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The polygon-circle paradox and convergence in thin plate theory

BY N. W. MURRAY†
Technische Universität, München

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Abstract. The solution for a simply supported many-sided polygonal plate does not agree with that for the corresponding circular plate. This paper describes the earlier work of Rao and Rajaiah on polygonal plates and then explains why best convergence of series solutions occurs when the boundary conditions are defined as

$$w = \nabla^2 w = 0.$$

Notation

D = operator in original problems,
$\tilde{D}$ = operator adjoint to D,
F_1, F_2 = boundary terms,
K = parameter which can be given arbitrary values,
s = number of sides of the regular polygon,
t = number of terms used in a truncated series,
u = eigenfunction (see equation (7)),
v = solution of adjoint problem,
w = lateral deflection of plate,
x, y = Cartesian coordinates,
$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$,
λ = eigenvalue (see equation (7)),
ν = Poisson's ratio.

The following suffices are also used:

m indicates mth eigenfunction or eigenvalue,
n indicates directional normal to plate boundary,
nn indicates double differentiation in direction normal to plate boundary,
t indicates direction along plate boundary,
tt indicates double differentiation in direction along plate boundary.

1. *Introduction.* The polygon-circle paradox in thin plate theory is concerned with the fact that the solutions for simply supported circular and regular polygonal plates do not agree when the number of sides of the polygon becomes large, even though at first sight one might expect them to do so.

† Present address: Department of Civil Engineering, Monash University, Clayton 3168, Victoria, Australia.

An extensive study of this paradox has been made by Rao and Rajaiah (1), (2), (3). They conclude that

(i) the problems of simply supported circular and regular polygonal plates are different because for the first case the boundary conditions are dependent upon Poisson's ratio ν and for the second case they are independent of it. This is because

$$M_n = -D[w_{nn} + \nu w_{tt}] \tag{1}$$

and

$$M_t = -D[w_{tt} + \nu w_{nn}]. \tag{2}$$

For the circular boundary $\quad w = M_n = 0 \quad \text{but} \quad M_t \neq 0 \quad$ (3)

so the boundary conditions can be written as

$$w = 0 \quad \text{and} \quad w_{nn} + \nu w_{tt} = 0. \tag{4}$$

For the regular polygonal boundary the edge is straight. Thus

$$w = M_n = M_t = 0, \tag{5}$$

so the boundary conditions can be written in the form

$$w = w_{nn} = w_{tt} = 0. \tag{6a}$$

These conditions may be coalesced into the following alternative form

$$w = 0 \quad \text{and} \quad w_{nn} + K w_{tt} = 0 \tag{6b}$$

where K may take any value.

(ii) At first sight this arbitrariness in K seems strange because, for a straight boundary, w_{tt} is identically zero and therefore one can argue that $w = w_{nn} = 0$ is all that is necessary. However, Rao and Rajaiah obtained series solutions for polygonal plates for different values of K and made an interesting discovery. For a given K value and a small number of terms t in the series, it was found that as the number of sides s was increased the solution apparently tended towards that for a circular plate whose Poisson's ratio was K. It was as if the truncated series solution could not distinguish between the circular plate problem (with $\nu = K$ in equation (4)) and the polygonal plate problem. Yet the theoretical analysis just described indicates that the solutions should have been different. When K was given the value 0·3 the error in the central deflection was 35 %, and when it was made zero, this error was 66 % so it is seen that serious errors can arise.

When s was made constant and t was increased it was found that the solution of a polygonal plate slowly converged away from the circular plate towards the exact solution. The rate of convergence was extremely slow as K decreased towards zero and the parameter s was given larger and larger values. The erroneous conclusions drawn by some authors from this behaviour of the series solutions has been discussed by Rao and Rajaiah. However, when the case $K = 1$ was studied it was found that convergence was very rapid even when s was very large. The authors did not attempt a mathematical explanation of this phenomenon but it is now offered in the next section.

2. *An explanation of why $K = 1$ gives best convergence.* A problem is self-adjoint if

(i) the original and the adjoint operators, D and $\tilde{D}$, respectively, coincide and

(ii) the original and the adjoint boundary conditions coincide.

When the problem is self-adjoint the solutions u_m of the equation

$$Du_m = \lambda_m u_m \tag{7}$$

and which satisfy the given boundary conditions have the following properties:

(i) u_m form a set of orthogonal functions in the given domain,

(ii) the eigenvalues λ_m are real,

(iii) the set of functions u_m is complete.

This first property means that a solution of the original problem can be expanded in terms of its eigenfunctions and the convergence is then most rapid.

It remains to demonstrate that for the flat plate problem:

(i) the original and adjoint operators coincide,

(ii) the original and adjoint boundary conditions coincide when $w_{nn}+w_{tt} = 0$, i.e., $\nabla^2 w = 0$.

This is easily done by using a 'process of liberation' described by Lanczos(4). Let v be the solution of the adjoint problem and Green's identity is then written as

$$vDw - w\tilde{D}v = \frac{\partial}{\partial x}[F_1(v,w)] + \frac{\partial}{\partial y}[F_2(v,w)], \tag{8}$$

where the operator of the original problem is written in the following form to preserve symmetry with respect to x and y,

$$D = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^2 \partial x^2} + \frac{\partial^4}{\partial y^4}, \tag{9}$$

and where F_1 and F_2 are functions which form a boundary term whose value should be zero.

To illustrate the method let us consider only one term on the left-hand side of equation (8)

$$\begin{aligned} v\frac{\partial^4 w}{\partial x^4} &= \frac{\partial}{\partial x}\left[v\frac{\partial^3 w}{\partial x^3}\right] - \frac{\partial v}{\partial x}\frac{\partial^3 w}{\partial x^3} \\ &= \frac{\partial}{\partial x}\left[v\frac{\partial^3 w}{\partial x^3} - \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2}\right] + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial x^2} \\ &= \frac{\partial}{\partial x}\left[v\frac{\partial^3 w}{\partial x^3} - \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial x}\right] - \frac{\partial^3 v}{\partial x^3}\frac{\partial w}{\partial x} \\ &= \frac{\partial}{\partial x}\left[v\frac{\partial^3 w}{\partial x^3} - \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial x} - \frac{\partial^3 v}{\partial x^3}w\right] + w\frac{\partial v^4}{\partial x^4}. \end{aligned} \tag{10}$$

The other terms in the expression vDw may be treated in a similar manner. After taking the 'liberated' terms such as $w\,\partial^4 v/\partial x^4$ (see equation (10)) to the left-hand side and summing, we obtain the particular case of Green's identity, viz.

$$\begin{aligned} &v\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^2 \partial x^2} + \frac{\partial^4 w}{\partial y^4}\right] - w\left[\frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial x^2\partial y^2} + \frac{\partial^4 v}{\partial y^2\partial x^2} + \frac{\partial^4 v}{\partial y^4}\right] \\ &= \frac{\partial}{\partial x}\left[v\frac{\partial^3 w}{\partial x^3} - \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial x} - \frac{\partial^3 v}{\partial x^3}w + v\frac{\partial^3 w}{\partial x\,\partial y^2} - \frac{\partial v}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial x} - \frac{\partial^3 v}{\partial x\,\partial y^2}w\right] \\ &\quad + \frac{\partial}{\partial y}\left[v\frac{\partial^3 w}{\partial y^3} - \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial y^2}\frac{\partial w}{\partial y} - \frac{\partial^3 v}{\partial y^3}w + v\frac{\partial^3 w}{\partial y\,\partial x^2} - \frac{\partial v}{\partial y}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial w}{\partial y} - \frac{\partial^3 v}{\partial y\,\partial x^2}w\right]. \end{aligned} \tag{11}$$

The terms in square brackets on the right can each be made to vanish when the boundary conditions

$$\nabla^2 w = 0 \quad \text{and} \quad \nabla^2 v = 0 \tag{12}$$

are chosen for the original and adjoint problems, respectively. We also note that

$$D = \tilde{D}, \tag{13}$$

so for the boundary condition

$$\nabla^2 w = 0$$

the problem is self-adjoint. This is equivalent to choosing $K = 1$ in the equation (6*b*) and it explains the rapid convergence noted by Rao and Rajaiah for that value of K. It is worth noticing that built-in boundary conditions, viz.,

$$w = v = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

also result in a self-adjoint problem of a similar nature. Elastically restrained plates do not directly yield a self-adjoint problem. This can also be seen on physical grounds because energy is transferred across the domain boundary during loading.

Finally, it is worth pointing out that the difficulties described here arise not only in plate theory but they can occur in many other boundary-value problems. They can often be overcome by a careful examination of boundary conditions.

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