

## Distributional equation in the limit of phase transition for fluids

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We study the convergence of a diffusive interface model to a sharp interface model. The model consists of the conservation of mass and momentum, where the mass undergoes a phase transition. The equations were considered in [W3] and in the diffuse case consist of the compressible Navier–Stokes system coupled with an Allen–Cahn equation. In the sharp interface limit a jump in the mass density as well as in the velocity occurs. The convergence of mass and momentum is considered in the distributional sense. The convergence of the free energy to a limit is shown in a separate paper. The procedure in this paper works also in other general situations.

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### 1. Introduction

We show by an example how phase field models converge to problems with sharp interface. Although we are interested in flow problems connected with phase field equations, we think that the approach introduced here is quite general, and applies also to other situations. Flow problems as considered here are treated in [AR], [AF], [DG], [FG], [LW], [LT] and existence theorems were achieved in [Ab], [AR], [LS], [R]. This shows that the underlying system of differential equations is of main interest.

The phase field model (see Section 3) is governed by the compressible Navier–Stokes system and the Allen–Cahn equation with a parameter  $\delta > 0$ . It models the interactive flow of a mixture of two different materials. Therefore we have two mass conservations for the two masses, and one momentum conservation for the sum of the two masses. The sum of the two mass conservation laws results in conservation of the total mass, that is, the sum of the two masses. Together with the momentum equation this is the compressible Navier–Stokes system. As remaining law we do not pick the difference of the two mass conservations, but instead we use the second mass law. This turns out to be the Allen–Cahn equation, which is the second equation of system (3.1) below. The first and third equations of (3.1) are the above mentioned compressible Navier–Stokes system.

The purpose of this paper is to take the Navier–Stokes/Allen–Cahn system and to study its limit behaviour as  $\delta \rightarrow 0$ . We show that in the distributional sense the equations converge. For explanation of these concepts we refer to the appendix.

The sharp interface problem concerns two fluids occupying  $\Omega_t^1$  and  $\Omega_t^2$  separated locally by a free boundary  $\Gamma_t$  (see Section 2). The two components interact on  $\Gamma$  by mass transfer with a reaction rate  $\tau_\delta$ . In fact, such an interface might be a thin layer, but looking at the phase transition from a certain distance, the transition zone will appear as a surface  $\Gamma$ .

Therefore, mathematically we consider the limit as  $\delta \rightarrow 0$  (see the proofs in Sections 5 and 6). In the case  $\delta > 0$  the independent variables are the two masses  $\rho_\delta^1$  and  $\rho_\delta^2$ , and the velocity  $v$ . With  $\rho = \rho_\delta^1 + \rho_\delta^2$  we have the following conservation laws for the two masses and the momentum:

$$\left. \begin{aligned} \partial_t \rho_\delta^1 + \operatorname{div}(\rho_\delta^1 v) &= \tau_\delta, \\ \partial_t \rho_\delta^2 + \operatorname{div}(\rho_\delta^2 v) &= -\tau_\delta, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + \Pi_\delta) &= \mathbf{f}_\delta. \end{aligned} \right\} \tag{1.1}$$

The quantities  $\tau_\delta$  and  $\Pi_\delta$  and how they depend on  $\delta$  are explained in Section 3, together with the mass equations in an equivalent form for  $\rho$  and  $\phi := (1/\rho)\rho_\delta^2$ , which we treat as order parameter, since the free energy depends on  $\nabla\phi$ .

It is shown in Sections 5 and 6 that under suitable assumptions the quantities under the derivatives converge pointwise to quantities in  $\Omega^m$ ,  $m = 1, 2$ , and  $\Gamma$ . These assumptions contain the special form of  $\tau_\delta$ ,

$$\begin{aligned} \tau_\delta &= \eta_\delta(\rho, \phi) \frac{\delta f_\delta}{\delta \phi}, & \eta_\delta(\rho, \phi) &= \frac{\eta_0(\rho, \phi)}{\delta}, \\ f_\delta &= \frac{1}{\delta} \rho W(\phi) + \delta h(\rho) \frac{|\nabla\phi|^2}{2} + U(\rho, \phi), \end{aligned} \tag{1.2}$$

where the  $\delta$ -scaling is essential.

The basis for this convergence is the distributional formulation, which requires the measures  $\mu_{\Omega^m}$  and  $\mu_\Gamma$  defined in (2.3). We arrive at the following set of equations:

$$\left. \begin{aligned} \partial_t(\rho^1 \mu_{\Omega^1}) + \operatorname{div}(\rho^1 v^1 \mu_{\Omega^1}) &= \tau \mu_\Gamma, \\ \partial_t(\rho^2 \mu_{\Omega^2}) + \operatorname{div}(\rho^2 v^2 \mu_{\Omega^2}) &= -\tau \mu_\Gamma, \\ \partial_t\left(\sum_m \rho^m v^m \mu_{\Omega^m}\right) + \operatorname{div}\left(\sum_m (\rho^m v^m \otimes v^m + \Pi^m) \mu_{\Omega^m} + \Pi^s \mu_\Gamma\right) &= \sum_m \mathbf{f}^m \mu_{\Omega^m}, \end{aligned} \right\} \tag{1.3}$$

where

$$\Pi^s = -\boldsymbol{\gamma}(\mathbb{I} - v \otimes v)$$

with  $v := v_{\Omega^1} = -v_{\Omega^2}$ , and where the surface tension  $\boldsymbol{\gamma}$  is given by

$$\begin{aligned} \boldsymbol{\gamma} &:= \int_{-\infty}^{\infty} \left( h(R^0) - \frac{a_2(R^0, \Phi^0)}{\tilde{a}(R^0, \Phi^0)} \frac{e_h(R^0)}{2} \right) |\partial_r \Phi^0|^2 \, dr \\ &= \int_{-\infty}^{\infty} \left( h(R_{M^0}^0) - \frac{a_2(R_{M^0}^0, \Phi_{M^0}^0)}{\tilde{a}(R_{M^0}^0, \Phi_{M^0}^0)} \frac{e_h(R_{M^0}^0)}{2} \right) |\partial_r \Phi_{M^0}^0|^2 \, dr \\ &= \boldsymbol{\gamma}(M^0), \quad M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2 \quad \text{on } \Gamma \end{aligned}$$

(see Section 6), where  $\lambda^1$  and  $\lambda^2$  are defined in 5.3. For the reaction rate  $\tau$  there is no additional formula except  $\tau = M^0$ , so that it is defined by the distributional equation

$$\partial_t\left(\sum_m \rho^m \mu_{\Omega^m}\right) + \operatorname{div}\left(\sum_m \rho^m v^m \mu_{\Omega^m}\right) = 0$$

via the strong equation

$$\boldsymbol{\tau} := -\rho^1(v^1 - v_\Gamma) \bullet \nu_{\Omega^1} + \rho^2(v^2 - v_\Gamma) \bullet \nu_{\Omega^2} \quad \text{on } \Gamma.$$

All other quantities are explained in 5.2 and 6.2. We mention that the standard way to describe the weak formulation of such problems makes use of test volumina (see e.g. [K], [S], [JR]). But we think that the usage of test functions is a more elegant way to formulate the system.

Besides these distributional equations there are additional boundary conditions at the interface  $\Gamma$ . They are derived in Section 7 and can be written as

$$\left. \begin{aligned} v_{\text{tan}}^1 &= v_{\text{tan}}^2, \\ \rho^1 &= g_1(\rho^1 \lambda^1), \\ \rho^2 &= g_2(\rho^2 \lambda^2), \end{aligned} \right\} \quad (1.4)$$

with  $g_1(M) := R_M^0(-\infty)$  and  $g_2(M) := R_M^0(+\infty)$ .

With these boundary conditions the description of the limit problem is complete, that is, the problem is completely determined by (1.3) and (1.4).

There are also equivalent forms of these additional conditions in (1.4); one version is given by

$$\left. \begin{aligned} v_{\text{tan}}^1 &= v_{\text{tan}}^2, \\ (v^1 - v^2) \bullet \nu &= \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \boldsymbol{\omega}(M^0), \\ G(\rho^1, \rho^2) &= 0, \end{aligned} \right\} \quad (1.5)$$

which one can find in Section 7.

The corresponding strong version of the above distributional equations is

$$\begin{aligned} \partial_t \rho^m + \operatorname{div}(\rho^m v^m) &= 0 \quad \text{in } \Omega^m, \quad m = 1, 2, \\ \rho^1(v^1 - v_\Gamma) \bullet \nu_{\Omega^1} + \rho^2(v^2 - v_\Gamma) \bullet \nu_{\Omega^2} &= 0 \quad \text{on } \Gamma, \\ \partial_t(\rho^m v^m) + \operatorname{div}(\rho^m v^m \otimes v^m + \Pi^m) &= \mathbf{f}^m \quad \text{in } \Omega^m, \quad m = 1, 2, \\ \operatorname{div} \Pi^s &= \sum_m (\rho^m(v^m - v_\Gamma) \bullet \nu_{\Omega^m} v^m + \Pi^m \nu_{\Omega^m}) \quad \text{on } \Gamma, \end{aligned}$$

where the last identity is the Delhaye condition [Del], which can also be written as

$$-\boldsymbol{\gamma} \kappa_\Gamma - \nabla^\Gamma \boldsymbol{\gamma} = M^0(v^2 - v^1) - (\Pi^2 - \Pi^1)v, \quad M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2 \quad (1.6)$$

taking the Laplace formula into account. This can also be written as

$$\nabla^\Gamma \boldsymbol{\gamma} = ((\Pi^2 - \Pi^1)v)_{\text{tan}}, \quad -\boldsymbol{\gamma} \kappa_\Gamma \bullet \nu + M^0(\lambda^2 - \lambda^1) = -v \bullet (\Pi^2 - \Pi^1)v.$$

Altogether, one describes the limit problem with conditions at the interface containing quantities, here  $\boldsymbol{\gamma}$ ,  $g_1$ , and  $g_2$ , which are described by the inner coordinate to the problem.

Therefore the phase field model enables one to derive and describe constitutive equations for the limit problem.

## 2. Sharp interface problem

We consider the interface problem in the distributional sense, since then it is in its natural form. This is because the space-time divergence, that is,  $\partial_t$  and  $\operatorname{div}$ , is an operation that really acts on the test function. Here the distributional formulation is also appropriate, because the interface comes by taking a limit of phase field equations (see Section 3).

The problem lives in a local domain  $\Omega \subset \mathbb{R} \times \mathbb{R}^n$  (for physical reasons one has to set  $n \leq 3$ , but in this paper  $n$  is arbitrary), which consists of two domains  $\Omega^1$  and  $\Omega^2$  and a smooth interface  $\Gamma$ , that is,

$$\Omega = \Omega^1 \cup \Gamma \cup \Omega^2. \quad (2.1)$$

We assume that  $\Gamma$  is timelike, so that  $\Omega_t = \Omega_t^1 \cup \Gamma_t \cup \Omega_t^2$  for each  $t$  with a smooth surface

$$\Gamma_t := \{x; (t, x) \in \Gamma\}. \quad (2.2)$$

Here  $\Gamma_t$  is the time slice of  $\Gamma$ , and  $\Omega_t^1$  and  $\Omega_t^2$  are defined in the same way. We define measures

$$\mu_{\Omega^m}(E) := \mathbf{L}^{n+1}(E \cap \Omega^m), \quad \mu_\Gamma(E) := \int_{\mathbb{R}} \mathbf{H}^{n-1}(\{x \in \Gamma_t; (t, x) \in E\}) \, d\mathbf{L}^1(t). \quad (2.3)$$

Note that

$$\mu_\Gamma = \frac{1}{\sqrt{1 + |v_\Gamma|^2}} \mathbf{H}^n \llcorner \Gamma, \quad (2.4)$$

where  $\mathbf{H}^n \llcorner \Gamma$  is the  $n$ -dimensional Hausdorff measure on  $\Gamma$ , and  $v_\Gamma$  is the normal surface velocity on  $\Gamma$ . We consider mass and momentum equations in the distributional sense, that is, the mass equations read

$$\begin{aligned} \partial_t(\rho^m \mu_{\Omega^m}) + \operatorname{div}(\rho^m v^m \mu_{\Omega^m}) &= \tau^m \mu_\Gamma \quad \text{for } m = 1, 2, \\ \tau^1 + \tau^2 &= 0 \quad (\tau := \tau^1 = -\tau^2), \end{aligned} \quad (2.5)$$

and the momentum equation is

$$\partial_t \left( \sum_m \rho^m v^m \mu_{\Omega^m} \right) + \operatorname{div} \left( \sum_m (\rho^m v^m \otimes v^m + \Pi^m) \mu_{\Omega^m} + \Pi^s \mu_\Gamma \right) = \sum_m \mathbf{f}^m \mu_{\Omega^m} + \mathbf{f}^s \mu_\Gamma. \quad (2.6)$$

This means that there is no mass at the interface, but there is a surface tension tensor  $\Pi^s$  on  $\Gamma$ . These equations are equivalent to versions in the strong sense (see [Al, Section 2]). By this we mean the mass conservation

$$\begin{aligned} \partial_t \rho^m + \operatorname{div}(\rho^m v^m) &= 0 && \text{in } \Omega^m, \\ -\tau &= \rho^1 (v^1 - v_\Gamma) \bullet v = \rho^2 (v^2 - v_\Gamma) \bullet v && \text{on } \Gamma \quad (v := v_{\Omega^1} = -v_{\Omega^2}), \end{aligned} \quad (2.7)$$

where  $v_{\Omega^m}$  is the outer normal of  $\Omega^m$  and  $v_\Gamma$  is the normal velocity of  $\Gamma$  (it is not a “velocity”, but rather a “normal velocity”), and the momentum equation

$$\begin{aligned} \partial_t \left( \sum_m \rho^m v^m \right) + \operatorname{div} \left( \sum_m (\rho^m v^m \otimes v^m + \Pi^m) \right) &= 0 && \text{in } \Omega^m, \\ \operatorname{div}^\Gamma \Pi^s &= \mathbf{f}^s + \sum_m \rho^m (v^m - v_\Gamma) \bullet v_{\Omega^m} v^m + \sum_m \Pi^m v_{\Omega^m} && \text{on } \Gamma. \end{aligned} \quad (2.8)$$

The last equation on  $\Gamma$  is the Delhaye interface condition (see [Del]). Another version of this condition in the situation of this paper is presented in (1.6).

Besides these interfacial conditions given by the distributional equations we have to assume additional conditions on the interface to ensure the existence of a unique local solution (see for example (1.4) and (1.5)). For a phase field model this additional information is included in the  $\delta$ -equation. These equations have to be extracted in the limit procedure; this is done in Section 7.

### 3. Phase field problem

In [W2] the flow of a mixture of two different materials has been considered, which is governed by the compressible Navier–Stokes system and the Allen–Cahn equation. It can be written as

$$\left. \begin{aligned} \partial_t \rho + \operatorname{div}(\rho v) &= 0, \\ \rho(\partial_t \phi + v \bullet \nabla \phi) &= -\boldsymbol{\tau}_\delta, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + \Pi_\delta) &= \mathbf{f}_\delta, \end{aligned} \right\} \quad (3.1)$$

where the term  $\mathbf{f}_\delta$  is an external force and is there for an arbitrary observer (see e.g. [AI]). The reaction rate  $\boldsymbol{\tau}_\delta$  fulfills (3.8) below, and  $\Pi_\delta$  is given by (3.5). The unknown functions  $\rho(t, x) > 0$ ,  $\phi(t, x) \in \mathbb{R}$ , and  $v(t, x) \in \mathbb{R}^n$  denote the total mass density, the mass fraction, and the velocity. We write

$$\begin{aligned} \rho &:= \rho_\delta^1 + \rho_\delta^2, & \phi_\delta^1 &:= \rho_\delta^1 / \rho, & \phi_\delta^2 &:= \rho_\delta^2 / \rho, & \phi &:= \phi_\delta^2, \\ \rho_\delta^1 &= (1 - \phi)\rho, & \rho_\delta^2 &= \phi\rho. \end{aligned} \quad (3.2)$$

This means that the mass densities  $(\rho_\delta^1, \rho_\delta^2)$  are equivalent variables to  $(\rho, \phi)$ . The reaction rate  $\boldsymbol{\tau}_\delta$  is a function of  $(\rho, \phi, \nabla \rho, \nabla \phi, \mathbf{D}^2 \phi)$ , and the total tension tensor  $\Pi_\delta$  is a function of  $(\rho, \phi, v, \nabla \phi, \mathbf{D}v)$ . The dependence on  $\nabla \phi$  can be understood as a dependence on a certain linear combination of the gradients of  $\rho_\delta^1$  and  $\rho_\delta^2$ , since

$$\nabla \phi = \frac{1}{\rho} (\phi_\delta^1 \nabla \rho_\delta^2 - \phi_\delta^2 \nabla \rho_\delta^1).$$

The system can be written as the mass conservation for each mass and the momentum conservation for the total mass:

$$\left. \begin{aligned} \partial_t \rho_\delta^1 + \operatorname{div}(\rho_\delta^1 v) &= \boldsymbol{\tau}_\delta, \\ \partial_t \rho_\delta^2 + \operatorname{div}(\rho_\delta^2 v) &= -\boldsymbol{\tau}_\delta, \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + \Pi_\delta) &= \mathbf{f}_\delta. \end{aligned} \right\} \quad (3.3)$$

So far the system has rarely been treated mathematically. In this connection we refer to [HS], [W1]. Concerning the systems we state the following

3.1. THEOREM With definition (3.2) the system (3.1) is equivalent to (3.3).

*Proof.* This has been shown in the appendix of [W2]. The conservation of momentum is the same in both systems. The sum of the first two equations of (3.3) gives the conservation of the total mass. The first equation of (3.3) reads  $\partial_t(\rho\phi) + \operatorname{div}(\rho\phi v) = -\boldsymbol{\tau}_\delta$  and conservation of the total mass turns it into  $\rho(\partial_t \phi + v \bullet \nabla \phi) = -\boldsymbol{\tau}_\delta$ .  $\square$

The quantities  $\boldsymbol{\tau}_\delta$  and  $\Pi_\delta$  are defined in the general case by the internal free energy density

$$f_\delta \equiv f_\delta(\rho, \phi, \nabla\phi) \quad (3.4)$$

and besides  $(Dv)^S$  by the Lamé coefficients  $a_1(\rho, \phi)$  and  $a_2(\rho, \phi)$ . The tensor  $\Pi_\delta$  has the form

$$\Pi_\delta \equiv \mathbb{P} - \mathbb{S}, \quad (3.5)$$

where  $\mathbb{P}$  is determined by the internal free energy  $f_\delta$ , containing the pressure  $p_{f_\delta}$ ,

$$\mathbb{P} \equiv \mathbb{P}(\rho, \phi, \nabla\phi) := p_{f_\delta} \mathbb{I} + \nabla\phi \otimes f_\delta{}'_{\nabla\phi}, \quad (3.6)$$

and the stress tensor

$$\mathbb{S} \equiv \mathbb{S}(\rho, \phi, (Dv)^S). \quad (3.7)$$

The mass transition rate  $\boldsymbol{\tau}_\delta$  is given by

$$\boldsymbol{\tau}_\delta \equiv \boldsymbol{\tau}_\delta(\rho, \phi, \nabla\rho, \nabla\phi, D^2\phi) := \eta_\delta(\rho, \phi) \frac{\delta f_\delta}{\delta\phi}. \quad (3.8)$$

The variation of a function  $g$  depending on  $(\rho, \phi, \nabla\phi)$  with respect to  $\phi$ , and the pressure  $p_g$ , are defined by

$$\frac{\delta g}{\delta\phi} := g'_\phi - \operatorname{div}(g'_{\nabla\phi}), \quad p_g := \rho g'_{\rho} - g. \quad (3.9)$$

Here  $g'_{\rho}$  and  $g'_{\nabla\phi}$  denote the derivatives of  $g$  with respect to the variable  $\rho$  and  $\nabla\phi$  (no new notation is introduced for these variables). The total free energy  $f$  includes the dynamical part and is given by

$$f \equiv f(\rho, \phi, v, \nabla\phi) = f_\delta(\rho, \phi, \nabla\phi) + \frac{\rho}{2}|v|^2. \quad (3.10)$$

For the total free energy  $f$  the following energy identity has been shown in [W2].

**3.2. THEOREM** The total free energy  $f$  defined in (3.10) together with the free energy flux

$$\boldsymbol{\psi} := f v + \Pi_\delta^T v - \dot{\phi} f'_{\nabla\phi}$$

satisfies

$$\partial_t f + \operatorname{div} \boldsymbol{\psi} - v \bullet \mathbf{f}_\delta = -\frac{1}{\rho} \boldsymbol{\tau}_\delta \frac{\delta f_\delta}{\delta\phi} - Dv : \mathbb{S} \leq 0. \quad (3.11)$$

Here for the inequality the assumptions in 3.3 are required. For every function  $g$  the total derivative is defined by

$$\dot{g} := (\partial_t + v \bullet \nabla)g = \partial_t g + v \bullet \nabla g.$$

*Proof.* Let  $(\rho, \phi, v)$  be a solution of system (3.1) (or (3.3)). For the dynamical part one computes

$$\partial_t \left( \frac{1}{2} \rho |v|^2 \right) + \operatorname{div} \left( \frac{1}{2} \rho |v|^2 v + \Pi_\delta^T v \right) = v \bullet \mathbf{f}_\delta + Dv : \Pi_\delta.$$

Then for a general total free energy flux  $\boldsymbol{\psi}$  with  $\boldsymbol{\psi} = f v + \boldsymbol{\psi}_0$  and  $f$  as in (3.10) one obtains

$$\begin{aligned} \partial_t f + \operatorname{div} \boldsymbol{\psi} &= \partial_t \left( f_\delta + \frac{1}{2} \rho |v|^2 \right) + \operatorname{div} \left( f_\delta v + \frac{1}{2} \rho |v|^2 v + \boldsymbol{\psi}_0 \right) \\ &= \partial_t f_\delta + \operatorname{div}(f_\delta v + \boldsymbol{\psi}_0 - \Pi_\delta^T v) + v \bullet \mathbf{f}_\delta + Dv : \Pi_\delta \\ &= \dot{f}_\delta + \operatorname{div}(\boldsymbol{\psi}_0 - \Pi_\delta^T v) + v \bullet \mathbf{f}_\delta + Dv : (f_\delta \mathbb{I} + \Pi_\delta). \end{aligned}$$

Now, for the free energy one gets

$$\begin{aligned} \dot{f}_\delta &= f_{\delta'\rho}\dot{\rho} + f_{\delta'\phi}\dot{\phi} + f_{\delta'\nabla\phi} \bullet (\nabla\phi)' \\ &= f_{\delta'\rho}\dot{\rho} + f_{\delta'\phi}\dot{\phi} + f_{\delta'\nabla\phi} \bullet \nabla\dot{\phi} - \text{D}v : (\nabla\phi \otimes f_{\delta'\nabla\phi}) \\ &= f_{\delta'\rho}\dot{\rho} + (f_{\delta'\phi} - \text{div}(f_{\delta'\nabla\phi}))\dot{\phi} + \text{div}(\dot{\phi}f_{\delta'\nabla\phi}) - \text{D}v : (\nabla\phi \otimes f_{\delta'\nabla\phi}). \end{aligned}$$

Therefore, using definition (3.9), one finally obtains

$$\begin{aligned} \partial_t f + \text{div} \psi &= \text{div}(\psi_0 - \Pi_\delta^T v + \dot{\phi}f_{\delta'\nabla\phi}) + v \bullet \mathbf{f}_\delta + f_{\delta'\rho}\dot{\rho} + \frac{\delta f_\delta}{\delta\phi} \dot{\phi} \\ &\quad + \text{D}v : (f_\delta \mathbb{I} - \nabla\phi \otimes f_{\delta'\nabla\phi} + \Pi_\delta). \end{aligned}$$

Inserting  $\dot{\rho}$  and  $\dot{\phi}$  from the mass equations, that is,  $\dot{\rho} = -\rho \text{div} v$  and  $\rho\dot{\phi} = -\boldsymbol{\tau}_\delta$ , one is led to

$$\begin{aligned} \partial_t f + \text{div} \psi &= \text{div}(\psi_0 - \Pi_\delta^T v + \dot{\phi}f_{\delta'\nabla\phi}) + v \bullet \mathbf{f}_\delta - \frac{1}{\rho} \boldsymbol{\tau}_\delta \frac{\delta f_\delta}{\delta\phi} \\ &\quad + \text{D}v : ((f_\delta - \rho f_{\delta'\rho})\mathbb{I} - \nabla\phi \otimes f_{\delta'\nabla\phi} + \Pi_\delta). \end{aligned}$$

Now, if the free energy flux  $\psi_0$  is chosen as in the assertion, and if the tensor  $\Pi_\delta$  is defined as in (3.5) and (3.6), one ends up with the identity (3.11). The inequality comes from 3.3 below.  $\square$

Let us now introduce the special representations for  $\mathbb{S}$  and  $f_\delta$ . The stress tensor  $\mathbb{S}$  in (3.7) is linear in  $(\text{D}v)^S$  and given by the classical formula

$$\mathbb{S} := a_1(\rho, \phi) \text{div} v \mathbb{I} + a_2(\rho, \phi) \left( (\text{D}v)^S - \frac{1}{n} \text{div} v \mathbb{I} \right) \quad (3.12)$$

with Lamé coefficients  $a_1$  and  $a_2$  depending on the mass quantities, that is,  $\rho$  and  $\phi$ . For the free energy density  $f_\delta$  we consider the following representation:

$$\begin{aligned} f_\delta(\rho, \phi, \nabla\phi) &:= \frac{1}{\delta} \rho W(\phi) + \delta h(\rho) \frac{|\nabla\phi|^2}{2} + U(\rho, \phi), \\ U_\phi(\rho, 0) &= 0, \quad U_\phi(\rho, 1) = 0, \\ W &\text{ has two local minima at 0 and 1.} \end{aligned} \quad (3.13)$$

There is no assumption on the values of  $W(0)$  and  $W(1)$ . The first variation with respect to  $\phi$  is

$$\frac{\delta f_\delta}{\delta\phi} = \frac{1}{\delta} \rho W_\phi(\phi) - \delta \text{div}(h(\rho)\nabla\phi) + \Psi_\phi(\rho, \phi). \quad (3.14)$$

The function  $W$  depending on  $(\rho, \phi)$  stands for a ‘‘double-well potential’’ and has two local minima in  $\phi$  at  $\phi = 0$  and  $\phi = 1$ . The data fulfil the following three assumptions.

**3.3. LEMMA** The entropy condition is satisfied if

$$\eta_\delta \geq 0 \quad \text{and} \quad a_1 \geq 0, a_2 \geq 0.$$

We assume the stronger conditions  $\eta_\delta > 0$ ,  $a_2 > 0$ . Hence we consider Newtonian flows in this paper.

**4. Asymptotic expansion**

We consider the case that the phase change happens in a small region around a smooth timelike surface  $\Gamma$ , that is, we assume that  $\Gamma$  is at least a  $C^2$ -interface. In a neighbourhood of  $\Gamma$  one introduces coordinates

$$(t, x) = (t, y + \delta r v(t, y)) \quad \text{with} \quad (t, y) \in \Gamma, r \in \mathbb{R}, \tag{4.1}$$

where  $v = v_{\Omega^1} = -v_{\Omega^2}$  on  $\Gamma$ . One considers a neighbourhood of  $\Gamma$  given by

$$\Gamma_\delta := \{(t, y + s v(t, y)); (t, y) \in \Gamma, |s| \leq \varepsilon_\delta\}, \tag{4.2}$$

where  $\varepsilon_\delta$  is chosen so that the solution outside the set  $\Gamma_\delta$  converges towards the outer solution in the open sets  $\Omega^1$  and  $\Omega^2$ , that is, the total domain is decomposed into the three sets

$$\Omega = \Omega_\delta^1 \cup \Gamma_\delta \cup \Omega_\delta^2 = \Omega^1 \cup \Gamma \cup \Omega^2,$$

where  $\overline{\Omega_\delta^m}$  converges to  $\overline{\Omega^m}$  locally in Hausdorff distance, and also  $\Gamma_\delta$  converges to  $\Gamma$  locally in Hausdorff distance (see Appendix, (A3)).

After the set  $\Gamma_\delta$  is stretched one has to consider in the  $(t, y, r)$ -coordinates with  $(t, y) \in \Gamma$  and  $r \in \mathbb{R}$  the inner expansion, which in our case satisfies the equations (4.5)–(4.7) below. Therefore  $\varepsilon_\delta \rightarrow 0$  as  $\delta \rightarrow 0$  and  $r_\delta := (1/\delta)\varepsilon_\delta \rightarrow \infty$  as  $\delta \rightarrow 0$ . Also we define

$$\Gamma_t := \{x; (t, x) \in \Gamma\}. \tag{4.3}$$

In a phase field model the function  $\phi$  stands for an order parameter depending on  $\delta > 0$ , here for the equations (3.1). Depending on this function we define for small  $\varepsilon > 0$  and all  $t$  an interfacial region

$$V_{\delta,\varepsilon}(t) := \{x; \varepsilon < \phi(t, x) < 1 - \varepsilon\}, \quad V_{\delta,\varepsilon} := \{(t, x); x \in V_{\delta,\varepsilon}(t)\}. \tag{4.4}$$

In the limit  $\delta \searrow 0$  for each  $\varepsilon > 0$  the set  $V_{\delta,\varepsilon}$  approaches the interface  $\Gamma$ . This is because  $\phi \rightarrow 0$  pointwise in  $\Omega^1$  and  $\phi \rightarrow 1$  pointwise in  $\Omega^2$ , therefore  $V_{\delta,\varepsilon} \subset \Gamma_\delta$  for small enough  $\delta$ , if  $\varepsilon$  is fixed.

Let  $x \in \Gamma_\delta$ . We denote by  $P_t(x)$  the projection of  $x$  onto  $\Gamma_t$ . The function  $s(t, x)$  denotes the signed distance from  $x$  to  $\Gamma_t$ , with  $s < 0$  in  $\Omega^1$  and  $s > 0$  in  $\Omega^2$ . Since  $\Gamma_{\delta,t}$  lies in a small neighbourhood around  $\Gamma_t$ , we assign to each point  $(t, x) \in \Gamma_\delta$  a unique pair  $(y(t, x), r(t, x))$  by

$$y(t, x) := P_t(x), \quad r(t, x) := \frac{1}{\delta} s(t, x),$$

$$s(t, x) := \begin{cases} -\text{dist}(x, \Gamma_t) & \text{for } (t, x) \in \Omega^1, \\ +\text{dist}(x, \Gamma_t) & \text{for } (t, x) \in \Omega^2. \end{cases}$$

With these definitions we calculate some first and second derivatives of the transformation:

$$\begin{aligned} \nabla s(t, x) &= v(t, P_t(x)), \\ \partial_t(P_t(x)) &= v_\Gamma(t, P_t(x)) + \mathcal{O}(\delta), \\ \partial_t s(t, x) &= -v_\Gamma(t, P_t(x)) \bullet v(t, P_t(x)), \\ D^2 s(t, x) &= D^\Gamma v(t, P_t(x)) + \mathcal{O}(\delta), \\ \text{trace}(D^2 s(t, x)) &= \Delta s(t, x) = -\kappa_\Gamma(t, P_t(x)) \bullet v(t, P_t(x)) + \mathcal{O}(\delta), \end{aligned}$$



where the differential operators  $D^\Gamma$  and  $\partial_t^\Gamma$  for a vector function  $w$  are defined by

$$D^\Gamma w(t, y) := \sum_{k=1}^{n-1} (\partial_{\tau_k} w(t, y)) \otimes \tau_k, \quad \partial_t^\Gamma w(t, y) := (\partial_t + v_\Gamma(t, y) \bullet \nabla)w(t, y),$$

where  $\tau_k, k = 1, \dots, n - 1$ , is an orthonormal system of the tangent space to  $\Gamma_t$  at  $y$ . The vector  $\nu_{\Omega^m}(t, y)$  is the unit outer normal vector to  $\Gamma_t$  at  $y$  with respect to  $\Omega_t^m$ , and  $\nu = \nu_{\Omega^1}$ . The function  $v_\Gamma(t, y)$  is the normal velocity vector of  $\Gamma_t$  and  $\kappa_\Gamma(t, y)$  is a normal vector denoting  $n$ -times the mean curvature of  $\Gamma_t$  at  $y$ .

Now we write the functions in the new coordinates  $(t, y, r)$  as

$$\begin{aligned} \rho(t, x) &= R(t, y(t, x), r(t, x)) = R(t, y(t, x), \frac{1}{\delta}s(t, x)), \\ \phi(t, x) &= \Phi(t, y(t, x), r(t, x)) = \Phi(t, y(t, x), \frac{1}{\delta}s(t, x)), \\ v(t, x) &= V(t, y(t, x), r(t, x)) = V(t, y(t, x), \frac{1}{\delta}s(t, x)). \end{aligned}$$

In [W2] the equations (3.1), that is, the mass conservation, the momentum conservation, and the Allen–Cahn equation, in these new inner variables are shown to be equivalent to

$$\frac{1}{\delta} \partial_r (R\Lambda) = \partial_t^\Gamma R + \operatorname{div}^\Gamma (RV) + \mathcal{O}(\delta), \tag{4.5}$$

$$\begin{aligned} \frac{\eta_0(R, \Phi)}{\delta} (RW'(\Phi) - \partial_r (h(R)\partial_r \Phi)) \\ = R\Lambda \partial_r \Phi - \eta_0(R, \Phi) (\Psi_\phi(R, \Phi) + \kappa_\Gamma \bullet v h(R)\partial_r \Phi) + \mathcal{O}(\delta), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \frac{1}{\delta} \left( \partial_r \left( e_h(R) \frac{|\partial_r \Phi|^2}{2} \right) v - \partial_r (a \partial_r V \bullet v) v - \partial_r \left( \frac{1}{2} a_2 \partial_r V \right) \right) \\ = \partial_r (R\Lambda V) - \partial_r (p_\psi) v - \nabla^\Gamma \left( p_h \frac{|\partial_r \Phi|^2}{2} \right) - \partial_r (h(R)\partial_r \Phi \nabla^\Gamma \Phi) \\ + \kappa_\Gamma \bullet v h(R) |\partial_r \Phi|^2 v + \nabla^\Gamma (\bar{a} \partial_r V \bullet v) + \partial_r (\bar{a} \operatorname{div}^\Gamma V) v \\ + \operatorname{div}^\Gamma \left( \frac{1}{2} a_2 v \otimes \partial_r V \right) + \partial_r \left( \frac{1}{2} a_2 \nabla^\Gamma V \right) v - \kappa_\Gamma \bullet v \frac{1}{2} a_2 \partial_r V + \mathcal{O}(\delta), \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} \Lambda &:= (v_\Gamma - V) \bullet v, \\ \eta_\delta(\rho, \phi) &= \frac{1}{\delta} \eta_0(\rho, \phi), \\ a &:= a_1 + \frac{n-2}{2n} a_2, \quad \bar{a} := a_1 - \frac{1}{n} a_2. \end{aligned} \tag{4.8}$$

The term  $\mathcal{O}(\delta)$  in (4.5)–(4.7) indicates that there are additional terms in the equation, which are estimated by  $\delta$ . In [W2] it is further shown that if one takes the *inner expansion* in  $\delta$ ,

$$\left. \begin{aligned} R(t, y, r) &= R^0(t, y, r) + \delta R^1(t, y, r) + \mathcal{O}(\delta^2), \\ \Phi(t, y, r) &= \Phi^0(t, y, r) + \delta \Phi^1(t, y, r) + \mathcal{O}(\delta^2), \\ V(t, y, r) &= V^0(t, y, r) + \delta V^1(t, y, r) + \mathcal{O}(\delta^2), \\ \text{where } R^0, R^1, \Phi^0, \Phi^1, V^0, V^1 &\text{ are bounded functions,} \end{aligned} \right\} \tag{4.9}$$

one derives from the equations in (4.5)–(4.7) the corresponding equations for  $(R^0, \Phi^0, V^0)$  and for  $(R^1, \Phi^1, V^1)$ , which are linear equations in  $(R^1, \Phi^1, V^1)$  with coefficients depending on  $(R^0, \Phi^0, V^0)$  (see [W3, Section 8]). Higher order equations are not required for the purpose of this paper.

We assume an *outer expansion*, for  $(t, x) \in \Omega^m, m = 1, 2$ ,

$$\left. \begin{aligned} \rho(t, x) &= \rho_m^0(t, x) + \delta\rho_m^1(t, x) + \mathcal{O}(\delta^2), \\ \phi(t, x) &= \phi_m^0(t, x) + \delta\phi_m^1(t, x) + \mathcal{O}(\delta^2), \quad \text{with } \phi_1^0 = 0, \phi_2^0 = 1, \\ v(t, x) &= v_m^0(t, x) + \delta v_m^1(t, x) + \mathcal{O}(\delta^2), \\ \mathbf{f}_\delta &= \mathbf{f}_m + \mathcal{O}(\delta), \end{aligned} \right\} \quad (4.10)$$

where  $\rho_m^0, \rho_m^1, \phi_m^1, v_m^0, v_m^1, \mathbf{f}_m$  are bounded functions.

Besides this, for the values of the expansion of  $(R, \Phi, V)$  there are boundary conditions at  $r = \pm\infty$ . These conditions come from the fact that in the region  $\delta r = s \approx \pm\varepsilon_\delta$  we have, for example for the values of  $v$ , the identity

$$V(t, y, r) = v(t, y + \delta r v(t, y)). \tag{4.11}$$

This identity implies, with the inner expansion (4.9) and the outer expansion (4.10) at  $x = y + \delta r v(t, y)$ , that

$$V^0(t, y, r) + \delta V^1(t, y, r) + \mathcal{O}(\delta^2) = v_2^0(t, y + \delta r v(t, y)) + \delta v_2^1(t, y + \delta r v(t, y)) + \mathcal{O}(\delta^2)$$

for  $r > 0$ , and analogously for  $r < 0$ . Now set  $r = r_\delta$ ,

$$V^0(t, y, r_\delta) + \mathcal{O}(\varepsilon_\delta) = v_2^0(t, y + \varepsilon_\delta v(t, y)) + \mathcal{O}(\delta), \quad r_\delta \rightarrow +\infty, \varepsilon_\delta := \delta r_\delta \rightarrow 0 \text{ as } \delta \rightarrow 0,$$

and obtain, as  $\delta \rightarrow 0$ ,

$$V^0(t, y, +\infty) = v_2^0(t, y), \quad V^0(t, y, -\infty) = v_1^0(t, y)$$

(the second identity follows in the same way). Similarly taking the derivative with respect to  $r$  in (4.11) one gets

$$\partial_r V(t, y, r) = \delta(v(t, y) \bullet \nabla)v(t, y + \delta r v(t, y)), \tag{4.12}$$

from which one deduces, by the same procedure as above, that  $\partial_r V^0(t, y, \pm\infty) = 0$  and

$$\partial_r V^1(t, y, -\infty) = (v(t, y) \bullet \nabla)v_1^0(t, y), \quad \partial_r V^1(t, y, +\infty) = (v(t, y) \bullet \nabla)v_2^0(t, y).$$

In the following sections we write  $v^m := v_m^0$  for  $m = 1, 2$ . The same holds for  $R$  and  $\Phi$ .

### 5. Mass conservation

The mass conservation for  $\rho_\delta^2 = \rho\phi$  in (3.3) is

$$\begin{aligned} \partial_t(\rho\phi) + \operatorname{div}(\rho\phi v) &= -\tau_\delta, \\ \tau_\delta &= \eta_\delta(\rho, \phi) \frac{\delta f_\delta}{\delta\phi}, \quad \eta_\delta(\rho, \phi) = \frac{\eta_0(\rho, \phi)}{\delta}, \quad \eta_0(\rho, \phi) > 0. \end{aligned}$$

In the distributional formulation this reads

$$\int_{\Omega} (\partial_t \zeta (\rho \phi) + \nabla \zeta \bullet (\rho \phi v) - \zeta \boldsymbol{\tau}_\delta) d\mathbf{L}^{n+1} = 0 \quad (5.1)$$

for  $\zeta \in C_0^\infty(\Omega; \mathbb{R})$ . The formulation of the mass conservation for  $\rho_\delta^1 = \rho(1 - \phi)$  looks similar (see the first equation of (3.3)).

We consider two classes of test functions in (5.1). The first choice gives as a result the ordinary differential equations one has to solve in the inner expansion. This result is then used in the second choice of the test functions. These test functions are chosen as functions of the global variables. Therefore one gets the equations of the outer expansion, and in addition a distributional equation across the interface. We show the following results when  $\delta \rightarrow 0$ , where the first result yields the  $1/\delta$ -term at the boundary.

5.1. THEOREM Assume (4.9) and (4.10). Then for  $(t, y) \in \Gamma$  we have in local coordinates

$$R^0 W'(\Phi^0) - \partial_r (h(R^0) \partial_r \Phi^0) = 0 \quad \text{for all } r \in \mathbb{R}.$$

This theorem is the version of the usual theorem on the zeroth order  $\Phi^0$  of the phase field. It is necessary to show the following result.

5.2. THEOREM Assume (4.9) and (4.10). Then as  $\delta \rightarrow 0$  the solution converges pointwise in the sense of distributions (see Appendix, (A1)) as follows:

$$\rho \phi \mathbf{L}^{n+1} \rightarrow \rho^2 \boldsymbol{\mu}_{\Omega^2}, \quad \rho \phi v \mathbf{L}^{n+1} \rightarrow \rho^2 v^2 \boldsymbol{\mu}_{\Omega^2}, \quad \boldsymbol{\tau}_\delta \mathbf{L}^{n+1} \rightarrow \boldsymbol{\tau} \boldsymbol{\mu}_\Gamma,$$

where

$$\begin{aligned} \boldsymbol{\tau} := & \int_{-\infty}^{+\infty} \eta_0(R^0, \Phi^0) (R^1 W'(R^0, \Phi^0) + R^0 W''(R^0, \Phi^0) \Phi^1 - \partial_r (h'(R^0) R^1 \partial_r \Phi^0) \\ & - \partial_r (h(R^0) \partial_r \Phi^1) + \kappa_\Gamma \bullet v h(R^0) \partial_r \Phi^0 + \Psi_\phi(R^0, \Phi^0)) dr. \end{aligned}$$

Therefore the limit equation is

$$\partial_t (\rho^2 \boldsymbol{\mu}_{\Omega^2}) + \operatorname{div}(\rho^2 v^2 \boldsymbol{\mu}_{\Omega^2}) = -\boldsymbol{\tau} \boldsymbol{\mu}_\Gamma.$$

Similarly one obtains the limit for the mass  $\rho^1$  in  $\Omega^1$  (see the end of this section).

The value of  $\boldsymbol{\tau}$  is uniquely determined, as shown in Section 8, but we have not been able to derive a constitutive equation for it, except

$$\boldsymbol{\tau} = M^0 \int_{-\infty}^{+\infty} \partial_r \Phi^0 dr = M^0,$$

which comes from inserting the definition of  $(R^1, \Phi^1)$ . Therefore this seems to define an arbitrary quantity  $\boldsymbol{\tau}$  given by the distributional equations only. This is in analogy to the arbitrary pressure value in the incompressible limit of the Navier–Stokes equations.

For the proofs we use local and global test functions.

*Local test function  $\zeta$* 

With the choice of a local test function  $\zeta = \xi$  with a  $C_0^\infty$ -function  $\xi$  around the free boundary we derive the well known first equation of the inner expansion (see (5.1)). This yields the  $1/\delta$ -term in (5.1). Explicitly we choose

$$\zeta(t, x) = \xi(t, y, r), \quad x = y + \delta r v(t, y), \quad (5.2)$$

where  $(t, y) \in \Gamma$ ,  $r \in \mathbb{R}$ , and  $v = v_{\Omega^2}$ . The support of  $r \mapsto \xi(t, y, r)$  is contained in a fixed interval  $[-r_\xi, r_\xi]$ , so that  $[-r_\xi, r_\xi] \subset [-r_\delta, r_\delta]$  for small  $\delta > 0$ . We compute the derivatives:

$$\begin{aligned} \partial_t \zeta &= \partial_t^\Gamma \xi - \frac{1}{\delta} v_\Gamma \bullet v \partial_r \xi + \mathcal{O}(\delta), \\ \nabla \zeta &= \nabla^\Gamma \xi + \frac{1}{\delta} \partial_r \xi v + \mathcal{O}(\delta), \end{aligned} \quad (5.3)$$

and we get, if  $\delta$  is small,

$$\begin{aligned} & \int_{\Omega} (\partial_t \zeta \cdot (\rho \phi) + \nabla \zeta \bullet (\rho \phi v) - \zeta \tau_\delta) \, dx \, dt \\ &= \int_{\mathbb{R}} \int_{-\varepsilon_\delta}^{+\varepsilon_\delta} \left\{ \int_{\Gamma_t} \left( \partial_t^\Gamma \xi \cdot \rho \phi + \nabla^\Gamma \xi \bullet (\rho \phi v) - \xi \tau_\delta + \frac{1}{\delta} \partial_r \xi \cdot \rho \phi (v - v_\Gamma) \bullet v \right. \right. \\ & \quad \left. \left. + \mathcal{O}(\delta) \chi_{\text{supp} \xi} \right) (1 + \mathcal{O}(s)) \, d\mathbf{H}^{n-1}(y) \right\} \, ds \, dt \\ &= \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \left\{ \int_{\Gamma_t} \left( \delta (\partial_t^\Gamma \xi \cdot \rho \phi + \nabla^\Gamma \xi \bullet (\rho \phi v)) - \xi \eta_0(\rho, \phi) \frac{\delta f_\delta}{\delta \phi} + \partial_r \xi \cdot \rho \phi (v - v_\Gamma) \bullet v \right. \right. \\ & \quad \left. \left. + \mathcal{O}(\delta^2) \chi_{\text{supp} \xi} \right) (1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \right\} \, dr \, dt \\ &= \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \left\{ \int_{\Gamma_t} \left( \xi \eta_0(\rho, \phi) \left( -\frac{1}{\delta} \rho W'(\phi) + \delta \operatorname{div}(h(\rho) \nabla \phi) + \Psi_\phi(\rho, \phi) \right) \right. \right. \\ & \quad \left. \left. + \mathcal{O}(1) \chi_{\text{supp} \xi} \right) (1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \right\} \, dr \, dt \\ &= \frac{1}{\delta} \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \left\{ \int_{\Gamma_t} \left( \xi \eta_0(R^0, \Phi^0) (-R^0 W'(\Phi^0) + \partial_r (h(R^0) \partial_r \Phi^0)) \right. \right. \\ & \quad \left. \left. + \mathcal{O}(\delta) \chi_{\text{supp} \xi} \right) (1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \right\} \, dr \, dt \\ &= \frac{1}{\delta} \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \int_{\Gamma_t} \xi \eta_0(R^0, \Phi^0) (-R^0 W'(\Phi^0) + \partial_r (h(R^0) \partial_r \Phi^0)) \, d\mathbf{H}^{n-1}(y) \, dr \, dt + \mathcal{O}(1). \end{aligned}$$

Then it follows that the  $1/\delta$ -term vanishes as  $\delta \searrow 0$ . Since  $\xi$  is arbitrary, using  $\eta_0 > 0$  one gets the identity in Theorem 5.1.

*Global test function  $\zeta$* 

We now choose test functions as functions of  $(t, x)$ . Since we claim that the terms converge in the sense of distributions, we have to choose independent test functions  $\alpha \in C_0^\infty(\Omega; \mathbb{R})$  and  $\beta \in$

$C_0^\infty(\Omega; \mathbb{R}^n)$ . We obtain

$$\begin{aligned} & \int_{\Omega} (\alpha\rho\phi + \beta \bullet (\rho\phi v) - \zeta\tau_\delta) \, dx \, dt \\ &= \int_{\Omega_\delta^2} (\alpha\rho\phi + \beta \bullet (\rho\phi v)) \, dx \, dt + \int_{\Gamma_\delta} (\alpha\rho\phi + \beta \bullet (\rho\phi v) - \zeta\tau_\delta) \, dx \, dt + o(1). \end{aligned} \quad (5.4)$$

Here we have used the particular form of  $\tau_\delta$  and that  $\phi \approx 1$  on  $\Omega_\delta^2$ ,  $\phi \approx 0$  on  $\Omega_\delta^1$  for  $\delta \searrow 0$ . This implies that  $\tau_\delta = o(1)$  in  $\Omega \setminus \Gamma_\delta$ . Since  $\rho$  and  $\phi$  are bounded and pointwise convergent with respect to the Lebesgue measure, we obtain further

$$\begin{aligned} & \int_{\Omega_\delta^2} (\alpha\rho\phi + \beta \bullet (\rho\phi v)) \, dx \, dt \rightarrow \int_{\Omega^2} (\alpha\rho^2 + \beta \bullet (\rho^2 v^2)) \, dx \, dt, \\ & \int_{\Gamma_\delta} (\alpha\rho\phi + \beta \bullet (\rho\phi v)) \, dx \, dt = \mathcal{O}(\varepsilon_\delta) \rightarrow 0, \end{aligned}$$

for  $\delta \searrow 0$ . And the  $\tau_\delta$ -term converges to

$$\begin{aligned} \int_{\Gamma_\delta} \zeta\tau_\delta \, dx \, dt &= \int_{\Gamma_\delta} \zeta \frac{1}{\delta} \eta_0(\rho, \phi) \frac{\delta f_\delta}{\delta\phi} \, dx \, dt \\ &= \int_{\Gamma_\delta} \zeta \frac{1}{\delta} \eta_0(\rho, \phi) \left( \frac{1}{\delta} \rho W'(\phi) - \delta \operatorname{div}(h(\rho)\nabla\phi) + \Psi_\phi(\rho, \phi) \right) \, dx \, dt \\ &= \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \int_{\Gamma_i} \zeta (\eta_0(R^0 + \delta R^1, \Phi^0 + \delta \Phi^1) + \mathcal{O}(\delta^2)) \\ &\quad \cdot \left\{ \frac{1}{\delta} (R^0 W'(\Phi^0) - \partial_r(h(R^0)\partial_r\Phi^0)) \right. \\ &\quad + (R^1 W'(\Phi^0) + R^0 W''(\Phi^0)\Phi^1 - \partial_r(h'(R^0)R^1\partial_r\Phi^0) \\ &\quad \left. - \partial_r(h(R^0)\partial_r\Phi^1) + \kappa_\Gamma \bullet \nu h(R^0)\partial_r\Phi^0 + \Psi_\phi(R^0, \Phi^0) \right\} \\ &\quad \cdot (1 + \mathcal{O}(\varepsilon_\delta)) \, d\mathbf{H}^{n-1}(y) \, dr \, dt. \end{aligned}$$

Thus, due to identity (5.1), we see that the  $1/\delta$ -term vanishes and that the expression for  $\delta \searrow 0$  converges to

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\Gamma_i} \zeta \int_{-\infty}^{+\infty} \eta_0(R^0, \Phi^0) (R^1 W'(\Phi^0) + R^0 W''(\Phi^0)\Phi^1 - \partial_r(h'(R^0)R^1\partial_r\Phi^0) \\ & \quad - \partial_r(h(R^0)\partial_r\Phi^1) + \kappa_\Gamma \bullet \nu h(R^0)\partial_r\Phi^0 + \Psi_\phi) \, dr \, d\mathbf{H}^{n-1}(y) \, dt, \end{aligned}$$

which is the result of Theorem 5.2.

*The two mass equations*

The strong version of the equation in 5.2 is

$$\begin{aligned} \partial_t \rho^2 + \operatorname{div}(\rho^2 v^2) &= 0 && \text{in } \Omega^2, \\ \tau &= \rho^2(v^2 - v_\Gamma) \bullet \nu_{\Omega^2} && \text{on } \Gamma. \end{aligned} \quad (5.5)$$

For  $\rho^1$  we obtain an analog with  $+\tau$  on the right hand side,

$$\partial_t(\rho^1 \mu_{\Omega^1}) + \operatorname{div}(\rho^1 v^1 \mu_{\Omega^1}) = \tau \mu_\Gamma, \tag{5.6}$$

or equivalently

$$\begin{aligned} \partial_t \rho^1 + \operatorname{div}(\rho^1 v^1) &= 0 && \text{in } \Omega^1, \\ \tau + \rho^1(v^1 - v_\Gamma) \bullet v_{\Omega^1} &= 0 && \text{on } \Gamma. \end{aligned} \tag{5.7}$$

From (5.5) and (5.7) we conclude that

$$\sum_{m=1}^2 \rho^m (v^m - v_\Gamma) \bullet v_{\Omega^m} = 0 \tag{5.8}$$

on  $\Gamma$  or

$$\rho^1(v^1 - v_\Gamma) \bullet v = \rho^2(v^2 - v_\Gamma) \bullet v \quad (v = v_{\Omega^1} = -v_{\Omega^2}).$$

This identity belongs to the conservation of the total mass, which in the distributional sense is the sum of the conservation of the individual masses and reads

$$\partial_t \left( \sum_m \rho^m \mu_{\Omega^m} \right) + \operatorname{div} \left( \sum_m \rho^m v^m \mu_{\Omega^m} \right) = 0. \tag{5.9}$$

The equation for the total mass of the phase field problem, which in the weak sense is

$$\int_{\Omega} (\partial_t \zeta \cdot \rho + \nabla \zeta \bullet (\rho v)) \, d\mathbf{L}^{n+1} = 0, \tag{5.10}$$

has one consequence in local coordinates, which occurs in a different  $\delta$ -term than in the Allen–Cahn equation.

**5.3. THEOREM** Assume (4.9) and (4.10). Then for  $(t, y) \in \Gamma$  we have in local coordinates

$$\partial_r (R^0 \Lambda^0) = 0 \quad \text{for all } r \in \mathbb{R}.$$

The boundary conditions for  $\Lambda^0 := (v_\Gamma - V^0) \bullet v$  are (without writing the arguments  $(t, y)$ )

$$\Lambda^0(-\infty) = \lambda^1 := (v_\Gamma - v^1) \bullet v, \quad \Lambda^0(+\infty) = \lambda^2 := (v_\Gamma - v^2) \bullet v.$$

For the proof we use local test functions  $\zeta = \xi$  with a  $C_0^\infty$ -function  $\xi$  around the free boundary. One infers from (5.10) that

$$\begin{aligned} 0 &= \int_{\Omega} \left( \left( \partial_t^\Gamma \xi - \frac{1}{\delta} v_\Gamma \bullet v \partial_r \xi \right) \rho + \left( \nabla^\Gamma \xi + \frac{1}{\delta} \partial_r \xi v \right) \bullet (\rho v) \right) \, d\mathbf{L}^{n+1} \\ &= \frac{1}{\delta} \int_{\mathbb{R}} \int_{-\varepsilon_\delta}^{+\varepsilon_\delta} \left\{ \int_{\Gamma_t} (-v_\Gamma \bullet v \partial_r \xi \cdot R^0 + \partial_r \xi v \bullet (R^0 V^0) + \mathcal{O}(\delta))(1 + \mathcal{O}(s)) \, d\mathbf{H}^{n-1}(y) \right\} \, ds \, dt \\ &= \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \left\{ \int_{\Gamma_t} (\partial_r \xi \cdot R^0 \cdot (V^0 - v_\Gamma) \bullet v + \mathcal{O}(\delta))(1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \right\} \, dr \, dt. \end{aligned}$$

Letting  $\delta \rightarrow 0$  one gets

$$0 = \int_{\mathbb{R}} \int_{-\infty}^{+\infty} \int_{\Gamma_t} \partial_r \xi \cdot R^0 \cdot (V^0 - v_\Gamma) \bullet v \, d\mathbf{H}^{n-1}(y) \, dr \, dt$$

and it follows that  $\partial_r (R^0 (V^0 - v_\Gamma) \bullet v) = 0$ .

## 6. Momentum conservation

The momentum conservation for  $v$  in system (3.3) is

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + \Pi_\delta) = \mathbf{f}_\delta,$$

where  $\mathbf{f}_\delta$  stands for an external term. In the distributional formulation this reads

$$\int_{\Omega} (\partial_t \zeta \bullet (\rho v) + \nabla \zeta : (\rho v \otimes v + \Pi_\delta) + \zeta \bullet \mathbf{f}_\delta) d\mathbf{L}^{n+1} = 0, \quad (6.1)$$

where we consider vector-valued test functions  $\zeta \in C_0^\infty(\Omega; \mathbb{R}^n)$ . We show the following results when  $\delta \rightarrow 0$ . The first result yields again, as for the mass conservation, the  $1/\delta$ -term at the boundary  $\Gamma$ .

6.1. THEOREM Assume (4.9) and (4.10). Then for  $(t, y) \in \Gamma$  we have in local coordinates

$$e_h(R^0) \frac{|\partial_r \Phi^0|^2}{2} = \tilde{a}(R^0, \Phi^0) \partial_r V^0 \bullet v \quad \text{for all } r \in \mathbb{R},$$

where

$$e_h := \rho h_{,\rho} + h \quad \text{and} \quad \tilde{a} := a_1 + \frac{n-1}{n} a_2 = a + \frac{1}{2} a_2. \quad (6.2)$$

This theorem is necessary to show the following result.

6.2. THEOREM Assume (4.9) and (4.10). Then as  $\delta \rightarrow 0$  the solution converges pointwise in the sense of distributions as follows:

$$\begin{aligned} \rho v \mathbf{L}^{n+1} &\rightarrow \sum_m \rho^m v^m \boldsymbol{\mu}_{\Omega^m}, \\ (\rho v \otimes v + \Pi_\delta) \mathbf{L}^{n+1} &\rightarrow \sum_m (\rho^m v^m \otimes v^m + \Pi^m) \boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma, \\ \mathbf{f}_\delta \mathbf{L}^{n+1} &\rightarrow \sum_m \mathbf{f}^m \boldsymbol{\mu}_{\Omega^m}. \end{aligned}$$

Here, with  $c^1 = 0$  in  $\Omega^1$  and  $c^2 = 1$  in  $\Omega^2$ ,

$$\begin{aligned} \Pi^m &:= p_\psi(\rho^m, c^m) \mathbb{I} - \mathbb{S}(\rho^m, c^m, (\nabla v^m)^S) \quad \text{in } \Omega^m, \\ \Pi^s &:= -\boldsymbol{\gamma} (\mathbb{I} - v \otimes v) \quad \text{on } \Gamma, \\ \boldsymbol{\gamma} &:= \int_{-\infty}^{\infty} (h(R^0) |\partial_r \Phi^0|^2 - a_2(R^0, \Phi^0) \partial_r V^0 \bullet v) dr \\ &= \int_{-\infty}^{\infty} \left( h(R^0) - \frac{a_2(R^0, \Phi^0)}{\tilde{a}(R^0, \Phi^0)} \frac{e_h(R^0)}{2} \right) |\partial_r \Phi^0|^2 dr \\ &= \int_{-\infty}^{\infty} \left( h(R_{M^0}^0) - \frac{a_2(R_{M^0}^0, \Phi_{M^0}^0)}{\tilde{a}(R_{M^0}^0, \Phi_{M^0}^0)} \frac{e_h(R_{M^0}^0)}{2} \right) |\partial_r \Phi_{M^0}^0|^2 dr \\ &= \boldsymbol{\gamma}(M^0), \quad M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2 \quad \text{on } \Gamma. \end{aligned} \quad (6.3)$$

Therefore the limit equation is

$$\partial_t \left( \sum_m \rho^m v^m \boldsymbol{\mu}_{\Omega^m} \right) + \operatorname{div} \left( \sum_m (\rho^m v^m \otimes v^m + \Pi^m) \boldsymbol{\mu}_{\Omega^m} + \Pi^s \boldsymbol{\mu}_\Gamma \right) = \sum_m \mathbf{f}^m \boldsymbol{\mu}_{\Omega^m}.$$

The equation (2.6) is satisfied with  $\mathbf{f}^s = 0$ . The strong formulation of this weak equation reads

$$\begin{aligned} \partial_t(\rho^m v^m) + \operatorname{div}(\rho^m v^m \otimes v^m + \Pi^m) &= \mathbf{f}^m && \text{in } \Omega^m, \quad m = 1, 2, \\ \operatorname{div}^\Gamma \Pi^s &= \sum_m \left( \rho^m (v^m - v_\Gamma) \bullet v_{\Omega^m} v^m + \Pi^m v_{\Omega^m} \right) && \text{on } \Gamma, \end{aligned}$$

and in addition the condition  $\Pi^s v = 0$  on  $\Gamma$  is satisfied by the above matrix  $\Pi^s$ . The function  $\boldsymbol{\gamma}$  is the surface tension, which is here given in terms of the local coordinates. We mention that  $\boldsymbol{\gamma}$  has no sign. The second representation of  $\boldsymbol{\gamma}$  follows from 6.1. The third representation is in advance of (7.4) in Section 7. It implies that  $\boldsymbol{\gamma}$  is a function of  $M^0$  alone.

For the proof of the theorems we consider again two classes of test functions, where the different meaning of these test functions is the same as for the mass.

#### Local test function $\zeta$

Consider test functions  $\zeta = \xi$  in the special case of local coordinates as in (4.1), that is,  $\zeta(t, x) = \xi(t, y, r)$  with compactly supported function  $(t, y, r) \mapsto \xi(t, y, r)$ , that is,  $\zeta$  has compact support in a small neighbourhood of  $\Gamma$  shrinking towards  $\Gamma$  when  $\delta \rightarrow 0$ . We conclude, using (5.3), that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \int_{\Omega_t} (\partial_t \zeta \bullet (\rho v) + \mathbf{D} \zeta : (\rho v \otimes v + \Pi_\delta) + \zeta \bullet \mathbf{f}_\delta) \, dx \, dt \\ &= \int_{\mathbb{R}} \int_{-\varepsilon_\delta}^{+\varepsilon_\delta} \int_{\Gamma_t} (\partial_t \zeta \bullet (\rho v) + \mathbf{D} \zeta : (\rho v \otimes v + \Pi_\delta) + \zeta \bullet \mathbf{f}_\delta) (1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \, ds \, dt \\ &= \int_{\mathbb{R}} \delta \int_{-r_\delta}^{+r_\delta} \int_{\Gamma_t} \left( \left( \partial_t^\Gamma \xi - \frac{1}{\delta} v_\Gamma \bullet v \partial_r \xi \right) \bullet (\rho v) + \left( \mathbf{D}^\Gamma \xi + \frac{1}{\delta} \partial_r \xi \otimes v \right) : (\rho v \otimes v + \Pi_\delta) + \xi \bullet \mathbf{f}_\delta \right) \\ &\quad \cdot (1 + \mathcal{O}(\delta)) \, d\mathbf{H}^{n-1}(y) \, dr \, dt. \end{aligned}$$

In a small neighbourhood of  $\Gamma$  we compute

$$\begin{aligned} \Pi_\delta &= \mathbb{P} - \mathbb{S} = \mathfrak{p}_\Psi \mathbb{I} + \frac{\delta}{2} \mathfrak{p}_h |\nabla \phi|^2 \mathbb{I} + \delta h \nabla \phi \otimes \nabla \phi - \left( a_1 - \frac{a_2}{n} \right) \operatorname{div} v \mathbb{I} - a_2 (\nabla v)^S \\ &= \frac{1}{\delta} \left( \frac{1}{2} \mathfrak{p}_h |\partial_r \Phi^0|^2 \mathbb{I} + h |\partial_r \Phi^0|^2 v \otimes v - \left( a_1 - \frac{a_2}{n} \right) v \bullet \partial_r V^0 \mathbb{I} - \frac{1}{2} a_2 (v \otimes \partial_r V^0 + \partial_r V^0 \otimes v) \right) \\ &\quad + \mathcal{O}(1), \end{aligned}$$

where the coefficients  $h$ ,  $a_1$ , and  $a_2$  have to be taken at the values  $(R^0, \Phi^0)$ . Since the main term is of order  $1/\delta$  we get for the above integral the value

$$\begin{aligned} &\int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \int_{\Gamma_t} (\partial_r \xi \otimes v) : \Pi_\delta \, d\mathbf{H}^{n-1}(y) \, dr \, dt + \mathcal{O}(1) \\ &= \int_{\mathbb{R}} \int_{-r_\delta}^{+r_\delta} \int_{\Gamma_t} \partial_r \xi \bullet (\Pi_\delta v) \, d\mathbf{H}^{n-1}(y) \, dr \, dt + \mathcal{O}(1). \end{aligned}$$

Now

$$\begin{aligned} \Pi_\delta v &= \frac{1}{\delta} \left( \frac{1}{2} e_h |\partial_r \Phi^0|^2 v - \left( a_1 + \frac{(n-2)a_2}{2n} \right) \partial_r V^0 \bullet v v - \frac{1}{2} a_2 \partial_r V^0 \right) + \mathcal{O}(1), \\ e_h &= \rho h \nu_\rho + h = \mathfrak{p}_h + 2h \quad (\mathfrak{p}_h \text{ defined in (3.9)}), \end{aligned}$$



and we finally end up with

$$\frac{1}{\delta} \int_{\mathbb{R}} \int_{-\infty}^{+\infty} \int_{\Gamma_t} \partial_r \xi \bullet \left( \frac{1}{2} e_h |\partial_r \Phi^0|^2 v - \left( a_1 + \frac{(n-2)a_2}{2n} \right) \partial_r V^0 \bullet v v - \frac{1}{2} a_2 \partial_r V^0 \right) d\mathbf{H}^{n-1}(y) dr dt + \mathcal{O}(1).$$

Therefore the  $1/\delta$ -term has to vanish, that is,

$$0 = \int_{\mathbb{R}} \int_{-\infty}^{+\infty} \int_{\Gamma_t} \partial_r \xi \bullet \left( \frac{1}{2} e_h |\partial_r \Phi^0|^2 v - \left( a_1 + \frac{(n-2)a_2}{2n} \right) \partial_r V^0 \bullet v v - \frac{1}{2} a_2 \partial_r V^0 \right) d\mathbf{H}^{n-1}(y) dr dt.$$

Therefore the term in brackets has to be constant in  $r$ . But since this term tends to 0 when  $r \rightarrow \pm\infty$ , the only possibility is

$$0 = \frac{1}{2} e_h |\partial_r \Phi^0|^2 v - \left( a_1 + \frac{(n-2)a_2}{2n} \right) \partial_r V^0 \bullet v v - \frac{1}{2} a_2 \partial_r V^0 \quad \text{for } r \in \mathbb{R}. \quad (6.4)$$

Multiplying this identity with tangential vectors  $\tau_k$  one finds, since  $a_2 > 0$ , that  $0 = \partial_r V^0 \bullet \tau_k = \partial_r (V^0 \bullet \tau_k)$ , and therefore one obtains the following

6.3. LEMMA We have

$$\partial_r V^0 \in \text{span}\{v\}, \quad (6.5)$$

$$V^0 \bullet \tau_k = \text{const} \quad \text{for tangential } \tau_k, k = 1, \dots, n-1, \quad (6.6)$$

where ‘‘const’’ means that this term is independent of  $r$ , but of course it depends on  $(t, y)$ .

Identity (6.4) with property (6.5) results in the pointwise equation

$$\frac{1}{2} e_h (R^0, \Phi^0) |\partial_r \Phi^0|^2 = \tilde{a} (R^0, \Phi^0) \partial_r V^0 \bullet v, \quad \tilde{a} := a_1 + \frac{(n-1)a_2}{n}, \quad (6.7)$$

which is 6.1. Besides this we have, using (6.5), the following representation for  $\Pi_\delta$ .

6.4. LEMMA For  $(t, y) \in \Gamma$  we obtain for all  $r \in \mathbb{R}$ , at  $(t, y + \delta r v(t, y))$ ,

$$\begin{aligned} \Pi_\delta &= \mathbb{P} - \mathbb{S} = p_\psi \mathbb{I} + \frac{\delta}{2} p_h |\nabla \phi|^2 \mathbb{I} + \delta h \nabla \phi \otimes \nabla \phi - \left( a_1 - \frac{a_2}{n} \right) \text{div } v \mathbb{I} - a_2 (\nabla v)^{\mathbb{S}} \\ &= \frac{1}{\delta} \left( \frac{1}{2} p_h |\partial_r \Phi^0|^2 \mathbb{I} + h |\partial_r \Phi^0|^2 v \otimes v - \left( a_1 - \frac{a_2}{n} \right) v \bullet \partial_r V^0 \mathbb{I} - a_2 v \bullet \partial_r V^0 v \otimes v \right) + \mathcal{O}(1) \end{aligned}$$

as  $\delta \rightarrow 0$ , where the coefficients have to be taken at appropriate arguments.

*Global test function  $\zeta$*

For the proof of Theorem 6.2 we compute, for test functions  $(t, x) \mapsto \alpha(t, x) \in \mathbb{R}^n$  and  $(t, x) \mapsto \beta(t, x) \in \mathbb{R}^{n \times n}$ ,

$$\begin{aligned} \int_{\Omega} (\alpha \bullet (\rho v) + \beta : (\rho v \otimes v + \Pi_\delta)) dx dt &= \sum_m \int_{\Omega_\delta^m} (\alpha \bullet (\rho v) + \beta : (\rho v \otimes v + \Pi_\delta)) dx dt \\ &\quad + \int_{\Gamma_\delta} (\alpha \bullet (\rho v) + \beta : (\rho v \otimes v)) dx dt + \int_{\Gamma_\delta} \beta : \Pi_\delta dx dt. \end{aligned}$$

The first line converges to the desired function, since in  $\Omega^m$ ,

$$\begin{aligned} \Pi_\delta &= p_\psi(\rho, \phi) \mathbb{I} - \mathbb{S}(\rho, \phi, (Dv)^S) + \mathcal{O}(\delta) \\ &\rightarrow p_\psi(\rho^m, c^m) \mathbb{I} - \mathbb{S}(\rho^m, c^m, (Dv^m)^S) \quad \text{as } \delta \rightarrow 0, \end{aligned}$$

where  $c^1 = 0$  in  $\Omega^1$  and  $c^2 = 1$  in  $\Omega^2$ . In the second line it is obvious, since the integrands are bounded functions, that

$$\int_{\Gamma_\delta} (\alpha \bullet (\rho v) + \beta : (\rho v \otimes v)) \, dx \, dt \rightarrow 0$$

as  $\delta \searrow 0$ . In the last summand on the right-hand side

$$\int_{\Gamma_\delta} \beta : \Pi_\delta \, dx \, dt = \int_{\mathbb{R}} \int_{\Gamma_t} \int_{-\varepsilon_\delta}^{+\varepsilon_\delta} (\beta : \Pi_\delta)(1 + \mathcal{O}(\varepsilon_\delta)) \, ds \, d\mathbf{H}^{n-1}(y) \, dt$$

we split the test function into its tangential part and its normal part by

$$\beta = \beta^\Gamma + (\beta v) \otimes v, \quad \beta^\Gamma := \sum_k (\beta \tau_k) \otimes \tau_k,$$

where  $\{\tau_1, \dots, \tau_{n-1}\}$  is an orthonormal basis of the tangent space to  $\Gamma_t$ , so that

$$\begin{aligned} \beta^\Gamma : \Pi_\delta &= \sum_k (\beta \tau_k) \bullet (\Pi_\delta \tau_k) = \beta : \Pi_\delta^\Gamma, \\ \beta : \Pi_\delta &= \beta^\Gamma : \Pi_\delta + (\beta v) \bullet (\Pi_\delta v) = \beta : (\Pi_\delta)^\Gamma + (\beta v) \bullet (\Pi_\delta v). \end{aligned}$$

We thus get the identity

$$\begin{aligned} \int_{\Gamma_\delta} \beta : \Pi_\delta \, dx \, dt &= \int_{\mathbb{R}} \int_{\Gamma_t} \delta \int_{-r_\delta}^{+r_\delta} (\beta : \Pi_\delta)(1 + \mathcal{O}(\varepsilon_\delta)) \, dr \, d\mathbf{H}^{n-1}(y) \, dt \\ &= \int_{\mathbb{R}} \int_{\Gamma_t} \int_{-r_\delta}^{+r_\delta} (\beta : (\delta \Pi_\delta)^\Gamma + (\beta v) \bullet (\delta \Pi_\delta v))(1 + \mathcal{O}(\varepsilon_\delta)) \, dr \, d\mathbf{H}^{n-1}(y) \, dt. \end{aligned}$$

We compute with the help of 6.4, as  $\delta \rightarrow 0$ ,

$$\begin{aligned} \delta \Pi_\delta v &\rightarrow \frac{1}{2} p_h |\partial_r \Phi^0|^2 v + h |\partial_r \Phi^0|^2 v - \left( a_1 - \frac{a_2}{n} \right) v \bullet \partial_r V^0 v - a_2 v \bullet \partial_r V^0 v \\ &= \left( \frac{1}{2} p_h + h \right) |\partial_r \Phi^0|^2 v - \left( a_1 + \frac{(n-1)a_2}{n} \right) \partial_r V^0 = 0 \end{aligned}$$

by 6.1, where the coefficients have to be evaluated at  $(R^0, \Phi^0)$ . This results in the fact that in the limit the normal component of the flux vanishes. We also derive from 6.4 that as  $\delta \rightarrow 0$ ,

$$\begin{aligned} (\delta \Pi_\delta)^\Gamma &= \delta \Pi_\delta - (\delta \Pi_\delta v) v = \delta \Pi_\delta (\mathbb{I} - v \otimes v) \\ &\rightarrow \left( \frac{1}{2} p_h |\partial_r \Phi^0|^2 - \left( a_1 - \frac{a_2}{n} \right) v \bullet \partial_r V^0 \right) (\mathbb{I} - v \otimes v) \\ &= \left( \frac{1}{2} p_h(R^0) |\partial_r \Phi^0|^2 - \left( a_1(R^0, \Phi^0) - \frac{a_2(R^0, \Phi^0)}{n} \right) v \bullet \partial_r V^0 \right) (\mathbb{I} - v \otimes v), \end{aligned}$$

and therefore

$$\int_{\Gamma_\delta} \beta : \Pi_\delta \, dx \, dt \rightarrow \int_{\mathbb{R}} \int_{\Gamma_t} \left( \int_{-\infty}^{+\infty} \left( \frac{1}{2} p_h(R^0) |\partial_r \Phi^0|^2 - (a_1(R^0, \Phi^0) - \frac{a_2(R^0, \Phi^0)}{n}) v \bullet \partial_r V^0 \right) dr \right) \beta : (\mathbb{I} - \nu \otimes \nu) \, d\mathbf{H}^{n-1}(y) \, dt.$$

Hence Theorem 6.2 is satisfied with  $\gamma$  being the integral over

$$\begin{aligned} & -\frac{1}{2} p_h(R^0) |\partial_r \Phi^0|^2 + \left( a_1(R^0, \Phi^0) - \frac{a_2(R^0, \Phi^0)}{n} \right) v \bullet \partial_r V^0 \\ & = h(R^0) |\partial_r \Phi^0|^2 - a_2(R^0, \Phi^0) \partial_r V^0 \bullet v = \left( h(R^0) - \frac{a_2(R^0, \Phi^0)}{\tilde{a}(R^0, \Phi^0)} \frac{e_h(R^0)}{2} \right) |\partial_r \Phi^0|^2, \end{aligned}$$

where these identities result from 6.1. If one writes the last expression in the variables  $(R^0, \Phi^0) = (R_{M^0}^0, \Phi_{M^0}^0)$  (see the system (7.4) in the next section), obtaining

$$\left( h(R_{M^0}^0) - \frac{a_2(R_{M^0}^0, \Phi_{M^0}^0)}{\tilde{a}(R_{M^0}^0, \Phi_{M^0}^0)} \frac{e_h(R_{M^0}^0)}{2} \right) |\partial_r \Phi_{M^0}^0|^2,$$

one gets the dependence

$$\gamma \equiv \gamma(M^0), \quad M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2 \quad \text{on } \Gamma.$$

### 7. Conditions on the interface

Besides the equations which are true on  $\Gamma$  from the distributional equations, there are additional boundary conditions, which are necessary for the full description of the sharp interface limit. In this section we show that these additional boundary conditions consist of constitutive equations which are a consequence of the equations in the inner variables proved in 5.1, 5.3, and 6.1 together with 6.3.

First it follows from 5.3 that there is an  $M^0 : \Gamma \rightarrow \mathbb{R}$  such that

$$M^0 = R^0 \Lambda^0 \quad \text{for all } r \in \mathbb{R}, \tag{7.1}$$

that is,  $M^0$  is a function of  $(t, y)$  only. Using the boundary conditions at  $r = \pm\infty$  one concludes

$$M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2, \quad \text{where } \lambda^m = (v_\Gamma - v^m) \bullet v \quad (v = v_{\Omega^1}), \tag{7.2}$$

a condition which is already contained in the distributional formulation of the total mass  $\rho^1 \mu_{\Omega^1} + \rho^2 \mu_{\Omega^2}$ . Therefore the following argument will be modulo the identity (7.1).

Next we exploit the inner momentum equation (6.4). It follows from (6.3) that  $(\partial_r V^0)_{\text{tan}} = 0$ , and therefore on  $\Gamma$ ,

$$v_{\text{tan}}^1 = v_{\text{tan}}^2 \quad (v_{\text{tan}}^m = v^m - v^m \bullet \nu \nu). \tag{7.3}$$

This is a very common equation, and says that the tangential part of the velocity is the same on the two sides.

Now we come to the normal component of the inner momentum equation, which is given by Theorem 6.1, and the inner version of the Allen–Cahn equation, which is Theorem 5.1,

$$\begin{aligned} R^0 W'(\Phi^0) - \partial_r(h(R^0)\partial_r\Phi^0) &= 0, \\ e_h(R^0)\frac{|\partial_r\Phi^0|^2}{2} - \tilde{a}(R^0, \Phi^0)\partial_r V^0 \bullet \nu &= 0. \end{aligned}$$

We repeat the argument of [W3]. From (7.1) we compute (remember that we impose  $R^0 > 0$ ) that  $(v_\Gamma - V^0) \bullet \nu = \Lambda^0 = M^0/R^0$ , hence

$$-\partial_r V^0 \bullet \nu = \partial_r \Lambda^0 = M^0 \partial_r \left( \frac{1}{R^0} \right).$$

Using this we rearrange 6.1 to obtain

$$\begin{aligned} R^0 W'(\Phi^0) - \partial_r(h(R^0)\partial_r\Phi^0) &= 0, \\ e_h(R^0)\frac{|\partial_r\Phi^0|^2}{2} + \tilde{a}(R^0, \Phi^0)M^0\partial_r\left(\frac{1}{R^0}\right) &= 0. \end{aligned}$$

These are two equations for the two variables  $R^0$  and  $\Phi^0$  having the value  $M^0$  as single parameter. The boundary conditions are

$$\begin{aligned} \Phi^0(-\infty) &= 0, & \Phi^0(+\infty) &= 1, \\ R^0(-\infty) &= \rho^1, & R^0(+\infty) &= \rho^2. \end{aligned}$$

We replace the Dirichlet data for  $R^0$  by Neumann data, which is possible since by the second differential equation we have  $\partial_r R^0 = \mathcal{O}(|\partial_r \Phi^0|^2)$ . Thus we obtain a system which depends only on  $M^0$ . We therefore replace in the differential system  $M^0$  by a general  $M$ , and denote by  $(R_M^0, \Phi_M^0)$  the solution satisfying

$$\begin{aligned} R_M^0 W'(\Phi_M^0) - \partial_r(h(R_M^0)\partial_r\Phi_M^0) &= 0, \\ e_h(R_M^0)\frac{|\partial_r\Phi_M^0|^2}{2} + \tilde{a}(R_M^0, \Phi_M^0)M\partial_r\left(\frac{1}{R_M^0}\right) &= 0, \\ \Phi_M^0(-\infty) &= 0, & \Phi_M^0(+\infty) &= 1, \\ \partial_r R_M^0(-\infty) &= 0, & \partial_r R_M^0(+\infty) &= 0. \end{aligned} \tag{7.4}$$

The existence of  $(R_M^0, \Phi_M^0)$  is shown in [W4], for  $h \equiv 1$  and for certain functions  $W$  which have two local minima, but with different heights, so that the mass transfer goes in a definite direction.

We then have  $(R^0, \Phi^0) = (R_{M^0}^0, \Phi_{M^0}^0)$ , and we conclude that

$$\rho^1 = R_{M^0}^0(-\infty) = R_{\rho^1\lambda^1}^0(-\infty), \quad \rho^2 = R_{M^0}^0(+\infty) = R_{\rho^2\lambda^2}^0(-\infty).$$

Thus we have two additional boundary conditions

$$\rho^1 = g_1(\rho^1\lambda^1), \quad \rho^2 = g_2(\rho^2\lambda^2) \tag{7.5}$$

with  $g_1(M) := R_M^0(-\infty)$  and  $g_2(M) := R_M^0(+\infty)$ . We mention that in the case where  $a_1, a_2$ , and  $h$  are homogeneous, it is shown in [W3] that the system above can be transformed into a system independent of  $M$ .

With (7.3) and (7.5) the additional conditions are complete. This can be seen in the following way. If the solution is known at one phase, say  $\Omega^1$ , with the velocity  $v^1$  pointing outwards from the domain, by which we mean that  $\lambda^1 < 0$ , that is,  $M^0 < 0$ , then from  $\rho^1$  and  $v^1$  the movement of the interface can be determined by  $\rho^1 = g_1(M^0)$  with  $M^0 = \rho^1 \lambda^1$  and  $v_\Gamma \bullet v = \lambda^1 + v^1 \bullet v$ . And from this one computes  $\rho^2 = g_2(M^0)$ ,  $\lambda^2 = M^0 / \rho^2$ , and  $v^2 \bullet v = v_\Gamma \bullet v - \lambda^2$ . Hence  $(\rho^2, v^2)$  is known for the hyperbolic mass equation in  $\Omega^2$ , for which  $v^2$  points inwards, and for the parabolic momentum equation in  $\Omega^2$ .

We mention that there are also equivalent forms of the conditions (7.3) and (7.5). Writing Theorem 6.1 as

$$\frac{e_h(R^0)}{\tilde{a}(R^0, \Phi^0)} \frac{|\partial_r \Phi^0|^2}{2} = \partial_r V^0 \bullet v$$

and integrating this over  $]-\infty, \infty[$  one obtains

$$\int_{-\infty}^{+\infty} \frac{e_h(R^0)}{\tilde{a}(R^0, \Phi^0)} \frac{|\partial_r \Phi^0|^2}{2} dr = [V^0]_{r=-\infty}^{r=+\infty} \bullet v = v^2 \bullet v - v^1 \bullet v.$$

If we denote the left hand integral by  $\omega$ ,

$$\omega := \int_{-\infty}^{+\infty} \frac{e_h(R^0)}{\tilde{a}(R^0, \Phi^0)} \frac{|\partial_r \Phi^0|^2}{2} dr = \int_{-\infty}^{+\infty} \frac{e_h(R_{M^0}^0)}{\tilde{a}(R_{M^0}^0, \Phi_{M^0}^0)} \frac{|\partial_r \Phi_{M^0}^0|^2}{2} dr, \quad (7.6)$$

we get the condition

$$v^2 \bullet v - v^1 \bullet v = \omega, \quad \omega \equiv \omega(M^0), \quad M^0 = \rho^1 \lambda^1 = \rho^2 \lambda^2 \quad \text{on } \Gamma. \quad (7.7)$$

And considering  $M$  as parametrization of a one-dimensional curve for  $(\rho^1, \rho^2)$  in  $\mathbb{R}^2$ , that is, the curve  $M \mapsto (g_1(M), g_2(M))$ , one is led to a function  $G$  satisfying

$$G(\rho^1, \rho^2) = 0. \quad (7.8)$$

That (7.7) and (7.8) are equivalent to the two constraints in (7.5) is not discussed here.

## 8. Uniqueness

It often occurs that in the limit equations some terms show up which are described by the inner expansion, in particular by higher order terms. The question is whether these terms are uniquely determined.

Here we have to deal with terms containing  $R^1, \Phi^1$ , and  $V^1$ . These quantities satisfy a linear problem, for which the Fredholm alternative can be applied. So we have to show that the term in the limit equation is independent of an element in the null space. Now  $(R^1, \Phi^1, V^1)$  have the form

$$\begin{aligned} (R^1, \Phi^1, V^1) &= (R_h^1, \Phi_h^1, V_h^1) + (R_p^1, \Phi_p^1, V_p^1), \\ (R_h^1, \Phi_h^1, V_h^1) &:= c(\partial_r R^0, \partial_r \Phi^0, \partial_r V^0), \end{aligned}$$

where  $c \in \mathbb{R}$ , and is the solution of the first order approximation of the system of inner differential equations (4.5) to (4.7) (see [W3, Section 8]). The kernel consists of the span of  $(\partial_r R^0, \partial_r \Phi^0, \partial_r V^0)$ . Here  $(R_p^1, \Phi_p^1, V_p^1)$  is the particular solution of the first order boundary value problem, which is unique. We have to show that  $\tau$  is independent of the homogeneous solution  $(R_h^1, \Phi_h^1, V_h^1)$ . This follows by 5.2, and

$$\begin{aligned} R_h^1 W'(\Phi^0) + R^0 W''(\Phi^0) \Phi_h^1 - \partial_r(h'(R^0) R_h^1 \partial_r \Phi^0) - \partial_r(h(R^0) \partial_r \Phi_h^1) \\ = c(\partial_r R^0 W'(\Phi^0) + R^0 W''(\Phi^0) \partial_r \Phi^0 - \partial_r(h'(R^0) \partial_r R^0 \partial_r \Phi^0) - \partial_r(h(R^0) \partial_r^2 \Phi^0)) \\ = c \partial_r(R^0 W'(\Phi^0) - \partial_r(h(R^0) \partial_r \Phi^0)) = 0 \end{aligned}$$

using 5.1, since the differential equation in 5.1 is an autonomous equation. Therefore we can write

$$\begin{aligned} \tau := \kappa_\Gamma \bullet v \int_{-\infty}^{+\infty} \eta_0(R^0, \Phi^0) h(R^0) \partial_r \Phi^0 \, dr \\ + \int_{-\infty}^{+\infty} \eta_0(R^0, \Phi^0) (\Psi_\phi(R^0, \Phi^0) + R_p^1 W'(\Phi^0) + R^0 W''(\Phi^0) \Phi_p^1 \\ - \partial_r(h'(R^0) R_p^1 \partial_r \Phi^0) - \partial_r(h(R^0) \partial_r \Phi_p^1)) \, dr \end{aligned}$$

This section clarifies the  $(R^1, \Phi^1)$  dependence of  $\tau$ , but separately it can be shown that  $\tau = M^0$ .

**9. Conclusion**

In this paper we show by an example how a phase field model converges to a problem with sharp interface. We think that the procedure in this paper works also for general phase field models.

We have chosen distributional equations, since they are the simplest possible way to describe the problem, in particular if it involves surfaces. Also from the physical point of view the weak formulation reflects more the nature, that is, phrasing it in terms of theoretical physics, the space-time divergence  $(\partial_r, \text{div})$  is the central ingredient of continuum physics. The formulation with test volumina (see e.g. [JR], [K], [S]) is equivalent to the formulation with test functions. If surfaces are involved, test functions are more flexible, because they avoid the complex representation which is necessary if the test volume intersects the surface.

Besides the differential equations which determine the system, it is also of interest how the free energy in the phase field model converges to a limit function. For the problem in this paper this will be studied in the forthcoming paper [AW]. There we also look at the special case where the minima of the double-well function are the same, that is,  $W(0) = W(1)$ . It follows from the equi-partition that in this case  $e_h = 0$ , that is,  $h(\rho) = \text{const}/\rho$ .

We mention that besides the distributional formulation (1.3) there are problems with two fluids and boundary conditions on  $\Gamma$  other than (1.4), for example the following:

$$\left. \begin{aligned} v^1 &= v^2, \\ (v^1 - v_\Gamma) \bullet v &= 0, \quad (v^2 - v_\Gamma) \bullet v = 0, \end{aligned} \right\} \tag{9.1}$$

at  $\Gamma$ . Of course, these equations, if they are the limit of diffuse problems, come from a different deduction than the one in this paper. The problem (1.3) with (9.1) is treated mathematically in [Den1]–[Den3].

## 10. Appendix

(A1) The sequence  $(T_m)_{m \in \mathbb{N}}$  of distributions converges “pointwise” to a distribution  $T$  if

$$\langle \zeta, T_m \rangle \rightarrow \langle \zeta, T \rangle \quad \text{as } m \rightarrow \infty \text{ for all } \zeta \text{ in } C_0^\infty(\Omega).$$

Usually there does not exist a special notation, if one considers measures  $\nu$  as distribution, say  $[\nu]$ . Therefore we say that the measures  $\nu_m$  converge “pointwise in the sense of distributions” to a measure  $\nu$  if  $[\nu_m]$  converge pointwise to  $[\nu]$  in the above sense.

(A2) The interface  $\Gamma \subset \mathbb{R} \times \mathbb{R}^n$  is called *timelike* if for all  $(t, x) \in \Gamma$  the  $n$ -dimensional tangent space  $T_{(t,x)}\Gamma$  is not equal to  $\{0\} \times \mathbb{R}^n$ .

(A3) Let  $(A_\delta)_{\delta>0}$  and  $A$  be compact subsets of  $\mathbb{R} \times \mathbb{R}^n$ . Then  $(A_\delta)_{\delta>0}$  converges to  $A$  as  $\delta \rightarrow 0$  with respect to the Hausdorff distance if for all  $\varepsilon > 0$ ,

$$(A_\delta \subset B_\varepsilon(A) \text{ and } A \subset B_\varepsilon(A_\delta)) \quad \text{for } \delta \text{ small enough.}$$

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