|x(t;t_0,x_0,\xi(\cdot)) - x(t;t_0,y_0,\eta(\cdot))\big| \le \rho_c(t-t_0)|x_0 - y_0| + \gamma_c \max_{t_0 \le s \le t} |\xi(s) - \eta(s)|\,.$$

This property combines essential features of the exponential stability in the sense of Lyapunov and the BIBO (bounded input - bounded output) stability in control theory (see, for instance, [11: p. 583]). It can be extracted from various other stability properties (see Section 4 below).

Note that if f is convergent near x^0, then

$$\big|x(t_0 + nT_c; t_0, x_0) - x^0(t_0 + nT_c)\big| \le q_c^n |x_0 - x^0(t_0)| \le q_c^n \delta_c$$

if $|x_0 - x^0(t_0)| < \delta_c$, therefore x^0 is σ-uniformly stable for $\sigma \le \delta_c$.

We also need to impose some growth condition on the perturbation. If f is globally Lipschitz continuous, by virtue of Gronwall's inequality the estimate

$$|x(t;t_0,x_0)| \le c_0 e^{c_1(t-t_0)}(1 + |x_0|)$$

holds for the solution of the unperturbed problem with some constants c_0, c_1. In order to obtain a corresponding estimate for the perturbed problem uniformly with respect to z_0, we want the growth condition

$$\textbf{(G)} \quad \begin{cases} \big|g\big(x(t),\ (\Gamma[t_0,z_0]Lx)\,(t)\big)\big| \le a_g|x(t)| + b_g \\ \text{for all } x \in W_{t_0,m}\,,\ (t_0,z_0) \in \mathbb{R} \times Z\,,\ t \ge t_0 \end{cases}$$

to be satisfied for some constants $a_g, b_g > 0$. Again we refer to [3] for an explicit statement of some sufficient conditions for this growth condition to hold.

We now formulate the main theorem.

Theorem 1. *Suppose that x^0 is an almost-periodic solution of system* (1.1) *where f is uniformly continuous in (t,x), satisfies a global Lipschitz condition in x and is convergent near x^0. Let g satisfy a global Lipschitz condition in x and z. Let Γ be a normal family with the threshold $\beta > 0$, assume that*

$$\operatorname{osc}_{-\infty,\infty}(Lx^0) > \beta\,, \tag{2.12}$$

and let, finally, the growth condition (G) *be satisfied. Then there exists $\varepsilon_0 > 0$ and $\sigma > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, the perturbed system* (1.2) - (1.3) *has a unique almost-periodic solution $y^\varepsilon = (x^\varepsilon, z^\varepsilon)$ satisfying $|x^\varepsilon(t) - x^0(t)| < \sigma$ for all $t \in \mathbb{R}$; this solution is σ-uniformly stable and enjoys the property*

$$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} |x^0(t) - x^\varepsilon(t)| = 0. \tag{2.13}$$

The proof of Theorem 1 is given in Section 3.

Indeed, a slightly stronger statement concerning uniqueness holds, as it is proved in Lemma 3.7 below: Every almost-periodic solution $\tilde{y}^\varepsilon = (\tilde{x}^\varepsilon, \tilde{z}^\varepsilon)$ of the perturbed system which is not identically equal to y^ε must satisfy

$$\inf_{t \in \mathbb{R}} |x^0(t) - \tilde{x}^\varepsilon(t)| \ge \sigma\,. \tag{2.14}$$

3. Proof of the main theorem

Throughout this section, we suppose that the assumptions of Theorem 1 hold, and we freely use the definitions and notations from the previous section.

3.1 Preliminary results. We begin with some preliminary results. We omit their proofs, since those are completely analogous to the ones given in the development from Lemma 2.1 to Corollary 2.3 in [3].

Lemma 3.1. *For every $\varepsilon \geq 0$, system* (1.2) - (1.3) *together with the initial conditions* (2.11) *has a unique solution*

$$y^\varepsilon = (x^\varepsilon, z^\varepsilon) = \big(x^\varepsilon(t; t_0, x_0, z_0), z^\varepsilon(t; t_0, x_0, z_0)\big) \qquad (t \geq t_0).$$

Note that the Volterra property of Γ is needed in the proof of Lemma 3.1.

Lemma 3.2. *There exists a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that the estimates*

$$\left.\begin{aligned} \left|\frac{d}{dt}(x^\varepsilon(t; t_0, x_0, z_0) - x^\varepsilon(t; t_0, x_1, z_1))\right| &\leq \gamma(t - t_0)\big(|x_0 - x_1| + \varepsilon\|z_0 - z_1\|_Z\big) \\ \left|\frac{d}{dt}(x^\varepsilon(t; t_0, x_0, z_0) - x(t; t_0, x_1))\right| &\leq \gamma(t - t_0)\big(|x_0 - x_1| + \varepsilon(1 + |x_1|)\big) \end{aligned}\right\}$$

hold for all $\varepsilon \geq 0$, $t \geq t_0$, $x_0, x_1 \in \mathbb{R}^d$ and $z_0, z_1 \in Z$.

Corollary 3.2. *There exist continuous functions $\gamma_x, \gamma_z, \gamma_w : \mathbb{R}_+ \to \mathbb{R}_+$ with*

$$\begin{aligned} \big|x^\varepsilon(t; t_0, x_0, z_0) - x^\varepsilon(t; t_0, x_1, z_1)\big| &\leq \gamma_x(t - t_0)\big(|x_0 - x_1| + \varepsilon\|z_0 - z_1\|_Z\big) \\ \big\|z^\varepsilon(t; t_0, x_0, z_0) - z^\varepsilon(t; t_0, x_1, z_1)\big\|_Z &\leq \gamma_z(t - t_0)\big(|x_0 - x_1| + \|z_0 - z_1\|_Z\big) \\ \big\|x^\varepsilon(\cdot; t_0, x_0, z_0) - x^\varepsilon(\cdot; t_0, x_1, z_1)\big\|_{t_0,t} &\leq \gamma_w(t - t_0)\big(|x_0 - x_1| + \varepsilon\|z_0 - z_1\|_Z\big) \\ \big\|x^\varepsilon(\cdot; t_0, x_0, z_0) - x(\cdot; t_0, x_1)\big\|_{t_0,t} &\leq \gamma_w(t - t_0)\big(|x_0 - x_1| + \varepsilon(1 + |x_1|)\big). \end{aligned}$$

Note that there is no ε in the rightmost term of the second inequality in the corollary above.

Corollary 3.2. *System* (1.2) - (1.3) *is well posed.*

3.2 The contraction property of the transition mapping. For a given function $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, we consider the initial value problems

$$\left.\begin{aligned} x' &= \varphi(t, x) + \varepsilon g(x, z(t)) \\ x(t_0) &= x_0 \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} z(t) &= (\Gamma[t_0, z(t_0)]Lx)(t) \\ z(t_0) &= z_0 \end{aligned}\right\}.$$

From its solution $y^\varepsilon = (x^\varepsilon, z^\varepsilon)$ we define the *transition mapping* $S^{\varphi,\varepsilon}_{t_0,t} : Y \to Y$ by

$$S^{\varphi,\varepsilon}_{t_0,t}(x_0, z_0) = y^\varepsilon(t; t_0, x_0, z_0) = \big(x^\varepsilon(t; t_0, x_0, z_0), z^\varepsilon(t; t_0, x_0, z_0)\big).$$

We consider the δ-neighbourhood of the given almost-periodic solution x^0 of the unperturbed system,

$$B(\delta, t) = \big\{\xi \in \mathbb{R}^d : |\xi - x^0(t)| < \delta\big\} \qquad (\delta > 0, t \in \mathbb{R}).$$

We will prove that $S^{f,\varepsilon}_{t_0,t_0+T}$ is a contraction for sufficiently large T.

Lemma 3.3. *There exists a $T_\beta > 0$ such that for all $T \geq T_\beta$ we have*

$$\beta_0 := \inf_{t\in\mathbb{R}} \operatorname{osc}_{t,t+T}(Lx^0) > \beta\,.$$

Proof. Fix $t_1, t_2 \in \mathbb{R}$ with $|Lx^0(t_1) - Lx^0(t_2)| = \beta + 4\eta$ for some $\eta > 0$. Set

$$T_\beta = 2\max\{l(\eta), |t_1 - t_2|\}\,.$$

Then for every $t \in \mathbb{R}$ and $T \geq T_\beta$, we find an η-almost period $\tau \in I_1 \cap I_2$ where $I_j = [t - t_j, t + T - t_j]$ $(j = 1, 2)$. Thus, $t_j + \tau \in [t, t+T]$ and $\beta_0 \geq |Lx^0(t_1 + \tau) - Lx^0(t_2 + \tau)| \geq \beta + 2\eta$ ∎

From the properties of the function q appearing in (2.8) it now follows that

$$q_\beta := \sup_{\substack{t_0\in\mathbb{R}\\ T\geq T_\beta}} q\big(\operatorname{osc}_{t_0,t_0+T}(Lx^0)\big) < 1\,. \tag{3.1}$$

Proposition 3.1. *There exists $T_* > 0$ such that for every $T \geq T_*$ and every $q_* > 0$ with*

$$\max\{q_c, q_\beta\} < q_* < 1 \tag{3.2}$$

there exist $\delta_* > 0$, $\varepsilon_* > 0$ and a metric d_* on $\mathbb{R}^d \times Z$ such that for every $0 < \varepsilon < \varepsilon_*$ and every $t_0 \in \mathbb{R}$ the transition mapping acts as

$$S^{f,\varepsilon}_{t_0,t_0+T} : B(\delta_*, t_0) \times Z \to B(\delta_*, t_0 + T) \times Z$$

and is a q_*-contraction with respect to d_*. Moreover,

$$\big|x^\varepsilon(t_0 + T; t_0, x_0, z_0) - x^0(t_0 + T)\big| \leq q_*|x_0 - x^0(t_0)| + C_*\varepsilon \tag{3.3}$$

holds with some constant C_* independent from $\varepsilon, t_0, x_0, z_0$.

Proof. Let δ_c be the constant appearing in Definition 2.4. We consider the perturbed solution $y^\varepsilon(t) = (x^\varepsilon(t), z^\varepsilon(t)) = S^{f,\varepsilon}_{t_0,t}(x_0, z_0)$ and define

$$M_\varepsilon(T) = \sup\Big\{|x^\varepsilon(t)| : t_0 \in \mathbb{R},\, z_0 \in Z,\, t \in [t_0, t_0 + T],\, x_0 \in B(\delta_c, t_0)\Big\}$$

$$G_\varepsilon(T) = a_g M_\varepsilon(T) + b_g\,.$$

The assumptions on f and g imply that M_ε and G_ε are continuous. The last estimate in Corollary 3.1 gives (note that x^0, being almost-periodic, is bounded)

$$M_\varepsilon(T) \leq \|x^0\|_\infty + \gamma_w(T)\big(\delta_c + \varepsilon(1 + \|x^0\|_\infty)\big)\,.$$

We now apply Definition 2.4 with $\xi(t) = \varepsilon g(y^\varepsilon(t))$ and $\eta \equiv 0$. We obtain that the inequality

$$\begin{aligned}&\big|x^\varepsilon(t_0 + T; t_0, x_0, z_0) - x^0(t_0 + T)\big|\\ &\qquad\leq \rho_c(T)|x_0 - x^0(t_0)| + \gamma_c\varepsilon \sup_{t\in[t_0,t_0+T]} |g(y^\varepsilon(t))|\end{aligned} \tag{3.4}$$

holds for $(x_0, z_0) \in B(\delta, t_0) \times Z$ if $\delta < \delta_c$ and $\varepsilon G_\varepsilon(T) < \varepsilon_c$. If, moreover, $\gamma_c \varepsilon a_g \le \frac{1}{2}$, then, because we can estimate the rightmost term in (3.4) by $G_\varepsilon(T)$, we can rearrange (3.4) to obtain the a priori estimate

$$M_\varepsilon(T) \le 2\left(\|x^0\|_\infty + \|\rho_c\|_\infty \delta_c + \frac{b_g}{2a_g}\right) =: M_\infty\,, \tag{3.5}$$

provided (3.4) is valid. Now fix any $T_1 > 0$, choose ε_1 such that

$$\sup_{\substack{T\in[0,T_1]\\ \varepsilon\le\varepsilon_1}} \varepsilon G_\varepsilon(T) \le \varepsilon_1 a_g\Big(\|x^0\|_\infty + \gamma_w(T)\big(\delta_c + \varepsilon_1(1+\|x^0\|_\infty)\big)\Big) + b_g \le \frac{\varepsilon_c}{2}\,. \tag{3.6}$$

Set now

$$\varepsilon_0 = \min\left\{\varepsilon_1, \frac{1}{2\gamma_c a_g}, \frac{\varepsilon_c}{2(a_g M_\infty + b_g)}\right\}. \tag{3.7}$$

Then for every $\varepsilon < \varepsilon_0$, we have $\sup_{T\ge T_1} \varepsilon G_\varepsilon(T) \le \frac{\varepsilon_c}{2}$; indeed it cannot occur that $\frac{\varepsilon_c}{2} < \varepsilon G_\varepsilon(T) \le \varepsilon_c$ due to (3.5) and (3.7), and $\varepsilon G_\varepsilon(T_1) \le \frac{\varepsilon_c}{2}$ by (3.6). Thus, estimate (3.4) holds for all $\delta < \delta_c$, $\varepsilon < \varepsilon_0$, $x_0 \in B(\delta, t_0)$, $z_0 \in Z$ and $t_0 \in \mathbb{R}$. If we now choose T_0 large enough and make ε_0 smaller if necessary, we see that

$$S^{f,\varepsilon}_{t_0,t_0+T}(B(\delta,t_0)) \subset B(\delta, t_0+T)$$

for all $\delta < \delta_c$, $\varepsilon < \varepsilon_0$, $t_0 \in \mathbb{R}$ and $T \ge T_0$. To derive the contraction property, define

$$T_* = \max\{T_0, T_\beta, T_c\}\,.$$

Let $t_0 \in \mathbb{R}$, $\varepsilon < \varepsilon_1$, $\delta < \delta_1$, $T \ge T_*$ and $(x_0, z_0), (x_1, z_1) \in B(\delta, t_0) \times Z$ be given. (Estimate (3.3) is then a consequence of (3.4).) We introduce the abbreviations

$$\left.\begin{aligned} x_0(t) &= x^\varepsilon(t; t_0, x_0, z_0) \\ x_1(t) &= x^\varepsilon(t; t_0, x_1, z_1)\end{aligned}\right\} \quad\text{and}\quad \left.\begin{aligned} z_0(t) &= z^\varepsilon(t; t_0, x_0, z_0) \\ z_1(t) &= z^\varepsilon(t; t_0, x_1, z_1)\end{aligned}\right\}.$$

Then

$$\left.\begin{aligned} x_0'(t) &= f(t, x_0(t)) + \xi(t) \\ x_1'(t) &= f(t, x_1(t)) + \eta(t)\end{aligned}\right\} \quad\text{with}\quad \left.\begin{aligned} \xi(t) &= \varepsilon\, g(x_0(t), z_0(t)) \\ \eta(t) &= \varepsilon\, g(x_1(t), z_1(t))\end{aligned}\right\}$$

holds for all $t \ge t_0$. Corollary 3.1 implies

$$|\xi(t) - \eta(t)| \le \varepsilon\gamma_*(T)\big(|x_0 - x_1| + \|z_0 - z_1\|_Z\big) \qquad (t \in [t_0, t_0+T])$$

for some function $\gamma_* : \mathbb{R}_+ \to \mathbb{R}_+$. Since f is convergent near x^0,

$$\begin{aligned} &\big|x_0(t_0+T) - x_1(t_0+T)\big| \\ &\qquad < \rho_c(T)|x_0 - x_1| + \gamma_c\gamma_*(T)\varepsilon\big(|x_0 - x_1| + \|z_0 - z_1\|_Z\big)\end{aligned} \tag{3.8}$$

holds if $\varepsilon \le \varepsilon_3$, where ε_3 is chosen such that

$$\varepsilon_3\gamma_*(T)\big(|x_0 - x_1| + \|z_0 - z_1\|_Z\big) \le \varepsilon_c\,.$$

To derive a corresponding estimate for $\|z_0(t_0+T) - z_1(t_0+T)\|_Z$, we use Property (N2) as follows. We first claim that

$$q\big(\operatorname{osc}_{t_0,t_0+T}(Lx^\varepsilon(\,\cdot\,;t_0,x_0,z_0))\big) \le q_\beta + \gamma(\varepsilon,\delta)\,,$$

where γ is a certain function with $\lim_{\varepsilon,\delta\to 0}\gamma(\varepsilon,\delta) = 0$. Indeed, this follows from the last estimate in Corollary 3.1 with $x_1 = x^0(t_0)$ (thus $x(\cdot;t_0,x_1) = x^0$), and from the continuity of q. Next, the use of the triangle inequality as well as of properties (N1) and (N2) yields

$$\begin{aligned}&\big\|z_0(t_0+T) - z_1(t_0+T)\big\|_Z\\ &\quad\le \gamma_u|L|\,\|x_0(\cdot) - x_1(\cdot)\|_{t_0,t_0+T} + (q_\beta + \gamma(\varepsilon,\delta))\|z_0 - z_1\|_Z\end{aligned}\,. \tag{3.9}$$

We now conclude from (3.8) and (3.9), using again the estimates of Corollary 3.1, that there exist constants $\gamma_1(T), \gamma_2(T), \gamma_3(T)$ depending only upon T with

$$\big|x_0(t_0+T) - x_1(t_0+T)\big| \le (q_c + \varepsilon\gamma_1(T))|x_0 - x_1| + \varepsilon\gamma_1(T)\|z_0 - z_1\|_Z \qquad 3.10)$$
$$\big\|z_0(t_0+T) - z_1(t_0+T)\big\|_Z \le \gamma_2(T)|x_0 - x_1| + (q_\beta + \gamma(\varepsilon,\delta) + \varepsilon\gamma_3(T))\|z_0 - z_1\|_Z.$$

An explicit inspection of the characteristic equation of the matrix

$$A_{\varepsilon,\delta} = \begin{pmatrix} q_c + \varepsilon\gamma_1(T) & \varepsilon\gamma_1(T) \\ \gamma_2(T) & q_\beta + \gamma(\varepsilon,\delta) + \varepsilon\gamma_3(T)\end{pmatrix} \tag{3.11}$$

shows that its spectral radius $r(A_{\varepsilon,\delta})$ satisfies

$$r(A_{\varepsilon,\delta}) = \max\{q_c, q_\beta\} + \alpha(\varepsilon,\delta), \qquad \lim_{\varepsilon,\delta\to 0}\alpha(\varepsilon,\delta) = 0\,.$$

Now we choose $\delta_* \le \delta_c$ and $\varepsilon_* \le \varepsilon_0$ small enough and a norm $\|\cdot\|_*$ on $\mathbb{R}^2$ such that $\|A_{\varepsilon,\delta}\|_* \le q_*$ holds for the associated operator norm if $\delta \le \delta_*$ and $\varepsilon \le \varepsilon_*$. Note that this choice may depend upon the specific value of T. We define the metric ρ_* on $\mathbb{R}^d \times Z$ by

$$\rho_*\big((x_0,z_0),(x_1,z_1)\big) = \big\|\big(|x_0 - x_1|, \|z_0 - z_1\|_Z\big)\big\|_*\,.$$

Then estimates (3.10) and (3.11) show that

$$\rho_*\Big(\big(x_0(t_0+T), z_0(t_0+T)\big), \big(x_1(t_0+T), z_1(t_0+T)\big)\Big) \le q_*\rho_*\big((x_0,z_0),(x_1,z_1)\big).$$

As the choice of the initial values (x_0,z_0) and (x_1,z_1) was arbitrary within $B(\delta_*,t_0)\times Z$, Proposition 3.1 is proved ∎

3.3 Completion of the proof. Before we continue with the proof of our main theorem, let us remark that the transition mapping has the semigroup property

$$S^{f,\varepsilon}_{t_0,t_2} = S^{f,\varepsilon}_{t_1,t_2} \circ S^{f,\varepsilon}_{t_0,t_1} \qquad (t_0 \le t_1 \le t_2)\,. \tag{3.12}$$

Since the family $\Gamma[t_0,z_0]$ is autonomous, we have, setting $f^h(t,x) = f(t-h,x)$,

$$S^{f^h,\varepsilon}_{t_0+h,t+h} = S^{f,\varepsilon}_{t_0,t} \qquad (t_0, h \in \mathbb{R},\, t \ge t_0)\,. \tag{3.13}$$

From now on we assume that $T \ge T_*$ has a fixed value, and T_*, δ_* and ε_* are chosen according to Proposition 3.1.

Lemma 3.4. *Let $z_* \in Z$ be given. Define the functions $y_n^\varepsilon = (x_n^\varepsilon, z_n^\varepsilon) : \mathbb{R} \to \mathbb{R}^d \times Z$ by*

$$y_n^\varepsilon(t) = S_{t-nT,t}^{f,\varepsilon}\big(x^0(t-nT), z_*\big).$$

Then these functions are uniformly bounded, that is, $\{y_n^\varepsilon(t) : t \in \mathbb{R},\, n \in \mathbb{N},\, \varepsilon < \varepsilon_*\}$ is bounded, and

$$y_*^\varepsilon(t) = \lim_{n\to\infty} y_n^\varepsilon(t) \tag{3.14}$$

exists uniformly in $t \in \mathbb{R}$.

Proof. Because of Proposition 3.1 and the semigroup property (3.12), we have $x_n^\varepsilon(t) \in B(\delta_*, t)$, thus y_n^ε is uniformly bounded by some constant C since x^0 and Z are bounded. We claim that $(y_n^\varepsilon(t))_{n\in\mathbb{N}}$ is a Cauchy sequence. Indeed, for any $m \geq n$ there holds

$$y_m^\varepsilon(t) = S_{t-nT,t}^{f,\varepsilon}\big(S_{t-mT,t-nT}^{f,\varepsilon}(x^0(t-mT), z_*)\big) \in B(\delta_*, t) \times Z\,,$$

so by Proposition 3.1 and the semigroup property (3.12)

$$d_*(y_n^\varepsilon(t), y_m^\varepsilon(t)) \leq q_*^n d_*\Big((x^0(t-nT), z_*), S_{t-mT,t-nT}^{f,\varepsilon}(x^0(t-mT), z_*)\Big) \leq C q_*^n$$

for some constant C, and the assertion follows ∎

Lemma 3.5. *For every $\varepsilon < \varepsilon_*$, the function y_*^ε is a solution of* (1.2) - (1.3) *in the sense of Definition* 2.1.

Proof. Let $t_0 \in \mathbb{R}$ and $t \geq t_0$ be arbitrary. It suffices to prove that

$$y_*^\varepsilon(t) = S_{t_0,t}^{f,\varepsilon}(y_*^\varepsilon(t_0)) \tag{3.15}$$

holds. Let $n \in \mathbb{N}$ and choose $m \geq n$ such that $t - mT \leq t_0 - nT$. Set

$$\left.\begin{array}{ll} w(s) = S_{t-mT,s}^{f,\varepsilon}(x^0(t-mT), z_*) & \quad (s \geq t-mT) \\ w_0(s) = S_{t_0-mT,s}^{f,\varepsilon}(x^0(t_0-mT), z_*) & \quad (s \geq t_0-mT) \end{array}\right\}.$$

Then

$$d_*\big(w(t_0), w_0(t_0)\big) \leq q_*^n d_*\big(w(t_0-nT), w_0(t_0-nT)\big) \leq C q_*^n\,. \tag{3.16}$$

Since

$$\left.\begin{array}{r} y_m^\varepsilon(t) = w(t) = S_{t_0,t}^{f,\varepsilon}(w(t_0)) \\ y_m^\varepsilon(t_0) = w_0(t_0) \end{array}\right\},$$

the uniform continuity of $S_{t_0,t}^{f,\varepsilon}$ and (3.16) yield

$$\lim_{m\to\infty} d_*\big(y_m^\varepsilon(t), S_{t_0,t}^{f,\varepsilon}(y_m^\varepsilon(t_0))\big) = 0\,,$$

from which (3.15) readily follows ∎

Lemma 3.6. *For every $\varepsilon < \varepsilon_*$, the function y_*^ε is almost-periodic.*

Proof. Let (h_n) be a sequence of real numbers. By Bochner's criterion it suffices to exhibit a subsequence (h_{n_k}) such that the sequence $(\tau^{h_{n_k}} y_*^\varepsilon)$ of shifts of y_*^ε is uniformly convergent as $k \to \infty$. We introduce the notation

$$f_k(t,x) = f(t - h_{n_k}, x)\,, \qquad \tau_k = \tau^{h_{n_k}}\,, \qquad x_k^0 = \tau_k x^0\,.$$

Since x^0 is almost-periodic and every such sequence (f_k) is equicontinuous and bounded on every subset $\mathbb{R} \times B$, B bounded, of $\mathbb{R} \times \mathbb{R}^d$, we can choose a subsequence (h_{n_k}) such that $x_k^0 \to \hat{x}$ uniformly for some continuous bounded $\hat{x} : \mathbb{R} \to \mathbb{R}^d$ and $f_k(t,x) \to \hat{f}(t,x)$ uniformly (on $\mathbb{R} \times B$ for every bounded B) for some function $\hat{f}$ which is again uniformly continuous in (t,x) and Lipschitz in x. We define functions $\hat{y}_n(t) : \mathbb{R} \to \mathbb{R}^d \times Z$ by

$$\hat{y}_n(t) = S^{\hat{f},\varepsilon}_{t-nT,t}\big(\hat{x}(t-nT), z_*\big).$$

By virtue of (3.13) and Proposition 3.1, the mappings

$$S^{f_k,\varepsilon}_{t_0,t_0+T} : B(\delta_*, t_0) \times Z \to B(\delta_*, t_0 + T) \times Z$$

are q_*-contractions for every $t_0 \in \mathbb{R}$; as they converge uniformly to $S^{\hat{f},\varepsilon}_{t_0,t_0+T}$, the same is true for the latter. Therefore, the proof of Lemma 3.4 also applies to prove that the limit

$$\hat{y}(t) = \lim_{n\to\infty} \hat{y}_n(t)$$

exists uniformly in t. We will show that $y_k = \tau_k y_*^\varepsilon$ converges uniformly to $\hat{y}$; this will complete the proof. Due to (3.13),

$$(\tau_k y_n^\varepsilon)(t) = S^{f_k,\varepsilon}_{t-nT,t}\big(x_k^0(t-nT), z_*\big).$$

We thus have

$$\begin{aligned}\hat{y}(t) - y_k(t) = {}& \big(\hat{y}(t) - \hat{y}_n(t)\big) + \big(\tau_k \circ (y_n^\varepsilon - y_*^\varepsilon)\big)(t) \\ &+ \Big(S^{\hat{f},\varepsilon}_{t-nT,t}\big(\hat{x}(t-nT), z_*\big) - S^{f_k,\varepsilon}_{t-nT,t}\big(x_k^0(t-nT), z_*\big)\Big).\end{aligned}$$

If we choose k and then n large enough, all three terms on the right-hand side of the last equation become smaller than any given $\eta > 0$, uniformly in t. Thus, y_k converges uniformly to $\hat{y}$ ∎

Recall that we have fixed T_*, q_*, δ_* and ε_* according to Proposition 3.1. Let us choose $\varepsilon_0 > 0$ such that

$$2\frac{C_*\varepsilon_0}{1-q_*} < \delta_* \qquad (\varepsilon_0 < \varepsilon_*).$$

Define also

$$\sigma = \frac{C_*\varepsilon_0}{1-q_*}.$$

Lemma 3.7. *For $\varepsilon < \varepsilon_0$ the solution $y_*^\varepsilon = (x_*^\varepsilon, z_*^\varepsilon)$ satisfies the estimate*

$$|x_*^\varepsilon(t) - x^0(t)| < \sigma \qquad (t \in \mathbb{R}), \tag{3.17}$$

it is σ-uniformly stable and enjoys property (2.13). *There is no other almost-periodic solution $y = (x, z)$ of* (1.2) - (1.3) *which satisfies the inequality $|x^\varepsilon(\tau) - x^0(\tau)| < \sigma$ for some $\tau > 0$.*

Proof. Let us fix $t \in \mathbb{R}$ and $n \in \mathbb{N}$, and define

$$d_j = x^\varepsilon\big(t - d_jT; t - nT, x^0(t - nT), z_*\big) - x^0(t - d_jT) \qquad (0 \le j \le n).$$

Then $d_n = 0$, and from (3.3) we conclude that $d_j \le q_* d_{j+1} + C_*\varepsilon$, thus

$$|x_n^\varepsilon(t) - x^0(t)| \le \frac{C_*\varepsilon}{1 - q_*}.$$

Passing to the limit $n \to \infty$, we see that

$$|x_*^\varepsilon(t) - x^0(t)| \le \frac{C_*\varepsilon}{1 - q_*}$$

holds by (3.14). Thus (3.17), and therefore also (2.13), follow. Since $\sigma < \delta_*$, the σ-uniform stability of y_*^ε follows from the q_*-contraction property of the transition mapping $S_{t_0,t_0+T}^{f,\varepsilon}$ over the set $B(\delta_*, t_0)$ (see Proposition 3.1).

It remains to prove the statement concerning uniqueness. Let $y = (x, z)$ be an almost-periodic solution of (1.2) - (1.3) satisfying $|x(\tau) - x^0(\tau)| < \sigma$ for some $\tau \in \mathbb{R}$. Then, in particular, both $x(\tau)$ and $x_*^\varepsilon(\tau)$ belong to $B(\delta_*, \tau)$ which again implies $|x(t) - x_*^\varepsilon(t)| \to 0$ as $t \to \infty$. That is, the difference $r(t) = x(t) - x_*^\varepsilon(t)$ tends to zero as $t \to \infty$; being the difference of two a lmost-periodic functions, it is again almost-periodic and thus identically zero. The lemma is completely proved ∎

The theorem is proved by Lemmas 3.5 - 3.7.

4. Applications

4.1 Global stability. We first discuss an application concerning the existence of a globally stable (that is, σ-uniformly stable for each $\sigma > 0$) almost-periodic solution. We consider the special case

$$f(t, x) = Ax + bF(t, c^T x)$$

of (1.1), where $A \in \mathbb{R}^{d,d}$, $b, c \in \mathbb{R}^d$, and the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is almost-periodic in t. (Here and in the following, ^{T} denotes the transpose of a vector respectively a matrix.) This equation arises for example in control theory when we use a nonlinear feedback $u = F(t, y)$ for the SISO (single input - single output) control system $x' = Ax + bu$, $y = c^T x$ (see, e.g., [10]).

The perturbed system (1.2) - (1.3) now reads as

$$x' = Ax + bF(t, c^T x) + \varepsilon g(x, z(t)) \tag{4.1}$$

$$z(t) = (\Gamma[t_0, z(t_0)]Lx)(t). \tag{4.2}$$

Suppose that A is a stable matrix which satisfies

$$\lambda_F \|G\| < 1\,, \tag{4.3}$$

where λ_F is a Lipschitz constant for F in y, and

$$\|G\| = \max_{-\infty<\omega<\infty} |G(i\omega)| = \max_{-\infty<\omega<\infty} |c^T(i\omega I - A)^{-1}b| \tag{4.4}$$

denotes the operator norm of the transfer function G of the linear system (A, b, c^T) in the frequency domain. (In equation (4.4) above, $i = \sqrt{-1}$, and I denotes the identity matrix in $\mathbb{R}^{d,d}$.) Under these conditions the unperturbed equation (1.1) has a unique almost-periodic solution $x^0 : \mathbb{R} \to \mathbb{R}^d$ which is globally asymptotically stable [12].

The following proposition shows that the same is true for the perturbed system (4.1) - (4.2).

Proposition 4.1. *Suppose that F is uniformly continuous and satisfies a global Lipschitz condition with respect to y such that* (4.3) *holds, suppose that g satisfies global Lipschitz conditions in x and z as well as the growth condition* (G). *Let Γ be a normal family with threshold $\beta > 0$, and let inequality* (2.12) *be valid for the unique almost-periodic globally asymptotically stable solution $x^0 : \mathbb{R} \to \mathbb{R}^d$ of (1.1). Then there exists $\varepsilon_0 > 0$ such that system* (4.1) - (4.2) *has, for every $0 < \varepsilon < \varepsilon_0$, a unique almost-periodic solution $y^\varepsilon = (x^\varepsilon, z^\varepsilon)$. This solution is globally stable and enjoys the property* (2.13).

The proof of the proposition will be given after Lemma 4.3.

Lemma 4.1. *Let $\lambda > 0$ such that $\lambda\|G\| < 1$. Then there exist positive numbers p, μ and γ such that for all functions $r, \zeta : [t_0, \infty) \to \mathbb{R}^d$ and $\alpha : [t_0, \infty) \to \mathbb{R}$ which satisfy $|\alpha(t)| \le \lambda |c^T r(t)|$ for all $t \ge t_0$, and*

$$r'(t) = Ar(t) + b\alpha(t) + \zeta(t)\,,$$

the estimate

$$|r(t)| \le \mu e^{-p(t-t_0)}|r(t_0)| + \gamma \max_{t_0 \le s \le t} |\zeta(s)|$$

holds for all $t \ge t_0$.

Proof. By Yakubovich's proof of the Bonjorno-Kalman-Yakubovich circle criterion (see, for instance, [10: p. 124/Lemma 6]) there exists a positive definite matrix P such that the differential inequality

$$\frac{d}{dt}\|r(t)\|_P \le -\lambda_0 \|r(t)\|_P + \gamma_0 |\zeta(t)|$$

holds for a.e. $t \ge t_0$; here $\|r\|_P = \sqrt{r^T P r}$ and λ_0, γ_0 are some positive constants. The assertion of the lemma now follows from Gronwall's inequality ∎

Corollary 4.1. *The function $f(t,x) = Ax + bF(t, c^T x)$ is convergent near x^0.*

Proof. By Definition 2.4 it suffices to show that there exists a bounded function $\rho_c : \mathbb{R}_+ \to \mathbb{R}_+$ with $\rho_c(T) \le q_c < 1$ for all $T \ge T_c$ such that, for all $t > t_0$, $t_0 \in \mathbb{R}$, the inequality

$$\big|x(t; t_0, x_0, \xi(\cdot)) - x(t; t_0, y_0, \eta(\cdot))\big| \le \rho_c(t - t_0)|x_0 - y_0| + \gamma_c \max_{t_0 \le s \le t} |\xi(s) - \eta(s)|$$

holds. Indeed, let some bounded functions $\xi = \xi(t)$ and $\eta = \eta(t)$ be given. We apply Lemma 4.1 with $\lambda = \lambda_F$ and

$$\left.\begin{aligned} r(t) &= x\big(t; t_0, x_0, \xi(\cdot)\big) - x\big(t; t_0, y_0, \eta(\cdot)\big) \\ \alpha(t) &= F\big(t, c^T x(t; t_0, x_0, \xi(\cdot))\big) - F\big(t, c^T x(t; t_0, y_0, \eta(\cdot))\big) \\ \zeta(t) &= \xi(s) - \eta(s) \end{aligned}\right\}$$

and conclude that the estimate

$$\big|x(t; t_0, x_0, \xi(\cdot)) - x(t; t_0, y_0, \eta(\cdot))\big| \le \mu e^{-p(t-t_0)}|x_0 - y_0| + \gamma \sup_{t_0 \le s \le t} |\xi(s) - \eta(s)|$$

holds for all $t_0 \in \mathbb{R}$, $t > t_0$ and $x_0, y_0 \in \mathbb{R}^d$. Thus we can define $\rho_c(s) = e^{-ps}$ and the corollary is proved ∎

Corollary 4.1 enables us to apply our main theorem which however does not yield immediately the global stability. To obtain the latter, we additionally need the following modification of Lemma 4.1.

Lemma 4.2. *Let $\lambda > 0$ such that $\lambda\|G\| < 1$, and let $\beta_0 > 0$. Then there exist positive numbers p, μ, γ and ε such that for all functions $r, \zeta : [t_0, \infty) \to \mathbb{R}^d$ and $\alpha : [t_0, \infty) \to \mathbb{R}$ which satisfy*

$$\left.\begin{aligned} |\alpha(t)| &\le \lambda |c^T r(t)| \\ r'(t) &= Ar(t) + b\alpha(t) + \zeta(t) \\ |\zeta(t)| &\le \varepsilon |r(t)| + \beta_0 \end{aligned}\right\}$$

for all $t \ge t_0$ the estimate

$$|r(t)| \le \mu e^{-p(t-t_0)}|r(t_0)| + \gamma\beta_0$$

holds for all $t \ge t_0$.

Proof. This assertion also follows immediately from [10: p. 124/Lemma 6] ∎

Lemma 4.3. *For each $\delta > 0$ and $R > 0$ there exists $\tau > 0$ and $\varepsilon_1 > 0$ such that the inequality*

$$\big|x^\varepsilon(t; t_0, x_0, z_0) - x^0(t)\big| < \delta$$

holds for all $t_0 \in \mathbb{R}$, $|x_0| < R$, $z_0 \in Z$, $\varepsilon < \varepsilon_1$ and $t > t_0 + \tau$.

Proof. Let us define

$$r(t) = x^{\varepsilon}(t; t_0, x_0, z_0) - x^0(t) .$$

Then r solves

$$r'(t) = Ar(\tau) + b\alpha(t) + \zeta(t) ,$$

where

$$\left.\begin{aligned} \alpha(t) &= F\big(t, c^T x^{\varepsilon}(t; t_0, x_0, z_0)\big) - F(t, c^T x^0(t)) \\ \zeta(t) &= \varepsilon g\big(x^{\varepsilon}(t; t_0, x_0, z_0), z^{\varepsilon}(t; t_0, x_0, z_0)\big) \end{aligned}\right\} .$$

We have

$$|\alpha(t)| \le \lambda_F |c^T r(t)| ,$$

and, due to property (G),

$$|\zeta(t)| \le \varepsilon\big(a_g |x^{\varepsilon}(t)| + b_g\big) \le \varepsilon\big(a_g |r(t)| + a_g \|x^0\|_\infty + b_g\big) .$$

The assertion now follows from Lemma 4.2 ∎

Proof of Proposition 4.1. By Corollary 4.1 and Theorem 1 we see that system (4.1) - (4.2) has, if ε is sufficiently small, a unique almost-periodic solution $y^{\varepsilon} = (x^{\varepsilon}, z^{\varepsilon})$ satisfying $|x^{\varepsilon}(t) - x^0(t)| < \sigma$ for all $t \in \mathbb{R}$ which is moreover σ-uniformly stable for small enough σ and enjoys property (2.11). Using Lemma 4.2 we now conclude that y^{ε} actually is σ-uniformly stable for arbitrarily large σ. The global uniqueness follows from the fact that every other almost-periodic solution must satisfy (2.14) ∎

Other kinds of frequency criteria [10] can be used in a similar way.

4.2 Smooth systems. We now consider a second application of our main theorem. Let us return to the general system (1.2) - (1.3). Suppose that the function f is smooth and that the unperturbed equation (1.1) has an almost-periodic solution $x^0 = x^0(t)$. Then we can consider the linearization of the system (1.1) along the trajectory x^0:

$$w' = A(t)w , \qquad A(t) = \partial_x f(t, x(t)) . \tag{4.5}$$

Proposition 4.2. *Suppose that $x^0 : \mathbb{R} \to \mathbb{R}^d$ is an almost-periodic solution of equation* (1.1)*. Suppose that the zero solution is the only solution of the linear equation* (4.5) *which is bounded on $(-\infty, 0)$, and that for each almost-periodic function $\xi : \mathbb{R} \to \mathbb{R}^d$ the equation $w' = A(t)w + \xi(t)$ has at least one bounded solution. Let g satisfy a local Lipschitz condition in x and z. Let the growth condition* (G) *be satisfied. Let Γ be a normal family with threshold $\beta > 0$ and assume that inequality* (2.12) *holds. Then there exist $\varepsilon_0, \sigma > 0$ such that system* (1.2) - (1.3) *has, for $0 < \varepsilon < \varepsilon_0$, a unique almost-periodic solution $y^{\varepsilon} = (x^{\varepsilon}, z^{\varepsilon})$ satisfying $|x^{\varepsilon}(t) - x^0(t)| < \sigma$ for all $t \in \mathbb{R}$; this solution is σ-uniformly stable and enjoys property* (2.13).

Proof. By Theorem 1 it suffices to show that the mapping f is convergent near x^0. Note, first, that equation (4.5) is asymptotically stable because the trivial solution is the only solution of (4.5) bounded on $(-\infty, 0)$. Further, this stability is exponential, because for each almost-periodic ξ equation (4.5) has at least one bounded solution (see [8: Theorems 2.3 and 3.2]). Considering [8: Theorem 11.1] we obtain the estimate

$$|r(t)| \leq \mu_c e^{-\lambda_c(t-t_0)}|r(t_0)| + \gamma_c \max_{t_0 \leq s \leq t} |\xi(s) - \eta(s)|$$

with

$$r(t) = x\big(t; t_0, x_0, \xi(\cdot)\big) - x\big(t; t_0, y_0, \eta(\cdot)\big)$$

for appropriate λ_c, μ_c and γ_c, if

$$\left.\begin{aligned} |x_0 - x^0(t_0)|, |y_0 - x^0(t_0)| &< \delta_c \\ |\xi(t)|, |\eta(t)| &\leq \varepsilon_c \end{aligned}\right\}$$

and δ_c, ε_c are sufficiently small. Therefore, f is convergent near x^0 ∎

We refer to [8] and the bibliography therein for the discussion of powerful methods to prove the solvability of the equation $w' = A(t)w + \xi(t)$ and the uniqueness on the half-line $(-\infty, 0)$ of the trivial solution of (4.5).

4.3 Further extensions. We have proved in the paper that under some technical conditions, hysteresis perturbations of the ordinary differential equation $x' = f(t, x)$ have asymptotically stable almost-periodic solutions, provided that such solutions exist for the unperturbed equation.

Instead of the class of all almost-periodic functions, it is possible to consider certain important and interesting subclasses such as

- Limit periodic functions, that is, almost-periodic functions for which the ratio of any pair of Fourier exponents is rational with a given set of exponents (see [2: p. 32]).
- Diagonal functions of periodic functions of several variables with a given set of periods (see [2: p. 36]).

There are natural extensions to Theorem 1 which guarantee that the almost-periodic solutions belong to a particular class, provided that the right-hand side f belongs to the same class with respect to t.

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