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State-dependent Medium Access Control for Resource-aware Networked Control Systems

Mohammadhossein Mamduhi

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Vorsitzender: Prof. Dr. Samarjit Chakraborty

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1. Prof. Dr.-Ing. Sandra Hirche
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Abstract

Networked Control Systems (NCS) are referred to distributed systems composed of a multitude of integrated elements such as sensors, actuators and controllers which often exploit a resource-constrained communication medium for exchanging data. On the one hand, the consideration of the shared constrained communication resources urges the design of efficient Medium Access Controllers (MAC) in order to arbitrate the channel access allocation procedure. On the other hand, rapid and day-by-day development of large-scale networked systems along with everlasting increase of information volume exchanged between distributed entities call for advent of new architectures to meet the real-time requirements of the control tasks in an operating networked system. Therefore, the introduction of communication networks for transmitting the sheer amount of data between spatially distributed entities in networked systems spurs the researchers to re-consider the traditional time-triggered control and scheduling schemes, and periodic sampling, and to look for more advanced techniques in order to achieve enhanced quality of control with more efficient consumption of limited resources (e.g., bandwidth and energy). However, control over shared communication resources often imposes several design challenges due to channel capacity limitations, network congestion, collisions, delays and data loss. These network-induced phenomena have often negative effects on the overall control performance and can even lead to instability.

Over the last decade, event-based sampling has shown to be an excellent design technique for networked systems under communication constraints outperforming the periodic rules in terms of resource consumption and leading to enhanced control performance. In the event-triggered paradigm, in contrast to the time-triggered sampling, the resources are used only when it is necessary, as specified through the event triggers. However, the available results in the literature often address certain aspects of the event-triggered control and scheduling synthesis for networked systems with resource constraints, under unrealistically simplified assumptions. As a result, a fully developed systematic design methodology is not yet presented, that guarantees the required quality of control, ensures efficient resource consumption, assures a robust characteristics against external uncertainties and communication unreliabilities, and fulfills the application expectations.

The aim of the present dissertation is to develop event-based control and scheduling strategies which dynamically take into account the real-time conditions and requirements of the distributed entities that are using the shared communication resources. Two major contributions of this dissertation are described as first, design of a scheduling protocol for a network of multiple sub-systems that are physically isolated and are coupled only through the shared communication network, and second the extension of control and scheduling architectures for a network of physically interconnected systems that are additionally coupled via a shared communication network. Within the first contribution, event-based prioritizing scheduling architectures are introduced, wherein the event triggers are state dependent. Comprehensive characteristic analyses and control/scheduling law synthesis are performed in both centralized and decentralized fashions. Then, the concept of state-dependent scheduling of scarce resources is implemented on physically interconnected networked systems, for a less general class of physical topologies. In both major parts, stability properties are thoroughly studied as the necessary aspect of an event-triggered networked system, as well as perfor-

mance analyses. Robustness of the proposed architectures is examined with respect to both systems' uncertainties and communication channel non-idealities. Finally, decentralization procedures are suggested to emphasize that the presented event-based techniques can be technically implemented on real networked systems.

Zusammenfassung

Als vernetzte Regelungssysteme (NCS) bezeichnet man verteilte Systeme, die sich aus einer Vielzahl an integrierten Komponenten, wie Sensoren, Aktuatoren und Reglern zusammensetzen, welche Daten über ein digitales Kommunikationsnetz austauschen. Die rasche Entwicklung immer größerer vernetzter Systeme zusammen mit dem fortwährenden Anwachsen des Informationsvolumens, das zwischen verteilten Einheiten ausgetauscht wird, verlangt nach der Erforschung innovativer Kommunikationsarchitekturen, um den Echtzeitanforderungen der Regelungsprozesse innerhalb des vernetzten Systems gerecht zu werden. Der Einsatz digitaler Kommunikationsnetze für NCS hat Wissenschaftler dazu angeregt neue Datenverarbeitungstechniken zu entwickeln, die über gewöhnliche zeitgesteuerte Mechanismen mit periodischer Abtastung hinausgehen und für eine effizientere Ausnutzung der Kommunikationsressourcen, wie der Datenbandbreite oder des Energievorrats, sorgen. Allerdings müssen beim Entwurf von Regelungsmethoden über ein mehrfach genutztes Kommunikationsnetz auf Grund der beschränkten Datenbandbreite, der Kommunikationsverzögerung, und des möglichen Datenverlusts auch neue Herausforderungen bewältigt werden. Diese kommunikations-bedingten Effekte haben oft einen negativen Einfluss auf die Regelgüte und können sogar dazu beitragen das Regelungssystem zu destabilisieren.

In den letzten 10 Jahren hat sich die ereignisbasierte Regelung als eine ausgezeichnete Entwurfsmethodik für vernetzte Systeme mit Kommunikationsbeschränkungen bewiesen. Ein Hauptmerkmal der ereignisbasierten Abtastung im Kontrast zu zeitgesteuerten Mechanismen besteht darin, dass Ressourcen nur dann beansprucht werden, wenn sie wirklich benötigt werden. Allerdings gehen bestehende Resultate in der Literatur zumeist von einer Reihe an vereinfachenden Annahmen an das vernetzte System aus. Aus diesem Grund bedarf es weiterer Anstrengungen für eine systematische Entwurfsmethodik, welche Garantien an die Regelgüte bei einer effizienten Nutzung von Ressourcen gibt und gleichzeitig robuste Merkmale gegenüber Kommunikationsunsicherheiten aufweist.

Ziel der vorliegenden Dissertation ist es, ereignisbasierte Regelungs- und Ressourcenzugriffsverfahren für verteilte Systeme, deren Daten über ein gemeinsames Kommunikationsnetz ausgetauscht werden, zu entwickeln, welche die Echtzeitbedingungen des vernetzten Systems dynamisch miteinbeziehen. Die beiden Hauptbeiträge dieser Dissertation sind wie folgt beschrieben. Als erstes werden Schedulingverfahren für mehrfache isolierte Regelungssysteme, dessen Rückkopplungen über ein geteiltes Kommunikationsnetz laufen, entworfen. Als zweites werden die Ergebnisse auf verkoppelte Regelungsarchitekturen erweitert, in denen neben der Überlagerung durch das gemeinsame Kommunikationsnetz auch Interaktionen auf Grund der physikalischen Kopplungen auftreten. In Bezug auf den ersten Beitrag werden ereignisbasierte priorisierende Zugriffsverfahren eingeführt, wobei der Ereignisgenerator vom Netzzustand abhängt. Deren Entwurf wird sowohl auf einen zentralisierten wie auch auf einen dezentralen Ansatz zurückgeführt. Danach wird das Konzept des zustandsabhängigen Ress-ourcenzugriffes auf gekoppelte dynamische Systeme für bestimmte Klassen von Kopplungsto-pologien ausgedehnt. In beiden Hauptbeiträgen werden eine detaillierte Stabilitätsanalyse sowie eine Untersuchung der Regelgüte des ereignisbasierten vernetzten Systems durchgeführt. Zudem wird die Robustheit der vorliegenden Architektur sowohl auf systembezogene wie auch auf kommunikationsbezogene Unsicherheiten untersucht. Abschließend werden dezentrale Ansätze für die ereignisbasierten Techniken erarbeitet, die

hervorheben, dass sich die vorgeschlagenen Verfahren in bestehenden Kommunikationsarchitekturen implementieren lassen.

Contents

Notations

Abbreviations

NCS	networked control system
LTI	linear time-invariant
i.i.d.	independent and identically distributed
LMSS	Lyapunov mean square stability
LSP	Lyapunov stability in probability
TOD	try-once-discard
TDMA	time-division multiple access
CSMA	carrier-sense multiple access
DAG	directed acyclic graph
PHR	positive Harris recurrent

Subscripts and Superscripts

Vectors and scalars

x_k	subscript $k \in \mathbb{Z}^+ \cup 0$ denotes time index
x^i	superscript $i \in \mathbb{N}$ denotes the corresponding control loop
$x_{[k,l]}$	sequence of variables x over time interval $[k,l]$, i.e. $\{x_k, x_{k+1}, \ldots, x_l\}$
$x_{>0}$	only positive values of x

Matrices

A_i	subscript $i \in \mathbb{N}$ denotes the corresponding control loop
A^n	superscript $n \in \mathbb{R}$ denotes the matrix power
$A^\top$	superscript $\top$ denotes the "transpose" operator

Frequently Used Variables

u_k	control signal

x_k	system state
w_k	system noise
e_k	error state
z_k	received signal at controller
k	time-step
A	system matrix
B	input matrix
C	output matrix
δ_k	scheduling variable
γ_k	dropout indicator
L	control gain
Δ	drift operator
J_k	per-time-step cost function
ν_k	waiting time
λ	error threshold
Q	weight matrix
η	communication cost
N	number of control loops, time horizon
Ω	probability sample space
$\mathscr{A}$	σ-algebra on Ω
P	probability measure on $\mathscr{A}$
$\mathscr{N}(\mu, W)$	Gaussian distribution with mean μ and covariance matrix W

Operators

x^T	superscript T denotes the "transpose" operator
$\mathrm{tr}(A)$	trace of square matrix A
I_n	identity matrix of size n
$\mathsf{E}[\cdot]$	expectation operator
$\mathsf{E}[\cdot\vert\cdot]$	conditional expectation operator
$\Vert x\Vert_l$	$l \in \mathbb{N}$ denotes l-norm of vector x
$\Vert x\Vert_Q$	$Q \in \mathbb{R}^{n\times n}$ denotes weighted 2-norm operator, i.e. $\Vert x\Vert_Q = \left(x^\mathsf{T} Q x\right)^{\frac{1}{2}}$
$\Vert A\Vert_2$	2-norm of matrix A
$\lceil\cdot\rceil$	ceiling operator
$\prod(\cdot)$	product operator
$\cup$	set union operator
$\wedge$	logical *and*-operator

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Introduction

Emerging networked control systems (NCS) such as smart industrial production lines, smart energy grids, and autonomous vehicular systems are all characterized by feedback control loops that are closed over a communication channel. The communication infrastructure is shared among those feedback control loops and potentially other applications. This poses novel challenges for the communication and control system design to support such networked control systems with stringent real-time requirements. Traditional design paradigms address the control and communication designs separately. This leads to a design framework within which no awareness of real-time control conditions are taken into account in governing communication resource consumption, and control strategies are also synthesized without accounting for the changing conditions and opportunities of the network infrastructure. A strictly separate design is known to lead to high conservatism and therewith to a low quality of control as well as to low efficiency and high cost in the context of resource usage. Therefore, in order to efficiently fulfill the tight control performance requirements, resource-aware control and control-aware communication strategies are assumed as the future of an NCS design. Recently, event-triggered control has been seen as a design framework which has the required potential of becoming the effective mechanism for such an inter-layer awareness, and therefore facilitates synthesizing efficient resource-aware control and control-aware communication strategies for large scale and networked systems. In the following, we first discuss the principles of event-triggered sampling in shared resource networked control systems, and then a short discussion about event-based control and scheduling problem in multiple-loop networked system is presented. This chapter is then followed by the related literature review and outline of the present dissertation.

1.1 Shared Resource NCS with Event-triggered Sampling

Since the early 2000s, there has been an increased interest to networked control approaches which take into account resource limitations in the communication network resulting in novel scheduling schemes for information transmission. Indeed, the efficient usage of available communication resources is a substantial requirement for networked control systems which must be addressed in parallel to the control design. Since the seminal work [2], many results including e.g., [3–9] have shown that event-triggered control outperforms time-triggered schemes by far; they achieve the same control performance despite the fact that they consume significantly less resources. In contrast to the time-triggering schemes that sample the systems by a fixed temporal schedule, event-based strategies call for a sample whenever a pre-specified event is triggered. Typically, the events depend on the state of the control system and can be even further adjusted depending on the scarcity of communication resources and channel situation [10]. This is another significant advantage over time-triggered schemes which are computationally very expensive to reconfigure and in general not scalable towards many sub-systems.

A different approach to the determination of the event-triggered scheme formulates its design as an optimal stochastic control problem [8, 9, 11–13]. These works show that event-triggered sampling outperforms time-triggered schemes also in a multi-loop networked system when using MAC schemes with idealized collision avoidance mechanisms. In fact, it is shown in [9, 10, 12] that threshold policies are optimal and that the certainty equivalence controller is optimal when the resource constraints are considered. Furthermore, there are significant functional advantages of event-triggered control policies over time-triggered ones such as scalability with respect to the number of control loops and adaptivity to structural changes in the system. However, the communication models in those works are idealized, assuming no collisions, data loss, and transmission delay as some of them foreseen in this dissertation. Moreover, sub-systems are only coupled by the common resource constraint and these works do not consider couplings in either the physical interaction or coupling in the cost function.

In the context of medium access strategies for networked control, prioritization has been introduced in [14] with the try-once-discard (TOD) protocol. Based on current measurement data, the protocol dynamically prioritizes transmissions by choosing the measurement with the largest discrepancy between its actual value and its estimate at the controller; the other measurements are discarded. TOD is a centralized MAC scheme, i.e. not scalable, and does not cope well with packet loss. In addition, stability criteria for such approaches often rely on deriving bounds on maximum inter-transmission times which is not applicable for intrinsically stochastic contention-based protocols such as carrier sense multiple access (CSMA). Stochastic protocols [15, 16] are shown to cope well with communication unreliability and can be implemented in decentralized fashion, i.e. they are scalable. However, so far the available designs are rather heuristic and primarily focus on stability rather than performance. Stochastic priority-based scheduling schemes for multiple control loops over a shared communication network are receiving more attention recently as they lead to more flexible resource allocation mechanisms capable of coping with typical challenges of realistic communication networks such as packet loss, collisions, and delay.

1.2 Multiple-loop Control and Scheduling Problem

A very relevant scenario for many application domains is multiple independent feedback control loops sharing the resource-limited communication infrastructure for closing the loop between the sensors and the controllers. Sharing the resource-limited communication here means that only a limited number of control loops can be closed at a time, thus a medium access control (MAC) mechanism decides which packets are forwarded over the communication network. It is well-known that current network scheduling solutions contribute to high end-to-end latency which significantly deteriorates the control performance and may even destabilize the overall networked control system. This calls for the advent of more intelligent design methodologies to fulfill the real-time control requirements. For the network schedules, this means to move from the traditional throughput oriented optimization of the network resources towards a real-time oriented communication to support networked control systems. For the control system, on the other hand, it means to become aware of the varying conditions and opportunities of the network infrastructure. At the moment, there exists no systematic approach for the joint design of the control and communication protocol under realistic settings. Therefore, two important questions in this context are as follows:

1. How should medium access be organized in a multiple-loop scenario such that the overall quality of control is maximized?
2. How should the controls be designed while taking current network conditions into account?

In this dissertation, we will mostly address the first question by proposing event-based channel access mechanisms which take the real-time control state into account. We take the scenario of an NCS consisting of multiple heterogeneous linear time-invariant control loops accessing one shared communication channel. We mainly study stability properties, performance efficiency and robustness of event-triggered scheduling architecture for such a networked system. Implementability of the proposed design is also comprehensively investigated in this dissertation.

The majority of the results have studied the event-triggered sampling for single loop networked control systems. Multiple-loop systems in which multiple control loops are coupled through shared communication resources have attained little attention so far. Exceptions can be found in e.g., [10, 11, 17, 18]. The synthesis of event-triggered control in single loop systems takes commonly an emulation-based approach, enabling us to choose a stabilizing continuous-time controller *a priori*. The control inputs are chosen according to this control law at triggering times and are either kept constant in between triggering times or adjusted only based on local model parameters and information history. By using concepts from Lyapunov theory and input-to-state stability, the event-trigger is chosen to be a threshold function of the measurements such that stability of the event-triggered control system is guaranteed [3, 4, 6, 7, 19, 20]. However, the mentioned approaches do not translate to the multi-loop shared resource case as targeted in this dissertation. In the multiple-loop scenario, the shared resource needs to be displayed as a sample-path constraint which complicates the analysis and design of event-triggered mechanisms considerably more compared to the single-loop NCSs.

1.3 State of the Art

There exist numerous research achievements related to the presented topics in this dissertation. On the one hand, many attempts have focused on developing the event-based sampling of control systems for more than a decade. On the other hand, there exists a vast amount of analytical and evaluative results on networked control systems tackling various design and analysis aspects. Applying event-triggered sampling on networked control systems and design of event-based control and scheduling laws have also been an attractive field of research for the researchers. However, most of the available results on event-triggered sampling of NCSs are either applicable on narrow classes of networked systems, or are derived under idealized assumptions which may not be feasible in real applications. In addition, a systematic quality-of-control-oriented design procedure for event-triggered networked control systems is missing in the literature. In this dissertation, we build a design framework for event-based control of networked control systems in a broader sense in terms of modeling, robustness and implementability, to enhance control performance. We will compare our obtained results with well-known developed approaches to evaluate performance enhancement under our design methodology. In what follows, we review some of the most important achievements in event-triggered control and control of networked systems and discuss the advantages and shortcomings of those works.

More than a decade ago, event-triggered sampling was first introduced in [2] as an effective mechanism to reduce the rate of data stream in a control loop. Afterwards, many attempts have been made to demonstrate the efficacy of event-triggered sampling compared to the traditional way of sampling the control systems with fixed temporal duration. It has been shown through several works that event-triggered sampling may generally lead to substantial reduction of computational processing without degrading the control performance [3,4,6,21–28]. With more in-depth analysis, it has also been demonstrated that event-triggered sampling can be applied successfully on networked systems wherein the control signals are transmitted through a wired or wireless communication channel [5, 17, 25, 29–35].

Different areas for the design of event-triggered control for large-scale systems, such as networked systems, multi-agent systems, and distributed systems, are considered. Among those, we will focus in this dissertation on the design of event-based control mechanisms for networked control systems under limited communication resources. Depending on how the overall networked system is characterized, many results exist in the literature incorporating the limited resources in the design of event-based control and scheduling laws. Hence, we provide a short overview of the available results in this area from each characterization perspective. The available works on event-based control synthesis under scarce communication resources, can broadly be categorized in one of the following groups:

1. single-loop vs. multiple-loop networked systems
2. linear control-loop vs. nonlinear control-loop networked systems
3. deterministic approaches vs. stochastic approaches

Most of the available works consider the event-trigger synthesis for single-loop networked systems under limited communication resources [23, 36–46]. In the mentioned works, the main focus is on appropriate event generation to reduce the communication cost and yet achieving the required control performance. Stability guarantees, as an essential property, are also investigated widely in those works. Compared to the single-loop scenario, multiple-loop networked control systems under communication constraints have attained little attention in the literature where the notable exceptions can be found in [10, 13, 17, 20, 29, 33, 34, 47–51]. The presented results in these works show that the event-triggered sampling is not always the best performing strategy for multiple-station networked systems, hence it should be employed with care. It is suggested by [17, 29] that event-triggering is considerably beneficial compared to the time-triggered sampling in CSMA based random channel arbitration. In contrast, it is shown in [47, 48] that within unslotted and slotted ALOHA, event-triggered sampling leads to performance degradation. The same authors later clarified in [13] that indeed event-triggering is not always beneficiary compared to the time-triggered sampling, and it leads to performance enhancement only within certain communication protocols. This indicates that the performance of event-triggered sampling is firmly dependent on the type of communication protocol. The results presented in [10, 33] suggest that the event-based approach can be effectively employed as a threshold policy to govern the channel access in limited resource networked systems. The results are then extended by deriving an optimal price-based scheduling mechanism for channel access arbitration, but at the expense of having a centralized network manager to optimally determine the communication prices. As a main shortcoming of the mentioned works, most of them have considered multiple scalar sub-systems in an NCS while the results available for sub-systems with higher dimensions are even less addressed. In this dissertation, we will focus on event-based sampling of multiple-loop networked control systems under communication resource limitations where the control loops may have higher dimensions.

Event-based sampling has been studied for networked systems comprising of either linear or nonlinear controlled sub-systems. The well-known results on event-based control and scheduling law synthesis, considering nonlinear networked control systems, are presented in [5, 15, 23, 52–57]. Different aspects of such networked systems are studied in these works, e.g., event generating for control and scheduling units, stability analysis, implementation of the event-triggered laws, and robustness analysis. In [23], the event-triggers are determined, according to the network-induced relative error, employing the input-to-state stability criterion for nonlinear networked systems. The authors of [54, 55] show $\mathscr{L}_p$ stability holds for nonlinear NCSs with bounded model disturbances operating under an event-triggered deterministic scheduling policy but for sufficiently small MATI. Lyapunov uniform global exponential (UGES) stability for nonlinear NCSs with exogenous disturbances and stochastic model of dropouts is addressed in [15]. A general framework incorporating communication constraints, varying transmission intervals and varying delays is presented in [53], where stability of nonlinear networked system, in terms of bounded maximum allowable transmission intervals, is guaranteed through Lyapunov-based methods. In [57] event-triggering rules for nonlinear networked control systems are synthesized for classes of NCSs governed by uniformly globally asymptotically stable (UGAS) protocols. A decentralized event-triggered approach for general distributed networked systems over broadcast channels in proposed in [5] where a station broadcasts its state information to the neighbors only when its local error exceeds a pre-specified threshold. Due to difficulty of performance analysis and optimization techniques in a nonlinear networked scenario, the mentioned works mostly studied stability properties of the networked system under event-based control and communication mechanisms. The available results in the area of networked systems with linear systems, however, have additionally considered more design aspects of the event-based strategies by precisely investigating overall network performance and implementability within standard communication protocols, see e.g., [12–14, 16, 20, 35, 46, 58–65]. Extensive stability analysis for networked systems with linear time invariant sub-systems are provided by employing either deterministic or stochastic concepts of stability, depending on the modeling paradigm [14, 16, 20, 62–67]. In addition, control performance under various scheduling policies in linear networked systems is investigated [46, 68–70]. The notion of periodic event-triggered control (PETC) is employed in [46] in order to obtain sub-optimal bounds on a quadratic cost function. In [68, 69], performance bounds are obtained for linear stochastic systems with quadratic value functions. Geometric bounds are also derived for infinite, yet countable, Markov chains in [71]. In the linear case, the event-triggers are often generated to maximize some performance metrics. This is not a trivial problem since the communication limitations need to be carefully considered, and they often appear as the constraint of the optimization problem, making the optimal synthesis of event-triggers generally infeasible to be analytically solved. Some results have shown that under simplifying assumptions the OP can be decomposed and therefore locally solved through Lagrange approach, e.g., [72], although a generic design of event-triggers in optimal fashion is not yet fully developed.

Similar to the classic control theory wherein the control systems can be characterized in either stochastic or deterministic fashions, networked systems over communication channels can also be categorized as stochastic or deterministic. Stochasticity in a networked system may appear specifically in several ways; via stochastic parameters in the sub-system's models, via stochastic exogenous inputs, and via stochastic communication channels. Standard deterministic control and scheduling mechanisms are often not very well compatible with stochastic systems due to the statistical nature of the latter system's behavior. Therefore, stochastic control and scheduling laws are developed in order to fulfill the requirements of non-deterministic systems. Stochastic stability notions usually consider the dynamic behavior of the system under consideration in a probabilistic sense. This means that a specific realization of the system behavior is not necessarily demonstrating the expected behavior. Consequently, in a stochastic networked system, the event triggers may also be characterized stochastically, e.g., they can be noise-driven. Hence, in terms of analysis and design, stochastic NCSs should be studied carefully by employing relevant mathematical toolboxes. Both stochastic and deterministic frameworks have been widely studied with the latter in more attention in the event-based scenario. Focusing on the results on event-based scheduling design for networked systems over shared communication resources, a well-known event-based deterministic scheduling law is proposed in [14] where the limited channel bandwidth is granted to the systems with the largest real-time estimation errors and the rest of transmission requests are discarded. This approach, which is capable of prioritized coordination of channel access, is called Maximum Error First (MEF) or Try Once Discard (TOD). Although TOD is a centralized scheduling approach which makes it not quite applicable for large-scale NCSs and multihop wireless networks [56], it is a well-established performance-efficient methodology to manage the channel access especially for small and medium size networks.

Stability of NCSs under the deterministic TOD protocol is shown in terms of global exponential stability assuming that the inter-transmission intervals, so called Maximal Allowable Transmission Interval (MATI), is sufficiently small [14, 16, 54, 55]. In [14], global exponential stability of multiple-packet transmission NCSs consisting of linear time-invariant systems employing TOD scheduling policy is addressed under three assumptions, i.e. the channel is error-free, no observation noise exists and no model disturbance deteriorates the system dynamics. The authors of [54, 55] show $\mathscr{L}_p$ stability holds for deterministic nonlinear NCSs with bounded model disturbances operating under TOD but for sufficiently small MATI. On the contrary, MATI does not apply to stochastic schemes with random access protocols as the intervals between consecutive transmissions cannot usually be upper bounded uniformly with probability one, which calls for new stability concepts for stochastic NCSs. Time-varying transmission delays and packet dropouts are considered in [16] and mean square stability is proved for LTI NCSs orchestrated by a proposed quadratic scheduling protocol of which TOD is shown to be a special realization. Lyapunov uniform global exponential stability (UGES) for nonlinear NCSs with exogenous disturbances and stochastic model of dropouts is addressed in [15]. The disadvantage of TOD and its deterministic variations is that they are centralized approaches and therefore not scalable. Moreover, these approaches are prone to stochastic noise and can cope with collisions only when predefined priority order is given, and hence they are not convenient for practice, e.g. wireless multihop networks. Stochastic scheduling methods on the other hand are developed and shown to be better matches for stochastic networked systems and randomized channels, especially for large scale networks [11, 49, 64, 73–77]. Moreover, non-deterministic scheduling mechanisms are demonstrated to be better options for contentious protocols, and they are easier to be implemented.

Robust analysis of control and scheduling designs for both stochastic and deterministic models of NCSs over non-ideal communication channels are also performed [66, 67, 78–80]. In [80,81], probabilistic models of dropout are used. In [82] the packet dropouts are modeled according to a Bernoulli process, where worst-case bounds for the number of consecutive dropouts are derived. In [83] the authors seek performance guarantees under networked induced delays and varying data packet rates in a probabilistic setting by modeling a packet dropping network as an erasure channel. Stability of time-invariant systems is studied in [66] for constant delays, while protocols with time-varying delays are discussed in [67]. A scheduling design considering uncertain delays in the communication channel is presented in [78].

In this dissertation, we are interested in developing design guidelines for event-triggered scheduling for stochastic networked systems with linear time-invariant control loops over limited communication resources. The main goal is to build an event-based scheduling design framework capable of managing scarce communication resources aiming at quality of control improvement, while it guarantees the essential properties such as stability and addresses robustness. Applicability of the proposed approaches is also addressed in this dissertation by introducing implementation procedures which take into account the limitations and challenges of real networked systems.

1.4 Outline and Contributions

In this dissertation, we address the event-based control and scheduling synthesis for multiple-loops stochastic networked control systems over shared communication resources. We start in Chapter 2 by having a precise look at structural properties of such networked systems and provide the modeling framework which will be used throughout this dissertation. Then, in Chapter 3, we introduce our first event-based scheduling architecture by proposing a pure probabilistic medium access control mechanism in centralized fashion. The proposed approach is then extended to a bi-character design in Chapter 4 where a deterministic threshold policy is embedded in the MAC architecture as a mechanism to determine the urgency of transmissions. We will discuss that the combined protocol is indeed superior to the pure probabilistic one in terms of control performance. Robustness of the proposed scheduling architecture is then investigated in Chapter 5 by considering both system uncertainties and channel imperfections. Chapter 6 presents the decentralization procedure to implement the proposed event-based scheduling mechanism in distributed fashion. There we assume to have access only to noisy sensor measurements, and further analyze the event-based scheduling over multi-hop channels. The similar approach is then shown to be applicable for a special class of interconnected networked control systems in Chapter 7, wherein the interconnection topology is supposed to follow directed acyclic graphs. The dissertation is finalized by concluding discussions and outlooks on some open research challenges. The major contributions within each chapter are outlined in more details as follows:

Chapter 2

In this chapter, we present the structural analysis and modeling of networked control systems (NCS) comprised of multitude of stochastic controlled sub-systems which share a communication medium subject to resource limitations. We introduce essential preliminary analyses and evaluations to pursue further discussions and derivations in the forthcoming chapters, where we address different aspects of a proper design for NCSs. First, we introduce the structural properties of NCSs with LTI stochastic control sub-systems in discrete framework. Separation property, as an important result enabling us to decompose the control and scheduling design procedure is then discussed. Independent synthesis of control strategies, in form of LQG controllers, as a well-known result employed in several research, e.g., [35, 59, 87], is in addition addressed. Then a brief but necessary introduction about the different scheduling scenarios is presented. This section is followed by derivation of the overall NCS dynamics under a general event-based scheduling design. It is shown that the network-induced error state is directly controlled by the scheduling variable, where these variables are determined by the event-based scheduling law. It is discussed that the error state can be modeled as a time-homogeneous, aperiodic, and ψ-irreducible Markov process. The results discussed in this chapter are partly from the author's own work in [84, 85], and partly from the available literature, most notably [33, 58, 59, 86].

Chapter 3

In this chapter, we address the efficient usage of scarce communication resources in networked control systems by introducing a novel stochastic scheduling scheme with capability of prioritizing the channel access based on current status of local control systems. More precisely, an error-dependent scheduler decides which transmission requests have the priority to be awarded the channel access. The priorities are assigned to each sub-system in a probabilistic fashion according to an error-dependent measure, such that higher chances of transmission is assigned to the sub-systems with greater error norms. The other transmission requests are blocked when the resource limit is reached. Due to the presence of stochastic noise and the coupling between sub-systems, the networked system under consideration requires novel methods to analyze its asymptotic behavior. Dynamics of the overall NCS can be described by the behavior of the so-called "network state", which can be modeled by a homogeneous Markov chain evolving in the uncountable state-space $\mathbb{R}^n$. We prove that the described NCS, arbitrated by the proposed stochastic scheduling protocol, is stochastically stable, according to the notion of "f-ergodicity". Furthermore, we derive analytic performance bounds for an average cost function comprised of an error-dependent quadratic term plus an incurred cost of communication. The results in this chapter are mainly from the author's own works in [84, 85, 94].

Chapter 4

In this chapter we develop the previously introduced probabilistic scheduling design in 3 to a bi-character policy comprised of both deterministic and probabilistic attributes. We discuss that the modified scheduler arbitrates the scarce communication resources more efficiently and observe how the deterministic feature of the scheduler may lead to enhance the overall control performance in comparison with pure probabilistic approaches. Given local error thresholds for each control loop, transmissions associated with sub-systems with lower error values than the pre-specified thresholds are deterministically discarded in order to make the channel less congested for those sub-systems with greater error values. In case the channel capacity is still sparse, then the scheduler allocates the communication channel probabilistically among all those eligible sub-systems based on a dynamic prioritized measure. Since the local errors are driven by the stochastic Gaussian noise process, transmissions occur randomly in an event-based fashion. This bi-character scheduling rule offers major advantages in comparison with purely deterministic or purely probabilistic architectures. In comparison with deterministic policies, the probabilistic nature of our protocol facilitates an approximate decentralized implementation. This will be addressed later in Chapter 6. In addition, by lowering the probability of channel access for sub-systems with lower local errors, the channel is made less congested for the sub-systems which are in more urgent status for transmission and consequently performance enhancement is attained. We prove that stochastic stability of the described NCS in Chapter 2 is preserved under the modified scheduler in terms of f-ergodicity of the overall network-induced error. Additionally, we derive uniform analytical performance bounds for an average cost function comprised of a quadratic error term and transmission penalty. The performance margins for the bi-character scheduling design are then analytically evaluated. Furthermore, it is concluded that the performance index is indeed a convex function of the error thresholds which consequently facilitates the search for sub-optimal uniform performance bounds. The results presented in this chapter are from the author's own works in [99, 100].

Chapter 5

In this chapter, we evaluate robustness properties of our introduced architecture. Control over shared communication resources often imposes various imperfections, such as capacity limitation, congestion, collisions, time delays and data loss, that impair the control performance and can even lead to instability. In practice, these challenges need to be carefully considered when designing control and scheduling strategies. In this chapter, we mainly take into account non-idealities in the communication channel to evaluate the robustness of the proposed design approach. More precisely, we address the control and scheduling designs for a multiple-loop stochastic NCS wherein the local sub-systems exchange their sensory data over a shared capacity-limited communication channel subject to data loss. Additionally, in a centralized design, it is often assumed that the controlling units have constantly access to global information from all distributed entities. This however is an ideal assumption because the sheer amount of information exchange cannot always be processed in timely manner, which in turn gives rise to delays. Therefore, scheduling approaches requiring complete information in every sampling times might not be feasible in practice due to the additional traffic imposed by the scheduler to coordinate among different control loops. A desired architecture should be capable of allocating resources efficiently even provided with partial information from local entities. We discuss that our proposed scheduling architecture is capable of effectively assigning the priorities in the absence of up-to-date error information from the networked entities. The presented theoretical and numerical results in this chapter are mainly from the author's own research in [99, 107].

Chapter 6

In this chapter, we discuss the implementation of the bi-character error-dependent scheduling approach in decentralized fashion for shared-resource multiple loop stochastic networked control systems. The results in this chapter correspond to three main contributions; first we assume that the local schedulers have access to the true error state of their corresponding local sub-systems. Then the results are extended to the case where the local event-triggers are defined assuming that only noisy sensor measurements are accessible and not the true error values. Finally, we address the problem of event-based medium access control for multi-link networks. In all mentioned contributions, the decentralized scheduling designs are assumed to combine deterministic and probabilistic attributes to efficiently allocate the limited communication resource among the control loops in an event-based fashion. Given local error thresholds, each control loop determines whether to compete for the channel access in a deterministic manner. Note that this process is performed locally within each control loop because the triggering condition is checked locally. Therefore, implementation of the probabilistic scheduling process in decentralized fashion is indeed discussed in this chapter. In the third section of this chapter, we assume that the channel access is determined based on probabilistic slotted ALOHA protocol, i.e. no state-dependent prioritization is considered to select the sub-systems for channel access. So, the approach is known to be decentralized. We demonstrate stochastic stability of the described NCSs under decentralized MAC architectures in terms of Lyapunov Stability in Probability (LSP). The presented discussions and results in this chapter are mainly from the author's work in [114, 115] and a collaborative work in [101].

Chapter 7

In this chapter, we extend our control and scheduling architectures to be applicable for a wider range of NCS models by considering the physical interconnections among the individual controlled sub-systems. Considering that the dynamics of a controlled sub-system is affected by the dynamic behavior of other sub-systems through physical links adds an extra coupling dimension to networked control systems, compared to the scenario wherein the coupling only appears in the communication channel. Due to complexity of analysis for very general interconnection models including bi-directional and cyclic interconnections, we identify a class of interconnected networked systems within which the state-dependent prioritizing scheduling policy can be implemented. We consider an interconnected NCS consisting of multiple heterogeneous stochastic LTI sub-systems where the physical interconnection is modeled by a directed acyclic graph (DAG). The sub-systems are controlled by a networked controller through a shared communication channel, which depending on the type of communication channel and information structure can be designed ranging from decentralized to distributed and to fully centralized control laws. In order to cope well with the expected transmission traffic, we employ a bi-character deterministic-probabilistic scheduling mechanism which dynamically assigns access priorities to each sub-system at each time-step according to an error-dependent priority measure. The sub-systems which are granted channel access then transmit their state information through the communication network. We prove stability of such interconnected networked systems under the proposed scheduling law in terms of f-ergodicity of overall network-induced error. Most of the presented analyses, results and discussions in this chapter are from the author's work in [119].

2

Structural Analysis of Multiple-Loop NCSs

In this chapter, we present the structural analysis and modeling of networked control systems (NCS) comprised of multitude of stochastic controlled sub-systems which share a communication medium subject to resource limitations. In an NCS where the individual sub-systems are assumed to be physically isolated, the coupling between them occurs solely within the shared communication channel. Design of networked control systems is often considered as the joint synthesis of control strategies and a network manager such that the quality of control, in overall sense, is maximized under resource limitations. However, a global solution which satisfies the desired operational requirements for a general model of networked control systems does not exist, to the best of our knowledge. Here, we look at NCSs consisting of multiple stochastic linear time-invariant sub-systems modeled in discrete time. Considering certain type of control units for individual sub-systems, we conclude an easier way of designing the network manager independent of control strategies. In the face of scarce communication resources, we need to design a network manager which requires the communication channel to transmit their data to designated stations, in order to arbitrate the channel access among all dynamic entities. We discuss how event-based strategies are beneficial for efficient arbitration of channel access.

This chapter is structured as follows. In Section 2.1 the NCS model which is largely employed in this dissertation is introduced and the structural properties are studied. We essentially look at the communication medium as the point of coupling between the local subsystems, and discuss the effects of channel limitations on the design of NCSs. We consider an ideal type of channel in this chapter, where the only constraint is in terms of limited capacity. Then, it is shown in Section 2.2 that, linearity of feedback control laws facilitates to design control and scheduling strategies independently. This property makes the control-scheduling co-design less complicated and furthermore enables one to employ vast varieties of control and scheduling architectures based on the real-time requirements of the NCS. Having the control and scheduling designs separated, we focus on the optimal control rules for each local sub-system in Section 2.3 in form of the linear-quadratic Gaussian (LQG) optimal control. Afterwards, we address various scheduling architectures for multiple-loop NCSs focusing on event-based strategies in Section 2.4.

2.1 Modeling and Structural Analysis of Stochastic Networked Control Systems

Throughout this dissertation, if not otherwise stated, we consider a networked control system composed of N heterogeneous controlled sub-systems which are exchanging sensory data over a shared communication network, as depicted in Figure 7.1. Each individual control loop consists of a linear time-invariant (LTI) stochastic plant $\mathscr{P}_i$, a linear stabilizing feedback controller $\mathscr{C}_i$, as well as sensors $\mathscr{S}_i$ which measure some system state. The communication channel is assumed to be subject to the capacity constraint such that not all N sub-systems can use the communication channel simultaneously. This leads to having several sub-systems which inevitably operate in open-loop at every time instance. To manage the channel access among the networked entities, an event-based scheduler decides when a pre-defined event is triggered for a certain sub-system to schedule that sub-subsystem for channel usage.

As all analysis in this dissertation is performed in discrete time frame, the LTI stochastic plant $\mathscr{P}^i$ is modeled in the state-space by the following linear stochastic difference equation:

$$x_{k+1}^i = A_i x_k^i + B_i u_k^i + w_k^i, \tag{2.1}$$

where $x_k^i \in \mathbb{R}^{n_i}$ describes the state vector of sub-system i with the vector size n_i, and the stochastic system noise $w_k^i \in \mathbb{R}^{n_i}$ is i.i.d. with $w_k^i \sim \mathscr{N}(0, W_i)$ at every time-step k, where $W_i > 0$ and is finite. For every subsystem $i \in \{1, \ldots, N\}$, the matrices $A_i \in \mathbb{R}^{n_i \times n_i}$ denote the system matrices associated with sub-system i and $B_i \in \mathbb{R}^{n_i \times m_i}$ describe the corresponding input matrices. The initial states x_0^i for all $i \in \{1, \ldots, N\}$ are allowed to be independent random variables chosen from arbitrary probability distributions with finite means and bounded variances. In addition, the random variables x_0^i and w_k^i for all operational time-steps k are assumed to be statistically independent. The initial state vector $x_0 = [x_0^{1^\mathsf{T}}, \ldots, x_0^{N^\mathsf{T}}]^\mathsf{T}$, together with the process noise sequence w_k, generate the probability space $(\Omega, \mathscr{A}, \mathsf{P})$, where Ω is the set of all possible outcomes, $\mathscr{A}$ is a σ-algebra of events where the corresponding probabilities are determined by the function P. It should be noted that the process noise w^i is randomly chosen based on the unsupported Gaussian distribution $\mathscr{N}(0, W_i)$, which instead results in an uncountable probability space wherein the set Ω contains uncountable outcomes. In addition, having unsupported distribution for the process noise w^i at every operational time-step eventuates that the state of a system i may take very large values at some time-steps even in the presence of a stabilizing feedback controller. Although the distribution of process noise is zero-mean, its realizations cover the range $(-\infty, \infty)$.

Every sub-system $i \in \{1, \ldots, N\}$ requires to transmit its sensory data to the corresponding controller $\mathscr{C}_i$ at each sample time k through the shared communication channel. In this

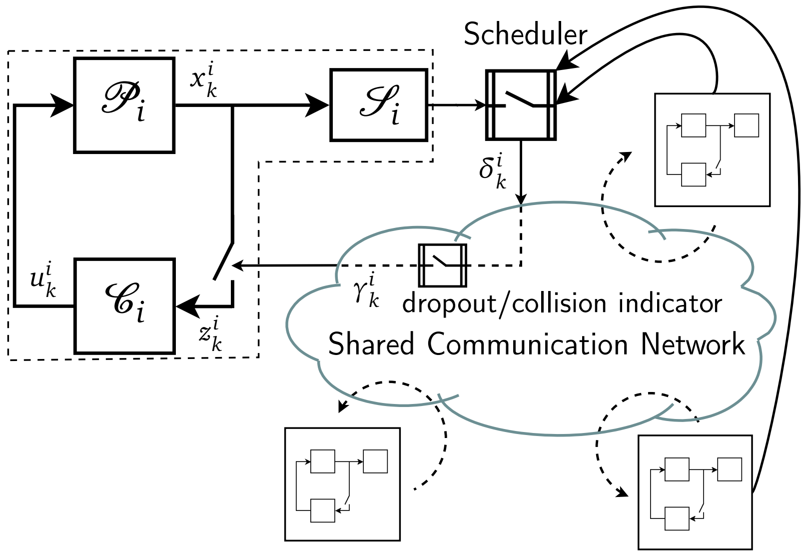

Figure 2.1: Schematic of a multiple loop NCS with shared communication channel.

dissertation, we focus on the type of NCSs within which only the links between the plants and their corresponding controllers are passing through shared communication networks. In other words, we assume that the control signals u_k^i are directly connected to their corresponding plants $\mathscr{P}_i$ and no shared channel exists between them. This is schematically shown in Figure 7.1. Furthermore, if not otherwise stated, it is assumed that if a transmission associated with sub-system i occurs at a time-step k, the true state vector x_k^i is accessible for the controller $\mathscr{C}_i$.

Since the capacity of the communication channel is constrained, only a limited number of sub-systems are allowed to transmit data through the channel at each time-step and the rest of the sub-systems have to wait and re-transmit at future times. To model this process, we introduce the binary variable $\delta_k^i \in \{0,1\}$ which represents whether a sub-system i is allowed to have access to the channel and consequently transmit state information to the controller $\mathscr{C}_i$ at a time-step k, as follows

$$\delta_k^i = \begin{cases} 1, & x_k^i \text{ sent through the channel,} \\ 0, & x_k^i \text{ blocked.} \end{cases}$$

Having the scheduling variable introduced, the capacity constraint can be expressed in terms of the number of maximally allowed open channel slots as follows:

$$\sum_{i=1}^{N} \delta_k^i = c < N. \tag{2.2}$$

The capacity constraint (2.2) needs to be fulfilled at an every single time-step k. The selected data packets are sent through the communication channel for transmission. A successful transmission, i.e. the pertaining data packet is not lost or collided, is acknowledged via the binary variable $\gamma_k^i \in \{0,1\}$

$$\gamma_k^i = \begin{cases} 1, & x_k^i \text{ successfully received} \\ 0, & x_k^i \text{ dropped.} \end{cases}$$

In case $\delta_k^i = 0$, then $\gamma_k^i = 0$. Depending on the introduced variables δ_k^i and γ_k^i, the received signal at the controller $\mathscr{C}_i$ at a time-step k is denoted by z_k^i and is defined as follows:

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge \gamma_k^i = 1 \\ \varnothing, & \text{otherwise.} \end{cases}$$

It is assumed that the controller of the i^{th} sub-system merely have access to the local knowledge. Therefore, the available information at the control station is A_i, B_i, W_i and the distribution of x_0^i. In case a transmission attempt from a sub-system is not successful at a given time-step k, i.e., either $\delta_k^i = 0$ or $\gamma_k^i = 0$, an *a priori* estimate of the system state x_k^i can be calculated based on local information available at control station $\mathscr{C}_i$, via a model-based estimator. To guarantee a well-behaved estimation, assume that the control law u_k^i is described by the following state feedback law

$$u_k^i = -L_i \mathsf{E}\left[x_k^i | Z_k^i\right], \tag{2.3}$$

where $Z_k^i = \{z_0^i, \ldots, z_k^i\}$ represents the i^{th} controller observation history. Therefore, if a stabilizing gain L_i can be found such that the closed-loop matrix $A_i - B_i L_i$ becomes asymptotically stable, then the well-behaved model-based estimation of system state x_k^i can be computed as follows:

$$\mathsf{E}\left[x_k^i | Z_k^i\right] = (A_i - B_i L_i) \mathsf{E}\left[x_{k-1}^i | Z_{k-1}^i\right], \tag{2.4}$$

with the initial condition $\mathsf{E}\left[x_0^i | Z_0^i\right] = 0$.

2.2 Control and Scheduling Architecture – Separation Property

In this section, we study the control and scheduling architectures for the described network setup in Section 2.1. First, remind that in this chapter we assume to have physically decoupled sub-systems, i.e. there exists no physical interconnection between the individual sub-systems and the coupling takes place only through the shared communication medium. Taken this, we assume each local sub-system is controlled by a state feedback controller which is itself updated at every time-step k by either the true state vector x_k^i (in case updated state information is successfully received, i.e. $\delta_k^i = 1$ and $\gamma_k^i = 1$) or by state estimates $\mathsf{E}[x_k^i]$ (in case $\delta_k^i = 0$ or $\gamma_k^i = 0$).

We need to investigate the influence of the constrained communication channel, as the coupling point between individual sub-systems of the NCS. It is apparent that in the absence of capacity constraint, all sub-systems are allowed to transmit their state information for their corresponding controllers in a timely manner. Consequently, the individual sub-systems would become decoupled and overall stability of the NCS would follow straightforwardly only if the local stabilizing feedback gains L_i exist for all decoupled sub-systems $i \in \{1,\dots,N\}$.

In the presence of the capacity constraint (2.2) however, stability of the individual sub-systems does not necessarily guarantee the overall NCS stability. We will discuss it in more details after describing the overall NCS state. In order to account for the capacity constraint and analytically investigate its influence on NCS dynamics, we introduce the one-step ahead *network-induced error* state e^i_k for sub-system i as the difference between the true state vector x^i_k and its estimate $\mathsf{E}\left[x^i_k|Z^i_k\right]$, which is computed at the controller side $\mathscr{C}_i$ at time-step k. Therefore, we define

$$e^i_k := x^i_k - \mathsf{E}\left[x^i_k \middle| Z^i_{k-1}\right], \tag{2.5}$$

where, $Z^i_k = \{z^i_0, z^i_1, \dots, z^i_k, \delta^i_0, \delta^i_1, \dots, \delta^i_k, \gamma^i_0, \gamma^i_1, \dots, \gamma^i_k\}$. In what follows, the dynamics of each individual sub-system i is derived by considering that e^i_k is also an associating state of i^{th}-system along with x^i_k. Therefore, let us define $[x^{i\top}_k, e^{i\top}_k]^\top$ as the aggregate state of sub-system i. Assume $\delta^i_k = 1$ and $\gamma^i_k = 1$, then the controller $\mathscr{C}_i$ will be updated with the true state vector x^i_k which incurs $u^i_k = -L_i x^i_k$. Therefore, for system state x^i we have from the plant dynamics (2.1)

$$\begin{aligned} x^i_{k+1} &= A_i x^i_k - B_i L_i x^i_k + w^i_k \\ &= (A_i - B_i L_i) x^i_k + w^i_k. \end{aligned}$$

For the error state e^i, the dynamics can be derived as follows:

$$\begin{aligned} e^i_{k+1} &= x^i_{k+1} - \mathsf{E}\left[x^i_{k+1}|Z^i_k\right] \\ &= A_i x^i_k - B_i L_i x^i_k + w^i_k - \mathsf{E}\left[A_i x^i_k - B_i L_i x^i_k + w^i_k | x^i_k\right] \\ &= (A_i - B_i L_i) x^i_k + w^i_k - (A_i - B_i L_i) x^i_k \\ &= w^i_k. \end{aligned}$$

If sub-system i does not successfully transmit at time-step k, i.e. whether $\delta^i_k = 0$ or $\gamma^i_k = 0$, then $z^i_k = \varnothing$ and consequently $u^i_k = -L_i \mathsf{E}\left[x^i_k \middle| Z^i_k\right]$. According to the definition of the estimation error, the equality $\mathsf{E}\left[x^i_k \middle| Z^i_k\right] = x^i_k - e^i_k$ follows. Substituting this in the control law given in (2.3) results in the following dynamics for system state x^i:

$$
\begin{aligned}
x_{k+1}^i &= A_i x_k^i - B_i L_i \,\mathsf{E}\left[x_k^i \middle| Z_k^i\right] + w_k^i \\
&= A_i x_k^i - B_i L_i (x_k^i - e_k^i) + w_k^i \\
&= (A_i - B_i L_i) x_k^i + B_i L_i e_k^i + w_k^i .
\end{aligned}
$$

Finally, for the error state e^i, we have

$$
\begin{aligned}
e_{k+1}^i =\, & A_i x_k^i - B_i L_i \mathsf{E}\left[x_k^i | Z_k^i\right] + w_k^i \\
& - \mathsf{E}\left[A_i x_k^i - B_i L_i \mathsf{E}\left[x_k^i | Z_k^i\right] + w_k^i | Z_k^i\right] \\
=\, & A_i x_k^i - B_i L_i \mathsf{E}\left[x_k^i | Z_k^i\right] + w_k^i - A_i \mathsf{E}\left[x_k^i | Z_k^i\right] + B_i L_i \mathsf{E}\left[x_k^i | Z_k^i\right] \\
=\, & A_i (x_k^i - \mathsf{E}\left[x_k^i | Z_k^i\right]) + w_k^i \\
=\, & A_i e_k^i + w_k^i .
\end{aligned}
$$

Considering both cases together, dynamics of sub-system i with state vector $[x_k^{i\mathsf{T}}, e_k^{i\mathsf{T}}]^\mathsf{T}$ can be expressed depending on the binary variables δ_k^i and γ_k^i as follows

$$
\begin{bmatrix} x_{k+1}^i \\ e_{k+1}^i \end{bmatrix} = \begin{bmatrix} A_i - B_i L_i & \left(1 - \delta_{k+1}^i \gamma_{k+1}^i\right) B_i L_i \\ 0 & \left(1 - \delta_{k+1}^i \gamma_{k+1}^i\right) A_i \end{bmatrix} \begin{bmatrix} x_k^i \\ e_k^i \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k^i . \tag{2.6}
$$

From (2.6), it can be seen that the aggregate state $[x_k^{i\mathsf{T}}, e_k^{i\mathsf{T}}]^\mathsf{T}$ has an upper-triangular dynamics within each sub-system $i \in \{1, \dots, N\}$. This confirms that the evolution of the error state e^i is independent of the system state x^i, though the evolution of x^i depends on how the error state e^i evolves. This clarifies the role of the communication constraint in stability guarantees of our NCS setup. Expression (2.6) demonstrates that the local stability of individual sub-systems is necessary but not sufficient to guarantee the overall NCS stability. In the other words, finding stabilizing feedback gain L_i for each individual sub-system $i \in \{1, \dots, N\}$ ensures that the closed loop matrix $A_i - B_i L_i$ is Hurwitz and hence the system state x^i is converging. However, convergence of the error state e^i is additionally required to be proven in order to guarantee stability, in terms of overall state convergence, of the individual sub-system i. From (2.6), we have for the system state x^i

$$
x_{k+1}^i = (A_i - B_i L_i)\, x_k^i + \left(1 - \delta_{k+1}^i \gamma_{k+1}^i\right) B_i L_i e_k^i + w_k^i , \tag{2.7}
$$

and for the error state e^i

$$
e_{k+1}^i = \left(1 - \delta_{k+1}^i \gamma_{k+1}^i\right) A_i e_k^i + w_k^i , \tag{2.8}
$$

where the ordering of decisions within one time period, i.e. $k \to k+1$, is assigned by the following sequence:

$$
\cdots \to e_k \to \delta_{k+1} \to \gamma_{k+1} \to z_{k+1} \to u_{k+1} \to e_{k+1} \to \cdots
$$

It is noteworthy that the error evolution at a certain time-step, e.g., $k+1$, depends on δ^i_{k+1} determined by the scheduler at the same time-step. This specifies that the scheduler first determines δ_{k+1} using the information at time-step k. Subsequently, the error at time-step $k+1$ attains a value provided by (2.8). Altogether, both δ_{k+1} and e_{k+1} depend on the information available at time-step k.

According to (2.7) if the closed loop matrix $(A_i - B_i L_i)$ is Hurwitz, i.e. by designing appropriate stabilizing feedback control law u^i, then it is sufficient to show that the error state e^i is also converging to achieve stability of sub-system i with the aggregate state $[x^{i\mathsf{T}}_k, e^{i\mathsf{T}}_k]^\mathsf{T}$. This is the direct consequence of the triangular dynamics shown in (2.6) which implies that evolution of error state e^i is independent from the system state behavior.

Another equivalent conclusion which can be drawn from the expressions (2.7) and (2.8) clarifies that the control law u^i does not affect the evolution of the error state e^i. Therefore, it can be synthesized independently such that the closed loop matrix $(A_i - B_i L_i)$ is stabilized. On the other hand, the scheduling variable δ^i only affects the error state dynamics, and does not have any role to play in designing the feedback gains L_i. This property, shown to hold for LTI systems, paves the way for separately designing the control and scheduling laws. A more remarkable result of the separation property between the control and scheduling laws is that an optimal co-design can be approached considering that a given global optimization problem can be decomposed into two independent optimization sub-problems, of which one is optimized by the control law, and the other by the scheduling law. Feasibility of such optimal design however depends firmly on solvability of the decomposed optimization sub-problems. In the next section, we first address the control law design.

2.3 Control Synthesis and Local Stability

In this section, we address the problem of control synthesis for the NCS model in Section 2.1. It should be reminded that in this chapter, we consider that if a transmission associated with sub-system i occurs, then the controller $\mathscr{C}_i$ receives the full state information x^i of corresponding plant $\mathscr{P}_i$. Mathematically speaking, we assume the following model for the plant dynamics and sensor measurements of local sub-system i:

$$\begin{aligned} x^i_{k+1} &= A_i x^i_k + B_i u^i_k + w^i_k \\ y^i_k &= x^i_k \end{aligned} \tag{2.9}$$

As discussed in the previous section, separation property exists between the control law and scheduling law synthesis for the NCS, which allows us to independently design each law and more importantly introduce independent index functions. It should be however noted that the separation property facilitates to compute the control law considering that the communication constraint is relaxed, i.e. we compute u^i_k assuming that the state information is provided for the controller in timely-manner. The effect of the communication constraint is reflected by the error-dependent term $B_i L_i e^i_k$ in the expression (2.6). This leads us to consider an emulation-based control design for the operational period of the described NCS in presence of the capacity limitation (2.2), in which the control inputs are chosen *a priori* and will be kept constant between the transmission times. This is an important assumption also valid for event-triggered control which often takes emulative approach presuming a stabilizing continuous time controller. The control inputs are chosen at triggering times and only update at the subsequent transmission times. We will extensively employ this property in the forthcoming chapters of this dissertation as we introduce event-based scheduling architectures.

Taking the bandwidth limitation (2.2) into account, we introduce the control strategy considering the fact that a local controller does not necessarily have access to the state information from its corresponding sub-system at every single sampling time. Thus, an emulation-based control law with the minimum requirements is supposed to be designed in order to steer the system state x^i to an asymptotic equilibrium. The minimum requirements for such a control law would be satisfied if the control law is linear and stabilizing. As already discussed, furthermore it is assumed that sensor and controller of the i^{th} sub-system are provided only by local knowledge, i.e. of A_i, B_i, W_i and the distribution of the initial state x_0^i. Therefore, we assume that the linear control law ϑ^i is described by the following measurable and causal mappings of the past observations:

$$u_k^i = \vartheta_k^i(Z_k^i) = -L_k^i \mathsf{E}\left[x_k^i | Z_k^i\right], \tag{2.10}$$

where $Z_k^i = \{z_0^i, \ldots, z_k^i\}$ is the i^{th} controller observation history, and L_k^i is any arbitrary stabilizing feedback gain which is computed at every time-step when updated state information arrive at the control station. In case a transmission request is blocked, i.e. $\delta_k^i = 0$, or the pertaining data packet is lost, i.e. $\gamma_k^i = 0$, an estimate of the system state x_k^i, i.e. $\mathsf{E}[x_k^i | Z_k^i]$ is computed via the model-based estimator introduced in (2.4). It is worth mentioning that upon arrival of a data packet, the corresponding controller will be updated by the true state vector x^i according to the noiseless measurement assumed in (2.9). We will further consider the noisy measurements which instead necessitates employment of a filter, such as Kalman filter at the control station, to estimate the state vector from noisy sensor measurements.

For the control law synthesis, it is already shown is Section 2.2, that the feedback gains L_i are to be found in order to make the closed loop matrix $(A_i - B_i L_i)$ Hurwitz, i.e., all its eigenvalues lie inside the unit circle. As the gains L_i do not influence the error state e^i, and additionally, the system dynamics involves Gaussian process noise w^i, the most practical choice of a cost function is linear quadratic Gaussian (LQG) for the control law synthesis. To this end, we consider the following LQG cost function over the finite time horizon T including the control input u^i and system state x^i in order to obtain the optimal stabilizing control gain L_i for the local sub-system i:

$$J_{x,u} = \mathsf{E}\left[x_T^{i^{\mathsf{T}}} S_i x_T + \sum_{k=0}^{T-1} x_k^{i^{\mathsf{T}}} F_i x_k^i + u_k^{i^{\mathsf{T}}} R_i u_k^i \right], \tag{2.11}$$

where, S_i and F_i are constant positive semi-definite matrices and R_i is a constant positive definite matrix. Therefore, the optimal stabilizing control gain L_k^i for every sub-system $i \in \{1, \ldots, N\}$ will be obtained as follows:

$$L_k^i = \left(B_i^\mathsf{T} P_{k+1}^i B_i + R_i\right)^{-1} B_i^\mathsf{T} P_{k+1}^i A_i, \tag{2.12}$$

where p_k^i is determined according to the following in-time-backward Riccati equation:

$$P_k^i = A_i^\mathsf{T} \left(P_{k+1}^i - P_{k+1}^i B_i \left(B_i^\mathsf{T} P_{k+1}^i B_i + R_i\right)^{-1} B_i^\mathsf{T} P_{k+1}^i\right) A_i + S_i, \tag{2.13}$$

and $P_T^i = S_i$.

We described the stabilizing linear controllers u^i's in (2.10) where control gains L_k^i's are derived in (2.12), according to the LQG cost function (2.11). This only guarantees asymptotic stability of the NCS if and only if every control unit $\mathscr{C}_i$, for $i \in \{1, \ldots, N\}$, is updated with state information at every sampling time. This is an immediate observation as relaxing the capacity constraint (2.2) leads to elimination of the error-dependent term $B_i L_k^i e_k^i$ since we have $\delta_k^i \gamma_k^i = 1$ for all i and all time-steps k. In the presence of the capacity limitation (2.2) however, stabilizing control law u^i is necessary but not sufficient to guarantee stability of the local sub-system i with the aggregate state $[x_k^{i^\mathsf{T}}, e_k^{i^\mathsf{T}}]^\mathsf{T}$. Convergence of the error state e^i is additionally essential to show stability of the sub-system i, subject to the constraining fact that the updates are arriving not necessarily in a time-regular fashion. To study evolution of the error state e^i, we should first notice that the scheduling variable δ^i plays an identical role for the error dynamics (2.8) as the control input u^i plays for the system state x^i in (2.7). In other words, δ^i is the control input which steers the error state in its state space. Therefore, dynamical behavior of the error state e^i is hinged on the scheduling law which decides the scheduling variables δ_k^i at each time-step k. Similar to the control law synthesis, we need first to determine the architecture of the scheduling law in order to investigate behavioral properties of the error state e^i, since the dynamics of the error state e^i depends on the sequence of variables δ_k^i, and γ_k^i. We address the scheduling design in the next section of this chapter, and subsequently introduce the concept of stability in the next chapter of this dissertation.

2.4 Event-triggered Scheduling Design

In this section, we comprehensively study different scheduling architectures for the described multi-loop NCS with a shared communication channel. As already stated, it is assumed that the shared communication channel is subject to the hard capacity constraint (2.2) which implies that not all N sub-systems can simultaneously have access to the channel at every time-step in order to transmit their state information to their corresponding controllers. As this constraint has to be satisfied at every time-step k, some of the transmission requests are inevitably blocked and the corresponding sub-systems remain open-loop between their own two consecutive transmission times.

A scheduling law for general shared resource NCSs can be designed in various forms. An appropriate architecture needs to be identified firstly according to the type of NCS under consideration, i.e. size of the NCS, communication medium infrastructure, type of local entities which are connected to the communication network and the application of such a networked

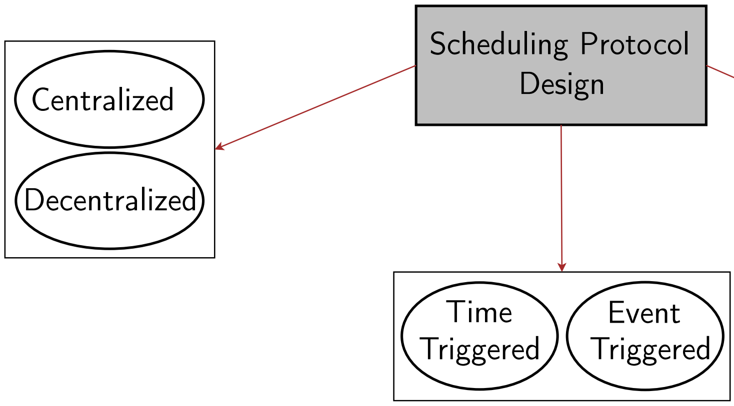

Figure 2.2: Categorization of the scheduling architectures.

system, and secondly based on the specific real time requirements. Different frameworks for scheduling architectures are available; the most important ones are centralized or decentralized, time-triggered or event-triggered, online or offline and deterministic or random access (probabilistic) methodologies. In Figure 2.2, different scheduling architectures are schematically categorized.

Time division multiple access (TDMA), frequency division multiple access (FDMA) and code division multiple access (CDMA) are well-known centralized scheduling architectures which are mostly preferred in small and medium-size networked control systems. As the main advantages of centralized scheduling approaches, they offer collision-free and precise channel access management often leading to higher throughput in comparison to decentralized strategies. Moreover, quality-of-service support and bandwidth arbitration is highly facilitated as those methods are capable of prioritizing channel access given either pre-determined priority order or dynamic prioritization based on the network and control systems' online conditions. However, centralized strategies lack flexibility and scalability due to their synchronous nature, which consequently make those approaches unfeasible-to-implement for large-scale networked systems. In addition, centralized protocols are characterized by a single point of failure, which can compromise the overall NCS performance.

Decentralized approaches, on the other hand, represent easy-to-install, low-cost and scalable scheduling architectures within which the transmission times of every node are determined locally or without the requirement of having global knowledge of all nodes between which the communication medium is shared. They are suitable for NCSs with a large number of connected entities. However, collisions take place inevitably within those protocols and need to be handled with care in the NCS design. They might be however insecure scheduling strategies as no global administration unit exits. In addition, collision avoidance mechanisms, e.g. the listen-before-talk scheme in CSMA-CA, call for all nodes to sense the channel permanently, which instead requires high energy consumption due to idle listening, overhearings, and message overhead.

The difference between event-triggered and time-triggered scheduling approaches is in the fact that the former takes into account the current situation of the communication channel or dynamics of the sub-systems in order to control the channel access while the latter approach schedules the communication resource according to some periodic offline rules. For example, TDMA is a time-triggered scheduling approach where the sequence of transmissions for a sub-system is determined *a priori*. The notable advantage of event-triggered approaches is the capability of real-time monitoring of the NCS and adjust the scheduling decisions accordingly for instance to prioritize the transmission requests from those nodes in urgent transmission state. Event-triggered scheduling policies consider a well-defined *event* as the trigger for a transmission rather than elapsing a fixed period of time. As many results over the recent years show [2,21,23,24,27], it is often more beneficial to sample a signal upon the occurrence of specific events, rather than fixed period of time, especially when dealing with scarce resources. In networked control systems, event-based control and scheduling schemes outperform the periodic (time-triggered) rules in terms of resource consumption while preserving the same level of control performance [11,25,30]. This motivates us to consider the event-triggered scheduling approach in this dissertation in order to appropriately manage the resource constraints imposed on the channel capacity. We will extensively discuss the event-based design and the specific design of our scheduling law in the next chapter.

The scheduling decisions may in addition be taken according to either deterministic or probabilistic rules. From well-known deterministic architectures, TDMA and Try-Once-Discard (TOD) can be mentioned, while CSMA or CSMA-CA are famous probabilistic channel access managers. Deterministic rules usually render better performance than their probabilistic counterparts, but the latter is more flexible in dealing with channel phenomena such as latency, congestion, data loss and collisions.

Throughout this dissertation, if not otherwise stated, we assume to employ an either event-based probabilistic or event-based bi-character (deterministic-probabilistic) scheduler, and we then study the implementation procedure of the mentioned schedulers in both centralized and decentralized fashions. We start by the centralized pure probabilistic and bi-character designs in Chapter 3, while the decentralized implementation of mentioned scheduling approaches is presented in Chapter 6.

As discussed in the previous section, the scheduling law plays the essential role in behavioral analysis of the error state e^i for each local sub-system i. Therefore, we define the binary scheduling variable δ^i_{k+1} at time-step $k+1$ as a measurable function $\varrho^i_k : \mathbb{R}^N \to \{0,1\}$ where $\varrho^i_k(\cdot)$ itself is a function of the exactly one time-step prior error state values e^j_k for all $j \in \{1,\ldots,N\}$ in the centralized framework, while $\varrho^i_k(\cdot)$ is function of only e^i_k in the decentralized design. Thus, in the centralized case we have[1]

$$\delta^i_{k+1} = \varrho^i_k(e^1_k,\ldots,e^N_k). \tag{2.14}$$

The scheduling rule (2.14) represents an error-dependent channel access mechanism which monitors, at the latest time-step, the conditions of each sub-system in order to decide the channel access scenario. Expression (2.14) resembles to the control law (2.10), though the scheduling law ϱ is not necessarily linear and depending on the information structure, i.e. centralized or decentralized, it depends on full NCS state information or local information, respectively.

In order to study stability of the overall NCS we need to define state variables of the networked system. We already discussed that the inclusion of the error e^i in i^{th} sub-system

[1] We investigate the decentralized scheduling design comprehensively in Chapter 6, and only discuss the centralized scenario in the present chapter.

state space is essential, along with the system state x^i, in order to take into account the coupling between the sub-systems which are caused by the capacity constraint (2.2). We define the overall NCS state by stacking the individual state vectors of all N sub-systems together in a vector, as follows:

$$[x_k^\top, e_k^\top]^\top := [x_k^{1^\top}, \ldots, x_k^{N^\top}, e_k^{1^\top}, \ldots, e_k^{N^\top}]^\top. \tag{2.15}$$

Having introduced (2.15), we can express the overall NCS dynamics as follows:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix}_{2n} = \begin{bmatrix} A-BL & (I_n - \Delta_{k+1}\Gamma_{k+1})BL \\ 0 & (I_n - \Delta_{k+1}\Gamma_{k+1})A \end{bmatrix}_{2n\times 2n} \begin{bmatrix} x_k \\ e_k \end{bmatrix}_{2n} + [w_k]_{2n}, \tag{2.16}$$

where, $A \in \mathbb{R}^{n\times n}$ is a block diagonal matrix consisting of the blocks $A_i\mathbb{R}^{n_i\times n_i}$ on the diagonal and zeros on the off-diagonal, $B \in \mathbb{R}^{n\times m}$ is block diagonal with entries $B_i \in \mathbb{R}^{n_i\times m_i}$, and gain matrix $L \in \mathbb{R}^{m\times n}$ has the blocks $L_i \in \mathbb{R}^{m_i\times n_i}$ on the diagonal entries. Moreover, $\Delta_{k+1}\Gamma_{k+1} \in \mathbb{R}^{N\times N}$ consists of the diagonal elements $\delta^i_{k+1}\gamma^i_{k+1} \in \{0,1\}$ and zero off-diagonals. It should be noted that, for the dimensions of the matrices to be appropriate, the multiplications $(1-\Delta_{k+1}\Gamma_{k+1})BL$ or $(1-\Delta_{k+1}\Gamma_{k+1})A$ in (2.16) are computed such that each of the N diagonal real binary variable $\delta^i_{k+1}\gamma^i_{k+1}$ are multiplied with the i^{th} block in the matrices BL and A. Since, $\sum_{i=1}^N n_i = n$ and $\sum_{i=1}^N m_i = m$, the above matrices are all well-defined and of appropriate dimensions. The aggregate noise variable in the compact form (2.16) is similarly defined as $[w_k]_{2n} := [w_k^{1^\top}, \ldots, w_k^{i^\top}, \ldots, w_k^{N^\top}, w_k^{1^\top}, \ldots, w_k^{i^\top}, \ldots, w_k^{N^\top}]^\top$.

Let us remind that we take emulation-based LQG control laws for individual sub-systems which stabilize their corresponding control loops in the absence of the channel capacity constraint (2.2). Therefore, to show stability of the overall NCS state (2.15), we need to investigate evolution properties of the overall network-induced error state $e_k \in \mathbb{R}^n$ which is defined as the stacked error vectors from all N control loops as follows:

$$e_k = [e_k^{1^\top}, \ldots, e_k^{N^\top}]^\top, \tag{2.17}$$

where $n = \sum_{i=1}^N n_i$ and n_i is the dimension of the i^{th} sub-system state or error vectors. The following Lemma characterizes the behavioral model of the network-induced error e_k, which is the cornerstone of the future stability analysis in this dissertation.

Lemma 2.1. The network-induced error e_k expressed in (2.17) is a time-homogeneous, aperiodic ψ-irreducible Markov chain[2].

Proof. The centralized scheduling policy introduced in (2.14), which can be regarded as the input for the error evolution (2.8), is a policy depending only on the most recent error values e_k when deciding on the scheduling variable δ_{k+1} at time-step $k+1$. Since the policy (2.14) is forgetful about the error values $e_m, m < k$, (2.17) is actually a Markov chain (see the formal definition of a Markov chain in *Definition* A.25 in Appendix A.7). Moreover, the Gaussian noise process w^i in (2.8) has a positive density function at any state e_k implying

[2]A comprehensive introductory about the definitions, concepts and terminologies of stochastic processes and Markov chains is presented in the Appendix A.6

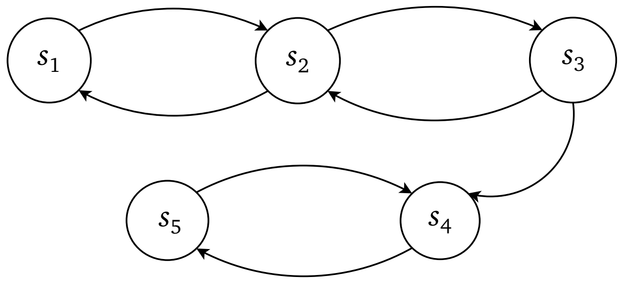

Figure 2.3: A reducible Markov chain with countable state space $\{s_1, s_2, s_3, s_4, s_5\}$.

that there is a transition probability for appropriate events $\mathscr{E} \in \mathscr{A}$, i.e.,

$$\mathsf{P}\left(e_{k+n} \in \mathscr{E} | e_m, m \leq k, e_k\right) = P^n(e_k, \mathscr{E}),$$

where $P^n(e_k, \mathscr{E})$ denotes the probability that the overall error state e_k enters the set $\mathscr{E}$ after exactly n transitions. The Markov chain e_k is time-homogeneous because the difference equation (2.4) is time-invariant and the noise process w^i is independent and identically distributed (i.i.d.) for $i = \{1, \ldots, N\}$ at every time-step k. This implies that the Markov chain evolves according to a stationary transition probability. Since the Gaussian noise distribution is absolutely continuous having a positive density function, it is furthermore concluded that the Markov chain e_k is aperiodic (see the formal definition of aperiodic Markov chains in *Definition* A.31 in Appendix A.7) and ψ-irreducible, where ψ represents a non-trivial measure on the general state space $\mathbb{R}^n$. The ψ-irreducibility ensures that every state of the state space is accessible for the Markov chain through finite number of transitions. □

Intuitively, the behavior of a stochastic process in an uncountable state space is characterized by several measures. First, a transition function denotes, at each transitional instance, the probability that the process evolves to a specific state. The transition probabilities are more comprehensible within countable state spaces, as there exist countable number of state destinations towards which the process can evolve with one transition. In uncountable spaces, if a stochastic process has access to every state, i.e. if the transition probability from every state to every other state is non-zero within finite transitions, then we call the process "irreducible". Figures 2.3 and 2.4 show graphical interpretations of a reducible and an irreducible Markov chain over a countable state space. A stochastic process is called "periodic" if the process can return to its initial state only at multiples of some integers larger than one, and it is called "aperiodic" otherwise. In Figure 2.5 two simple periodic and aperiodic Markov chains are schematically illustrated. The time homogeneity of a stochastic process emphasizes that the evolution of the process state is time-independent, i.e. the transition probability between every two states is independent of the time instance.

To measure the performance of the various designs of scheduler, we define the following per-time-step cost function for all sub-systems $i \in \{1, \ldots, N\}$:

$$J_{e_k} = \sum_{i=1}^{N} e_k^{i^\top} Q_k^i e_k^i + \eta_k \delta_k^i := \sum_{i=1}^{N} \|e_k^i\|_{Q_k^i}^2 + \eta_k \delta_k^i, \tag{2.18}$$

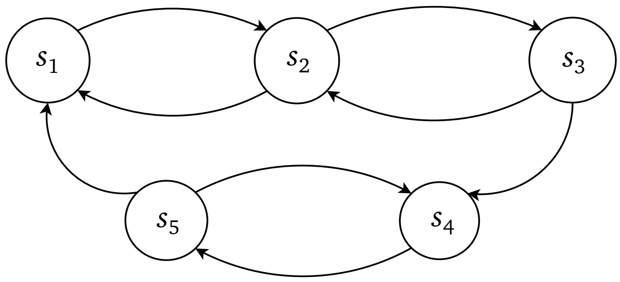

Figure 2.4: An irreducible Markov chain with countable state space $\{s_1, s_2, s_3, s_4, s_5\}$.

where the non-negative parameter η_k denotes the cost of channel utilization at time k which is assume to be equal for every sub-system, and Q_k^i is a symmetric positive semi-definite weight matrix associated with i^{th} loop at time-step k.

The cost function in (2.18) expresses that, at a time-step k, the total penalty corresponds to the shared communication medium among the sub-systems is the weighted quadratic size of the estimation error of all sub-systems plus η_k which is the cost incurred due to occupying a channel slot at a time-step k.

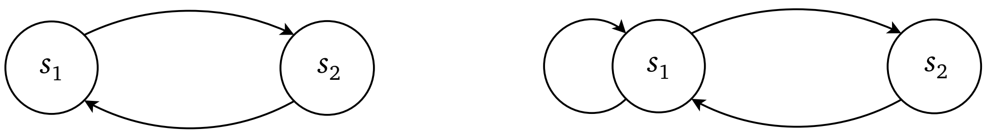

Figure 2.5: Periodic (left) and aperiodic (right) Markov chains with state space $\{s_1, s_2\}$.

2.5 Summary

In this chapter, we investigated the structural characteristics of multiple-loop stochastic networked control systems subject to scarcity of the shared communication resources. The general control and scheduling architectures are comprehensively discussed in this chapter, and the separation property is confirmed to hold within the considered NCS framework. Afterwards, the essential system states and variables to be observed and governed are identified, and consequently the overall model for the described NCS is derived. More precisely, the aggregation of the local system states and error states are considered as the overall NCS state, and then the overall networked dynamics is represented in state space, taking into account the communication constraints. Stability of the overall NCS is shown to be dependent on both control and scheduling scenarios which can be independently synthesized due to the existence of the separation principle. Finally, we analyzed the properties of the state dynamics and demonstrated that the error state, which is dynamically steered by the scheduling variables, is a homogeneous, aperiodic, and ψ-irreducible Markov process evolving in the uncountable state space $\mathbb{R}^n$, where n is the dimension of the networked state. The results of this chapter are essential in stability analysis and performance evaluation in the rest of this dissertation, and therefore will be recalled frequently in the forthcoming chapters.

2.6 Contributions

The presented results in this chapter are essential preliminary analyses and evaluations to pursue further discussions and derivations in the forthcoming chapters, where we address different aspects of a proper design for NCSs such as stability and performance analysis, robustness, and implementation procedure. The results discussed in this chapter are partly from the author's own work in [84,85], and partly from the available literature, most notably [33, 58, 59, 86].

The structural properties of NCSs with LTI stochastic control sub-systems in discrete framework, presented in Section 2.1, are basically from [33, 84, 85].

Separation property, as an important result enabling us to decompose the control and scheduling design procedure, is the direct consequence of the discussed structural properties. Independent synthesis of control strategies, in form of the LQG controllers as discussed in Section 2.2, is a well-known result which is employed in several research, e.g., [35, 59, 87].

In Section 2.4, a brief but necessary introduction about the different scheduling scenarios is presented. More comprehensive discussions can be found in, e.g., [29, 88–90]. This section is followed by derivation of the overall NCS dynamics under a general event-based scheduling design. It is shown that the network-induced error state is directly controlled by the scheduling variable, where these variables are determined by the event-based scheduling law. It is discussed that the error state can be modeled as a time-homogeneous, aperiodic, and ψ-irreducible Markov process. Extensive discussions about the properties of a Markov process are presented in [86]. Throughout this dissertation, we mainly exploit the terminologies, definitions, concepts and theorems about stochastic processes discussed in [86].

3

Centralized State-dependent Scheduling Design

The design of efficient event-based scheduling policies to use the expensive and limited resources is a momentous and still widely open theme in the area of networked control systems. The importance of event selection and scheduling process is more affirmed when the resources such as communication, energy, and computation are sparse. In this chapter, we address the efficient usage of scarce communication resources in networked control systems, composed of multiple LTI heterogeneous stochastic control-loops over a shared constrained communication channel, by introducing a novel stochastic scheduling scheme with capability of prioritizing the channel access based on current status of local control systems. More precisely, an error-dependent scheduler decides which transmission requests have the priority to be awarded the channel access. The priorities are assigned to each sub-system in a probabilistic fashion according to an error-dependent measure, such that higher chances of transmission is assigned to the sub-systems with greater error norms. The other transmission requests are blocked when the resource limit is reached.

Due to the presence of stochastic noise and the coupling between sub-systems, the networked system under consideration requires novel methods to analyze its asymptotic behavior. Dynamics of the overall NCS can be described by the behavior of the so-called "network state", which can be modeled by a homogeneous Markov chain evolving in the uncountable state-space $\mathbb{R}^n$. We prove that the described NCS, arbitrated by the proposed stochastic scheduling protocol, is stochastically stable, according to the notion of "f-ergodicity".

Furthermore, we derive analytic performance bounds for an average cost function comprised of an error-dependent quadratic term plus an incurred cost of communication.

This chapter is organized as follows. We introduce the scheduling architecture and its structural properties in Section 3.1. The overall networked model is then studied under the employment of the probabilistic scheduling rule in Section 3.2, and the NCS dynamics is shown to be Markovian. Section 3.3 presents a comprehensive analysis of stability for the described NCSs. Afterwards, we validate our stability and performance claims in Section 3.4 with simulation results which illustrate an improved performance in terms of the average mean-square error compared to the conventional protocols, such as TDMA and CSMA.

3.1 Pure Probabilistic Event-based Prioritized Scheduling Law

It is comprehensively discussed in Chapter 2 that the separation property exists between the control and scheduling strategies for the considered NCS described in (2.1)-(2.4). In addition, the form of the scheduling law in centralized fashion is shown to be given as (2.14), which represents an event-based scheduling scenario wherein the transmission times are chosen dynamically based on the real-time conditions of sub-systems.

A well-known deterministic version of such an event-based centralized error-dependent scheduling approach is proposed in [14]. The introduced scheduling scenario is called *maximum error first-try once discard* (MEF-TOD). According to the MEF-TOD scheduling rule, the sub-system with the highest discrepancy between its true and estimated state value, at the current transmission instant, wins the channel slot competition among all those sub-systems which are attempting to transmit through that transmission slot, and all the other transmission requests are discarded. In the mentioned initiating work and also further research inline with the proposed concept, e.g. [15, 16, 54, 55], the MEF-TOD scheduling approach is implemented on various NCS models such as NCSs consisting of linear and non-linear system dynamics with deterministic model uncertainties over either deterministic or stochastic channels. (see Chapter 1 for a comprehensive literature review.). However, MEF-TOD is a centralized approach which makes it rather inapplicable for large-scale NCSs and multihop wireless networks. Moreover, the applicability of this approach has not been investigated for NCSs with stochastic sub-systems, to the best of the author's knowledge. Furthermore, MEF-TOD as a pure deterministic scheduling law, is shown to be inflexible when dealing with channel imperfections such as packet loss and delay. In what follows, we introduce a probabilistic scheduling mechanism which resembles the MEF-TOD approach, but in a randomized fashion with a bias. Rather than determining which sub-systems should transmit in a deterministic manner, the proposed scheduler prioritizes the transmission requests based on the size of the error states, and then assigns channel access probabilities accordingly. Then, the sub-systems, which will be awarded the channel access, are determined by a randomized mechanism. The main advantage of such a probabilistic design is that it can be implemented in a distributed fashion depending only on the local information of each station. We will discuss this extensively in Chapter 6. Moreover, within this framework, we show that the channel phenomena such as packet dropouts and delay can be effectively addressed, as will be later discussed in Chapter 5.

As the first step towards designing state-dependent scheduling policies, we propose in this section a state-dependent pure probabilistic scheduling protocol for NCSs comprised of multiple heterogeneous LTI control loops sharing a capacity-limited communication network to transmit their sensory data to their corresponding controllers. The scheduler first assigns a probability to each sub-system according to a centralized error-dependent priority measure. Then, the medium access is granted to those sub-systems in a way that sub-systems with higher priorities have higher chances of channel access. However, it is not guaranteed that a sub-system with a lower priority is not allowed to transmit in the presence of another sub-system with a higher priority. The importance of studying probabilistic approaches lies within the fact that these approaches are theoretically more general than their deterministic counterparts. In fact, with the design parameters we introduce in this chapter, deterministic version of prioritizing scheduler can be almost-surely realized from the proposed probabilistic design.

Recalling from the centralized scheduling law (2.14), we introduce the following prioritized error-dependent probabilistic scheduling rule by defining the probability of transmission for a sub-system i at a time-step $k+1$, given that the scheduler is provided the information about the error states e_k^j, i.e., the events, from all $j \in \{1, \ldots, N\}$:

$$\mathsf{P}\left[\delta_{k+1}^i = 1 \middle| e_k^j, j \in \{1, \ldots, N\}\right] = \frac{\|e_k^i\|_2^p}{\sum_{j=1}^N \|e_k^j\|_2^p}, \tag{3.1}$$

where, $p \geq 1$ is an integer. Scheduling policy (3.1) is a probability measure at every time-step, as the probabilities of transmissions for all sub-systems sum up to one, i.e.,

$$\sum_{i=1}^{N} \mathsf{P}\left[\delta_{k+1}^i = 1 \middle| e_k^j, j \in \{1, \ldots, N\}\right] = 1.$$

It is apparent that the above probability measure is supported on the semi-infinite interval $[0, \infty)$ at each time-step k. While the scheduling policy (3.1) determines only the channel access probabilities, it does not determine which sub-systems eventually transmit. Since the scheduling law (3.1) depends only on the latest error state values, the proposed rule follows the general form given in (2.14), which in turn ensures that the Markov property of the network-induced error state (2.17) remains valid. It it noteworthy that, in this chapter, we assume channel perfection, i.e., each data packet which is granted the channel access is assumed to be successfully received with zero latency by its corresponding controller.

In order to determine the sub-systems which finally transmit at each time-step, a randomized process is employed wherein the selection mechanism is not uniform, but biased according to the assigned probabilities given in (3.1). Each sub-system associated with its assigned probability takes part in a biased randomization, and the random outcome determines which sub-systems will have access to the channel. As a simple illustrative example, consider an NCS with only two sub-systems, i.e., $N = 2$, competing for the sole communication slot, i.e., $c = 1$. As a realization, assume that the priorities are assigned by the scheduler to be 0.8 and 0.2. Therefore, the biased randomization process is tossing an unfair coin where the probabilities of having head and tail are 80% and 20%, respectively. One sub-system will finally transmit based on the outcome of tossing the coin. Hence, it is not guaranteed that the sub-system with higher priority transmits, though it is more likely. It is worth mentioning that the scheduling rule (2.14) follows a collision-free policy due to the fact that the scheduling unit determines the sub-systems which eventually transmit.

Recalling the local error dynamics (2.8), one can conclude that even after a successful transmission of a local sub-system i at a time-step k, i.e., $\delta^i_k \gamma^i_k = 1$, the estimation error does not reset to zero, but to the noise value, i.e., $e^i_k = w^i_{k-1}$. Therefore, in case $\delta^i_k \gamma^i_k = 1$, we can compute the probability that the same sub-system i re-transmits again at the next time-step $k+1$, as follows:

$$\mathsf{P}\left[\delta^i_{k+1} = 1 \middle| e^j_k, j \in \{1,\dots,N\}\right] = \frac{\|w^i_{k-1}\|^p_2}{\|w^i_{k-1}\|^p_2 + \sum_{j=1,j\neq i}^{N} \|e^j_k\|^p_2}$$

This indicates that the sub-system i would have a non-zero chance of successive transmission at the next time-step $k+1$ if $w^i_{k-1} \neq 0$, and this chance might not be negligible if the random noise realization w^i_{k-1} is large. This is possible as the noise values are assumed to be randomly drawn from a zero-mean Gaussian distribution with unsupported range $(-\infty, \infty)$. Due to the existence of such a noise sequence, it is essential that the scheduling law (2.14) forgets about the history of a sub-system's dynamics and the previous transmission occasions, and decides the transmission order only based on the current conditions of the overall networked system. Mathematically speaking, this feature of the scheduling rule (2.14) enables us to model the evolution of the network-induced error (2.17) with Markov chains.

Remark 3.1. The power $p \geq 1$ in the probabilistic scheduling law (2.14) is indeed a design parameter which adjusts the relation between the size of the error of a sub-system i, i.e. $\|e^i_k\|_2$ and its chance of having access to the channel at the subsequent time-step $k+1$, i.e. $\mathsf{P}\left[\delta^i_{k+1} = 1\right]$. Having $p \to \infty$ incurs the sub-system with the highest error *almost surely* transmits, which indicates a similar behavior as MEF-TOD approach.

3.2 Overall NCS Model with Markovian Error State

Having introduced the probabilistic error-dependent prioritizing scheduling scheme (3.1) in this section, we derive the overall NCS model for this specific choice of channel scheduler. First of all, remind that the communication channel in this chapter is assumed to be ideal, i.e., if a data packet is scheduled for transmission, it will be received by its corresponding control station without any transmission delay. Thus, $\gamma^i_k = 1$ for all sub-systems $i \in \{1,\dots,N\}$ for whom $\delta^i_k = 1$, at every time-step k. We consider to have an NCS composed of N heterogeneous control loops each with the dynamics and measurements described by (2.9). The local control laws are taken to be emulation-based and the feedback gains are designed such that the LQG cost function (2.11) is minimized. As already discussed, within the emulation-based framework, the feedback gains (2.12) are kept constant between the transmission times. Therefore, the overall NCS dynamics with the state vector $[x_k^\top, e_k^\top]^\top$ defined in (2.15) can be expressed as follows:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix}_{2n} = \begin{bmatrix} A - BL & (I_n - \Delta_{k+1})BL \\ 0_{n\times n} & (I_n - \Delta_{k+1})A \end{bmatrix}_{2n\times 2n} \begin{bmatrix} x_k \\ e_k \end{bmatrix}_{2n} + [w_k]_{2n}, \tag{3.2}$$

with

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_N \end{bmatrix}_{n\times n} \quad , \quad BL = \begin{bmatrix} B_1 L_1 & & 0 \\ & \ddots & \\ 0 & & B_N L_N \end{bmatrix}_{n\times n}$$

$$\Delta_{k+1} = \begin{bmatrix} \delta^1_{k+1} & & 0 \\ & \ddots & \\ 0 & & \delta^N_{k+1} \end{bmatrix}_{n\times n} \quad , \quad [w_k]_{2n} = [w_k^{1^\top}, \ldots, w_k^{N^\top}, w_k^{1^\top}, \ldots, w_k^{N^\top}]^\top$$

The scheduling variables δ^i_{k+1} are determined according to the described biased random process, where the assigned bias terms (priorities) are computed at each time-step according to the rule (3.1). The rest of the discussions in the Section 2.4 of Chapter 2 remain valid. We will recall state representation (3.2) in the next section, wherein comprehensive stability discussions for the described NCS model are provided.

3.3 Stability Analysis

In this section, first the concept of f-ergodicity, as a stochastic stability criteria which might be employed to analyze the asymptotic behavior of stochastic networked control systems, is introduced, and the selection of this notion of stability is justified. The necessary preliminaries of stochastic stability are then provided, and some essential modifications are presented. In the second part of this section, stochastic stability of the described NCS under the capacity constraint (2.2), and the proposed control and scheduling policies (2.10) (3.1), is shown in terms of f-ergodicity.

3.3.1 Stochastic Stability Concepts and Preliminaries

Generally speaking, fundamental properties of stochastic systems are often not analogous to their deterministic counterparts due to the random nature of their behavior. Usually, properties of stochastic systems are investigated by looking at the average behavior of sufficiently large sample sets, as one single sample might have quite different output behavior from another realization, under exactly the same input signals and initial values. Therefore, it is conventional to use stability notions which observe the dynamic behavior of the stochastic systems under consideration in their corresponding probability spaces. Stability of stochastic systems are mainly investigated within three different frameworks; namely *stability in probability* (the weakest), *mean stability*, and *almost-sure stability* (the strongest) [91]. Various Lyapunov methods are of primary stability tests for deterministic systems, and they have been largely developed over the last decades. Analogous stochastic certificates have also been presented for stochastic processes. Within these Lyapunov-based stochastic notions of stability, one should typically search for an appropriate Lyapunov function, such that the drift of the Lyapunov function, in average, converges to either some compact sets, or in a stronger sense, to zero. In [86], comprehensive discussions about stability of stochastic processes, in

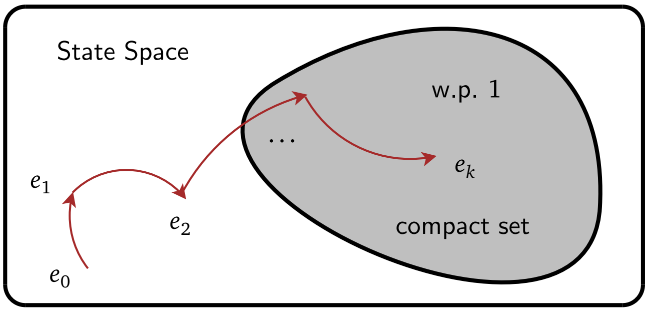

Figure 3.1: Illustration of an ergodic process in an uncountable state space.

different types of state spaces, are provided[1].
Since the control loops in the networked system of our interest are disturbed by stochastic process noise $[w_k]$, the network state $[x_k^\top, e_k^\top]^\top$ behaves stochastically, as described in (3.2). This urges us to employ an appropriate, i.e., the strongest, stochastic stability concept accordingly. Stability concept employed in this chapter is the notion of f-ergodicity. Using this concept, one can study the long-term average behavior of a stochastic process.

As already discussed, the network-induced error state e_k defined in (2.17) is a homogeneous Markov chain evolving in the uncountable state-space $\mathbb{R}^n$. This practically incurs that there exists a non-zero transition probability between each two states of the state-space $\mathbb{R}^n$. In the other words, the error Markov chain may evolve freely over the entire state-space from every state to every other, with a finite number of transitions. Stability notions in uncountable state spaces often generalize their counterparts in countable spaces, where the latter, stability concepts are rather more straightforward due to countability of the events in the σ-algebra [86]. One of those stochastic stability concepts which can be extended to be applied for uncountable state-spaces, is *ergodicity*, and is introduced in the following:

Definition 3.1. A random process is called *ergodic* if the time-average of its events over one sample sequence of transitions represents the behavior of the process over the entire state-space over which the process may evolve.

It concludes from the definition 3.1 that, if a process is ergodic, its state evolves over its corresponding state-space according to an invariant finite-moment probability measure.

Having discussed the different frameworks for stability analysis of stochastic processes, ergodicity can be considered as a concept of mean stability. One of the most common variations of such stability concept is Lyapunov mean square stability (LMSS) which automatically holds, if ergodicity is certified. The concept of ergodicity is graphically depicted in Figure 3.1.

In order to discuss the necessary conditions incurring ergodicity for a stochastic process with a Markov state, we first introduce the following concepts of stochastic processes, and properties of Markov chains.

Definition 3.2. [92] Let the Markov chain $\Phi = (\Phi_0, \Phi_1, \dots)$ evolve in some general state space X, with the individual random variables measurable with respect to some known σ-algebra $\mathscr{B}(X)$. Then Φ is said to be *positive Harris recurrent* (PHR) if

[1] In this dissertation, more comprehensive discussions about probability theory, stochastic processes and concept of stochastic stability for Markov chains are presented in Appendices A.6 and A.7

1. There exists a σ-algebra measure $\nu(B) > 0$ for a set $B \in \mathscr{B}$ such that for all initial states $\Phi_0 \in X$
$$P(\Phi_k \in B, k < \infty) = 1,$$
2. Φ admits a unique invariant probability measure.

Intuitively, definition 3.2 expresses that if a state of a PHR Markov chain leaves a set $B \in \mathscr{B}$ with non-zero probability, then the state returns to B after finite transitions with probability one. It is clear that set B is a compact set, as otherwise leaving the set would be mathematically meaningless.

Proposition 3.1. *[86] Let $f \geq 1$ be a real-valued function in $\mathbb{R}^n$. A Markov chain Φ is said to be f-ergodic, if one of the followings hold:*

1. *Φ is positive Harris recurrent with the unique invariant probability measure π,*
2. *the expectation $\pi(f) := \int f(\Phi_k)\pi(d\Phi_k)$ is finite,*
3. *$\lim_{k\to\infty} \|P^k(\Phi_0, .) - \pi\|_f = 0$, for every initial value $\Phi_0 \in X$, where $\|\nu\|_f = \sup_{|g|\leq f} |\nu(g)|$.*

The following definition introduces the notion of Markov chain gradient (drift) in discrete time, with respect to a real-valued function of states.

Definition 3.3 (Drift for Markov chains)**.** Let $V : \mathbb{R}^n \to [0, +\infty)$ be a real-valued function and Φ be a Markov chain. The drift operator Δ is defined for any non-negative measurable function V as follows

$$\Delta V(\Phi_k) = \mathsf{E}[V(\Phi_{k+1})|\Phi_k] - V(\Phi_k), \ \ \Phi_k \in \mathbb{R}^n. \tag{3.3}$$

Definition 3.4. A subset $C \in \mathscr{B}(X)$ is called ν-petite if a non-trivial measure ν on $\mathscr{B}(X)$ exists such that for all $x \in C$, and $B \in \mathscr{B}(X)$, the sampled chain Φ_a satisfies

$$K(x, B) \geq \nu(B),$$

where $K(x, B) := \sum_0^\infty P^n(x, B)a(n)$ is the probability transition kernel of the sampled chain with sampling distribution a.

Definition 3.5. A subset $C \in \mathscr{B}(X)$ of the measurable space $(X, \mathscr{B})$ is called ν-small if a non-trivial measure ν on $\mathscr{B}(X)$ and $k > 0$ exists such that for all $x \in C$, and $B \in \mathscr{B}(X)$

$$P^k(x, B) \geq \nu(B).$$

Proposition 3.2. *[86, 5.5.2] If a subset $C \in \mathscr{B}(X)$ is ν-small, then C is petite.*

Proposition 3.3. *[86, 6.3.3] Assume X is a linear state-space model, then every compact subset of X is small.*

One may conclude from definitions 3.4 and 3.5, and propositions 3.2 and 3.3, that every compact subset of linear state spaces is petite. Having the overview of the necessary concepts and terminologies, the following theorem summarizes f-ergodicity of Markov chains in general state spaces [86, Ch. 14].

Theorem 3.1 (f-Norm Ergodic Theorem). *Suppose that the Markov chain Φ is ψ-irreducible and aperiodic and let $f(\Phi) \geq 1$ be a real-valued function in $\mathbb{R}^n$. If a petite set $\mathscr{D}$ and a non-negative real-valued function V exists such that $\Delta V(\Phi) \leq f(\Phi)$ for every $\Phi \in \mathbb{R}^n \backslash \mathscr{D}$ and $\Delta V < \infty$ for $\Phi \in \mathscr{D}$, then the Markov chain Φ is said to be f-ergodic.*

In summary, f-ergodicity confirms that the Markov state evolves according to an invariant finite-variance measure over entire state-space. This ensures that the Markov chain is a stationary process, and guarantees if the Markov state leaves some compact subsets of σ-algebra $\mathscr{B}(X)$, it returns to these subsets in finite time with probability one.

3.3.2 Stability Results

In this section, we address stochastic stability of the shared resource multiple-loop NCS described in (2.1)-(2.4), under the pure probabilistic scheduling policy (3.1). As discussed in the previous chapter, assuming that the emulation-based control laws are pre-designed according to (2.3), the overall NCS stability will be achieved if the network-induced error Markov chain e_k (2.17) is shown to be stable. Convergence of the error state e_k, which is shown to be a homogeneous, aperiodic and ψ-irreducible Markov chain, is ensured in this section in terms of f-ergodicity. In what follows, we first introduce a necessary modification to the drift operator Δ in definition 3.3, as a direct consequence of having the capacity constraint (2.2). Afterwards, we invoke *Theorem* 3.1 in order to infer f-ergodicity of the error state e_k which eventually guarantees overall stability of the described NCS.

First of all, we select the following non-negative real-valued state-dependent Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$ at time-step k:

$$V(e_k) = \sum_{i=1}^{N} \left(e_k^{i^\mathsf{T}} e_k^i \right)^{\frac{p}{2}} = \sum_{i=1}^{N} \| e_k^i \|_2^p, \tag{3.4}$$

where N is the number of local control loops. The expected value of the above Lyapunov function is indeed the summation of p-moments of probability distributions corresponding to each local error state of the individual sub-systems $\{1, \ldots, N\}$. This is an often-standard Lyapunov candidate for systems evolving on probability spaces. However, due to the characteristics of the selected Lyapunov function (3.4), and the capacity constraint (2.2), f-ergodicity of the error Markov chain (2.17) cannot always be guaranteed by employing the one-step transition drift ΔV in (3.3), i.e., for $k \to k+1$. We illustrate this observation for $N = 2$ by constructing the following example. Furthermore, through this illustrative example, we see how the drift should be modified to become suitable for invoking *Theorem* 3.1 within our problem setup.

Illustrative example: Consider an NCS composed of two identical scalar sub-systems, parameterized by $A_1 = A_2 = A$, with the plants described as (2.1). Assume that the communication channel has capacity $c = 1$, meaning that only one sub-system is allowed to transmit at every time-step. For the purpose of illustration, assume the initial error values at time-step k are also identical, i.e., $e_k^1 = e_k^2 = \bar{e}_k$, and $p = 2$. The noise distributions are assumed to be i.i.d. for each sub-system and the random noise values are chosen from $\mathcal{N}(0, W)$. Based on the scheduling rule (3.1), the transmission chance for each sub-system is $\frac{1}{2}$ at the next time-step $k+1$, because the initial error values e_k^1 and e_k^2 are equal. In order to check if the error transition from $k \to k+1$ admits an f-ergodic process, we invoke *Theorem* 3.1 and employ the Lyapunov function (3.4). Having $e_k = [e_k^1 \; e_k^2]^\top$, it follows from the drift operator in *Definition* 3.3 that:

$$\begin{aligned}\Delta V(e_k) &= \mathsf{E}[V(e_{k+1})|e_k] - V(e_k)\\ &= \mathsf{E}\left[\|e_{k+1}^1\|_2^2 + \|e_{k+1}^2\|_2^2 | \bar{e}_k\right] - 2\|\bar{e}_k\|_2^2\\ &= \mathsf{E}\left[\|(1-\delta_{k+1}^1)A\bar{e}_k + w_k^1\|_2^2\right] + \mathsf{E}\left[\|(1-\delta_{k+1}^2)A\bar{e}_k + w_k^2\|_2^2|\bar{e}_k\right] - 2\|\bar{e}_k\|_2^2.\end{aligned}$$

Each sub-system has 50% chance of transmission, therefore the drift will be reduced to

$$\begin{aligned}\Delta V(e_k) = \quad & \underbrace{\frac{1}{2}\left[\mathsf{E}\left[\|w_k^1\|_2^2\right] + \mathsf{E}\left[\|A\bar{e}_k + w_k^2\|_2^2|e_k\right]\right]}_{\text{sub-system 1 transmits at time-step } k+1 \text{ with probability } \frac{1}{2}}\\ + \quad & \underbrace{\frac{1}{2}\left[\mathsf{E}\left[\|A\bar{e}_k + w_k^1\|_2^2|\bar{e}_k\right] + \mathsf{E}\left[\|w_k^2\|_2^2\right]\right]}_{\text{sub-system 2 transmits at time-step } k+1 \text{ with probability } \frac{1}{2}} \quad -2\|\bar{e}_k\|_2^2\\ &= \frac{1}{2}\left[W + \mathsf{E}\left[\|A\bar{e}_k\|_2^2|\bar{e}_k\right] + W\right] + \frac{1}{2}\left[\mathsf{E}\left[\|A\bar{e}_k\|_2^2|\bar{e}_k\right] + W + W\right] - 2\|\bar{e}_k\|_2^2\\ &= 2W + \|A\bar{e}_k\|_2^2 - 2\|\bar{e}_k\|_2^2,\end{aligned}$$

where $W > 0$ is the variance of the Gaussian distribution $\mathcal{N}$. It is clear that for $A > \sqrt{2}$, the above drift is positive, which violates the necessary condition in *Theorem* 3.1 on a non-compact set. Hence, this example illustrates that f-ergodicity of the error Markov chain e_k is not always guaranteed under the scheduling policy (3.1) if the capacity is constrained. Now, we apply *Theorem* 3.1, however this time by computing the drift over two time-steps, i.e., $k \to k+2$. Assume that the scheduler has decided to grant the channel access to sub-system 1 at time-step $k+1$, given the initial values $e_k^1 = e_k^2 = \bar{e}_k$. This concludes that $e_{k+1}^1 = w_k^1$, and $e_{k+1}^2 = A\bar{e}_k + w_k^2$. Computing the drift over two time-steps concludes:

$$\begin{aligned}\Delta V(e_k, 2) &= \mathsf{E}[V(e_{k+2})|e_k] - V(e_k)\\ &= \mathsf{E}\left[\|e_{k+2}^1\|_2^2 + \|e_{k+2}^2\|_2^2|\bar{e}_k\right] - 2\|\bar{e}_k\|_2^2\\ &= \mathsf{E}\left[\|(1-\delta_{k+2}^1)Ae_{k+1}^1 + w_{k+1}^1\|_2^2|\bar{e}_k\right] + \mathsf{E}\left[\|(1-\delta_{k+2}^2)Ae_{k+1}^2 + w_{k+1}^2\|_2^2|\bar{e}_k\right] - 2\|\bar{e}_k\|_2^2\\ &= \mathsf{E}\left[\|(1-\delta_{k+2}^1)Aw_k^1\|_2^2\right] + W + \mathsf{E}\left[\|(1-\delta_{k+2}^2)Ae_{k+1}^2\|_2^2|\bar{e}_k\right] + W - 2\|\bar{e}_k\|_2^2\\ &\le \|A\|_2^2 W + W + \mathsf{E}\left[\|(1-\delta_{k+2}^2)Ae_{k+1}^2\|_2^2|\bar{e}_k\right] + W - 2\|\bar{e}_k\|_2^2\\ &= \left(\|A\|_2^2 + 2\right)W + \mathsf{E}\left[(1-\delta_{k+2}^2)\|Ae_{k+1}^2\|_2^2|\bar{e}_k\right] - 2\|\bar{e}_k\|_2^2.\end{aligned}$$

The inequality above is ensured knowing $\mathsf{E}\left[1-\delta_{k+2}^1\right] \le 1$. Now we compute $\mathsf{E}\left[1-\delta_{k+2}^2\right]$, which represents the expectation that sub-system 2 transmits, if $\delta_{k+2}^2 = 1$, or does not transmit if $\delta^2_{k+2}=0$, at time-step $k+2$, after not being awarded the channel access at time $k+1$. According to the law of iterated expectations[2], we have for the drift operator $\Delta V(e_k,2)$,

$$
\begin{aligned}
\Delta V(e_k,2) &\leq \big(\|A\|_2^2+2\big)W + \mathsf{E}\big[\mathsf{E}\big[\big(1-\delta^2_{k+2}\big)\|Ae^2_{k+1}\|_2^2|e_{k+1}\big]\big|\bar{e}_k\big]-2\|\bar{e}_k\|_2^2 \\
&\leq \big(\|A\|_2^2+2\big)W + \|A\|_2^2\,\mathsf{E}\big[\mathsf{E}\big[\big(1-\delta^2_{k+2}\big)|e_{k+1}\big]\|e^2_{k+1}\|_2^2\big|\bar{e}_k\big]-2\|\bar{e}_k\|_2^2 \\
&\leq \big(\|A\|_2^2+2\big)W + \|A\|_2^2\,\mathsf{E}\left[\frac{\|e^1_{k+1}\|_2^2}{\|e^1_{k+1}\|_2^2+\|e^2_{k+1}\|_2^2}\,\|e^2_{k+1}\|_2^2\Big|\bar{e}_k\right]-2\|\bar{e}_k\|_2^2 \\
&= \big(\|A\|_2^2+2\big)W + \|A\|_2^2\,\mathsf{E}\left[\frac{\|w^1_{k}\|_2^2}{\|w^1_{k}\|_2^2+\|e^2_{k+1}\|_2^2}\,\|e^2_{k+1}\|_2^2\Big|\bar{e}_k\right]-2\|\bar{e}_k\|_2^2 \\
&\leq \big(\|A\|_2^2+2\big)W + \|A\|_2^2\,\mathsf{E}\left[\frac{\|w^1_{k}\|_2^2}{\|e^2_{k+1}\|_2^2}\,\|e^2_{k+1}\|_2^2\Big|e_k\right]-2\|\bar{e}_k\|_2^2 \\
&= \big(\|A\|_2^2+2\big)W + \|A\|_2^2 W-2\|\bar{e}_k\|_2^2 \\
&= 2\big(\|A\|_2^2+1\big)W-2\|\bar{e}_k\|_2^2.
\end{aligned}
$$

In line with *Theorem* 3.1, we define $f(e_k) = 2\big[\epsilon\|\bar{e}_k\|_2^2 - W\big(\|A\|_2^2+1\big)\big]$ where $\epsilon \in (0,1]$. Therefore, we can find a small set (identically compact set since we have LTI NCS model) $\mathcal{D}$ such that for $\bar{e}_k \in \mathbb{R}^n/\mathcal{D}$, the condition $f \geq 1$ is satisfied, and $\Delta V(e_k,2) \leq f$. In addition, it is obvious that $\Delta V(e_k,2) < \infty$ for $\bar{e}_k \in \mathcal{D}$, and therefore *Theorem* 3.1 holds and the error Markov chain e_k is f-ergodic.

Intuitively, only after all sub-systems have non-zero chances to transmit, a negative drift ΔV over some interval of interest can be guaranteed, if the channel access is supposed to be scheduled according to the law (3.1). Having non-zero transmission probabilities strongly depends on the number of sub-systems N and the channel capacity c. As we observed in the illustrative example, where $N=2$ and $c=1$, the minimum length of the interval over which a negative drift of the error Markov chain could be surely achieved is $\frac{N}{c}=2$. This observation suggests that f-ergodicity can be shown by employing the original one-step transition drift operator (3.3), under the scheduling policy (3.1) only if the communication channel has enough capacity for all sub-systems at least to have chances of transmission. It is very important to note that, having non-zero chance of transmission does not guarantee an eventual transmission. As we will discuss in the following, overall network-induced error e_k defined in (2.17) can be shown to be converging only if all its components, which are indeed the local error states from all sub-systems, are in expectation recurrent.

Proposition 3.4. *Consider the multiple-loop NCS described in (2.1)-(2.4), where the channel access is scheduled according to the error-dependent probabilistic law (3.1). Over all initial*

[2] The *law of iterated expectations*, also known as *towering property* and *law of total expectation* is an important toolbox to compute the expectation of a random variable which is conditioned on iterative random variables. One form of the law states that if X is a random variable with $\mathsf{E}[X]<\infty$, and the value of a conditioning variable y_2 is determined by that of another conditioning variable y_1, then

$$\mathsf{E}[X|y_1]=\mathsf{E}[\mathsf{E}[X|y_2]|y_1].$$

conditions x_0, and for all $A_i \in \mathbb{R}^{n_i \times n_i}$ and W_i's, f-ergodicity of the error Markov chain (2.17) can sufficiently be guaranteed by computing the drift of the Lyapunov function (3.4) over time intervals with lengths equal or greater than $\lceil \frac{N}{c} \rceil$.

Proof. Consider $\frac{N}{c} > 1$, which implies the integer $\lceil \frac{N}{c} \rceil \geq 2$, and assume that the dynamics of the network-induced error Markov chain e_k is monitored while transiting over an interval of length $l \leq \lceil \frac{N}{c} \rceil - 1$. Over such an interval, there exists exactly cl transmission possibilities. It is straightforward to calculate that

$$cl \leq c\left(\lceil \frac{N}{c} \rceil - 1\right) < N, \tag{3.5}$$

where, the last inequality follows as $\lceil \frac{N}{c} \rceil - 1 < \frac{N}{c}$. From (3.5), it deterministically concludes that at least one sub-system remains without a transmission and consequently open-loop over the entire interval of length l (at most $N - c$ sub-systems may remain open-loop if the same c sub-systems transmit subsequently at every time-step). This means that, at least one sub-system does not have any chance of transmission due to scarcity of possibilities. As the policy (3.1) randomizes the channel access, one can imagine a certain sub-system i with unstable plant $\mathscr{P}_i$, with its initial state outside a compact set in the state-space, is the sub-system which has not transmitted over the interval with length l. Therefore, the error of this sub-system, i.e. e_k^i does not reset over the mentioned interval and furthermore it is amplified by the system matrix A_i, as described in (2.8). The term e_k^i then becomes an increasing component of the aggregate network-induced error e_k in (2.17) when computing the drift operator over the interval of length l which results in violation of the conditions in *Theorem* 3.1. $\square$

Remark 3.2. It should be noted that calculation of the drift of the Lyapunov function (3.4) over time intervals with minimum length $\lceil \frac{N}{c} \rceil$ provides only a sufficient condition, not necessary, to guarantee a negative drift. In the other words, having a positive drift over any interval of length shorter than $\lceil \frac{N}{c} \rceil$, nothing can be said about f-ergodicity of the error state. Intuitively, the drift value depends not only the time interval over which it is computed, but also on the system parameters x_0^i's, A_i's and W_i's. In fact, over the mentioned interval, a negative drift is ensured for all values of x_0^i's, A_i's and W_i's, however, negative drifts over shorter intervals can also be achieved for certain values of those system-related parameters.

Remark 3.3. The immediate conclusion from *Proposition* 3.4 is that even if a single state of a Markov process is not recurrent, i.e. it is evanescent, the overall process is not guaranteed to be f-ergodic. In our described NCS setting, even if a single error state which is steered by scheduling law and additionally is driven by the stochastic noise, has zero probability of recurrency, then existence of a stationary distribution for the overall network-induced error cannot be guaranteed.

The above proposition stimulates us to investigate the f-ergodicity of the error Markov chain (2.17) by computing the drift over the interval with minimum length $l = \lceil \frac{N}{c} \rceil$. It is worth noting that ergodicity is an asymptotic property of Markov chains, hence showing ergodic according to a negative drift over an interval implies that ergodicity will be achieved by defining the drift over longer intervals [93]. In other words, the ergodicity certificate remains valid by employing the drift operator over a longer horizon than $\lceil \frac{N}{c} \rceil$.
It should be stressed that having the minimum length of the interval equals $\lceil \frac{N}{c} \rceil$ does not

assure that all N sub-systems eventually transmit over the period. In fact, this might not even be required form the application point of view to have enough transmission possibilities for all sub-systems to transmit exactly once during every $\lceil \frac{N}{c} \rceil$ time intervals. Imagine that, one or some of the sub-systems in an NCS may require to have more frequent transmissions than the other sub-systems due to, for example, their respective plant instability. Considering the overall NCS performance as the primary metric to be maximized, those sub-systems are better to transmit more regularly. This results in a situation that not all N sub-systems transmit at exactly N transmission possibilities, as some sub-systems are required to transmit more occasionally than the others.

For the sake of simplicity, especially in stability proof, and avoid unnecessary complications, we assume the following capacity constraint at every time-step k:

$$\sum_{i=1}^{N} \delta_k^i = 1. \tag{3.6}$$

This concludes that at each time-step, exactly one sub-system is allowed to transmit. The following results are readily extendable for $\sum_{i=1}^{N} \delta_k^i = c > 1$, for the single-hop scenario, where the c dedicated communication links are assigned exclusively to c users at a time without collision. Having this said, and according to the proposition 3.4, we consider an interval of length N to compute the drift, in order to study f-ergodicity of the network-induced error e_k. Considering the initial time-step as k, we then infer f-ergodicity by computing the drift over the interval $[k, k+N]$, and accordingly modify the drift definition in (3.3). The modified drift operator, so-called *multi-step drift*, is defined over the interval $[k, k+N]$ with length N as follows:

Definition 3.6 (Multi-step drift for Markov chains)**.** Let $V : \mathbb{R}^n \to [0, +\infty)$ be a real-valued function and Φ a Markov chain evolving on the state space $\mathbb{R}^n$. The multi-step drift operator $\Delta V(\Phi_k, l)$ is defined over an interval with length l for any measurable function V as

$$\Delta V(\Phi_k, l) = \mathsf{E}[V(\Phi_{k+l}) | \Phi_k] - V(\Phi_k). \tag{3.7}$$

According to the Definition 3.6, and the communication constraint (3.6), the multi-step drift of the introduced Lyapunov function (3.4) can be expressed as

$$\Delta V(e_k, N) = \mathsf{E}\left[\sum_{i=1}^{N} \|e_{k+N}^i\|_2^p \Big| e_k \right] - \sum_{i=1}^{N} \|e_k^i\|_2^p. \tag{3.8}$$

Now we are ready to state the main theorem of this chapter, which guarantees stability of the overall NCS of our interest. Before proceeding to the theorem, we express the following result, which will be recalled frequently through the proof of the theorem. It is straightforward to show that the error state e_{k+N}^i of sub-system i, at the final time-step of the interval $[k, k+N]$, can be expressed as function of an earlier error state $e^i_{k+r'_i}$, where $r'_i \in [0, N-1]$, by stepping backward from time-step $k+N$ to the time-step $k+r'_i$:

$$e^i_{k+N} = \prod_{d=r'_i+1}^{N} \left(1-\delta^i_{k+d}\right) A_i^{N-r'_i} e^i_{k+r'_i} + \sum_{r=r'_i}^{N-1} \left[\prod_{d=r+2}^{N} \left(1-\delta^i_{k+d}\right) A_i^{N-r-1} w^i_{k+r} \right], \tag{3.9}$$

where, d is the scheduling counter, and we define $\prod_{d=N+1}^{N}\left(1-\delta^i_{k+d}\right) := 1$.

Theorem 3.2. *Consider an NCS consisting of N heterogeneous LTI stochastic sub-systems modeled as (2.1), and a transmission channel subject to the constraint (3.6), and the control, estimation, and scheduling laws given by (2.3), (2.4), and (3.1), respectively. Then, the network-induced error Markov chain (2.17) is f-ergodic.*

Proof. To study f-ergodicity of the error state e_k, we assume that our networked system has operated from the initial time-step k until time $k+N-1$ under the policy (3.1). Then, the last time-step of the interval, i.e. $k+N$, is scheduled taking into account all possible scenarios that might have happened over the previous time-steps $[k, k+N-1]$. Therefore, we define two disjoint and complementary sets of sub-systems, namely $\mathcal{G}$ and $\bar{\mathcal{G}}$, as follows:

- $\mathcal{G}$: contains sub-systems having transmitted at least once over the interval $[k, k+N-1]$, i.e., $\delta^{i\in\mathcal{G}}_{k+d} = 1$ for at least one $d \in \{0, 1, \ldots, N-1\}$,
- $\bar{\mathcal{G}}$: contain sub-systems having no transmission over the interval $[k, k+N-1]$, i.e., $\delta^{i\in\mathcal{G}}_{k+d} = 0$ for all $d \in \{0, 1, \ldots, N-1\}$,

where, $\mathcal{G} \cup \bar{\mathcal{G}} = N$. Depending on, firstly, when a sub-system i is granted the channel access, and secondly, the noise realizations w^i over the time interval $[k, k+N-1]$, the local error state e^i_{k+N-1} might have been entered an arbitrary but compact set with boundaries M_i or might be outside, i.e., being unbounded with respect to that arbitrary set. Therefore, we may, from another perspective, classify all sub-systems $\{1, \ldots, N\}$ to two complementary and mutually exclusive sets of sub-systems, as follows:

1 : After $N-1$ time-steps, i^{th} error state is entered an arbitrary compact set $\mathbf{M}_i$ with boundary M_i, with $\|e^i_{k+N-1}\|_2^p \leq M_i$,

2 : After $N-1$ time-steps, i^{th} error state is outside an arbitrary compact set $\mathbf{M}_i$ with boundary M_i, with $\|e^i_{k+N-1}\|_2^p > M_i$.

In other words, the first case denotes that the local error state of a sub-system i has been evolved, such that, after $N-1$ time-steps it ended up inside a compact set with the boundary M_i, while in the latter, the error state stays outside of a compact set with the known boundary M_i. This is schematically illustrated in Figure 3.2. It is noteworthy that an unbounded variable is not necessarily infinite. The latter classification, compared to dividing the sub-systems into the introduced sets $\mathcal{G}$ and $\bar{\mathcal{G}}$, is indeed clarifies the statistical independence

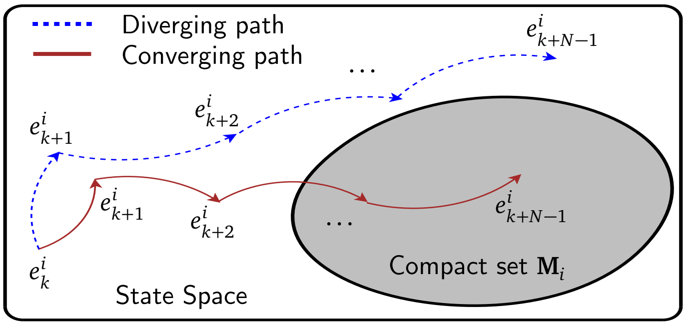

Figure 3.2: Sample evolution of error state from time k till time $k+N-1$. (if $e^i_{k+N-1} \in \mathbf{M}_i$, then $\|e^i_{k+N-1}\|_2^p \leq M_i$, otherwise $\|e^i_{k+N-1}\|_2^p > M_i$.)

between the transmission sequence and the sequence of noise realizations, over the interval $[k, k+N-1]$. Intuitively, there is no guarantee for a sub-system which has transmitted just recently not to have a noticeable error value in the subsequent time-step, due to possibly having a large noise value. On the other hand, even if a sub-system does not transmit for a quite long period, its error does not necessarily becomes very large, because the noise realization may take mathematically opposite values.

Considering the sets $\mathcal{G}$ and $\bar{\mathcal{G}}$, along with the latter classification with respect to being either inside or outside of an arbitrary compact set, we can divide the entire state-space of our Markov chain into the following three complementary and mutually exclusive sub-state-spaces. We show the f-ergodicity of the Markov chain within each of those cases to conclude the f-ergodicity over the entire state space $\mathbb{R}^n$. The error state, corresponding to a sub-system i, is characterized by exactly one of the following three cases:

c_1 A sub-system i satisfies the condition $\|e^i_{k+N-1}\|_2^p \leq M_i$ and sub-system i has either transmitted over the time interval $[k, k+N-1]$ or not, i.e. either $i \in \mathcal{G}$ or $i \in \bar{\mathcal{G}}$,

c_2 A sub-system i satisfies the condition $\|e^i_{k+N-1}\|_2^p > M_i$ and sub-system i has transmitted at least once over the time interval $[k, k+N-1]$, i.e. $i \in \mathcal{G}$,

c_3 A sub-system i satisfies the condition $\|e^i_{k+N-1}\|_2^p > M_i$ and sub-system i has never transmitted over the time interval $[k, k+N-1]$, i.e. $i \in \bar{\mathcal{G}}$.

It is required by the multi-step drift (3.8) to compute the expectation of p-powered 2-norm of the error vector e^i_{k+N}, for general integer p. Recalling the expression (3.9), and exploiting the *binomial Theorem* for a sub-system i, we can break down (3.8) to partial expectations in order to simplify the computations as follows:

$$
\begin{aligned}
&\mathsf{E}\big[V(e^i_{k+N})\big] = \mathsf{E}\big[\|e^i_{k+N}\|_2^p\big] \\
&= \mathsf{E}\left[\left\| \prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r_i'} e^i_{k+r_i'} + \sum_{r=r_i'}^{N-1} \left[\prod_{d=r+2}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r-1} w^i_{k+r} \right] \right\|_2^p \right] \\
&= \mathsf{E}\left[\left\| \prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r_i'} e^i_{k+r_i'} \right\|_2^p \right] + \mathsf{E}\left[\left\| \sum_{r=r_i'}^{N-1} \prod_{d=r+2}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r-1} w^i_{k+r} \right\|_2^p \right] \qquad (3.10) \\
&+ \sum_{\substack{k_1+k_2=p \\ 1\le k_1,k_2<p}} \frac{p!}{k_1!\,k_2!} \mathsf{E}\left[\left\| \prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r_i'} e^i_{k+r_i'} \right\|_2^{k_1} \left\| \sum_{r=r_i'}^{N-1} \prod_{d=r+2}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r-1} w^i_{k+r} \right\|_2^{k_2} \right].
\end{aligned}
$$

Statistical independence of $e^i_{k+r_i'}$ and w^i_{k+r} for $r \in [r_i', N-1]$ implies that the last line in the above expression can be reduced to

$$
\begin{aligned}
&\sum_{\substack{k_1+k_2=p \\ 1\le k_1,k_2<p}} \frac{p!}{k_1!\,k_2!} \mathsf{E}\left[\left\| \prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r_i'} e^i_{k+r_i'} \right\|_2^{k_1} \left\| \sum_{r=r_i'}^{N-1} \prod_{d=r+2}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r-1} w^i_{k+r} \right\|_2^{k_2} \right] \le \\
&\sum_{\substack{k_1+k_2=p \\ 1\le k_1,k_2<p}} \frac{p!}{k_1!\,k_2!} \|A_i^{N-r_i'}\|_2^{k_1} \mathsf{E}\left[\prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) \|e^i_{k+r_i'}\|_2^{k_1} \right] \sum_{r=r_i'}^{N-1} \mathsf{E}\left[\prod_{d=r+2}^{N} \big(1-\delta^i_{k+d}\big) \|A_i^{N-r-1} w^i_{k+r}\|_2^{k_2} \right]
\end{aligned}
\qquad (3.11)
$$

where, all the terms in the latest expression above are bounded for arbitrary finite values of p, k_1, k_2, and A_i, except $\mathsf{E}\Big[\prod_{d=r_i'+1}^{N}\big(1-\delta^i_{k+d}\big)\,\|e^i_{k+r_i'}\|_2^{k_1}\Big]$. However, if the first p-powered term in the expression (3.10) is shown to be bounded, then the boundedness of the aforementioned k_1-powered term is automatically followed as $k_1 < p$. Extending the same derivations as (3.10), but for all sub-systems $i \in \{1, \dots, N\}$, we have

$$
\begin{aligned}
\mathsf{E}\left[V(e_{k+N}) | e_k\right] &= \sum_{i=1}^{N} \mathsf{E}\left[\|e^i_{k+N}\|_2^p | e_k\right] \\
&\le \sum_{i=1}^{N} \mathsf{E}\left[\left\| \prod_{d=r_i'+1}^{N} \big(1-\delta^i_{k+d}\big) A_i^{N-r_i'} e^i_{k+r_i'} \right\|_2^p \Big| e_k \right] + g_i^{b+}(w^i) + g_i^{+}(e^i, w^i), \qquad (3.12)
\end{aligned}
$$

where, the noise-dependent $g_i^{b+}(w^i)$ stands for the positive and bounded term $\mathsf{E}\Big[\big\|\sum_{r=r_i'}^{N-1}\prod_{d=r+2}^{N}\big(1-\delta^i_{k+d}\big)A_i^{N-r-1}w^i_{k+r}\big\|_2^p\Big]$ in (3.10), while noise and error-dependent term $g_i^{+}(e^i, w^i)$ stands for the positive but conditionally-bounded term (3.11).

Starting from the first case c_1, it is straightforward to show the conditions in *Theorem* 3.1 hold when employing the multi-step drift introduced in (3.8). From (3.12), with $r_i' = N-1$, it yields for all sub-systems $i \in c_1$

$$\begin{aligned}\Delta V(e_k^{c_1},N) &\leq \sum_{i\in c_1} \mathsf{E}\big[(1-\delta^i_{k+N})\|A_i e^i_{k+N-1}\|_2^p \big| e_k^{c_1}\big] + \sum_{i\in c_1}\mathsf{E}\big[\|w^i_{k+N-1}\|_2^p\big] + g_i^+(e^i,w^i) - \sum_{i\in c_1}\|e^i_k\|_2^p \\ &\leq \sum_{i\in c_1}\|A_i\|_2^p \mathsf{E}\big[\|e^i_{k+N-1}\|_2^p\big| e_k^{c_1}\big] + \sum_{i\in c_1}\mathsf{E}\big[\|w^i_{k+N-1}\|_2^p\big] + g_i^+(e^i,w^i) - \sum_{i\in c_1}\|e^i_k\|_2^p, \end{aligned} \tag{3.13}$$

where in this case, $g_i^+(e^i,w^i)$ can be rewritten according to (3.11) as

$$\begin{aligned} g_i^+(e^i,w^i) &= \sum_{1\leq k_1,k_2<p}^{k_1+k_2=p} \frac{p!}{k_1!\,k_2!}\|A_i\|_2^{k_1}\mathsf{E}\Big[\big(1-\delta^i_{k+N}\big)\|e^i_{k+N-1}\|_2^{k_1}\Big]\mathsf{E}\big[\|w^i_{k+N-1}\|_2^{k_2}\big] \\ &\leq \sum_{1\leq k_1,k_2<p}^{k_1+k_2=p} \frac{p!}{k_1!\,k_2!}\|A_i\|_2^{k_1}\mathsf{E}\Big[\|e^i_{k+N-1}\|_2^{k_1}\Big]\mathsf{E}\Big[\|w^i_{k+N-1}\|_2^{k_2}\Big]. \end{aligned}$$

In case c_1, the condition $\|e^i_{k+N-1}\|_2^p \leq M_i$ holds for all sub-systems $i \in c_1$. Therefore, from (3.13), we can derive the following upper bound for $\Delta V(e_k^{c_1},N)$ as follows:

$$\begin{aligned}\Delta V(e_k^{c_1},N) \leq& \sum_{i\in c_1} M_i\|A_i\|_2^p + \sum_{i\in c_1}\mathsf{E}\Big[\|w^i_{k+N-1}\|_2^p\Big] \\ &+ \sum_{1\leq k_1,k_2<p}^{k_1+k_2=p}\frac{p!}{k_1!\,k_2!}M_i^{\frac{k_1}{p}}\|A_i\|_2^{k_1}\mathsf{E}\big[\|w^i_{k+N-1}\|_2^{k_2}\big] - \sum_{i\in c_1}\|e^i_k\|_2^p. \end{aligned} \tag{3.14}$$

For a sub-system i characterized by case c_2, it is assumed that at least a transmission is occurred. Assume that the latest transmission was at a time-step $k + r_i' + 1$, with $r_i' \in [0, N-2]$. This means $\delta^i_{k+r_i'+1} = 1$. Thus, employing (3.12) for the case c_2, the drift operator $\Delta V(e_k^{c_2},N)$ reduces to

$$\begin{aligned}\Delta V(e_k^{c_2},N) &\leq \sum_{i\in c_2} g_i^{b+}(w^i) - \sum_{i\in c_2}\|e^i_k\|_2^p \\ &\leq \sum_{i\in c_2}\sum_{r=r_i'}^{N-1}\|A_i^{N-r-1}\|_2^p\,\mathsf{E}\Big[\|w^i_{k+r}\|_2^p\Big] - \sum_{i\in c_2}\|e^i_k\|_2^p. \end{aligned} \tag{3.15}$$

Within the third case c_3, the condition $\delta^i_{k'} = 0$ hold for all sub-systems $i \in c_3$ and at all time-steps $k' \in [k, k+N-1]$. Therefore, setting $r_i' = 0$ and knowing that the initial values e^i_k's are given, expression (3.12) can be rewritten as follows:

$$
\begin{aligned}
\mathsf{E}\left[V(e^{c_3}_{k+N})|e_k\right] \leq & \sum_{i\in c_3}\left[\|A_i^N\|_2^p\|e_k^i\|_2^p + \sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1}w_{k+r}^i\|_2^p\right]\right] \\
& + \sum_{i\in c_3}\left[\sum_{1\leq k_1,k_2<p}^{k_1+k_2=p}\frac{p!}{k_1!\,k_2!}\|A_i^N\|_2^{k_1}\|e_k^i\|_2^{k_1}\sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1}w_{k+r}^i\|_2^{k_2}\right]\right] \\
\leq & \sum_{i\in c_3}\left[\|A_i^N\|_2^p V(e_k) + \sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1}w_{k+r}^i\|_2^p\right]\right] \\
& + \sum_{i\in c_3}\left[\sum_{1\leq k_1,k_2<p}^{k_1+k_2=p}\frac{p!}{k_1!\,k_2!}\|A_i^N\|_2^{k_1}V(e_k)\sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1}w_{k+r}^i\|_2^{k_2}\right]\right],
\end{aligned}
\tag{3.16}
$$

where, the last line holds given that $\|e_k^i\|_2 \geq 1$ to ensure $V(e_k) \geq \|e_k^i\|_2^{k_1}$ (if $\|e_k^i\|_2 < 1$, then f-ergodicity is already guaranteed.). Since the three introduced cases $c_1 - c_3$ are complementary and disjoint, we can assign probabilities $\mathsf{P}_{c_1} - \mathsf{P}_{c_3}$ to each of those cases, with $\mathsf{P}_{c_1} - \mathsf{P}_{c_3}$ denoting the probabilities that a sub-system i ends up in the corresponding case over the interval $[k, k+N]$. Hence, the multi-step drift operator (3.8) needs to be rewritten as the summation of the partial drift operators for each case $c_1 - c_3$, associated with their corresponding probabilities $\mathsf{P}_{c_1} - \mathsf{P}_{c_3}$, as follows:

$$
\Delta V(e_k, N) = \sum_{c_l, l=1}^{3}\mathsf{P}_{c_l}\,\mathsf{E}\left[\sum_{i\in c_l}\|e_{k+N}^i\|_2^p|e_k\right] - \sum_{i=1}^{N}\|e_k^i\|_2^p. \tag{3.17}
$$

We will see later in the proof that, except P_{c_3}, incorporation of the probabilities P_{c_1} and P_{c_2} is not necessary to show f-ergodicity of the overall error state e_k, though computing them results in having less conservative stability margins. To compute P_{c_3}, we follow the described scenario. If an arbitrary sub-system j is supposed to stay in the set c_3, then j does not transmit over the entire interval $[k, k+N]$. Then, there exists another sub-system, let us say i, which transmits at the final time-step $k+N$, where i has transmitted before, at least for one time-step over the interval $[k, k+N-1]$. Assume $k+r_i'$ is the latest time-step before time-step $k+N$ that sub-system i has transmitted, i.e. $r_i' \in [0, N-1]$, and $\delta_{k+r_i'}^i = 1$. Then, the probability that sub-system i re-transmits at time-step $k+N$, in the presence of the sub-system j which has never transmitted over the time interval $[k, k+N-1]$, and in addition satisfies the condition $\|e_{k+N-1}^j\|_2^p > M_j$, can be computed as follows:

$$
\begin{aligned}
& \mathsf{P}\left[\delta_{k+N}^i = 1\middle|\delta_{k+r_i'}^i = 1, \delta_{k'}^j = 0, \|e_{k'}^j\|_2^p > M_j,\ \forall k' \in [k, k+N]\right] \\
& = \mathsf{E}\left[\mathsf{P}\left[\delta_{k+N}^i = 1\middle|e_k\right]\middle|\delta_{k+r_i'}^i = 1, \delta_{k'}^j = 0, \|e_{k'}^j\|_2^p > M_j,\ \forall k' \in [k, k+N]\right] \\
& = \mathsf{E}\left[\frac{\|e_{k+N-1}^i\|_2^p}{\sum_{l=1}^N\|e_{k+N-1}^l\|_2^p}\middle|\delta_{k+r_i'}^i = 1, \delta_{k'}^j = 0, \|e_{k'}^j\|_2^p > M_j,\ \forall k' \in [k, k+N]\right]
\end{aligned}
\tag{3.18}
$$

Since sub-system i has had an earlier transmission at $k + r'_i$, we know from (3.9) that

$$e^i_{k+N-1} = \sum_{r=r'_i}^{N-2} A_i^{N-r-1} w^i_{k+r}.$$

Therefore, (3.18) is reduced to

$$\begin{aligned}
&\mathsf{E}\left[\frac{\|\sum_{r=r'_i}^{N-2} A_i^{N-r-1} w^i_{k+r}\|_2^p}{\sum_{l_1\in c_1}\|e^{l_1}_{k+N-1}\|_2^p + \sum_{l_2\in c_2\cup c_3}\|e^{l_2}_{k+N-1}\|_2^p}\Big|\delta^j_{k'}=0, \|e^j_{k'}\|_2^p > M_j,\ \forall k' \in [k, k+N]\right]\\
&\le \mathsf{E}\left[\frac{\|\sum_{r=r'_i}^{N-2} A_i^{N-r-1} w^i_{k+r}\|_2^p}{\sum_{l_2\in c_2\cup c_3}\|e^{l_2}_{k+N-1}\|_2^p}\Big|\delta^j_{k'}=0, \|e^j_{k'}\|_2^p > M_j,\ \forall k' \in [k, k+N]\right]\\
&\le \mathsf{E}\left[\frac{\|\sum_{r=r'_i}^{N-2} A_i^{N-r-1} w^i_{k+r}\|_2^p}{\sum_{l_2\in c_2\cup c_3} M_{l-2}}\right] = \frac{\sum_{r=r'_i}^{N-2}\mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^p\right]}{\sum_{l_2\in c_2\cup c_3} M_{l_2}} = \mathsf{P}_{c_3}\,.
\end{aligned} \tag{3.19}$$

Since the expression in the numerator, i.e., $\sum_{r=r'_i}^{N-2}\mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^p\right]$ is finite, one can infer from (3.19) that the probability of a subsequent transmission of a certain sub-system i, in the presence of other sub-systems $j \in c_3$, can be made arbitrarily close to zero by considering appropriate compact set boundaries M_{l_2}'s.

Recalling the expressions (3.14), (3.15), (3.16), and (3.19), we can rewrite the multi-step drift operator (3.17) as follows:

$$
\begin{aligned}
\Delta V(e_k, N) &\leq \mathsf{E}\left[\sum_{i\in c_1} \|e^i_{k+N}\|_2^p | e_k\right] + \mathsf{E}\left[\sum_{i\in c_2} \|e^i_{k+N}\|_2^p | e_k\right] + \mathsf{P}_{c_3}\,\mathsf{E}\left[\sum_{i\in c_3} \|e^i_{k+N}\|_2^p | e_k\right] - \sum_{i=1}^{N} \|e^i_k\|_2^p \\
&\leq \sum_{i\in c_1} M_i \|A_i\|_2^p + \sum_{i\in c_1} \mathsf{E}\left[\|w^i_{k+N-1}\|_2^p\right] + \sum_{1\leq k_1,k_2<p}^{k_1+k_2=p} \frac{p!}{k_1!\,k_2!} M_i^{\frac{k_1}{p}} \|A_i\|_2^{k_1} \mathsf{E}\left[\|w^i_{k+N-1}\|_2^{k_2}\right] \\
&+ \sum_{i\in c_2}\sum_{r=r'_i}^{N-1} \|A_i^{N-r-1}\|_2^p\, \mathsf{E}\left[\|w^i_{k+r}\|_2^p\right] + \mathsf{P}_{c_3} V(e_k) \sum_{i\in c_3}\left[\|A_i^N\|_2^p + \sum_{r=0}^{N-1} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^p\right]\right] \\
&+ \mathsf{P}_{c_3} V(e_k) \sum_{i\in c_3}\left[\sum_{1\leq k_1,k_2<p}^{k_1+k_2=p} \frac{p!}{k_1!\,k_2!} \|A_i^N\|_2^{k_1} \sum_{r=0}^{N-1} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^{k_2}\right]\right] - V(e_k) \\
&= \underbrace{\mathsf{P}_{c_3} \sum_{i\in c_3}\left[\|A_i^N\|_2^p + \sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^p\right] + \sum_{1\leq k_1,k_2<p}^{k_1+k_2=p} \frac{p!}{k_1!k_2!} \|A_i^N\|_2^{k_1} \sum_{r=0}^{N-1}\mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|_2^{k_2}\right]\right]}_{\xi^{b^+}_{c_3}} V(e_k) \\
&+ \xi^{b^+}_{c_1,c_2} - V(e_k) \\
&= \left[\mathsf{P}_{c_3}\, \xi^{b^+}_{c_3} - 1\right] V(e_k) + \xi^{b^+}_{c_1,c_2}
\end{aligned}
$$

where, $\xi^{b^+}_{c_1,c_2}$ stands for the first four positive and bounded expressions immediately after the inequality sign above, correspond to the cases c_1 and c_2. The positive and bounded $\xi^{b^+}_{c_3}$ corresponds to the case c_3, and is shown in the above expression.
We define the real-valued function $f(e_k) = \varepsilon_f V(e_k) - \xi^{b^+}_{c_1,c_2}$, where $\varepsilon_f \in (0,1]$. Since the noise-dependent expression in the numerator of P_{c_3} is positive and constant, therefore, we can find appropriate compact sets with boundaries M_{l_2}'s and ε_f such that $\left[\mathsf{P}_{c_3}\xi^{b^+}_{c_3} - 1\right] \leq -\varepsilon_f$ which in turn implies that $\Delta V(e_k, N) \leq -f(e_k)$. On the other hand, since $\xi^{b^+}_{c_1,c_2}$ is constant, positive, and independent of the error state e^i, we can find appropriate ε_f and compact set $\mathscr{D}$ such that $f(e_k) \geq 1$, for $e_k \notin \mathscr{D}$. Moreover, the condition $\Delta V(e_k, N) < \infty$, when e_k is staying inside the compact set $\mathscr{D}$, holds since $\xi^{b^+}_{c_1,c_2}$ and N are both positive and finite. Thus, according to the *Theorem* 3.1, f-ergodicity of the error state e_k is readily followed, which ensures that the error state e_k is expected to converge towards the compact set $\mathscr{D}$ after N time-steps, starting from any initial values. □

After the presented lengthy proof, *Theorem* 3.2 ensures that the network-induced error state e_k is f-ergodic if the scheduling policy (3.1) is employed for channel access arbitration. This conveys that e_k is evolving according to an invariant finite-moment probability measure over the entire state-space $\mathbb{R}^n$. Taking this, together with the stabilizing feedback control laws (2.10), stochastic stability of the overall NCS with the aggregate state vector $[x_k^\top, e_k^\top]^\top$, under the capacity constraint (3.6), is readily guaranteed in terms of f-ergodicity.

Remark 3.4. The probability of observing the first and second cases c_1 and c_2 can also be computed considering the scheduling policy (3.1). This results in having less conservative stability margin in terms of smaller compact set $\mathscr{D}$, which itself is indicating the region of attraction for the error state. However, as the derived upper bounds for $\mathsf{E}\big[V(e^{c_1}_{k+N})\big]$ and $\mathsf{E}\big[V(e^{c_2}_{k+N})\big]$, in (3.14) and (3.15), respectively, are independent of the initial values, it is not theoretically required to compute P_{c_1} and P_{c_2}. Thus, we considered them as unity leading to more conservative upper-bounds.

3.4 Numerical Evaluations

Here in this section, we present numerical results to validate our stability claims and show that our proposed scheduling approach defined in (3.1) leads to performance enhancement compared to the conventional approaches. To make the numerical evaluations, we consider networked control systems with different number of scalar sub-systems. To take into account the heterogeneity of sub-systems, we assume within each NCS setup, to have two heterogeneous classes of sub-systems each containing finite and equal number of homogeneous sub-systems. The first class of sub-systems includes control loops with unstable plants, i.e. $A_1 > 1$, and the second class contains control loops with stable process, i.e. $A_2 < 1$. The system parameters are assumed to be $A_1 = 1.25$, $B_1 = 1$ for the first class[3] and $A_2 = 0.75$, $B_2 = 1$, for the second class. In both classes, the system states initiate from zero, i.e. $x_0^1 = x_0^2 = 0$ and the noise process is randomly realized by $\mathscr{N}(0,1)$. The channel capacity is set to be one, i.e. $c = 1$, meaning that only one sub-system is allowed to transmit at each time-step. This yields for the NCSs with $N > 2$ that, at every time-step, we have open-loop sub-systems with unstable processes. Recall that the separation property exists between the control law and the scheduling law synthesis. Thus, in order to evaluate solely on the performance of our proposed scheduling architecture, we assume to have local stabilizing dead-beat control laws with $L_i = \frac{A_i}{B_i}$ for each class $i \in \{1,2\}$. In addition, a model-based estimator, as given by (2.4), is assumed to be employed by the control side in order to estimate the system state x^i in case the expected transmission is blocked.

Figure 3.3 illustrates the performance comparisons between the proposed probabilistic prioritized event-based (denoted as "PEB" in Figure 3.3) scheduler (3.1) for different p powers and some of the practical and well-established scheduling methods. The simulations are performed for different NCS setups with the number of sub-systems $N \in \{2,4,6,8,10\}$. Since the error Markov chain (2.17) is noise driven, we look at the mean variance of the network-induced error per sub-system, which indicates the variance of the average distribution upon which the local error of a sub-system evolves. The averages are calculated by their empirical means through Monte Carlo simulations over a horizon of 100 000 samples. The lower bound is determined by relaxing the initial problem to have no resource constraint, but instead restrains the total average transmission rate per time-step to be 1. This can be calculated through a bi-level approach, discussed comprehensively in [33].

In order to compare our results with the conventional scheduling scenarios, we have chosen two channel scheduling schemes, the deterministic time-triggered TDMA protocol and the probabilistic random CSMA protocol. The time division multiple access (TDMA) (also

[3] We simply denote the system matrices of all homogeneous sub-systems included in the first class as A_1 and in the second class as A_2.

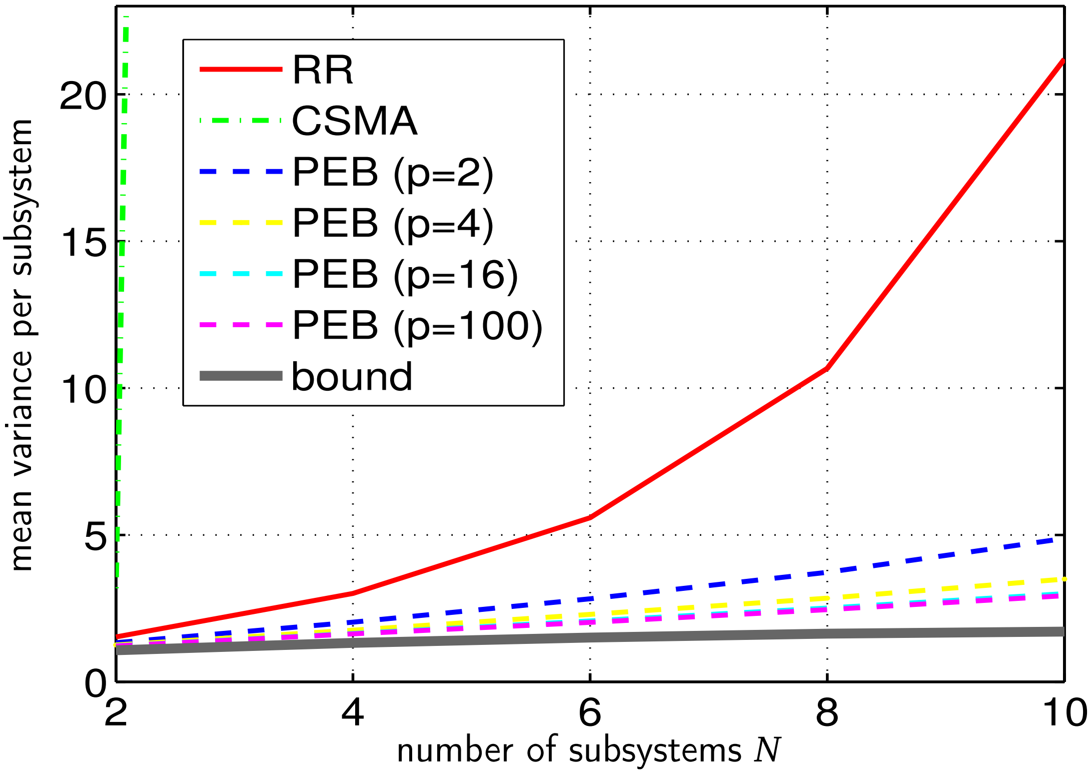

Figure 3.3: Comparison of the mean steady-state variance of e_k^i for various protocols and number of subsystems.

called "Round Robin" or "RR") is a periodic channel access scheme, where sub-systems use the communication channel based on a pre-given token ring, within which each sub-system transmits exactly once over every N time-steps interval. Therefore, in for example the NCS setup with $N = 4$, every sub-system transmits once every 4 time-steps. The idealized carrier sense multi access (CSMA) protocol operates in probabilistic fashion, and in its original definition, the probability that the communication channel is awarded to a sub-system, is fixed at each time and is not dependent on real-time situation of the NCS. In fact the transmission probability for each sub-system equals $\frac{1}{N}$ at each time-step. The described TDMA and CSMA approaches are schematically shown in Figure 3.4 and Figure 3.5.

Figure 3.3 illustrates that with increasing number of sub-systems competing for the communication channel, i.e. N, the performance gap between the PEB scheduler and the idealized CSMA and TDMA protocols becomes more evident. Unlike the TDMA and idealized CSMA approaches, the error variance under the PEB scheduler (3.1) grows slowly as the number of sub-systems N increases, and deviates moderately from the lower bound[4]. This suggests that the event-based protocol (3.1) is better fit for NCSs with a large number of connected sub-systems, especially when the shared communication resources are limited. According to the simulation results in Figure 3.3, the idealized CSMA protocol admits an

[4]Although, we have no optimality claim, the comparisons between the proposed event-based policy and more efficient token rings (near-optimal) for TDMA approach and non-uniform probabilities for CSMA scheme are presented in the next chapter.

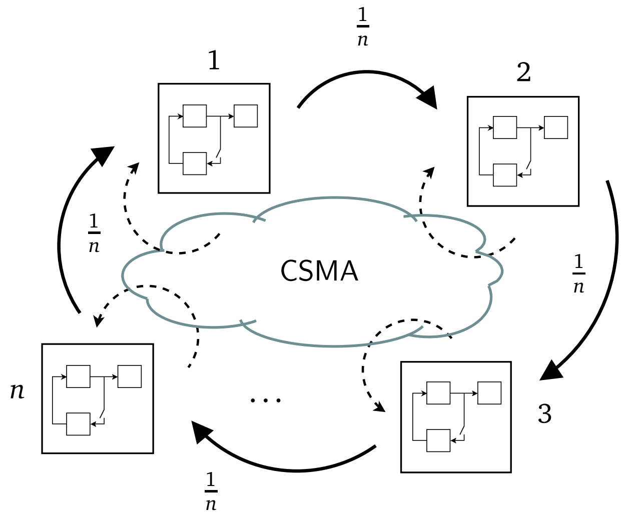

Figure 3.4: Carrier sense multiple access protocol. In this idealized case, each sub-system has equal probability of having access to the channel.

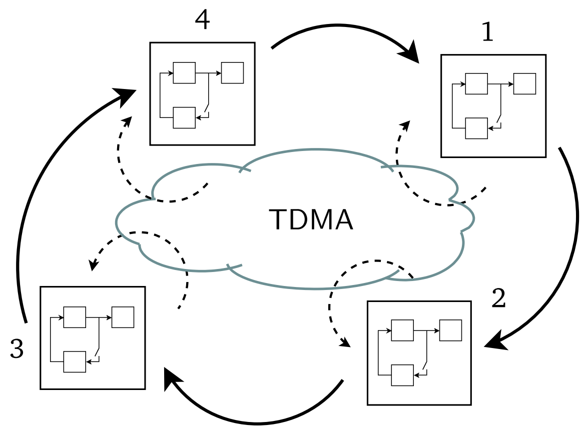

Figure 3.5: Time division multiple access protocol. The pre-given token ring in this example is $\{\ldots, 1, 2, 3, 4, 1, 2, 3, 4, \ldots\}$.

acceptable performance only for the NCS with $N = 2$. For $N \geq 6$, the average error variance takes values of magnitude 10^{15} which suggests an unbounded variance and therefore non-convergent evolution. This is in accordance with Theorem 2 in [33], where the stability condition is shown to be violated for $N \geq 6$ for CSMA channel scheduling scheme.

Figure 3.3 also shows performance efficiency of the scheduler (3.1) improves by increasing the power p. This is an expected observation as with increasing p, sub-systems with greater local error norms will have higher chances to be granted the channel access. In case $p \to \infty$, the sub-system with the highest error norm will *almost surely* transmit at every time-step.

3.5 Summary

In this chapter, we introduced a novel probabilistic event-based scheduling architecture which takes into account the real-time dynamic requirement of each local sub-system in order to assign channel access priorities. The scheduler has a centralized form, but we later show a decentralized counterpart can be realized which possesses similar characteristics. We briefly introduced the MEF-TOD approach which is a well-established deterministic event-based scheduler, and we summarized its advantages and disadvantages. Then, we proposed a probabilistic event-based scheduling approach which is similarly capable of prioritizing the channel access via monitoring the network-induced error dynamics. The overall model of an NCS with the proposed event-based scheduling mechanism and the LQG control law, is then represented. Afterwards, extensive stability analysis for multi-loop NCSs consisting of multiple heterogeneous LTI stochastic sub-systems sharing limited communication resources is presented. The considered concept of stochastic stability, i.e. f-ergodicity, is justified and the interpretation of stability certificates is clarified. Finally, the theoretical derivations are validated through Monte Carlo simulations for different NCS setups. To demonstrate the performance efficiency of our proposed scheduling design, we made comparisons with conventional protocols such as TDMA and CSMA and the performance discrepancy is shown to be considerable, especially as the size of the NCS grows.

3.6 Contributions

The presented results and analyses in this chapter are mainly focused on studying stochastic stability of NCSs consisting of stochastic LTI control loops. Similar analyses will be presented in the forthcoming chapters when we discuss the bi-character (deterministic/probabilistic) scheduling architecture and even the decentralized version of the proposed event-based prioritizing scheme. Therefore, we opted to have a detailed stability analysis so that we can recall some of them in those chapters. Stability proof in this chapter is partially presented in some of the author's own works, i.e., [84, 85, 94]. Stochastic stability of networked control systems by means of Markov state ergodicity are in addition employed in some other works, e.g. [33, 95]. However, the presented stability results in this chapter are original and unique due to the formulation of overall networked system, event-based error-dependent scheduling architecture, and the hard capacity constraint.

The preliminaries presented in sub-section 3.3.1 contain essential definitions, propositions and theorems to investigate stability of the described NCS. The mentioned concepts are well-established results for Markov chains and stochastic processes and could be found in standard literature, e.g. in [86, 96, 97], where we mostly used [86].

The numerical evaluations presented in Section 3.4, involves an illustration, i.e. Figure 3.3, which is already presented in two of the author's own publications in [84, 94].

4

Centralized Bi-character State-dependent Scheduling Design

Extending the results in Chapter 3, in this chapter we develop the previously introduced probabilistic scheduling design to a bi-character policy comprised of both deterministic and probabilistic attributes. We discuss that the modified scheduler arbitrates the scarce communication resources more efficiently and observe how the deterministic feature of the scheduler may lead to enhance the overall control performance in comparison with pure probabilistic approaches. The bi-character scheduling policy operates as follows. Given local error thresholds for each control loop, transmissions associated with sub-systems with lower error values than the pre-specified thresholds are deterministically discarded in order to make the channel less congested for those sub-systems with greater error values. The thresholds are indeed a powerful design parameter which can be tuned appropriately in order to use the resources efficiently. In case the channel capacity is still sparse for all those sub-systems with the corresponding errors exceeding their local error thresholds, then the scheduler allocates the communication channel probabilistically among all those eligible sub-systems based on a prioritized measure. Since the local errors are driven by the stochastic Gaussian noise process, transmissions occur randomly in an event-based fashion. The introduced bi-character scheduling rule offers major advantages in comparison with purely deterministic or purely probabilistic architectures. In comparison with deterministic policies, the probabilistic nature of our protocol facilitates an approximate decentralized implementation. This will be addressed later in Chapter 6. In addition, by lowering the probability of channel access for sub-systems with lower local errors, the channel is made less congested for the sub-systems which are in more urgent status for transmission and consequently performance enhancement is attained.

We prove that stochastic stability of the described NCS in Chapter 2 is preserved under the modified scheduler in terms of f-ergodicity of the overall network-induced error, which is modeled as a homogeneous Markov chain. Additionally, we derive uniform analytical performance bounds for the similar average cost function in Chapter 3, comprised of a quadratic error term and transmission penalty. The performance margins for the bi-character scheduling design are then analytically evaluated. Furthermore, it is concluded that the performance index is indeed a convex function of the error thresholds which consequently facilitates the search for sub-optimal uniform performance bounds. In addition, the improved performance claims in terms of average error variances are corroborated by numerical results.

This chapter is structured as follows. We introduce the bi-character event-based scheduling rule for NCSs comprised of multiple heterogeneous control loops exchanging data over a shared communication network in Section 4.1. Stability analysis is presented in Section 4.2 via stochastic stability notion of f-ergodicity of the overall network state. In Section 4.3 the analytical performance bounds are derived for the modified scheduling law, and a comparison is made with those bounds correspond to the pure probabilistic policy illustrating the performance enhancement. Simulation results in Section 4.4 illustrate that the modified scheduling architecture presents a reduction in the aggregate network-induced error variance compared to pure probabilistic, time-triggered and random access scheduling policies, especially as the number of control loops increases.

4.1 Event-based Deterministic-Probabilistic Prioritized Scheduling Law

Recalling the discussions in Chapter 3, here in this section we present a modified scheduling rule, compared to the pure probabilistic mechanism introduced in (3.1). It is illustrated in Section 3.1 that the scheduling law (2.14) can take a form of a probability distribution function. Taking a step further, we show in this chapter that the scheduling architecture may possesses both probabilistic and deterministic characteristics. In this section, we introduce an error-dependent centralized scheduling rule that dynamically prioritizes the channel access for a network of multiple stochastic LTI sub-systems exchanging information over a shared communication channel. The prioritization process, unlike the introduced pure probabilistic law (3.1), is performed in two phases; first according to a local deterministic threshold mechanism, and second based on a probabilistic biased mechanism. After comprehensive stability and performance analysis, we show in this chapter that incorporation of the deterministic feature to the scheduling law enhances the performance of the overall NCS, while stochastic stability of the networked system remains valid.

Similar to the previous chapters, assume that the shared communication channel is subject to the capacity constraint as not all sub-systems can simultaneously use the channel at every time-step, i.e. the hard constraint (2.2) should be fulfilled at every time-step k,

$$\sum_{i=1}^{N} \delta_k^i = c < N.$$

Therefore, at every time-step, some of the transmission requests are discarded and the corresponding control loops remain open while they can only try to get access to the channel in further time-steps. The following centralized error-dependent scheduling rule defines the probability of channel access for each sub-system at a time-step $k+1$, given the event information at the prior time-step k, according to a deterministic-probabilistic measure:

$$\mathsf{P}[\delta^i_{k+1} = 1 | e^j_k, \lambda_i] = \begin{cases} 0 & \|e^i_k\|^2_{Q^i} \le \lambda_i \\ 1 & \|e^i_k\|^2_{Q^i} > \lambda_i \wedge j_\lambda \le c \\ \frac{\|e^i_k\|^2_{Q^i}}{\sum_{j\in n_{\lambda,k}} \|e^j_k\|^2_{Q^j}} & \|e^i_k\|^2_{Q^i} > \lambda_i \wedge j_\lambda > c, \end{cases} \tag{4.1}$$

where, $\lambda_i \in \mathbb{R}^+ \cup \{0\}$ is the local error threshold for sub-system i, and $n_{\lambda,k} \le N$ denotes the number of sub-systems satisfying the threshold condition $\|e^j_k\|^2_{Q^j} > \lambda_j$ at time-step k. In addition, $Q^i \in \mathbb{R}^{n_i \times n_i}$ represents a positive definite weight matrix associated with sub-system i. It is clear that the above probability distribution is supported on the semi-infinite interval $[0, \infty)$. Moreover, it is straightforward to see that (4.1) is a probability measure as the assigned probabilities to all possible events sums up to one.

The bi-character policy (4.1) states that the process of blocking transmission requests at every time-step is performed in two independent phases. Initially, according to the first argument in the scheduling law (4.1), if the weighted square norm of local error of a sub-system i is less than or equal to the given threshold λ_i, then no transmission request associated with sub-system i will be submitted, i.e. if $\|e^i_k\|^2_{Q^i} \le \lambda_i$ at a time-step k, then $\delta^i_{k+1} = 0$, for the subsequent time-step $k+1$. It is worth noting that this decision is taken locally within each sub-system, and the decision to either submit a transmission request or not is totally deterministic.

Every sub-system which satisfies the condition $\|e^i_k\|^2_{Q^i} > \lambda_i$, would submit a transmission request to the centralized scheduler for time-step $k+1$. Then, if $n_{\lambda,k} \le c$, all those requests are through and the corresponding sub-systems transmit, as it can be seen from the second argument of the policy (4.1). Otherwise, if $n_{\lambda,k} > c$, then the scheduler is switched from deterministic state to probabilistic and subsequently the priorities are assigned to each transmission request, according to the third argument of (4.1), and the channel is allocated randomly until the capacity is reached. This feature of the scheduler (4.1) is reminiscent of the pure probabilistic scheduler in (3.1). Other transmission requests, which are not awarded the channel access, are therefore discarded and their corresponding loops remain open at least for one time-step. As already discussed in Section 3.1, if $n_{\lambda,k} > c$, then there is no guarantee that a sub-system with higher priority certainly transmits ahead of a sub-system with lower priority, although it is more likely. Intuitively, the involvement of the deterministic feature in the scheduling rule (4.1) facilitates more efficient allocation of the limited channel capacity among eligible sub-systems by deterministically excluding the sub-systems for whom a transmission is not crucial at that certain sampling time. Therefore, employing the scheduler (4.1) concludes that, at every time-step, some sub-systems do not transmit because they are not eligible to transmit (i.e., the first argument of (4.1)), but there might be also some sub-systems that do not transmit due to the limited capacity even though they were eligible for transmission.

Similar to the previous chapter and without affecting the generality of the approach presented herein, we assume $c = 1$ in the interest of analyses and derivations brevity. Therefore, for every $k \geq 0$ we have

$$\sum_{i=1}^{N} \delta_k^i = 1.$$

The following results, however, can easily be extended towards $\sum_{i=1}^{N} \delta_k^i = c < N$, where $c > 1$. Moreover, be reminded that in this chapter, we assume no packet dropout occurs, i.e., if $\delta_{k'}^i = 1$ at some time k', then certainly $\gamma_{k'}^i = 1$.

Remark 4.1. The design parameters λ_i's and Q^i's for $i \in \{1, \ldots, N\}$ play crucial roles in the efficacy of the bi-character scheduling rule (4.1), and therefore their appropriate tuning is of great importance. The error thresholds λ_i's appear in the deterministic part of the scheduling process to recognize the transmission eligibility of each control loop, while the weight matrices Q^i's directly affect the priorities through the assigned access probabilities. In the other words, λ_i's determine the acceptable level of local error below which a transmission is not necessary, and Q^i's specify how often a sub-system needs data transmission.

Remark 4.2. According to (4.1), in case $n_{\lambda,k} > c$, the channel access is granted according to the biased randomization with the bias terms (priorities) determined as in the third argument of (4.1). Unlike the deterministic MEF-TOD rule, where the highest priorities certainly transmit, the randomization in (4.1) allows a sub-system with lower priority to have transmission chance. This randomization provides a flexible design framework enabling us to tune the scheduling parameters appropriately to achieve desired properties, such as dealing with noisy systems, and implementing the scheduler approximately in decentralized fashion, as we will see in Chapter 6. Furthermore, it facilitates the investigation of data loss in the communication channel where comprehensive discussions are presented in Chapter 5. In addition, by slightly deviating from the current scheduler, we can tune the error thresholds not locally but centrally in a network manager unit such that the highest priorities transmit *almost surely*. Detailed discussions about this is out of scope of this dissertation, however a similar scenario is proposed in [72].

4.2 Stability Analysis

In this section, we address stability analysis for NCSs under the introduced bi-character scheduling design (4.1). Firstly, we state that the overall NCS model presented in Section 3.2 remain unchanged, so we assume to have the overall NCS state dynamics as expressed in (3.2). Moreover, in the following stability analysis, we frequently recall the preliminary definitions, propositions and theorems presented in Section 3.3.1.

It should be noted that, the modified bi-character scheduling rule (4.1) is a more general policy than the pure probabilistic law (3.1), because setting the error thresholds λ_i's to zero and the weight matrices Q^i's to identity for all $i \in \{1, \ldots, N\}$ results in having the pure probabilistic policy (3.1) in Chapter 3 for $p = 2$. Therefore, the illustrative example discussed in Section 3.3.2 is still valid and can similarly be referenced in this section to justify the need for modification of the one-step drift operator defined in (3.3).

In order to employ *Theorem* 3.1 to show stochastic stability for the modified scheduling law, first, we select the following error-dependent non-negative real-valued Lyapunov candidate $V : \mathbb{R}^n \to \mathbb{R}$:

$$V(e_k) = \sum_{i=1}^{N} e_k^{i^\mathsf{T}} Q^i e_k^i = \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2. \tag{4.2}$$

The Lyapunov function (4.2) is a special form of (3.4), when $p = 2$. Therefore, similar to the proof of *Theorem* 3.2, and assuming that the capacity constraint (3.6) holds, we consider the multi-step drift operator (3.7) over the time horizon of N time-steps, as follows:

$$\Delta V(e_k, N) = \mathsf{E}\left[\sum_{i=1}^{N} \|e_{k+N}^i\|_{Q^i}^2 \middle| e_k\right] - \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2. \tag{4.3}$$

Theorem 4.1. *Consider an NCS consisting of N heterogeneous LTI stochastic sub-systems modeled as (2.1), and a transmission channel subject to the constraint (3.6), and the control, estimation and scheduling laws given by (2.3), (2.4) and (4.1), respectively. Then for any positive λ_i's and positive definite Q^i's the Markov chain (2.17) is f-ergodic.*

Proof. See Appendix A.1. □

Remark 4.3. *Theorem* 4.1 assures that the Markov chain (2.17) visits a compact set $\mathcal{D}_f \subset \mathbb{R}^n$ at most every N time-steps for any $\lambda_i > 0$ and $Q^i \geq 0$. The boundary of $\mathcal{D}_f$ however varies with the design parameters λ_i's, Q^i's, N and system matrices A_i. It is inevitable that large N and limited capacity c results in quite large boundaries for the error variance.

Corollary 4.1. The multi-loop NCS described in (2.1)-(2.8) with the overall network state $[x_k^\mathsf{T}, e_k^\mathsf{T}]^\mathsf{T}$ under the scheduling law (4.1) is Lyapunov mean square stable.

Before proceeding to the proof of the corollary, we present the definition for *Lyapunov mean square stability* (LMSS):

Definition 4.1. [91] An LTI system with state vector X_k is said to possess Lyapunov mean square stability (LMSS) if given $\varepsilon > 0$, there exists $\rho(\varepsilon)$ such that $\|X_0\|_2 < \rho$ implies

$$\sup_{k\geq 0} \mathsf{E}\left[\|X_k\|_2^2\right] \leq \varepsilon.$$

Proof of Corollary 4.1. Assume $X_k = [x_k^\mathsf{T}, e_k^\mathsf{T}]^\mathsf{T}$. Then the Lyapunov mean square stability of the overall NCS state is achieved if $\sum_{c_l} \mathsf{P}_{i\in c_l} \mathsf{E}\left[\|e_k^i\|_2^2\right] \leq \varepsilon$ for all $c_l \in \{c_1, c_2, c_3\}$, assuming that the stabilizing control gains L_i exist for all $i \in \{1, \ldots, N\}$. Uniform upper-bounds for $\mathsf{E}[\|e_k^{i\in c_l}\|_2^2]$ for cases $\{c_1, c_2, l_1^{c_3}\}$ are derived in (A.3)-(A.5) over intervals with length N, considering $\mathsf{P}_{c_l} = 1$, as follows:

$$\sum_{i\in c_1} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 | e_k\right] \leq \sum_{c_1} \lambda_i \|A_i\|_2^2 + \mathrm{tr}(Q^i W_i), \tag{4.4}$$

$$\sum_{i\in c_2} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big] \leq \sum_{i\in c_2}\sum_{r=r'_i}^{N} \|A_i^{N-r}\|_2^2\,\mathrm{tr}(Q^iW_i), \tag{4.5}$$

$$\sum_{i\in l_1^{c_3}} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big] \leq \sum_{i\in l_1^{c_3}}\left[\|A_i^{N-r'_i}\|_2^2\lambda_i + \sum_{r=r'_i}^{N-1}\|A_i^{N-r-1}\|_2^2\,\mathrm{tr}(Q^iW_i)\right]. \tag{4.6}$$

For the case $l_2^{c_3}$, the uniform upper-bound can be obtained employing (A.6) and (A.8), as

$$\sum_{i\in l_2^{c_3}} \mathsf{P}_{i\in l_2^{c_3}}\,\mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big]$$
$$\leq \sum_{j\in l_2^{c_3}}\|A_j^N\|_2^2\sum_{r=\bar{r}}^{N-1}\|A_i^{N-r}\|_2^2\,\mathrm{tr}(Q^iW_i) + \frac{\sum_{r=\bar{r}}^{N-1}\|A_i^{N-r}\|_2^2\,\mathrm{tr}(Q^iW_i)}{\sum_{j\in c_2}\lambda_j + \sum_{j\in l_1^{c_3}}\lambda_j}\sum_{j\in l_2^{c_3}}\sum_{r=1}^{N}\mathrm{tr}(Q^jW_j)\|A_j^{N-r}\|_2^2. \tag{4.7}$$

It concludes from (4.4)-(4.7) that the overall error expectation remains bounded over every time interval of length N, and the proof then readily follows. □

Remark 4.4. It should be noted that LMSS implies f-ergodicity but not vice-versa since the latter holds not only with quadratic Lyapunov function but also with any appropriate Lyapunov variations which fulfill the conditions in *Theorem* 3.1.

4.3 Analytic Performance Bounds and Design Methods

In the previous section, we showed f-ergodicity of the network-induced error Markov chain (2.17) ensuring that the error state evolves according to a unique invariant probability distribution (see *Definition* 3.1). Moreover, *Theorem* 4.1 confirmed that the error state converges to some compact subset of the state space $\mathbb{R}^n$ at most every N time-steps. In this section, we investigate the performance of the proposed bi-character error-dependent scheduling policy (4.1) by obtaining analytical uniform performance bounds for the average of the per-time-step cost function J_{e_k} introduced in (2.18). Indeed, the obtained performance bounds in this section are determining the boundaries of the compact set towards which the error state e_k converges. In what follows, we first derive the uniform performance bounds for the similar NCS setup as in *Theorem* 4.1, and then we discuss the design methods in order to achieve the desired performance margins.

4.3.1 Analytic Performance Bounds

Here we recall the per-time-step cost function J_{e_k} as follows:

$$J_{e_k} = \sum_{i=1}^{N} e_k^{i^\mathsf{T}} Q_k^i e_k^i + \eta_k\delta_k^i := \sum_{i=1}^{N}\|e_k^i\|^2_{Q_k^i} + \eta_k\delta_k^i, \tag{4.8}$$

where, the non-negative parameter η_k denotes the cost of channel utilization at time k. The average cost associated by the introduced per-time-step cost function J_{e_k} in (4.8), over a finite horizon with length T can be expressed as:

$$J_{\text{ave}} = \sup_{e_k} \frac{1}{T} \sum_{k=0}^{T-1} \mathsf{E}\left[J_{e_k}\right] \tag{4.9}$$

In order to obtain an analytic upper bound for the average cost function J_{ave} in (4.9), statement of the following lemma is essential.

Lemma 4.1. [98] Let e_k be a Markov chain with general state space X. Introduce $J_{e_k} : X \to \mathbb{R}$ and define a measurable function $h : X \to \mathbb{R}$. Define the average cost J_{ave} as

$$J_{\text{ave}} = \lim_{n\to\infty} \sup \frac{1}{n} \sum_{k=0}^{n-1} \mathsf{E}\left[J_{e_k}\right].$$

If $h(e_k) \geq 0$ for all $e_k \in X$, then

$$J_{\text{ave}} \leq \sup_{e_k \in X} \left\{J_{e_k} + \mathsf{E}\left[h(e_{k+1}) \,|\, e_k\right] - h(e_k)\right\}.$$

Recall that we are interested in intervals of length N while Lemma 4.1 provides the upper bound for the average cost function over only one-step transitions, i.e. $k \to k+1$. To take this into account, and due to the fact that the error state e_k is an aperiodic, time-homogeneous, and ψ-irreducible Markov chain evolving in uncountable state-space $\mathbb{R}^n$, one can always generate a sampled Markov chain from the original chain which takes the states of the original Markov chain at time-steps $\{0, N, 2N, \ldots\}$. It is straightforward to show that ψ-irreducibility and aperiodicity of the original Markov chain are carried over to the generated sampled chain. Moreover, time-homogeneity of the original Markov chain implies time-homogeneity of the constructed Markov chain [86, Chapter 1]. Therefore, we can rewrite the upper bound for the average cost J_{ave} from Lemma 4.1 over a multi-step interval $[k, k+N]$, as follows:

$$J_{\text{ave}} \leq \sup_{e_k \in X} \left\{J_{e_k} + \mathsf{E}\left[h(e_{k+N}) \,|\, e_k\right] - h(e_k)\right\}. \tag{4.10}$$

We introduce the non-negative quadratic function $h(e_k) = \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2$. Recalling the per-time-step cost (4.8), the upper bound for the average cost (4.10) is reduced to

$$
\begin{aligned}
J_{\text{ave}} &\leq \sup_{e_k \in X} \left[J_{e_k} + \mathsf{E}\left[h\left(e_{k+N}\right)|e_k\right] - \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2 \right] \\
&= \sup_{e_k \in X} \left[\mathsf{E}\left[h\left(e_{k+N}\right)|e_k\right] + \sum_{i=1}^{N} \eta_k \delta_k^i \right] \\
&= \sup_{e_k \in X} \sum_{i=1}^{N} \left[\mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 |e_k \right] + \eta_k \delta_k^i \right].
\end{aligned} \tag{4.11}
$$

Similarity between (4.11) and the expectation of the Lyapunov candidate introduced in (4.2) can be noticed, which instead facilitates the following derivations in this section.

Theorem 4.2. *Consider the NCS described in Theorem 4.1. Then the average cost*

$$
J_{ave} = \sup_{e_k} \frac{1}{N} \sum_{k=0}^{N-1} \mathsf{E}\left[J_{e_k}\right]
$$

is uniformly upper bounded as a function of system matrices A_i, error thresholds λ_i, weight matrices Q^i, number of sub-systems N, and noise variance W_i of all $i \in \{1, \ldots, N\}$, over all initial conditions.

Proof. We derive the upper bounds for the average cost (4.11) for each case c_1, c_2 and sub-cases $l_1^{c_3}$ and $l_2^{c_3}$. Since the uniform upper bounds for $\sum_{i \in c_l} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 |e_k \right]$ are obtained in the proof of *Theorem* 4.1, we only need to take into account the communication penalty in (4.11) whenever a transmission happens.
Consider that some sub-systems i belong to the case c_1. Then, the upper-bound in (A.3) for $\sum_{i \in c_1} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 |e_k \right]$ is valid. Therefore, it follows from (4.11) that

$$
J_{\text{ave}}^{i \in c_1} \leq \sum_{i \in c_1} \lambda_i \|A_i\|_2^2 + n_{\delta_{\tilde{k}}^i} \eta_{\tilde{k}} + \operatorname{tr}(Q^i W_i), \tag{4.12}
$$

where, $n_{\delta_{\tilde{k}}^i}$ is the number of transmissions associated with sub-systems i at arbitrary time-steps $\tilde{k} \in [k, k+N]$.
Similarly, for sub-systems $i \in c_2$, we recall (A.4) and therefore (4.11) reduces to

$$
J_{\text{ave}}^{i \in c_2} \leq \sum_{i \in c_2} \sum_{r=r_i'}^{N} \operatorname{tr}(Q^i W_i) \|A_i^{N-r}\|_2^2 + n_{\delta_{\tilde{k}}^i} \eta_{\tilde{k}}. \tag{4.13}
$$

Recalling that the sub-systems belonging to c_3 never transmit, therefore $n_{\delta_{\tilde{k}}^i} = 0$ for all $i \in c_3$. Both upper bounds derived in (A.5) and (A.6) for the sub-cases $l_1^{c_3}$ and $l_2^{c_3}$ are still valid. Thus, for $i \in l_1^{c_3}$, we have

$$
J_{\text{ave}}^{i \in l_1^{c_3}} \leq \sum_{i \in l_1^{c_3}} \left[\lambda_i \|A_i^{N-r_i'}\|_2^2 + \sum_{r=r_i'}^{N-1} \operatorname{tr}(Q^i W_i) \|A_i^{N-r-1}\|_2^2 \right]. \tag{4.14}
$$

For sub-systems $j \in l_2^{c_3}$, the error-dependent (non-uniform) upper bound holds from (A.6):

$$J_{\text{ave}}^{j\in l_2^{c_3}} \leq \sup_{e_k} \sum_{j\in l_2^{c_3}} \left[\|A_j^N\|_2^2 \, V(e_k^{j\in l_2^{c_3}}) + \sum_{r=1}^{N} \text{tr}(Q^j W_j) \|A_j^{N-r}\|_2^2 \right]. \tag{4.15}$$

Since we have already derived uniform upper-bounds for $J_{\text{ave}}^{i\in c_1}$, $J_{\text{ave}}^{i\in c_2}$, and $J_{\text{ave}}^{i\in 1_1^{c_3}}$ in (4.12)-(4.14), respectively, the average cost function J_{e_k} can be rewritten to incorporate the probability of occurrence $\mathsf{P}_{l_2^{c_3}}$ as follows:

$$J_{\text{ave}} \leq J_{\text{ave}}^{i\in c_1} + J_{\text{ave}}^{i\in c_2} + J_{\text{ave}}^{i\in 1_1^{c_3}} + \mathsf{P}_{l_2^{c_3}} J_{\text{ave}}^{i\in l_2^{c_3}}, \tag{4.16}$$

where, probability of occurrence $\mathsf{P}_{l_2^{c_3}}$ derived in (A.8) is still valid. Considering again the worst-case scenario, and from (4.15) we have for all $j \in l_2^{c_3}$

$$\begin{aligned}
\mathsf{P}_{l_2^{c_3}} J_{\text{ave}}^{j\in l_2^{c_3}} \leq\; & \sup_{e_k\in l_2^{c_3}} \frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i)}{\sum_{j\in c_2} \lambda_j + \sum_{j\in l_1^{c_3}} \lambda_j + V(e_k^{j\in l_2^{c_3}})} \sum_{j\in l_2^{c_3}} \|A_j^N\|_2^2 \, V(e_k^{j\in l_2^{c_3}}) \\
& + \sup_{e_k\in l_2^{c_3}} \frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i)}{\sum_{j\in c_2} \lambda_j + \sum_{j\in l_1^{c_3}} \lambda_j + \sum_{j\in l_2^{c_3}} \|e_k^j\|_{Q^j}^2} \sum_{j\in l_2^{c_3}} \sum_{r=1}^{N} \text{tr}(Q^j W_j)\|A_j^{N-r}\|_2^2 \\
\leq\; & \sum_{j\in l_2^{c_3}} \|A_j^N\|_2^2 \sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i) \\
& + \frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i)}{\sum_{j\in c_2} \lambda_j + \sum_{j\in l_1^{c_3}} \lambda_j} \sum_{j\in l_2^{c_3}} \sum_{r=1}^{N} \text{tr}(Q^j W_j)\|A_j^{N-r}\|_2^2.
\end{aligned} \tag{4.17}$$

Finally, employing the uniform upper-bounds (4.12)-(4.17) for all cases $c_1 - c_3$ and according to (4.16), the average cost J_{ave} remains bounded uniformly, as follows:

$$\begin{aligned}
J_{\text{ave}} \leq\; & \sum_{i\in c_1} \lambda_i \|A_i\|_2^2 + n_{\delta_{\tilde{k}}^i} \eta_{\tilde{k}} + \text{tr}(Q^i W_i) \\
& + \sum_{i\in c_2} \sum_{r=r_i'}^{N} \text{tr}(Q^i W_i)\|A_i^{N-r}\|_2^2 + n_{\delta_{\tilde{k}}^i} \eta_{\tilde{k}} + \sum_{i\in l_1^{c_3}} \left[\lambda_i \|A_i^{N-r_i'}\|_2^2 + \sum_{r=r_i'}^{N-1} \text{tr}(Q^i W_i)\|A_i^{N-r-1}\|_2^2 \right] \\
& + \sum_{j\in l_2^{c_3}} \|A_j^N\|_2^2 \sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i) + \frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|_2^2 \,\text{tr}(Q^i W_i)}{\sum_{j\in c_2} \lambda_j + \sum_{j\in l_1^{c_3}} \lambda_j} \sum_{j\in l_2^{c_3}} \sum_{r=1}^{N} \text{tr}(Q^j W_j)\|A_j^{N-r}\|_2^2.
\end{aligned} \tag{4.18}$$

The bound on J_{ave} is independent of any initial values, however it is conservative due to multiple upper-bounding. The expression (4.18) represents the boundaries of the compact set towards which the error state e_k is expected to converge at least every N time-steps. This concludes the proof. □

Remark 4.5. To calculate (4.17), we employed the occurrence probability $\mathsf{P}_{l_2^{c_3}}$ computed in (A.8). Remind that we assumed the worst-case scenario, i.e. $\|e_{k'}^{j\in l_2^{c_3}}\|_{Q^j}^2 \le \|e_{k'+1}^{j\in l_2^{c_3}}\|_{Q^j}^2$ while computing (A.8). Analytically, this worst-case assumption places a condition on the associated noise process in the error dynamics, for sub-systems belong to the sub-case $l_2^{c_3}$. We already showed that $\|e_{k+N}^j\|_{Q^j}^2$ for $j \in l_2^{c_3}$ can be written as

$$\|e_{k+N}^j\|_{Q^j}^2 = \Big\|A_j^N e_k^j + \sum_{r=0}^{N-1} A_j^{N-r-1} w_{k+r}^j \Big\|_{Q^j}^2.$$

In fact, to satisfy the worst-case scenario, the distribution of the noise-dependent vector $\sum_{r=0}^{N-1} A_j^{N-r-1} w_{k+r}^j = A_j^{N-1} w_k^j + \ldots + w_{k+N-1}^j := W\left(A_j, w^j\right)$ should be restricted to the parts of the multi-dimensional Gaussian distribution that enlarge the term $A_j^N e_k^j$. To facilitate the derivations, let us assume without loss of generality to have the matrices A_j in Jordan form. The worst case occurs when the noise-dependent and error-dependent terms have the same signs element-wise. Due to the symmetry of $W\left(A_j, w^j\right)$, both positive and negative parts of the distribution turn out to have the same values. Since the noise variables are independent and $W\left(A_j, w^j\right)$ has a zero-mean multi-dimensional Gaussian distribution with covariance matrix $\Sigma = \left(A_j^{N-1} + \ldots + A_j + I\right) C_j$, employing the law of *unconscious statistician*, we can compute $\mathsf{E}\Big[\|W\left(A_j, w^j\right)\|_{Q^j}^2 | W \ge 0\Big]$ as follows:

$$\begin{aligned} &\mathsf{E}\Big[\|W\left(A_j, w^j\right)\|_{Q^j}^2 | W \ge 0\Big] \\ &\quad = \frac{\|Q^j\|_2^2}{\sqrt{(2\pi)^{n_j}|\Sigma|}} \int_0^{\infty} \cdots \int_0^{\infty} \|w\|_2^2 \exp\left(\frac{-w^T \Sigma^{-1} w}{2}\right) dw, \end{aligned} \tag{4.19}$$

where w is the n_j-dimensional noise-dependent random vector, and $|\Sigma|$ is the determinant of the covariance matrix.

Remark 4.6. It can be seen that the upper bounds derived in (4.12)-(4.17) are hyperbolic/convex functions of λ_i. The convexity of these upper bounds streamlines the search for the unique minimizing value of λ_i. After finding the minimizing values of the error thresholds λ_i, a sub-optimal upper bound for the average cost (4.11) is obtained. We drop the full derivation of the minimizing error thresholds to avoid very lengthy expressions coming out of taking derivative from the already-lengthy expression (4.18).

4.3.2 Design Methods

The obtained upper-bound for the average cost function J_{ave} in (4.18) represents the boundaries of the compact set $\mathscr{D}_f$ introduced in *Theorem* 4.1 as the convergent set for the error state e_k. As discussed in *Theorem* 4.1, f-ergodicity of the network-induced error e_k is ensured under no restrictive assumption on the parameters such as the system matrices A_i, noise covariances $W_i < \infty$, number of sub-systems $N < \infty$, error thresholds λ_i, and weight matrices Q^i, for all sub-systems $i \in \{1, \ldots, N\}$. The obtained results are showed furthermore under the worst-case channel capacity assumption, i.e. only one possibility of transmission is available at every time-step. Although, the boundedness of the average cost function, and f-ergodicity of the network-induced error Markov chain are shown under very mild assumptions, the results are conservative. For example, the boundaries of the compact set $\mathscr{D}_f$ may become very large, though finite, which in reality is definitely not desirable. Here we discuss how to achieve desired performance bounds, and study the roles of different NCS parameters affecting the obtained bound on the cost function.

As it is expected, the bound on average cost J_{ave} in (4.18) increases if the noise covariance W_i increases. This suggests that if the random noise is less certain to be around the expected value, which is zero, then the expected compact set towards which the error state converges has extended boundaries. In addition, having more unstable sub-systems, i.e. larger eigenvalues for the system matrices A_i, results in having expanded performance bounds, which is intuitively comprehensible. Assuming that the channel capacity is fixed, if the number of subsystems N increases, then the bound on J_{ave} would also increase, according to (4.18). This is also an expected observation because if $\frac{c}{N}$ decreases while c is kept constant, the competition for limited communication resources is inevitably intestified, which consequently results in longer time periods one sub-system should wait until the channel becomes free. The dependency of the upper-bound for J_{ave} on the channel capacity c is not reflected in (4.18), because according to (3.6), $c = 1$. However, it is foreseeable that increasing the capacity c results in tighter boundaries for the convergent compact set $\mathscr{D}_f$. Mathematically speaking, if we consider to have c possibilities for transmission at every time-step, one of the changes to the upper-bound (4.18) would be replacing N with $\frac{N}{c}$ which results in having tighter bounds on J_{ave}.

As already discussed when introducing the scheduling policy (4.1), the local parameters λ_i's and Q^i's tune the deterministic threshold and assigned priorities, respectively. The weight matrices Q^i indeed determine the importance of transmissions for a certain sub-system, and therefore they are designed to assign the priorities in favor of those sub-systems with higher operational importance. Increasing them would increase the chance the corresponding subsystem has to be granted the access to the channel. Setting the weight matrix very high for one certain sub-system results in that sub-system to have expectedly much more frequent transmissions than the others.

The dependency of the upper-bound (4.18) on the local error thresholds λ_i's has a hyperbolic nature. We will illustrate this convex dependency numerically later in the simulation results. In fact, setting all the error thresholds to zero results in the pure probabilistic scheduling policy (3.1), introduced in the Chapter 3. Although it is shown that the f-ergodicity remains valid under the policy (3.1), we will see in the next section that the performance of the bi-character policy (4.1) outperforms that of (3.1). Intuitively, setting the threshold to zero allows all sub-systems $i \in \{1, \ldots, N\}$ to take part in the channel competition at every time-step. Since the channel access is supposed to be eventually granted through a randomized mechanism, even those sub-systems with negligible error values have chances to use the channel and therefore occupy the limited transmission possibilities. Hence, the channel becomes busy and other sub-systems with higher errors remain open-loop and should wait for next time-steps to transmit, which leads to increase in the aggregate error variance. On the other hand, setting those thresholds very high means that more sub-systems with relatively high errors (but still below their corresponding highly set thresholds) are excluded from the

channel competition and they remain deterministically open-loop until their errors exceed the corresponding thresholds. This scenario also leads to increased aggregate error variance. Therefore, there should be optimal error thresholds which maintain the balance between the aggregate error variance and giving the chance to transmit to the sub-systems which are in more stringent real-time conditions.
For example, the upper-bound (4.17) for the sub-systems belonging to the sub-case $l_2^{c_3}$ ensures that the partial average cost $\mathsf{P}_{l_2^{c_3}} J_{\text{ave}}^{j \in l_2^{c_3}}$ can be made small by increasing the error thresholds λ_j's for those sub-systems which are competing against the sub-systems belonging to the sub-case $l_2^{c_3}$, i.e. $j \in c_2$ and $j \in l_1^{c_3}$, and decreasing Q^j's. However, the cost cannot be made arbitrarily small due to the first term which depends on the constant system parameters. It intuitively explains that, despite having sub-systems operating in open-loop and sparsity of the communication resources, which might cause an even unstable sub-system with large error waiting for channel access for quite some time, the aggregate error remains bounded.

4.4 Numerical Evaluations

In this section, the performance of our proposed bi-character scheduler is investigated and compared with conventional scheduling policies such as TDMA and CSMA. We also demonstrate that the deterministic feature of the scheduler yields performance improvements in comparison with the pure stochastic scheduler introduced in the preceding chapter.

Similar to the results in Section 3.4, we assume an NCS comprised of two heterogeneous classes each consisting of homogeneous scalar sub-systems. The first class includes $\frac{N}{2}$ control loops with unstable plants and parameters $A_1 = 1.25$, $B_1 = 1$, while the second class contains $\frac{N}{2}$ loops with stable processes with parameters $A_2 = 0.75$, and $B_2 = 1$. We select $x_0^i = 0$ for each sub-system $i \in \{1, \ldots, N\}$, and $w_k^i \sim \mathcal{N}(0, 1)$. To stabilize the local sub-systems, we choose deadbeat control laws $L_i = A_i$, and the model-based observer (2.4). For simplicity, we select $Q^i = I$ for all $i \in \{1, \ldots, N\}$.

For illustrative purposes, we assume that the local error thresholds are equal for all sub-systems connected to the communication channel in one specific NCS setup. We drop the subscript i and simply denote the thresholds λ in the rest of this section. Figure 4.1 illustrates how the variance of the aggregate error (2.17) changes with respect to the local thresholds λ, for NCSs with different number of sub-systems, employing the bi-character scheduling policy (4.1). The results are in accordance with *Remark* 4.6 confirming that the variance of aggregate error is indeed a convex function of error thresholds λ. This facilitates tuning the error thresholds optimally in order to minimize the aggregate error variance. As it can be seen, the optimal value of λ monotonically increases by increasing the number of sub-systems N whom are sharing the sole transmission channel. This is an expected observation as the competition to access the sole channel slot intensifies with N growing.

By selecting the optimal error thresholds, Figure 4.2 provides the comparisons in terms of average error variance between our bi-character policy and TDMA, CSMA, and the event-triggered threshold policy proposed in [33] for various setups of NCSs with different number of sub-systems $N \in \{2, 4, 6, 8, 10, 20\}$, subject to the capacity constraint (3.6). Note that for $N > 2$, we have more unstable sub-systems than the available transmission slots meaning

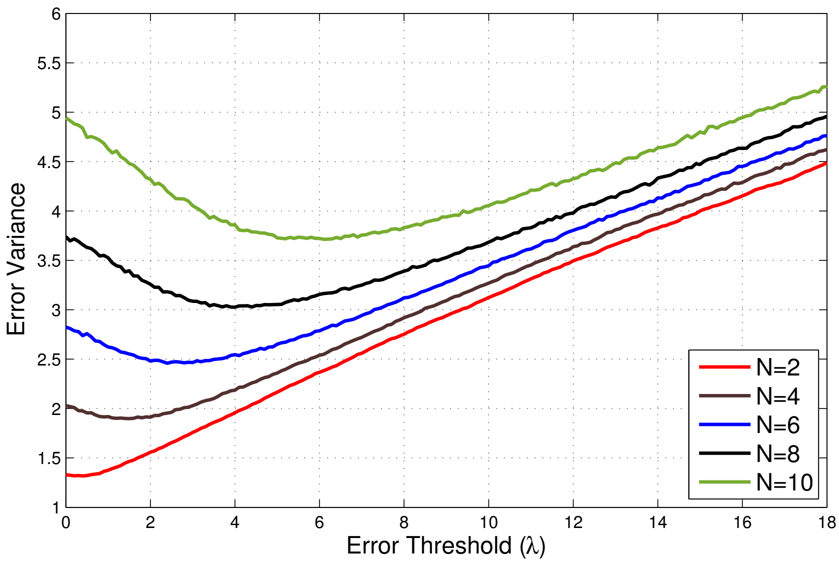

Figure 4.1: Aggregate error variance vs. error thresholds for NCSs with $N \in \{2,4,6,8,10\}$.

that at least one unstable sub-system has to be open-loop at every time-step. The variances in Figure 4.2 are calculated via Monte Carlo simulations over a horizon of 2×10^5 samples. The lower bound is determined by an optimal methodology developed in [33] that replaces the hard constraint on the number of transmissions per time-step, i.e. (3.6), by an average transmission rate constraint. For comparable results, we disregard the communication penalty in Figure 4.2 by considering $\eta = 0$. Non-zero communication penalty results in a constant (average) scale up in all curves of Figure 4.2. We will however consider the penalties when computing the bound for the average cost function J_{ave}.

The selected error thresholds are the optimal values for each NCS setup taken from the Figure 4.1, and the same values are used in Figure 4.2. The optimal values for the thresholds can also be obtained from the analytic expression for the average cost J_{ave} in (4.18), however in the expense of conservatism. The deterministic feature of the bi-character policy (4.1) can be removed to obtain the pure probabilistic scheduler, as introduced in (3.1), by considering all sub-systems for the channel access competition at every time-step [84], i.e. by setting $\lambda = 0$.

To compare our results fairly with the TDMA scheme, we derive the optimal pattern for the token ring by brute force search over a window of finite time-steps. Recall that, optimal TDMA pattern over infinite horizon is NP-hard problem. We search for the patterns among all permutations which result in the minimum average error variance over the considered window for each NSC setup $N \in \{2,4,6,8,10,20\}$. The search for the optimal TDMA pattern is however not extendable to greater number of sub-systems and longer time windows. For example, the brute force search for an NCS with $N = 4$ over 9 time-steps lasts nearly 11 hours on a 3.90 GHz 4690 Core i5 CPU to return the optimal token ring. Furthermore, the optimal pattern is highly sensitive with respect to the system parameters and the token ring changes the order of transmissions considerably with rather moderate change of e.g. system matrices A_i. In addition, we compare the performance of our bi-character approach with the CSMA scheme assuming that transmission probability for a sub-system i with system matrix A_i is

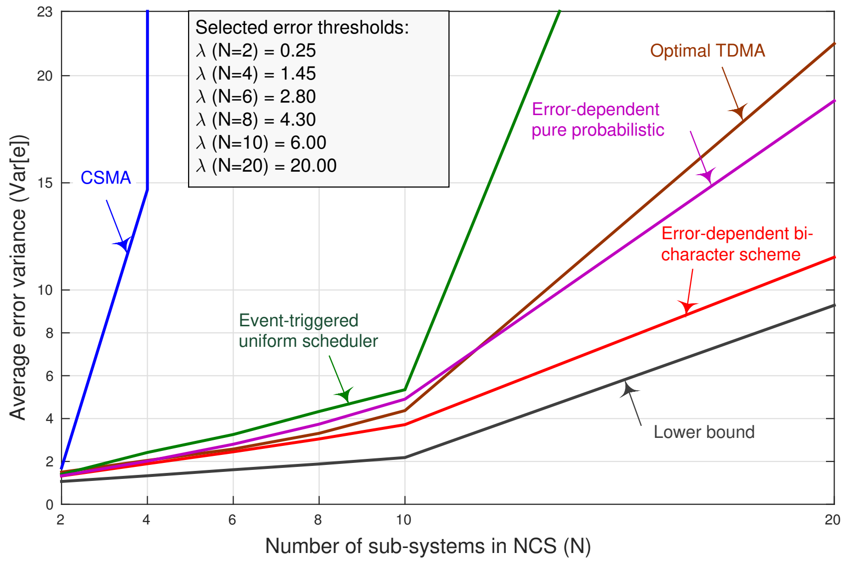

Figure 4.2: Comparison of the average error variance vs. number of control loops for different scheduling policies.

pre-given by $\frac{A_i^2}{\sum_{j=1}^{N} A_j^2}$ at each time-step. As it can be seen in Figure 4.2, employing the CSMA protocol results in an acceptable performance only up to $N = 4$. For $N = 6$ the variance takes the value of magnitude 8×10^3. This is expected as the CSMA policy allocates the channel statically resulting in probable non-transmission state over a long horizon for an unstable system. As the number of sub-systems increases, the performance efficiency of our scheduler becomes more evident in comparison to the (near)-optimal TDMA and CSMA schemes. To show that the deterministic feature of the bi-character policy (4.1) leads to enhanced overall NCS performance, we additionally plot the error variance under the pure probabilistic approach, which can be simply obtained by considering $\lambda = 0$ for all sub-systems in (4.1). Expectedly, the aggregate error variance grows compared to the bi-character scheduler, as sub-systems with small errors might use the channel in the presence of those with large error values.

Furthermore, the scheduling approach proposed in [33] suggests a bi-character scheme which is identical to our proposed approach (4.1) in deterministic part, but alters the error-dependent prioritization by a non-prioritized (uniform) assigning of channel access probabilities (green curve in Figure 4.2). Indeed, if the number of qualified sub-systems for transmission, i.e. those who exceed their respective thresholds, is greater than the channel capacity, say c, then the scheduler randomly selects c sub-systems out of the qualified ones according to the uniform distribution. It can be seen that our approach outperforms all mentioned scheduling schemes, in terms of the variance of network-induced error, especially when the size of the NCS increases. More interestingly, the performance of the bi-character approach closely follows the theoretical lower bound obtained from the optimal approach and does not diverge considerably from it even by increasing number of sub-systems.

Table 4.1 provides analytic upper bounds for the average cost function J_{ave} in (4.11) via the expression (4.18), considering NCSs with $N = \{2, 4, 6, 8, 10\}$. The upper bounds are calculated with the minimizing error thresholds derived from the analytical expression (4.18) and are considered to be equal for all sub-systems. Since the analytic upper-bound (4.18) is conservative, the minimizing thresholds are not necessarily equal to the optimal thresholds obtained from simulation results. In this table, we consider non-zero communication penalty and for simplicity, we assume $\eta_i = \lambda$, which means that if the weighted error norm of a sub-system is below its corresponding threshold, then having a transmission is more costly than being excluded from channel access competition and consequently remaining open-loop. In order to compare the analytic bound for J_{ave} with the simulation results, we also computed the aggregate error variance with the new error thresholds, as shown in Table 4.1, by Monte Carlo simulations.

Table 4.1: Simulative mean variances vs. analytic upper bounds for the average cost (4.18) with costly communication ($\eta_i = \lambda$)

Number of sub-systems (N)	2	4	6	8	10
Error threshold (λ)	0.28	6	10	15	18
Simulative mean variance	1.33	2.53	3.43	4.50	5.26
Upper bound on J_{ave}	3.12	3.99	6.14	12.28	25.60

It can be seen from Table 4.1 that the analytic bound for J_{ave} is conservative in comparison with the simulation results. This is expected as the upper-bound (4.18) is derived for the worst-case scenario and in addition we computed it via multiple upper-bounding. In addition, the simulative aggregate error variances in Table 4.1, where the thresholds are obtained from the conservative bound (4.18), are higher in comparison with that of Figure 4.2, wherein the averages are calculated with the optimal thresholds in Figure 4.1. This is again show the conservatism of the analytic bound (4.18) and is in accordance of our expectations. Note that the analytic cost values shown in Table 4.1 are derived considering non-zero communication penalty, while the communication penalty is not considered in the stimulative results. The absence of the communication penalty in the average cost (4.18) decreases the overall cost as demonstrated in the Table 4.2.

Table 4.2: Simulative mean variances vs. analytic upper bounds for the average cost (4.18) with cost-less communication ($\eta_i = 0$)

Number of sub-systems (N)	2	4	6	8	10
Error threshold (λ)	0.28	6	10	15	18
Simulative mean variance	1.33	2.53	3.43	4.50	5.26
Analytic upper bounds on J_{ave}	1.65	3.25	5.30	11.35	24.70

4.5 Summary

In this chapter, we first introduced a novel bi-character error-dependent scheduling mechanism for multiple-loop NCSs under capacity constraints, with the scheduler possessing both deterministic and probabilistic features. Introducing local error thresholds for each sub-system, those with lower error values than their corresponding thresholds are deterministically kept out of channel access competition. This typically decreases the congestion in channel access requests and is in favor of those sub-systems with greater error values. The deterministic feature of the bi-character scheduler is shown to be a useful addition to the pure probabilistic scheme introduced in Chapter 3. The error thresholds, which determine which sub-systems are allowed to submit transmission requests, can be tuned appropriately in order to use the communication resource more efficiently. Afterwards, if the channel capacity is still not enough for all those sub-systems who have submitted a transmission request, then the probabilistic mechanism of the bi-character scheduler is triggered and it allocates the communication channel among all those sub-systems according to a prioritized randomization. We addressed stochastic stability under the bi-character policy in terms of f-ergodicity, and later derived analytic uniform performance bound for an average cost function, comprised of the network-induced error variance and communication penalty. Although the analytic bound for the average cost is shown to be moderately conservative, it provides a measure of the compact set towards which the error Markov chain is expected to converge, irrespective of the initial values. Our theoretical claims are validated through simulations and the efficacy of our proposed bi-character scheme is illustrated when being compared with the conventional scheduling architectures. Moreover, the simulation results suggest that the addition of deterministic feature is highly beneficial compared to pure probabilistic scheduling mechanism.

4.6 Contributions

The presented results in this chapter are partially from the author's own work in [99, 100]. Main efforts in this chapter are focused on stability analysis and performance evaluation of an error-dependent bi-character scheduling policy which deterministically expels some of the sub-systems from channel access competition based on a local threshold policy, before allocating the channel in a biased probabilistic fashion among the remaining sub-systems.

Similar works in the area of real-time scheduling of resource constrained NCSs are accomplished, e.g. see [33, 50, 101–103]. As discussed in Section 4.4, our presented results differ from the analysis in [33] as the authors of the mentioned work consider a uniform randomization to allocate the communication channel, and the performance of our biased randomized approach is shown to be superior. Moreover, the same threshold policy as in (4.1) is employed in [101] to allocate a multi-hop communication medium among sub-systems based on the slotted ALOHA MAC scheme. A state-dependent local scheduling architecture is also presented in [50] allocating the channel in a contention-based scenario depending on the local measurement information and transmission history. However, non of the mentioned works considered the error-dependency in two phases, i.e. probabilistic and deterministic, so in that sense our presented results are completely original.

In the numerical section, the illustrative Figure 4.2 and Tables 4.1 and 4.2 are already presented in the author's own work [99]. Moreover, Figure 4.1 is presented in [100].

5

Scheduling over Non-Ideal Channels

Control over shared communication resources often imposes various imperfections, such as capacity limitation, congestion, collisions, time delays and data loss, that impair the control performance and can even lead to instability of the overall networked system. In real applications of networked systems, we inevitably have to deal with one or some of those mentioned challenges which need to be carefully considered when designing control and scheduling strategies. In this chapter, we take into account non-idealities in the communication channel to evaluate the robustness of the proposed design approaches with respect to the mentioned phenomena. More precisely, we address the control and scheduling designs for a multiple-loop stochastic NCS wherein the local sub-systems exchange their sensory data over a shared capacity-limited communication channel subject to data loss. Additionally, in a centralized design, it is often assumed that the controlling units have constantly access to global information from all distributed entities. In the preceding chapters, we assume that the centralized scheduler receives network-induced error information from all sub-systems which are requesting for a transmission. This however is an ideal assumption because this sheer amount of information exchange cannot always be processed in timely manner, which in turn gives rise to delays. Therefore, scheduling approaches requiring complete information in every sampling times might not be feasible in practice due to the additional traffic imposed by the scheduler to coordinate among different control loops. A desired architecture should be capable of allocating resources efficiently even provided with partial information from local entities. In this chapter, we also address the problem of scheduling with incomplete information and we show that our proposed scheduling approach is indeed robust with respect to the lack of information from the entities who use the resources.

This chapter is organized as follows. We first introduce the challenges that may arise from having a non-ideal shared communication channel among multiple control loops in Section 5.1. We then focus on the data packet loss in communication channels in Section 5.2 and show stability guarantees under event-based scheduling law over lossy channels. Section 5.3 investigates the effect of incomplete information for a centralized scheduler and presents a modified architecture in order to cope with not having access to the central information in a timely manner. Numerical evaluations in Section 5.4 illustrate that the proposed scheduling approaches can indeed guarantee the required robustness with respect to those phenomena.

5.1 Non-Idealities in Shared Communication Channels

An ideal communication channel is usually referred to a data transmission medium over which a data packet[1], which is scheduled for channel usage, will be transferred without end-to-end latency, and will surely be received by the intended station. The non-idealities in a communication medium may come from two different sources; one from the physical limitations of the communication infrastructure and the other one might be resulted from the employed medium access control (MAC) mechanism. Network congestion is a direct consequence of limited communication bandwidth and inevitably occurs when the incoming data traffic from sending stations exceeds the outgoing traffic. Network congestion leads to reduction of throughput, while results in an increase in dropouts and latency. Usually network protocols employ compensatory mechanisms, such as re-transmission back-off technique in CSMA-CA (carrier-sense multiple access with collision avoidance), to reduce the effects of network congestion. Although packet dropouts are often occur due to network congestion, they may caused by other factors such as erroneous network links, malfunctioning network hardware, weak-power wireless signals from stations located far, or even intentionally through e.g. dynamic source routing (DSR) routing protocol.
Apart from the aforementioned channel imperfections, we may face data packet collisions which are caused if more than one sending stations attempt to transmit data packets through a single channel link, simultaneously. It can be concluded from the definition of packet collisions that the occurrence of such phenomenon is tightly coupled with the channel access mechanism. We typically categorize various media access control strategies into two groups, so called, *contention-based* protocols and *contention-free* protocols. The former type of MAC protocols facilitate the situation where multiple stations are able to share a communication medium without pre-coordination. In fact, data transmission in contention-based MAC protocols may occur at any time and the channel is typically granted based on the first-come, first-served scenario. It is obvious that within such protocols, collisions may happen which can lead to successive packet dropouts and consequently can impair the network performance. The most famous contention-based communication medium is Ethernet which is designed to make a network shared among multiple computers. In case a collision is detected, the collided packets are either dropped or sent back to their corresponding stations to be re-transmitted, e.g. CSMA-CD (carrier-sense multiple access with collision detection). On the other hand, contention-free MAC protocols coordinate the channel access for every single station which requests to have a transmission. Most famous types of contention-free MAC protocols are time-division multiple access (TDMA), code-division multiple access (CDMA),

[1] Here in this dissertation, when we talk about data and communication channel, we refer to data with packet-switched network format which can be transmitted through single-hop shared or multi-hop simultaneous channels. Therefore, traditional bit-stream communication is not treated in this research.

and frequency-division multiple access (FDMA) where they pre-determine a time-duration, code sequence or frequency range over which one station should use the shared communication channel. Due to the pre-given schedule, this group of protocols represent collision-free MAC strategies.

The appropriate type of MAC protocol to be employed is tightly dependent on the specific application of the networked system, number of users or stations, and even geographical conditions. Contention-based MAC strategies offer distributed implementation, represent easy-to-install, low-cost and scalable channel access design which are essentially suitable for networked systems with a large number of users. However, collisions take place inevitably within these protocols and need to be handled with care as an important design criteria. Contention-based protocols in addition might be less secure compared to contention-free counterparts, as no global administration unit exits. In addition, collision avoidance mechanisms, e.g. the listen-before-talk scheme, call for all nodes to sense the channel permanently, which instead requires high energy consumption due to idle listening, overhearings, and message overhead. Contention-free protocols on the other hand offer collision-free and precise channel scheduling with higher throughput compared to contention-based protocols. Moreover, they consume less energy compared to e.g. carrier sense multiple access with collision avoidance (CSMA-CA), where each node senses the channel permanently [104]. Furthermore, QoS support and bandwidth arbitration is facilitated as they can prioritize channel access. However, their major drawback is lack of flexibility and scalability which make them not quite suitable for large-scale networks due to their synchronous nature.

In the following sections, we will address the problem that whether event-based data scheduling over communication channels subject to bandwidth limitations, packet dropouts and lack of centralized information is beneficial. The considered type of scheduling mechanism in this chapter is centralized, hence contention-free. Data collisions will be studied later in the Chapter 6 when the decentralized contention-based scheduling mechanism is discussed. In what follows, we first address the scheduling synthesis under packet dropouts (without considering incomplete information for the scheduler) and then investigate the capability of our introduced scheduling approach under information sparsity considering ideal channel characteristics. However, the obtained results can be combined as these two phenomena are independent from each other.

5.2 Data Scheduling with Packet Dropouts

We consider here in this section, the previously described NCS formulation in Chapter 4. We assume an NCS comprised of N heterogeneous LTI control loops, with stochastic plants described by the difference equation (2.1), coupled through a non-ideal shared communication network subject to the capacity constraint (2.2). The control process is assumed to take an emulation-based form and the local controllers are synthesized according to the LQG framework, as discussed in Chapter 2 through expressions (2.10)-(2.13). The distributed model-based estimators are also designed as expressed in (2.4). Moreover, the scheduling mechanism in this section has the identical profile as the bi-character error-dependent prioritizing architecture which is comprehensively introduced in Section 4.1. In addition to the mentioned characteristics of the local sub-systems and the communication network, we assume in this section that the communication channel is subject to data packet dropouts. Further in Section 5.3, we analyze the event-based scheduling design assuming that the scheduler does not have access to the updated events information from every sub-system at every time-step.

Assume that the centralized error-dependent bi-character scheduling law introduced in (4.1) is employed as the contention-free mechanism to allocate the limited communication resource among the sub-systems. To take into account the packet loss, we need to monitor if a packet is successfully received by the end-station or not. Assume that the successful transmission of a sub-system i which has been granted the channel access by the scheduler, at a time-step k, is acknowledged by the binary-valued signal γ_k^i as follows:

$$\gamma_k^i = \begin{cases} 1, & x_k^i \text{ successfully received} \\ 0, & x_k^i \text{ dropped.} \end{cases}$$

Therefore, the aggregate state vector $[x_k^{i\mathsf{T}}, e_k^{i\mathsf{T}}]^\mathsf{T}$, corresponds to sub-system i, can be described as (2.6), and therefore the overall NCS state vector $[x_k^\mathsf{T}, e_k^\mathsf{T}]^\mathsf{T}$ is expressed as follows:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix}_{2n} = \begin{bmatrix} A-BL & (I_n-\Delta_{k+1})BL \\ 0_{n\times n} & (I_n-\Delta_{k+1})A \end{bmatrix}_{2n\times 2n} \begin{bmatrix} x_k \\ e_k \end{bmatrix}_{2n} + [w_k]_{2n}, \tag{5.1}$$

where, the parameters A, BL, and $[w_k]_{2n}$ are already described in (3.2), and,

$$\Delta_{k+1} = \begin{bmatrix} \delta_{k+1}^1\gamma_{k+1}^1 & & 0 \\ & \ddots & \\ 0 & & \delta_{k+1}^N\gamma_{k+1}^N \end{bmatrix}_{n\times n}.$$

The dropout model considered in this chapter is deterministic, i.e. we assume the communication channel experiences m dropouts over a finite length time-interval. Then we address how stability margins change with respect to the number of dropouts. However, counterpart stochastic models for packet dropouts, i.e. assigning a dropout probability to each sub-system which is awarded the channel access at every time-step, may similarly be considered, see e.g. [105, 106]. In what follows, we show that stochastic stability of NCSs over shared communication network which are subject to packet dropouts can be shown employing event-based data scheduling.

5.2.1 Stability Analysis

Here, we should remind that the preliminary definitions, propositions and theorems presented in Section 3.3.1 are all valid so we may frequently recall them in this section. Moreover, the scheduling mechanism and the related parameters such as error thresholds and weight matrices follow the bi-character event-based architecture introduced in (4.1). In addition, the illustrative example presented in Section 3.3.2 to insist the need for modification of the one-step drift operator $\Delta V(e_k)$ defined in (3.3) can be repeated in this section again, to justify the use of the multi-step drift operator.

First, we select the error-dependent non-negative real-valued Lyapunov candidate $V : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$V(e_k) = \sum_{i=1}^{N} e_k^{i\mathsf{T}} Q^i e_k^i = \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2 .$$

Then it is straightforward to check that, under the capacity constraint (3.6), i.e. only one data packet is allowed to be sent through the communication channel at every time-step, if a packet which is scheduled for transmission is dropped in the communication channel, then the length of the interval over which *Theorem* thm:driftcrit can be employed, needs to be increased by one. Indeed, we can imagine that, instead of having a transmission from the scheduled real sub-system, a virtual sub-system has transmitted. Therefore, similar to the proof of *Theorem* 3.2, and assuming that over the interval $[k, k+N]$, m packets are dropped, we can modify the multi-step drift operator (3.7) over the time horizon of $N+m$ time-steps, as follows:

$$\Delta V(e_k, N+m) = \mathsf{E}\left[\sum_{i=1}^{N} \|e_{k+N+m}^i\|_{Q^i}^2 \Big| e_k\right] - \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2 . \tag{5.2}$$

It should be recalled that ergodicity is an asymptotic type of stability, thus if it holds according to a negative drift over an interval, then it holds by defining the drift over any longer interval.

Theorem 5.1. *Consider an NCS consisting of N heterogeneous LTI stochastic sub-systems modeled as (2.1), and a transmission channel subject to the constraint (3.6), and the control, estimation and scheduling laws given by (2.3), (2.4) and (4.1), respectively. Assume that the communication channel experiences $m \in \mathbb{N} \cup \{0\}$ packet dropouts. Define the multi-step drift (5.2) over any time interval of length $N+m$. Then for any positive λ_i's and positive definite Q^i's the Markov chain (2.17) is f-ergodic.*

Proof. See Appendix A.2. □

Remark 5.1. Similar stability guarantees can be provided for a probabilistic model of packet dropouts, i.e. by assigning a probability of dropping out to each data packet which is granted the channel access. Comprehensive stability analysis for this scenario is not brought in this dissertation, but it follows the same ideas for the deterministic packer dropout scenario.

Remark 5.2. Uniform performance bounds for the average cost function J_{ave} introduced in (4.11) can similarly be obtained when packet dropouts are taken into account. As it is shown for stability proof, we can take the same scenario and extend the horizon from N to $N+m$ and obtain initial-value-independent upper bounds for J_{ave}. The upper bounds would expectedly be greater than that of the case without packet dropouts, as derived in Chapter 4.

5.3 Scheduling with Incomplete Event Information

So far, in the design of centralized event-triggered scheduling mechanisms in this dissertation, we have assumed that the scheduler is provided with the updated event information, in a timely manner, from all sub-systems who decide to participate in the channel access competition. In this section however, we take into account the possibility that the event information from one or more than one of the sub-systems, which are requesting for channel access, are received by the scheduler with some delays.

To clarify further analyses while taking into account the delays in sending event information for the centralized scheduler, we should here mention that it is assumed that the NCS of interest is associated with two different communication media for data transmission. One is the shared communication channel over which the sub-systems transmit their state information to their corresponding controllers (if they are awarded the access by the scheduler). This communication channel is subject to the capacity constraint $c < N$, as well as the possibility that the scheduled data packets are dropped (see Section 5.2 of the present chapter.). We assume that the aforementioned communication channel is delay-free, i.e. if a sub-system is awarded the access at some time k', and if the data packet is not dropped (i.e. $\theta^i_{k'} = 1$), then the i^{th} controller will be immediately updated with the transmitted state information.

There is, in addition, another communication channel established between the sub-systems and the centralized scheduling unit over which the local sub-systems who intend to transmit, send their event information (weighted error norms in this dissertation) to the scheduler. This channel is assumed to have no capacity constraint since the amount of data being exchanged over this channel is much lower than the data traffic load over the former communication channel. This channel, however, is supposed to be subject to the time-varying delay $d^i_{k'}$. This declares that despite a transmission request from a local sub-system i is submitted at some time k', the required error information needed for error-dependent scheduling is received by the scheduler with some delay time $d^i_{k'}$. These delays might be induced by scheduler overhead or from congestion in the corresponding channel if a lot of sub-systems send their information in one time-step. Consequently, the scheduler will consider that request as soon as it is received. Delay time $d^i_{k'}$ is assumed to be a multiple of sampling time and can vary time to time for each sub-system, but is assumed to be finite. From now on we make difference between these two channels by denoting the former channel by *communication channel*, and the latter by *scheduling channel*.

It is straightforward to check that only the scheduling policy is going to be affected by these introduced delays and not the difference equation (2.8). Assuming that not all sub-systems can simultaneously transmit according to (2.2), the following bi-character error-dependent rule defines the channel access probability for each sub-system $i \in \{1, \ldots, N\}$ at time-step $k+1$, given possibly the outdated event information form one or more of those eligible sub-systems, according to a deterministic-probabilistic probability measure:

$$\mathsf{P}[\delta^i_{k+1} = 1 | e^j_k, \lambda_j] = \begin{cases} 0, & \|e^i_k\|^2_{Q^i} \leq \lambda_i \\ 1, & \|e^i_{k-d^i_k}\|^2_{Q^i} > \lambda_i \wedge n_{\lambda,k} \leq c \\ \frac{\|e^i_{k-d^i_k}\|^2_{Q^i}}{\sum_{n_{\lambda,k}} \|e^j_{k-d^j_k}\|^2_{Q^j}}, & \|e^i_{k-d^i_k}\|^2_{Q^i} > \lambda_i \wedge n_{\lambda,k} > c, \end{cases} \tag{5.3}$$

where, λ_i's represent the local error thresholds for sub-systems $i \in \{1, \ldots, N\}$, d^i_k denotes the

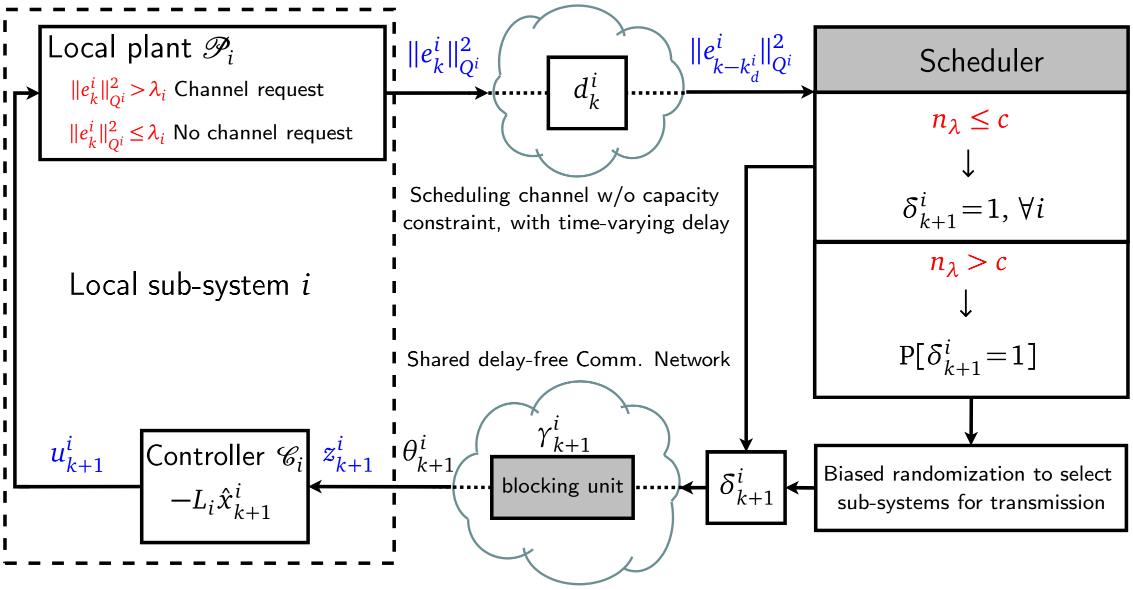

Figure 5.1: Schematic of centralized error-dependent scheduling mechanism with channel imperfections.

delay time in transmission request submission at time-step k, and Q^i's are symmetric positive definite weight matrices. Moreover, $n_{\lambda,k}$ denotes the number of qualified sub-systems for channel access competition, while $c < N$ denotes the channel capacity.

As the local errors are known to their corresponding control loops, the first argument of policy (5.3) is checked locally within each sub-system to decide whether a transmission request should be submitted. If the condition $\|e_k^i\|_{Q^i}^2 > \lambda_i$ holds, then, a transmission request from sub-system i is submitted at time-step k to the scheduler. Depending on the delay time d_k^i, the scheduler takes the channel access request into account either in the same time step (in case $d_k^i = 0$) or later on (in case $d_k^i \neq 0$). It can be seen from (5.3) that only the norm of error is required for the scheduler to assign the priorities, and not the whole error vector. If a sub-system is awarded the channel access, either from a transmission request at that current time-step or from a delayed request, the corresponding controller will be updated with the latest state vector. If $n_{\lambda,k} \leq c$, then all of the received requests will be allowed to transmit as seen from the second argument of (5.3). Otherwise, the channel is allocated probabilistically until the capacity is reached, while the other transmission requests are blocked. The described NCS with both communication channel and scheduling channel, employing the scheduling process under the delay time d_k^i is schematically depicted in Figure 5.1.

5.3.1 Stability Analysis

Reminding that the preliminary definitions, propositions and theorems discussed in Section 3.3.1 are valid, we invoke *Theorem* 3.1 in order to show that the modified event-based scheduling law (5.3) preserves f-ergodicity of the network-induced error e_k, if the multi-step drift is defined over the time-interval $[k, k+N]$.

Theorem 5.2. *Consider an NCS consisting of N heterogeneous LTI stochastic sub-systems modeled as (2.1), and a transmission channel subject to the constraint (3.6), and the control and estimation processes are given by (2.3) and (2.4), respectively. Suppose that the transmission request from an arbitrary sub-system i is received by the scheduler with delay time $d_k^i < N$. Assuming that the channel access is scheduled by (5.3), then the Markov chain (2.17) is f-ergodic.*

Proof. See Appendix A.2. ☐

Remark 5.3. In case $d_{k'}^i \geq N$, i.e. the error of system i remains unknown to the scheduler for the entire interval $[k, k+N]$, f-ergodicity of the Markov chain cannot necessarily be guaranteed by computing the drift over $[k, k+N]$. Instead, one can extend the length of the interval to $2N$, $3N$, etc. to show ergodicity, assuming that $d_{k'}^i$ is finite.

5.4 Numerical Evaluations

In this section, we numerically validate the robustness claims of our bi-character approach with respect to the possibility of packet dropouts, and scheduling with incomplete event information. For the results to be comparable with the numerical evaluations in the previous chapters, where we assume pure probabilistic and bi-character policies but with ideal communication channels, we consider similar NCS setup as employed in Section 3.4. Therefore, we assume NCSs with different number of sub-systems $N \in \{2, 4, 6, 8, 10\}$, where each NCS is comprised of two heterogeneous classes each consisting of $\frac{N}{2}$ homogeneous scalar sub-systems. The first class includes unstable control loops with parameters $A_1 = 1.25$, $B_1 = 1$, while the second class contains control loops with stable processes where the parameters are $A_2 = 0.75$, and $B_2 = 1$. We select $x_0^i = 0$ for each sub-system $i \in \{1, \ldots, N\}$, and $w_k^i \sim \mathcal{N}(0, 1)$. Local sub-systems are assumed to be stabilized, in case a transmission is successfully received, by deadbeat control laws $L_i = A_i$, and the model-based observer is given as (2.4). For illustrative purposes, we select $Q^i = I$ for all $i \in \{1, \ldots, N\}$, which suggests that all sub-systems have identical pre-given priorities.

We compare the performance of our bi-character scheduling mechanism under channel phenomena such as data loss and delay in transmission request submission, with the pure probabilistic policy proposed in Chapter 3, and the bi-character policy for ideal communication channels presented in Chapter 4. The error thresholds λ are assumed to be equal for all sub-systems in a certain NCS setup, and they are selected optimally according to the Figure 4.1, for the ideal channel assumptions. The selected values for different NCS setups are shown in Figure 5.2

Figure 5.2 demonstrates the simulation results providing the comparison between different strategies for NCSs with different number of sub-systems $N \in \{2, 4, 6, 8, 10\}$. The communication constraint (3.6) asserts that for NCSs with $N > 2$, we have more unstable sub-systems than the available transmission slots per time-step ($c = 1$). The averages are calculated via Monte Carlo simulations over a horizon of 5×10^5. The lower bound is determined by an optimal methodology developed in [33] that replaces the hard constraint on the number of transmissions per time-step by an average transmission rate constraint (the grey solid curve in Figure 5.2). As already discussed, employing the bi-character design leads to

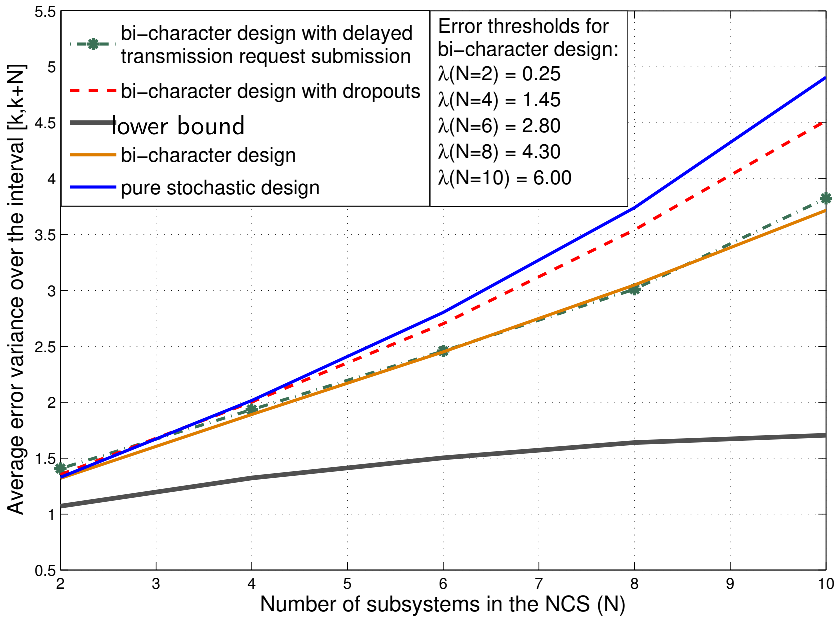

Figure 5.2: Comparison of the average error variance vs. number of control loops for different scheduling policies.

improved average error variance compared to the pure probabilistic protocol, (see Section 4.4).

The average error variances are calculated, for the same thresholds λ, considering one packet dropout over every N time-steps, for an unstable control loop (red-dotted curve in Figure 5.2). The sub-system for which the corresponding packet is supposed to be dropped and the time-step at which the packet dropout happens, are chosen randomly. The error variance expectedly increases compared to the case without packet dropouts (yellow solid curve in Figure 5.2). We also simulate the average error variances considering a randomly chosen delay time $d_k^i \in [0, N-1]$ for two sub-systems; one stable and one unstable (green-dotted curve in Figure 5.2). As it can be seen in Figure 5.2, the error variance is slightly different compared to the ideal case where $d_k^i = 0$. This observation is expected since a delayed transmission request might lead to a higher chance of channel access for a specific sub-system (as the corresponding error might decrease over time). Moreover, even if a transmission request with the highest priority is received undelayed, an eventual transmission is not guaranteed due to the probabilistic character of the scheduling policy. Thus the average variances when receiving the transmission requests with delay remain close to the undelayed scenario.

5.5 Summary

In this chapter, we investigated the applicability of the event-triggered prioritizing bi-character scheduling rule when the shared communication channel does not guarantee that a scheduled data packet will be received by the receiving end station, i.e., the channel is subject to packet dropouts. In addition, we analyzed robustness of the scheduling architecture assuming that the event information is not timely updated, and the scheduler has to cope

with outdated channel requests. Considering a deterministic model for packet dropouts, it is shown, both theoretically and numerically, that the margins of stability are enlarged with increasing number of dropouts, (compared to stability margins in Chapter 4). As a retransmissions is not allowed in case a packet is dropped, this conclusion is expected, since a dropout is equivalent to have N open-loop sub-systems. We proved that the network-induced error remains f-ergodic, even if the scheduled data packets are dropped, but over a lengthier time horizon.
Furthermore, it is of great importance to evaluate the robustness of the proposed centralized scheduling approach with respect to availability of up-to-date event information. Since the deterministic part of the scheduler can be implemented locally within each sub-system, the centralized event information is needed only for assigning the priorities. However, channel access is eventually granted according to a biased randomization, and not according to the priorities. Of course a sub-system with high priority is more likely to be awarded the channel access, but within our approach this is not deterministically guaranteed. We showed theoretically that f-ergodicity holds in case the scheduler is not updated perfectly, and consequently the priorities are assigned based on outdated event information. Through numerical evaluations, it is shown that performance of the scheduler remains in the vicinity of the case which the scheduler is updated routinely. Expectedly, increasing the number of outdated event information from channel requesting sub-systems, especially from unstable sub-systems, results in further increase in the error variance, as discussed in *Remark* A.1.

5.6 Contributions

The presented theoretical and numerical results in this chapter are partly from the author's own research in [99, 107]. The main contribution of this chapter is to evaluate robustness of event-triggered prioritizing bi-character scheduling law with respect to packet dropouts and lack of up-to-date centralized event information for the scheduler. Numerous works have considered different aspects of non-ideal communication channels, where among all [79,80,82,83,108] are addressing the problem of data packet loss either within deterministic or probabilistic frameworks. In [80, 82, 108], probabilistic models of dropout are used where in the latter two the dropouts are modeled according to the Bernoulli process. In the deterministic framework [79], worst-case bounds for the number of consecutive dropouts are derived. In [83] the authors seek performance guarantees under network-induced delays and varying data packet rates in a probabilistic setting, by modeling a packet dropping network as an erasure channel.

We discussed that our proposed scheduling architecture is capable of coping with data packet dropouts in a deterministic fashion. Moreover, the scheduler is capable of effectively assigning the priorities in the absence of up-to-date error information from the networked entities. As the proposed scheduling approach in this dissertation is original, comprehensive robust analysis is essential. It is worth mentioning that, in centralized framework collisions do not take place, however, we will extensively discuss about collisions in the next chapter, when we introduce the decentralized event-based scheduling mechanism.

Numerical evaluations validate our claims about performance margins through the illustration Figure 5.2, which is taken from the author's own work in [107].

6

Decentralized Implementation of State-Dependent Scheduling Law

Scheduling protocols can be realized in a centralized or distributed fashion. Time division multiple access (TDMA), frequency division multiple access (FDMA), code division multiple access (CDMA), and Try Once Discard (TOD) are well-known centralized protocols preferred in, for instance, medium-size wireless sensor networks. Centralized approaches offer collision-free and precise channel scheduling with higher throughput compared to decentralized protocols. Moreover, the offline centralized approaches consume less energy compared to e.g. carrier sense multiple access with collision avoidance (CSMA-CA), where each node needs to sense the channel permanently. Furthermore, QoS support and bandwidth arbitration is facilitated as they can prioritize channel access. However, they lack flexibility and scalability and therefore are not well-suitable for large-scale networks due to their synchronous nature. Furthermore, centralized protocols are characterized by a single point of failure, which can compromise the NCS performance. Distributed approaches however, represent easy-to-install, low-cost and scalable scheduling architectures suitable for NCSs with a large number of nodes. However, collisions take place inevitably within distributed protocols and need to be handled with care in the NCS design. In terms of overall operational security, they might be insecure as no global administration unit exits. In addition, collision avoidance mechanisms, e.g. the listen-before-talk scheme, call for all nodes to sense the channel permanently, which instead requires high energy consumption due to idle listening, overhearings, and message overhead. Altogether, both centralized and distributed scheduling mechanisms are practical depending on the application purpose and therefore need to be comprehensively investigated.

In this chapter, we discuss the implementation of the bi-character error-dependent scheduling approach in decentralized fashion for multiple loop stochastic networked control systems in which the individual control loops are coupled through a shared communication channel. The results in this chapter correspond to three main contributions; first in Section 6.1, we assume that the local schedulers have access to the true error state of their corresponding local sub-systems. Then the results are extended in Section 6.4 to the case where the local event-triggers are defined assuming that only noisy sensor measurements are accessible and not the true error values. Finally, we address the problem of event-based medium access control for multi-hop networks in Section 6.5. In this section, we assume that true state values are accessible, i.e. similar to the assumptions in Section 6.1, however the results can be generalized with almost the same procedure as in Section 6.4. In all mentioned contributions, the decentralized scheduling designs are assumed to combine deterministic and probabilistic attributes to efficiently allocate the limited communication resource among the control loops in an event-based fashion. Given local error thresholds, each control loop determines whether to compete for the channel access in a deterministic manner. Note that this process is performed locally within each control loop because the triggering condition is checked locally. Therefore, implementation of the probabilistic scheduling process in decentralized fashion is indeed discussed in this chapter. It should be mentioned here that, in the third section of this chapter, we assume that the channel access is determined based on probabilistic slotted ALOHA protocol, i.e. no state-dependent prioritization is considered to select the sub-systems for channel access. So, the approach is known to be decentralized. We demonstrate stochastic stability of the described NCSs under decentralized MAC architectures in terms of Lyapunov Stability in Probability (LSP). The numerical results certify our stability claim and illustrate that our approach improves resource utilization and reduces the network-induced error variance in comparison with time-triggered and uniform random access scheduling policies.

This chapter is organized as follows. We present the decentralized bi-character error-dependent scheduling architecture under the assumption that perfect state information is accessible for event-triggers, and discuss the structural properties of such systems in Section 6.1. In addition, the implementation procedure of the proposed scheduling scenario is addressed for practical NCS realizations. In Section 6.2 stochastic stability of the described capacity-limited NCSs under the introduced decentralized scheduling rule is addressed by the concept of Lyapunov stability in probability. A comprehensive analysis shows that the collision rate can be effectively controlled by the design parameters which consequently leads to the performance enhancement. Numerical simulations in Section 6.3 validate our stability assertion and corroborate the improved performance claims. Then the same sequence of analyses is performed in Section 6.4 wherein we relax the assumption on having access to the perfect state information, and instead assume that only noisy sensor measurements are available. Finally, Section 6.5 presents stability and numerical evaluations of a bi-character scheduling design in form a threshold policy complemented by slotted ALOHA MAC protocol in a multi-hop channel scenario.

6.1 Event-based Decentralized Prioritized Scheduling Design

In Chapter 3 we introduced the pure probabilistic event-based scheduling rule where it follows by incorporation of the deterministic feature to the scheduler in Chapter 4. Both of the proposed scheduling approaches are intrinsically centralized which assumes that the event triggers, in our case the weighted error norms, from all sub-systems involved in channel access competition, are provided for the scheduler at every time-step. Therefore, the scheduling process for those chapters has the form of a centralized control unit where all sub-systems are connected to and update their event information in a timely manner. This approach, as earlier discussed, might not always be desirable due to e.g. far distance between the sending stations, high amount of data which needs to be transmitted and analyzed by the centralized scheduling unit, extra transmission costs, and even privacy issues. Decentralized control and scheduling mechanisms, which they introduce local decision making units capable of execution of control and scheduling inputs with only having access to the local information, are appropriate architectures for those networked systems. In what follows we introduce a decentralized event-triggered error-dependent prioritizing scheduling mechanism which is capable of efficiently allocating scarce communication resources among multiple control entities according to their local real-time conditions.
Recalling expression (2.14) as the mapping from the centralized information set $\{e_k^1, \ldots, e_k^N\}$ to the binary scheduling variable δ_{k+1}^i for every sub-systems $i \in \{1, \ldots, N\}$, we can state the decentralized error-dependent counterpart mapping as follows:

$$\delta_{k+1}^i = \varrho_k^i(e_k^i), \tag{6.1}$$

where, the scheduling law ϱ_k^i might be either deterministic, probabilistic, or a combination of both, but is allowed to be solely dependent on local information of sub-system i. We will discuss the specific form of the scheduling rule further in this section.
The sub-systems characteristics, i.e. the dynamics of the processes, local control laws and local estimators are assumed to be identical to the described model in Chapter 3 and Chapter 4. Therefor, the overall NCS model derived in Section 3.2 will be used throughout this section as well. Therefore, we assume to have the overall NCS state dynamics expressed in (3.2), with the only difference to be the selection of the scheduling variables δ_{k+1}^i's, and of course the effect of collisions should be taken into account, as a direct consequence of having decentralized local schedulers. A multiple-loop NCS with decentralized scheduling mechanism is schematically depicted in Figure 6.1.

In order to introduce the decentralized scheduling mechanism, it is essential to have a closer look at the communication channel. First of all, we assume that the operational time scale of the communication channel, denoted as *micro time slots*, is much finer than that of the local control systems, denoted as *macro time slots*. Lets define T as the sampling period of the control systems. Therefore, we assume between two consecutive macro time slots $k \to k+1$, the micro slots are distributed as $\{kT, kT+\tau, \ldots, kT+(h-1)\tau, kT+h\tau\}$, where τ is the temporal duration of each micro slot and $h \gg 1$ denotes the number of

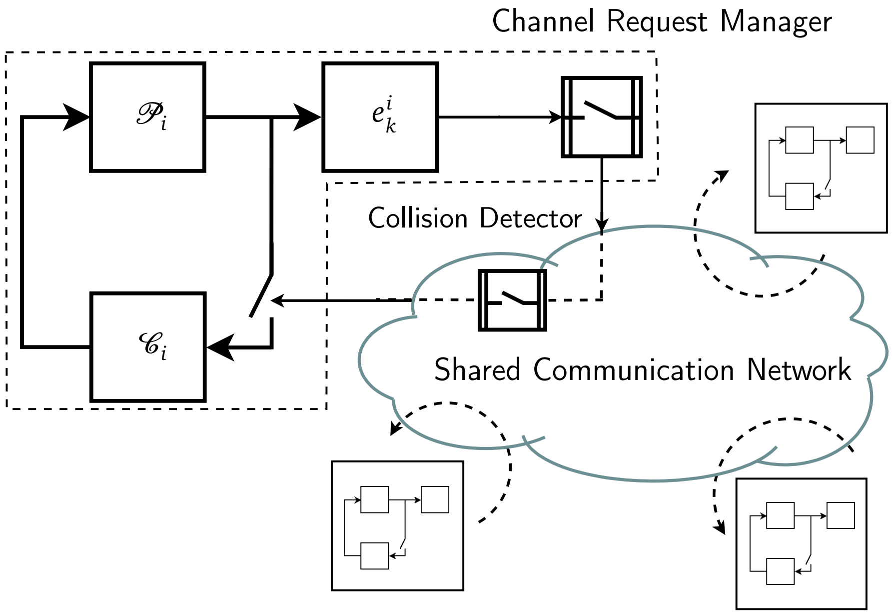

Figure 6.1: A multiple-loop NCS with a shared communication channel and decentralized scheduling mechanism.

micro slots within one macro slot, hence $T = h\tau$. Considering the current communication and telecommunication standards in terms of bandwidth and speed, and compare it with proper sampling rate of LTI control systems, it is quite reasonable to assume $h \gg 1$. One data packet can be transmitted through the communication channel starting from every micro time slot $\{1,\ldots,h\}$. As a simplifying assumption, we consider that a data packet will be received by the corresponding controller immediately after being transmitted, therefore even if the packet is sent at time slot $(h-1)\tau$, it is assumed to be received before starting the next macro time slot.

We aim for a decentralized approach, i.e. the i^{th} sub-system is provided with only local information A_i, B_i, W_i, λ_i, Z_k^i and the distribution of x_0^i, where λ_i is the local error threshold and $Z_k^i = \{z_0^i, \ldots, z_k^i\}$. We remind that

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \wedge \gamma_k^i = 1 \\ \varnothing, & \text{otherwise.} \end{cases}$$

Every sub-system $i \in \{1,\ldots,N\}$ has the knowledge about the latest local error $e_{k'}^i$ at an arbitrary time-step k' as well as λ_i to decide whether to compete for channel access at the subsequent time-step $k'+1$. Remind that the deterministic feature of the centralized bi-character scheduling rule (4.1) is a local threshold policy, as every sub-system can check whether its weighted error norm exceeds the local threshold. Therefore, the deterministic scheduling process introduced in Chapter 4, can be carried over to the decentralized scenario without any modification. Hence, if the weighted square norm of the latest time-step error of a sub-system i is smaller or equal to the given local threshold, sub-system i will be deterministically excluded from the channel access competition. Thus, we have

$$\mathsf{P}[\delta^i_{k'+1} = 1 | e^i_{k'}] = 0 \quad \text{if} \quad \|e^i_{k'}\|^2_{Q^i} \leq \lambda_i. \tag{6.2}$$

As already discussed, the weighting matrix Q^i contains characteristic information about each local sub-system which specifies how frequent a sub-system needs data transmission. For example, an unstable system usually needs more data transmission than a stable system and consequently is more sensitive to the size of error. To account for this, the corresponding weight matrices can be selected to be larger for unstable sub-systems to give them higher chance of transmission than the stable ones.
Each sub-system i which becomes eligible to compete for the channel access at time-step $k'+1$, satisfies the following condition:

$$\|e^i_{k'}\|^2_{Q^i} > \lambda_i. \tag{6.3}$$

After determining whether a sub-system is eligible for transmission through the deterministic threshold policy (6.2), the allocation of the channel is performed via a decentralized probabilistic mechanism. The mechanism is error-dependent in the way that the sub-systems with larger error values have higher chances of being granted the channel access. In the absence of centralized information, our proposed probabilistic mechanism determines the transmission order locally within each sub-system. For the purpose of simplicity in derivations, we assume that only one sub-system can transmit at multiples of τ between every subsequent time-steps $k \to k+1$, i.e.

$$\sum_{i=1}^{N} \delta^i_k = 1. \tag{6.4}$$

The provided results can be readily extended for $\sum_{i=1}^{N} \delta^i_k = c < N$ assuming that the c channels are single-hop. The multi-hop scenario will be treated later in Section 6.5.

Assume that each station which is about to send a data packet over the communication channel is capable of sensing or hearing the carrier to see if there exists free bandwidth. The idea behind the decentralized scheduling law is then straightforward; an eligible sub-system for channel access at some time-step $k'+1$ locally inspects whether the channel is free or occupied. If the channel is sensed as *free* (idle), then the data packet is sent, otherwise, if the channel is sensed as *occupied* (busy), the sub-system backs off. We ideally assume that upon granting the channel access to a sub-system, the data packet will be received instantly. The channel is also assumed to be error-free which is not a visionary assumption due to the negligible error rate of existing high-speed communication technologies and high reliability guaranteed by error detection and correction mechanisms.
Before presenting our proposition for the decentralized scheduling law, we first introduce integer random variables $\nu^i_{k'} \in \{\tau, 2\tau, \ldots, (h-1)\tau\}$, called *waiting times*. The waiting times denote the time duration a transmission-eligible sub-system i, which satisfies the condition (6.3), waits before listening to the channel. In the other words, sub-system i starts sensing the channel at time $k'h + \nu^i_{k'}$. If the channel is sensed as free, sub-system i immediately transmits, i.e. at communication time slot $k'h + \nu^i_{k'}$. Otherwise, it backs

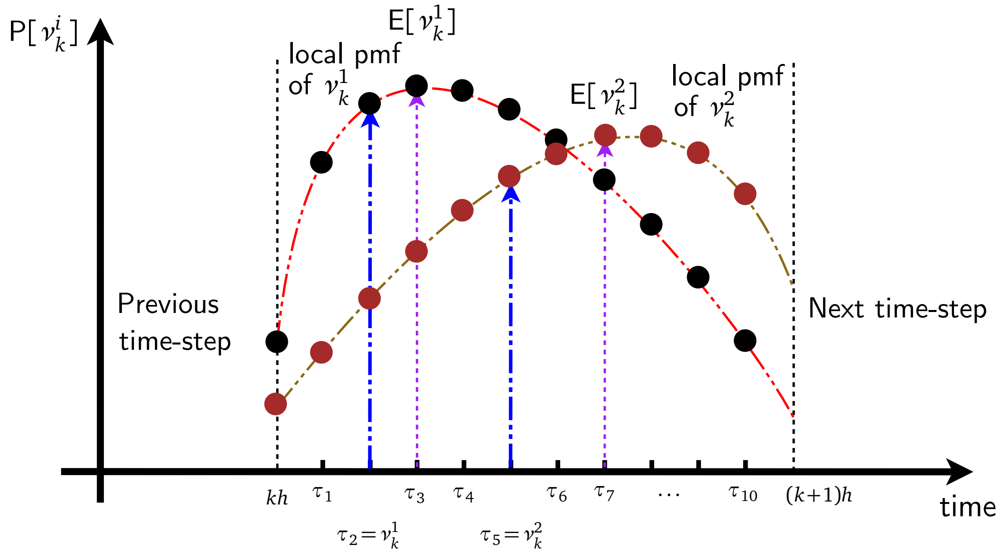

Figure 6.2: Two waiting times ν_k^1 and ν_k^2 are selected according to the local probability mass functions with error-dependent means E[ν_k^1] and E[ν_k^2].

off and does not attempt to re-transmit again until the end of that time-step. Indeed, we assume that each sub-system attempts only once for transmission at every time-step k', and if the attempt fails, that sub-system remains open-loop and tries to transmit at further time-steps $\{k'+1, k'+2, \ldots\}$. It is worth mentioning that the dependency of random waiting times, which is a MAC layer parameter, on error values from the application layer (i.e. the control layer), couples the medium access to the application. In the following, we propose for a sub-system i that the waiting time $\nu_{k'}^i$ to be chosen randomly from an arbitrary but finite-variance and concave probability mass function with the given error-dependent mean

$$\mathsf{E}[\,\nu_{k'}^i\,] = \frac{\alpha_{k'}^i}{\|e_{k'}^i\|_{Q^i}^2}. \tag{6.5}$$

where, $\alpha_{k'}^i \in \mathbb{R}^+$ is a tuning parameter, and $\mathsf{E}[\,\nu_{k'}^i\,] \in (0, \alpha_{k'}^i \lambda_i^{-1})$ according to the assumption (6.3). The random variables $\nu_{k'}^i$ for each sub-system are chosen according to their corresponding local probability mass functions. The concavity of the local distribution emphasizes the prioritized character as it ensures that the random waiting times are chosen with higher probabilities around the mean (6.5). The local distribution functions are defined on the interval $[k'h, (k'+1)h]$ and for the next time-step the same procedure repeats with the updated local error values. Having said that, each sub-system i randomly chooses its $\nu_{k'}^i \in [k'h + \tau_1, \ldots, k'h + \tau_h]$. Figure 6.2 depicts a schematic of probability mass functions for two sub-systems competing for sole channel slot within one time-step.

The intuition behind the error-dependent mean (6.5) is that if a sub-system is characterized by a large error value at a specific time-step k', then the mean of the local distribution, from which the waiting time is chosen, becomes small. Consequently, the waiting time is more likely to be shorter which eventually increases the chance of sensing the channel as free. Finally, depending on the randomly chosen waiting times, the sub-system with the shortest waiting time is the only sub-system which senses the channel as free, and transmits.

The other candidates have to wait for next time steps, when the same process repeats again depending on the updated local error values. This happens due to the constraint (6.4) that only one data packet is allowed to be sent through the channel.
It should be noted that the mean value $\mathsf{E}[\nu^i_{k'}]$ may not coincide exactly with a sampling instance $\tau_s \in [\tau_1, \tau_h]$ of the discrete local distributions because $\frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}}$ may take any values and not necessarily the multiples of τ. To take this into account, we select the closest time instance to the value $\frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}}$ to be the mean of the discrete distributions, i.e. we choose either τ_s or τ_{s+1}, where $\tau_s < \frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}} < \tau_{s+1}$. The assumption $h \gg 1$ ensures that the distance $\frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}} - \tau_s$ or $\tau_{s+1} - \frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}}$ is negligible in comparison with h.
As the introduced policy is decentralized, the possibility of packet collisions should also be accounted for. Within the presented scenario, a collision occurs if at least two sub-systems choose exactly identical waiting times and start sensing the channel simultaneously. We assume that each sub-system is informed by the *collision detection unit*, in case a collision occurs. Moreover, assuming that if a collision is detected, the collided packets are all dropped, a collision can be treated as a dropout in the communication channel. As already discussed for the packet loss in Chapter 5, at every time-step k, a successful transmission is reported via the binary variable γ_k:

$$\gamma^i_k = \begin{cases} 1, & x^i_k \text{ successfully received} \\ 0, & x^i_k \text{ dropped.} \end{cases} \tag{6.6}$$

Similarly, the dynamics of the local error state e^i_k becomes

$$e^i_{k+1} = \left(1 - \theta^i_{k+1}\right) A_i e^i_k + w^i_k, \tag{6.7}$$

where $\theta^i_k = \delta^i_k \gamma^i_k$. We can achieve such a design within a TCP-like protocol, where the acknowledgment of a successful transmission is sent over an error-free reverse link to each sub-system.
Indeed, having the scheduling law in probabilistic fashion is beneficial in decreasing the rate of collisions. First, if two or more sub-systems end up having identical error values, then $\mathsf{E}[\nu^i_{k'}] = \mathsf{E}[\nu^j_{k'}]$, which within our proposed probabilistic mechanism, does not necessarily imply $\nu^i_{k'} = \nu^j_{k'}$. Moreover, the probabilistic scheduling law excels when having homogeneous sub-systems with similar dynamics. Usually, the noise values are negligible with respect to the systematic error, then considering the waiting time in absolute form would increase the chance of collisions.
Assume that $\mathcal{G}_{k'}$ denotes the set of eligible sub-systems for transmission at the next time-step $k' + 1$, i.e.

$$j \in \mathcal{G}_{k'} \quad \text{if} \quad \|e^j_{k'}\|^2_{Q^j} > \lambda_j. \tag{6.8}$$

Considering the communication constraint (6.4), the probability that a sub-system $j \in \mathcal{G}_{k'}$

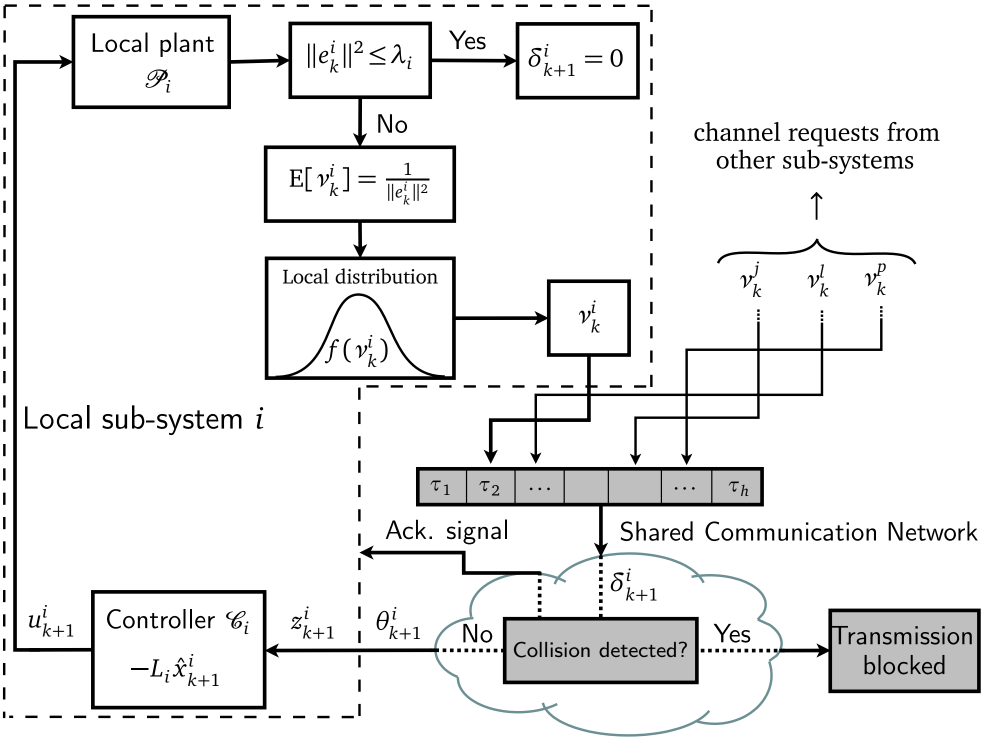

Figure 6.3: Multiple-loop NCS with error-dependent decentralized scheduling algorithm and collision detection unit.

transmits at time-step $k' + 1$ can be expressed as

$$\mathrm{P}[\delta_{k'+1}^j = 1 | e_k^j] = \mathrm{P}[\nu_{k'}^j < \nu_{k'}^l], \qquad \forall l \in \mathscr{G}_{k'}, l \neq j. \tag{6.9}$$

Basically, if a sub-system j transmits then it ensures that its waiting time has been shorter than all other eligible sub-systems $l \in \mathscr{G}_{k'}, l \neq j$, i.e. $\nu_{k'}^j < \nu_{k'}^l$. The sub-systems that find the channel occupied, might try to transmit in future time-steps adhering to the similar procedure, if they are eligible depending on their updated error values. The decentralized algorithm is schematically shown in Figure 6.3 for a multi-loop NCS with collision detection unit.

6.2 Stability Analysis

In this section, we study stability of multiple-loop NCSs with shared communication networks subject to the capacity constraint (6.4), and the proposed decentralized scheduling policy (6.2) and (6.9). Essentially, we show Lyapunov stability in probability (LSP) for the described NCS. Before proceeding to the main stability results, we state some preliminaries about LSP.

6.2.1 Preliminaries

Definition 6.1. (Lyapunov Stability in Probability (LSP), [91]) A linear system with state vector x_k possesses LSP if given $\varepsilon, \varepsilon' > 0$, exists $\rho(\varepsilon, \varepsilon') > 0$ such that $|x_0| < \rho$ implies

$$\lim_{k\to\infty} \sup \mathsf{P}\left[x_k^\mathsf{T} x_k \geq \varepsilon'\right] \leq \varepsilon. \tag{6.10}$$

The following lemma shows that, considering the separation property discussed in Chapter 2, the LSP is achievable by solely investigating the aggregate error state e_k.

Lemma 6.1. For a shared resource multiple-loop NCS with the plants, controllers, estimators, and state described by (2.1)-(2.6), the condition in (6.10) is equivalent to

$$\lim_{k\to\infty} \sup \mathsf{P}\left[e_k^\mathsf{T} e_k \geq \xi'\right] \leq \xi, \tag{6.11}$$

where $\xi' > 0$ and the constant ξ fulfills $0 \leq \xi \leq \varepsilon$.

Proof. We know that the system state x_k^i for each control loop i evolves as

$$x_{k+1}^i = (A_i - B_i L_i)x_k^i + (1-\theta_{k+1}^i)B_i L_i e_k^i + w_k^i. \tag{6.12}$$

As already discussed in Chapter 2, the evolution of the error e_k^i is independent of the system state x_k^i within each individual control loop $i \in \{1,\dots,N\}$. Furthermore, by assumption, the emulative control law (2.3) ensures the closed-loop matrix $(A_i - B_i L_i)$ is Hurwitz. Together with the assumption that the initial value x_0^i is randomly chosen from a bounded variance distribution, it follows that the system state x_k^i is converging. In addition, the disturbance process w_k^i is i.i.d. according to $\mathcal{N}(0, W_i)$, and is bounded in probability. Thus, showing $\lim_{k\to\infty} \sup \mathsf{P}\left[e_k^{i^\mathsf{T}} e_k^i \geq \xi_i'\right] \leq \xi_i$ ensures existence of constants ε_i and $\varepsilon_i' > 0$ such that $\lim_{k\to\infty} \sup \mathsf{P}\left[x_k^{i^\mathsf{T}} x_k^i \geq \varepsilon_i'\right] \leq \varepsilon_i$.
As individual control loops operate independently, we take the aggregate NCS state (x_k, e_k). Then, the existence of ξ and $\xi' > 0$ such that $\lim_{k\to\infty} \sup \mathsf{P}\left[e_k^\mathsf{T} e_k \geq \xi'\right] \leq \xi$, implies existence of ε and $\varepsilon' > 0$ such that $\lim_{k\to\infty} \sup \mathsf{P}\left[x_k^\mathsf{T} x_k \geq \varepsilon'\right] \leq \varepsilon$, and the proof then readily follows. $\square$

Remark 6.1. One can recall from *Lemma* 6.2 the separation property between the control law and scheduling law synthesis discussed comprehensively in Chapter 2. Indeed, existence of the stabilizing control law ensures LSP of the system states, and therefore showing the network-induced error also possesses LSP is enough to ensure that the aggregate state $[x_k^{i^\mathsf{T}}, e_k^{i^\mathsf{T}}]^\mathsf{T}$ is Lyapunov stable in probability.

As expected values are more straightforward in pursuing further analyses than probabilities, we employ the *Markov's inequality* which ensures for $\xi' > 0$

$$\mathsf{P}\left[e_k^\mathsf{T} e_k \geq \xi'\right] \leq \frac{\mathsf{E}\left[e_k^\mathsf{T} e_k\right]}{\xi'}. \tag{6.13}$$

The above inequality confirms that showing the error is uniformly bounded in expectation ensures finding appropriate ξ and $\xi' > 0$ such that (6.11) is satisfied for arbitrary $\rho(\xi', \xi)$. Therefore, from now on, we focus on deriving upper-bound for the expectation of aggregate weighted quadratic error norm:

$$\mathsf{E}\left[e_k^{\mathsf{T}} Q e_k\right] = \sum_{i=1}^{N} \mathsf{E}\left[e_k^{i\,\mathsf{T}} Q^i e_k^i\right] = \sum_{i=1}^{N} \mathsf{E}\left[\|e_k^i\|_{Q^i}^2\right], \tag{6.14}$$

where $Q = \mathrm{diag}(Q^i)$. Incorporating the weight matrices, the condition (6.11) can be modified as

$$\lim_{k\to\infty} \sup \mathsf{P}\left[e_k^{\mathsf{T}} Q e_k \geq \bar{\xi}'\right] \leq \bar{\xi}. \tag{6.15}$$

Due to the capacity constraint (6.4), the boundedness of (6.14) cannot always be shown over one step transition. Similar to the Chapter 3 and Chapter 4, this observation can be shown via constructing the following illustrative example.

Illustrative example Consider an NCS composed of two identical scalar sub-systems competing for the sole channel slot. For the illustrative purposes, assume $Q^1 = Q^2 = 1$, $\alpha_k^1 = \alpha_k^2 = 1$, and $\lambda_1 = \lambda_2 = \bar{\lambda}$. In addition, assume that both sub-systems start from identical initial value at time-step k, i.e. $e_k^1 = e_k^2 = \bar{e}_k$. To avoid triviality, consider that the condition (6.3) is fulfilled and consequently both sub-systems are eligible for channel access competition at the time-strep $k+1$. Further, assume that both sub-systems have identical local distributions to choose their waiting times ν_{k+1}^1 and ν_{k+1}^2 from. According to (6.5), both distributions have the same means $\frac{1}{\|\bar{e}_k\|_2^2}$. Thus, the transmission chance for each sub-system is $\frac{1}{2}$. Therefore, from (6.7) with $e_k = [e_k^1 \; e_k^2]^{\mathsf{T}}$, it follows that

$$\begin{aligned}
\sum_{i=1,2} \mathsf{E}\left[\|e_{k+1}^i\|_2^2\right] &= \sum_{i=1,2} \mathsf{E}\left[\|\left(1-\theta_{k+1}^i\right)A_i e_k^i + w_k^i\|_2^2\right] \\
&= \frac{1}{2}\left(\mathsf{E}[\|A\bar{e}_k + w_k^1\|_2^2 | e_k] + \mathsf{E}[\|w_k^2\|_2^2]\right) \\
&+ \frac{1}{2}\left(\mathsf{E}[\|A\bar{e}_k + w_k^2\|_2^2 | e_k] + \mathsf{E}[\|w_k^1\|_2^2]\right) = \mathrm{tr}(W_1) + \mathrm{tr}(W_2) + \|A\bar{e}_k\|_2^2,
\end{aligned}$$

which is not bounded in expectation for arbitrary $\bar{e}_k$ and system matrix A. As already discussed, between two consecutive transmissions of each sub-system, they operate in open-loop. Hence, in general, the respective local errors are expected to grow. Thus to obtain boundedness of error state, all sub-systems need to have transmission chances. Due to the constraint (6.4), an interval of length N provides enough transmission possibilities.

6.2.2 Stability Analysis – Lyapunov Stability in Probability

After expressing all the necessary preliminaries, in this section we aim to show that the proposed decentralized scheduling policy introduced in (6.2) and (6.9), leads to Lyapunov stability in probability for the overall NCS. The stability analysis takes almost similar procedure as already discussed in Chapter 3 and Chapter 4 by dividing the entire state-space to sub-spaces and then analyze the error state behavior in each of them. The change in the notion of stochastic stability, i.e. from f-ergodicity to LSP, when employing decentralized

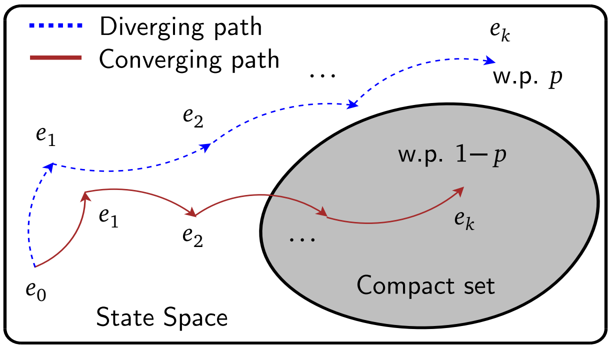

Figure 6.4: Illustration of an LSP process in an uncountable state space.

scheduling rule is inevitable due to the possibility of successive packet collisions. Indeed, f-ergodicity provides tighter stability margins than LSP, thus if a stochastic system is shown to be f-ergodic, then it possesses LSP as well, but not the vice-versa. Figure 6.4 depicts a schematic of an LSP process evolving in an arbitrary state-space. Despite the fact that in our NCS scenario, packet collision has the same effect of packet dropout, i.e. both end up with the channel being blocked, these two are resulted from different sources. A collision happens as the direct consequence of MAC design while packet dropouts may occur due to hardware malfunctioning and excessive channel traffic.

Similar to the previous chapters, we let the NCS of interest operate over the interval $[k, k+N]$, with arbitrary initial state e_k. Similar to the proof of *Theorem* 3.2, we assume that the network-induced error evolves from the initial time-step k until $k+N-1$ and we predict its evolution considering all the possible scenarios under the introduced decentralized scheduling policy over the interval $[k, k+N-1]$. Then, looking at time-step $k+N$, we show the aggregate error state e_{k+N} fulfills (6.15). Stability of the overall NCS with aggregate network state $[x_k, e_k]$ is summarized in the following theorem:

Theorem 6.1. *Consider an NCS with N heterogeneous LTI control loops, with the plants given by (2.1), sharing a communication channel subject to the constraint (6.4). Given the control laws (2.3) and scheduling laws (6.2) and (6.9), the aggregate network state $[x_k, e_k]$ is Lyapunov stable in probability for any positive λ_i's and α_k^i's and any positive definite Q^i's.*

Proof. Similar to the proof of the *Theorem* 4.1, we divide the sub-systems $i \in \{1, \dots, N\}$ at time-step $k+N-1$ into three disjoint but complementary sets $c_1 - c_3$ covering the entire state space $\mathbb{R}^n$ as follows:
Sub-system i:

c_1: has either successfully transmitted or not within the past $N-1$ time-steps, and is in set $\bar{\mathcal{G}}_{k+N-1}$, i.e.

$$i \in \bar{\mathcal{G}}_{k+N-1} \quad \Rightarrow \quad \|e^i_{k+N-1}\|^2_{Q^i} \leq \lambda_i,$$

c_2: has successfully transmitted at least once within the past $N-1$ time-steps, and is in set $\mathcal{G}_{k+N-1}$, i.e.

$$\exists k' \in [k, k+N-1] : \theta^i_{k'} = 1 \;\text{ and }\; \|e^i_{k+N-1}\|^2_{Q^i} > \lambda_i,$$

$l_1^{c_3}$ has not transmitted within the past $N-1$ time-steps, but has been in $\bar{\mathcal{G}}$ at least once, the last occurred at a time-step $k+r'_i$, with $r'_i \in [0, \dots, N-2]$,

$l_2^{c_3}$ has not transmitted within the past $N-1$ time-steps, and has been in $\mathscr{G}$ for all time-steps $[k, k+N-1]$.

where,

$$i \in \begin{cases} \mathscr{G}_{k'} & \text{if} \quad \|e^i_{k'}\|^2_{Q^i} > \lambda_i, \\ \bar{\mathscr{G}}_{k'} & \text{if} \quad \|e^i_{k'}\|^2_{Q^i} \leq \lambda_i, \end{cases} \tag{6.16}$$

satisfying $\mathscr{G}_{k'} \cup \bar{\mathscr{G}}_{k'} = N$. According to (6.2), sub-systems belonging to $\mathscr{G}_{k'}$ are considered for transmission at time-step $k'+1$.

Suggested by the expression (6.13), we need to study the boundedness of error expectation over the interval $[k, k+N]$ for all cases c_1-c_3. Since, the cases are complementary and mutually exclusive, we can calculate the probability that each case may happen, and express (6.14) as

$$\sum_{i=1}^{N} \mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i}\right] = \sum_{i \in c_l} \mathsf{P}_{c_l}\, \mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i} | c_l, \mathsf{P}_{c_l}\right], \qquad \text{for} \quad c_l \in \{c_1, c_2, c_3\}, \tag{6.17}$$

where $\sum_{l=1}^{3} \mathsf{P}_{c_l} = 1$. Assuming $\mathsf{P}_{c_1} = \mathsf{P}_{c_2} = \mathsf{P}_{l_1^{c_3}} = 1$, we can make the upper-bound for (6.17) as follows:

$$\sum_{i=1}^{N} \mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i}\right] \leq \sum_{i \in c_1, c_2, l_1^{c_3}} \mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i} | c_l\right] + \sum_{i \in l_2^{c_3}} \mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i} | l_2^{c_3}, \mathsf{P}_{l_2^{c_3}}\right]. \tag{6.18}$$

As we comprehensively discussed in the proof of the *Theorem* 4.1, the boundedness of the expectation of error norm of a sub-system i, i.e. $\mathsf{E}\left[\|e^i_{k+N}\|^i_{Q^i}\right]$ which belongs to the cases c_1, c_2, and $l_1^{c_3}$ can be shown regardless of the scheduling law. This can be seen from the derived upper-bound expressions (A.3), (A.4), and (A.5) for the cases c_1 and c_2 and the sub-case $l_1^{c_3}$, respectively. Relying on those upper-bounds for the mentioned cases, we have for those sub-systems i belong to c_1 from (A.3) that,

$$\sum_{i \in c_1} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i} | e^i_k\right] \leq \sum_{i \in c_1} \|A_i\|^2_2 \lambda_i + \mathrm{tr}(Q^i W_i).$$

Therefore, the condition (6.15) is fulfilled by assuming $\bar{\xi}' > \sum_{i \in c_1} \|A_i\|^2_2 \lambda_i + \mathsf{E}\left[\|w^i_{k+N-1}\|^2_{Q^i}\right]$, and $\bar{\xi} = \frac{\sum_{i \in c_1} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i} | e^i_k\right]}{\bar{\xi}'} < 1$. The LSP then readily follows for sub-systems $i \in c_1$.

For sub-systems $i \in c_2$, we have the upper-bound on the error norm expectation according to the expression (A.4), as follows:

$$\sum_{i \in c_2} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i} | e^i_k\right] \leq \sum_{i \in c_2} \sum_{r = r'_i}^{N} \|A_i^{N-r}\|^2_2 \, \mathrm{tr}(Q^i W_i).$$

Hence, the condition (6.15) becomes satisfied considering $\bar{\xi}' > \sum_{i \in c_2} \sum_{r=r_i'}^{N} \|A_i^{N-r}\|_2^2 \operatorname{tr}(Q^i W_i)$, and $\bar{\xi} = \frac{\sum_{c_2} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 | e_k^i\right]}{\bar{\xi}'} < 1$.

For sub-systems belonging to the sub-case $l_1^{c_3}$, we have the following upper-bound from (A.5):

$$\sum_{i \in l_1^{c_3}} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 | e_k^i\right] \leq \sum_{i \in l_1^{c_3}} \left[\|A_i^{N-r_i'}\|_2^2 \lambda_i + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)\right].$$

The necessary condition (6.15), ensuring the LSP for $i \in l_1^{c_3}$, is therefore satisfied by choosing $\bar{\xi}' > \sum_{i \in l_1^{c_3}} \left[\|A_i^{N-r_i'}\|_2^2 \lambda_i + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)\right]$, and $\bar{\xi} = \frac{\sum_{l_1^{c_3}} \mathsf{E}\left[\|e_{k+N}^i\|_{Q^i}^2 | e_k^i\right]}{\bar{\xi}'} < 1$.

The sub-systems $j \in l_2^{c_3}$ have always been candidates for channel access, i.e. $j \in \mathscr{G}_{[k,k+N-1]}$. Hence, $\|e_{k'}^j\|_{Q^j}^2 > \lambda_j$ for all $k' \in [k, k+N-1]$. Hence, from (6.7), we conclude that

$$\begin{aligned}\sum_{j \in l_2^{c_3}} \mathsf{E}\left[\|e_{k+N}^j\|_{Q^j}^2 | e_k^j\right] &= \sum_{j \in l_2^{c_3}} \mathsf{E}\left[\|(1-\theta_{k+N}^j) A_j e_{k+N-1}^j + w_{k+N-1}^j\|_{Q^j}^2 | e_k^j\right] \\ &\leq \sum_{j \in l_2^{c_3}} \|A_j\|_2^2 \mathsf{E}\left[\|A_j e_{k+N-1}^j\|_{Q^j}^2 | e_k^j\right] + \operatorname{tr}(Q^j W_j). \end{aligned} \tag{6.19}$$

Expression (6.19) is not uniformly bounded since the term e_{k+N-1}^j in (6.19) is not upper bounded according to (6.16). However, as discussed before, we can calculate the probability that a sub-system ends up in the sub-case $l_2^{c_3}$ for the entire interval according to the decentralized scheduling policy (6.9). To calculate $\mathsf{P}_{l_2^{c_3}}$ we need to consider the possibility of packet collisions. As it is already explained, if a collision is acknowledged, then the channel will not be awarded to any sub-system at that specific time-step. Collisions may happen at all time-steps, therefore there is a non-zero probability that all the scheduled data packets collide. What follows is simple; all sub-systems operate in open-loop at all time steps, since no transmission would occur. Within the sub-case $l_2^{c_3}$, we investigate two collision scenarios over the interval of interest, i.e. $[k, k+N]$:

1. there has been at least one successful transmission over the interval $[k, k+N]$,

2. there has been no successful transmission over the entire interval $[k, k+N]$.

Starting with the first scenario, we assume that whenever a collision is detected and consequently all the sub-systems are expelled from having access to the channel, a virtual control loop has successfully transmitted. This means that at the time-step that the collision is detected, N real sub-systems and one virtual one share the communication channel while channel is awarded to the virtual sub-system. The virtual loops are assumed to have the same discrete LTI dynamics as described in (2.1). Since we need to have at least one successful transmission, the worst case situation entails that the channel may experience

$m \le N-1$ collisions over the interval $[k, k+N]$. Hence, at time-step $k+N$ we have N real and m virtual sub-systems competing for channel access, where all virtual ones have already transmitted exactly once. Since, we have $N+m$ sub-systems, we need to extend our interval from $[k, k+N]$ to $[k, k_m]$, where $k_m := k+N+m$. Introduction of the virtual loops is merely a justification for extending the time interval and plays no more role in stability analysis. If a sub-system $j \in l_2^{c_3}$ successfully transmits at time-step k_m, then its error norm becomes bounded in expectation. Otherwise, if j never successfully transmits over the interval $[k, k_m]$, then there exists another sub-system, say $i \in \{1, \ldots, N+m\}$, which transmits at time-step k_m while i has already transmitted successfully at least once. Let $k+\bar{r}$ denote the most recent time-step at which $\theta^i_{k+\bar{r}} = 1$ for $\bar{r} \le N+m-1$. Based on the decentralized scheduling law (6.9), the probability that sub-system i re-transmits at the final time-step k_m, in the presence of the sub-system $j \in l_2^{c_3}$ can be computed as follows:

$$\begin{aligned}
&\mathsf{P}[\theta^i_{k_m} = 1 | e^i_k, \theta^i_{k+\bar{r}} = 1, \theta^j_{k'} = 0, \quad \forall k' \in [k, k_m]] \\
&\quad = \mathsf{P}[\nu^j_{k_m-1} > \nu^i_{k_m-1} | e^i_k, \theta^i_{k+\bar{r}} = 1, \theta^j_{k'} = 0, \quad \forall k' \in [k, k_m]] \\
&\quad \le \frac{\mathsf{E}[\nu^j_{k_m-1} | e^i_k, \theta^j_{k'} = 0, \quad \forall k' \in [k, k_m]]}{s^i_{k_m-1}\tau},
\end{aligned}$$

where, the last upper-bound follows from the *Markov's inequality* considering that sub-system i is chosen the positive constant waiting time $\nu^i_{k_m-1} = s^i_{k_m-1}\tau$, with $s^i_{k_m-1} \in \{1, \ldots, h-1\}$. From the *law of iterated expectation*, it follows that

$$\begin{aligned}
\frac{\mathsf{E}[\nu^j_{k_m-1} | \theta^j_{k'} = 0 \quad \forall k' \in [k, k_m]]}{s^i_{k_m-1}\tau} &= \frac{\mathsf{E}[\mathsf{E}[\nu^j_{k_m-1} | e^j_{k_m-1}] | \theta^j_{k'} = 0 \quad \forall k' \in [k, k_m]]}{s^i_{k_m-1}\tau} \\
&= \frac{\alpha^j_{k_m-1}}{s^i_{k_m-1}\tau \|e^j_{k_m-1}\|^j_{Q^j}},
\end{aligned} \tag{6.20}$$

where the last equality follows from the assumption (6.5) on the mean of the local probability mass functions. The expression (6.20) confirms that having large error values corresponding to sub-systems $j \in l_2^{c_3}$, with no prior transmission over the entire interval $[k, k_m]$, reduces the probability of re-transmission of a sub-system $i \notin l_2^{c_3}$ with at least one prior successful transmission.

Having the time interval extended to $[k, k_m]$, and furthermore considering the expression (6.19) for the error expectation of a sub-system $j \in l_2^{c_3}$, it follows from (6.17) that

$$\sum_{j\in l_2^{c_3}} \mathsf{P}_{l_2^{c_3}}\, \mathsf{E}\Big[\|e^j_{k_m}\|^2_{Q^j}|e^j_k\Big] = \sum_{j\in l_2^{c_3}} \mathsf{P}[\nu^j_{k_m-1} > \nu^i_{k_m-1}|\theta^i_{k+\bar{r}}=1]\mathsf{E}\Big[\|e^j_{k_m}\|^2_{Q^j}|e^j_k\Big]$$
$$\leq \sum_{j\in l_2^{c_3}} \frac{\alpha^j_{k_m-1}\|A_j\|_2^2\|e^j_{k_m-1}\|^2_{Q^j}}{s^i_{k_m-1}\tau\|e^j_{k_m-1}\|^2_{Q^j}} + \frac{\alpha^j_{k_m-1}\,\mathrm{tr}(Q^jW_j)}{s^i_{k_m-1}\tau\|e^j_{k_m-1}\|^2_{Q^j}}$$
$$\leq \sum_{j\in l_2^{c_3}} \frac{\alpha^j_{k_m-1}\|A_j\|_2^2}{s^i_{k_m-1}\tau} + \frac{\alpha^j_{k_m-1}\,\mathrm{tr}(Q^jW_j)}{\lambda_j s^i_{k_m-1}\tau}, \tag{6.21}$$

where the last inequality follows from (6.16) ensuring that $\|e^j_{k_m-1}\|^2_{Q^j} > \lambda_j$ for every sub-system $j \in l_2^{c_3}$.
Since (6.21) is a uniform upper-bound independent of initial values e^j_k, LSP condition (6.15) holds by selecting $\bar{\xi} = \frac{\sum_{j\in l_2^{c_3}} \mathsf{E}\left[\|e^j_{k_m}\|^2_{Q^j}|e^j_k\right]}{\bar{\xi}'} < 1$ and $\bar{\xi}' > \sum_{j\in l_2^{c_3}} \frac{\alpha^j_{k_m-1}\|A_j\|_2^2}{s^i_{k_m-1}\tau} + \frac{\alpha^j_{k_m-1}\,\mathrm{tr}(Q^jW_j)}{\lambda_j s^i_{k_m-1}\tau}$, over the interval $[k, k_m]$.
Expression (6.21) can be made small by appropriately tuning λ_j's, $\alpha^j_{k_m-1}$'s, and Q^j's. However, it should be noted that there exist always trade-offs, e.g. increasing the threshold λ_j, on the one hand decreases (6.21), but on the other hand keeps the sub-system j out of channel access competition for higher error values. The same can be said about $\alpha^j_{k_m-1}$ as decreasing it leads to tighter bounds in (6.21), however, it also cause higher chances of collisions as the mean of the distributions concentrate on smaller areas of multiple micro time slots. Therefore, tuning the scheduling parameters should be carefully performed.
It is shown earlier that condition (6.15) holds over time interval $[k, k+N]$ for the cases c_1, c_2, and sub-case $l_1^{c_3}$, which implies that they stay bounded in expectation over longer finite horizons, i.e. $[k, k_m]$. Thus, rewriting (6.18) over the extended interval $[k, k_m]$ yields,

$$\sum_{i=1}^{N} \mathsf{E}\Big[\|e^i_{k+N}\|^i_{Q^i}\Big] \leq \sum_{i\in c_1}\|A_i\|_2^2\lambda_i + \mathrm{tr}(Q^iW_i) + \sum_{i\in c_2}\sum_{r=r'_i}^{N}\|A_i^{N-r}\|_2^2\,\mathrm{tr}(Q^iW_i)$$
$$+\sum_{i\in l_1^{c_3}}\left[\|A_i^{N-r'_i}\|_2^2\lambda_i + \sum_{r=r'_i}^{N-1}\|A_i^{N-r-1}\|_2^2\,\mathrm{tr}(Q^iW_i)\right] + \alpha^j_{k_m-1}\sum_{j\in l_2^{c_3}}\frac{\|A_j\|_2^2}{s^i_{k_m-1}\tau} + \frac{\mathrm{tr}(Q^jW_j)}{\lambda_j s^i_{k_m-1}\tau}.$$

Therefore, assuming that at least one successful transmission happens over $[k, k+N]$, the above expression ensures that the error Markov chain e_k satisfies (6.15) over the interval $[k, k_m]$, which in turn affirms that the overall NCS possesses LSP.

The second collision scenario prevents us to employ the same procedure by computing the probability $\mathsf{P}_{l_2^{c_3}}$ to ensure (6.15) for sub-case $l_2^{c_3}$. The reason is that within this scenario no transmission happens over $[k, k+N]$ due to successive collisions. Consequently, all sub-systems remain open-loop over the entire interval which leads to instability of the overall NCS state in Lyapunov mean-square stability sense.
To infer (6.15) for the scenario of having successive collisions, we should calculate the probability that at least two sub-systems select identical waiting times at every time-step. Assume

that a sub-system i has selected $\nu^i_{k'} = s^i_{k'}\tau$ at some time-step k', with $s^i_{k'} \in \{1,\dots,h-1\}$. To calculate the probability of a collision, the probability that a sub-system j selects exactly the identical waiting time $\nu^j_{k'} = s^i_{k'}\tau$ should be calculated. We know that

$$\mathsf{E}[\nu^j_{k'}] = \sum_{m=1}^{h-1} m\tau \ \mathsf{P}(\nu^j_{k'} = m\tau).$$

Therefore, the probability that $\nu^j_{k'} = s^i_{k'}\tau$, can be computed as

$$\begin{aligned}\mathsf{P}(\nu^j_{k'} = s^i_{k'}\tau) &= \frac{1}{s^i_{k'}\tau}\left[\mathsf{E}[\nu^j_{k'}] - \sum_{m=1, m\neq s^i_{k'}}^{h-1} m\tau \ \mathsf{P}(\nu^j_{k'} = m\tau)\right] \\ &< \frac{1}{s^i_{k'}\tau}\left[\frac{\alpha^j_{k'}}{\lambda_j} - \sum_{m=1, m\neq s^i_{k'}}^{h-1} m\tau \ \mathsf{P}(\nu^j_{k'} = m\tau)\right] \leq \frac{\alpha^j_{k'}}{s^i_{k'}\tau\lambda_j}.\end{aligned}$$

The obtained probability above should be extended for every permutations of sub-systems i and j which may collide at every time-step $k' \in [k, k+N]$. Therefore, we can find the upper-bound for the probability of occurring successive collisions over the entire interval $[k, k+N]$ as follows:

$$\mathsf{P}\left[\sum_{i=1}^{N} \theta^i_{k'} = 0, \forall k' \in [k, k+N]\right] \leq \prod_{k'=k}^{k+N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\alpha^j_{k'}}{s^i_{k'}\tau\lambda_j}. \tag{6.22}$$

From the expression (3.9), it follows that if no sub-system transmits at all, we have

$$\sum_{i=1}^{N} \|e^i_{k+N}\|^2_{Q^i} = \sum_{i=1}^{N} \|A_i^N e^i_k + \sum_{r=0}^{N-1} A_i^{N-r-1} w^i_{k+r}\|^2_{Q^i}.$$

Therefore, to infer (6.15), we choose $\bar{\xi}' = \sum_{i=1}^{N} \|A_i^N e^i_k + \sum_{r=0}^{N-1} A_i^{N-r-1} w^i_{k+r}\|^2_{Q^i} > 0$. Then

$$\sup_{e_k} \mathsf{P}\left[\sum_{i=1}^{N} \|e^i_{k+N}\|^2_{Q^i} \geq \bar{\xi}'\right] < \prod_{k'=k}^{k+N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\alpha^j_{k'}}{s^i_{k'}\tau\lambda_j}, \tag{6.23}$$

for an arbitrary $\rho(\bar{\xi}', \bar{\xi})$, and hence LSP of the overall NCS is readily obtained according to (6.15). □

Remark 6.2. One can infer from (6.23) that, by having extremely large thresholds λ_i, very small values of $\alpha^j_{k'}$ or infinite number of communication samples h, the probability of having consecutive collisions converges to zero. This is expected as by enlarging the error thresholds, we are keeping more sub-systems out of the channel access competition which leads to less congestion and consequently less collision. In addition, as $h \to \infty$, the chance of collisions converges to zero because the chance that at least two sub-systems choose identical waiting times becomes infinitesimal. We also discussed that decreasing $\alpha^j_{k'}$ leads to decrease in the successive collision rate. However, setting the thresholds extremely large causes that the corresponding sub-systems do not practically attempt for channel access, as according to (6.2) the error values need to pass very large values to be eligible for transmission. This increases the aggregate error variance. In addition, $h \to \infty$ is a non-realistic assumption considering the current communication technologies. Therefore, these trade-offs should always be taken into account. Note that, excluding the second collision scenario, the stronger stability notion, in terms of Lyapunov mean square stability (LMSS), will be obtained [91].

Remark 6.3. As the number of sub-systems occupying the communication channel increases, the probability of collisions grows. Nevertheless, we still can show LSP as long as the ratio h/N is larger than zero, i.e. N is finite. It is to be expected , however, that the performance of the overall networked system decreases as the ratio h/N decreases.

Remark 6.4. A more efficient way to treat the collisions is to adopt the persistent transmission policy. Assume two sub-systems choose identical waiting times $\nu^i_{k'} = \nu^j_{k'} = k'h + \bar{\tau}$, leading to a collision. A re-transmission can be considered within the interval $[k'h + \bar{\tau}, k'h + \tau_h]$ repeating the same procedure of selecting new waiting times. This approach is more efficient especially for large scale NCSs.

6.3 Numerical Evaluations

In this section, we numerically evaluate stability margins for an NCS composed of multiple heterogeneous LTI control loops which are sharing a band-limited communication network under our proposed decentralized scheduling mechanism introduced in (6.2) and (6.9). Moreover, we show performance efficiency of the decentralized scheduler when it is compared with conventional approaches such as TDMA and idealized CSMA protocols. Furthermore, we compare our results with those obtained from the bi-character centralized and pure probabilistic scheduling policies introduced in [99] and [94], respectively.

Similar to the numerical sections in the previous chapters, we consider an NCS with N scalar sub-systems divided into two heterogeneous classes; $\frac{N}{2}$ homogeneous control loops with unstable and $\frac{N}{2}$ homogeneous control loops with stable processes. The system parameters are as already introduced, i.e. $A_1 = 1.25$, $B_1 = 1$ for the unstable class, and $A_2 = 0.75$, $B_2 = 1$ for the stable class. The initial state is chosen as $x^1_0 = x^2_0 = 0$, and the noise sequence is given by the i.i.d. process $w^i_k \sim \mathcal{N}(0, 1)$. To stabilize the sub-systems, we similarly choose deadbeat control laws $L_i = A_i$, and we select $Q^i = I$ for both classes. In addition, we select the design parameter λ_i's according to the Table (6.1).

Table 6.1: Selected error thresholds and the number of collisions.

Number of plants (N)	2	4	6	8	10
Error threshold (λ)	0.25	1.45	2.80	4.30	6.00
Collisions in 2×10^5 samples	252	972	2094	3535	5009
Collisions (%)	0.126	0.486	1.047	1.767	2.504

Figure 6.5 provides the comparisons in terms of aggregate error variance between our proposed decentralized bi-character policy and the related protocols for NCSs with different

number of sub-systems $N \in \{2, 4, 6, 8, 10\}$ subject to the constraint (6.4). This ensures that for $N > 2$, at least one unstable sub-system operates in open-loop at each time-step. As for the time-triggered TDMA approach, we consider periodic transmissions for each control loop with periods of exactly N time-steps. Idealized CSMA operates statically such that the chance of transmission is $\frac{1}{N}$ for each sub-system at each time-step. The averages are calculated via Monte Carlo simulations over a horizon of 2×10^5 samples. For comparison we also consider the lower bound by relaxing the hard capacity constraint (6.4) to a soft average transmission rate [33]. For simplicity, we calculate the error variances by considering equal local error thresholds λ_i's for all sub-systems in an NCS setup, according to Table 6.1. The increase of λ as N increases follows from assuming fixed channel capacity. The number of micro slots within one macro slot is assumed to be $h = 150$. The local parameters α_k^i's are also assumed to be constant for all time-steps and they are $\alpha_k^i = 150$ for all $i \in \{1, \ldots, N\}$ with $N \in \{2, 4, 6, 8, 10\}$. In addition, the local waiting times are chosen randomly from the Poisson distributions for each sub-system at every time-step, where the mean of the Poisson distributions are time varying according to (6.5). The number of detected collisions, under the employment of the proposed decentralized scheduling law is shown in Table 6.1. It can be seen that, although collisions inevitably happen, the collision rate is rather low which indicates that the decentralized policy can be tuned properly to avoid high collision rates.

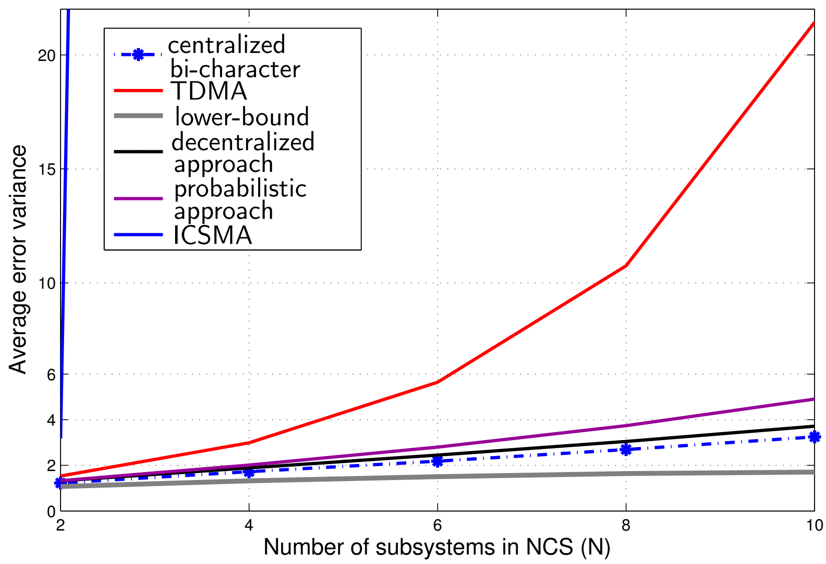

Figure 6.5: Comparison of the average error variance vs. the number of control loops for different scheduling policies.

The idealized CSMA protocol (blue curve in Figure 6.5), as also demonstrated in previous sections, results in an acceptable performance only for $N = 2$, while the bi-character approaches, centralized and decentralized, outperform TDMA (red curve in Figure 6.5). The performance gap increases with increasing number of sub-systems N. The bi-character centralized scheduler, drawn as the dotted blue curve in Figure 6.5, slightly outperforms the bi-character decentralized law (black curve in Figure 6.5), according to Figure 6.5. This observation is expected though due to the absence of packet collisions within centralized approaches. Comparing the performance of the decentralized policy with that of the pure probabilistic scheduler shows that the performance can become enhanced under the former protocol. This is however not a firm conclusion, as we usually expect to get better performance out of centralized designs. The reason for having an improved performance in Figure 6.5 from the decentralized approach is that the pure probabilistic approach is probably not the optimal centralized way of arbitrating the communication channel.

6.4 Scheduling Design with Noisy Measurements

So far in this chapter, it is assumed that perfect state information is accessible for the event-triggers. In many real situations however, only noisy measurements correlated to state variables are accessible, therefore realistic design of control and scheduling mechanisms for networked control systems (NCS) often entails dealing with noise-deteriorated sensor measurements. In this section, we extend our proposed decentralized event-triggered medium access control (MAC) for a network of multiple control loops with noisy sensor measurements. In contrast to the previous sections of this chapter, the event triggers are considered as the discrepancy between the sensor measurements and the estimated state, thus no need to have access to the perfect state information. Local noise-prone sensor stations measure system states and use the shared medium in order to transmit those sensory data to their respective control units. We first express the NCS model and the state estimation problem.

Each control loop $i \in \{1,\dots,N\}$ in the NCS is composed of an LTI stochastic process $\mathscr{P}_i$, sensor $\mathscr{S}_i$, and a control unit which itself includes a Kalman filter $\mathscr{F}_i$ and a feedback controller $\mathscr{C}_i$. The controlled sub-system i can be described by the following discrete-time stochastic difference equation

$$\begin{aligned} x^i_{k+1} &= A_i x^i_k + B_i u^i_k + w^i_k, \\ y^i_k &= C_i x^i_k + v^i_k, \end{aligned} \tag{6.24}$$

where, $x^i_k \in \mathbb{R}^{n_i}$, $u^i_k \in \mathbb{R}^{m_i}$, $A_i \in \mathbb{R}^{n_i \times n_i}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$ describe the i^{th} sub-system state vector, control input, system matrix, and input matrix, respectively. For the ease of derivations, we assume the output matrix is square, i.e. $C_i \in \mathbb{R}^{n_i \times n_i}$, and C_i^{-1} exists. In addition, the pair (A_i, B_i) is assumed to be controllable, and (A_i, C_i) to be observable. System noise $w^i \in \mathbb{R}^{n_i}$, and measurement noise $v^i \in \mathbb{R}^{q_i}$ are i.i.d. random sequences with realization $w^i_k \sim \mathscr{N}(0, W_i)$ and $v^i_k \sim \mathscr{N}(0, V_i)$, at each time-step k. Due to capacity limitation of the communication channel, scheduling units are integrated in sensor stations to determine whether a transmission is feasible. The scheduling decision at each time-step k is taken locally, and is denoted by the binary variable $\delta^i_k \in \{0,1\}$, as

$$\delta^i_k = \begin{cases} 1, & y^i_k \text{ sent through the channel} \\ 0, & y^i_k \text{ blocked.} \end{cases}$$

We assume a loss-less network, i.e. there is no externally caused packet loss. However, packets from different control loops may collide during the medium access. Acknowledgments of successful transmissions are broadcasted via an error-free reserved link back to the sub-systems. We assume the link to be of the broadcast type so every station will be informed if a collision takes place. We assume that if a collision is detected, the involved sub-systems are discarded with no chance of re-transmission. At every time-step k, the acknowledgement signal $\gamma_k^i \in \{0,1\}$, received by every sub-system i, is described as follows:

$$\gamma_k^i = \begin{cases} 1, & y_k^i \text{ successfully received,} \\ 0, & y_k^i \text{ collided.} \end{cases}$$

For the expressions (6.28) and (6.29) to represent the collisions, δ_{k+1}^i should be replaced by θ_{k+1}^i, where $\theta_{k+1}^i = \delta_{k+1}^i \gamma_{k+1}^i$.

The system state values are estimated by the local Kalman filters at the control units, and the control inputs are then computed by the local feedback controllers. Thus, the signal received by the control unit $\mathscr{C}_i$ at a time-step k, i.e. z_k^i, can be expressed based on new event-triggers as

$$z_k^i = \begin{cases} y_k^i, & \text{if } \delta_k^i \gamma_k^i = 1 \\ \varnothing, & \text{if } \delta_k^i \gamma_k^i = 0. \end{cases}$$

We define $\mathscr{I}_k^i = \{\theta_0^i, y_0^i, \dots, \theta_k^i, y_k^i\}$ as the history of received signals at the control side $\mathscr{C}_i$, where $\theta_k^i = \delta_k^i \gamma_k^i$. Then, the control input u_k^i is computed by the following state-feedback law described by a measurable and causal mapping of past observations as

$$u_k^i = -L_i \,\mathsf{E}[x_k^i | \mathscr{I}_k^i] = -L_i \hat{x}_k^i, \tag{6.25}$$

where, $L_i \in \mathbb{R}^{m_i \times n_i}$ is a stabilizing feedback gain. Depending on whether new sensory measurements are arriving, we compute the state estimate $\hat{x}_k^i$ via the Kalman filter [109], as

$$\hat{x}_k^i = \hat{x}_k^{i-} + \theta_k^i K_k^i (y_k^i - C_i \hat{x}_k^{i-}) \tag{6.26}$$

with the *a priori* state estimate $\hat{x}_k^{i-}$, and estimate error covariance $P_k^{i-} = \mathsf{E}\left[(x_k^i - \hat{x}_k^{i-})(x_k^i - \hat{x}_k^{i-})^\mathsf{T}\right]$ given as follows

$$\begin{aligned} \hat{x}_{k+1}^{i-} &= A_i \hat{x}_k^i + B_i u_k^i, \\ P_{k+1}^{i-} &= A_i P_k^i A_i^\mathsf{T} + W_i, \end{aligned}$$

and the following optimal Kalman gain K_k^i and the *a posteriori* estimate error covariance $P_k^i = \mathsf{E}\left[(x_k^i - \hat{x}_k^i)(x_k^i - \hat{x}_k^i)^\mathsf{T}\right]$:

$$\begin{aligned} K_k^i &= P_k^{i-} C_i^\mathsf{T} (C_i P_k^{i-} C_i^\mathsf{T} + V_i)^{-1}, \\ P_k^i &= P_k^{i-} - K_k^i (C_i P_k^{i-} C_i^\mathsf{T} + V_i) K_k^{i^\mathsf{T}}, \end{aligned}$$

with initial conditions $\hat{x}_0^i = 0$ and $P_0^i = 0$. It follows from the expressions (6.24)-(6.26) that $\theta_k^i = 0$ leads to a model-based estimate of x_k^i, i.e. $\hat{x}_k^i = \hat{x}_k^{i-} = (A_i - B_i L_i)\,\hat{x}_{k-1}^i$. The estimate is well-behaved as the gain L_i is stabilizing, and (A_i, B_i) is stabilizable.

We introduce the network-induced error e_k^i for each sub-system $i \in \{1, \ldots, N\}$, at every time-step k as the difference between measurements and state estimates at the controller,

$$e_k^i := y_k^i - \hat{y}_k^i, \tag{6.27}$$

where, $\hat{y}_k^i = C_i \hat{x}_k^i$. Recalling (6.24)-(6.27), it is straightforward to conclude the network-induced error state e_k^i evolves according to the following stochastic difference equation

$$e_{k+1}^i = \left(I_{n_i} - \theta_{k+1}^i C_i K_{k+1}^i\right)\left[C_i A_i C_i^{-1}(e_k^i - v_k^i) + C_i w_k^i + v_{k+1}^i\right]. \tag{6.28}$$

The system state dynamics can similarly be obtained as follows

$$x_{k+1}^i = (A_i - B_i L_i) x_k^i + B_i L_i C_i^{-1}\left(e_k^i - v_k^i\right) + w_k^i. \tag{6.29}$$

Immediate conclusion from (6.28) and (6.29) is that evolution of the network-induced error state e_k^i is independent of the system state x_k^i. This asserts the separation property holds between the control law and scheduling law synthesis. Thus, we take an emulation-based scenario and choose a stabilizing controller *a priori*. The control inputs are computed according to this law at triggering times and kept constant in between triggering times. Stabilizability of pairs (A_i, B_i) ensures the closed-loop matrix $(A_i - B_i L_i)$ is stable. Then it follows from (6.29) that the aggregate state of sub-system i, i.e. $[x_k^{i^\mathsf{T}}\ e_k^{i^\mathsf{T}}]^\mathsf{T}$ is stable if e_k^i is convergent.

Remark 6.5. In this paper we assume that output matrices are invertible. Relaxing this assumption urges the inclusion of additional observers which results in an additional observation error. Since the pairs (A_i, C_i) are observable for all i's, stabilizing observer gains exist such that stability of the closed-loop system, including the observer state, is ensured in the absence of the capacity constraint. It is straightforward to show that, the evolution of network-induced error defined in (6.27) remains independent from the observer state and hence the scheduling process remains unaffected. The relaxation to arbitrary output matrices C_i, i.e. true output feedback control, is not considered here due to ease of derivations. The essential techniques to show stability, however, extend to this case as well.

We consider the same channel modeling as introduced in Section 6.1, i.e. the sampling interval of the communication network (micro slots), is much finer than that of the operational time scale of control systems (macro slots). Assuming that T is the sampling period of the control systems, we assume that between two consecutive macro time slots $k \to k+1$, the micro slots are distributed as $\{kT, kT + \tau, \ldots, kT + (h-1)\tau, kT + h\tau\}$, where τ is the temporal duration of each micro slot and $h \gg 1$ denotes the number of micro slots within one macro slot, hence $T = h\tau$. The sampling delays corresponding to transmission time can be ignored assuming high enough channel throughput. Having a decentralized architecture, the i^{th} control and scheduling units are assumed to be provided with local information A_i, B_i, C_i, W_i, V_i, L_i, K^i_k, λ_i, Q^i, I^i_k and the distribution of x^i_0. With no need of accessing true state value $x^i_{k'}$, every scheduling unit only receives the latest local error state $e^i_{k'}$ defined in (6.27). The measurement vector $y^i_{k'+1}$ is forwarded for transmission at the subsequent time $k'+1$ only if the square weighted norm of $e^i_{k'}$ exceeds the local threshold λ_i, otherwise $y^i_{k'+1}$ will be discarded. In the other words,

$$\mathsf{P}[\delta^i_{k'+1} = 1 \big| e^i_{k'}] = 0 \quad \text{if} \quad \|e^i_{k'}\|^2_{Q^i} \leq \lambda_i. \tag{6.30}$$

The set of sensor stations cleared to transmit at time-step $k'+1$ is denoted by $\mathcal{G}_{k'+1}$, i.e. for each $i \in \{1,\ldots,N\}$, $i \in \mathcal{G}_{k'+1}$ if $\|e^i_{k'}\|^2_{Q^i} > \lambda_i$. Again, we assume that only one sensor station can transmit over every macro time-step, satisfying eq. (6.4). Assuming that transmitted data will be received instantly, a transmission can be commenced at any micro slot τ before reaching the end of the macro slot, i.e. $\{0,\tau,2\tau,\ldots,(h-1)\tau\}$. In order to determine the commence time, randomly selected waiting times $\nu^i_{k'} \in \{0,\tau,2\tau,\ldots,(h-1)\tau\}$ are chosen independently for each sub-system and at each time-step. As already discussed, they denote the time duration a sensor station $i \in \mathcal{G}_{k'+1}$ waits and then starts sensing the channel. Measurement vector $y^i_{k'+1}$ is transmitted only if the channel is sensed as idle. Otherwise, a back off time of the length $h\tau - \nu^i_{k'}$ is considered, which ensures that station i does not attempt a re-transmission over the macro slot $k' \to k'+1$. The waiting times are selected according to independent local discrete distributions associating with finite-variance concave probability mass functions (*pmf*), where the set of all possible outcomes is restricted to the micro slots between $k' \to k'+1$. Similar to the discussions in Section 6.1, the expected value of a local *pmf* is supposed to be error-dependent as follows:

$$\mathsf{E}[\nu^i_{k'}] = \frac{\alpha^i_{k'}}{\|e^i_{k'}\|^2_{Q^i}}, \tag{6.31}$$

where, $\alpha^i_{k'}$ is the scheduling parameter which is tuned appropriately to avoid high collision rates. The sensor station associated with the shortest waiting time transmits, and the other stations are discarded. Therefore, considering (6.4), the probability that sensor station $i \in \mathcal{G}_{k'+1}$ transmits at time-step $k'+1$ can be written as follows:

$$\mathsf{P}[\delta^i_{k'+1} = 1 \big| e^i_{k'}] = \mathsf{P}[\nu^i_{k'} < \nu^l_{k'}], \quad \forall l \in \mathcal{G}_{k'+1}, l \neq i. \tag{6.32}$$

Remark 6.6. Selecting the waiting times randomly, on the one hand reduces the collision rate by introducing the communication micro-frames as a collision avoidance mechanism. On the other hand, increasing the traffic corresponds to sub-systems with high error values leads to concentration of local *pmf*s across early micro slots, which increases the collision rate. To avoid so, parameter α^i_k can be properly adjusted to place the mean of the *pmf*s farther than the congested area. To tune α^i_k, broadcasted channel feedback, containing information about the latest traffic and collisions, can be used.

6.4.1 Stability Analysis

In this section, we study stability properties of the multi-loop networked control systems, subject to noise-prone sensor measurements, under the proposed event-based decentralized MAC scheme. The following analysis follows the same lines as already presented in Section 6.2. The major difference between the results in this sections and Section 6.2 is the change of the event triggers, which consequently results in different evolution dynamics for the system state and error state. Comparing eq. (6.7) and (6.28), the influence of having access to the noisy measurements, instead of having access to the true state values, can be seen. It requires more efforts to analyze stability with new definition of the event triggers, however this is a more applicable assumption, as in many situations assuming to have access to the perfect state values is not realistic.
As also discussed in Section 6.2, the system state x_k^i, defined in (6.29) remains stable considering the emulation-based strategy, only if the error state e_k^i in (6.28) is stable. We similarly take the concept of Lyapunov stability in probability (LSP) to show stochastic stability of the error state e_k^i. It should be noted that, the proposed MAC design in this section is a generalized approach compared to the presented MAC architecture in Section 6.1. It is an evident conclusion since the introduced event trigger in Section 6.1 is a special case of the event trigger considered in this section, i.e. by assuming C_i as identity, and no measurement noise. The results obtained in *Lemma* 6.1 can be extended for the new error and system state evolution, as follows:

Lemma 6.2. For a control loop with state vector $[x_k^{i^\mathsf{T}}\; e_k^{i^\mathsf{T}}]^\mathsf{T}$, where the error and systems states are described in (6.28) and (6.29), respectively, the LSP condition (6.10) is equivalently satisfied if exists $\xi_i' > 0$ and $0 \le \xi_i \le \varepsilon_i$ such that

$$\lim_{k\to\infty} \sup \mathsf{P}\left[e_k^{i^\mathsf{T}} e_k^i \ge \xi_i'\right] \le \xi_i.$$

Proof. It is shown that the system state x_k^i within each control loop i evolves according to (6.29), where the closed-loop matrix $(A_i - B_i L_i)$ is Hurwitz since stabilizing gains L_i's exist. In addition, we know that the error-dependent term $B_i L_i C_i^{-1}\left(e_k^i - v_k^i\right)$ is independent of x_k^i, according to (6.28). It follows from the *Markov's inequality* that for $\zeta_i' > 0$

$$\mathsf{P}\left[(e_k^i - v_k^i)^\mathsf{T}(e_k^i - v_k^i) \ge \zeta_i'\right] \le \frac{\mathsf{E}\left[(e_k^i - v_k^i)^\mathsf{T}(e_k^i - v_k^i)\right]}{\zeta_i'}$$
$$= \frac{\mathsf{E}\left[\|e_k^i - v_k^i\|_2^2\right]}{\zeta_i'} \le \frac{\mathsf{E}\left[\|e_k^i\|_2^2\right] + \mathrm{tr}(V_i) + 2\sqrt{\mathrm{tr}(V_i)\mathsf{E}\left[\|e_k^i\|_2^2\right]}}{\zeta_i'}$$

where the last inequality follows from $\mathsf{E}\left[\|e_k^i - v_k^i\|_2^2\right] = \mathsf{E}\left[\|e_k^i\|_2^2\right] + \mathsf{E}\left[\|v_k^i\|_2^2\right] - 2Cov(e_k^{i^\mathsf{T}}, v_k^{i^\mathsf{T}})$, and $Cov(e_k^{i^\mathsf{T}}, v_k^{i^\mathsf{T}}) \le \sqrt{Var(v_k^i)Var(e_k^i)}$. Together with the assumption that x_0^i is selected from a bounded moment distribution, and w_k^i is i.i.d. process according to $\mathcal{N}(0, W_i)$, it follows that the closed-loop dynamics $(A_i - B_i L_i)x_k^i + w_k^i$ is bounded in expectation, and consequently possesses LSP, according to the *Markov's inequality*. Thus, showing $\mathsf{E}\left[e_k^{i^\mathsf{T}} e_k^i\right] =$

$\mathsf{E}\big[\|e_k^i\|_2^2\big]$ is bounded, i.e. exists $\xi_i, \xi_i' > 0$ such that $\lim_{k\to\infty}\sup \mathsf{P}\big[e_k^{i^\mathsf{T}} e_k^i \geq \xi_i'\big] \leq \xi_i$, ensures $\lim_{k\to\infty}\sup \mathsf{P}\big[(e_k^i - v_k^i)^\mathsf{T}(e_k^i - v_k^i) \geq \zeta_i'\big] \leq \zeta_i$ with $\zeta_i = \frac{\mathsf{E}\big[\|e_k^i\|_2^2\big]+\mathrm{tr}(V_i)+2\sqrt{\mathrm{tr}(V_i)\,\mathsf{E}\big[\|e_k^i\|_2^2\big]}}{\zeta_i'}$. This implies boundedness of $\mathsf{E}\big[x_{k+1}^{i^\mathsf{T}} x_{k+1}^i\big]$ which assures $\varepsilon_i \geq \xi_i$ and $\varepsilon_i' > 0$ exist such that $\lim_{k\to\infty}\sup \mathsf{P}\big[x_{k+1}^{i^\mathsf{T}} x_{k+1}^i \geq \varepsilon_i'\big] \leq \varepsilon_i$, and LSP of the aggregate state $[x_k^{i^\mathsf{T}}\ e_k^{i^\mathsf{T}}]^\mathsf{T}$ follows. □

Note that the random variables e_k^i and v_k^i are statistically dependent, according to the expression (6.28), and hence the error state e_k^i is not inherently Markovian. This is in line with the fact that network-induced error in (6.28), with event-triggers (6.27), is seldomly characterized by Markov chains. However, Markovianity of a process state is not a necessity to achieve LSP, see *Definition* 6.1. We further show that the statistical dependence of e_k^i and v_k^i can be treated to show LSP of the aggregate state $[x_k^{i^\mathsf{T}}\ e_k^{i^\mathsf{T}}]^\mathsf{T}$ within each sub-system.

Remark 6.7. Although, Markovianity of a process state is not a necessary property to achieve LSP, it is quite beneficiary when studying the real-time system behavior via Lyapunov approaches. We briefly suggest here that the error state e_k^i and measurement noise process v_k^i can be grouped as one state to represent a newly generated Markovian stochastic process. However this approach should be followed carefully as the grouped state is not observable since the noise measurement is not observable. For detained information see [86].

Similar discussions as in Section 6.2 can be followed here again to conclude that the LSP condition is equivalent to the expression (6.15). Employing *Markov's inequality*, we can instead state the following:

$$\mathsf{P}\left[e_k^\mathsf{T} Q e_k \geq \bar{\xi}'\right] \leq \frac{\mathsf{E}\left[e_k^\mathsf{T} Q e_k\right]}{\bar{\xi}'} = \frac{\sum_{i=1}^N \mathsf{E}\left[\|e_k^i\|_{Q^i}^2\right]}{\bar{\xi}'}. \tag{6.33}$$

In addition, the illustrative example constructed in Section 6.2 to motivate the multi-step interval, is still valid here. Therefore, without modifying the illustrative example, and assuming constraint (6.4), we study stability for the even-based decentralized MAC architecture with the new event triggers, in terms of LSP by monitoring the error state evolution over an interval with length N.

Theorem 6.2. *Consider a multi-loop NCS consisting of N heterogeneous LTI stochastic subsystems modeled as in (6.24), sharing a communication channel subject to the constraint (6.4). Under the emulative control, estimation and scheduling laws given by (2.3), (6.26), (6.30), and (6.32), respectively, the overall NCS state $[x_k^T, e_k^T]^T$ is Lyapunov stable in probability, for any positive λ_i's and α_k^i's, and any positive definite Q^i's.*

Proof. See Appendix A.3. □

Remark 6.8. A comparison between stability notion for centralized scheduling discussed in Chapter 4, i.e. f-ergodicity, and the concept of Lyapunov stability in probability employed in this chapter clarifies that an f-ergodic process is also Lyapunov stable in probability. However, a stochastic process which is Lyapunov stable in probability is not necessarily f-ergodic.

Table 6.2: Decentralized and uniform bi-character scheduling

Number of plants (N)	2	4	6	8	10	12
decentralized bi-character						
Error threshold (λ)	0.01	0.10	0.40	0.70	0.88	0.98
aggregate error variance	1.08	1.50	1.96	2.50	3.17	4.20
collision rate (%)	0.78	0.89	1.12	1.35	1.58	1.88
uniform bi-character						
Error threshold (λ)	0.10	2.10	3.80	6.00	11.20	14.00
aggregate error variance	0.90	1.91	2.79	3.79	5.10	7.61
collision rate (%)	0.19	0.19	0.24	0.30	0.27	0.35

6.4.2 Performance Evaluations

In this section we evaluate performance of the proposed decentralized event-triggered MAC protocol in terms of the overall network-induced error variance and collision rate considering the new event-triggers. To show our approach efficacy, comparisons are made with the known protocols such as CSMA, TDMA, and the try-once-discard (TOD). Furthermore, we underscore the benefit of error dependency in selecting the waiting times on the collision rate and the resulting average error variance by providing a comparison with a uniform mechanism for selecting the waiting times.

We consider the same networked model as described in Section 6.3. So we quickly summarize that we consider NCSs with two heterogeneous classes $\{cl_1, cl_2\}$ of scalar sub-systems with state-space representations $(A_{cl_1}, B_{cl_1}, C_{cl_1}, D_{cl_1}) = (1.25, 1, 1, 0)$, and $(A_{cl_2}, B_{cl_2}, C_{cl_2}, D_{cl_2}) = (0.75, 1, 1, 0)$. Each sub-system is controlled via a dead beat control law, and we select $Q^i = I$ for all sub-systems. The initial values are set as $x_0^{cl_1} = x_0^{cl_2} = 0$. The process noise and the measurement noise are assumed to be Gaussian with $w_k^i \sim \mathcal{N}(0, 1)$ and $v_k^i \sim \mathcal{N}(0, 0.1)$, respectively. We conduct a Monte Carlo simulation with 2×10^5 samples and assume a channel capacity of $c = 1$. The simulations are performed for a varying number of connected sub-systems $N \in \{2, 4, 6, 8, 10, 12\}$, with $\frac{N}{2}$ sub-systems belong to each class. Moreover, in every simulated NCS setting the same error thresholds λ are applied for all sub-systems. The number of micro slots is set to $h = 150$ and the waiting times in every macro slot are chosen from Poisson distributions with error-dependent mean (6.5). To tune the scaling factor $\alpha_{k'}^i$, we use the feedback signal broadcasted from the channel, according to the following empirical law

$$\alpha_{k'}^i = q_t n_{t,k'-1} + q_c n_{c,k'-1}, \tag{6.34}$$

where we consider the traffic $n_{t,k-1}$, i.e. the number of systems trying to transmit, and the collision rate $n_{c,k-1}$ from the previous time step. The weights q_t and q_c scale the importance of the respective channel information.

Table 6.2 shows the resulting aggregate error variance of our proposed decentralized MAC for different number of sub-systems N, with the error thresholds λ and the scaling weights

$q_t = 60$ and $q_c = 34$. Furthermore, Table 6.2 provides the collision rates, which are shown to be low due to a proper parameterization of the Poisson distributions. The second part of the table shows the results of a uniform distributed realization, i.e. sub-systems are deterministically excluded from channel access if their error is underneath the threshold λ and otherwise waiting times are chosen based on a uniform distribution. As expected, this leads to less collision rates since the waiting times are selected by identical probabilities across the entire macro slot, but at the same time to increased error variances since the channel access is not prioritized.

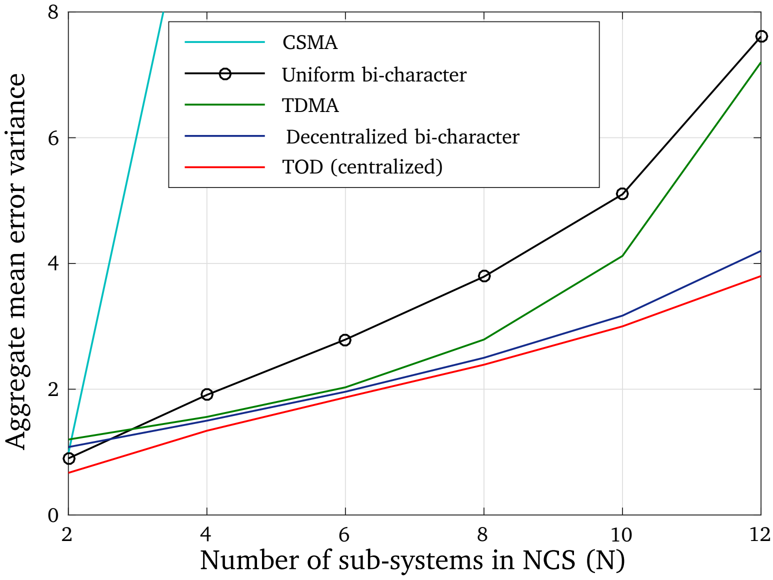

Figure 6.6: Comparison of the aggregate error variances for a varying number of subsystems and different MAC protocol realizations

The comparison with other protocols is illustrated in Figure 6.6. The TOD approach, which is a centralized scheduling realization expectedly leads to the best performance and can be considered as the lower bound. For the CSMA realization the constant channel access probabilities are assigned to each sub-system depending on the system matrices, i.e. $\frac{\|A_i\|_2^2}{\sum_{j=1}^{N}\|A_j\|_2^2}$. This approach illustrates an acceptable performance only for $N=2$. The performance of our decentralized MAC protocol, even for higher numbers of sub-systems, follows that of the centralized TOD closely. Moreover, it outperforms the TDMA, where a constant time-based access order for the connected systems is assumed where only unstable plants are part of the transmission schedule. Moreover, the uniform bi-character comparison shows a decrease in the collision rates, but error variances can not compete with the introduced approach for a high number of sub-systems.

Figure 6.7 shows the impact of the distribution scaling on the collision rate and the aggregate error variance. It shows the probability density of the waiting time over one macro slot for $N=8$. The red *pdf*, where mean of the scaling factor $\alpha_{k'}^{i}$ equals 77.5, the selected waiting times are concentrated at the beginning of the macro slot, and leads to a collision rate of 3.47%, and error variance of 2.71. The result can be improved by increasing the mean of $\alpha_{k'}^{i}$ to 163.00. The consequence is a better utilization of the whole channel sampling range in one time-step, which leads to less collisions (1.35%) and lower error variance of 2.50.

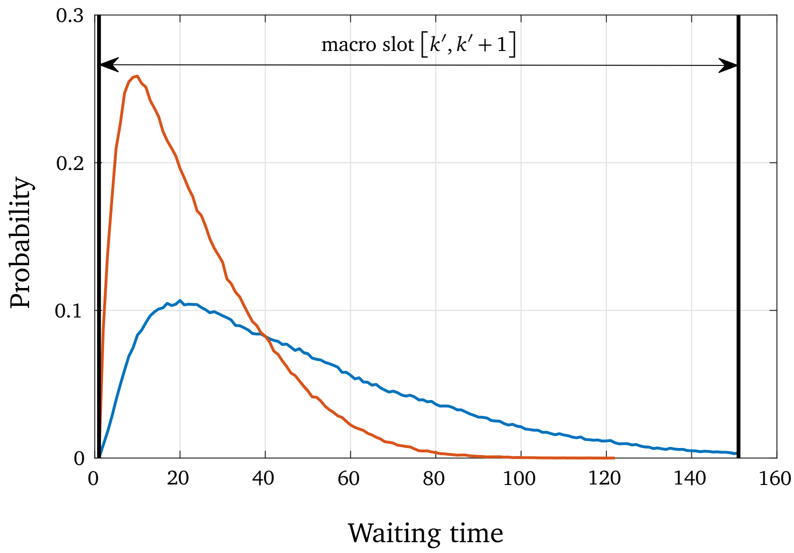

Figure 6.7: *pdf* of resulting waiting times with mean value $\alpha^i_{k'} = 77.51$ and $\alpha^i_{k'} = 163.00$, for NCS with $N = 8$ sub-systems during one macro slot.

6.5 Event-based Scheduling for NCSs with Multi-link Communication Network

So far in this chapter, we addressed decentralized event-based MAC architectures for single-hop channels, i.e. there is only one link for the transmitting station to send its data packet. In this section, we extend our results to multiple link scenario and we introduce a MAC design to arbitrate the contention-based channel access based on multi-channel slotted ALOHA protocol. We analyze the behavior of the multi-channel slotted ALOHA medium access, considering an event-based networked control system consisting of multiple LTI control sub-systems as the communication endpoints. First, a local threshold-based scheduler determines whether a sub-system is eligible for a transmission attempt. Afterwards, according to the multi-channel slotted ALOHA protocol, each eligible sub-system selects one of the multiple transmission channels randomly to send its own data packet. Stability of the resulting NCS over the multi-channel slotted ALOHA is discussed in terms of Lyapunov stability in probability (LSP). We evaluate the performance of the event-based scheduler, and further propose an improvement to it by incorporating adaptive thresholds. In the modified scheduler design, network and control systems are coupled via the knowledge of the network state, and each local scheduler adapts its threshold based on the available network resources. Numerically, we demonstrate that an adaptive choice of the transmission threshold is beneficiary compared to the non-adaptive static design.

6.5.1 Network modeling and structural analysis

We consider the similar networked model considered in Section 6.1, i.e. the sub-systems dynamics, control laws and estimators are described as in (2.1), (2.3), and (2.4). The local scheduler situated at each local control loop decides to access the medium at every time-step

k only if the following threshold inequality holds:

$$\|e_k^i\|_2 > \lambda_i, \tag{6.35}$$

where, λ_i is the local error threshold for sub-system i. Therefore, if (6.35) is satisfied at some time-step k, then the corresponding sub-system is eligible for transmission at the next time-step $k+1$. Otherwise, it is deterministically excluded from the channel access, i.e.

$$\mathsf{P}[\delta_{k+1}^i = 1 | e_k^i] = \begin{cases} 0, & \text{if } \|e_k^i\|_2 \leq \lambda_i \\ 1, & \text{otherwise.} \end{cases} \tag{6.36}$$

According to (6.36), all sub-systems with error norms greater than their corresponding local thresholds randomly select one of the channels to transmit. If there is no collision with other eligible sub-systems, data packets will be sent. Otherwise, in case at least two sub-systems select the same transmission link to transmit, no packet would go through. Note that the deployed scheduling policy (6.36) is not explicitly dependent on whether the transmission has been successful or it has collided, therefore, channel sensing of acknowledgments are not necessary for the policy's realization.

The communication network model is restricted to the Medium Access Layer (MAC) and is represented by a multi-channel slotted ALOHA protocol [110], see Figure 6.8. As the most common practical example, we can refer to LTE-based system and its Random Access Channel [111], while mappings to different single-channel wireless or even bus systems can also be imagined. In every time slot, we assume M non-overlapping transmission channels are available, where $M \geq 2$. The information about the available number of channels is assumed to be known to all sub-systems in the beginning of each time slot.

For the sake of simplicity, we assume that the communication time slots are equal in duration to the control sampling periods, and that all sub-systems' control periods are synchronized. Thus, in every control period we have M available transmission channels, meaning:

$$\sum_{i=1}^{N} \theta_k^i \leq M, \tag{6.37}$$

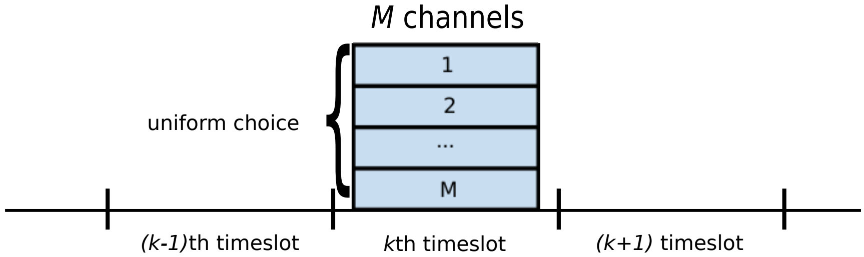

Figure 6.8: Multi-channel slotted ALOHA. One time slot is assumed equal to a control period of any sub-system. A channel can represent a frequency, code [1] or time domain transmission opportunity, depending on the communication technology in use.

where, $\theta_k^i = \delta_k^i \gamma_k^i$, and γ_k^i is the collision indicator and introduced in (6.6). According to the slotted ALOHA protocol, if a packet is scheduled for transmission, it will be sent through one of M channels, which is randomly chosen. We denote the set of sub-systems which are eligible for transmission at time-step $k+1$ as $\mathcal{G}_k$. Then, the probability of successful transmission for a given eligible sub-system time-step at time $k+1$ is calculated as:

$$\mathsf{P}\left[\gamma_{k+1}^i = 1 \middle| \|e_k^i\|_2 > \lambda_i\right] = \left(\frac{M-1}{M}\right)^{g_k}, \tag{6.38}$$

where g_k is the cardinality of the set $\mathcal{G}_k$.

The transmission threshold λ_i is directly affecting both the error of the corresponding sub-system, and the arrival rate of the transmission requests for network access. Since the network is modeled by slotted ALOHA mechanism, too high arrival rate of requests will result in a high collision rate and consequently degrades the performance of the overall networked system, significantly. Following this observation, our hypothesis is that adapting λ_i to network state, leads to the enhancement of control performance. This motivates us to further modify the introduced local event-based resource-aware scheduling design with an adaptive choice of the error thresholds. We illustrate numerically the efficacy of our proposed approach in terms of reducing the average networked-induced error variance, and show the superiority of the adaptive event-based scheduler compared to the scheduling design with non-adaptive thresholds.

6.5.2 Stability Analysis

In this section, we study stability of multiple-loop NCSs with shared multi-channel communication networks subject to the constraint (6.37), and the introduced threshold-based decentralized scheduling policy (6.36). Similar to the previous sections, we describe the overall network state at some time-step k by the aggregation of the system states x_k^i, and error states e_k^i from all sub-systems $i \in \{1, \ldots, N\}$, i.e. $\left[x_k^\mathsf{T}, e_k^\mathsf{T}\right]^\mathsf{T}$, where $x_k = \left[x_k^{1^\mathsf{T}}, \ldots, x_k^{N^\mathsf{T}}\right]^\mathsf{T}$ and $e_k = \left[e_k^{1^\mathsf{T}}, \ldots, e_k^{N^\mathsf{T}}\right]^\mathsf{T}$.

Following stability analysis follows the similar procedure as in Section 6.2. The major difference is the constraint on the channel capacity, and the MAC protocol, which is ALOHA-based in this section, and error-dependent prioritizing in Section 6.2.

We already discussed in Section 6.2 that the system state x_k^i, defined in (2.7) remains stable assuming the stabilizing control laws are pre-designed, only if the error state e_k^i in (6.7) is stable. We similarly take the concept of Lyapunov stability in probability (LSP) to show convergence of the error state e_k^i. It should be mentioned that the conclusion obtained from *Lemma* 6.1 remains valid for the problem framework in this section. With similar discussions as in Section 6.2, it is concluded that the LSP condition is equivalent to the expressions (6.15). Employing *Markov's inequality*, we can instead use the expression (6.13). In addition, the illustrative example in Section 6.2 to justify the multi-step interval for studying LSP, can be constructed here again according to the new ALOHA-based scheduling law. Recall that to obtain boundedness of error expectation, we need to look at an interval of time-steps over which, given the channel capacity constraint (6.37), all sub-systems have non-zero chances of transmission. Since, we have multiple transmission links, one can infer that an interval of length $\left\lfloor \frac{N}{M-1} \right\rfloor$ provides enough transmission possibilities for an NCS of N sub-systems and M available transmission channels. For stability analysis, we assume the worst case scenario by considering the minimum number of available transmission channels, i.e. $M = 2$. This yields that the minimum length of the interval over which LSP is investigated equals N.

Theorem 6.3. *Consider an NCS with N heterogeneous LTI control sub-systems, with the plants given by (2.1), sharing a multi-channel communication network with two available transmission channels per time-step. Given the control law (2.3), the estimation process (2.4) and threshold policy (6.36), the NCS of interest is Lyapunov stable in probability under slotted ALOHA MAC protocol.*

Proof. See Appendix A.3. □

Remark 6.9. In *Theorem* 6.3, stability is guaranteed considering no prioritization in channel arbitration, since slotted ALOHA is a uniform random arbitration mechanism. However, any prioritized mechanism, which allocate the channel not uniformly but based on some priority measure, can also be considered. The design of such mechanism should be performed very carefully since improper prioritization may lead to higher collision rate than uniform arbitration.

6.5.3 Performance Evaluations

In this section, we evaluate the performance of a threshold-based scheduler over multi-channel slotted aloha. Both communication and control-related aspects are investigated.
For the simulations, we consider the similar evaluation setup as introduced in Section 6.3. The number of transmission channels at each time-slot, unless stated otherwise, is considered to be $M = 10$. It is worth mentioning that not only stability or instability of a plant determines the urge of a transmission, but also system noise, which is modeled as a random process evolves according to unsupported Gaussian distribution, influences the threshold-based policy. Therefore, it is not guaranteed that if a plant is stable, then it is asymptotically stable even if no transmission is associated with that sub-system. Due to presence of noise, a sub-system with stable plant might become in more urgent situation for transmission than a sub-system with unstable plant.
To evaluate control performance, we study the average error variance of N sub-systems:

$$\Sigma = \frac{1}{N} \sum_{i=1}^{N} \mathsf{var}[e_k^i] \tag{6.39}$$

From the communication point of view, we use two metrics. First one is average channel utilization, commonly known as throughput T, defined as:

$$T = \frac{\mathsf{E}[n_p^s]}{M}, \tag{6.40}$$

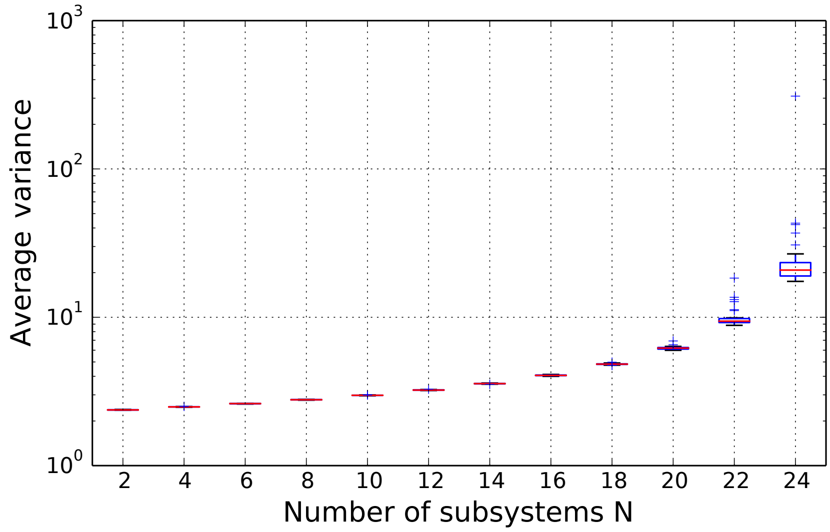

Figure 6.9: Average error variance var$[e_k^i]$ vs. number of sub-systems N (30 runs): $M = 10$, $\lambda' = 2$.

where $\mathsf{E}[n_p^s]$ is expected number of successful transmissions per time-slot. Ratio of collided packets is used as the second performance metric, and is defined as:

$$r_{coll} = \frac{\mathsf{E}[n_p^c]}{\mathsf{E}[n_p^c] + \mathsf{E}[n_p^s]}, \tag{6.41}$$

where $\mathsf{E}[n_p^c]$ is the expected number of collided transmissions per time-slot.
The transmission threshold λ_i is considered homogeneous for all N sub-systems throughout the simulation i.e. $\lambda_i = \lambda_j$, for all $i, j \in N$. To simplify the notations, we denote it by λ'.

Static Threshold Scheduler

First, we consider a scheduler where the transmission threshold is chosen independent of the number of transmission channels M. Figure 6.9 demonstrates the average error variances versus the increasing number of sub-systems N. We observe a non-linear growth of the error variance, and higher confidence interval for the resulting variance over multiple runs. The growth of the error variance can be explained by looking at Figure 6.11: with the increasing number of sub-systems we see an increase in collision rate. Since for the unstable sub-systems, the error accumulates exponentially with every collision, linear increase in collisions results in a non-linear increase in the variance of the error. Furthermore, we observe in Figure 6.11, that the shape of the throughput curve (solid blue curve) corresponds to the commonly known dependency for multi- and single-channel slotted aloha with Poisson distribution arrival rate [1]. The highest value $T \approx 1/e \approx 0.368$ is achieved at $N = 26$.

Figure 6.10 shows the dependency of the error variance on the transmission threshold λ'. As we observe, and it is inline with the hypothesis we have stated previously, the dependency is convex. In the one hand, with λ' close to 0, the transmission is attempted almost at every time-step, thus, causing many collisions and shifting the throughput T operating region as in Figure 6.11 to the right. The collisions, in turn, further increase the $\|e_k^i\|_2$ for all unstable

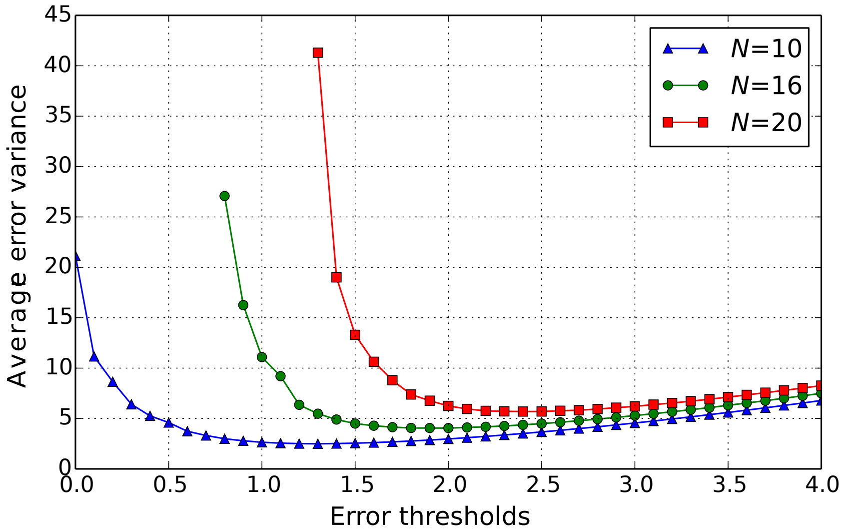

Figure 6.10: Average error variance var$[e_k^i]$ vs. λ'. $M = 10$.

sub-systems with $A_i > 1$, which results is further increase in the number of access attempts. As expected, the average error variance grows. On the other hand, if λ' is chosen too high, the increase in the error variance is caused by the underutilized communication medium (throughput T low). Thus, it is observed that there exists an optimal value for λ' in a given NCS scenario defined by N, M.

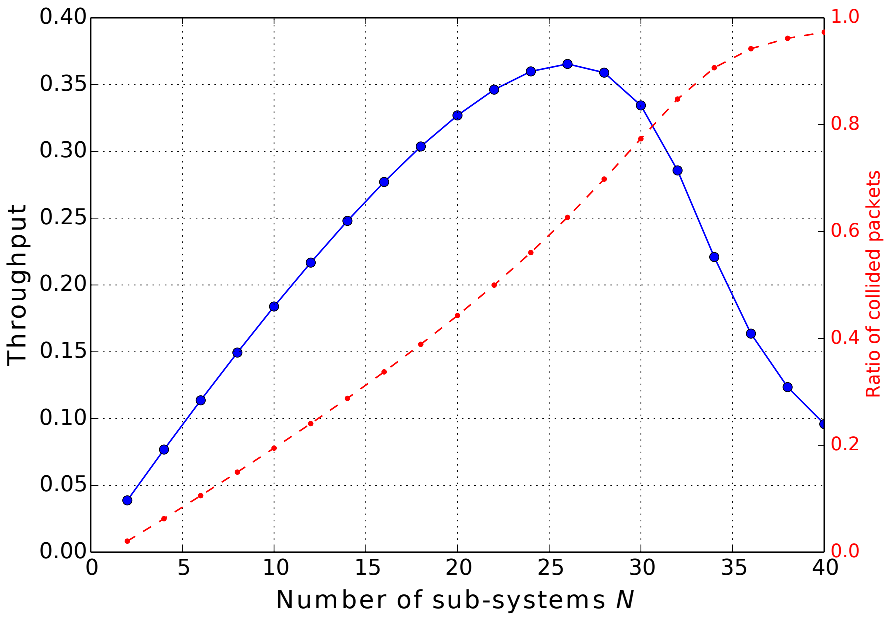

Figure 6.11: Average throughput and collision rate vs. number of sub-systems N. $M = 10$

Table 6.3: Optimal $\lambda' = f(N, M)$.

	N						
	4	6	8	10	12	14	16
$M = 5$	1.0	1.5	2.0	2.4	3.5	5.2	8.1
$M = 10$	0.6	0.8	1.0	1.2	1.4	1.6	1.8

Scheduler with Threshold Adaptation

Following the observation about the existence of an optimal λ', we propose an improvement to the threshold design defined in (6.42). Namely, we use the knowledge about network state M and the number of present sub-systems N, in order to select the λ' such that the performance is improved:

$$\lambda' = f(M), \tag{6.42}$$

where higher number of channels results in a higher λ'. Numerically obtained values for λ', depending on M, and N are summarized in Table 6.3.

This approach is mainly beneficiary for varying number of the available channels M. For simplicity, we model the number of channels as a random variable with two possible values $M \in \{M_1, M_2\}$, $M_1 < M_2$, with:

$$\mathsf{P}[M = M_1] = 1 - \mathsf{P}[M = M_2]. \tag{6.43}$$

These two states can represent presence or absence of a background traffic with reserved channels, for example, as described in [112, 113]. Although we consider only two states for M, the proposed scheduling design is extendable for a more general case of multiple states. In the evaluation scenario $M_1 = 5$ and $M_2 = 10$, and $\mathsf{P}[M = M_1] = 0.5$ are considered.

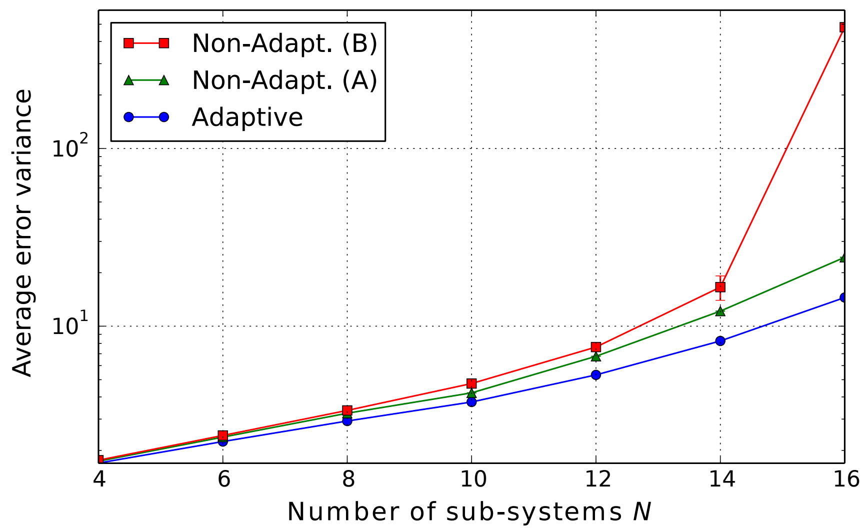

Figure 6.12: Average error variance vs. number of sub-systems N for three cases: Adaptive λ', Non-Adaptive (λ' optimal for M_1 channels), Non-Adaptive (λ' optimal for M_2 channels). $M_1 = 5$, $M_2 = 10$, $\mathsf{P}[M = M_2] = 0.5$.

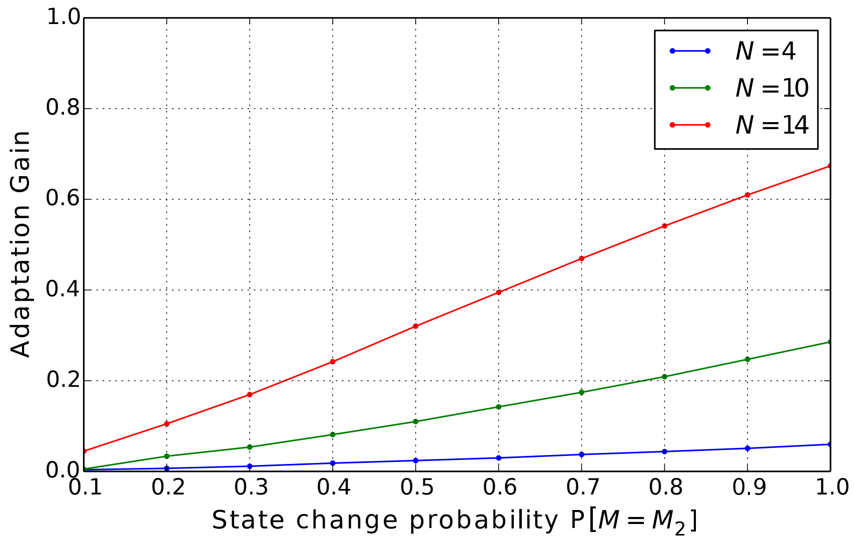

Figure 6.13: Adaptation gain G_{adap} vs. Probability of the "good" channel $\mathsf{P}[M = M2]$ for $N \in \{4, 10, 14\}$. $M_1 = 5$, $M_2 = 10$.

For comparisons between the static and adaptive scheduler, we consider two simulative scenarios for the static scheduler: **(A)** λ' is statically set to minimize the error variance for $M = M_1$, and **(B)** λ' is statically set to minimize the error variance for $M = M_2$, for a given number of sub-systems N. The comparisons are demonstrated in Figure 6.12. It is observed that the adaptive scheduler outperforms the non-adaptive one in terms of average error variance. It also shows that the first static scenario, i.e. **(A)**, which is optimizing the threshold for the lower number of channels M_2, is performing noticeably better than scenario **(B)**, which is optimizing the threshold for the higher number of channels M_1. This observation is supported by Figure 6.10 illustrating that the slope of the curves on the left side of the optimal point is much higher than that of the right side of it. This shows that over-utilization is more deteriorating, in terms of average error variance, than under-utilization.

To evaluate the influence of adaptation on the performance, we introduce Σ_{na} and Σ_a as the average error variances for non-adaptive and adaptive schedulers, respectively. Then, the adaptation gain can be defined as follows:

$$G_{adap} = \frac{\Sigma_{na} - \Sigma_a}{\Sigma_{na}}. \tag{6.44}$$

The results are depicted in Figure 6.13. The parameter $\mathsf{P}[M = M_2]$ is a measure of how frequently the network state changes. For $\mathsf{P}[M = M_2] = 0.1$ almost no change take place, hence, both schedulers are close in terms of performance. Moreover, for $\mathsf{P}[M = M_2] = 1$, although no change in network state takes place, the default state of the channel is $M = M_2$. Thus, the static scheduler is not optimal for all time-slots. For the network state changing every second time, the adaptive scheduler is able to reduce the error variance by up to 30%.

6.6 Summary

So far in this dissertation, we discussed event-triggered centralized scheduling mechanisms for shared resource NCSs, assuming the event information from all sub-systems are accessible. In this chapter, we first introduced an event-based decentralized bi-character scheduling mechanism which is capable of prioritizing the channel access among sub-systems based on only local event information. The scheduling architecture possesses both deterministic and probabilistic features, where the former feature aims at having less congested competition for scarce resources, and the latter feature provides a flexible design framework enabling us to properly adjust the scheduling parameters to avoid high collision rate.

In the second part of this chapter, we addressed the decentralized scheduling problem considering that noise-prone sensors measure the system states. The previous results were all based on the fact that perfect state information is accessible for the scheduler, which is not always a realistic assumption. Thus, we opted to design the event triggers not based on the true error values, but depending on estimated values based on noisy sensor measurements. Through numerical evaluations, it is shown that the new event triggers are sufficiently convenient to be employed to prioritize the channel access. It is moreover shown that performance of the overall NCS under the new event triggers closely follows the optimal centralized protocol.

Finally, in third part of this chapter, we analyzed the applicability of the decentralized bi-character policy for NCSs with multi-channel networks. For the ease of analysis, we assumed a threshold-based scheduler and then we employed the non-prioritized slotted ALOHA MAC protocol to assign the available channels to transmission requests.

For all the decentralized scenarios introduced in this chapter, stability of the stochastic NCS is addressed in terms of Lyapunov stability in probability. Despite the fact that collisions are unavoidable in decentralized scenarios, we showed that appropriate tuning of the scheduling parameters leads to acceptable collision rates. It is concluded in this chapter that the error-dependent scheduling design is locally implementable for NCSs with realistic assumptions, and is capable of improving the overall performance compared to the existing approaches.

6.7 Contributions

The presented discussions and results in this chapter are mainly from the author's work in [114,115] and a collaborative work in [101]. We demonstrated in this chapter that stochastic stability of multiple-loop NCSs can be preserved under the employment of decentralized error-dependent scheduling policies. Often in the literature, the channel access is assigned randomly among the sub-systems which leads to decreased performance due to the possibility of awarding the channel to a system with low real-time priority. We showed in this chapter that the randomization in awarding the channel access may become biased based on real-time requirements of local control entities. We discussed that event-based approaches can effectively be applied on networked systems even in the absence of having access to the perfect state information [115], and in addition in case of multi-link channels [101].

Decentralized scheduling mechanisms are an attractive field of research in the area of NCSs, e.g., [32,33,47,48,58,116,117]. Often, in the mentioned works, events are defined in terms

of deterministic thresholds, and after that the contention is resolved randomly. In [33], an optimal threshold policy is introduced for the design of scheduling laws in the absence of collisions. The communication resource is afterwards allocated with a CSMA-based approach for a single-hop scenario. In [47,48] the event-based approaches are employed accompanied with unslotted and slotted ALOHA MAC protocols, respectively. However, to the best of our knowledge, the error dependency is not carried over to the randomization process when it comes to contention-based channel allocation. The design of such scheduling mechanisms however should be performed with maximum care due to collisions. In fact, careless design of biased randomization leads to high collision rates which lowers the performance, sometimes even lower compared to performance of unbiased randomization.
The illustrations presented in this chapter, i.e. Figure 6.5 and Table 6.1 are taken from the author's own publication in [114]. Moreover, the figures and tables presented in Section 6.4.2 are from the author's research submitted in [115]. In addition, the plots and tables presented in Section 6.5.3, are presented in a collaborative work published in [101].

7

Control and Scheduling Design for Interconnected NCSs

Interconnected networked control systems represent a system class with application examples ranging from infrastructure systems, e.g. distributed electrical power grids, and water/gas distribution systems to mechatronic systems like large-scale telescopes, shape-adaptive aircraft wings and, adaptive mechanical structures. The unifying property is that several components are physically interconnected.

In this chapter, we extend our control and scheduling architectures to be applicable for a wider range of NCS models by considering the physical interconnections among the individual controlled sub-systems. Considering that the dynamics of a controlled sub-system is affected by the dynamic behavior of other sub-systems through physical links adds an extra coupling dimension to networked control systems and inevitably makes the architecture more complex to be properly designed. Compared to the scenario of NCSs consisting of physically isolated sub-systems where the coupling only appears in the communication channel, in this chapter we have to deal with an additional coupling point which occurs through the physical interconnection links. More precisely, we address the problem of event-based data scheduling and control for physically interconnected networked control systems over capacity-limited communication resources. Due to complexity of analysis for very general interconnection models including bi-directional and cyclic interconnections, we identify a class of interconnected networked systems within which the state-dependent prioritizing scheduling policy can be implemented. We consider an interconnected NCS consisting of multiple heterogeneous stochastic LTI sub-systems where the physical interconnection is modeled by a directed acyclic graph (DAG). DAGs are used to model systems with some sort of hierarchy, e.g. security systems, vehicle platoons or traffic networks. The sub-systems are controlled by a networked controller through a shared communication channel, which depending on the type of communication channel and information structure can be designed ranging from decentralized to distributed and to fully centralized control laws. In order to cope well with the expected transmission traffic, we employ a bi-character deterministic-probabilistic scheduling mechanism which dynamically assigns access priorities to each sub-system at each time-step according to an error-dependent priority measure. The sub-systems which are granted channel access then transmit their state information through the communication network. We prove stability of such interconnected networked systems under the proposed scheduling law in terms of f-ergodicity of overall network-induced error. Simulation results illustrate the proposed approach and show a reduction in the error variance compared to time-triggered TDMA and uniform random-access scheduling policies such as CSMA.

This chapter is structured as follows. We present the model of an interconnected NCS and introduce the considered physical interconnection model and information pattern in Section 7.1. Moreover, we investigate the structural properties of the overall interconnected NCS under DAG physical topology. We then discuss the control and estimation processes in Section 7.2 and derive the necessary conditions on the information propagation pattern. Section 7.3 represents the scheduling process and discusses the influence of state-dependency in decision making process on the network performance. Under the given assumptions on interconnection structure and information pattern, we employ a method based on cascading in Section 7.4, which leads to stability guarantee in terms of f-ergodicity. The improved performance claims and simulation results are validated by numerical results in Section 7.5.

7.1 NCS Model, Interconnection Architecture, and Information Pattern

So far in this dissertation, we have discussed the control and scheduling mechanisms for multiple-loop networked control systems where the control-loops are supposed to be physically isolated. In the other words, the only point of coupling between the sub-systems is in the shared communication channel. Each local control loop in that scenario is supposed to be steered by a local controller and local system states x^i's for all $i \in \{1, \ldots, N\}$ are also independent from each other. In this section, we introduce the physical interconnections as an extra point of coupling between the control entities of an NCS. The physical interconnections bring extra complexity to design methodologies and consequently complicate the analyses, as we will notice further in this chapter. Therefore, we focus on a special class of interconnected NCSs and apply our design propositions on that specific class of systems.
Consider networked control systems composed of N heterogeneous LTI controlled sub-systems which are physically interconnected and they are additionally coupled through a shared communication network, as depicted in Figure 7.1. Each controlled sub-system consists of a stochastic discrete time LTI plant $\mathcal{P}_i$ and a linear feedback controller $\mathcal{C}_i$, where the communication network is installed between the plant and controller as the medium for state information to be transmitted from the plant to the controller.

To have a better understanding of the terminologies which are used in this chapter, we introduce the following: A connected graph with the set of vertices $\mathcal{V}$ and set of directed edges $\mathcal{E}$ is represented by $\mathcal{G}_c(\mathcal{V}, \mathcal{E})$. A node $j \in \mathcal{V}$ is called an *affecting node* if at least one

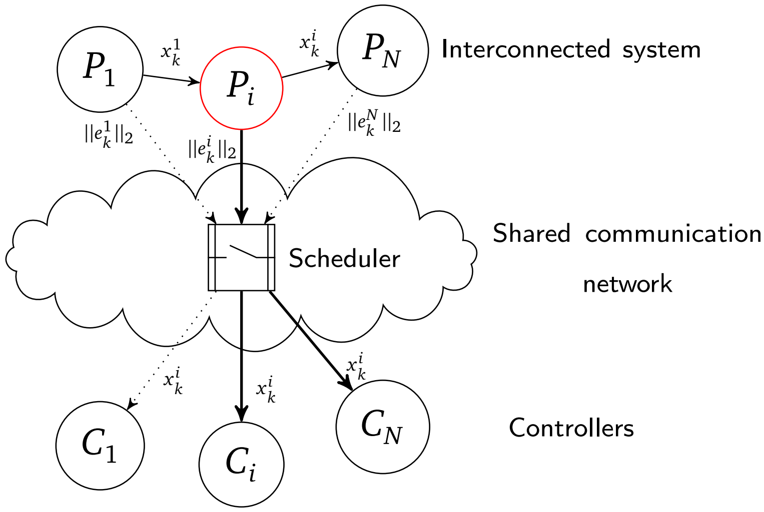

Figure 7.1: A physically interconnected NCS over a shared communication network with error-dependent centralized scheduler.

node i exists between which a directed path from j to i is established. Node i is called an *affected node*. If the path's length is one, node j is a *direct neighbor* of node i. A node $l \in \mathcal{V}$ is called *multi-hop neighbor* of node i if a directed path of length greater than one exists from l to i.

Having the terminologies introduced, we assume that the transmitted data from a specific node across the communication channel is accessible for all corresponding affected sub-systems (nodes). The control unit $\mathscr{C}_i$ computes the control input for sub-system i utilizing the state estimates according to the topology of the distributed control law. Therefore, control unit $\mathscr{C}_i$ corresponding to sub-system (node) i has access to the transmitted information from all direct and multi-hop neighbors of node i. To allocate the capacity-limited channel among sub-systems, a scheduler decides whether the i^{th} state $x_k^i \in \mathbb{R}^{n_i}$ is an event to be scheduled for channel access. A directed acyclic graph (DAG) $\mathscr{G}_c$ represents the physical interconnections between the sub-systems $i \in \{1,\dots,N\}$, wherein every sub-system i is represented by a node. This formulation defines a class of interconnected networked systems in which there exist no cycles and two nodes cannot be the respective neighbors of each other. An edge from node j to node i indicates that the dynamics of i^{th} node is directly affected by node j^{th}. We define the set of all direct and multi-hop neighbors of a node i as $\tilde{S}_n^i$, while the set of only direct neighbors of node i is denoted by S_n^i.

Considering the physical interconnections, dynamics of an LTI stochastic plant $\mathscr{P}_i$ can be expressed according to the following stochastic difference equation:

$$x_{k+1}^i = A_i x_k^i + B_i u_k^i + \sum_{j\in S_n^i} A_{ij} x_k^j + w_k^i, \tag{7.1}$$

where the noise sequence $w_k^i \in \mathbb{R}^{n_i}$ is i.i.d. with $\mathcal{N}(0, W_i)$ at each time-step k. The constant matrices A_i, A_{ij} and B_i are all from appropriate dimensions, such that $A_i \in \mathbb{R}^{n_i\times n_i}, A_{ij} \in \mathbb{R}^{n_i\times n_j}, B_i \in \mathbb{R}^{n_i\times m_i}$ describe system matrices, interconnection matrices and input matrices for every $i \in \{1,\dots,N\}$, respectively. The initial state x_0^i is supposed to be

randomly chosen from an arbitrary distribution with bounded variance. According to DAG properties, if node j is either a direct or multi-hop neighbor of node i, i.e. $A_{ij} \neq [0]_{n_i \times n_j}$, then node i is not a direct nor a multi-hop neighbor of node j, i.e. $A_{ji} = [0]_{n_j \times n_i}$. The initial state $x_0 := [x_0^{1\top}, \ldots, x_0^{N\top}]^\top$, together with the noise sequence w_k, generate the probability space $(\Omega, \mathscr{A}, \mathsf{P})$, where Ω is the set of all possible outcomes, $\mathscr{A}$ represents the events with probabilities determined by the function P.
Concatenation of the system state from all sub-systems leads to the overall networked system state as

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{7.2}$$

where the aggregate state $x = [x^{1^\top}, \ldots, x^{N^\top}]^\top \in \mathbb{R}^n$, aggregate control input $u = [u^{1^\top}, \ldots, u^{N^\top}]^\top \in \mathbb{R}^m$, overall system matrix $A \in \mathbb{R}^{n \times n}$ consists of the blocks A_i on the diagonal, and A_{ij} on the off-diagonal, and the overall control matrix $B \in \mathbb{R}^{n \times m}$ is block-diagonal with the entries B_i. Clearly, $\sum_{i=1}^N n_i = n$ and $\sum_{i=1}^N m_i = m$.

Recall that we confine our focus on a special class of interconnected NCSs where the structure of interconnections follows directed-acyclic graphs (DAGs). Figure 7.2 illustrates a typical DAG with eight nodes, where the only-affecting and only-affected nodes are designated by blue circles and red circles, respectively.

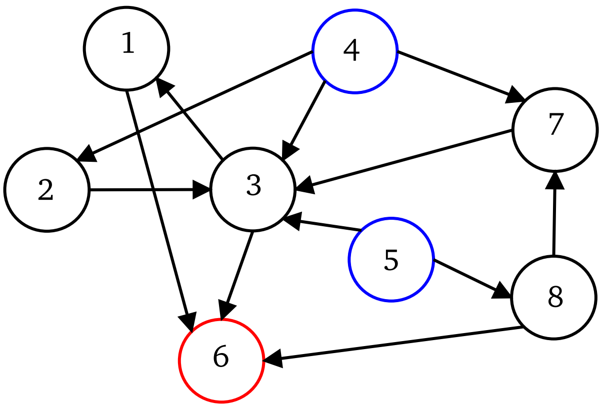

Figure 7.2: A DAG with eight nodes and twelve directed edges. A node j is direct neighbor of node i if there exists an edge from node j to node i.

Two main properties of DAGs are as follows:

1. There exists no cycle which ensures that the Markov property is not violated,
2. If node j is either a direct or multi-hop neighbor of node i, i.e. $j \in \tilde{S}_n^i$, then node i is not a direct nor a multi-hop neighbor of node j, i.e. $i \notin \tilde{S}_n^j$.

These two properties play an essential role in analyzing stability of the overall interconnected NCS, as we will see later in this chapter. We can re-arrange the nodes and edges of all DAGs to have a better understanding of the interconnection representation. Having DAGs, we are always able to divide the nodes into different layers, from the layer including

$$S_n^1 = \{3\}$$
$$S_n^2 = \{4\}$$
$$S_n^3 = \{2, 4, 5, 7\}$$
$$S_n^4 = \varnothing$$
$$S_n^5 = \varnothing$$
$$S_n^6 = \{1, 3, 8\}$$
$$S_n^7 = \{4, 8\}$$
$$S_n^8 = \{5\}$$

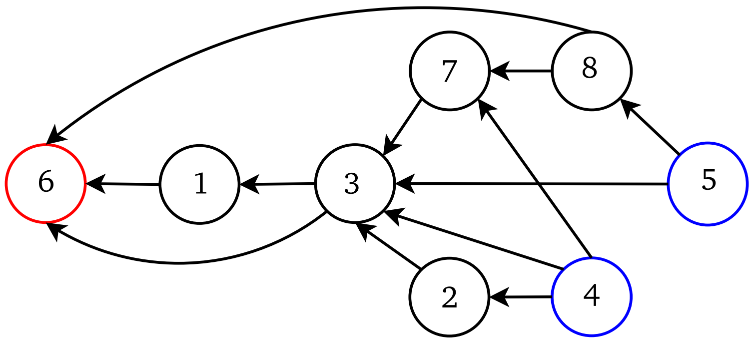

Figure 7.3: A DAG with eight nodes and twelve directed edges. Node 6 is the only-affected node while nodes 4 and 5 are the only-affecting nodes.

only-affecting nodes to the layer containing only-affected nodes. In Figure. 7.3 the identical DAG from Figure 7.2 is redrawn from the highest layer containing the only-affecting nodes (node 4 and 5) to the lowest layer containing the only-affected nodes (node 6).

It is clear from (7.1) that the only-affecting nodes possess only local dynamics as they are not affected by any other node in the graph (see nodes 4 and 5 in Figure 7.3). This property enables us to look at the overall interconnected NCS from the highest layer to the lowest such that the sub-systems (nodes) belonging to the lower layers possess cascaded dynamics.

7.2 Control Synthesis and Estimation Process

So far in this chapter, we described the physical interconnection topology, which follows DAGs. In this section, we address control law synthesis for such interconnected NCSs. First of all, it should be noticed that the control law synthesis is tightly coupled with the information topology, and not the physical interconnection. Although the interconnection topology in this chapter is restricted to be of DAG type, it is not necessary for the control laws to follow such assumption. However, we need to impose a requirement on the available set of information for the control unit of a sub-system (node). Under this assumption, the control laws are then allowed to be designed in centralized, distributed or decentralized fashion, depending on the information topology. The assumption is as follows:

Assumption 7.1. Control unit of a sub-system i, including the model-based estimator and controller, is required to have access to the real-time information transmitted across the shared communication channel from all its direct or indirect neighbors. In addition, we assume that i^{th} control unit has access to the system information of its direct or indirect neighbors $j \in \tilde{S}_n^i$, such as A_j, B_j, A_{ij}, L_{ij} and the distribution of W_j and x_0^j. If j is an indirect neighbor of node i, then the information about all A_{ql} for those nodes q and l which are connecting the nodes i and j should be accessible for node i.

Under the given assumption, let the concatenated networked system (7.2) be driven by a state feedback control $u_k = [u_k^{1^\top}, \ldots, u_k^{N^\top}]^\top$. According to the minimum requirement on

the information pattern, each node has access to the information about its own affecting nodes in the interconnection graph. The affecting nodes of a certain node i include direct and multi-hop neighbors of node i. Thus, the local control unit $\mathscr{C}_i$ can exploit this available information to compute the control signal u_k^i. This leads to a distributed control law having the same DAG structure as the interconnection graph. It is worth noting that control laws do not necessarily need to have distributed form and can still be designed decentrally as long as they are stabilizing. For the purpose of simplicity and notational convenience, we assume that the control laws are designed in distributed fashion with a structure that is identical to that of physical interconnection DAG. This ensures that the control units exploit the information they have access to, according to the assumption 7.1. Furthermore, this assumption simplifies the theoretical representations in this chapter, while the results are readily extendable to arbitrary information structures.
The control law u_k^i is updated with true state values x_k^i and x_k^j if all direct neighbors $j \in S_n^i$ transmit at time-step k along with the i^{th} node itself. In case of a non-transmission from either the node i or a direct neighboring node j, the estimates $\hat{x}_k^i$ or $\hat{x}_k^j$ are computed by a model-based estimator:

$$u_k^i = -L_i \mathsf{E}[x_k^i | \mathscr{I}_k^i] - \sum_{j \in \tilde{S}_n^i} L_{ij} \mathsf{E}[x_k^j | \mathscr{I}_k^i], \tag{7.3}$$

where $\mathscr{I}_k^i$ is the information set available at node i at a time-step k, which according to assumption 7.1, is as follows:

$$\mathscr{I}_k^i = \{z_0^i, \ldots, z_{k-1}^i\} \cup_{j \in \tilde{S}_n^i} \{z_0^j, \ldots, z_{k-1}^j\} \tag{7.4}$$

with, L_i and L_{ij} representing the feedback gains. Depending on the scheduling variable δ_k^i for each sub-system i, the received information at the control side $\mathscr{C}_i$, and in addition at all control stations for which node i is either a direct or an indirect neighbor, is

$$z_k^i = \begin{cases} x_k^i, & \delta_k^i = 1 \\ \varnothing, & \delta_k^i = 0. \end{cases}$$

In what follows, we calculate the estimate of a neighboring node's state vector, given the available information at node i. Assuming $j \in S_n^i$, we have,

$$\begin{aligned} \mathsf{E}\left[x_{k+1}^i | \mathscr{I}_k^i\right] &= \mathsf{E}\left[A_i x_k^i + B_i u_k^i + \sum_{j \in S_n^i} A_{ij} x_k^j + w_k^i \middle| \mathscr{I}_k^i\right] \\ &= (A_i - B_i L_i)\, \mathsf{E}\left[x_k^i \middle| \mathscr{I}_k^i, \mathscr{I}_k^{i,-i}\right] + \sum_{j \in S_n^i} \left(A_{ij} - B_i L_{ij}\right) \mathsf{E}\left[x_k^j \middle| \mathscr{I}_k^i\right]. \end{aligned}$$

Therefore, the estimate of the state vector x_k^j computed at node i is required to have an expectation of the local error state x^i_{k+1}. We know,

$$\begin{aligned}\mathsf{E}\left[x^j_{k+1}|\mathscr{I}^i_k\right] &= \mathsf{E}\left[A_j x^j_k + B_j u^j_k + \sum_{l\in S^j_n} A_{jl} x^l_k + w^j_k \middle| \mathscr{I}^i_k\right] \\ &= \left(A_j - B_j L_j\right)\mathsf{E}\left[x^j_k \middle| \mathscr{I}^i_k\right] + \sum_{l\in S^j_n}\left(A_{jl} - B_j L_{jl}\right)\mathsf{E}\left[x^l_k \middle| \mathscr{I}^i_k\right].\end{aligned}$$

It can be straightforwardly checked that the state estimate $\hat{x}^j_k$ of a node $j \in S^i_n$ computed at node i, i.e. $\mathsf{E}\left[x^j_k|\mathscr{I}^i_k\right]$ coincides with the estimation of x^j_k computed at node j, i.e. $\mathsf{E}\left[x^j_k|\mathscr{I}^j_k\right]$. This directly follows from the fact that, according to the DAG interconnection structure, if $j \in S^i_n$, the information set $\mathscr{I}^i_k$ includes the information set $\mathscr{I}^j_k$, i.e. $\mathscr{I}^j_k \subset \mathscr{I}^i_k$. In the other words, $\mathsf{E}\left[x^j_{k+1}|\mathscr{I}^j_k\right] = \mathsf{E}\left[x^j_{k+1}|\mathscr{I}^i_k\right]$. Therefore, we have

$$\mathsf{E}\left[x^i_k|\mathscr{I}^i_k\right] = (A_i - B_i L_i)\mathsf{E}\left[x^i_{k-1}|\mathscr{I}^i_{k-1}\right] + \sum_{j\in S^i_n}(A_{ij} - B_i L_{ij})\mathsf{E}\left[x^j_{k-1}|\mathscr{I}^j_{k-1}\right], \tag{7.5}$$

with initial distribution $\mathsf{E}\left[x^i_0|\mathscr{I}^i_0\right] = \mathsf{E}[x^j_0|\mathscr{I}^j_0] = 0$.

Assuming stabilizability of the concatenated system (7.2), i.e. pair (A, B) is stabilizable, an emulation-based control law, e.g. with the approach proposed in [118], is enough to ensure that the feedback gain L, consisting of blocks L_i on the diagonal and L_{ij} on the off-diagonal, is stabilizing. Therefore, the closed-loop matrix $(A - BL)$ is Hurwitz.

The network-induced error correspond to a node i at time-step k is defined as the difference between actual and estimated state values:

$$e^i_k := x^i_k - \mathsf{E}\left[x^i_k|\mathscr{I}^i_{k-1}\right]. \tag{7.6}$$

Define the aggregate state $\left[x^{i\top}_k, e^{i\top}_k\right]^\top$ at node i. Then, considering identical DAGs for interconnection structure and information topology, dynamics of the i^{th} node can be expressed as below, following the equations (7.1)-(7.6):

$$\begin{aligned}\begin{bmatrix} x^i_{k+1} \\ e^i_{k+1}\end{bmatrix} &= \begin{bmatrix}(A_i - B_i L_i) & (1-\delta^i_k)B_i L_i \\ 0 & (1-\delta^i_k)A_i\end{bmatrix}\begin{bmatrix} x^i_k \\ e^i_k\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} w^i_k \\ &+ \sum_{j\in S^i_n}\begin{bmatrix}\left(A_{ij} - B_i L_{ij}\right) & (1-\delta^j_k)B_i L_{ij} \\ 0 & (1-\delta^j_k)A_{ij}\end{bmatrix}\begin{bmatrix} x^j_k \\ e^j_k\end{bmatrix}.\end{aligned} \tag{7.7}$$

As it can be seen, (7.7) demonstrates a triangular dynamics which ensures that the evolution of the local error state e^i_k is independent of the local system state x^i_k, and the neighboring states x^j_k for all $j \in \tilde{S}^i_n$. However, the error state e^i_k is affected through physical interconnections and therefore dynamics of e^i_k is not independent from the error states of the i^{th} sub-system's neighboring nodes.

It should be noticed that, if the information topology does not follow the same DAG that of interconnection structure, the triangularity of the expression (7.7) is not violated, instead extra terms appear on the first rows of the block-square matrices in expression (7.7).

7.3 State-Dependent Scheduling Law

Having discussed the control synthesis in distributed fashion, we consider the emulation-based framework assuming that the feedback gain L, consisting of blocks L_i on the diagonal and blocks L_{ij} on the off-diagonal, stabilizes the system state (7.2), in the absence of capacity constraints. Hence, we focus on the scheduling design with limited communication resources. Due to the capacity limitation of the shared communication channel, not all sub-systems can transmit simultaneously. Therefore, the scheduling unit decides which sub-systems use the channel at each time-step k. The scheduling decision variable has, similar to the previous chapters, the following binary form

$$\delta_k^i = \begin{cases} 1, & x_k^i \text{ sent through the channel} \\ 0, & x_k^i \text{ blocked.} \end{cases}$$

As discussed in the previous section, if the channel is granted to a sub-system i, at a time-step k, then x_k^i is sent across the channel and is accessible for all sub-systems whom are affected by sub-system i through the physical interconnections. To determine the scheduling decision, we propose an event-based law which dynamically prioritizes the limited communication resources among the sub-systems in an interconnected NCS. Assume the communication channel has the capacity constraint $c < N$. Then, the following bi-character error-dependent scheduling rule defines the probability of channel access for each node i at a time-step $k+1$ given the error values e_k^j and error thresholds λ_j for all $j \in \{1, \dots, N\}$ at exactly the most recent time-step k:

$$\mathsf{P}[\delta_{k+1}^i = 1 | e_k^j, \lambda_j] = \begin{cases} 0 & \|e_k^i\|_{Q^i}^2 \leq \lambda_i \\ 1 & \|e_k^i\|_{Q^i}^2 > \lambda_i \wedge n_{\lambda,k} \leq c \\ \frac{\|e_k^i\|_{Q^i}^2}{\sum_{n_{\lambda,k}} \|e_k^j\|_{Q^i}^2} & \|e_k^i\|_{Q^i}^2 > \lambda_i \wedge n_{\lambda,k} > c \end{cases} \tag{7.8}$$

where λ_i is the local error threshold for node i. The number of nodes satisfying $\|e_k^i\|_{Q^i}^2 > \lambda_i$ at time-step k is denoted by $n_{\lambda,k}$, and Q^i is a positive definite weight matrix. It is clear that the probability distribution above is supported on interval $[0, \infty)$.

The introduced scheduling law is characterized by both deterministic and probabilistic features. First, if $\|e_k^i\|_{Q^i}^2 \leq \lambda_i$, then the corresponding node is deterministically excluded from channel access competition. For those being qualified for a transmission, in case $n_{\lambda,k} > c$, then c nodes are randomly selected such that the ones with larger errors have more chances of being selected, as indicated in the last line of (7.8). We assume that the scheduler is updated with error norms $\|e_k^i\|_{Q^i}^2 \in \mathbb{R}$ from all nodes i at every time-step k. This can be realized by considering a low-capacity channel between the scheduler and sub-systems to send those N real-valued data to the scheduler.

The randomization in the last line of (7.8) allows us to implement the scheduling policy approximately in decentralized fashion. A similar decentralization approach is presented in [114] for physically isolated NCSs, however for interconnected NCSs the decentralization should be performed with more care, which is out of scope of this dissertation.
From (7.7), it can be concluded that the local error state e_k^i evolves as follows:

$$e_{k+1}^i = \left(1-\delta_{k+1}^i\right)A_i e_k^i + \sum_{j\in S_n^i}\left(1-\delta_{k+1}^j\right)A_{ij}e_k^j + w_k^i. \tag{7.9}$$

We define the aggregate network-induced error state $e_k \in \mathbb{R}^n$ by stacking the local error vectors from all sub-systems $i \in \{1,\ldots,N\}$ as follows:

$$e_k = [e_k^{1\mathsf{T}},\ldots,e_k^{N\mathsf{T}}]^\mathsf{T}, \tag{7.10}$$

The discussions in *Lemma* 2.1 can be repeated here again to conclude that the aggregate error vector (7.10) induced by the network scheduled by the introduced law (7.8) is a homogeneous, aperiodic and ψ-irreducible Markov process. It should be noted that, the Markov property, in its original definition, holds only assuming the interconnections follow the DAG structure.

7.4 Stability Analysis under DAG Interconnection Architecture

In this section, we address stochastic stability of the described interconnected NCSs under the proposed event-triggered scheduling mechanism. As we already discussed in Section 7.2, the aggregate state $[x_k^{i\mathsf{T}}, e_k^{i\mathsf{T}}]^\mathsf{T}$ has triangular dynamics within each sub-system i implying that the evolution of the local error state e_k^i is independent of the system state x_k^i and neighboring states x_k^j for all $j \in S_n^i$. Employing the emulation-based control strategy which leads to have stable closed-loop control systems, expression (7.9) suggests that convergence of the network-induced error state e_k introduced in (7.10) is sufficient to show the overall stability of the interconnected NCS, in terms of state convergence. We discussed that the error state e_k is a time-homogeneous aperiodic and ψ-irreducible Markov chain. This motivates us to evaluate the behavior of e_k stochastically in terms of convergence to a compact set. In fact, having the concatenated system (7.2) stabilized in the absence of the communication constraint, the triangularity of (7.7) allows us to invoke *Theorem* 3.1 to analyze the stochastic behavior of the error state e_k, which itself is steered by the scheduling input, according to expression (7.9). In the following stability analysis, we frequently recall the preliminary definitions, propositions and theorems presented in Section 3.3.1.
In order to invoke *Theorem* 3.1, we first define a non-negative real-valued Lyapunov function $V(e_k): \mathbb{R}^n \to \mathbb{R}^+$ as follows:

$$V(e_k) = \sum_{i=1}^{N} e_k^{i\mathsf{T}} Q^i e_k^i = \sum_{i=1}^{N} \|e_k^i\|_{Q^i}^2. \tag{7.11}$$

Due to characteristics of the selected function (7.11), employing the drift operator ΔV, introduced in *Definition* 3.3, over one transition step, i.e. $k \to k+1$, for the Markov chain (7.10) becomes too conservative. Similar to the previous chapters, we construct the following example to illustrate this observation:

Illustrative Example: Consider an interconnected NCS consisting of two identical scalar sub-systems competing for the sole channel slot at each time-step. The parameters are assumed to be $A_1 = A_2 = A$, $A_{12} \neq 0$ which implies, according to the DAG properties that $A_{21} = 0$. For illustration purposes, assume $Q_1 = Q_2 = 1$ and identical initial values $e_k^1 = e_k^2 = \bar{e}_k > \lambda_1 = \lambda_2$. Thus, starting from time-step k, the transmission chance for each node at time-step $k+1$ is $\frac{1}{2}$ according to (7.8). From (7.9) and (7.11), the one-step drift (3.3) can be calculated as follows:

$$\begin{aligned}
\Delta V(e_k) &= \mathsf{E}[V(e_{k+1})|e_k] - V(e_k) \\
&= \frac{1}{2}\left(\|A_{12}\bar{e}_k\|_2^2 + \mathsf{E}\left[\|w_k^1\|_2^2\right] + \|A\bar{e}_k\|_2^2 + \mathsf{E}\left[\|w_k^2\|_2^2\right]\right) \\
&+ \frac{1}{2}\left(\|A\bar{e}_k\|_2^2 + \mathsf{E}\left[\|w_k^1\|_2^2\right] + \mathsf{E}\left[\|w_k^2\|_2^2\right]\right) - 2\|\bar{e}_k\|_2^2 \\
&= \|A\bar{e}_k\|_2^2 + \frac{1}{2}\|A_{12}\bar{e}_k\|_2^2 + \mathrm{tr}(W_1) + \mathrm{tr}(W_2) - 2\|\bar{e}_k\|_2^2.
\end{aligned}$$

For $A > \sqrt{2}$ or $A_{12} > 2$, the drift is not converging, which violates the condition in *Theorem* 3.1. To obtain a negative drift, as it is extensively discussed in earlier chapters, all sub-systems need to have chances of transmission. Having the communication channel subject to the capacity constraint $\sum_{i=1}^{N} \delta_k^i = 1$, an interval with length N ensures that enough transmission possibilities are provided, although it does not guarantee that a sub-system certainly transmits over that interval. To that end, to infer f-ergodicity, we exploit the modified multi-step-drift operator over an N time-step interval, e.g. over the interval $[k, k+N]$, as already introduced in (3.3):

$$\Delta V(e_k, N) = \mathsf{E}[V(e_{k+N})|e_k] - V(e_k), \quad e_k \in \mathbb{R}^n. \tag{7.12}$$

We will see in the main theorem of this chapter that f-ergodicity of the Markov chain (7.10) is recovered by employing (7.12), under the scheduling policy (7.8) considering that the interconnection topology follows the DAG pattern.

Before proceeding to the theorem, it is essential to observe how the error of a sub-system i in an interconnected NCS evolves over an interval. Employing (7.9), the error at the final time of the interval $[k, k+N]$, i.e. e_{k+N}^i, can be written as a function of an error value $e_{k+r_i}^i$ at an arbitrary prior time-step $k + r_i \in [k, k+N]$, as follows:

$$
\begin{aligned}
e^i_{k+N} = & \prod_{\alpha=r_i+1}^{N} \left(1-\delta^i_{k+\alpha}\right) A_i^{N-r_i} e^i_{k+r_i} \\
& \underbrace{+ \sum_{r=r_i}^{N-1} \left[\prod_{\alpha=r+2}^{N} \left(1-\delta^i_{k+\alpha}\right) A_i^{N-r-1}\, w^i_{k+r} \right]}_{\text{Corresponds to local dynamics of node } i} \\
& + \sum_{j\in S_n^i} \left[\sum_{\beta=0}^{N-r_i-1} \prod_{\gamma=r_i+1}^{r_i+\beta+1} (1-\delta^j_{k+\gamma}) \prod_{\kappa=r_i+\beta+2}^{N} (1-\delta^i_{k+\kappa})\, A_i^{\bar{\beta}_1}\, A_{ij}\, A_j^{\beta} \right] e^j_{k+r_i} \\
& + \sum_{j\in S_n^i} \left[\sum_{\beta=0}^{N-r_i-2} \prod_{\gamma=r_i+2}^{r_i+\beta+2} (1-\delta^j_{k+\gamma}) \prod_{\kappa=r_i+\beta+3}^{N} (1-\delta^i_{k+\kappa})\, A_i^{\bar{\beta}_2}\, A_{ij}\, A_j^{\beta} \right] w^j_{k+r_i} \\
& + \sum_{j\in S_n^i} \left[\sum_{\beta=0}^{N-r_i-3} \prod_{\gamma=r_i+3}^{r_i+\beta+3} (1-\delta^j_{k+\gamma}) \prod_{\kappa=r_i+\beta+4}^{N} (1-\delta^i_{k+\kappa})\, A_i^{\bar{\beta}_3}\, A_{ij}\, A_j^{\beta} \right] w^j_{k+r_i+1} \\
& + \dots \\
& \underbrace{\begin{aligned} & + \sum_{j\in S_n^i} \left[\sum_{\beta=0}^{1} \prod_{\gamma=N-1}^{N+\beta-1} (1-\delta^j_{k+\gamma}) \prod_{\kappa=N+\beta}^{N} (1-\delta^i_{k+\kappa})\, A_i^{1-\beta}\, A_{ij}\, A_j^{\beta} \right] w^j_{k+N-3} \\ & + \sum_{j\in S_n^i} (1-\delta^j_{k+N})\, A_{ij}\, w^j_{k+N-2} \end{aligned}}_{\text{The effect of direct neighbors } j\in S_n^i} \\
& + \sum_{j\in S_n^i} \sum_{l\in S_n^j} \mathscr{F}_e(\delta^i,\delta^j,\delta^l,A_i,A_j,A_l,A_{ij},A_{jl})\, e^l_{k+r_i} \\
& + \sum_{j\in S_n^i} \sum_{l\in S_n^j} \mathscr{F}_w(\delta^i,\delta^j,\delta^l,A_i,A_j,A_l,A_{ij},A_{jl})\, w^l_{[k+r_i:k+N-3]} \\
& + \dots \\
& \underbrace{+ \sum_{j\in S_n^i} \sum_{l\in S_n^j} \dots \sum_{o\in S_n^t} (1-\delta^j_{k+N})\,(1-\delta^l_{k+N-1}) \dots (1-\delta^o_{k+r_i+1})\, \bar{A}_{io}\, e^o_{k+r_i}}_{\text{The effect of multi-hop neighbors of node } i}
\end{aligned}
\tag{7.13}
$$

where, $\bar{\beta}_1 = N - r_i - \beta - 1$, $\bar{\beta}_2 = N - r_i - \beta - 2$, and $\bar{A}_{io} = A_{ij}A_{jl}\dots A_{to}$. The two matrices $\mathscr{F}_e$ and $\mathscr{F}_w$ are not given explicitly due to space limitations, but they represent the effect of two-hop neighbors $l \in S_n^j$. [1] In addition, having a finite number of sub-systems ensures that the last term in (7.13) includes a finite number of summation operators.

We know that $A_{ij}e_k^j$ represents the interconnection effect from node $j \in S_n^i$ on the error state e_k^i, according to error evolution (7.9). In addition, e_k^j contains the interconnection effects from its own neighboring nodes $l \in S_n^j$, which are not necessarily direct neighbors of the node i. Therefore, to have a comprehensive understanding of how the error of a node i evolves, it does not suffice to observe only dynamics and transmission times of the i^{th} node and its direct neighbors. It is also essential to take into account the interconnection effects from the indirect neighbors. Therefore, as it is suggested by the expression (7.13), it is extremely complex to analyze and expect the behavior of a single node of an NCS under the very general form of interconnection topology. Indeed, in general form of interconnection structure, which is represented by connected graphs with circles and bi-direction edges, to analyze the behavior of a single node i, the behavior of all nodes need to be observed at all sampling times. With these explanations, selection of DAGs as the interconnection model between the networked sub-systems in this Chapter can be better justified.
As already reviewed in Section 7.1, a DAG represents a graph with no cycle and in addition if a node j is either a direct or multi-hop neighbor of node i, i.e. in our notation convention, $j \in \tilde{S}_n^i$, then node i is neither a direct nor a multi-hop neighbor of node j, i.e. $i \notin \tilde{S}_n^j$. Having these two properties, we showed that a DAG can always be divided into hierarchies, from the highest including only-affecting nodes to the lowest containing only-affected nodes. Moreover, it can be concluded from (7.13) that the only-affecting nodes possess local dynamics as they are not affected by any other node. This is a powerful conclusion which enables us to analyze (7.13) by initially looking at the only-affecting nodes. The interconnection effect from the only-affecting nodes can be regarded as cascaded effect. Note that, the interconnection effects from the higher layers to the lower layers can be represented as cascaded effect only due to the DAG properties. In fact, it is ensured through the non-existence of cycles in a DAG that the interconnection effect from an affecting node is not circulating back to that node again through the circles in the graph. Considering an interconnected NCS as a cascade system is quite beneficial because the cascade of f-ergodic systems is f-ergodic. Therefore, showing that the only affecting nodes are stable in terms of f-ergodicity affirms that f-ergodicity of their neighboring nodes (if they are regardless of the neighboring effects) is not violated through the interconnection with an f-ergodic affecting node. In fact, given an only-affecting node j which admits an f-ergodic error state, then f-ergodicity of an affected node i is preserved if j is a neighbor of i. In Figure 7.3 for example, nodes 4 and 5 are only-affecting. Assuming that these nodes, which are not affected through the interconnections, are f-ergodic, then nodes 2 and 8 are also f-ergodic if and only if their respective

[1] The explicit forms of $\mathscr{F}_e$ and $\mathscr{F}_w$ for special case $r_i = N-3$ are as follows:

$$\begin{aligned}
\mathscr{F}_e &= (1-\delta_{k+N}^i)(1-\delta_{k+N-1}^j)(1-\delta_{k+N-2}^l)A_iA_{ij}A_{jl} \\
&\quad + (1-\delta_{k+N}^j)(1-\delta_{k+N-1}^j)(1-\delta_{k+N-2}^l)A_{ij}A_jA_{jl} \\
&\quad + (1-\delta_{k+N}^j)(1-\delta_{k+N-1}^l)(1-\delta_{k+N-2}^l)A_{ij}A_{jl}A_l \\
\mathscr{F}_w &= (1-\delta_{k+N}^j)(1-\delta_{k+N-1}^l)A_{ij}A_{jl}.
\end{aligned}$$

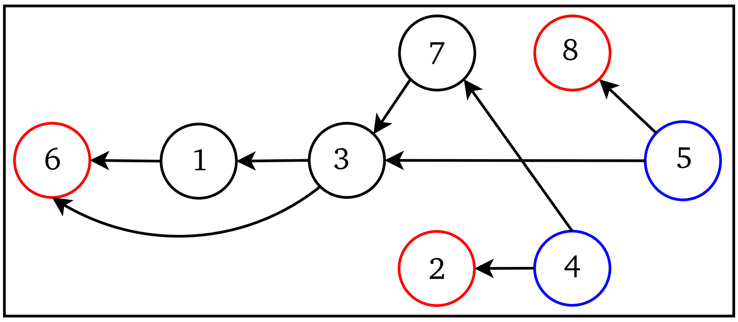

Figure 7.4: The DAG interconnection structure for the simulated NCS.

local dynamics (excluding the interconnection effect from nodes 4 and 5) are stable in terms of f-ergodicity. Similarly, stability of nodes 4 and 8 ensure stochastic stability of node 7, if and only if the local dynamics of node 7 is stable. We can continue this scenario until reaching the only-affected node 6. Following this approach enables us to look at the error state of each node independent of the neighboring effects. Now we are ready to present the main theorem of this paper.

Theorem 7.1. *Consider an interconnected NCS with N heterogeneous LTI stochastic sub-systems given in (7.1) and a communication channel subject to the capacity constraint $\sum_{i=1}^{N} \delta_k^i = 1$, and the control, estimation and scheduling laws given by (7.3), (7.5), and (7.8), respectively. Let the interconnections between sub-systems be modeled by a DAG $\mathscr{G}_c(\mathscr{V}, \mathscr{E})$. Then for any positive real λ_i's and positive definite Q^i's the Markov chain (7.10) is f-ergodic.*

Proof. See Appendix A.4. □

Remark 7.1. For general interconnection structures with undirected edges and cycles, the ergodicity of the error (3.9) cannot be guaranteed. This follows from the fact that the error of a node i at some prior time-steps might appear in its dynamics again in future through the neighboring nodes, which violates the Markov property. However, the DAG assumption on the control structure can be relaxed, which in turn introduces extra coupling terms in the system state dynamics x_k^i, i.e. in the first rows of the squared matrices in (7.7).

7.5 Numerical Evaluations

In this section, we evaluate the performance of our event-triggered bi-character scheduler for an NCS composed of both stable and unstable sub-systems which are physically interconnected according to a DAG structure. First, we illustrate that the variance of network-induced error states for each sub-system (node) remain bounded on average. Then, the performance of our proposed scheduling mechanism is compared with that of TDMA and CSMA policies. To illustrate the simulation results, we consider an interconnected NCS comprised of eight scalar sub-systems competing for the sole channel slot at each time-step. The DAG interconnection between sub-systems is depicted in Figure 7.4. In Figure 7.4 the red and blue nodes represent only-affected and only-affecting nodes in the NCS, respectively.

For the sake of illustration, we assume that the NCS is composed of two classes of homogeneous sub-systems. First class includes stable sub-systems and the second class contains

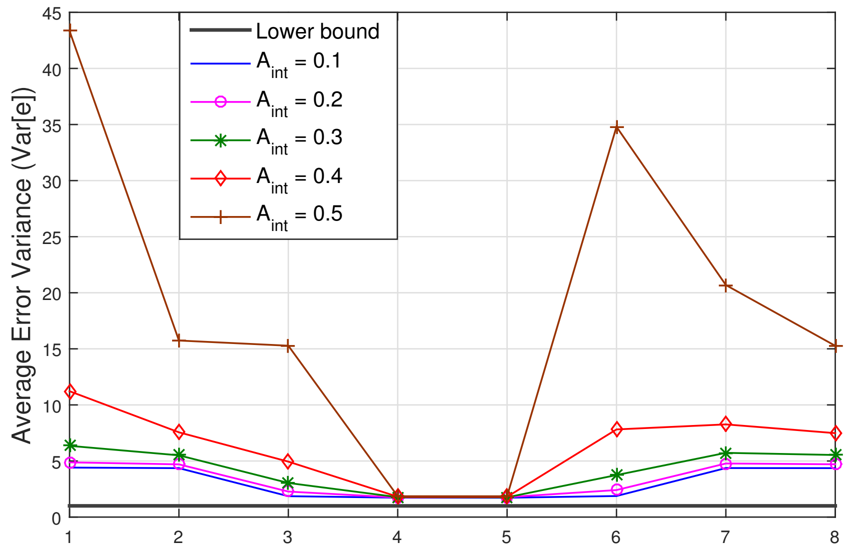

Figure 7.5: Error variances for each node of the DAG in Figure 7.4.

unstable sub-systems. System parameters within each class are assumed to be identical for all sub-systems. We consider that the nodes $\{3,4,5,6\}$ represent stable sub-systems with parameters $A_1 = 0.7$, $B_1 = 1$ and $\lambda_1 = 7$. The second class includes the remaining four sub-systems represented by the nodes $\{1,2,7,8\}$ with parameters $A_2 = 1.2$, $B_2 = 1$ and $\lambda_2 = 7$. Both classes are assumed to start from zero initial condition $x_0^1 = x_0^2 = 0$, and the noise sequences w_k^i is i.i.d. according to $\mathcal{N}(0,1)$. In addition, we assume to have a distributed control policy, meaning that each sub-system is steered by a state feedback controller having stabilizing gains L_i and L_{ij} from all neighboring nodes j. As it is ensured that a sub-system is closed-loop stable if it transmits, we can remove the effect of the system states, and focus on the error behavior over time. To do so, we choose a deadbeat control law to stabilize local sub-systems, having the form $L_i = A_i$ and $L_{ij} = A_{ij}$ for all i,j with the model-based observer (7.5). For the sake of simplicity, we select $Q_1 = Q_2 = I$ for both classes $\{1,2\}$. In addition, we assume that all interconnection strengths A_{ij}, if they exist, are identical, and are denoted by A_{int}. The following simulations results are derived under the scheduling law (7.8).

Figure 7.5 provides the average error variances for all nodes $\{1,\ldots,8\}$, under the interconnection structure depicted in Figure 7.4, for different interconnection strengths $A_{\text{int}} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$. It is worth mentioning that, since we have four sub-systems with unstable plants while only one transmission slot per time-step is available, at least three unstable sub-systems operate in open-loop at each time-step. The averages are calculated via Monte Carlo simulation over the horizon of 5×10^4 samples. The lower bound is derived by removing the capacity constraint, i.e. every sub-system transmits at every time-step. The simulation results indicate that the average error variances increase with increasing interconnection strength A_{int}. This is indeed an expected conclusion, since increasing A_{int} implies that the behavior of a sub-system is under increasingly influence of its direct and indirect neighboring sub-systems, which one or some of them might be unstable.

The error variances corresponding to nodes 4 and 5, which are the only-affecting nodes and consequently are under no neighboring effects, change only slightly with increasing A_{int}. The

slight increase in their corresponding variances follows from having the event-based scheduler which takes into account the real-time error values of each node in order to allocate the channel. In fact by increasing A_{int}, transmission chances for nodes 4 and 5 decrease because the other nodes which are under neighboring effect, would have an increased error and therefore higher chances of having channel access. As the channel access probabilities for the affected nodes increase, nodes 4 and 5 transmit less occasionally over time, which generally leads to an slight increase of their error values.
The error variance corresponding to the node 6 grows rapidly by increasing A_{int}, which results from the nodes 6 being the most affected node. In addition, the higher growth rate of error variances for the nodes $\{1,2,7,8\}$ follows from having unstable plants.

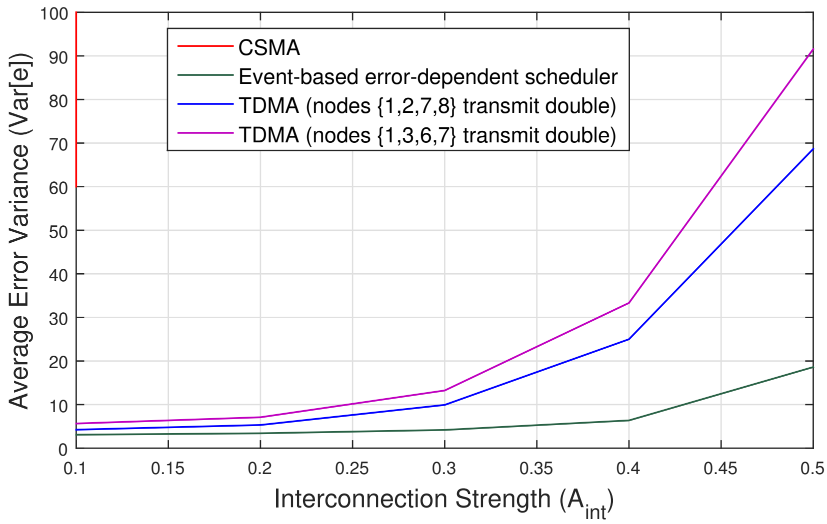

Figure 7.6: Comparison of error variances for different scheduling protocols.

In Figure 7.6, performance of our proposed event-triggered policy is compared with that of TDMA and uniform CSMA protocols, which are time-triggered and random access schemes, respectively. For the results to be fairer when compared with the TDMA approach, we consider two types of token rings. In the first scenario, all unstable sub-systems which are designated by nodes $\{1,2,7,8\}$ in Figure 7.4, are prioritized in an offline fashion by having double transmission occasions compared to the stable sub-systems, i.e. nodes $\{3,4,5,6\}$. Therefore, the token ring for this scenario repeats every twelve time-steps over which stable sub-systems transmit at four occasions and unstable ones transmit at eight occasions. The performance of this scenario is depicted by the solid blue curve in Figure 7.6. Within the next TDMA scenario, we assume that the sub-systems which are under the greatest interconnection effects transmit double than the rest of the sub-systems. To that end, nodes $\{1,3,6,7\}$ which represent the four most affected sub-systems transmit twice than the nodes $\{2,4,5,8\}$. This scenario is shown in Figure 7.6 by the solid violet curve. It can be seen that the former scenario provides better performance in terms of average aggregate error variance, and the performance gap becomes more evident as the interconnection strength increases. This is however an expected observation, because more transmission from unstable sub-systems mean more resetting in their respective errors which are also affecting the dynamics of other sub-systems through interconnections. Therefore, it is conjectured that having more transmission from

unstable nodes is more beneficial than more transmissions from most affecting nodes. This is however not a rule, as the performance of the TDMA approach is tightly coupled with the system parameters, such as system matrices A_i, interconnection matrices A_{ij}, noise variables, channel capacity, and the number of sub-systems.
Finding the global optimal pattern for the TDMA approach over infinite horizon is an NP-hard problem, while searching for near optimal patterns over finite horizon requires an exhaustive brute force search. Not only near optimal TDMA patterns are computationally expensive to be induced, but the patterns also are not extendable to different NCS setups as the tokens are highly dependent on system parameters, especially for interconnected systems. It can be seen from Figure 7.6 that the event-based prioritized scheme outperforms both TDMA scenarios in terms of average error variance. Moreover, the superiority of our proposed scheduling protocol is noticeable when the interconnection strength increases.
In addition, a comparison with the idealized CSMA protocol, which is a uniform random access scheme, is illustrated in Figure fig:plot2. Considering the uniform access probability $\frac{A_i^2}{\sum_{j=1}^{N} A_j^2}$ for each node $i \in \{1, \ldots, 8\}$ at each time-step, the average error variance for a low interconnection strength $A_{\text{int}} = 0.1$ has the order of 10^8. This is expected since employing CSMA results in a probable non-transmission state for a node with rapidly growing error.

7.6 Summary

In this chapter we addressed the event-triggered scheduling design for NCSs consisting of multiple heterogeneous control loops which are not only coupled through a shared communication medium, but are also physically interconnected. Design paradigms for these types of networked systems are often difficult due to the complex nature of their dynamic behavior which is not solely dependent on the characteristics of a local sub-system but also on the neighboring ones. We considered a special class of interconnected networked systems and proposed a centralized error-dependent scheduling mechanism capable of allocating the channel among physically interconnected control loops in a prioritized fashion.
Within this class of interconnected NCSs, the physical interconnections are assumed to follow DAG structure. This assumption implies two essential properties which enable us to look at the NCS as a cascaded system, from the highest hierarchy including the only-affecting nodes, to the lowest layer containing the only-affected nodes. These properties are as follows: 1) on a DAG, there exists no cycle, 2) if a node is a direct or indirect neighbor of another node, then the latter is neither a direct nor an indirect neighbor of the former node.
Under those properties, the Markov property of the network-induced error state is shown to be preserved, and furthermore, we proved f-ergodicity of each sub-system's local error state. Since, cascade of f-ergodic processes admits an f-ergodic cascaded process, stochastic stability of the overall interconnected NCS is concluded from convergence of local error states. Although the interconnection structure is assumed to follow DAGs, the control structure is allowed to have centralized, distributed, or decentralized architectures, depending on information topology. If the control of a sub-system has access to the information transmitted across the communication channel, e.g. having a broadcast channel, then the controllers can be synthesized centrally. In case, the control unit of a node has limited information access,

then distributed design is feasible. Otherwise, decentralized control can be applied.
Numerical evaluations illustrated that theoretical stability claim, in terms of boundedness of error variance for each node, is valid. Furthermore, we showed that the event-triggered approach leads to an enhanced performance compared with typical scheduling approaches such as TDMA and CSMA.

7.7 Contributions

Most of the presented analyses, results and discussions in this chapter are from the author's work in [119]. Figure 7.1 and graph representations in Figure 7.2 and Figure 7.3 demonstrated in Section 7.1 are also from the mentioned work. In addition, Figure 7.5, presented in Section 7.5, is from the same work by the author of this dissertation. The major efforts in this chapter are devoted to showing the efficacy of event-triggered scheduling rules for physically interconnected networked control systems. We first introduced a special class of interconnected NCSs wherein the interconnection topology is assumed to be a directed acyclic graph. Afterwards, we proposed control and scheduling laws taking into account the information pattern and physical interconnection topology. Focusing on the scheduling architecture and its performance efficiency, we proposed a centralized error-dependent scheduling mechanism possessing both deterministic and probabilistic features. The scheduler receives the error norms from each sub-system to decide which sub-system should transmit at each time-step. The probabilistic character of the proposed scheduler will be beneficial to implement it approximately in decentralized fashion. We already demonstrated this for physically isolated NCSs in this dissertation in Chapter 5. The same procedure is conjectured to be applicable, with more care about the interconnections, for the physically interconnected NCSs.
There exist numerous works on the distributed control of interconnected systems, see e.g. [118,120–123]. To the best of our knowledge, however, the event-based scheduling for interconnected NCSs such that the scheduler takes into account the real-time requirements of each control entity, has attracted very little attention, see some notable works in [49, 117, 124]. This area of research on NCSs is currently wide open and calls for the advent methodologies and novel applicable design approaches.
As an early step towards the event-triggered scheduling for interconnected NCSs, we discussed in this chapter that event-based approaches can preserve stability of interconnected networked systems, at least for special classes. The main advantage of employing event-based laws is their superior performance compared to the conventional static scheduling approaches, such as time-triggered and random access schemes.

8

Conclusions and Outlook

In this dissertation, the notion of state-dependent medium access control (MAC) protocols for multiple-loop networked control systems under the influence of limited communication resources is examined. We proposed error-dependent MAC architectures in various forms to comply with divers design challenges, for networked systems comprised of multiple stochastic linear time-invariant control loops sharing a common communication network. The novelty in the MAC design lies in the incorporation of error states in the MAC layer, where the error dependency appears in both deterministic and probabilistic fashions. MAC architectures with deterministic character often excel in efficient allocation of the scarce communication resources among the network entities. However, they usually lack the required flexibility to cope with the wider range of design challenges as deterministic mechanisms often come short with respect to scalability and robustness. In addition, implementation of deterministic approaches which are capable of prioritizing network access based on the real-time conditions of the channel consumers' conditions, are often computationally very expensive. Therefore, to take care of deterministic approaches' deficiencies, yet exploiting the offered benefits, we employed the concept of bi-character event-based scheduling architectures. First, the sub-systems which are eligible for having access to the communication channel are identified through an error-dependent deterministic threshold policy. Afterwards, the qualified sub-systems would compete for the scarce transmission opportunities and the competition is eventually decided according to a biased randomization. The bias term is error-dependent making the randomization process biased with respect to the real-time situations of the transmission-qualified sub-systems. In other words, higher channel access probabilities are assigned to sub-systems for with more stringent real-time conditions. In this dissertation, we have focused on stability guarantees and efficiency of the proposed MAC architectures. Since the considered control loops are driven by stochastic noise processes, we employed notions of stochastic stability, such as ergodicity, mean square stability, and Lyapunov stability in probability, to achieve stability certificates. We also demonstrated that event-triggered bi-character architectures are capable of being adjusted appropriately in order to excel in resource allocation problem under various design challenges such as data loss, and collisions. In addition, distributed implementation of the mentioned MAC strategies is thoroughly discussed to emphasize the applicability of the event-based MAC approaches. In what follows, we briefly summarize the main highlights and conclusions of each chapter of this dissertation.

8.1 Conclusions

Chapter 2

In this chapter, we studied the structural properties of multiple-loop networked control systems under resource constraints, considering feedback control laws and general form of scheduling architecture. The main conclusion in this chapter is the existence of separation property between the control law and scheduling law synthesis within the considered NCS framework. This facilitates designing appropriate mechanisms for generating the control inputs and allocating the scarce communication resources, independently. The control laws are synthesized via local state feedback controllers and it is shown that if the state information is updated in timely fashion, i.e. in the absence of communication constraints, the local control loops are stabilized. Furthermore, the event-based scheduling mechanism is discussed, and the events are introduced as the discrepancy between the real and estimated state values for each local control side. Finally, the overall NCS state is introduced as the aggregation of system states and error states from all sub-systems, and the NCS dynamics is derived.

Chapter 3

In this chapter, we introduced a pure probabilistic event-triggered scheduling mechanism for multi-loop NCSs constrained by capacity limitation. Within the proposed architecture, the channel access priorities are determined for each sub-system by a centralize scheduler such that higher access probabilities are assigned to the sub-systems with higher estimation errors. The final decision on which sub-systems transmit is taken via a biased randomization. The proposed scheduling scheme can be thought as the probabilistic counterpart of the deterministic MEF-TOD approach, which awards the channel to the sub-systems with highest errors. Stochastic stability of the overall NCS is studied under the proposed MAC design in terms of f-ergodicity of the overall network-induced error, which instead guarantees stability of system states under the capacity constraints. It is illustrated numerically that the suggested approach outperforms the conventional protocols such as TDMA and CSMA, in terms of aggregate error variance, and behaves robustly with growing size of the NCS.

Chapter 4

In this chapter, we introduced a bi-character MAC scheme by complementing the pure probabilistic scheduling approach with a deterministic mechanism. Within the new scheduling design, every sub-system initially decides whether to take part in channel access competition. This decision is taken according to a deterministic threshold policy such that the local events are compared with their pre-designed thresholds and the ones exceeding the thresholds are forwarded for transmission. Then, the access probabilities are assigned to each eligible sub-system and the biased randomization determines which sub-systems eventually transmit. The deterministic feature of the scheduling process makes the access competition less intense by excluding the sub-systems which are not in stringent transmission state. We showed that the separation property remains valid under the modified scheduler. Stochastic stability of the overall networked system is also guaranteed in terms of f-ergodicity. In addition, theoretical uniform upper-bounds are derived for an average cost function comprising of transmission cost and variance of the aggregate error state. Numerical comparison with pure probabilistic mechanism illustrate performance enhancement when employing the bi-character design.

Chapter 5

In chapter 5, we investigated robustness properties of the bi-character scheduling policy with respect to non-idealities of the communication channel. In addition, we analyzed the situation which the event-based scheduler is not updated with the latest event information in a timely manner. It is shown that the probabilistic feature of the bi-character scheduler is providing the required design flexibility to cope with such imperfections. Although the overall NCS performance is degraded under the mentioned non-idealities, stochastic stability is still guaranteed, but with larger margins. It is additionally addressed that the centralized scheduler is capable of allocating the scarce communication resources among sub-systems efficiently even if the event information is not updated regularly, or if the transmission requests are outdated. Packet dropouts are also shown to be dealt with by the bi-character scheduling law, considering a deterministic model of data packet dropouts.

Chapter 6

In this chapter, we introduced the procedure to implement the bi-character event-based scheduling policy in decentralized fashion. In many applications, centralized information is not accessible for control or scheduling units. To take this into account, we proposed a systematic scheduling approach capable of prioritizing the channel access based on the local real-time system states. First, it is locally determined that whether a sub-system is eligible to transmit, according to a deterministic threshold policy. Then, if a transmission attempt is approved, a random waiting time is selected from a local probability mass function with error-dependent expected value. After elapsing this time, the sub-system senses the channel and transmits, if the channel is sensed as idle, and backs off otherwise. Furthermore, we propose a similar approach but assuming that the local scheduling units have access only to the noise-deteriorated sensor measurements. Finally, we study event-based scheduling for multi-link channels where each transmitting node randomly selects one of the available channel links. We assume that the randomization is uniform though, i.e. a link is selected according to the ALOHA strategy. For the mentioned scenarios, stochastic stability is guaranteed by the notion of Lyapunov stability in probability, and it is further shown that appropriate tuning of the scheduling parameters is instrumental in lowering the collision rates.

Chapter 7

In this chapter, the concept of event-based data scheduling is extended for networked systems comprised of physically interconnected sub-systems. Due to complexity of the physically coupled NCSs, we analyzed the event-based scheduling architecture only for a special class of interconnection topology, wherein the interconnections follow directed acyclic graphs. Within DAG structure, a sub-system is allowed to be the neighbor of another sub-system only if the latter itself is not a direct or indirect neighbor of the former sub-system. This assumption facilitates the derivation of stability certificates by ensuring that Markov property remains intact under the interconnection effects. Moreover, under DAG structure, every path in the graph starting from the only-affecting node to the only-affected node, can be viewed as cascade of nodes where the previous one affects the one after, but not the vice-versa. Having this property, we can guarantee stochastic stability by looking at only local dynamics of each sub-system. Relaxing the assumption on the DAG structure results in the contravention of the Markov property in its original definition, and more importantly, cascading dynamics.

8.2 Outlook

Over more than a decade, numerous research is conducted on event-triggered approaches as an efficient methodology to govern control systems, especially for large-scale networked and cyber physical systems. Various practical and theoretical aspects of the event-triggered sampling have been addressed in the literature, yet the topic is still increasingly attentive. In this dissertation, we have focused towards a fundamental understanding of event-triggered methodology for efficient synthesis of medium access control strategies for networked control systems. However, there exist a vast amount of open challenges which need to be examined with care. Open problems are to be addressed to support the event-triggered methodology as a powerful approach for modern control and decision making topics. The so-far-developed results in this area suggest that event-triggered control has a great potential to excel in future of control research and application.
Several research directions can be followed in order to further develop the presented results in this dissertation. We summarize the main future research paths in the followings:

Optimal design of event-triggered prioritizing bi-character scheduling law

In the bi-character event-based scheduling design, the error thresholds are introduced as the local deterministic measures to determine whether a transmission request should be forwarded. It is shown numerically that the average variance of the aggregate error state is a convex function of the error thresholds. Furthermore, it was illustrated that the upper-bounds for the average aggregate error variance are also convex with respect to those thresholds. However, the upper-bounds are conservative and even if the optimal thresholds are derived from those upper-bounds, they are not global optimal values. This raises the challenge that whether the optimal error thresholds are theoretically attainable. Some works have considered deriving the optimal event triggers under simplifying assumptions, however in those works, the probabilistic channel arbitration takes the form of a uniform randomization.

State-dependent scheduling design for multi-link communication channels

In this dissertation, it is often assumed that the communication channel is composed of a single transmission link, or multiple dedicated channels. This ensures that there would not be any conflicts in selecting a transmission channel. However, this might not be a realistic scenario especially for wireless networks where the nodes usually select a channel randomly and consequently, collisions may take place. We tackled the challenge by proposing a threshold-based scheduling mechanism for a multi-link channel NCS, however, the eventual transmissions are determined through the non-prioritized slotted ALOHA mechanism. This raises the question that how an event-based prioritizing scheduling architecture can be implemented for multi-channel networks such that the collisions can be effectively taken care of.

General interconnection topology for physically coupled sub-systems

In chapter 7, we addressed the state-dependent scheduling for multiple-loop networked control systems wherein the control loops are coupled not only in the shared communication channel, but also through the physical interconnection links. However, the considered interconnection structure is assumed to be confined to the directed acyclic graph (DAG) topology. Although it is unlikely that the very same results we obtained for that special class of interconnected NCSs can be repeated for the generic interconnected structure, it is crucial to classify the other interconnection topologies and search for appropriate solution frameworks to extend the concept of event-based prioritizing scheduling for such highly coupled systems.

Realization of the event-based scheduling within available communication technology

The design methodologies proposed in this work would be of added excellency if they are applied to real experiments and available communication technologies. What we have shown in this dissertation, is an initial step to draw the attentions that state-dependency can be employed as a powerful concept to achieve enhanced quality of control. We made the first attempt by showing that our propositions are implementable in distributed fashion. The next step can be realizing the state-dependency for well-known schemes, e.g. the error dependency can be well accommodated in the IEEE 802 protocols through the back-off exponents. It is conjectured that, rather than increasing or decreasing the lengths of the back-off windows exponentially, it might be beneficial to determine the back-off windows according to real-time conditions of transmitting node, e.g. by some suitably defined error states.

Observability of stochastic NCSs with limited capacity

Considering the overall NCS as a single control system, a unique criterion for observability of such a control system, especially when the channel access is orchestrated by a stochastic governing unit, is missing in the literature. In fact, assuming that state of the overall networked system is the aggregation of local states from local sub-systems, then the observability firmly depends on the scheduling output. The structural conditions under which an overall NCS is called observable is still an open research question. Furthermore, if the scheduling mechanism possesses a stochastic feature, it is even more challenging to formulate observability. It is conjectured that under such a scenario, steady-state probabilities for observability can be provided, however, to our best knowledge, there exist no approved result on that.

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A

Appendices

A.1 Appendix to Chapter 4

A.1.1 Proof of Theorem 4.1

Proof. To study stochastic stability of the overall NCS state dynamics (3.2), we again assume that the NCS has operated over time-steps k to $k+N-1$ under the bi-character policy (4.1). Then, the last time-step $k+N$ is scheduled considering all possible scenarios that might have happened in the probability space $\mathbb{R}^n$ over horizon $[k, k+N-1]$. To this end, we define at every arbitrary time-step $k' \in [k, k+N]$, two time-varying disjoint sets of sub-systems, namely $S^1_{k'}$ and $S^2_{k'}$, such that every $i \in \{1, \dots, N\}$ is categorized in one set as follows:

$$i \in \begin{cases} S^1_{k'} & \text{if } \|e^i_{k'}\|^2_{Q^i} \leq \lambda_i \\ S^2_{k'} & \text{if } \|e^i_{k'}\|^2_{Q^i} > \lambda_i \end{cases}, \tag{A.1}$$

where it is straightforward to check $S^1_{k'} \cup S^2_{k'} = N$. According to (A.1), all eligible sub-systems for transmission at time-step $k'+1$ are included in the set $S^2_{k'}$, while $S^1_{k'}$ contains those which are deterministically excluded from channel access competition. These two sets are time-varying because the sub-systems might transit between them during time transition $k' \to k'+1$. However, not only a transmission necessarily results in error decrement, or on the other hand a non-transmission necessarily results in error increment, but also the random noise process with range $(-\infty, +\infty)$ might decrease or increase the error. Therefore, sub-systems' inclusion in either set $S^1_{k'}$ or $S^2_{k'}$ depends on both transmission occurrence and realization of the random noise process at time k'. To take this into account and due to the bi-character scheduling law (4.1), we discern three complementary and mutually exclusive cases to characterize the transition of a sub-system i up to time-step $k+N-1$ as:

A subsystem $i \in \{1, \dots, N\}$:

c_1: has either transmitted or not within the past $N-1$ time-steps, and $i \in S^1_{k+N-1}$, i.e.,

$$i \in S^1_{k+N-1} \quad \Rightarrow \quad \|e^i_{k+N-1}\|^2_{Q^i} \leq \lambda_i,$$

c_2: has transmitted at least once within the past $N-1$ time-steps, latest at $k+r'_i \in [k, k+N-1]$, and $i \in S^2_{k+N-1}$, i.e.,

$$\exists k' \in [k, k+N-1] : \delta^i_{k+r'_i} = 1 \text{ and } \delta^i_{[k+r'_i+1,\dots,k+N-1]} = 0 \text{ and } \|e^i_{k+N-1}\|^2_{Q^i} > \lambda_i,$$

c_3: has not transmitted within the past $N-1$ time-steps, and $i \in S^2_{k+N-1}$, i.e.,

$$\forall k' \in [k, k+N-1] : \delta^i_{k'} = 0 \text{ and } \|e^i_{k+N-1}\|^2_{Q^i} > \lambda_i.$$

Each subsystem is characterized by exactly one of the above disjoint cases, thus cardinality of the union of sub-systems belonging to c_1, c_2 and c_3 equals N.

We apply *Theorem* 3.1 for each introduced case c_1, c_2 and c_3 employing Lyapunov function (4.2). The multi-step drift (4.3) is split into partial drifts $\Delta V(e^{i\in c_l}_k, N)$ for each case $c_l, l \in \{1,2,3\}$, as follows:

$$\Delta V(e^{i\in c_l}_k, N) = \sum_{i\in c_l} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i} | e_k\Big] - V(e^{i\in c_l}_k), \tag{A.2}$$

where $V(e^{i\in c_l}_k) = \sum_{i\in c_l} \|e^i_k\|^2_{Q^i}$.

According to definition of the first case c_1, sub-systems $i \in c_1$ belong to the set S^1_{k+N-1}, which immediately implies $\|e^i_{k+N-1}\|^2_{Q^i} \le \lambda_i$, whether they have already transmitted or not. This implies $\delta^i_{k+N} = 0$ for all $i \in c_1$. Therefore, we have

$$\begin{aligned}\sum_{i\in c_1} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i} | e_k\Big] &= \sum_{i\in c_1} \mathsf{E}\Big[\|A_i e^i_{k+N-1} + w^i_{k+N-1}\|^2_{Q^i} | e_k\Big] \\ &\le \sum_{i\in c_1} \|A_i\|^2_2 \mathsf{E}\Big[\|e^i_{k+N-1}\|^2_{Q^i} | e_k\Big] + \text{tr}(Q^i W_i),\end{aligned}$$

where the above inequality is ensured according to the Cauchy-Schwarz rule. Since $\|e^i_{k+N-1}\|^2_{Q^i} \le \lambda_i$ for all $i \in c_1$, it follows that

$$\sum_{c_1} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i} | e_k\Big] \le \sum_{c_1} \lambda_i \|A_i\|^2_2 + \text{tr}(Q^i W_i). \tag{A.3}$$

According to (A.2), the partial drift for c_1 becomes

$$\Delta V(e^{i\in c_1}_k, N) \le \sum_{i\in c_1} \lambda_i \|A_i\|^2_2 + \text{tr}(Q^i W_i) - V(e^{c_1}_k).$$

Therefore, we define $f_{c_1}(e^{c_1}_k) = \epsilon_1 V(e^{c_1}_k) - \xi^{b^+}_1$, with positive and bounded term $\xi^{b^+}_1 = \sum_{c_1} \lambda_i \|A_i\|^2_2 + \text{tr}(Q^i W_i)$, and $\epsilon_1 \in (0,1]$. As already discussed in *Proposition* 3.3, compact sets of $\mathbb{R}^n$ are all small. Thus, we can find a small set $\mathcal{D}_1$ and a constant ϵ_1 such that $f_{c_1} \ge 1$ and $\Delta V(e^{i\in c_1}_k, N) \le -f_{c_1}$ for $e^{c_1}_k \notin \mathcal{D}_1$.

For a sub-system $i \in c_2$, it is assumed that the latest transmission has occurred at time-step $k + r'_i$, where $r'_i \in [0, N-1]$. This yields $\delta^i_{k+r'_i} = 1$ and $\delta^i_{[k+r'_i+1,\dots,k+N-1]} = 0$. Recalling the expression (3.9), and following the statistical independence of the noise sequence w^i_{k+r} and error vector $e^i_{k+r'_i}$, we have

$$\begin{aligned}\sum_{i\in c_2}\mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big] &= \sum_{i\in c_2}\mathsf{E}\left[\Big\|\prod_{d=r'_i}^{N}\big(1-\delta^i_{k+d}\big)A_i^{N-r'_i+1}e^i_{k+r'_i-1}\Big\|^2_{Q^i}|e_k\right]\\ &+\sum_{i\in c_2}\mathsf{E}\left[\Big\|\sum_{r=r'_i}^{N}\prod_{d=r+1}^{N}\big[1-\delta^i_{k+d}\big]A_i^{N-r}w^i_{k+r-1}\Big\|^2_{Q^i}\right]\\ &= \sum_{i\in c_2}\mathsf{E}\left[\Big\|\sum_{r=r'_i}^{N}A_i^{N-r}w^i_{k+r-1}\Big\|^2_{Q^i}\right]\\ &\le \sum_{i\in c_2}\sum_{r=r'_i}^{N}\|A_i^{N-r}\|^2_2\,\mathrm{tr}(Q^iW_i). \end{aligned} \tag{A.4}$$

From the multi-step drift definition (A.2), it follows

$$\Delta V(e_k^{i\in c_2},N)\le\sum_{i\in c_2}\sum_{r=r'_i}^{N}\|A_i^{N-r}\|^2_2\,\mathrm{tr}(Q^iW_i)-V(e_k^{c_2}).$$

We similarly define $f_{c_2}(e_k^{c_2}) = \epsilon_2 V(e_k^{c_2}) - \xi_2^{b^+}$, where positive and bounded term $\xi_2^{b^+} = \|A_i^{N-r}\|^2_2\,\mathrm{tr}(Q^iW_i)$, and $\epsilon_2 \in (0,1]$. Then we can find a small set $\mathscr{D}_2 \subset \mathbb{R}^n$ and a constant ϵ_2 such that $f_{c_2} \ge 1$ and $\Delta V(e_k^{c_2},N) \le -f_{c_2}$ for $e_k^{c_2} \notin \mathscr{D}_2$.

Sub-systems included in the third case are all submitting transmission request at the final time-step $k+N$, since $\|e^i_{k+N-1}\|^2_{Q^i} > \lambda_i$ for all $i \in c_3$. Moreover, they have never transmitted before, i.e. $\sum_{i\in c_3}\delta^i_{k'} = 0$ for all time-steps $k' \in [k, k+N-1]$. To invoke *Theorem* 3.1 in the third case, we split the case c_3 into two complementary and disjoint sub-cases, as follows:

$l_1^{c_3}$ A sub-system $i \in c_3$ has not transmitted within the past $N-1$ time steps, but has been in the set $S^1_{k+r'_i}$ at least once over that period, the latest at a time $k+r'_i \in [k, k+N-2]$,

$l_2^{c_3}$ A sub-system $i \in c_3$ has not transmitted within the past $N-1$ time-steps, and has been in the set $S^2_{\bar{k}}$ for all $\bar{k} \in [k, k+N-1]$.

Recall that sub-systems $i \in c_3$ belong to S^2_{k+N-1} at time-step $k+N-1$. For sub-system belonging to the sub-case $l_1^{c_3}$, suppose that $k+r'_i$ has been the last time-step for which $i \in S^1_{k+r'_i}$, which implies $\|e^i_{k+r'_i}\|^2_{Q^i} \le \lambda_i$. Knowing that $\delta^i_{k'} = 0$ for all $i \in c_3$ up to time-step $k+N$, and also recalling (3.9), we reach to

$$\sum_{i\in l_1^{c_3}} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big] = \sum_{i\in l_1^{c_3}} \mathsf{E}\left[\Big\|A_i^{N-r_i'} e^i_{k+r_i'} + \sum_{r=r_i'}^{N-1} A_i^{N-r-1} w^i_{k+r}\Big\|^2_{Q^i}\Big| e_k\right]$$
$$\leq \sum_{i\in l_1^{c_3}} \left[\|A_i^{N-r_i'}\|_2^2\, \mathsf{E}\Big[\|e^i_{k+r_i'}\|^2_{Q^i}|e_k\Big] + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2 \mathsf{E}\Big[\|w^i_{k+r}\|^2_{Q^i}\Big]\right]$$
$$\leq \sum_{i\in l_1^{c_3}} \left[\|A_i^{N-r_i'}\|_2^2 \lambda_i + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2\, \mathrm{tr}(Q^i W_i)\right]. \tag{A.5}$$

Substitute (A.5) in the partial drift (A.2), we get for the sub-case $l_1^{c_3}$

$$\Delta V(e_k^{i\in l_1^{c_3}}, N) \leq \sum_{i\in l_1^{c_3}} \left[\|A_i^{N-r_i'}\|_2^2 \lambda_i + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2\, \mathrm{tr}(Q^i W_i)\right] - V(e_k^{l_1^{c_3}}).$$

Define $f_{l_1^{c_3}}(e_k^{l_1^{c_3}}) = \epsilon_{l_1^{c_3}} V(e_k^{l_1^{c_3}}) - \xi^{b+}_{l_1^{c_3}}$, where $\xi^{b+}_{l_1^{c_3}} = \sum_{i\in l_1^{c_3}} \Big[\|A_i^{N-r_i'}\|_2^2 \lambda_i + \sum_{r=r_i'}^{N-1} \|A_i^{N-r-1}\|_2^2\, \mathrm{tr}(Q^i W_i)\Big]$, and $\epsilon_{l_1^{c_3}} \in (0,1]$. Hence, we can find an appropriate small set $\mathcal{D}_{l_1^{c_3}} \subset \mathbb{R}^n$ and an $\epsilon_{l_1^{c_3}}$ such that $f_{l_1^{c_3}} \geq 1$, and $\Delta V(e_k^{l_1^{c_3}}, N) \leq -f_{l_1^{c_3}}$ for $e_k^{l_1^{c_3}} \notin \mathcal{D}_{l_1^{c_3}}$.

In sub-case $l_2^{c_3}$, the belonging sub-systems i are always eligible candidates for channel access because $i \in S^2_{\bar{k}}$ for all $\bar{k} \in [k, k+N-1]$, although they have never transmitted due to the probabilistic nature of the scheduling policy (4.1), thus $\delta^i_{\bar{k}} = 0$ for all $i \in l_2^{c_3}$ and for all $\bar{k} \in [k, k+N-1]$. Knowing that $\|e^i_{k'}\|^2_{Q_i} > \lambda_i$ for all time-steps $k' \in [k, k+N-1]$, and recalling (3.9) with $r_i' = 0$, we reach to

$$\sum_{i\in l_2^{c_3}} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}|e_k\Big] = \sum_{i\in l_2^{c_3}} \mathsf{E}\left[\Big\|A_i^N e_k^i + \sum_{r=0}^{N-1}\big[A_i^{N-r-1} w^i_{k+r}\big]\Big\|^2_{Q^i}\Big| e_k\right]$$
$$\leq \sum_{i\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \mathsf{E}\Big[\|w^i_{k+r}\|^2_{Q^i}\Big] + \|A_i^N\|_2^2 \|e_k^i\|^2_{Q^i}$$
$$\leq \sum_{i\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2\, \mathrm{tr}(Q^i W_i) + \|A_i^N\|_2^2 V(e_k^{l_2^{c_3}}). \tag{A.6}$$

It follows from (A.6) that the upper-bound for $\sum_{i\in l_2^{c_3}} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}\Big]$ is dependent on the initial values e_k via the last term $V(e_k^{l_2^{c_3}})$, unlike the other cases c_1 and c_2 and the sub-case $l_1^{c_3}$ for which we calculated the uniform upper bounds for $\sum_{i\in c_l} \mathsf{E}\Big[\|e^i_{k+N}\|^2_{Q^i}\Big]$. As the introduced cases and sub-cases are mutually exclusive and complementary for the state-space $\mathbb{R}^n$, we can calculate the probability P_{c_l} for each case $c_1 - c_3$ according to the scheduling policy (4.1). Recalling from Section 3.3.2, the multi-step drift operator (3.8) can be expressed as follows when the probabilities P_{c_l} are associated:

$$\Delta V(e_k, N) = \sum_{c_l, l=1}^{3} \mathsf{P}_{c_l} \mathsf{E}\left[\sum_{i \in c_l} \|e^i_{k+N}\|^2_{Q^i} | e_k\right] - \sum_{i=1}^{N} \|e^i_k\|^2_{Q^i}, \tag{A.7}$$

where, P_{c_l} represents the occurrence probability of each case $\{c_1, c_2, c_3\}$, and $\sum_{l=1}^{3} \mathsf{P}_{c_j} = 1$. As the only sub-case for which we are not able to find a uniform upper-bound without calculating its occurrence probability is $l_2^{c_3}$, it suffices to calculate $\mathsf{P}_{l_2^{c_3}}$ in order to invoke *Theorem* 3.1. Recall that the length of the interval of interest equals N. The intuition is simple; if one sub-system, say j, is not granted the channel access during the entire interval $[k, k+N]$ while j has always been an eligible candidate, i.e. $j \in l_2^{c_3}$, then certainly exists another sub-system, say i, which transmits more than once (recall that all sub-systems belonging to c_3 never transmit over $[k, k+N-1]$). Let $k + \bar{r}$ denotes the most recent time-step at which sub-system i has transmitted, i.e. $\delta^i_{k+\bar{r}} = 1$. The probability that the very same sub-system i re-transmits at $k+N$, i.e. $\delta^i_{k+N} = 1$, in expense of blocking the sub-system $j \in l_2^{c_3}$ with no prior transmission at all time steps $\bar{k} \in [k, k+N-1]$, can be computed as

$$\begin{aligned}
&\mathsf{P}\left[\delta^i_{k+N} = 1 | \delta^i_{k+\bar{r}} = 1, \delta^j_{\bar{k}} = 0, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j, \forall \bar{k} \in [k, k+N-1]\right] \\
&\quad = \mathsf{E}\left[\mathsf{P}[\delta^i_{k+N} = 1 | e_k] | \delta^i_{k+\bar{r}} = 1, \delta^j_{\bar{k}} = 0, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j\right] \\
&\quad = \mathsf{E}\left[\frac{\|e^i_{k+N-1}\|^2_{Q^i}}{\sum_{j \in S^2} \|e^j_{k+N-1}\|^2_{Q^j}} \Big| \delta^i_{k+\bar{r}} = 1, \delta^j_{\bar{k}} = 0, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j\right],
\end{aligned}$$

where, the second line in above expression is obtained according to the *law of iterated expectations*. Here we consider a worst-case scenario within the sub-case $l_2^{c_3}$ such that the sub-system $j \in l_2^{c_3}$ have monotonically increasing error norms, i.e. between two consecutive time-steps $k' \to k'+1$, we assume $\|e^{j \in l_2^{c_3}}_{k'}\|^2_{Q^j} \leq \|e^{j \in l_2^{c_3}}_{k'+1}\|^2_{Q^j}$. Then, we arrive at

$$\begin{aligned}
&\mathsf{E}\left[\frac{\|e^i_{k+N-1}\|^2_{Q^i}}{\sum_{j \in S^2} \|e^j_{k+N-1}\|^2_{Q^j}} \Big| \delta^i_{k+\bar{r}} = 1, \delta^j_{\bar{k}} = 0, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j, \|e^{j \in l_2^{c_3}}_{k'}\|^2_{Q^j} \leq \|e^{j \in l_2^{c_3}}_{k'+1}\|^2_{Q^j}\right] \\
&= \mathsf{E}\left[\frac{\|e^i_{k+N-1}\|^2_{Q^i}}{\sum_{j \in c_2} \|e^j_{k+N-1}\|^2_{Q^j} + \sum_{j \in c_3} \|e^j_{k+N-1}\|^2_{Q^j}} \Big| \delta^i_{k+\bar{r}} = 1, \delta^j_{\bar{k}} = 0, \|e^{j \in l_2^{c_3}}_{k'}\|^2_{Q^j} \leq \|e^{j \in l_2^{c_3}}_{k'+1}\|^2_{Q^j}\right] \\
&\leq \mathsf{E}\left[\frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|^2_2 \operatorname{tr}(Q^i W_i)}{\sum_{j \in c_2} \lambda_j + \sum_{j \in l_1^{c_3}} \|e^j_{k+N-1}\|^2_{Q^j} + \sum_{j \in l_2^{c_3}} \|e^j_{k+N-1}\|^2_{Q^j}} \Big| \|e^{j \in l_2^{c_3}}_{k'}\|^2_{Q^j} \leq \|e^{j \in l_2^{c_3}}_{k'+1}\|^2_{Q^j}\right] \\
&\leq \mathsf{E}\left[\frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|^2_2 \operatorname{tr}(Q^i W_i)}{\sum_{j \in c_2} \lambda_j + \sum_{j \in l_1^{c_3}} \lambda_j + \sum_{j \in l_2^{c_3}} \|e^j_k\|^2_{Q^j}} \Big| e_k\right] \\
&= \frac{\sum_{r=\bar{r}}^{N-1} \|A_i^{N-r}\|^2_2 \operatorname{tr}(Q^i W_i)}{\sum_{j \in c_2} \lambda_j + \sum_{j \in l_1^{c_3}} \lambda_j + \sum_{j \in l_2^{c_3}} \|e^j_k\|^2_{Q^j}} = \mathsf{P}_{l_2^{c_3}}.
\end{aligned} \tag{A.8}$$

From (A.8) we can infer that the probability of subsequent transmissions for a certain sub-system i, in presence of sub-systems j with large error norms and without prior transmissions, can be made arbitrarily close to zero by tuning λ_j's and Q^j's. Intuitively, when a sub-system $j \in l_2^{c_3}$, its chance of transmission at the final time-step $k+N$ can be increased by raising the error thresholds λ_j of the other eligible-for-transmission candidates belonging to the cases c_2 and $l_1^{c_3}$ (recall that the sub-systems in set c_1 are not eligible candidates for transmission at time-step $K+N$.). By increasing the error threshold correspond to the aforementioned sub-systems $j \in c_2$ and $j \in l_1^{c_3}$, more of them are left out of the channel competition in favor of the sub-systems $j \in l_2^{c_3}$.
Recalling the upper-bound (A.6), and applying the probability (A.8) for the case $l_2^{c_3}$, we can find the upper bound for the partial drift $\Delta V(e_k^{j\in l_2^{c_3}}, N)$ as follows:

$$\begin{aligned}\Delta V(e_k^{j\in l_2^{c_3}}, N) &= \mathsf{P}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \mathsf{E}\Big[\|e^j_{k+N}\|^2_{Q^j}|e_k\Big] - V(e_k^{l_2^{c_3}}) \\ &\leq \mathsf{P}_{l_2^{c_3}} \left[\sum_{j\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_j^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j) + \|A_j^N\|_2^2 V(e_k^{l_2^{c_3}})\right] - V(e_k^{l_2^{c_3}}) \\ &\leq \left[\mathsf{P}_{l_2^{c_3}} \sum_{l_2^{c_3}} \|A_j^N\|_2^2 - 1\right] V(e_k^{l_2^{c_3}}) + \mathsf{P}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_j^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j). \end{aligned} \tag{A.9}$$

Similar to the previous cases, we now define $f_{l_2^{c_3}}(e_k^{l_2^{c_3}}) = \epsilon_{l_2^{c_3}} V(e_k^{l_2^{c_3}}) - \xi^{b+}_{l_2^{c_3}}$, where $\xi^{b+}_{l_2^{c_3}} = \mathsf{P}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_j^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j)$, and $\epsilon_{l_2^{c_3}} \in (0,1]$. It can be seen that we can find a small set $\mathscr{D}_{l_2^{c_3}}$ and $\epsilon_{l_2^{c_3}}$ such that $f_{l_2^{c_3}} \geq 1$, and $\Delta V(e_k^{j\in l_2^{c_3}}, N) \leq -f_{l_2^{c_3}}$ for $e_k^{l_2^{c_3}} \notin \mathscr{D}_{l_2^{c_3}}$. Intuitively, for large values of $e_k^{l_2^{c_3}}$, we have large $V(e_k^{l_2^{c_3}})$ and small $\mathsf{P}_{l_2^{c_3}}$, so $\mathscr{D}_{l_2^{c_3}}$ can be determined such that the condition $f_{l_2^{c_3}} \geq 1$ satisfies. On the other hand, to have a decreasing drift, the coefficient of $V(e_k^{l_2^{c_3}})$ in (A.9), i.e. $\Big[\mathsf{P}_{l_2^{c_3}} \sum_{l_2^{c_3}} \|A_j^N\|_2^2 - 1\Big]$ should be kept negative, which instead means that the small set $\mathscr{D}_{l_2^{c_3}}$ should be determined to be sufficiently large.

So far, the partial drifts for all cases c_1–c_3 are shown to be convergent. Therefore, the small set $\mathscr{D}_f \subset \mathbb{R}^n$ can be selected as $\mathscr{D}_f = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_{l_1^{c_3}} \cup \mathscr{D}_{l_2^{c_3}}$ ensuring that real-valued functions $f(e_k) \geq 1$ exist such that $\Delta V(e_k, N) \leq -\min f(e_k)$ for $e_k \in \mathbb{R}^n \backslash \mathscr{D}_f$, and $\Delta V(e_k, N) < \infty$ for $e_k \in \mathscr{D}_f$. This confirms the condition in *Theorem* 3.1 holds for multi-step drift (A.7), which readily proves that the Markov chain (2.17) is f-ergodic. □

A.2 Appendix to Chapter 5

A.2.1 Proof of Theorem 5.1

Proof. The cases c_1–c_3, and $l_1^{c_3}, l_2^{c_3}$, introduced in the proof of *Theorem* 4.1, can be considered again in this proof, if only the word "transmission" is replaced by "successful transmission". Thus the cases can be re-expressed as follows:
A sub-system $i \in \{1, \ldots, N\}$:

c_1: has either successfully transmitted or not within the past $N-1$ time-steps, and $i \in S^1_{k+N-1}$, i.e.,

$$i \in S^1_{k+N-1} \quad \Rightarrow \quad \|e^i_{k+N-1}\|^2_{Q^i} \leq \lambda_i,$$

c_2: has successfully transmitted at least once within the past $N-1$ time-steps, latest at $k+r'_i \in [k, k+N-1]$, and $i \in S^2_{k+N-1}$, i.e.,

$$\exists k' \in [k, k+N-1] : \delta^i_{k+r'_i}\gamma^i_{k+r'_i} = 1 \text{ and } \delta^i_{[k+r'_i+1,\dots,k+N-1]} = 0 \text{ and } \|e^i_{k+N-1}\|^2_{Q^i} > \lambda_i,$$

$l^{c_3}_1$ A sub-system $i \in c_3$ has not successfully transmitted within the past $N-1$ time steps, but has been in the set $S^1_{k+r'_i}$ at least once over that period, the latest at a time $k+r'_i \in [k, k+N-2]$,

$l^{c_3}_2$ A sub-system $i \in c_3$ has not successfully transmitted within the past $N-1$ time-steps, and has been in the set $S^2_{\bar{k}}$ for all $\bar{k} \in [k, k+N-1]$.

It is straightforward to check that we merely need to investigate the sub-case $l^{c_3}_2$, as the other cases can be uniformly upper bounded independently of the scheduling process. This conclusion can be observed through the previously obtained expressions (A.3), (A.4), and (A.5) for the cases c_1 and c_2 and the sub-case $l^{c_3}_1$, respectively. The intuition is that within those cases there exist *a priori* knowledge about the error Markov chain at a specific prior time-step. In fact, it is known that either the square norm of error of a specific sub-system i is either below the threshold λ_i or gets reset at some time-steps. Since the error evolves as a homogeneous Markov chain, driven by the variance-bounded Gaussian noise process, it suffices to know that the Markov chain has been bounded at a prior time-step in order to show boundedness of the error expectation in the future over a finite horizon. Remind that for the sub-systems belonging to the set c_2, at least a successful transmission should occur, thus if a sub-system is scheduled for transmission and, at all transmission times, the corresponding packets are dropped, then that sub-system non longer belongs to the set c_2.

Recall the inequality (A.6) for the expectation of square weighted error norm correspond to sub-systems belonging to the sub-case $l^{c_3}_2$:

$$\sum_{i \in l^{c_3}_2} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i} | e_k\right] \leq \sum_{i \in l^{c_3}_2} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|^2_2 \operatorname{tr}(Q^i W_i) + \|A_i^N\|^2_2 V(e_k^{l^{c_3}_2}). \tag{A.10}$$

We similarly calculate the probability that a sub-system which belongs to the sub-case $l^{c_3}_2$, remains in that sub-case for the final time-step $k+N$, considering that m scheduled packets for transmission are dropped over the time-interval $[k, k+N]$. The idea to employ the f-norm ergodic theorem is whenever a packet is dropped, we assume that a virtual control loop has successfully transmitted instead of the sub-system for whom the scheduled data packet is dropped. Basically, when a dropout occurs, it is assumed that N real and one virtual sub-systems share the communication channel, and the channel is awarded to the virtual sub-system. The virtual control loops are assumed to have the same discrete LTI dynamics as described in (2.8) with appropriately chosen system parameters. Over the interval $[k, k+N]$, we assume having as many virtual loops connected to the NCS as the dropped packets. Notice that this scenario only affects the scheduling mechanism, as we expect a dropout does affect the assigned probabilities. Hence, extending the interval length over which we evaluate the Markov chain behavior, is merely the consequence of having m dropouts, which requires lengthier intervals in order to imagine non-zero chances of transmission for every real sub-system. The virtual sub-systems incur no extra cost in the average cost function, and the increased cost is the direct consequence of having no transmission when a scheduled data packet is dropped.

Let the communication channel experiences m dropouts over the interval $[k, k+N-1]$ such that at least one transmission has been successful. Thus, at time-step $k+N$ we have N real and m virtual sub-systems, where all virtual sub-systems have transmitted. Similar to the proof of *Theorem* 4.1, the probability that a sub-system i which has successfully transmitted at least once at earlier time-steps, re-transmits at time-step $k+N+m$ in the presence of sub-systems $j \in l_2^{c_3}$, while m packets are already dropped can be computed as follows:

$$\begin{aligned}
&\mathsf{P}[\delta^i_{k+N+m} = 1 | \delta^i_{k+\bar{r}} = 1, e^j_k, m, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j] \\
&\quad = \mathsf{E}\left[\frac{\|e^i_{k+k_{d_m}}\|^2_{Q^i}}{\sum_{j\in S^2} \|e^j_{k+k_{d_m}}\|^2_{Q^j}} \Big| e^j_k, m, \delta^i_{k+\bar{r}} = 1, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j \right] \\
&\quad \leq \mathsf{E}\left[\frac{\| \sum_{r=\bar{r}}^{N+m-1} A_i^{N-r} w^i_{k+r-1}\|^2_{Q^i}}{\sum_{j\in c_2} \|e^j_{k+k_{d_m}}\|^2_{Q^j} + \sum_{j\in c_3} \|e^j_{k+k_{d_m}}\|^2_{Q^j}} \Big| e^j_k, m, \delta^i_{k+\bar{r}} = 1, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j \right] \\
&\quad \leq \frac{\sum_{r=\bar{r}}^{N+m-1} \mathsf{E}\left[\|A_i^{N-r} w^i_{k+r-1}\|^2_{Q^i}\right]}{\sum_{j\in c_2} \lambda_j + \sum_{j\in l_1^{c_3}} \lambda_j + \sum_{j\in l_2^{c_3}} \|e^j_k\|^2_{Q^j}} = \tilde{\mathsf{P}}_{l_2^{c_3}}
\end{aligned} \tag{A.11}$$

where, $k_{d_m} = N+m-1$. The upper-bound (A.11) is obtained considering the worst-case scenario, i.e. $\|e^{j\in l_2^{c_3}}_{k'}\|^2_{Q^j} \leq \|e^{j\in l_2^{c_3}}_{k'+1}\|^2_{Q^j}$ for all time-steps $k' \in [k, k+N+m-1]$. In addition, we take the inequality $\sum_{l_1^{c_3}} \|e^j_{k+N+m-1}\|^2_{Q^j} > \sum_{l_1^{c_3}} \lambda_j$, which results from inclusion in the sub-case $l_1^{c_3}$. It can be seen from (A.11) that in case of having m packets being dropped, the probability that a sub-system $j \in l_2^{c_3}$ is expelled from transmission can still be made arbitrarily close to zero, but over a larger horizon $[k, k+N+m]$. Recall that the f-ergodicity of the Markov chain (2.17) for the other cases c_1, c_2, and $l_1^{c_3}$ are shown according to a negative drift over N time-step horizon, which ensures that f-ergodicity holds for those cases even if the drift is defined over any longer interval. Finally, employing (5.2), we calculating the $N+m$-step drift for the sub-case $l_2^{c_3}$ as follows:

$$\begin{aligned}
\Delta V(e_k^{j\in l_2^{c_3}}, N+m) &= \tilde{\mathsf{P}}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \mathsf{E}\left[\|e^j_{k+N+m}\|^2_{Q^j} | e_k\right] - V(e_k^{l_2^{c_3}}) \\
&\leq \left[\tilde{\mathsf{P}}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \|A_j^{N+m}\|_2^2 - 1 \right] V(e_k^{l_2^{c_3}}) + \tilde{\mathsf{P}}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \sum_{r=0}^{N+m-1} \|A_j^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j)
\end{aligned}$$

Define $\tilde{f}_{l_2^{c_3}} = \tilde{\epsilon}_{l_2^{c_3}} V(e_k^{l_2^{c_3}}) - \tilde{\xi}^{b^+}_{l_2^{c_3}}$, where $\tilde{\xi}^{b^+}_{l_2^{c_3}} = \tilde{\mathsf{P}}_{l_2^{c_3}} \sum_{j\in l_2^{c_3}} \sum_{r=0}^{N+m-1} \|A_j^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j)$ and $\tilde{\epsilon}_{l_2^{c_3}} \in (0,1]$. Then, we can find a small set $\tilde{\mathscr{D}}_{l_2^{c_3}}$ such that $\tilde{f}_{l_2^{c_3}} \geq 1$, and $\Delta V(e_k^{l_2^{c_3}}, N+m) \leq -\tilde{f}_{l_2^{c_3}}$ for $e_k^{l_2^{c_3}} \notin \tilde{\mathscr{D}}_{l_2^{c_3}}$. This confirms that the conditions in *Theorem* 3.1 hold for the sub-systems belonging to the sub-case $l_2^{c_3}$ over the horizon $[k, k+N+m]$. Hence, the f-norm ergodic theorem ensures the ergodicity of the Markov chain (2.17) with the drift defined over the interval of length $N+m$, in case of m packet dropouts, and the proof then readily follows. □

A.2.2 Proof of Theorem 5.2

Proof. Similar to the proof of *Theorem* 5.1, we can define the cases $c_1 - c_3$ and sub-cases $l_1^{c_3}$ and $l_2^{c_3}$, and it is then straightforward to check that only sub-case $l_2^{c_3}$ needs to be re-considered when the time-delays are incorporated in the scheduling design. This follows since the delay variables d_k^i's only appear in the scheduling law and solely affect the assigned priorities. The obtained uniform upper-bounds (A.3), (A.4), and (A.5) for the weighted error variance at time $k+N$ for the cases c_1 and c_2 and sub-case $l_1^{c_3}$, remain unchanged within the new scheduling set up in this theorem.
For the sub-case $l_2^{c_3}$, we already obtained the following bound on the expectation of the weighted error norm in N time-step ahead as follows:

$$\sum_{j\in l_2^{c_3}} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i}|e_k\right] \leq \sum_{j\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j) + \|A_i^N\|_2^2 V(e_k^{l_2^{c_3}})$$

Dependency of the above expression on the initial values e_k^i via the last term implies that an appropriate $f_{l_2^{c_3}}$ to invoke *Theorem* 3.1 cannot be found such that the conditions of the theorem are fulfilled. In a similar approach and due to the fact that the considered cases are mutually exclusive, we calculate the probability that a sub-system ends up belonging to the sub-case $l_2^{c_3}$ over the entire interval $[k, k+N]$, according to the scheduling policy (5.3). Moreover, assume that no packet dropout happen.
Assume that the scheduler is about to assign the priorities at the final time-step $k+N$, while $e^i_{k+N-d^i_{k+N-1}-1}$ is the latest received event information at the scheduler from sub-system i, with $d^i_{k+N-1} < N$. Recall that the sub-systems in the set c_1 are not eligible for transmission at time-step $k+N$, thus if a sub-system $i \notin l_2^{c_3}$ transmits in the presence of sub-systems $j \in l_2^{c_3}$, then $i \in c_2$ or $i \in l_1^{c_3}$. We consider the worst-case scenario by assuming that the sub-systems included in $l_2^{c_3}$ satisfy $\|e^j_{k'}\|^2_{Q^j} \leq \|e^j_{k'+1}\|^2_{Q^j}$. In fact, the worst case scenario considers $j \in l_2^{c_3}$ with their respective errors monotonically increasing from one time-step to the next. Assume $k+\bar{r} < k+N$ is the latest time-step at which sub-system $i \in c_2$ has transmitted i.e. $\delta^i_{k+\bar{r}} = 1$. Thus, the probability of re-transmission for sub-system $i \in c_2$ considering delayed information update is computed as follows:

$$\begin{aligned}
&\mathsf{P}[\delta^i_{k+N} = 1|\delta^i_{k+\bar{r}} = 1, d^j_{k+N-1}, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j] \\
&\quad = \mathsf{E}\left[\frac{\|e^j_{k+k_{d_i}}\|^2_{Q^i}}{\sum_{j\in c_2}\|e^i_{k+k_{d_i}}\|^2_{Q^i} + \sum_{j\in c_3}\|e^j_{k+k_{d_j}}\|^2_{Q^j}}\Big|\delta^i_{k+\bar{r}} = 1, d^j_{k+N-1}, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j\right] \\
&\quad \leq \mathsf{E}\left[\frac{\|\sum_{r=\bar{r}}^{N-1} A_i^{k_{d_i}-r} w^i_{k+r}\|^2_{Q^i}}{\sum_{j\in c_2}\lambda_j + \sum_{j\in l_1^{c_3}}\|e^j_{k+k_{d_j}}\|^2_{Q^j} + \sum_{j\in l_2^{c_3}}\|e^j_{k+k_{d_j}}\|^2_{Q^j}}\Big|\delta^i_{k+\bar{r}} = 1, d^j_{k+N-1}, \|e^j_{\bar{k}}\|^2_{Q^j} > \lambda_j\right] \\
&\quad \leq \frac{\sum_{r=\bar{r}}^{N-1}\|A_i^{k_{d_i}-r}\|_2^2 \operatorname{tr}(Q^i W_i)}{\sum_{c_2}\lambda_i + \sum_{l_1^{c_3}}\lambda_j + \sum_{l_2^{c_3}}\|e^j_k\|^2_{Q^j}} = \mathsf{P}_{l_2^{c_3}}.
\end{aligned} \tag{A.12}$$

where, $k_{d_i} = N - d^i_{k+N-1}$. From (A.12) one infers that the probability of subsequent transmission for a certain sub-system, in the presence of other competitors with large errors and without prior transmissions, can be made arbitrarily close to zero by tuning λ_j's and Q^j's.

Intuitively, in case $d^i_{k'} \neq 0$, the error thresholds should be increased further (in comparison with the up-to-date information case $d^i_{k'} = 0$) in order to make the probability of happening the sub-case $l_2^{c_3}$ arbitrarily close to zero.
Accompanying the obtained probability $\mathsf{P}_{l_2^{c_3}}$, it follows for sub-case $l_2^{c_3}$

$$\mathsf{P}_{l_2^{c_3}} \Delta V(e_k^{j \in l_2^{c_3}}, N) \leq \left[\mathsf{P}_{l_2^{c_3}} \|A_j^N\|_2^2 - 1\right] V(e_k^{l_2^{c_3}}) + \xi^{b+}_{l_2^{c_3}},$$

where $\xi^{b+}_{l_2^{c_3}} = \mathsf{P}_{l_2^{c_3}} \sum_{j \in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^j W_j)$. We now define $f_{l_2^{c_3}} = \epsilon_{l_2^{c_3}} V(e_k^{l_2^{c_3}}) - \xi^{b+}_{l_2^{c_3}}$, with $\epsilon_{l_2^{c_3}} \in (0,1]$. Similarly, We can find a small set $\mathscr{D}_{l_2^{c_3}}$ and $\epsilon_{l_2^{c_3}}$ such that $f_{l_2^{c_3}} \geq 1$, and $\Delta V(e_k^{l_2^{c_3}}, N) \leq -f_{l_2^{c_3}}$ for $e_k^{l_2^{c_3}} \notin \tilde{\mathscr{D}}_{l_2^{c_3}}$. This confirms that the condition in *Theorem* 3.1 holds, and f-ergodicity of Markov chain (2.17) is then readily followed. ☐

Remark A.1. It is worth observing how the probability (A.12) changes with delay time $d^i_{k'}$ depending on whether the delay corresponds to a stable sub-system or an unstable one. If the plant $\mathscr{P}^i$ is stable, then longer delay results in larger values in the numerator. Thus, we need larger error thresholds λ_i for $i \in c_2$ and λ_j for $j \in l_1^{c_3}$, in comparison with $d^i_{k'} = 0$, in the denominator to make $\mathsf{P}_{l_2^{c_3}}$ arbitrarily close to zero. This observation is expected as a stable plant often has lower chance to access the channel in the presence of sub-systems with unstable plants and no prior transmission.

A.3 Appendix to Chapter 6

A.3.1 Proof of Theorem 6.2

Proof. As discussed in *Lemma* 6.2, LSP of the overall NCS state $[x_k^\mathsf{T}, e_k^\mathsf{T}]^\mathsf{T}$ follows if the aggregate error state e_k possesses LSP. Furthermore, it is shown in (6.33) that LSP of the error state defined in (6.28) can alternatively be investigated by searching for a uniform upper-bound for the expectation of the weighted error norm. In addition, we discussed that LSP is supposed to be shown by monitoring the evolution of error state over an interval with length N, i.e. assuming time-step k as an arbitrary initial sample time, convergence of error state is addressed over the interval $[k, k+N]$. The following expression enables us to evaluate dynamic behavior of a local error state e_k^i while transiting up to time-step $k+N$, by stepping backward in time to any prior time $k+r_i$, with $r_i \in [0, N-1]$:

$$\begin{aligned} e^i_{k+N} &= \prod\nolimits_{d_1=0}^{N-r_i-1} \left(\Delta^i_{k+N-d_1}\bar{A}^c_i\right)\left(e^i_{k+r_i} - v^i_{k+r_i}\right) \\ &\quad + \sum\nolimits_{q_1=r_i}^{N-1} \prod\nolimits_{d_2=0}^{N-q_1-2} \left(\Delta^i_{k+N-d_2}\bar{A}^c_i\right)\Delta^i_{k+q_1+1} C_i w^i_{k+q_1} \\ &\quad + \sum\nolimits_{q_2=r_i+1}^{N-1} \prod\nolimits_{d_3=0}^{N-q_2-1} \left(\Delta^i_{k+N-d_3}\bar{A}^c_i\right)\left(\Delta^i_{k+q_2} - I\right) v^i_{k+q_2} + \Delta^i_{k+N} v^i_{k+N}, \end{aligned} \tag{A.13}$$

where, $\Delta^i_{k'} = I - \theta^i_{k'} C_i K^i_{k'}$, and $\bar{A}^c_i = C_i A_i C_i^{-1}$, and we define $\prod_0^{-1}(\cdot) := 1$, and $\sum_N^{N-1}(\cdot) := 0$. We first analyze the case that successive collisions occur over the interval $[k, k+N]$, i.e. $\theta^i_{k'} = 0$ for all $i \in \{1, \dots, N\}$ at all time-steps $k' \in [k, k+N]$. Let $r_i = 0$, then from (A.13) we have $e^i_{k+N} = \bar{A}_i^{c^{N-r_i-1}}\left(e^i_k - v^i_k\right) + \sum_{q_1=0}^{N-1} \bar{A}_i^{c^{N-q_1-2}} C_i w^i_{k+q_1} + v^i_{k+N}$. The probability that such a scenario occurs is computed in (6.22). Setting $\bar{\xi}' = \sum_{i=1}^N \|\bar{A}_i^{c^{N-r_i-1}}\left(e^i_k - v^i_k\right) + \sum_{q_1=0}^{N-1}\bar{A}_i^{c^{N-q_1-2}} C_i w^i_{k+q_1} + v^i_{k+N}\|^2_{Q^i}$, it follows from (6.15)

$$\sup_{e_k} \mathsf{P}\left[\sum_{i=1}^{N} \|e^i_{k+N}\|^2_{Q^i} \geq \bar{\xi}'\right] < \prod_{k'=k}^{k+N} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \frac{\alpha^j_{k'}}{s^i_{k'} \tau \lambda_j},$$

which ensures Lyapunov stability with probability (6.22). In the rest of this proof, we assume that not all data packets collide over the interval $[k, k+N]$. Behavior of a sub-system i while transiting from k to $k+N$ can be characterized by one of the following three complementary and mutually exclusive occurrences, covering the entire state space. First, a sub-system i is in the set $\bar{\mathscr{G}}_{k'} := \mathbf{N} - \mathscr{G}_{k'}$, at least in one occasion over $[k, k+N]$, where $\mathbf{N}$ is the set of all sub-systems $\{1, \dots, N\}$ and $\mathscr{G}_{k'}$ is introduced in (6.8). Second, a sub-system i is always in $\mathscr{G}_{k'}$, but has successfully transmitted at least once over $[k, k+N]$. Third, over the same period, a sub-system i is always in the set $\mathscr{G}_{k'}$, and has never successfully transmitted.
For the first scenario, denoted as s_1, assume that the time-step $k+r_i$ is the latest occasion such that $i \in \bar{\mathscr{G}}_{k+r_i}$. This implies $\|e^i_{k+r_i}\|^2_{Q^i} \leq \lambda_i$. From (A.13), and due to mutual statistical independence of $e^i_{k+r_i}$, $w^i_{k+q_1}$, $v^i_{k+q_2}$, and v^i_{k+N}, and in addition, noting that $I - C_i K^i_{k'}$ is positive definite, the following upper-bound can be obtained for all $i \in s_1$

$$\begin{aligned} \sum\nolimits_{i\in s_1} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i} | e^i_k\right] &\leq \sum\nolimits_{i\in s_1} \|\bar{A}_i^{c^{N-r_i-1}}\|^2_2 \, \mathsf{E}\left[\|e^i_{k+r_i} - v^i_{k+r_i}\|^2_{Q^i} | e^i_k\right] \\ &\quad + \sum\nolimits_{i\in s_1} \sum\nolimits_{q_1=r_i}^{N-1} \mathrm{tr}(Q^i W_i) \|\bar{A}_i^{c^{N-q_1-2}} C_i\|^2_2 + \mathrm{tr}(Q^i V_i) \\ &\leq \sum\nolimits_{i\in s_1} \|\bar{A}_i^{c^{N-r_i-1}}\|^2_2 \left[\lambda_i + \mathrm{tr}(Q^i V_i) + 2\sqrt{\mathrm{tr}(Q^i V_i)\lambda_i}\right] + \zeta^i_b \end{aligned} \tag{A.14}$$

where, ζ_b^i stands for the second line of the above expression. LSP condition (6.15) is then fulfilled with $\bar{\xi}'$ selected larger than (A.14) and $\bar{\xi} = \frac{\sum_{i\in s_1} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i}|e^i_k\right]}{\bar{\xi}'} < 1$.
For sub-systems i experiencing the second scenario, denoted as s_2, assume $k+r_i$ is the latest time-step of a corresponding successful transmission, i.e. $\theta^i_{k+r_i} = 1$. Hence, $\theta^i_{k+r_{ii}} = 0$ for all $r_i < r_{ii} \leq N$. It follows from (A.13)

$$\begin{aligned}\sum\nolimits_{i\in s_2} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i}|e^i_k\right] \leq & \sum\nolimits_{i\in s_2} \|\bar{A}_i^{c^{N-r_i-1}}\|_2^2\, \mathsf{E}\left[\|e^i_{k+r_i} - \nu^i_{k+r_i}\|^2_{Q^i}|e^i_k\right] \\ & + \sum\nolimits_{i\in s_2}\sum\nolimits_{q_1=r_i}^{N-1} \mathrm{tr}(Q^i W_i)\|\bar{A}_i^{c^{N-q_1-2}} C_i\|_2^2 + \mathrm{tr}(Q^i V_i) \\ \leq & \sum\nolimits_{i\in s_2} \|\bar{A}_i^{c^{N-r_i-1}}\|_2^2\, \mathrm{tr}(Q^i P^i_{k+r_i} C_i^\top C_i) + \zeta_b^i,\end{aligned} \tag{A.15}$$

where, $P^i_{k+r_i}$ is the *a posteriori* covariance matrix of estimation error $x^i_{k+r_i} - \hat{x}^i_{k+r_i}$. Upper-bound (A.15) follows from the Kalman filter properties ensuring that if $\theta^i_{k+r_i} = 1$, the followings hold:

$$\begin{aligned}\mathsf{E}\left[e^i_{k+r_i} - \nu^i_{k+r_i}\right] &= C_i\, \mathsf{E}\left[x^i_{k+r_i} - \hat{x}^i_{k+r_i}\right] = 0, \\ \mathsf{E}\left[\|e^i_{k+r_i} - \nu^i_{k+r_i}\|^2_{Q^i}\right] &= \mathrm{tr}(Q^i Cov[e^i_{k+r_i} - \nu^i_{k+r_i}]).\end{aligned}$$

Thus, LSP condition (6.15) satisfies with $\bar{\xi}'$ selected to be larger than (A.15) and $\bar{\xi} = \frac{\sum_{i\in s_2} \mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i}|e^i_k\right]}{\bar{\xi}'} < 1$.
Denoting the third scenario as s_3, all sub-systems $i \in s_3$ never transmit over the interval $[k, k+N]$ while all of them are in $\mathcal{G}_{k'}$ for all $k' \in [k, k+N]$. This implies that a uniform upper-bound for $\mathsf{E}\left[\|e^i_{k+N}\|^2_{Q^i}\right]$ cannot be found since $\|e^i_{k'}\|^2_{Q^i} > \lambda_i$ for all k'. However, as the introduced scenarios $s_1 - s_3$ are mutually exclusive, we can compute the probability that a sub-system i remains in the set s_3 for all time-steps $[k, k+N]$, according to the MAC protocol (6.32). Assuming not all transmission attempts fail due to collisions, a sub-system $i \in s_3$ never transmits successfully due to either never finding the channel free, or having collisions whenever the channel is sensed idle. Recall the similar scenario of having virtual sub-systems, discussed in the proof of *Theorem* 6.1, and let the channel experiences $m < N$ collisions over the interval $[k, k+N]$. Having $N+m$ sub-systems requires to extend our operational interval to $[k, k_m]$, where $k_m = k+N+m$. For a sub-system i to be in s_3, it is required that another sub-system, say j, which has already transmitted, re-transmits at the final time k_m. Let $k+r_j$ denote the latest step at which $\theta^j_{k+r_j} = 1$ before the final step k_m. The probability that sub-system j re-transmits at k_m, in the presence of sub-system i with no prior transmission, can be calculated as

$$\begin{aligned}\mathsf{P}[\theta^j_{k_m} &= 1|\theta^j_{k+r_j} = 1, \theta^i_{k'} = 0, \quad \forall k' \in [k, k_m]] \\ &= \mathsf{P}[\nu^i_{k_m-1} > \nu^j_{k_m-1}|\theta^j_{k+r_j} = 1, \theta^i_{k'} = 0 \quad \forall k' \in [k, k_m]] \\ &\leq \frac{\mathsf{E}[\nu^i_{k_m-1}|\theta^i_{k'} = 0, \quad \forall k' \in [k, k_m]]}{s^j_{k_m-1}\tau},\end{aligned} \tag{A.16}$$

where, the last expression follows from *Markov's inequality*, and $\nu^j_{k_m-1}=s^j_{k_m-1}\tau$ for arbitrary $s^j_{k_m-1}\in\{1,\dots,h-1\}$. Exploiting the *law of iterated expectation*, we can compute (A.16), given the latest error vector $e^i_{k_m-1}$, and from (6.31), as

$$\frac{\mathsf{E}[\mathsf{E}[\nu^i_{k_m-1}|e^i_{k_m-1}]|\theta^i_{k'}=0]}{s^j_{k_m-1}\tau}=\frac{\alpha^i_{k_m-1}}{s^j_{k_m-1}\tau\|e^i_{k_m-1}\|^2_{Q^i}}=\mathsf{P}_{s_3}\,. \tag{A.17}$$

Incorporating the probability obtained in (A.17), we have for a sub-system $i\in s_3$ that

$$\begin{aligned}
\sum\nolimits_{i\in s_3}\mathsf{P}_{s_3}\,\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}|e^i_{k_m-1}\Big]&=\sum\nolimits_{i\in s_3}\mathsf{P}[\nu^i_{k_m-1}>\nu^j_{k_m-1}|\theta^i_{k'}=0,\forall k']\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}|e^i_{k_m-1}\Big]\\
&\le\sum\nolimits_{i\in s_3}\frac{\alpha^i_{k_m-1}\|\bar{A}^c_i\|^2_2\|e^i_{k_m-1}\|^2_{Q^i}}{s^j_{k_m-1}\tau\|e^i_{k_m-1}\|^2_{Q^i}}+\sum\nolimits_{i\in s_3}\frac{\alpha^i_{k_m-1}\big[\big[\|\bar{A}^c_i\|^2_2+1\big]\mathrm{tr}(Q^iV_i)+\|C_i\|^2_2\,\mathrm{tr}(Q^iW_i)\big]}{s^j_{k_m-1}\tau\|e^i_{k_m-1}\|^2_{Q^i}}\\
&\le\sum\nolimits_{i\in s_3}\frac{\alpha^i_{k_m-1}\|\bar{A}^c_i\|^2_2}{s^j_{k_m-1}\tau}+\sum\nolimits_{i\in s_3}\frac{\alpha^i_{k_m-1}\big[\big[\|\bar{A}^c_i\|^2_2+1\big]\mathrm{tr}(Q^iV_i)+\|C_i\|^2_2\,\mathrm{tr}(Q^iW_i)\big]}{s^j_{k_m-1}\tau\lambda_i},
\end{aligned} \tag{A.18}$$

where, the last line follows from $\|e^i_{k_m-1}\|^2_{Q^i}>\lambda_i$ for all $i\in s_3$. Incorporating the probability of occurrence for each scenario, and extending the interval to $[k,k_m]$, (6.14) becomes

$$\sum\nolimits_{i=1}^{N+m}\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}\Big]\le\sum\nolimits_{i\in s_1}\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}\Big]+\sum\nolimits_{i\in s_2}\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}\Big]+\sum\nolimits_{i\in s_3}\mathsf{P}_{s_3}\,\mathsf{E}\Big[\|e^i_{k_m}\|^2_{Q^i}\Big].$$

We know that LSP holds for the error state e_k over any longer interval than $[k,k+N]$. Moreover for the last scenario s_3, since (A.18) is uniformly bounded, LSP condition (6.15) will be satisfied over $[k,k_m]$ by setting $\bar{\xi}'$ greater than (A.18) and selecting $\bar{\xi}=\frac{\sum_{i\in s_3}\mathsf{P}_{s_3}\mathsf{E}\big[\|e^i_{k_m}\|^2_{Q^i}|e^i_k\big]}{\bar{\xi}'}<1$. This ensures the aggregate error state e_k possesses LSP while transiting over a finite interval $[k,k_m]$ with m collisions. Together with *Lemma* 6.2, LSP of the overall networked state $[x_k^\top\ e_k^\top]^\top$ readily follows. ☐

A.3.2 Proof of Theorem 6.3

Proof. To show LSP, let the NCS of interest operate over the interval $[k,k+N]$ of length N, starting from time-step k with arbitrary initial state e_k. We take the similar approach as in the proof of *Theorem* 6.2 and depending on whether the condition (6.35) is satisfied at every time-step k', we divide the sub-systems $i\in\{1,\dots,N\}$ into two complementary and disjoint sets as follows:

$$i\in\begin{cases}\mathcal{G}_{k'} & \|e^i_{k'}\|_2>\lambda_i,\\ \bar{\mathcal{G}}_{k'} & \|e^i_{k'}\|_2\le\lambda_i,\end{cases} \tag{A.19}$$

where $\mathcal{G}_{k'}\cup\bar{\mathcal{G}}_{k'}=N$. In accordance with the slotted ALOHA policy, if a sub-system i is eligible for transmission at some time $k'+1$, i.e. if $\|e^i_{k'}\|_2>\lambda_i$, then i selects one of the available transmission channels in uniform random fashion. If no other transmission-eligible sub-system $j\neq i$ selects that certain transmission channel, then sub-system i successfully transmits. Otherwise, a collision occurs and collided packets are dropped. The corresponding sub-systems then have to wait until the next time-step, i.e. $k'+2$, to transmit, only if the inequality (6.35) is satisfied with the updated error at time $k+1$. To address stability, we discern four complementary and mutually exclusive cases, covering the entire state space over which the error state e_k evolves until time-step $k+N-1$ as follows:

Sub-system i:

c_1: has either successfully transmitted or not within the past $N-1$ time-steps, and is in set $\bar{\mathcal{G}}_{k+N-1}$, i.e.

$$i \in \bar{\mathcal{G}}_{k+N-1} \quad \Rightarrow \quad \|e^i_{k+N-1}\|_2 \leq \lambda_i,$$

c_2: has successfully transmitted at least once within the past $N-1$ time-steps, and is in set $\mathcal{G}_{k+N-1}$, i.e.

$$\exists k' \in [k, k+N-1] : \theta^i_{k'} = 1 \text{ and } \|e^i_{k+N-1}\|_2 > \lambda_i,$$

c_3: has not successfully transmitted within the past N–1 time-steps, and is in the set $\mathcal{G}_{k+N-1}$, but has been in the set $\bar{\mathcal{G}}_{k'}$ at least once at a time-step $k' \in [k, k+N-2]$, i.e.

$$\forall k' \in [k, k+N-1] : \theta^i_{k'} = 0 \text{ and } \|e^i_{k'}\|_2 \leq \lambda_i.$$

c_4: has not successfully transmitted within the past $N-1$ time-steps, and has always been in the set $\mathcal{G}_{k'}$ for all time-steps $k' \in [k, k+N-1]$, i.e.

$$\forall k' \in [k, k+N-1] : \theta^i_{k'} = 0 \text{ and } \|e^i_{k'}\|_2 > \lambda_i.$$

Introducing the above cases, we study the boundedness of error norm expectation over the interval $[k, k+N]$ for cases c_1-c_4. Since, the cases are complementary and mutually exclusive, i.e. each sub-system belongs exactly to one of the cases c_1-c_4, we can express (6.14) as

$$\sum\nolimits_{i=1}^{N} \mathsf{E}\left[\|e^i_{k+N}\|_2^2\right] = \sum\nolimits_{i\in c_l}^{l=1,2,3,4} \mathsf{E}\left[\|e^i_{k+N}\|_2^2 | c_l\right]. \tag{A.20}$$

Suppose that some sub-systems i belong to c_1. Since $i \in \bar{\mathcal{G}}_{k+N-1}$, it follows from (A.19) that $\|e^i_{k+N-1}\| \leq \lambda_i$. Thus, those sub-systems are not eligible for transmission at time-step $k+N$, i.e. $\theta^i_{k+N} = 0$. Then, it follows from (6.7) and (A.20) that

$$\begin{aligned}
\sum_{i\in c_1} \mathsf{E}\left[\|e^i_{k+N}\|_2^2 | e_k\right] &= \sum_{i\in c_1} \mathsf{E}\left[\|A_i e^i_{k+N-1} + w^i_{k+N-1}\|_2^2 | e_k\right] \\
&\leq \sum_{i\in c_1} \|A_i\|_2^2 \mathsf{E}\left[\|e^i_{k+N-1}\|_2^2 | e_k\right] + \mathsf{E}\left[\|w^i_{k+N-1}\|_2^2\right] \\
&\leq \sum_{i\in c_1} \lambda_i^2 \|A_i\|_2^2 + \operatorname{tr}(W_i).
\end{aligned} \tag{A.21}$$

This fulfills the condition (6.15) with $\bar{\xi}' > \sum_{c_1} \lambda_i^2 \|A_i\|_2^2 + \operatorname{tr}(W_i)$, and $\bar{\xi} = \frac{\sum_{c_1} \mathsf{E}\left[\|e^i_{k+N}\|_2^2 | e_k\right]}{\bar{\xi}'} < 1$.

For some $i \in c_2$, let a successful transmission occur at time-step $k+r_i$, where $r_i \in [1, N-1]$, i.e. $\theta^i_{k+r_i} = 1$. We can express e^i_{k+N} as a function of the error at time $k+r_i-1$ as

$$\begin{aligned}
e^i_{k+N} &= \prod\nolimits_{j=r_i}^{N} \left(1 - \theta^i_{k+j}\right) A_i^{N-r_i+1} e^i_{k+r_i-1} \\
&\quad + \sum\nolimits_{r=r_i}^{N} \left[\prod\nolimits_{j=r+1}^{N} \left(1 - \theta^i_{k+j}\right) A_i^{N-r} w^i_{k+r-1}\right],
\end{aligned} \tag{A.22}$$

where we define $\prod_{N+1}^{N}(1-\theta^i_{k+j}) := 1$. The first term of the above equality vanishes as $\theta^i_{k+r_i}=1$. By statistical independence of w^i_{k+r-1} and θ^i_{k+j}, it follows from (A.22)

$$\sum_{i\in c_2}\mathsf{E}\big[\|e^i_{k+N}\|_2^2|e_k\big]=\sum_{i\in c_2}\mathsf{E}\left[\|\sum_{r=r_i'}^{N}\prod_{j=r+1}^{N}\big[1-\theta^i_{k+j}\big]A_i^{N-r}w^i_{k+r-1}\|_2^2\right]$$
$$\leq\sum_{i\in c_2}\sum_{r=r_i'}^{N}\mathsf{E}\big[\|A_i^{N-r}w^i_{k+r-1}\|_2^2\big]\leq\sum_{i\in c_2}\sum_{r=r_i'}^{N}\|A_i^{N-r}\|_2^2\,\mathrm{tr}(W_i). \tag{A.23}$$

Hence, the condition (6.15) is satisfied considering $\bar{\xi}'$ chosen to be larger than (A.4), and $\bar{\xi}=\frac{\sum_{c_2}\mathsf{E}\big[\|e^i_{k+N}\|_2^2|e_k\big]}{\bar{\xi}'}<1$. Note that we assume to have only two transmission channels per time slot in this proof, therefore if the number of sub-systems which are eligible for transmission at a specific time-step is greater than two, and one sub-system belongs to c_2, then the rest of sub-systems belong to either set c_3 or c_4. This means that, one successful transmission occurs through one of the two available channels, while the other eligible sub-systems will not successfully transmit.

For the case c_3, assume that the $k+r_i$ is the last time-step for sub-systems $i\in c_3$ that $i\in\bar{\mathcal{G}}_{k+r_i}$, which in turn implies that $\|e^i_{k+r_i}\|_2\leq\lambda_i$. Recall that the sub-systems $i\in c_3$ belong to $\mathcal{G}_{k+N-1}$. Knowing that $\theta^i_{k'}=0$ for $i\in c_3$ for all $k'\in[k,k+N-1]$, we reach

$$\sum_{i\in c_3}\mathsf{E}\big[\|e^i_{k+N}\|_2^2|e_k\big]\leq\sum_{i\in c_3}\left[\lambda_i^2\|A_i^{N-r_i}\|_2^2+\sum_{r=r_i}^{N-1}\|A_i^{N-r-1}\|_2^2\,\mathrm{tr}(W_i)\right]. \tag{A.24}$$

The condition (6.15) is met by selecting $\bar{\xi}'$ larger than the uniform upper bound (A.24), and $\bar{\xi}=\frac{\sum_{c_3}\mathsf{E}\big[\|e^i_{k+N}\|_2^2|e_k\big]}{\bar{\xi}'}<1$.

The sub-systems $i\in c_4$ have always been candidates for channel access, but they have never transmitted, which indicates that every single attempt from those sub-systems resulted in a collision. Hence, $\|e^i_{k'}\|_2>\lambda_i$ for all $k'\in[k,k+N-1]$ while $\theta^i_{k'}=0$. To show LSP in this case, we consider the worst case scenario by assuming that every attempts to access one of the two available channels results in collisions and consequently no successful transmission would happen over the entire period $[k,k+N]$. Generally, the probability that such a scenario happens for M available transmission channels can be calculated as follows:

$$\mathsf{P}_{fail}=\prod_{t=k}^{k+N}\mathsf{P}^t_{fail}, \tag{A.25}$$

where P^k_{fail} denotes the probability that all sub-systems collide in a given slot k over M available channels. Let the number of sub-systems eligible for transmission in a given slot k is denoted by $g_k=|\mathcal{G}_{k-1}|$. Thus, the probability P^k_{fail} can be derived as:

$$\mathsf{P}^k_{fail}=\frac{M^{g_k}-m_{1s}}{M^{g_k}}, \tag{A.26}$$

where M^{g_k} is the total number of possible channel selections for all transmitting sub-systems at time k, and m_{1s} is the number of outcomes with at least one successful transmission.

The probability that of one specific sub-system to succeed is given in (6.38). As it can be any of g_k sub-systems, and they can be successful with any channel, total number of such outcomes can be computed as follows:

$$g_k M \left(\frac{M-1}{M}\right)^{g_k} \tag{A.27}$$

Now, by inclusion-exclusion principle, the probabilities of two successful transmissions in the slot k are counted in (A.27) twice. There are exactly 2! ways for two success matches, and they can occur for any pair of channels and for any pair of sub-systems, resulting in

$$2!\binom{M}{2}\binom{g_k}{2}(M-2)^{g_k-2} \tag{A.28}$$

possible outcomes. We still need to subtract the number of outcomes with three successes, and so forth. Thus, at the end, we can derive m_{1s} as follows:

$$m_{1s} = \sum_{j=1}^{min(g_k,M)} (-1)^{j+1} \cdot j!\binom{M}{j}\binom{g_k}{j}(M-j)^{g_k-j}. \tag{A.29}$$

Using expression (A.29) in (A.26), the probability that all transmissions fail in one slot becomes:

$$\mathsf{P}_{fail}^k = \frac{M^{g_k} + \sum\limits_{j=1}^{min(g_k,M)} (-1)^j \cdot j!\binom{M}{j}\binom{g_k}{j}(M-j)^{g_k-j}}{M^{g_k}}, \tag{A.30}$$

Note that for any given slot, maximum number of eligible for transmission sub-systems is at most N, thus, $g_k \leq N$. Therefore, we can derive the upper bound on the P_{fail} as:

$$\mathsf{P}_{fail} \leq (\mathsf{P}_{fail}^k)^N, \tag{A.31}$$

with:

$$\mathsf{P}_{fail}^k \leq \frac{M^N + \sum\limits_{j=1}^{min(N,M)} (-1)^j \cdot j!\binom{M}{j}\binom{N}{j}(M-j)^{N-j}}{M^N}. \tag{A.32}$$

From (A.22), if no sub-system transmits over the N-step horizon, we can choose $\bar{\xi}' = \sum_{i=1}^N \|A_i^N e_k^i + \sum_{r=1}^N A_i^{N-r} w_{k+r-1}^i\|_2^2 > 0$, which implies

$$\sup_{e_k} \mathsf{P}\left[\sum_{i=1}^N \|e_{k+N}^i\|_2^2 \geq \bar{\xi}'\right] < \mathsf{P}_{fail}, \tag{A.33}$$

for an arbitrary $\rho(\bar{\xi}', \bar{\xi})$, and LSP of the overall NCS then readily follows according to (6.15). □

A.4 Appendix to Chapter 7

A.4.1 Proof of Theorem 7.1

Proof. According to the earlier discussions about the node layers under the DAG interconnection structure, we study convergence of the error state e_k^i for each sub-system considering only its local dynamics, i.e. we look at the first two terms in (7.13):

$$e_{k+N}^{i,\text{local}} = \prod_{\alpha=r_i+1}^{N} \left(1-\delta_{k+\alpha}^i\right) A_i^{N-r_i} e_{k+r_i}^i + \sum_{r=r_i}^{N-1} \left[\prod_{\alpha=r+2}^{N} \left(1-\delta_{k+\alpha}^i\right) A_i^{N-r-1} w_{k+r}^i \right]. \tag{A.34}$$

We follow a similar approach as in Chapter 5 by assuming that, over an interval with length N, each sub-system has operated from time-step k to $k+N-1$ utilizing the policy (7.8). Then, the last time $k+N$ is scheduled considering all possible scenarios which might have happened over the entire state-space during the transitional interval $[k, k+N-1]$. Note that, according to (7.8), the scheduling unit receives true error values from each node and decides the priorities considering the interconnections as well. Therefore, the error values in (7.8) are given by the expression (7.13). We define at every time-step $k' \in [k, k+N]$, two time-varying disjoint sets $S_{k'}^1$ and $S_{k'}^2$, such that for every node $i \in \{1, \dots, N\}$

$$i \in \begin{cases} S_{k'}^1 & \text{if } \|e_{k'}^i\|_{Q^i}^2 \leq \lambda_i \\ S_{k'}^2 & \text{if } \|e_{k'}^i\|_{Q^i}^2 > \lambda_i \end{cases}, \tag{A.35}$$

with $S_{k'}^1 \cup S_{k'}^2 = N$. Eligible nodes to transmit at time $k'+1$ are included in $S_{k'}^2$, while $S_{k'}^1$ contains the excluded nodes from channel access competition. In order to schedule the final time-step $k+N$, we discern three complementary and disjoint cases for a node i which is assumed to have evolved until time-step $k+N-1$, as follows:

c_1: Node i has either transmitted or not within the past $N-1$ steps, and $i \in S_{k+N-1}^1$, i.e.

$$i \in S_{k+N-1}^1 \quad \Rightarrow \quad \|e_{k+N-1}^i\|_{Q^i}^2 \leq \lambda_i,$$

c_2: Node i has transmitted at least once within the past $N-1$ steps, and $i \in S_{k+N-1}^2$, i.e.

$$\exists k' \in [k, k+N-1] : \delta_{k'}^i = 1 \text{ and } \|e_{k+N-1}^i\|_{Q^i}^2 > \lambda_i,$$

c_3: Node i has not transmitted within the past $N-1$ time-steps, and is in set S_{k+N-1}^2, i.e.

$$\forall k' \in [k, k+N-1] : \delta_{k'}^i = 0 \text{ and } \|e_{k+N-1}^i\|_{Q^i}^2 > \lambda_i.$$

Each node is characterized by exactly one of the above cases during transition from k to $k+N-1$, thus the cardinality of the union of sub-systems belonging to c_1, c_2 and c_3 equals N. We similarly apply *Theorem* 3.1 to the cases $c_1 - c_3$ employing the drift (7.12) and Lyapunov function (7.11). To show f-ergodicity of only the local part of the error state e_k^i, the N-step drift (7.12) can be split into partial drifts as follows:

$$\Delta V(e_k^{i \in c_l}, N) = \sum_{i \in c_l} \mathsf{E}\left[\|e_{k+N}^{i,\text{local}}\|_{Q^i}^2 | e_k \right] - V(e_k^{c_l}), \tag{A.36}$$

where $V(e_k^{c_l}) = \sum_{i\in c_l} \|e_k^i\|^2_{Q^i}$, for $\forall i \in c_l$ and $l \in \{1,2,3\}$.

For a sub-system i belonging to the case c_1, we know $\|e^i_{k+N-1}\|^2_{Q^i} \leq \lambda_i$. This ensures that $\delta^i_{k+N} = 0$ for $\forall i \in c_1$. Setting $r_i = N-1$ in (A.34), yields

$$\begin{aligned}
\sum_{i\in c_1} \mathsf{E}\left[\|e^{i,\text{local}}_{k+N}\|^2_{Q^i}|e_k\right] &= \sum_{i\in c_1} \mathsf{E}\left[\|A_i e^i_{k+N-1} + w^i_{k+N-1}\|^2_{Q^i}|e_k\right] \\
&\leq \sum_{i\in c_1} \|A_i\|^2_2 \mathsf{E}\left[\|e^i_{k+N-1}\|^2_{Q^i}|e_k\right] + \mathsf{E}\left[\|w^i_{k+N-1}\|^2_{Q^i}\right] \\
&\leq \sum_{i\in c_1} \|A_i\|^2_2 \lambda_i + \text{tr}(Q^i W_i). \qquad \text{(A.37)}
\end{aligned}$$

It should be noticed that in the local error dynamics (A.34), the prior error state $e^i_{k+r_i}$ contains the interconnection effects as is shown in (7.13). That is why in obtaining (A.37), we are allowed to exploit the inequality $\|e^i_{k+N-1}\|^2_{Q^i} \leq \lambda_i$, which is comparing the actual error value, not the local value of the error state, with the given error threshold. Having (A.37), the following uniform upper-bound for the partial drift (A.36) can be derived:

$$\Delta V(e_k^{i\in c_1}, N) \leq \sum_{i\in c_1} \|A_i\|^2_2 \lambda_i + \text{tr}(Q^i W_i) - V(e_k^{c_1}).$$

Defining $f_{c_1}(e_k) = \epsilon_1 V(e_k^{c_1}) - \xi_1^{b^+}$, where $\xi_1^{b^+} = \sum_{i\in c_1} \|A_i\|^2_2 \lambda_i + \text{tr}(Q^i W_i)$, and $\epsilon_1 \in (0,1]$, we can find a small set $\mathscr{D}_1$ and constant ϵ_1 such that $f_{c_1} \geq 1$ and $\Delta V(e_k^{c_1}, N) \leq -f_{c_1}$ for $e_k^{c_1} \notin \mathscr{D}_1$.

For a sub-system $i \in c_2$, we assume that the last transmission has taken place at time-step $k + r_i$, i.e. $\delta^i_{k+r_i} = 1$. Considering the statistical independence of the system noise sequence w^i_{k+r} and error state $e^i_{k+r_i-1}$, it follows from (A.34) that

$$\begin{aligned}
\sum_{i\in c_2} \mathsf{E}\left[\|e^{i,\text{local}}_{k+N}\|^2_{Q^i}|e_k\right] &= \sum_{i\in c_2}\sum_{r=r_i}^{N-1} \mathsf{E}\left[\|A_i^{N-r-1} w^i_{k+r}\|^2_{Q^i}\right] \\
&\leq \sum_{i\in c_2}\sum_{r=r_i}^{N-1} \|A_i^{N-r-1}\|^2_2 \,\text{tr}(Q^i W_i). \qquad \text{(A.38)}
\end{aligned}$$

From (A.36), the following upper-bound for the partial drift can be obtained

$$\Delta V(e_k^{c_2}, N) \leq \sum_{i\in c_2}\sum_{r=r_i}^{N-1} \|A_i^{N-r-1}\|^2_2 \,\text{tr}(Q^i W_i) - V(e_k^{c_2}).$$

Similar to the case c_1, define $f_{c_2}(e_k) = \epsilon_2 V(e_k^{c_2}) - \xi_2^{b^+}$, where $\xi_2^{b^+} = \sum_{i\in c_2}\sum_{r=r_i}^{N-1} \|A_i^{N-r-1}\|^2_2 \,\text{tr}(Q^i W_i)$, and $\epsilon_2 \in (0,1]$. We can then find a small set $\mathscr{D}_2 \subset \mathbb{R}^n$ and constant ϵ_2 such that $f_{c_2} \geq 1$ and $\Delta V(e_k^{c_2}, N) \leq -f_{c_2}$, for $e_k^{c_2} \notin \mathscr{D}_2$.

The sub-systems i belonging to the set c_3 are eligible for channel access at the final time-step $k + N$ because $i \in S^2_{k+N-1}$. To infer f-ergodicity, we split the case c_3 into two complementary and disjoint sub-cases as follows:

$l_1^{c_3}$ node i has not transmitted within the past $N-1$ time-steps, but has been in the set S^1 at least once, latest at a time-step $k+r_i \in [k, k+N-2]$,

$l_2^{c_3}$ node i has not transmitted within the past $N-1$ time-steps, and has been in S^2 for all time-steps $[k, k+N-1]$.

A sub-system $i \in l_1^{c_3}$, has been in the set $S^1_{k+r_i}$ for the last time, which implies $\|e^i_{k+r_i}\|^2_{Q^i} \leq \lambda_i$. Knowing that for all $i \in l_1^{c_3}$, we have $\delta^i_{\bar{k}} = 0$ for $\bar{k} \in [k, k+N-1]$, it follows that

$$\sum_{i\in l_1^{c_3}} \mathsf{E}\Big[\|e^{i,\text{local}}_{k+N}\|^2_{Q^i} | e_k\Big] \leq \sum_{i\in l_1^{c_3}} \left[\|A_i^{N-r_i}\|_2^2 \lambda_i + \sum_{r=r_i}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)\right]. \tag{A.39}$$

Define $f_{l_1^{c_3}}(e_k) = \epsilon_{l_1^{c_3}} V(e_k^{l_1^{c_3}}) - \xi^{b^+}_{l_1^{c_3}}$, where $\xi^{b^+}_{l_1^{c_3}} = \sum_{i\in l_1^{c_3}} \Big[\|A_i^{N-r_i}\|_2^2 \lambda_i + \sum_{r=r_i}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)\Big]$, with $\epsilon_{l_1^{c_3}} \in (0,1]$. Thus, we can find a small set $\mathscr{D}_{l_1^{c_3}}$ and $\epsilon_{l_1^{c_3}}$ such that $f_{l_1^{c_3}} \geq 1$, and $\Delta V(e_k^{l_1^{c_3}}, N) \leq -f_{l_1^{c_3}}$, for $e_k^{l_1^{c_3}} \notin \mathscr{D}_{l_1^{c_3}}$.

For sub-systems belonging to the sub-case $l_2^{c_3}$, we know $\|e^i_{\bar{k}}\|^2_{Q^i} > \lambda_i$ for all time-steps $\bar{k} \in [k, k+N-1]$. From (A.34), with setting $r_i = 0$, we have

$$\begin{aligned}
\sum_{i\in l_2^{c_3}} \mathsf{E}\Big[\|e^{i,\text{local}}_{k+N}\|^2_{Q^i} | e_k\Big] &\leq \sum_{i\in l_2^{c_3}} \mathsf{E}\left[\|A_i^N e_k^i + \sum_{r=0}^{N-1} \big[A_i^{N-r-1} w^i_{k+r}\big]\|^2_{Q^i} | e_k\right] \\
&\leq \sum_{i\in l_2^{c_3}} \|A_i^N\|_2^2 \|e_k^i\|^2_{Q^i} + \mathsf{E}\left[\|\sum_{r=0}^{N-1} \big[A_i^{N-r-1} w^i_{k+r}\big]\|^2_{Q^i}\right] \\
&\leq \sum_{i\in l_2^{c_3}} \|A_i^N\|_2^2 V(e_k^{l_2^{c_3}}) + \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)
\end{aligned} \tag{A.40}$$

Expression (A.40) depends on e_k^i via the term $V(e_k^{l_2^{c_3}})$, thus it is not uniformly upper bounded for arbitrary initial values. As the considered cases cannot happen all together, we calculate the probability that a sub-system $i \in l_2^{c_3}$ does not transmit at the final time $k+N$, and instead another sub-system, which has had at least one prior transmission, re-transmits. Recall that the length of the interval equals N. Thus, if a node, say i, does not transmit for all time-steps $\bar{k} \in [k, k+N]$, then inevitably another node, say j, transmits more than once. Let $k+r_j$ denote the latest time-step at which node j transmitted, i.e. $\delta^j_{k+r_j} = 1$. Inevitably, node j which is qualified for transmission at time-step $k+N$ belongs to the set c_2. Thus, the probability that $j \in c_2$ re-transmits at $k+N$, in the presence of node $i \in l_2^{c_3}$ is

$$\begin{aligned}
&\mathsf{P}[\delta^j_{k+N} = 1 | \delta^j_{k+r_j} = 1, \delta^i_{\bar{k}} = 0, \|e^i_{\bar{k}}\|^2_{Q^i} > \lambda_i] \\
&\quad = \mathsf{E}\Big[\mathsf{P}[\delta^j_{k+N} = 1 | e_k] \Big| \delta^j_{k+r_j} = 1, \delta^i_{\bar{k}} = 0, \|e^i_{\bar{k}}\|^2_{Q^i} > \lambda_i\Big] \\
&\quad = \mathsf{E}\left[\frac{\|e^j_{k+N-1}\|^2_{Q_j}}{\sum_{i\in S^2} \|e^i_{k+N-1}\|^2_{Q^i}} \Big| \delta^j_{k+r_j} = 1, \delta^i_{\bar{k}} = 0, \|e^i_{\bar{k}}\|^2_{Q^i} > \lambda_i\right].
\end{aligned}$$

So far, f-ergodicity of the error states for sub-systems belonging to the cases c_1, c_2, and sub-case $l_1^{c_3}$ is shown without calculating the probability of occurrence of those cases. For sub-systems included in the sub-case $l_2^{c_3}$ however, f-ergodicity cannot be shown without incorporating the probability of occurrence. Remember that the scheduler assigns the priorities based on true error values (7.13) and not the local values (A.34). Although we study stability of overall networked system by separately looking at each node, the interconnections affect the chance of transmission for a special node. Consider the worst-case scenario which entails $\|e_{k'}^{l_2^{c_3}}\|_{Q_i}^2 \leq \|e_{k'+1}^{l_2^{c_3}}\|_{Q_i}^2$, for all $i \in l_2^{c_3}$. Recalling that for nodes $q \in c_1$, we have $\delta_{k+N}^q = 0$, we reach

$$
\begin{aligned}
\mathsf{P}_{l_2^{c_3}} &= \mathsf{P}[\delta_{k+N}^j = 1 | \delta_{k+r_j}^j = 1, \delta_{\bar{k}}^i = 0, \|e_{\bar{k}}^i\|_{Q^i}^2 > \lambda_i] \\
&= \mathsf{E}\left[\frac{\|e_{k+N-1}^j\|_{Q_j}^2}{\|e_{k+N-1}^j\|_{Q_j}^2 + \sum_{i\in\{c_2,c_3\}}^{i\neq j} \|e_{k+N-1}^i\|_{Q^i}^2} \Big| \delta_{k+r_j}^j = 1, \delta_{\bar{k}}^i = 0, \|e_{\bar{k}}^i\|_{Q^i}^2 > \lambda_i \right] \\
&\leq \mathsf{E}\left[\frac{\|e_{k+N-1}^j\|_{Q_j}^2}{\|e_{k+N-1}^j\|_{Q_j}^2 + \sum_{i\in c_2}^{i\neq j} \lambda_i + \sum_{i\in l_1^{c_3}} \lambda_i + \sum_{i\in l_2^{c_3}} \lambda_i} \Big| \delta_{k+r_j}^j = 1, \delta_{\bar{k}}^i = 0, \|e_{\bar{k}}^i\|_{Q^i}^2 > \lambda_i \right].
\end{aligned}
$$

Due to the linearity of control loops and the error dynamics (7.9) and since $\|e_k^j\|_{Q^j}$ is finite for all nodes $j \in \{1, \dots, N\}$, we conclude $\|e_{k+N-1}^j\|_{Q_j}^2$ is finite. Thus, by selecting appropriate λ_i's and Q^i's we can theoretically make the latest inequality arbitrarily close to zero.

Intuitively, the probability of subsequent transmissions for a certain node, in presence of nodes with large errors and without prior transmissions, can be made arbitrarily close to zero by tuning λ_i's and Q^i's. In fact, by increasing λ_j for $j \notin l_2^{c_3}$, more of them are left out of channel competition in favor of nodes in $l_2^{c_3}$. The N-step drift (7.12) can be rewritten in terms of partial drifts incorporating their occurrence probabilities as

$$
\Delta V(e_k^{c_l}, N) = \sum_{c_l} \mathsf{P}_{c_l} \mathsf{E}\left[\|e_{k+N}^{i,\text{local}}\|_{Q^i}^2 \right] - V(e_k^{c_l}), \tag{A.41}
$$

where, P_{c_l} is the occurrence probability of a case c_l, such that $\sum_{c_l} \mathsf{P}_{c_l} = 1$ for $l \in \{1, 2, 3\}$. Therefore, the overall drift can be expressed as

$$
\begin{aligned}
\Delta V(e_k, N) &\leq \Delta V(e_k^{i\in c_1}, N) + \Delta V(e_k^{i\in c_2}, N) + \Delta V(e_k^{i\in l_1^{c_3}}, N) + \sum_{i\in l_2^{c_3}} \mathsf{P}_{l_2^{c_3}} \mathsf{E}\left[\|e_{k+N}^{i,\text{local}}\|_{Q^i}^2 \right] - V(e_k^{l_2^{c_3}}) \\
&\leq -\left[f_{c_1} + f_{c_2} + f_{l_1^{c_3}} + f_{l_2^{c_3}} \right] = -f(e_k),
\end{aligned}
$$

where, we assume in the above inequality that $\mathsf{P}_{c_1} = \mathsf{P}_{c_2} = \mathsf{P}_{l_1^{c_3}} = 1$. Thus for the sub-case $l_2^{c_3}$, and with (A.40), we have

$$
\begin{aligned}
\Delta V(e_k^{l_2^{c_3}}, N) &= \mathsf{P}_{l_2^{c_3}} \left[\sum_{i\in l_2^{c_3}} \mathsf{E}\left[\|e_{k+N}^{i,\text{local}}\|_{Q^i}^2 | e_k \right] \right] - V(e_k^{l_2^{c_3}}) \\
&\leq \left[\mathsf{P}_{l_2^{c_3}} \sum_{i\in l_2^{c_3}} \|A_i^N\|_2^2 - 1 \right] V(e_k^{l_2^{c_3}}) + \mathsf{P}_{l_2^{c_3}} \sum_{i\in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i).
\end{aligned}
$$

Define $f_{l_2^{c_3}} = \epsilon_{l_2^{c_3}} V(e_k^{l_2^{c_3}}) - \mathsf{P}_{l_2^{c_3}} \sum_{i \in l_2^{c_3}} \sum_{r=0}^{N-1} \|A_i^{N-r-1}\|_2^2 \operatorname{tr}(Q^i W_i)$, with $\epsilon_{l_2^{c_3}} \in (0,1]$. We can then find small set $\mathscr{D}_{l_2^{c_3}}$ and $\epsilon_{l_2^{c_3}}$ such that $f_{l_2^{c_3}} \geq 1$, and $\Delta V(e_k^{l_2^{c_3}}, N) \leq -f_{l_2^{c_3}}$, for $e_k^{l_2^{c_3}} \notin \mathscr{D}_{l_2^{c_3}}$.

We have shown that the conditions of *Theorem* 3.1 hold separately for each case c_l. As the cases are complementary and disjoint, we define the small set $\mathscr{D}_f \subset \mathbb{R}^n$ and $\epsilon_f \in (0,1]$ such that $f(e_k) \geq 1$, and $\Delta V(e_k, N) \leq -f(e_k)$, for $e_k \notin \mathscr{D}_f$. This confirms that *Theorem* 3.1 holds for the overall drift (A.41), which proves the Markov chain (A.34) is f-ergodic. This yields that the random error values for each sub-system are selected from bounded variance distributions. This proves the f-ergodicity of local error state (A.34) and consequently the f-ergodicity of the cascade system represented in (7.13). $\square$

A.5 Matrix Algebra

In this appendix, we will briefly review some of the basic concepts and principles in matrix algebra which are extensively used throughout this dissertation. We remind some of the crucial matrix algebraic operations, and further we will have an overview of norm operators. The following presentations can be found in [125–127], or in similar reference textbooks.

A.5.1 Matrix principles and Operations

In the following definitions, unless otherwise stated, a $m \times n$ matrix A, consisting of m rows and n columns of real-valued elements, is represented by $A_{m\times n}$. If $m = 1$, then A is called a *row vector*, and if $n = 1$, A is a *column vector*.

Definition A.1 (Matrix Transpose)**.** Transpose of the m-by-n matrix $A_{m\times n}$ is denoted by $A^\top$, and is a n-by-m matrix whose rows and columns are the columns and rows of the matrix $A_{m\times n}$, respectively. If a matrix A equals its transpose, i.e. $A = A^\top$, then A is called a *symmetric* matrix. It is clear that a symmetric matrix is always a square matrix.

Definition A.2 (Positive Definite (Semi-Definite) Matrix)**.** A symmetric matrix $A_{n\times n}$ is positive definite (positive semi-definite) if and only if all eigenvalues of A are positive (non-negative).

Definition A.3 (Trace Operator)**.** Trace of the square matrix $A_{n\times n}$ is the summation of the elements a_{ii}, i.e. the elements on the main diagonal. Trace operator is not defined for non-square matrices.

Trace operator is an invariant linear mapping with the following properties:

1. $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
2. $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$, for constant and scalar c.
3. $\mathrm{tr}(A) = \mathrm{tr}(A^\top)$.
4. $\mathrm{tr}(A_{n\times n}) = \sum_{i=1}^{n} \lambda_i$, where λ_i's are the eigenvalues (real and complex) of matrix A.
5. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, for $A_{m\times n}$ and $B_{n\times m}$.
6. Trace operator is invariant under the cyclic permutations, i.e. $\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$. Generally, non-cyclic permutations are not allowed, i.e. $\mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$, for general A, B, and C matrices of appropriate dimensions.

Definition A.4 (Hurwitz Matrix)**.** A square matrix $A_{n\times n}$ is called *Hurwitz* if all eigenvalues of A have strictly negative real parts.

Definition A.5 (Matrix Power)**.** If p is a positive integer, then for $A_{n\times n}$, A^p is the p times product of matrix A, and is an n-by-n matrix, i.e. $A^p = \underbrace{A \ldots A}_{p-\text{times}}$.

A.5.2 Norm Operators

Definition A.6 (Norm Definition)**.** A norm operator T is a linear mapping from a vector space V to the non-negative real values space $\mathbb{R}^+ \cup \{0\}$, i.e. $T : V \to \mathbb{R}^+ \cup \{0\}$.

Assume u and v are two real-valued vectors with appropriate dimensions. The norm operator then admits the following properties:

1. $T(v) \geq 0$.
2. If $T(v) = 0$, then v is the zero vector, i.e. the vector with all zero elements.
3. $T(-v) = T(v)$.
4. $T(u+v) \leq T(u) + T(v)$.
5. $T(u-v) \geq |T(u) - T(v)|$, where $|\cdot|$ is the absolute value operator.
6. If c is a scalar, then $T(cv) = |c|T(v)$.

Definition A.7 (Euclidean Norm)**.** Assume a real-valued vector $v = [v_1, v_2, \ldots, v_n]^\top \in \mathbb{R}^n$. Then, the Euclidean norm of v is defined as

$$\|v\|_2 := \left(v_1^2 + v_2^2 + \ldots + v_n^2\right)^{\frac{1}{2}}.$$

This norm is also known as *2-norm*. The Euclidean norm can equivalently be expressed as

$$\|v\|_2 := \left(v^\top v\right)^{\frac{1}{2}}.$$

The general L_p-norm of the vector v is defined as follows:

$$\|v\|_p := \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}.$$

Norm operator can similarly be applied on matrices. However, there exist different definitions for norm of a matrix. In below, we introduce the most important two, so-called *induced norm* and *Frobenius norm*.

Definition A.8 (Frobenius Matrix Norm)**.** Consider a $m \times n$ matrix $A_{m\times n}$, with real-valued entries a_{ij}, for $i = \{1, \ldots, m\}$ and $j = \{1, \ldots, n\}$. Then the Frobenius (2-norm) of matrix A is defined as follows:

$$\|A\|_F := \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}.$$

Frobenius norm of matrix $A_{m\times n}$ can be equivalently stated as

$$\|A\|_F = \left(\mathrm{tr}(A^\top A)\right)^{\frac{1}{2}}.$$

Reminding the definition of singular values of a matrix, the Frobenius norm can simply be expressed as

$$\|A\|_F = \left(\sum_{l=1}^{\min\{m,n\}} \sigma_l^2 \right)^{\frac{1}{2}},$$

where, σ_l's are the singular values of matrix A.
Generally, the p-norm operator, known also as *Schatten norm*, is defined as

$$\|A\|_p = \left(\sum_{l=1}^{\min\{m,n\}} \sigma_l^p \right)^{\frac{1}{p}}.$$

Definition A.9 (Induced Matrix Norm)**.** Assume v is any vector with non-zero p-norm $\|v\|_p \neq 0$. Then the induced norm of matrix A with appropriate dimension is defined as

$$\|A\|_p = \sup_{v \neq 0} \frac{\|Av\|_p}{\|v\|_p}.$$

For the special case $p = 2$, we have

$$\|A\|_2 = \left(\lambda_{max}(A^\top A)\right)^{\frac{1}{2}} = \sigma_{\max}(A).$$

It is straightforward to conclude from the matrix norm definitions above that

$$\|A\|_2 = \sigma_{\max}(A) \leq \left(\sum_{l=1}^{\min\{m,n\}} \sigma_l^2 \right)^{\frac{1}{2}} = \|A\|_F.$$

Thus, Frobenius norm of a matrix is an upper bound for its induced norm.
In the following, we summarize the properties of matrix Frobenius norms. Assume A and B are two real-valued matrices with appropriate dimensions. Then

1. $\|A\|_2 \geq 0$.
2. If $\|A\|_2 = 0$, then A is the zero matrix.
3. $\|A+B\|_2 \leq \|A\|_2 + \|B\|_2$.
4. If A and B are square matrices, then $\|AB\|_2 \leq \|A\|_2\|B\|_2$.
5. $\|cA\|_2 = |c|\|A\|_2$ for all scalars c.

A.6 Probability Theory and Random Variables

Throughout this dissertation, we considered systems whose dynamics are affected with stochastic noise processes and random disturbances. This urges us to analyze the system behavior in probabilistic sense. In this appendix, we intend to review the basic concepts of probability spaces and then we summarize properties of random variables. The presented notions, definitions, and concepts are mainly from the textbooks [128, 129], and can also be found in similar references.

A.6.1 Probability Theory

In this part, we provide the essential definitions of measure theory and then we define probability spaces.

Definition A.10 (σ-algebra). A σ-algebra $\mathscr{A}$ defined on a set Ω is a set containing subsets of Ω including the empty set.

Definition A.11 (Measurable Space). A pair $(\Omega, \mathscr{A})$, where $\mathscr{A}$ is a σ-algebra on Ω, is called a *measurable space*.

Definition A.12 (Measure). Assume that the pair $(\Omega, \mathscr{A})$ represent a measurable space. Then, a non-negative set function $\mu : \mathscr{A} \to [0, \infty]$ is called a *measure* on the measurable space $(\Omega, \mathscr{A})$, if μ is countably additive, i.e. if $\mathscr{B} = \{B_i\}_{i=1}^{\infty} \subset \mathscr{A}$ is a countable collection of mutually disjoint sets B_i, then

$$\mu\left(\cup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu\left(B_i\right).$$

Definition A.13 (Probability Measure). A measure μ defined on the measurable space $(\Omega, \mathscr{A})$ is called a *probability measure* if $\mu(\Omega) = 1$.

Definition A.14 (Probability Space). A probability space is represented by three essential components so-called *sample space* Ω, *σ-algebra* or *event space* $\mathscr{A}$, and *probability measure* P. This three-tuple, often presented as $(\Omega, \mathscr{A}, \mathsf{P})$, are defined in the followings:

sample space Ω is a non-empty set including all possible outcomes in a process or experiment, where an outcome is defined as the output of a single run of the experiment.

σ-algebra $\mathscr{A}$ is a collection of subsets of the sample space Ω. The event space $\mathscr{A}$ contains the outcomes of the process that we expect to happen.

probability measure $\mathsf{P} : \mathscr{A} \to [0, 1]$ is a measure on $(\Omega, \mathscr{A})$, which assigns a probability in the range of $[0, 1]$ to each of the outcomes of the σ-algebra $\mathscr{A}$.

Having above definition, we can summarize the properties of probability spaces as follows:

1. The σ-algebra $\mathscr{A}$ is a subset of 2^{Ω}, i.e. $\mathscr{A} \subset 2^{\Omega}$.

2. The sample set Ω is contained in $\mathscr{A}$, i.e. $\Omega \in \mathscr{A}$.

3. $\mathsf{P}(\Omega) = 1$, and $\mathsf{P}(\emptyset) = 0$.

4. If $\mathscr{B} \in \mathscr{A}$, then $\mathscr{B}^c \in \mathscr{A}$, where c denotes the complement of set $\mathscr{B}$, i.e. $\mathscr{B}^c = \Omega \setminus \mathscr{B}$.

5. Assume $\mathscr{B}_1$ and $\mathscr{B}_2$ are two sets of events in the σ-algebra $\mathscr{A}$. If $\mathscr{B}_1 \subset \mathscr{B}_2$, then $\mathsf{P}(\mathscr{B}_1) \leq \mathsf{P}(\mathscr{B}_2)$.

Definition A.15 (Complete Probability Space)**.** A probability space $(\Omega, \mathscr{A}, \mathsf{P})$ is called *complete* if every subset of every set $\mathscr{B} \in \mathscr{A}$, with $\mathsf{P}(B) = 0$, is included in the σ-algebra $\mathscr{A}$.

A.6.2 Random variables

In this part, we present the definitions of random variables, expectation and conditional expectation, and then summarize the properties of those operators. The presented properties have been extensively used throughout this dissertation.

Definition A.16 (Random Variable)**.** Let the triplet $(\Omega, \mathscr{A}, \mathsf{P})$ represent a probability space, and $\mathscr{B}$ is a measurable space. A measurable function $x : \Omega \to \mathscr{B}$ is called a *random variable*. If $\mathscr{B} = \mathbb{R}$, then x is called a *real-valued* random variable.

Definition A.17 (Generated σ-algebra)**.** Assume x is a random variable defined on a probability space $(\Omega, \mathscr{A}, \mathsf{P})$. Then, the smallest σ-algebra on which x is measurable, can be obtained as the intersection of all σ-algebras on which x is measurable. This σ-algebra which is denoted as $\sigma(x)$, is called *generated* σ-algebra by x.

Definition A.18 (Expected Value)**.** Let x be a random variable defined on a probability space $(\Omega, \mathscr{A}, \mathsf{P})$. The expectation of x is defined as follows

$$\mathsf{E}[x] = \int_{\Omega} x(\omega)\, \mathsf{P}(d\omega).$$

Expected value is a linear operator with the following properties which are stated without proof: (in the followings, X and Y are assumed to be random variables with finite expectations.)

1. $\mathsf{E}[c] = c$, for every constant c.

2. If $X \leq Y$, then $\mathsf{E}[X] \leq \mathsf{E}[Y]$.

3. $\mathsf{E}[X + Y] = \mathsf{E}[X] + \mathsf{E}[Y]$.

4. $\mathsf{E}[XY] = \mathsf{E}[X]\,\mathsf{E}[Y]$, only if X and Y are uncorrelated, i.e. $Cov(X, Y) = 0$.

Definition A.19 (Markov's Inequality)**.** Consider x as a non-negative random variable with $\mathsf{E}[x] < \infty$. For every $a > 0$, it follows

$$\mathsf{P}(x > a) \leq \frac{\mathsf{E}[x]}{a}.$$

Definition A.20 (Jensen's Inequality)**.** Assume h is a convex function and X is a random vector. Then,

$$\mathsf{E}[h(X)] \geq h(\mathsf{E}[X]).$$

If h is concave, then

$$\mathsf{E}[h(X)] \leq h(\mathsf{E}[X]).$$

Definition A.21 (Cauchy-Schwartz Inequality)**.** Assume X and Y are two random variables with finite standard deviations σ_X and σ_Y, respectively. Then

$$Cov(X,Y) \leq \sigma_X \sigma_Y = \sqrt{Var(X)Var(Y)}.$$

Definition A.22 (Conditional Expectation)**.** Let $X : \Omega \to \mathbb{R}^n$ be a random variable defined on the probability space $(\Omega, \mathscr{A}, \mathsf{P})$ with $\mathsf{E}[|X|] < \infty$. Assume $\sigma(Y) \subset \mathscr{A}$ is the Y-generated σ-algebra on the same probability space $(\Omega, \mathscr{A}, \mathsf{P})$, where Y is another random variable with $\mathsf{E}[|Y|] < \infty$. The conditional expectation of X given Y, i.e. $\mathsf{E}[X|Y]$, is defined as the *almost surely* unique mapping which fulfills the following two conditions

1. $\mathsf{E}[X|Y]$ is $\sigma(Y)$ measurable.
2. $\int_Y E[X|Y] d\,\mathsf{P} = \int_Y X d\,\mathsf{P}$, for all $Y \in \sigma(Y)$.

Conditional expectations are also linear operators with the following properties which are expressed without proof. Assume X, Y and Z are integrable random variables, then

1. Assume $X > Y$ *almost surely*, then $\mathsf{E}[X|Z] > \mathsf{E}[Y|Z]$.
2. $\mathsf{E}[aX + bY|Z] = a\,\mathsf{E}[X|Z] + b\,\mathsf{E}[Y|Z]$, for every $a, b \in \mathbb{R}$.
3. $\mathsf{E}[\mathsf{E}[X|Z]|Z] = \mathsf{E}[X|Z]$.
4. $\mathsf{E}[X|Z] = \mathsf{E}[X]$, if X and Z are independent random variables.
5. $\mathsf{E}[\mathsf{E}[X|Y]|Z] = \mathsf{E}[X|Z] = \mathsf{E}[\mathsf{E}[X|Z]|Y]$, if Z is $\sigma(Y)$-measurable.
6. $\mathsf{E}[XZ|Y] = \mathsf{E}[X|Y]Z$, if Z is $\sigma(Y)$-measurable, and $\mathsf{E}[|XZ|] < \infty$.

A.7 Stability of Markov Processes

Throughout this dissertation, we used Markov chains as an appropriate model framework for the system evolution over uncountable state spaces. This appendix gives a brief look to the basics and principles of Markov processes evolving on uncountable state spaces. Furthermore, the notion of stability for stochastic Markovian processes over uncountable state spaces is summarized. The presentations in this appendix are mainly from the textbook [86].

A.7.1 Markov Chains

In this part, we present some essential preliminaries about Markov chains and summarize the properties of Markovian processes evolving over uncountable state spaces.

Definition A.23 (Countable vs. General State Spaces)**.** Let $\mathscr{S}$ be a state space. Then,

- $\mathscr{S}$ is called a *countable* state space, if it contains finite or countable number of elements and is discrete, with $\mathscr{A}(\mathscr{S})$ the σ-algebra of all subsets of $\mathscr{S}$.
- $\mathscr{S}$ is called a *general* state space, if $\mathscr{S}$ is assigned a countably generated σ-algebra $\mathscr{A}(\mathscr{S})$.

Definition A.24 (Transition Probability Kernel)**.** Let $\mathscr{S}$ be a general state space with countably generated σ-algebra $\mathscr{A}(\mathscr{S})$. We call $P = \{P(s,A), s \in \mathscr{S}, A \in \mathscr{A}(\mathscr{S})\}$ a transition probability kernel if the followings hold:

1. $P(\cdot,A)$ is non-negative and measurable on state space $\mathscr{S}$, for each $A \in \mathscr{A}(\mathscr{S})$,
2. $P(s,\cdot)$ is a probability measure on the σ-algebra $\mathscr{A}(\mathscr{S})$, for each $s \in \mathscr{S}$.

Theorem A.1. *Let $P = \{P(s,A), s \in \mathscr{S}, A \in \mathscr{A}(\mathscr{S})\}$ denote any transition probability kernel on general state space $\mathscr{S}$ with the σ-algebra $\mathscr{A}(\mathscr{S})$. For any initial measure μ on $\mathscr{A}(\mathscr{S})$, a stochastic process $\mathbf{\Phi} = \{\Phi_0, \Phi_1, \ldots\}$ exists on the probability space $(\Omega, \mathscr{F}, \mathsf{P}_\mu)$, where $\Omega = \prod_{i=0}^{\infty} \mathscr{S}_i$ is a product space, measurable with respect to the product σ-algebra $\mathscr{F} = \vee_{i=0}^{\infty} \mathscr{A}(\mathscr{S}_i)$, with the probability of the event $\mathbf{\Phi} \in A$ given by $\mathsf{P}_\mu(A)$, where P_μ is a probability measure on $\mathscr{F}$, such that*

$$\mathsf{P}_\mu(\Phi_0 \in A_0, \Phi_1 \in A_1, \ldots \Phi_n \in A_n) = \int_{z_0 \in A_0} \cdots \int_{z_{n-1} \in A_{n-1}} \mu(dz_0) P(z_0, dz_1) \ldots P(z_{n-1}, A_n), \tag{A.42}$$

for every measurable $A_i \subset \mathscr{S}_i$, $i = 0, 1, \ldots, n$, and for any n.

Proof. See [86], Chapter 3. □

Definition A.25 (Markov Chains on General State Spaces)**.** A stochastic process $\mathbf{\Phi} = \{\Phi_0, \Phi_1, \ldots\}$ defined on $(\Omega, \mathscr{F})$ is called *time homogeneous Markov chain* with transition probability kernel $P(s,A)$ and initial distribution μ, if the finite dimensional distributions of $\mathbf{\Phi}$ fulfill (A.42) for every positive integer n. ([86], Chapter 3).

Definition A.26 (n-step Transition Probability Kernel). Let $\mathscr{S}$ be a general state space equipped with σ-algebra $\mathscr{A}(\mathscr{S})$. Then the n-step transition probability kernel $\{P^n(s,A), s \in \mathscr{S}, A \in \mathscr{A}(\mathscr{S})\}$ is defined as

$$P^n(s,A) = \int_{\mathscr{S}} P(s,dz)P^{n-1}(z,A), \qquad s \in \mathscr{S}, A \in \mathscr{A}(\mathscr{S}), \tag{A.43}$$

where, we set,

$$P^0(s,A) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$

Proposition A.1 (Markov Property). *Let $\mathbf{\Phi} = \{\Phi_0, \Phi_1, \ldots\}$ be a Markov chain defined on $(\Omega, \mathscr{F})$ equipped with the initial measure μ. Assume $h : \Omega \to \mathbb{R}$ is a measurable and bounded function. Then for any positive integer n, the Markov property follows as*

$$\mathsf{E}_\mu \left[h\left(\Phi_{n+1}, \Phi_{n+2}, \ldots\right) | \Phi_0, \ldots, \Phi_n, \Phi_n = s \right] = \mathsf{E}_\Phi \left[h(\Phi_1, \Phi_2, \ldots) \right].$$

According to *Proposition* A.1, the Markov property implies that the evolution of a Markov state is forgetful of all its past values except the most immediate value.

Definition A.27 (Occupation Times, and Return Times). Let $\mathscr{S}$ be a general state space equipped with σ-algebra $\mathscr{A}(\mathscr{S})$. Then for any $A \in \mathscr{A}(\mathscr{S})$,

1. the occupation time, refers to the measurable function $\eta_A : \Omega \to \mathbb{Z}^+ \cup \{\infty\}$, which denotes the number of visits to set A by the Markov chain $\mathbf{\Phi} = \{\Phi_1, \Phi_2, \ldots\}$, i.e.

$$\eta_A := \sum_{n=1}^{\infty} \mathbb{I}\{\Phi_n \in A\}.$$

2. the return time, refers to the measurable function $\tau_A : \Omega \to \mathbb{Z}^+ \cup \{\infty\}$, which denotes the first return time to the set A by the Markov chain $\mathbf{\Phi} = \{\Phi_1, \Phi_2, \ldots\}$, i.e.

$$\tau_A := min\{n \geq 1 : \Phi_n \in A\}.$$

A.7.2 Stochastic Stability

Stability analysis of Markov chains evolving over uncountable state spaces requires additional care compared to the developed stability approaches for Markov chains on countable spaces. Here in this part, we briefly review the essential modifications and extensions on those results to be applicable for analyzing the behavior of Markov chains over uncountable state spaces.

Definition A.28 (φ-Irreducible Markov Chains). Let the Markov chain $\mathbf{\Phi}$ be defined on $(\mathscr{S}, \mathscr{A})$, and φ be a measure on the σ-algebra $\mathscr{A}(\mathscr{S})$. Then $\mathbf{\Phi}$ is said to be φ-irreducible, if for every $s \in \mathscr{S}$, $\varphi(A) > 0$ implies

$$\mathsf{P}_s(\tau_A < \infty) > 0.$$

The measure φ is then called the *irreducibility measure* of the Markov chain $\mathbf{\Phi}$.

The definition of φ-irreducibility concludes that the entire state space of a φ-irreducible Markov Chain can be reached via finite number of transitions, regardless of the initial state. We will see in what follows that this property plays an essential role in stability analysis of Markov chains on uncountable state spaces.

Theorem A.2 (ψ-Irreducible Markov Chains). *If a Markov chain* $\mathbf{\Phi}$ *defined on* $(\mathscr{S}, \mathscr{A})$ *is* φ*-irreducible, then a unique maximal irreducibility measure* ψ *exists on* $\mathscr{A}(\mathscr{S})$ *such that*

1. $\psi \succ \varphi$,
2. *Markov chain* $\mathbf{\Phi}$ *is* φ'*-irreducible for any other measure* φ'*, if and only if* $\psi \succ \varphi'$,
3. $\psi(A) > 0$ *implies* $\mathsf{P}_s(\tau_A < \infty) > 0$*, for every* $s \in \mathscr{S}$,
4. *If* $\psi(A) = 0$*, then* $\psi(\bar{A}) = 0$*, where* $\bar{A} := \{z : \mathsf{P}_z(\tau_A < \infty) > 0\}$.

Proof. See [86], Chapter 4. □

Definition A.29 (ν-Petite Sets). A subset$A \in \mathscr{A}(\mathscr{S})$ is called ν-petite if a non-trivial measure ν on $\mathscr{A}(\mathscr{S})$ exists such that for all $s \in A$, and $B \in \mathscr{A}(\mathscr{S})$, the sampled chain Φ_a satisfies

$$K(s, B) \geq \nu(B),$$

where $K(s, B) := \sum_0^\infty P^n(s, B)a(n)$ is the probability transition kernel of the sampled chain with sampling distribution a.

Definition A.30 (ν-Small Sets). A subset $A \in \mathscr{A}(\mathscr{S})$ of the measurable space $(\mathscr{S}, \mathscr{A})$ is called ν-small if a non-trivial measure ν on $\mathscr{A}(\mathscr{S})$ and $k > 0$ exists such that for all $s \in A$, and $B \in \mathscr{A}(\mathscr{S})$

$$P^k(s, B) \geq \nu(B).$$

Proposition A.2. *[86, 5.5.2] If a subset* $A \in \mathscr{A}(\mathscr{S})$ *is* ν*-small, then* A *is petite.*

Proposition A.3. *[86, 6.3.3] Assume* X *is a linear state-space model, then every compact subset of* X *is small.*

It can be concluded from definitions A.29 and A.30, and propositions A.2 and A.3, that every compact subset of a linear state space is petite.

Definition A.31 (Aperiodic and Strongly Aperiodic Markov Chain). Let $\mathbf{\Phi}$ be a φ-irreducible Markov chain. The Markov chain $\mathbf{\Phi}$ is called *aperiodic* if the largest common period for which a d-cycle occurs equals one. The chain $\mathbf{\Phi}$ is called *strongly aperiodic* if exists a ν-small set A such that $\nu(A) > 0$.

In what follows, we introduce three notions of stochastic stability which can be employed to analyze the asymptotic behavior of Markov chains in uncountable state spaces.

Definition A.32 (Harris Recurrence). A φ-irreducible Markov chain $\Phi = (\Phi_0, \Phi_1, \ldots)$ is called *Harris recurrent* if for every $s \in \mathscr{S}$, and every $A \in \mathscr{A}(\mathscr{S})$ with $\varphi(A) > 0$

$$\mathsf{P}_s(\tau_A < \infty) = 1.$$

Definition A.33 (Positive Harris Recurrence). Let the Markov chain $\Phi = (\Phi_0, \Phi_1, \ldots)$ evolve in some general state space $\mathscr{S}$, with the individual random variables measurable with respect to some known σ-algebra $\mathscr{A}(\mathscr{S})$. Then Φ is said to be *positive Harris recurrent* (PHR) if

1. There exists a σ-algebra measure $\nu(A) > 0$ for a set $A \in \mathscr{A}$ such that for all initial states $\Phi_0 \in \mathscr{S}$
$$P(\Phi_k \in A, k < \infty) = 1,$$
2. Φ admits a finite invariant probability measure.

Intuitively, definition 3.2 states that if a state of a PHR Markov chain leaves a subset $A \in \mathscr{A}$ with non-zero probability, then the state returns to A after finite transitions with probability one.

Proposition A.4. *Let $f \geq 1$ be a real-valued function in $\mathbb{R}^n$. A Markov chain Φ is said to be f-ergodic, if one of the followings hold:*

1. *Φ is positive Harris recurrent with the unique invariant probability measure π,*
2. *the expectation $\pi(f) := \int f(\Phi_k)\pi(d\Phi_k)$ is finite,*
3. $\lim_{k\to\infty} \|P^k(\Phi_0, .) - \pi\|_f = 0$, *for every initial value $\Phi_0 \in X$, where $\|\nu\|_f = \sup_{|g|\leq f} |\nu(g)|$.*

The following definition introduces the notion of Markov chain gradient (drift) in discrete time, with respect to a real-valued function of states.

Definition A.34 (Drift for Markov chains). Let $V : \mathbb{R}^n \to [0, +\infty)$ be a real-valued function and Φ be a Markov chain. The drift operator Δ is defined for any non-negative measurable function V as follows

$$\Delta V(\Phi_n) = \mathsf{E}[V(\Phi_{n+1})|\Phi_n] - V(\Phi_n), \;\; \Phi_n \in \mathbb{R}^n. \tag{A.44}$$

Theorem A.3 (Foster's Criterion). *Let Φ be a φ-irreducible Markov chain. If there exists a small set $A \in \mathscr{A}(\mathscr{S})$ such that for any non-negative measurable function V on $\mathscr{S}$*

$$\Delta V(\Phi) \leq -1, \qquad \Phi \in \mathscr{S} \setminus A,$$

and $\Delta V(\Phi) < \infty$ for $\Phi \in A$, Markov chain $\boldsymbol{\Phi}$ is positive Harris recurrent.